CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 1

Lecture 09: Discrete Structures

[Cartesian product of finite sets; Union and intersection; Principle of inclusion-exclusion; Set operations; Set identities]

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Recap:: Discrete structures

- Discrete structures examples:
  - (i) sets (collections of objects) and multisets;
    (ii) combinations (built from sets; unordered collections of objects to be used in counting);
    (iii) relations (sets of ordered pairs representing relationships between objects);
    (iv) graphs (sets of vertices and edges connecting vertices);
    (v) sequences (ordered lists of elements, as well as special type of functions expressing relationships among elements);
    (vi) matrices.

Notation table

Symbol / Notation	Meaning
$\cdot$	Arithmetic and algebraic multiplication in number theory.
$\times$	Multiply operation for Cartesian product in set theory.
$\cup$, $\cup_{i=1}^{n}$	Union operation of two, more (arbitrary no.) sets in set theory resp.
$\cap$, $\cap_{i=1}^{n}$	Intersection operation of two, more (arbitrary) sets in set theory resp.
$\vee$, $\vee_{i=1}^{n}$	Disjunction of two, more (arbitrary no.) assertions resp.
$\wedge$, $\wedge_{i=1}^{n}$	Conjunction of two, more (arbitrary no.) assertions resp.
$\setminus$	Set subtraction of two sets in set theory.
Δ	Symmetric difference of two sets in set theory.
$\bar{A}$	Complement of set A in set theory.
$\mathcal{U}$	Universal set (or universe) in set theory.
$\emptyset$	Empty set in set theory.

Set

- Set operations:
  - Product property:: Cartesian product of finite sets $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$:
    $\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n = \prod_{i=1}^{n} \mathbf{A}_i = \{(a_1, a_2, \cdots, a_n) \mid a_i \in \mathbf{A}_i,\ i = 1,2,\cdots,n\}$.
    where, $(a_1, a_2, \cdots, a_n)$ = ordered n-tuple.
    - Property:: Ordered n-tuple: ordered collection of n elements.
    - Product equality property:: Equality of two ordered n-tuples $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n)$: $(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n)$, if and only if $a_i = b_i$ for all $i = 1,2,\cdots,n$.
  - Property:: Cartesian product of finite sets $\mathbf{A}$ and $\mathbf{B}$: set of all ordered pairs (a, b), i.e., $\mathbf{A} \times \mathbf{B} = \{(a, b) \mid (a \in \mathbf{A}) \wedge (b \in \mathbf{B})\}$.
    - Property:: Ordered pair: ordered 2-tuple.

Set

- Set operations:
  - Property:: Cartesian product of finite set $\mathbf{A}^m$:
    $\mathbf{A}^m = \{(a_1, a_2, \cdots, a_m) \mid a_i \in \mathbf{A},\ i = 1,2,\cdots,m\}$.
  - Property:: $|\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n| = |\mathbf{A}_1| \cdot |\mathbf{A}_2| \cdot \ldots \cdot |\mathbf{A}_n| = \prod_{i=1}^{n} |\mathbf{A}_i|$.
  - Property:: $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$, if $\mathbf{A} = \emptyset$ or $\mathbf{B} = \emptyset$ (so, $\mathbf{A} \times \mathbf{B} = \emptyset$) or $\mathbf{A} = \mathbf{B}$.
  - Relation property:: Relation from set $\mathbf{A}$ to set $\mathbf{B}$: subset of Cartesian product $\mathbf{A} \times \mathbf{B}$, more specifically called binary relation, denoted by $\mathcal{R}$.
  - Property:: Elements of relation $\mathcal{R}$ from set $\mathbf{A}$ to set $\mathbf{B}$: ordered pairs, where first element to belong to $\mathbf{A}$ and second to $\mathbf{B}$.

Set

- Set operations:
  - Union property:: Union of finite family of sets: $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \cdots \cup \mathbf{A}_n =$
    $\bigcup_{i=1}^{n} \mathbf{A}_i = \{x \mid \bigvee_{i=1}^{n} (x \in \mathbf{A}_i)\} = \{x \mid \exists i (x \in \mathbf{A}_i),\ i = 1,2,\cdots,n\}$.
  - Generalization property:: For indexed set $\mathbf{S}$, such that $i \in \mathbf{S}$, union of finite family of sets: $\bigcup_{i \in \mathbf{S}} \mathbf{A}_i = \{x \mid \exists i \in \mathbf{S}\ (x \in \mathbf{A}_i)\}$.
  - Property:: Union of two sets $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{A} \cup \mathbf{B} = \{x \mid (x \in \mathbf{A}) \vee (x \in \mathbf{B})\}$.
  - Property:: Union of infinite family of sets: $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \cdots \cup \mathbf{A}_n \cup \cdots =$
    $\bigcup_{i=1}^{\infty} \mathbf{A}_i = \{x \mid \bigvee_{i=1}^{\infty} (x \in \mathbf{A}_i)\}$.

Set

- Set operations:
  - Intersection property:: Intersection of finite family of sets: $\mathbf{A}_1 \cap \mathbf{A}_2 \cap \cdots \cap \mathbf{A}_n = \bigcap_{i=1}^{n} \mathbf{A}_i = \{x \mid \bigwedge_{i=1}^{n}(x \in \mathbf{A}_i)\} = \{x \mid \forall i(x \in \mathbf{A}_i),$ $i = 1,2,\cdots,n\}$.
  - Generalization property:: For indexed set $\mathbf{S}$, such that $i \in \mathbf{S}$, intersection of finite family of sets: $\bigcap_{i \in \mathbf{S}} \mathbf{A}_i = \{x \mid \forall i \in \mathbf{S}\ (x \in \mathbf{A}_i)\}$.
  - Intersection property:: Intersection of two sets $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{A} \cap \mathbf{B} = \{x \mid (x \in \mathbf{A}) \wedge (x \in \mathbf{B})\}$.
  - Property:: Intersection of infinite family of sets: $\mathbf{A}_1 \cap \mathbf{A}_2 \cap \cdots \cap \mathbf{A}_n \cap \cdots = \bigcap_{i=1}^{\infty} \mathbf{A}_i = \{x \mid \bigwedge_{i=1}^{\infty}(x \in \mathbf{A}_i)\}$.

Set

- Set operations:
  - Property:: Principle of inclusion-exclusion: Generalization of following two-set relation — $|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$.
  - Disjoint property:: If $\mathbf{A} \cap \mathbf{B} = \emptyset$, then $\mathbf{A}$ and $\mathbf{B}$ to become disjoint.
  - Set subtraction property:: Set subtraction of two finite sets $\mathbf{A}$ and $\mathbf{B}$ = Difference between $\mathbf{A}$ and $\mathbf{B}$ = complement of $\mathbf{B}$ with respect to $\mathbf{A}$: $\mathbf{A} \setminus \mathbf{B} = \mathbf{A} - \mathbf{B} = \{x \mid (x \in \mathbf{A}) \wedge (x \notin \mathbf{B})\}$.
  - Successor property:: Successor of $\mathbf{A}$: $\mathrm{succ}(\mathbf{A}) = \mathbf{A} \cup \{\mathbf{A}\}$.

Set

- Set operations:
  - Symmetric difference property:: Symmetric difference of $\mathbf{A}$ and $\mathbf{B}$ (also disjunctive union): $\mathbf{A} \,\Delta\, \mathbf{B} = \mathbf{A} \ominus \mathbf{B} = \mathbf{A} \oplus \mathbf{B} = \{x \mid (x \in \mathbf{A}) \oplus (x \in \mathbf{B})\}$.
  Exclusive disjunction
  - Property:: $\mathbf{A} \,\Delta\, \mathbf{B} = (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A})$ $= (\mathbf{A} \cup \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{B})$.
  - Complement property:: Complement of $\mathbf{A}$ (denoted by $\overline{\mathbf{A}}$, also by $\mathbf{A}'$, or $\mathbf{A}^{\mathrm{C}}$): $\overline{\mathbf{A}} = \{x \in \mathcal{U} \mid x \notin \mathbf{A}\} = \mathcal{U} \setminus \mathbf{A}$.
  - Property:: Complement of $\mathbf{A}$ with respect to superset $\mathbf{B}$ (such that $\mathbf{A} \subseteq \mathbf{B}$): $\{x \in \mathbf{B} \mid x \notin \mathbf{A}\}$.
  - Property:: $\mathbf{A} \setminus \mathbf{B} = \mathbf{A} \cap \overline{\mathbf{B}}$.

Set

- Set operations: examples.
  - Example-1:: for sets $\mathbf{A} = \{1, 2\}$, $\mathbf{B} = \{a, b, c\}$, their Cartesian products $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$:

    $$\mathbf{A} \times \mathbf{B} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$
    $$\mathbf{B} \times \mathbf{A} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

    Also, $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$,
    $\mathbf{A}^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$,
    $\mathbf{B}^2 = \mathbf{B} \times \mathbf{B} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$.
    Further, $|\mathbf{A} \times \mathbf{B}| = |\mathbf{B} \times \mathbf{A}| = |\mathbf{A}| \cdot |\mathbf{B}| = 6$;
    $|\mathbf{A}^2| = |\mathbf{A}| \cdot |\mathbf{A}| = 4$; $|\mathbf{A}^3| = |\mathbf{A}| \cdot |\mathbf{A}| \cdot |\mathbf{A}| = 8$; $|\mathbf{B}^2| = |\mathbf{B}| \cdot |\mathbf{B}| = 9$.

Set

- Set operations: examples.
  - Example-2:: for sets $\mathbf{A} = \{1, 3, 5\}$, and $\mathbf{B} = \{1, 2, 3\}$,
    $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A} = \{1, 2, 3, 5\}$; $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A} = \{1, 3\}$;
    $\mathbf{A} \setminus \mathbf{B} = \{5\}$; $\mathbf{B} \setminus \mathbf{A} = \{2\}$; $\mathbf{A} \,\Delta\, \mathbf{B} = \mathbf{B} \,\Delta\, \mathbf{A} = \{2, 5\}$.
  - Example-3:: for $\mathbf{A} = \{1, 3, 5, 7, 9\}$, and $\mathbf{B} = \{2, 4, 6, 8, 10\}$, $\mathbf{A} \cap \mathbf{B} = \emptyset$, so, $\mathbf{A}$ and $\mathbf{B}$ to be disjoint.
    $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$;
    $\mathbf{A} \setminus \mathbf{B} = \{1, 3, 5, 7, 9\}$; $\mathbf{B} \setminus \mathbf{A} = \{2, 4, 6, 8, 10\}$;
    $\mathbf{A} \,\Delta\, \mathbf{B} = \mathbf{B} \,\Delta\, \mathbf{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Set

- Set operations: examples.
  - Example-4:: for set $\mathbf{A} = \{a, e, i, o, u\}$, and universal set $\boldsymbol{\mathcal{U}}$ = {set of letters of English alphabet}, so
    $\overline{\mathbf{A}} = \mathbf{A}^{\mathrm{C}} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.
  - Example-5:: for set $\mathbf{A}_i = \{i, i+1, i+2, i+3, \dots\}$, where $i = 1, 2, \cdots, n$,
    $\bigcup_{i=1}^{n} \mathbf{A}_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, i+3, \dots\} = \{1, 2, 3, \dots\}$, and
    $\bigcap_{i=1}^{n} \mathbf{A}_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, i+3, \dots\} = \{n, n+1, n+2, \dots\} = \mathbf{A}_n$.
  - Example-6:: for set $\mathbf{B}_i = \{1, 2, 3, \dots, i\}$, where $i = 1, 2, 3, \cdots, \infty$,
    $\bigcup_{i=1}^{\infty} \mathbf{B}_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \dots, i\} = \{1, 2, 3, \dots\} = \mathbb{Z}^+$, and
    $\bigcap_{i=1}^{\infty} \mathbf{B}_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \dots, i\} = \{1\} = \mathbf{B}_1$.

Set

- Computer representation of set: techniques for storing elements of set, so as to carry out set operations efficiently.
  - Bit string representation: storing elements of given set $\mathbf{A}$, based on arbitrary ordering of elements of $\boldsymbol{\mathcal{U}}$, assuming $\boldsymbol{\mathcal{U}}$ to be finite (and of reasonable size not larger than memory capacity of computer), i.e., $\underbrace{b_1 b_2 \cdots b_n}_{n \text{ bits for } \mathbf{A}}$, where $|\mathbf{A}| = n$ and $b_i = \begin{cases} 1, & \text{if } a_i \in \mathbf{A} \\ 0, & \text{otherwise} \end{cases}$ for $i = 1,2, \ldots, n$.
    - Benefit: easy to find complements of sets and unions, intersections and differences of sets.

Set

- Set operation identity: logically equal resultant sets.
  - Set identities → Laws of algebra of set theory.
  - Proof of each law based on 3 proof methods —

Methods of Proving Set Identities	
Description	***Method***
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side (for any arbitrary element $x \in \boldsymbol{\mathcal{U}}$)
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Laws of algebra of set theory

Idempotent laws:	(III.1a) $\mathbf{A} \cup \mathbf{A} = \mathbf{A}$	(III.1b) $\mathbf{A} \cap \mathbf{A} = \mathbf{A}$
Identity laws:	(III.2a) $\mathbf{A} \cup \emptyset = \mathbf{A}$	(III.2b) $\mathbf{A} \cap \boldsymbol{\mathcal{U}} = \mathbf{A}$
Domination laws:	(III.3a) $\mathbf{A} \cup \boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}}$	(III.3b) $\mathbf{A} \cap \emptyset = \emptyset$
Associative laws:	(III.4a) $(\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C} = \mathbf{A} \cup (\mathbf{B} \cup \mathbf{C})$	(III.4b) $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$
Commutative laws:	(III.5a) $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$	(III.5b) $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$
Distributive laws:	(III.6a) $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$	(III.6b) $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$
Involution law:	(III.7) $\overline{(\overline{\mathbf{A}})} = \mathbf{A}$, $(\mathbf{A}^C)^C = \mathbf{A}$	(also complementation law)
De Morgan's laws:	(III.8a) $\overline{(\mathbf{A} \cup \mathbf{B})} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}}$, $(\mathbf{A} \cup \mathbf{B})^C = (\mathbf{A})^C \cap (\mathbf{B})^C$	(III.8b) $\overline{(\mathbf{A} \cap \mathbf{B})} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$, $(\mathbf{A} \cap \mathbf{B})^C = (\mathbf{A})^C \cup (\mathbf{B})^C$

Laws of algebra of set theory

Complement laws:	(III.9a) $\mathbf{A} \cup \overline{\mathbf{A}} = \boldsymbol{\mathcal{U}}$, $\mathbf{A} \cup \mathbf{A}^{\mathrm{C}} = \boldsymbol{\mathcal{U}}$	(III.9c) $\mathbf{A} \cap \overline{\mathbf{A}} = \emptyset$, $\mathbf{A} \cap \mathbf{A}^{\mathrm{C}} = \emptyset$
	(III.9b) $\overline{\boldsymbol{\mathcal{U}}} = \emptyset$, $\boldsymbol{\mathcal{U}}^{\mathrm{C}} = \emptyset$	(III.9d) $\overline{\emptyset} = \boldsymbol{\mathcal{U}}$, $\emptyset^{\mathrm{C}} = \boldsymbol{\mathcal{U}}$
Absorption laws:	(III.10a) $\mathbf{A} \cup (\mathbf{A} \cap \mathbf{B}) = \mathbf{A}$	(III.10b) $\mathbf{A} \cap (\mathbf{A} \cup \mathbf{B}) = \mathbf{A}$

Set

- Set operation identity: examples.
  - Example-1:: Proof for law-(III.8b): $\overline{(\mathbf{A} \cap \mathbf{B})} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$.

  Proof: using subset method [one set shown to be subset of other].

  (1) Proving $\overline{(\mathbf{A} \cap \mathbf{B})} \subseteq \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$, i.e., if $x \in \overline{(\mathbf{A} \cap \mathbf{B})}$, then $x \in (\overline{\mathbf{A}} \cup \overline{\mathbf{B}})$.

  Let $x \in \overline{(\mathbf{A} \cap \mathbf{B})}$. By definition of complement, $x \notin (\mathbf{A} \cap \mathbf{B})$, i.e., $\neg\big((x \in \mathbf{A}) \wedge (x \in \mathbf{B})\big)$ to be TRUE as per definition of intersection.

  By De Morgan's law for propositions in (I.10b), $\neg(x \in \mathbf{A}) \vee \neg(x \in \mathbf{B})$.

  Using definition of negation of propositions, $(x \notin \mathbf{A})$ or $(x \notin \mathbf{B})$, i.e., $(x \in \overline{\mathbf{A}})$ or $(x \in \overline{\mathbf{B}})$ as per definition of complement of set.

  So, by definition of union, $x \in (\overline{\mathbf{A}} \cup \overline{\mathbf{B}})$.

(contd. to next slide)

Set

- Set operation identity: examples.
  - Example-1 proof (contd.)::

  (2) Proving $\overline{\mathbf{A}} \cup \overline{\mathbf{B}} \subseteq \overline{(\mathbf{A} \cap \mathbf{B})}$, i.e., if $x \in (\overline{\mathbf{A}} \cup \overline{\mathbf{B}})$, then $x \in \overline{(\mathbf{A} \cap \mathbf{B})}$.

  Let $x \in (\overline{\mathbf{A}} \cup \overline{\mathbf{B}})$. By definition of union, $(x \in \overline{\mathbf{A}})$ or $(x \in \overline{\mathbf{B}})$.

  As per definition of complement of set, $(x \notin \mathbf{A})$ or $(x \notin \mathbf{B})$. So, proposition $\neg(x \in \mathbf{A}) \vee \neg(x \in \mathbf{B})$ to become TRUE.

  By De Morgan's law for propositions in (I.10b), $\neg\big((x \in \mathbf{A}) \wedge (x \in \mathbf{B})\big)$ to be TRUE. Then, as per definition of intersection, $\neg\big(x \in (\mathbf{A} \cap \mathbf{B})\big)$.

  So, by definition of complement, $x \in \overline{(\mathbf{A} \cap \mathbf{B})}$.

  Combining (1) and (2), $\overline{(\mathbf{A} \cap \mathbf{B})} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$. ■

Set

- Set operation identity: examples.
  - Example-2:: Same as Example-1, i.e., law-(III.8b): $\overline{(\mathbf{A} \cap \mathbf{B})} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$.

Proof: using compact form of subset method.

$\overline{(\mathbf{A} \cap \mathbf{B})} = \{x \mid x \notin (\mathbf{A} \cap \mathbf{B})\}$ by set-builder of set complement
$= \{x \mid \neg(x \in (\mathbf{A} \cap \mathbf{B}))\}$ by definition of set "not belong to"
$= \{x \mid \neg((x \in \mathbf{A}) \wedge (x \in \mathbf{B}))\}$ by definition of set intersection
$= \{x \mid \neg(x \in \mathbf{A}) \vee \neg(x \in \mathbf{B})\}$ by De Morgan's law in (I.10b)
$= \{x \mid (x \notin \mathbf{A}) \vee (x \notin \mathbf{B})\}$ by definition of set "not belong to"
$= \{x \mid (x \in \overline{\mathbf{A}}) \vee (x \in \overline{\mathbf{B}})\}$ by definition of set complement
$= \{x \mid (x \in (\overline{\mathbf{A}} \cup \overline{\mathbf{B}}))\} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$ by definition of set union. ■

Set

- Set operation identity: examples.
  - Example-3:: Proof for law-(III.6b): $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$.

Proof: using compact subset method.

$\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = \{x \mid x \in (\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}))\}$ by set-builder of set intersection
$= \{x \mid (x \in \mathbf{A}) \wedge (x \in (\mathbf{B} \cup \mathbf{C}))\}$ by definition of set intersection
$= \{x \mid (x \in \mathbf{A}) \wedge ((x \in \mathbf{B}) \vee (x \in \mathbf{C}))\}$ by definition of set union
$= \{x \mid ((x \in \mathbf{A}) \wedge (x \in \mathbf{B})) \vee ((x \in \mathbf{A}) \vee (x \in \mathbf{C}))\}$ by (I.6b)
$= \{x \mid (x \in (\mathbf{A} \cap \mathbf{B})) \vee (x \in (\mathbf{A} \cap \mathbf{C}))\}$ by definition of intersection
$= \{x \mid x \in ((\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C}))\}$ by definition of set union
$= (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$ by set-builder of set union. ■

Set

- Set operation identity: examples.
  - Example-4:: Proof for law-(III.6b): $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$.

Proof: using membership table method.

Adjacent table: membership table for given combinations of sets. Same 5th and 8th columns. ■

$\mathbf{A}$	$\mathbf{B}$	$\mathbf{C}$	$\mathbf{B \cup C}$	$\mathbf{A \cap (B \cup C)}$	$\mathbf{A \cap B}$	$\mathbf{A \cap C}$	$\mathbf{(A \cap B) \cup (A \cap C)}$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Set

- Set operation identity: examples.
  - Example-5:: Proof for: $\overline{(\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}))} = (\overline{\mathbf{C}} \cup \overline{\mathbf{B}}) \cap \overline{\mathbf{A}}$.

  Proof: apply existing set identities to prove new identities.

  $\overline{(\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}))} = \overline{\mathbf{A}} \cap \overline{(\mathbf{B} \cap \mathbf{C})}$ by De Morgan's law in (III.8a)

  $= \overline{\mathbf{A}} \cap (\overline{\mathbf{B}} \cup \overline{\mathbf{C}})$ by De Morgan's law in (III.8b)

  $= (\overline{\mathbf{B}} \cup \overline{\mathbf{C}}) \cap \overline{\mathbf{A}}$ by commutative law in (III.5b)

  $= (\overline{\mathbf{C}} \cup \overline{\mathbf{B}}) \cap \overline{\mathbf{A}}$ by commutative law in (III.5a). ■

Summary

- Focus: Set operations.
- Cartesian product of finite sets, with examples.
- Union and intersection of finite sets, and related theorems with examples.
- Principle of inclusion-exclusion in set theory.
- Set operations, with examples.
- Set identities, i.e., laws of algebra of set theory.
- Set operation identity examples.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Cartesian product of sets:: [Ros21]:128-130.
- Union of sets:: [Ros21]:133-134,139-141.
- Intersection of sets:: [Ros21]:134,139-141.
- Principle of inclusion-exclusion in set theory:: [Ros21]:134.
- Set operations:: [Ros21]:135-136.
- Set identities:: [Ros21]:136-139.

Lecture Exercises: Problem 1 [Ref: Gate 2015, Q.2, p.1 (Set-3)]

Suppose $\boldsymbol{U}$ is the power set of the set $\boldsymbol{S} = \{1,2,3,4,5,6\}$. For any $\boldsymbol{T} \in \boldsymbol{U}$, let $|\boldsymbol{T}|$ denote the number of elements in $\boldsymbol{T}$ and $\boldsymbol{T}'$ denote the complement of $\boldsymbol{T}$. For any $\boldsymbol{T}, \boldsymbol{R} \in \boldsymbol{U}$, let $\boldsymbol{T} \setminus \boldsymbol{R}$ be the set of all elements in $\boldsymbol{T}$ which are not in $\boldsymbol{R}$. Which one of the following is TRUE?

(a) $\forall \boldsymbol{X} \in \boldsymbol{U}\ (|\boldsymbol{X}| = |\boldsymbol{X}'|)$.
(b) $\exists \boldsymbol{X} \in \boldsymbol{U}\ \exists \boldsymbol{Y} \in \boldsymbol{U}\ (|\boldsymbol{X}| = 5, |\boldsymbol{Y}| = 5$ and $\boldsymbol{X} \cap \boldsymbol{Y} = \emptyset)$.
(c) $\forall \boldsymbol{X} \in \boldsymbol{U}\ \forall \boldsymbol{Y} \in \boldsymbol{U}\ (|\boldsymbol{X}| = 2, |\boldsymbol{Y}| = 3$ and $\boldsymbol{X} \setminus \boldsymbol{Y} = \emptyset)$.
(d) $\forall \boldsymbol{X} \in \boldsymbol{U}\ \forall \boldsymbol{Y} \in \boldsymbol{U}\ (\boldsymbol{X} \setminus \boldsymbol{Y} = \boldsymbol{Y}' \setminus \boldsymbol{X}')$.

Lecture Exercises: Problem 1 Ans

(D) Let arbitrary variables $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{U}$.

Then, L.H.S.: $\boldsymbol{X} \setminus \boldsymbol{Y} = \boldsymbol{X} \cap \boldsymbol{Y}'$ by set subtraction property
Also, R.H.S.: $\boldsymbol{Y}' \setminus \boldsymbol{X}' = \boldsymbol{Y}' \cap (\boldsymbol{X}')'$ by set subtraction property
$= \boldsymbol{Y}' \cap \boldsymbol{X}$ by involution law (III.7)
$= \boldsymbol{X} \cap \boldsymbol{Y}'$ by commutative law (III.5b)

So, $(\boldsymbol{X} \setminus \boldsymbol{Y} = \boldsymbol{Y}' \setminus \boldsymbol{X}')$. ①

After applying universal generalization (II.8) on ① for both variables successively, $\forall \boldsymbol{X} \in \boldsymbol{U}\ \forall \boldsymbol{Y} \in \boldsymbol{U}\ (\boldsymbol{X} \setminus \boldsymbol{Y} = \boldsymbol{Y}' \setminus \boldsymbol{X}') \equiv \forall \boldsymbol{Y} \in \boldsymbol{U}\ \forall \boldsymbol{X} \in \boldsymbol{U}\ (\boldsymbol{X} \setminus \boldsymbol{Y} = \boldsymbol{Y}' \setminus \boldsymbol{X}')$ (as per nested universal quantification ordering property).

- What about options (A), (B), (C)?