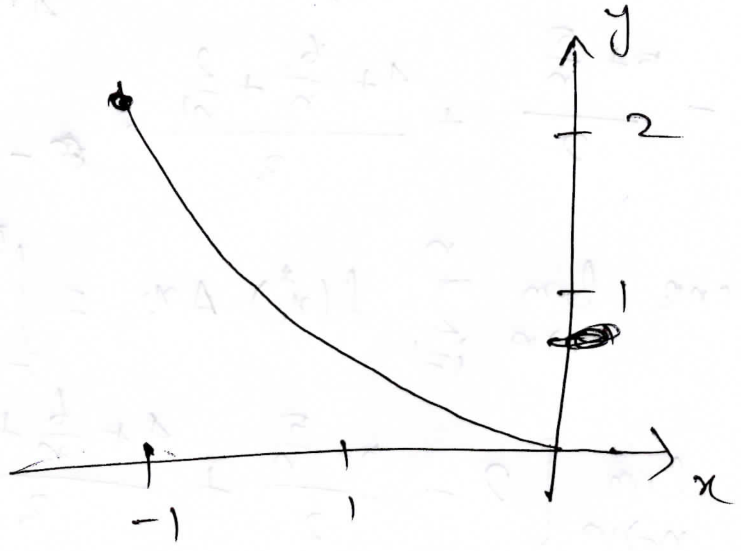
Tutorial 4

(1) $f(x) = x^2 - x^3$



Let $\Delta x = \dfrac{0-(-1)}{n} = \dfrac{1}{n}$

$x_i^* = c_i = -1 + i\,\Delta x = -1 + \dfrac{i}{n}$.

The ~~right hand~~ Riemann sum is

$$\cancel{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{n} x_i^*}$$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{n} \left(x_i^{*2} - x_i^{*3}\right)\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left[\left(-1+\frac{i}{n}\right)^2 - \left(-1+\frac{i}{n}\right)^3\right]\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left[\left(1 - \frac{2i}{n} + \frac{i^2}{n^2}\right) - \left(\frac{i^3}{n^3} - \frac{3i^2}{n^2} + \frac{3i}{n} - 1\right)\right]\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left[2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right]\frac{1}{n}$$

$$= \sum_{i=1}^{n}\frac{2}{n} - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{2}{n}(n) - \frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= 2 - \frac{5n+5}{2n} + \frac{4n^2+6n+2}{3n^2} - \frac{n^2+2n+1}{4n^2}$$ (2)

$$= 2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}$$

Hence $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \int_{-1}^{0} f(x)\,dx$

$$= \lim_{n\to\infty}\left[2 - \frac{5+\frac{5}{n}}{2} + \frac{4+\frac{6}{n}+\frac{2}{n^2}}{3} - \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}\right]$$

$$= 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}.$$

(2) $f(x) = \sqrt{x+8}$ increasing on $[0,1]$

$\Rightarrow \max f = f(1) = \sqrt{1+8} = 3$

$\min f = f(0) = \sqrt{0+8} = 2\sqrt{2}$

[$f'(x) = \frac{1}{2\sqrt{x+8}}$ ~~not defined~~ It has no critical points with $[0,1]$ So it attends its max. and min. at end points.]

Therefore $(b-a)\min f \le \int_a^b f(x)\,dx \le \max f\,(b-a)$

So $(1-0)\min f \le \int_0^1 \sqrt{x+8}\,dx \le (1-0)\max f$

$\Rightarrow 2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$

(3) $y = x\int_2^{x^2} \sin t^3\,dt$

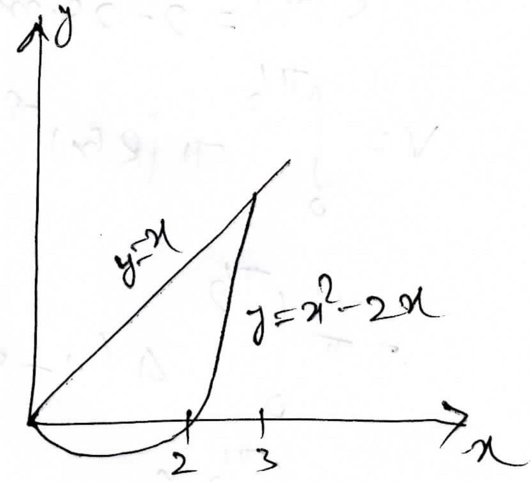
$$\Rightarrow \frac{dy}{dx} = x\cdot \underbrace{\frac{d}{dx}\left(\int_2^{x^2} \sin t^3\,dt\right)}_{\text{use fundamental th}^m} + 1\cdot\int_2^{x^2} \sin t^3\,dt$$

$$= x\cdot \sin(x^2)^3\,\frac{d}{dx}(x^2) + \int_2^{x^2} \sin t^3\,dt = 2x^2\sin x^6 + \int_2^{x^2} \sin t^3\,dt.$$

4 (i) Limits of integration

$x^2 - 2x = x \Rightarrow x^2 = 3x$

$\Rightarrow x(x-3) = 0 \Rightarrow x = 0$ and $x = 3$.

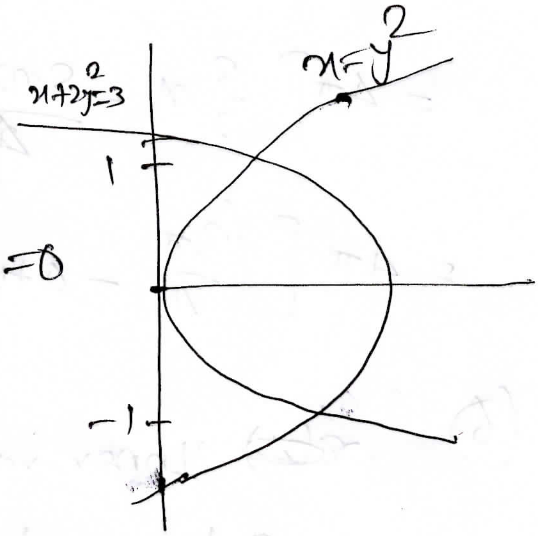


$f(x) - g(x) = x - (x^2 - 2x) = 3x - x^2$

$$\text{Area} = \int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - 9 = \frac{27-18}{2} = \frac{9}{2}.$$

(ii) Limits of integration.



$x = y^2$ and $x = 3 - 2y^2$

$\Rightarrow y^2 = 3 - 2y^2 \Rightarrow 3y^3 = 3 \Rightarrow 3(y-1)(y+1) = 0$

$\Rightarrow y = 1$ and $= -1$.

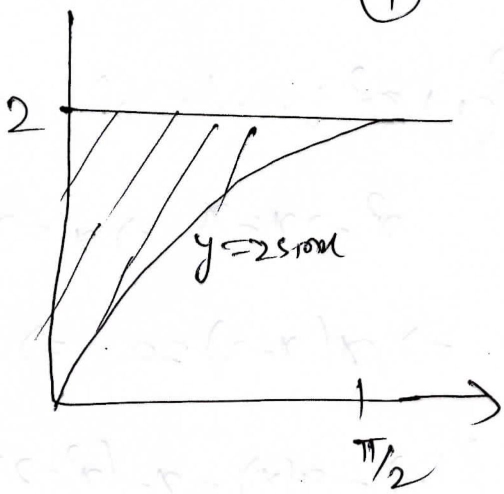
$f(y) - g(y) = (3 - 2y^2) - y^2$

$= 3 - 3y^2 = 3(1 - y^2)$

$$\text{Area} = 3\int_{-1}^{1} (1 - y^2)\,dy = 3\left[y - \frac{y^3}{3}\right]_{-1}^{1}$$

$$= 3\left(1 - \frac{1}{3}\right) - 3\left(-1 + \frac{1}{3}\right) = 3 \cdot 2\left(1 - \frac{1}{3}\right) = 4.$$

⑤ $R(x) = 2 - 2\sin x = 2(1-\sin x)$



$$V = \int_0^{\pi/2} \pi [R(x)]^2 \, dx$$

$$= \pi \int_0^{\pi/2} 4(1-\sin x)^2 \, dx$$

$$= 4\pi \int_0^{\pi/2} (1 + \sin^2 x - 2\sin x) \, dx$$

$$= 4\pi \int_0^{\pi/2} \left[1 + \frac{1-\cos 2x}{2} - 2\sin x\right] dx$$

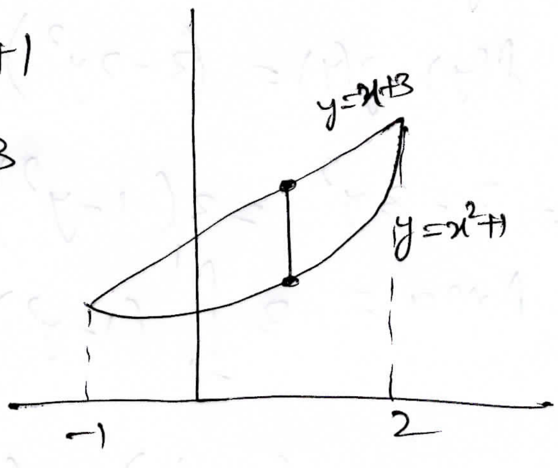
$$= 4\pi \int_0^{\pi/2} \left[\frac{3}{2} - \frac{\cos 2x}{2} - 2\sin x\right] dx$$

$$= 4\pi \left[\frac{3}{2}x - \frac{\sin 2x}{4} + 2\cos x\right]_0^{\pi/2}$$

$$= 4\pi \left[\left(\frac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 2)\right] = \pi(3\pi - 8)$$

⑥ Inner radii $r(x) = x^2 + 1$

Outer radii $R(x) = x + 3$



$$V = \int_{-1}^{1} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \pi \int_{-1}^{2} \left[(x+3)^2 - (x^2+1)^2\right] dx$$

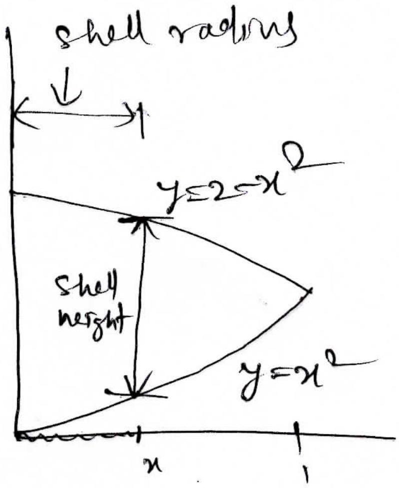
$=$

Point of intersection

$x^2 + 1 = x + 3 \Rightarrow x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\Rightarrow x = -1$ and 2



(7) Point of intersection

$2-x^2=x^2$

$\Rightarrow 2x^2=2 \Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

$x=-1$ is not considered.

$$V=\int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$=\int_0^1 2\pi x\left[(2-x^2)-x^2\right]dx$$

$$=2\pi\int_0^1 x(2-2x^2)\,dx$$

$$=4\pi\int_0^1 (x-x^3)\,dx$$