Tutorial ~~6~~

1. Find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist (By using polar coordinates).

   (a) $f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$

   (b) $f(x, y) = \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$

   (c) $f(x, y) = \frac{2x}{x^2 + x + y^2}$

   (d) $f(x, y) = \tan^{-1}\left(\frac{|x| + |y|}{x^2 + y^2}\right)$

   (e) $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

2. Show that the following functions have no limit as $(x, y) \to (0, 0)$.

   (a) $f(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$

   (b) $f(x, y) = \frac{x^4}{x^4 + y^2}$

   (c) $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$

   (d) $f(x, y) = \frac{x^2}{x^2 - y}$

   (e) $f(x, y) = \frac{xy}{|xy|}$

3. Show that the following limits do not exist.

   (a) $\lim_{(x,y) \to (0,0)} \frac{xy^2 - 1}{y - 1}$

   (b) $\lim_{(x,y) \to (1,-1)} \frac{xy + 1}{x^2 - y^2}$

4. Use (ϵ, δ) definition for the following problems. Given $f(x, y)$ and a positive ϵ, in each of the following problems, show that there exists a $\delta > 0$ such that for all (x, y), $\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - f(0, 0)| < \epsilon$.

   (a) $f(x, y) = x^2 + y^2, \epsilon = 0.01$

   (b) $f(x, y) = \frac{y}{x^2+1}, \epsilon = 0.05$

   (c) $f(x, y) = \frac{x+y}{x^2+1}, \epsilon = 0.01$

   (d) $f(x, y) = \frac{x+y}{2+\cos(x)}, \epsilon = 0.02$

5. Discuss the continuity of the following functions at $(0, 0)$.

   (a) $$f(x, y) = \begin{cases} \frac{(xy)^2}{x^3+y^3} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

   (b) $$f(x, y) = \begin{cases} \frac{e^{xy}}{x^2+1} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

6. $$f(x, y) = \begin{cases} x \sin(\frac{1}{x}) + y \sin(\frac{1}{y}) & x \neq 0,\ y \neq 0 \\ x \sin(\frac{1}{x}) & x \neq 0,\ y = 0 \\ y \sin(\frac{1}{y}) & x = 0,\ y \neq 0 \end{cases}$$

   Is the above function continuous at the origin?