Tutorial-8 Solution

Q.1 (1) $f(x,y) = e^{x^2+y^2-4x}$

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= (2x-4)\,e^{x^2+y^2-4x} = 0 \Rightarrow (2x-4)=0\\ \frac{\partial f}{\partial y} &= 2y\,e^{x^2+y^2-4x} = 0 \Rightarrow 2y=0\end{aligned}\right\}$$

$x=2,\ y=0$

The critical point is $(2,0)$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(2,0)} = \left[2e^{x^2+y^2-4x} + (2x-4)^2 e^{x^2+y^2-4x}\right]_{(2,0)}$$

$$= 2e^{-4} = \frac{2}{e^4}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(2,0)} = \left[2e^{x^2+y^2-4x} + 4y^2 e^{x^2+y^2-4x}\right]_{(2,0)}$$

$$= 2e^{-4} = \frac{2}{e^4}$$

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(2,0)} = \left[(2x-4)\cdot 2y\, e^{x^2+y^2-4x}\right]_{(2,0)} = 0$$

$$\left. f_{xx}\cdot f_{yy} - f_{xy}^2\right|_{(2,0)} = \frac{4}{e^8} > 0$$

$$\left. f_{xx}\right|_{(2,0)} = \frac{2}{e^4} > 0$$

So $f(x,y)$ has a local minimum at $(2,0)$

$$f(2,0) = \frac{1}{e^4}$$

(ii) $f(x,y) = \ln(x+y) + x^2y$

Sol^n $f_x(x,y) = 2x + \frac{1}{x+y} = 0$

$f_y(x,y) = \frac{1}{x+y} - 1 = 0 \Rightarrow \frac{1}{x+y} = 1$

$2x+1=0 \Rightarrow x = -\frac{1}{2}$

$y = 1-(-\frac{1}{2}) = \frac{3}{2}$

$(-\frac{1}{2}, \frac{3}{2})$ is the critical point.

$f_{xx}\Big|_{(-\frac{1}{2},\frac{3}{2})} = 2 - \frac{1}{(x+y)^2}\Big]_{(-\frac{1}{2},\frac{3}{2})} = 2 - \frac{1}{1} = 2-1 = 1$

$f_{yy}\Big|_{(-\frac{1}{2},\frac{3}{2})} = \frac{-1}{(x+y)^2}\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1$

$f_{xy}\Big|_{(-\frac{1}{2},\frac{3}{2})} = \frac{-1}{(x+y)^2}\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1$

$f_{xx}f_{yy} - (f_{xy})^2\Big|_{(-\frac{1}{2},\frac{3}{2})} = -1-1 = -2 < 0$

So $(-\frac{1}{2}, \frac{3}{2})$ is a saddle point of $f(x,y)$.

(iii) $f(x,y) = 1 - \sqrt[3]{x^2+y^2}$

Ans $f_x(x,y) = \frac{-2x}{3(x^2+y^2)^{2/3}} = 0$

$f_y(x,y) = \frac{-2y}{3(x^2+y^2)^{2/3}} = 0$ } No solution to the system.

However we must also consider where the partial derivatives are undefined. This occurs when $x=0, y=0$.

So critical point is $(0,0)$.

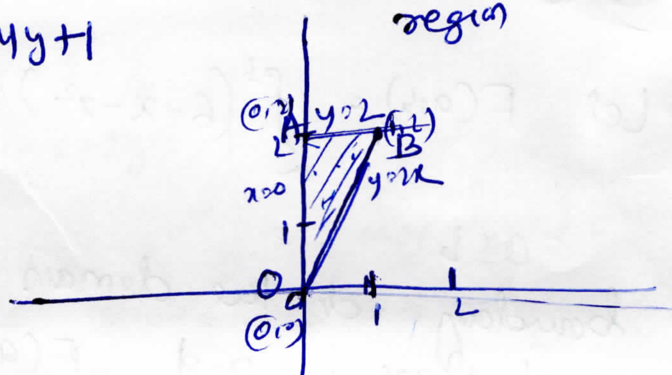
We can not use 2nd derivative test on the partial derivatives are not defined at $(0,0)$.

See $f(0,0) = 1$

$f(x,y) = 1 - \sqrt[3]{x^2+y^2} \leq 1$ for all (x,y)

$\Rightarrow (0,0)$ is a local maximum.

$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$



(i) <u>On OA</u>, $f(x,y) = f(0,y) = y^2 - 4y + 1$ on $0 \le y \le 2$

$f'(0,y) = 2y - 4 = 0 \Rightarrow y = 2$

$f(0,0) = 1$, $f(0,2) = -3$

(ii) <u>On AB</u>, $f(x,y) = f(x,2) = 2x^2 - 4x - 3$ on $0 \le x \le 1$

$f'(x,2) = 4x - 4 = 0 \Rightarrow x = 1$

$f(0,2) = -3$ and $f(1,2) = -5$

(iii) <u>On OB</u>, $f(x,y) = f(x,2x) = 6x^2 - 12x + 1$ on $0 \le x \le 1$

end points $(0,0)$, $(1,2)$

$f(0,0) = 1$, $f(1,2) = -5$

(iv) <u>For interior points</u>

$$\left.\begin{aligned} f_x(x,y) &= 4x - 4 = 0 \\ f_y(x,y) &= 2y - 4 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

$(1,2)$ is not an interior point.

So for (i), (ii), (iii), (iv), absolute maximum is 1 at $(0,0)$. and absolute minimum is -5 at $(1,2)$.

Q.3 Sol^n Let $F(a,b) = \int_a^b (6-x-x^2)\,dx$

The boundary of the domain of F is the line $a=b$ ($a \le b$) in the ab-plane, and $F(a,a)=0$.

So F is identically 0 on the boundary of its domain.

For interior critical points we have:

$$\frac{\partial F}{\partial a} = -(6-a-a^2)=0 \Rightarrow a = -3, 2$$

$$\frac{\partial F}{\partial b} = (6-b-b^2)=0 \Rightarrow b = -3, 2.$$

Since $a \le b$, there is only one interior critical point $(-3,2)$.

$$F_{aa}\Big|_{(-3,2)} = -(-1-2a)\Big|_{(-3,2)} = -(-1-2(-3)) = -5 < 0$$

$$F_{bb}\Big|_{(-3,2)} = (-1-2b)\Big|_{(-3,2)} = -1-2\cdot 2 = -5$$

$$F_{ab}\Big|_{(-3,2)} = 0$$

$$F_{aa}F_{bb} - (F_{ab})^2\Big|_{(-3,2)} = 25 > 0$$

at $F_{aa}\Big|_{(-3,2)} = -5 < 0$

So $(-3,2)$ is a point of local maximum.

So at $a=-3$, $b=2$ the funct $\int_a^b (6-x-x^2)\,dx$ has its largest value.

Solution

$T(x,y) = x^2+2y^2-x$

$$\left.\begin{array}{l} T_x = 2x-1 = 0 \\ T_y = 4y = 0 \end{array}\right\} \Rightarrow y=0,\ x=\frac{1}{2}$$

$(\frac{1}{2}, 0)$ is interior point.

$T(\frac{1}{2}, 0) = -\frac{1}{4}$

On the boundary $x^2+y^2=1$

$T(x,y) = x^2+2(1-x^2)-x = -x^2-x+2$ for $-1 \le x \le 1$

$\Rightarrow T'(x,y) = -2x-1=0 \Rightarrow x=-\frac{1}{2},\ y=\pm\frac{\sqrt{3}}{2}$

$T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{9}{4}$

$T(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{9}{4}$

$x=-1, y=0$

And $T(-1,0) = 1-(-1) = 2$

$x=1, y=0$, $T(1,0) = 1-1=0$

So the hottest is $\frac{9}{4}°$ at $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and at $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

the coldest is $-\frac{1}{4}°$ at $(\frac{1}{2}, 0)$.

1. 1

~~Find all critical points for the function $f(x,y) = x^2 - 6xy^2 + y^4$ and classify each as yielding a relative maxima, a relative~~

5. Verify that $(0,0)$ is the only critical point of the function $f(x,y) = x^2 - 6xy^2 + y^4$ and that the Hessian or discriminant $f_{xx} f_{yy} - f_{xy}^2 = 0$ at this critical point. Show algebraically that the critical point gives a saddle point.

Sol^n

The value of the function at $(0,0)$ is $f(0,0) = 0$. There is a relative minimum at $(0,0)$ if $f(x,y) \geq 0$ in some disc around $(0,0)$, there is a relative maximum if $f(x,y) \leq 0$ in some such disc.

- We first note that the values of $f(x,y) \geq 0$ along the x-axis and the y-axis away from the origin.
- On the parabola $x = y^2$, values of the function are

$$f(x,y) = (y^2)^2 - 6(y^2)y^2 + y^4 = -4y^4 \leq 0$$

Thus every open disc centered at $(0,0)$ contains points (x,y) where $f(x,y) \geq 0$ and contains points (x,y) where $f(x,y) \leq 0$. Thus $(0,0)$ yields a saddle point.

6.

Using Taylor's formula, for any $K \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \cdots - \frac{1}{2K}x^{2K} < \log(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2K+1}x^{2K+1}$$

Pf By Taylor's formula, $\exists c \in (0, x)$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

$f(x) = \log(1+x)$ $\quad f(0) = 0$ $\quad f'''(x) = \dfrac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f'(x) = \dfrac{1}{1+x}$ $\quad f'(0) = 1$ $\quad f^{iv}(x) = \dfrac{-6}{(1+x)^4}$ $\quad f^{iv}(0) = -6$.

$f''(x) = -\dfrac{1}{(1+x)^2}$ $\quad f''(0) = -1$

So

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}$$

Let $x > 0$. Then for $n = 2K$

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots - \frac{1}{2K}x^{2K} + \frac{(-1)^{2K}}{2K+1}\frac{x^{2K+1}}{(1+c)^{2K+1}}$$

As $R_{2K} = \dfrac{1}{2K+1}\dfrac{x^{2K+1}}{(1+c)^{2K+1}} > 0$ (as $x > 0 \Rightarrow c > 0$)

So $f(x) = \log(1+x) > x - \dfrac{x^2}{2} + \cdots - \dfrac{1}{2K}x^{2K}$ ——— (1)

Let

Let $x > 0$. Then for $n = 2k+1$

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{1}{2k+1}x^{2k+1} + \frac{(-1)^{2k+1}}{2k+2}\frac{x^{2k+2}}{(1+c)^{2k+2}}$$

As $R_{2k+1} = \frac{-1}{2k+2}\frac{x^{2k+2}}{(1+c)^{2k+2}} < 0$ (as $x>0$ so $c>0$)

$\Rightarrow$ ~~$f(x) = \log(1+x)$ $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{1}{2k+1}x^{2k+1}$~~

$$\Rightarrow x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{1}{2k+1}x^{2k+1}$$

$$> x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + \frac{1}{2k+1}x^{2k+1} + \frac{(-1)^{2k+1}}{2k+2}\frac{x^{2k+2}}{(1+c)^{2k+2}}$$

$$= \log(1+x). \qquad \text{— (2)}$$

from (1) and (2) it is proved.

7 The function $f(x,y) = x^2 - xy + y^2$ is approximated by a first degree Taylor's polynomial about the point $(2,3)$. Find a square $|x-2| < \delta$, $|y-3| < \sigma$ with centre at $(2,3)$ such that the error of approximation is less than or equal to 0.1. ~~in magnitude for all~~

Soln we have

$f_x = 2x - y$, $f_y = 2y - x$, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 2$.

The maximum error in the first degree approximation is $|R| \le \frac{M}{2}\left(|x-2| + |y-3|\right)^2$

where $M = \max\{|f_{xx}|, |f_{xy}|, |f_{yy}|\} = \max\{2, 1, 2\} = 2$.

we also have $|x-2| < \delta$, $|y-3| < \delta$.

Therefore we want to determine δ s.t.

$$|R| \leq \frac{2}{2}(\delta+\delta)^2 = 4\delta^2 < 0.1$$

i.e., $\delta < \sqrt{0.025}$