CS34110 Discrete Mathematics and Graph Theory

**UNIT – III, Module – 2**

# Lecture 30: Graph/Tree Connectivity

[ Cut vertex, cut edge; Separable, nonseparable graphs; Block; Connectivity; Vertex cut, connectivity $\kappa$; $k$-connected graph; Connectedness theorems ]

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## Notation table

| Symbol / Notation | Meaning |
|---|---|
| $\omega(\cdot)$ | Number of components of graph. E.g., $\omega(\mathcal{G})$: for graph $\mathcal{G}$. |
| $\kappa(\cdot)$ | Minimum cardinality of vertex cut of graph. E.g., $\kappa(\mathcal{G})$: for graph $\mathcal{G}$. |
| $\lambda(\cdot)$ | Minimum cardinality of edge cut of graph. E.g., $\lambda(\mathcal{G})$: for graph $\mathcal{G}$. |
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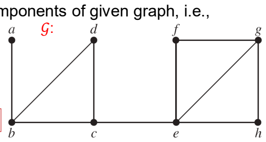
## Graph connectedness

- Graph connectedness: how much chance of disconnections in connected undirected graph.
  - Property:: Cut vertex in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: vertex $v \in \mathbf{V}$, such that $\omega(\mathcal{G} - v) > \omega(\mathcal{G})$, where $\mathcal{G} - v$ = deletion of $v$ from $\mathcal{G}$, and $\omega(\cdot)$ = number of components of given graph, i.e., removal of such vertex (and all incident edges) from given graph to produce more components of that graph.

Cut vertices: $b$, $c$, $e$

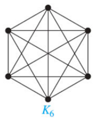



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

## Graph connectedness

- Graph connectedness:
  - Property:: Cut vertex: also called cut point, cut node, articulation point.
  - Property:: $(\mathcal{G} - v) \subset \mathcal{G}$, $(\mathcal{G} - v)$ → disconnected.
  - Property:: No cut vertex in $K_n$ $(n \geq 3)$.
  - Property:: Nonseparable graph: connected nontrivial graph $\mathcal{G}$ without any cut vertex.

Nonseparable graph →




[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]



## Graph connectedness

- Graph connectedness:
  - Property:: Separable graph: connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with at least one cut vertex.
  - Property:: In graph representing some computer network, cut vertex = essential router (essential: not allowed to fail, so that all computers to be able to communicate).



## Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: Vertex $v \in \mathbf{V}$ in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to become cut vertex, if and only if presence of two vertices $u_1, u_2 \in \mathbf{V}$ in $\mathcal{G}$ such that $u_1 \neq v, u_2 \neq v$ and every path between $u_1$ and $u_2$ to pass through $v$.

  Proof: (A) (Cut vertex $v$) → (every path between $u_1$ and $u_2$ through $v$).
  (Cut vertex $v$) → $(\mathcal{G} - v)$ disconnected and $\omega(\mathcal{G} - v) \geq 2$, i.e., at least 2 components of $\mathcal{G}$ after deletion of $v$.
  Let $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$ be two such components.
  Idea: To partition $\mathbf{V} \setminus \{v\}$ into nonempty, disjoint $\mathbf{V}_1$, $\mathbf{V}_2$.

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## Graph connectedness

- Graph connectedness:
  Proof of ⟨Theorem⟩ contd.:
  Two possibilities: (i) $u_1, u_2 \in \mathbf{V}_1$ or $u_1, u_2 \in \mathbf{V}_2$, i.e., $u_1, u_2$ in same component $\mathcal{G}_1$ or $\mathcal{G}_2$; (ii) $u_1 \in \mathbf{V}_1$ and $u_2 \in \mathbf{V}_2$, or $u_2 \in \mathbf{V}_1$ and $u_1 \in \mathbf{V}_2$, i.e., $u_1, u_2$ in different components $\mathcal{G}_1$ and $\mathcal{G}_2$.
  For possibility (i): since $\mathcal{G}_1$ or $\mathcal{G}_2$ connected components of connected $\mathcal{G}$, so every path between $u_1$ and $u_2$ to pass through $v$.
  For possibility (ii): since $u_1 \in \mathbf{V}_1$ and $u_2 \in \mathbf{V}_2$ (also applicable for $u_2 \in \mathbf{V}_1$ and $u_1 \in \mathbf{V}_2$), and $\mathbf{V}_1$, $\mathbf{V}_2$ disjoint due to deletion of $v$, so again, every path between $u_1$ and $u_2$ to pass through $v$.

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## Graph connectedness

- Graph connectedness:
  Proof of ⟨Theorem⟩ contd-2.:
  Hence, for all scenarios, where $u_1, u_2, v \in \mathbf{V}$ and $u_1 \neq v, u_2 \neq v$, (Cut vertex $v$) → (every path between $u_1$ and $u_2$ through $v$).
  (B) (Every path between $u_1$ and $u_2$ through $v$) → (cut vertex $v$).
  ($v$ present on every path in $\mathcal{G}$ joining $u_1$ and $u_2$) → (no path possible joining $u_1$ and $u_2$ in $\mathcal{G} - v$) → ($\mathcal{G} - v$ disconnected).
  So, $v$ to become cut vertex in $\mathcal{G}$.
  Hence, for all scenarios, where $u_1, u_2, v \in \mathbf{V}$ and $u_1 \neq v, u_2 \neq v$, (Every path between $u_1$ and $u_2$ through $v$) → (cut vertex $v$). ∎



## Graph connectedness

- Graph connectedness:
  - Property:: Next theorem to be generalization of previous theorem.
  - Property:: ⟨Theorem⟩: Vertex $v \in \mathbf{V}$ in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to become cut vertex, if and only if possibility of partition of set $\mathbf{V} \setminus \{v\}$ in $\mathcal{G}$ into two subsets $\mathbf{V}_1$ and $\mathbf{V}_2$ such that for any vertices $u_1 \in \mathbf{V}_1$, $u_2 \in \mathbf{V}_2$, every path between $u_1$ and $u_2$ to pass through $v$.

  Proof: Similar to previous theorem's proof.

## Graph connectedness

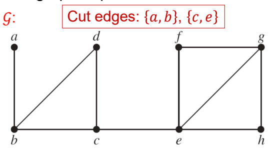
- Graph connectedness:
  - Property:: ⟨Theorem⟩: Every nontrivial connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to contain at least two non-cut vertices.



## Graph connectedness

- Graph connectedness:
  - Property:: Cut edge in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: edge $e \in \mathbf{E}$, such that $\omega(\mathcal{G} - e) > \omega(\mathcal{G})$, where $\mathcal{G} - e$ = deletion of $e$ from $\mathcal{G}$, i.e., removal of such edge from given graph to produce more components of that graph.
  - Property:: Cut edge: also called bridge.

$\mathcal{G}$: Cut edges: $\{a, b\}$, $\{c, e\}$




[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]



## Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: Edge $e \in \mathbf{E}$ in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to become cut edge, if and only if each of following conditions to hold —
    (i) no circuit of $\mathcal{G}$ to contain $e$;
    (ii) presence of two vertices $u_1, u_2 \in \mathbf{V}$ in $\mathcal{G}$ such that every path between $u_1$ and $u_2$ to pass through $e$;
    (iii) possibility of partition of set $\mathbf{V}$ into two subsets $\mathbf{V}_1$ and $\mathbf{V}_2$ such that for any vertices $u_1 \in \mathbf{V}_1$, $u_2 \in \mathbf{V}_2$, every path between $u_1$ and $u_2$ to pass through $e$. [generalization of condition (ii)]

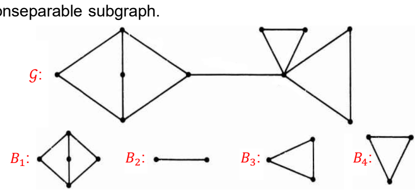
## Graph connectedness

- Graph connectedness:
  - Property:: In graph representing some computer network, cut edge = essential link (essential: not allowed to fail, so that all computers to be able to communicate).



## Graph connectedness

- Graph connectedness:
  - Property:: Block of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: maximal nonseparable subgraph.




[Ref: Frank Harary, *Graph Theory*, Addison-Wesley, Reading, MA, USA, 1969.]



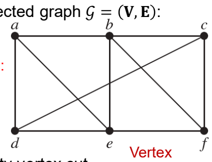
## Graph connectedness

- Graph connectedness:
  - Property:: Separable graph to generate more than one block.

## Graph connectivity

- Graph connectivity: determining connectivity invariants of given graph, based on cut vertex, cut edge, block etc.
  - Property:: Vertex cut of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: vertex subset $\mathbf{v} \subseteq \mathbf{V}$, if $\mathcal{G} - \mathbf{v}$ to become disconnected or trivial.
  - Property:: Vertex cut: also called separating set.
  - Property:: Every connected graph, except complete graph, to have nonempty vertex cut.

$\mathcal{G}$:




Vertex cut: $\{b, c, e\}$

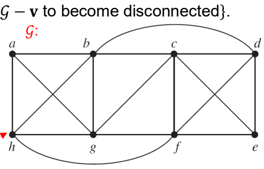
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]



## Graph connectivity

- Graph connectivity:
  - Property:: Vertex connectivity of noncomplete connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: minimum cardinality of vertex cut of $\mathcal{G}$, denoted by $\kappa(\mathcal{G})$, i.e., $\kappa(\mathcal{G}) = \min\{|\mathbf{v}| \mid \mathbf{v} \subseteq \mathbf{V},\ \ \mathcal{G} - \mathbf{v}$ to become disconnected$\}$.
  - Property:: Vertex connectivity: one form of connectivity invariant.
  - Property:: Vertex connectivity: also called connectivity, or point connectivity.

$\mathcal{G}$:




Vertex cut: $\{b, c, f\}$; $\kappa(\mathcal{G}) = 3$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]



## Graph connectivity

- Graph connectivity:
  - Property:: $\kappa(\mathcal{G}') = 0$, if $\mathcal{G}'$ disconnected graph.
  - Property:: $\kappa(\mathcal{G}) = 1$, for connected graph $\mathcal{G}$ with cut vertex.
  - Property:: $\kappa(K_n) = n - 1$, for complete graph $K_n$ $(n > 0)$ [reason: $n - 1$ number of vertices removed to produce trivial graph (graph with single vertex), as no vertex cuts in $K_n$, because after removing any subset of vertices and corresponding all incident edges in $K_n$ still remaining some complete graph].
  - Property:: For any graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$, $0 \leq \kappa(\mathcal{G}) \leq n - 1$.
  - Property:: Larger $\kappa(\mathcal{G})$ → more connectedness in graph $\mathcal{G}$.

## Graph connectivity

- Graph connectivity:
  - Property:: $k$-connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: if $\kappa(\mathcal{G}) \geq k$.
  - Property:: $k$-connected graph: also called $k$-vertex-connected.
  - Property:: 1-connected graph → connected nontrivial graph.
  - Property:: 2-connected (or biconnected) graph → nonseparable graph with at least three vertices (in other words, block with more than one edge).
  - Property:: $K_2$ → block, not 2-connected.
  - Property:: $k$-connected graph → $j$-connected graph for all $j$ with $0 \leq j \leq k$.



## Summary

- Focus: Graph connectedness and connectivity.
- Graph connectedness and connectivity.
- Cut vertex in graph, with properties and examples.
- Nonseparable graph, with examples.
- Cut edge in graph, with examples.
- Block, with examples.
- Vertex cut of graph, with properties and examples.
- Vertex connectivity of graph, with properties and examples.
- $k$-connected graph, with properties.
- Graph connectedness related theorems, with proofs.



## References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.