CS34110 Discrete Mathematics and Graph Theory

UNIT – IV, Module – 2

Lecture 37: Graph Coloring

[Graph coloring problem; Chromatic number; p-partite graph; Chromatic partitioning; Uniquely colorable; Coloring theorems; Four color problem]

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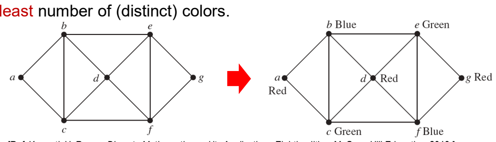
Notation table

Symbol / Notation	Meaning
$\chi(\mathcal{G})$	Chromatic number of graph $\mathcal{G}$.
$\chi'(\mathcal{G})$	Edge chromatic number of graph $\mathcal{G}$.

Graph coloring

- Graph coloring of simple graph: assignment of color to each vertex of graph, so that no two adjacent vertices assigned same color.
  - Property:: Graph coloring problem: coloring of simple graph using least number of (distinct) colors.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

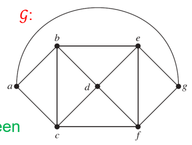
- Graph coloring:
  - Property:: Chromatic number of graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: denoted by $\chi(\mathcal{G})$; $\chi(\mathcal{G})$ = least number of (distinct) colors needed for coloring of $\mathcal{G}$.
  - Property:: Chromatic number of planar graph $\mathcal{G}$: minimum number of (distinct) colors required to color planar map so that no two adjacent regions assigned same color.
  - Property:: Steps to determine $\chi(\mathcal{G}) = k$ (say) — (i) to show that coloring of $\mathcal{G}$ possible with k (distinct) colors, by constructing such a coloring; (ii) to show that no coloring of $\mathcal{G}$ possible using fewer than k colors.

Graph coloring

- Graph coloring:
  - Property:: Chromatically k-critical graph: connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, such that $(\chi(\mathcal{G}) = k) \wedge \left(\forall e \in \mathbf{E}\left(\chi(\mathcal{G} - e) = (k-1)\right)\right)$.

Graph coloring

- Graph coloring:
  - Example-1:: Chromatic number of graph $\mathcal{G}$:



At least 4 colors required to assign different colors to neighbor vertices a, b, c, g of $\mathcal{G}$.
Assignment scenario: red to a, blue to b, green to c, brown to g.
Then, d to be colored red, as d adjacent to blue b, green c.
Then, e to be colored green, because e adjacent only to blue b, red d, brown g.

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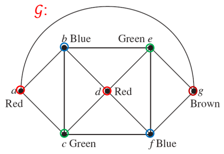
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

- Graph coloring:
  - Example-1 contd.::
    Then, f to be colored blue, because f adjacent only to green c, red d, green e, brown g.
    Completing coloring, $\chi(\mathcal{G}) = 4$.

$\mathcal{G}$:

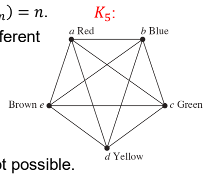


[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

- Graph coloring:
  - Example-2:: Chromatic number of K_n: $\chi(K_n) = n$.
    Coloring by n colors possible to assign different colors to vertices of K_n.
    Assigning same color to two different vertices of K_n not possible, as every vertex adjacent to every other vertex.
    So, coloring of K_n by less than n colors not possible.

K_5:

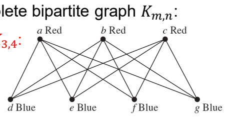


[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

- Graph coloring:
  - Example-3:: Chromatic number of complete bipartite graph $K_{m,n}$:
    $\chi(K_{m,n}) = 2$.
    In simple bipartite graph, possibility to assign one of 2 different colors to each vertex, so that no two adjacent vertices assigned same color. Less than 2 colors not possible.
    Then, coloring of set of m vertices with one color, and coloring of set of n vertices with second color, and no two adjacent vertices of same color.

$K_{3,4}$:



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

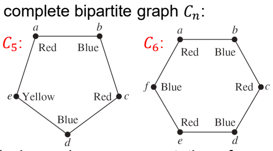
- Graph coloring:
  - Example-4:: Chromatic number of complete bipartite graph C_n:

$\chi(C_n) = \begin{cases} 2, & n \text{ even}, n \geq 4 \\ 3, & n \text{ odd}, n \geq 3. \end{cases}$

Case 1: n even —

To construct coloring, vertex v_1 to be colored by c_1.

C_5: C_6:



Proceeding around C_n clockwise (using a planar representation of C_n) and coloring vertex v_2 (adjacent to v_1) by c_2 (where $c_1 \neq c_2$).

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[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph coloring

- Graph coloring:
  - Example-4 contd.::

Coloring vertex v_3 (adjacent to v_2 but not to v_1) again by c_1.
Coloring vertex v_4 (adjacent to v_3 but not to v_2) again by c_2.
Continuing in this fashion, c_1 to become color for odd vertices and c_2 to become color for even vertices.
Then, coloring vertex v_n (adjacent to v_{n-1} and v_1, but not to v_{n-2}) again by c_2. Finally, graph coloring completed with 2 colors.
Also, not possible to color C_n by less than 2 colors.

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Graph coloring

- Graph coloring:
  - Example-4 contd-2.::

Case 2: n odd —
To construct coloring, vertex v_1 (and subsequently all odd-numbered vertices, like previous case) to be colored by c_1, as non-adjacent.
Then, vertex v_2 (adjacent to v_1 and v_3) to be colored by c_2, and so subsequently all even-numbered vertices, as non-adjacent.
But, for odd-numbered vertex v_n, coloring v_n (adjacent to even-numbered v_{n-1} and odd-numbered v_1) not possible to be colored by either c_1 or c_2.

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Graph coloring

- Graph coloring:
  - Example-4 contd-3.::

    So, third color c_3 required,

    Finally, graph coloring completed with 3 colors.

    Also, not possible to color C_n by less than 3 colors.

Graph coloring

- Graph coloring:
  - Property:: Edge coloring of graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: assignment of colors to edges of $\mathcal{G}$, so that different colors to be assigned to edges incident with common vertex.
  - Property:: Edge chromatic number of graph $\mathcal{G}$: denoted by $\chi'(\mathcal{G})$;

    $\chi'(\mathcal{G})$ = smallest number of colors used in edge coloring of $\mathcal{G}$.

Graph coloring

- Graph coloring:
  - Property:: ⟨Theorem⟩: Every tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ with two or more vertices to be 2-chromatic (or, bichromatic).

Graph coloring

- Graph coloring:
  - Property:: Not every 2-chromatic graph to become tree.
  - Property:: ⟨Theorem⟩: Graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with at least one edge to be 2-chromatic if and only if no circuits of odd length to be present in $\mathcal{G}$.

  Proof: Since at least one edge in $\mathcal{G}$, $\mathcal{G}$ not possible to be 1-chromatic.
  Two cases to prove — (A) No odd-length circuits → 2-chromatic;
  (B) 2-chromatic → No odd-length circuits.
  (A) Let $\mathcal{G}$ be connected graph with circuits of only even lengths, i.e., $\neg$(No odd−length circuits in $\mathcal{G}$).
  Consider a spanning tree $\mathcal{T}$ in $\mathcal{G}$.

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Graph coloring

- Graph coloring:

  Proof of ⟨Theorem⟩ contd.:
  Using proper coloring procedure, coloring of $\mathcal{T}$ with two colors, c_1 and c_2, possible based on earlier theorem.
  Idea: adding of chords one by one to $\mathcal{T}$, and coloring of added end vertices in such chords.
  On adding any chord h to $\mathcal{T}$, other end vertex of h, if not already added and colored in $\mathcal{T}$, to be colored by either c_1 or c_2, ensuring different color from adjacent colored vertices of $\mathcal{T}$.
  Thus, at end of adding/coloring all chords in $\mathcal{T}$, $\mathcal{T}$ to become $\mathcal{G}$.

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Graph coloring

- Graph coloring:

  Proof of ⟨Theorem⟩ contd-2.:
  Since no circuits of odd length present in $\mathcal{G}$, end vertices of every chord being added to $\mathcal{T}$, to be differently colored in $\mathcal{T}$.
  So, coloring of $\mathcal{G}$ complete with two colors, c_1 and c_2, with no adjacent vertices having same color, i.e., $\mathcal{G}$ become 2-chromatic.
  (B) Let one circuit of odd length present in $\mathcal{G}$, i.e., $\neg$(No odd−length circuits in $\mathcal{G}$). Also given that $\mathcal{G}$ 2-chromatic.
  For coloring of this odd-length circuit, at least three colors required.
  Then, by contraposition, 2-chromatic → No odd-length circuits. ■

Graph coloring

- Graph coloring:
  - Property:: ⟨Theorem⟩: In graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $\chi(\mathcal{G}) \leq 1 + \triangle(\mathcal{G})$, where $\triangle(\mathcal{G}) = \max\{\deg(v) \mid v \in \mathbf{V}\}$.

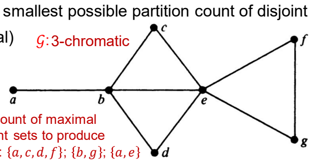
Graph coloring

- Graph coloring:
  - Property:: p-partite graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: possibility to decompose $\mathbf{V}$ into p disjoint subsets $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_p$, such that no edge $e \in \mathbf{E}$ joining vertices in same subset.
  - Property:: χ-chromatic graph to be p-partite, if and only if $\chi \leq p$.

Graph coloring

- Chromatic partitioning of graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: partitioning of vertex set $\mathbf{V}$ of $\mathcal{G}$ into p disjoint subsets $\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_p$, based on coloring.
  - Property:: Chromatic partitioning of simple, connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: partitioning of $\mathbf{V}$ into smallest possible partition count of disjoint (but need not to be maximal) independent sets.

$\mathcal{G}$: 3-chromatic

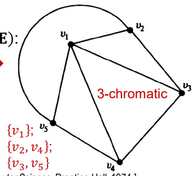


Minimum count of maximal independent sets to produce $\mathbf{V}$ on union: $\{a, c, d, f\}$; $\{b, g\}$; $\{a, e\}$

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph coloring

- Chromatic partitioning:
  - Property:: ⟨Theorem⟩: For χ-chromatic graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices, $\beta \geq \frac{n}{\chi}$ (or $\beta \cdot \chi \geq n$) where $\beta(\mathcal{G}) =$ independence number of $\mathcal{G}$.
  - Property:: Uniquely colorable graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: only one chromatic partition of $\mathcal{G}$ possible. →



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph coloring

- Four color problem:
  - Property:: ⟨Four Color Conjecture⟩: Chromatic number of planar graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be no greater than four.
    [Alternate statement: four colors to become sufficient for coloring any atlas (or, a map on a plane) such that different colors for countries with common boundaries.]

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.