CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 15: Counting

[Pigeonhole principle, proof; Generalized pigeonhole principle, proof; Properties; Practical applicability]

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Pigeonhole principle

- Pigeonhole principle: general principle, in which for more pigeons than pigeonholes, at least one pigeonhole to MUST hold at least two pigeons roosting in it.
  - Applicability: demonstrating possibly unexpected results in finite as well as infinite sets —
    (i) in any lossless compression algorithm, compressing some inputs, in turn, result in making some other inputs larger.

[Ref: https://en.wikipedia.org/wiki/Pigeonhole_principle.]

Pigeonhole principle

- Pigeonhole principle:
  - Property:: ⟨Theorem⟩ [Pigeonhole principle]: If $(k+1)$ or more objects to be placed into k boxes (where, $k \in \mathbb{Z}^+$), then at least one box to contain two or more objects.

  Proof: [Proof by contraposition] Let $\neg q$ be "no k boxes with > 1 object," where q be "at least one of k boxes with > 1 object."
  Then, "k boxes to hold at most k objects." Denoting it by $\neg p$, where p be "$\geq (k+1)$ objects into k boxes $\equiv (k+1)$ or more objects in k boxes."
  $\therefore \neg q \rightarrow \neg p$. So, $p \rightarrow q$. ■
  - Property:: Pigeonhole principle also called Dirichlet drawer principle.

Pigeonhole principle

- Pigeonhole principle:
  - Property:: ⟨Theorem⟩: Function $f\colon \mathbf{A} \to \mathbf{B}$ from set $\mathbf{A}$ with $|\mathbf{A}| = k+1$ or more elements to set $\mathbf{B}$ with $|\mathbf{B}| = k$ elements to be not one-to-one (i.e., injective).

  Proof: For each $y \in \mathbf{B}$, y to be assigned to one of boxes to hold all elements $x \in \mathbf{A}$, such that $f(x) = y$.
  Given, domain $\mathbf{A}$ to contain $k+1$ or more elements, and codomain $\mathbf{B}$ to contain only k elements.
  Then, according to pigeonhole principle, one of k boxes to contain two or more elements x of $\mathbf{A}$. Thus, f not one-to-one. ∎

Pigeonhole principle

- Pigeonhole principle:
  - Property:: ⟨Theorem⟩ [Generalized Pigeonhole principle]: For $N, k \in \mathbb{Z}^+$, if N elements to be placed into k boxes, then at least one box to hold at least $\lceil N/k \rceil$ elements. [Based on ceiling function]

  Proof: [Proof by contraposition] Let p, q be defined from above.
  Let $\neg q$ be "no box to hold more than $\lceil N/k \rceil - 1$ elements."
  So, at most total number of elements for k boxes $= k \cdot (\lceil N/k \rceil - 1) < k \cdot \left((N/k + 1) - 1\right) = N$, based on inequality theorem $\lceil N/k \rceil < (N/k + 1)$.
  Then, "k boxes to hold less than N elements;" conclusively, $\neg p$.
  $\therefore \neg q \to \neg p$. So, $p \to q$. ∎

Pigeonhole principle

- Pigeonhole principle:
  - Property:: ⟨Theorem⟩ [Generalized pigeonhole principle (alternate)]: For $k, m \in \mathbb{Z}^+$, if N $(= k \cdot m + 1)$ or more elements to be distributed among k sets, then pigeonhole principle asserting that at least one of k sets to hold at least $(m+1)$ elements.
  - Property:: ⟨Theorem⟩ [Generalized Pigeonhole principle alternate-2]: For $N, k \in \mathbb{Z}^+$, if N elements to be placed into k boxes, then at least one box to hold at least $\lfloor (N-1)/k \rfloor + 1$ elements. [Based on floor function]

Pigeonhole principle

- Pigeonhole principle examples:
  - Example-1:: as known, number of distinct birth dates = 366.
    So, in any group of 367 people, mandatorily at least two of 367 persons with same birthday, according to pigeonhole principle.
  - Example-2:: given, any exam to be marked on scale from 0 to 100 marks, i.e., scores of 101 distinct marks.
    So, for any class of students, number of students in that class to guarantee at least two students to score same marks on final exam = 101 + 1 = 102, according to pigeonhole principle.

Pigeonhole principle

- Pigeonhole principle examples:
  - Example-3:: determining minimum value of total number of objects N, such that at least r of N objects to be in one of k boxes when distributed among k boxes. [reverse of alt gen. pigeonhole principle]
    As per generalized pigeonhole principle, with N objects, at least one box to hold at least $\lceil N/_k \rceil$ objects. Then, for given problem, $r \leq \lceil N/_k \rceil$.
    $\therefore r < N/_k + 1$, based on inequality theorem. $\therefore N > k \cdot (r-1)$.
    So, minimum value of $N = k \cdot (r-1) + 1$.
    To further argue that even lesser value of N NOT possible, attempt to avoid r in all of k boxes resulting in at most $r-1$ objects in all.

Pigeonhole principle

- Pigeonhole principle examples:
  - Example-4:: Given, in discrete math class with seven possible grades A+, A, B, C, D, P, F, and requirement of at least 8 students to receive same grade in any category, to find minimum class strength.
    Given, $k = 7$, $r = 8$. So, according to Example-3,
    minimum value of class strength $N = k \cdot (r-1) + 1 = 50$.
  - Example-5:: Given, deck of 52 cards (assume shuffled), need for selecting how many N cards, so that at least 3 cards of same suit?
    Considering four boxes, one for each suit of cards, then cards chosen, and such cards to be placed in box reserved for that suit.

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Pigeonhole principle

- Pigeonhole principle examples:
  - Example-5 contd.
    Then, $k = 4$, $r = 3$. So, according to Example-3,
    minimum number of cards to be selected $N = k \cdot (r - 1) + 1 = 9$.
    [Note: If 8 cards selected → possibility of two cards of each suit.
    So, more than eight cards needed.
    Another way to think: after eighth card carefully chosen, no way to avoid having third card of some suit.]

Pigeonhole principle

- Pigeonhole principle practical applicability:
  - Property:: ⟨Theorem⟩: Among any group of $(n + 1)$ positive integers not exceeding value by $2 \cdot n$, mandatory presence of one integer in that group to be divisible by another integer in same group, for $n \in \mathbb{Z}^+$.

  Proof: Writing each of $(n + 1)$ positive integers $a_1, a_2, \ldots, a_{n+1}$ as product of power of 2 and some odd integer. In other words, for $j = 1,2, \ldots, n + 1$, $a_j = 2^{k_j} \cdot b_j$, where $k_j \in \mathbb{N}$, and b_j = odd positive integer.
  As per given condition: each of $b_1, b_2, \ldots, b_{n+1}$ to be less than $2 \cdot n$.
  Use of proven result: only n odd positive integers possible to be less than $2 \cdot n$.

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Pigeonhole principle

- Pigeonhole principle practical applicability:

  Proof of ⟨Theorem⟩ contd.
  Then, according to the pigeonhole principle, two of $b_1, b_2, \ldots, b_{n+1}$ MUST be equal. ∴ for two distinct integers i and j $(i, j = 1,2, \ldots, n + 1)$, possibility of $b_i = b_j$. Let b be common value of b_i and b_j.
  Then, $a_i = 2^{k_i} \cdot b$ and $a_j = 2^{k_j} \cdot b$.
  Now, if $k_i < k_j$, then $(2^{k_j} \cdot b) > (2^{k_i} \cdot b)$. So, a_j divisible by a_i.
  Again, if $k_i > k_j$, then $(2^{k_i} \cdot b) > (2^{k_j} \cdot b)$. So, a_i divisible by a_j.
  Combining above result to complete proof. ■

Pigeonhole principle

- Pigeonhole principle practical applicability:
  - Property:: ⟨Erdös−Szekerés theorem⟩: For $n \in \mathbb{Z}^+$, every sequence of (n^2+1) distinct real numbers to contain subsequence of length $(n+1)$ and of nature either strictly increasing or strictly decreasing. [Note: above theorem also applicable to integers.]

  Proof: Proof by contradiction.

  Let q be "no strictly increasing and strictly decreasing subsequences of length $(n+1)$."

  Let $a_1, a_2, \ldots, a_{n^2+1}$ be terms of given sequence of (n^2+1) distinct real numbers.

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Pigeonhole principle

- Pigeonhole principle practical applicability:

  Proof of ⟨Theorem⟩ contd.

  Associating ordered pair with each term of this sequence, i.e., for k-th term a_k, associating (i_k, d_k) to a_k, where i_k = length of longest strictly increasing subsequence starting at a_k, d_k = length of longest strictly decreasing subsequence also starting at a_k, $k = 1,2,\ldots,(n^2+1)$, $i_k, d_k \in \mathbb{Z}^+$.

  Assuming q to be TRUE, both $i_k \leq n$, and $d_k \leq n$.

  Applying product rule, number of possible distinct ordered pairs associated to $a_k = n^2$, i.e., n^2 distinct ordered pairs for (i_k, d_k).

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Pigeonhole principle

- Pigeonhole principle practical applicability:

  Proof of ⟨Theorem⟩ contd-2.

  Then, considering (n^2+1) ordered pairs for (i_k, d_k), two of them MUST be equal (according to the pigeonhole principle), i.e.

  $\exists a_s, a_t \big((i_s = i_t) \wedge (d_s = d_t)\big)$, where $(s < t)$, terms $a_s, a_t \in \{a_{n^2+1}\}$.

  Due to distinct terms in sequence $\{a_{n^2+1}\}$, either $a_s < a_t$, or $a_s > a_t$.

  Case $(a_s < a_t)$: Because $(a_s < a_t)$ and $(i_s = i_t)$, possibility to build strictly increasing subsequence of length $(i_t + 1)$ starting at a_s, by (step-1:) taking a_s, followed by (step-2:) strictly increasing subsequence of length i_t beginning at a_t.

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Pigeonhole principle

- Pigeonhole principle practical applicability:
  Proof of ⟨Theorem⟩ contd-3.
  So, possibility for $i_k > n$, i.e. possibility of $\neg(i_k \leq n)$.
  Case $(a_s > a_t)$: Because $(a_s > a_t)$ and $(d_s = d_t)$, possibility to build strictly decreasing subsequence of length $(d_t + 1)$ starting at a_s, by (step-1:) taking a_s, followed by (step-2) strictly decreasing subsequence of length d_t beginning at a_t.
  So, possibility for $d_k > n$, i.e. possibility of $\neg(d_k \leq n)$.
  Joining both cases, $\neg(i_k \leq n) \vee \neg(d_k \leq n) \equiv \neg\big((i_k \leq n) \wedge (d_k \leq n)\big)$
  $\equiv \neg q$ to be TRUE.

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Pigeonhole principle

- Pigeonhole principle practical applicability:
  Proof of ⟨Theorem⟩ contd-4.
  ∴ Considering q to be TRUE, established $\neg q$ to be TRUE.
  A contradiction. So, wrong assumption at start.
  Hence, possibility of strictly increasing or strictly decreasing subsequences of length $(n + 1)$. ■
  - Example:: For sequence 8,11,9,1,4,6,12,10,5,7 of 10 $(= 3^2 + 1)$ terms, Possibility of at least 1 strictly increasing or decreasing subsequence of length $(3 + 1 =)$ 4. In fact, such subsequences present: **1,4,6,12**; 1,4,6,7; 1,4,6,10; 1,4,5,7; 11,9,6,5.

Pigeonhole principle

- Pigeonhole principle practical applicability:
  - Property:: ⟨Theorem⟩: A real number to become rational, if and only if repeating decimal expansion of that real number possible.

Summary

- Focus: Pigeonhole principle.
- Pigeonhole principle in simple form, with proof.
- Simple examples of pigeonhole principle.
- Generalized pigeonhole principle theorems, with proofs.
- Properties of pigeonhole principle.
- Practical applicability of pigeonhole principle, with related theorems and proofs.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip17] Seymour Lipschutz, Marc L. Lipson, Varsha H. Patil, *Discrete Mathematics (Schaum's Outlines)*, Revised Third edition, McGraw-Hill Education, 2017.

Further Reading

- Pigeonhole principle:: [Ros21]:420-422.
- Generalized pigeonhole principle:: [Ros21]:422-424.
- Practical applicability of pigeonhole principle:: [Ros21]:424-426.

Lecture Exercises: Problem 1 [Ref: Gate 2005, Q.44]

What is the minimum number of ordered pairs of non-negative numbers that should be chosen to ensure that there are two pairs (a, b) and (c, d) in the chosen set such that, $a \equiv c \bmod 3$ and $b \equiv d \bmod 5$?

(a) 4.
(b) 6.
(c) 16.
(d) 24.

Lecture Exercises: Problem 1 Ans

- Given, ordered pairs (i.e., 2-tuples) of non-negative numbers.
- Let any such 2-tuple be $(x, y) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$, where $x \in \mathbb{N}$ and $y \in \mathbb{N}$.
- Then, as per given criteria,
  $x \bmod 3 = \{0, 1, 2\} = \mathbf{A}$ (where forming set $\mathbf{A}$ can be taken as task-1).
  $y \bmod 5 = \{0, 1, 2, 3, 4\} = \mathbf{B}$ (where forming set $\mathbf{B}$ taken as task-2).
- Since performing both tasks are required for this job, and since task-2 is independently performed in relation to task-1, so as per product rule, count of 2-tuples (x, y) with above criteria $= |x \bmod 3| \cdot |y \bmod 5| = 15$.
- Now, as per given problem, function $f: \mathbb{N}^2 \rightarrow \mathbf{A} \times \mathbf{B}$ is mapping $\aleph_0^2$ count of elements in domain to 15 number of elements in codomain.
- Then, pigeon: elements of $\mathbb{N}^2$, pigeonhole: elements of $\mathbf{A} \times \mathbf{B}$, where number of pigeonholes = 15.

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Lecture Exercises: Problem 1 Ans contd.

- Now, as per pigeonhole principle, for two 2-tuples from $\mathbb{N}^2$ to be mapped to one of 15 elements of $\mathbf{A} \times \mathbf{B}$, at least (15+1)=16 number of 2-tuples from $\mathbb{N}^2$ is to be chosen.
- So, correct choice is (c).