CS34110 Discrete Mathematics and Graph Theory

UNIT – IV, Module – 1

Lecture 36: Graph Planarity

[Corollaries of Euler's Formula; Planarity of K_5, $K_{3,3}$ revisited; Kuratowski's theorem; Crossing number; Thickness]

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Graph planarity

- Euler's formula:
  - Property:: ⟨Corollary of Euler's Formula⟩: If $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be connected planar simple graph with $|\mathbf{E}| = e$ edges, $|\mathbf{V}| = v$ vertices and r regions, where $v \geq 3$ and $e \geq 2$, then $2 \cdot e \geq 3 \cdot r$. [Planarity implying "$2 \cdot e \geq 3 \cdot r$"]

  Proof: Given $\mathcal{G}$ simple, so — (i) not possible that multiple edges produce regions of degree two, and (ii) not possible that loops produce regions of degree one.
  So, degree of any region from $\mathcal{G}$ to be at least three, including degree of unbounded region to be at least three due to $v \geq 3$. ①

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Graph planarity

- Euler's formula:

  Proof of ⟨Corollary of Euler's Formula⟩ contd.:
  Because each edge in $\mathcal{G}$ to be traced out exactly twice while drawing (either in case of tracing two different neighboring regions, or twice in case of tracing same region), so
  sum of degrees of all regions = twice number of edges in $\mathcal{G}$. ②
  As r number of regions given, so by combining ① and ②,
  $2 \cdot e = \sum_{\forall R} \deg(R) \geq 3 \cdot r$, or, $\frac{2}{3} \cdot e \geq r$. ■

Graph planarity

- Euler's formula:
  - Property:: ⟨Corollary of Euler's Formula⟩: If $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be connected planar simple graph with $|\mathbf{E}| = e$ edges and $|\mathbf{V}| = v$ vertices, where $v \geq 3$, then $e \leq 3 \cdot v - 6$. [Planarity implying "$e \leq 3 \cdot v - 6$"]

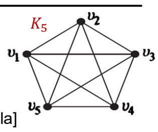
  Proof: Given $\mathcal{G}$ simple, so — (i) not possible that multiple edges produce regions of degree two, and (ii) not possible that loops produce regions of degree one.
  So, degree of any region from $\mathcal{G}$ to be at least three, including degree of unbounded region to be at least three due to $v \geq 3$. ①
  Let r number of regions produced when $\mathcal{G}$ drawn in plane. (contd. to next slide)

Graph planarity

- Euler's formula:

  Proof of ⟨Corollary of Euler's Formula⟩ contd.:
  Because each edge in $\mathcal{G}$ to be traced out exactly twice while drawing (either in case of tracing two different neighboring regions, or twice in case of tracing same region), so
  sum of degrees of all regions = twice number of edges in $\mathcal{G}$. ②
  Combining ① and ②, $2 \cdot e = \sum_{\forall R} \deg(R) \geq 3 \cdot r$, or, $\frac{2}{3} \cdot e \geq r$. ③
  Putting Euler's Formula $r = e - v + 2$ in ③ to produce
  $e - v + 2 \leq \frac{2}{3} \cdot e$, or, $\frac{1}{3} \cdot e \leq v - 2$, or, $e \leq 3 \cdot v - 6$. ■

Graph planarity



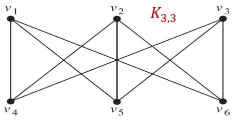
- Euler's formula:
  - Property:: ⟨Theorem: Planarity of K_5 revisited⟩: K_5 nonplanar.

  Proof: [By contraposition, Corollary of Euler's Formula]
  In $K_5 = K_n$, $|\mathbf{V}| = v = n = 5$ vertices, and $|\mathbf{E}| = e = \frac{1}{2} \cdot n \cdot (n-1) = 10$ edges. So, as per Corollary of Euler's Formula, $3 \cdot v - 6 = 9$, and thus $e = 10 \nleq 9 = 3 \cdot v - 6$, or $e \nleq 3 \cdot v - 6$, showing Corollary of Euler's Formula not satisfied for K_5.
  Then, $\neg(e \leq 3 \cdot v - 6) \rightarrow \neg$ Planarity [contrapositive of Corollary of Euler's Formula]. Hence, K_5 nonplanar. ■

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Euler's formula:
  - Property:: Converse of Corollary of Euler's Formula: "$e \leq 3 \cdot v - 6$" $\rightarrow$ Planarity.
    Converse NOT ALWAYS TRUE.
  - E.g., $K_{3,3}$ nonplanar, but in $K_{3,3}$, $|\mathbf{V}| = v = 6$ vertices, and $|\mathbf{E}| = e = 9$ edges.
    Then, $e = 9 \leq 12 = 3 \cdot v - 6$.
    So, "$e \leq 3 \cdot v - 6$" $\nrightarrow$ Planarity.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Euler's formula:
  - Property:: ⟨Corollary of Euler's Formula⟩: If $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be connected planar simple graph with $|\mathbf{E}| = e$ edges and $|\mathbf{V}| = v$ vertices, where $v \geq 3$, and no circuit of length three present in $\mathcal{G}$, then $e \leq 2 \cdot v - 4$.
    [(Planarity) ∧ (no circuit of length three) implying "$e \leq 2 \cdot v - 4$"]

  Proof: Given $\mathcal{G}$ simple, so — (i) not possible that multiple edges produce regions of degree two, and (ii) not possible that loops produce regions of degree one.
  Also given no circuit of length three present in $\mathcal{G}$, and so not possible to produce regions of degree three.

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Graph planarity

- Euler's formula:

  Proof of ⟨Corollary of Euler's Formula⟩ contd.:
  So, degree of any region from $\mathcal{G}$ to be at least four, including degree of unbounded region to be at least four. ①
  Let r number of regions produced when $\mathcal{G}$ drawn in plane.
  Because each edge in $\mathcal{G}$ to be traced out exactly twice while drawing (either in case of tracing two different neighboring regions, or twice in case of tracing same region), so
  sum of degrees of all regions = twice number of edges in $\mathcal{G}$. ②

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Graph planarity

- Euler's formula:

  Proof of ⟨Corollary of Euler's Formula⟩ contd-2.:

  Combining ① and ②, $2 \cdot e = \sum_{\forall R} \deg(R) \geq 4 \cdot r$, or, $\frac{1}{2} \cdot e \geq r$. ③

  Putting Euler's Formula $r = e - v + 2$ in ③ to produce

  $e - v + 2 \leq \frac{1}{2} \cdot e$, or, $\frac{1}{2} \cdot e \leq v - 2$, or, $e \leq 2 \cdot v - 4$. ■

Graph planarity

- Euler's formula:
  - Property:: ⟨Theorem: Planarity of $K_{3,3}$ revisited⟩: $K_{3,3}$ nonplanar.

  Proof: In $K_{3,3}$, $|\mathbf{V}| = v = 6$ vertices, $|\mathbf{E}| = e = 9$ edges.

  Also no circuit in $K_{3,3}$, due to bipartite nature of $K_{3,3}$.

  So, as per Corollary of Euler's Formula, $2 \cdot v - 4 = 8$, and

  thus, $e = 9 \nleq 8 = 2 \cdot v - 4$, i.e., $\neg(e \leq 2 \cdot v - 4)$.

  Hence, $K_{3,3}$ nonplanar, as $\neg(e \leq 2 \cdot v - 4) \rightarrow \neg$ Planarity

  [contrapositive of (Planarity) ∧ (no circuit of length three) →

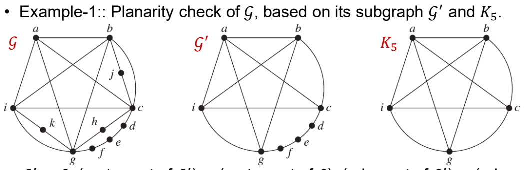
  $(e \leq 2 \cdot v - 4)$]. ■

Graph planarity

- Kuratowski's theorem: necessary and sufficient condition for graph nonplanarity.
  - Property:: Graph to be nonplanar if it containing either K_5 or $K_{3,3}$ as subgraph.
  - Property:: ⟨Kuratowski's theorem⟩: Graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be nonplanar, if and only if $\mathcal{G}$ to contain some subgraph homeomorphic to K_5 or $K_{3,3}$.

Graph planarity

- Kuratowski's theorem:
  - Example-1:: Planarity check of $\mathcal{G}$, based on its subgraph $\mathcal{G}'$ and K_5.



$\mathcal{G}' \subset \mathcal{G}$: (vertex set of $\mathcal{G}'$) $\subseteq$ (vertex set of $\mathcal{G}$); (edge set of $\mathcal{G}'$) $\subseteq$ (edge set of $\mathcal{G}$); all edges and corresponding vertices of $\mathcal{G}'$ present in $\mathcal{G}$.

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Graph planarity

- Kuratowski's theorem:
  - Example-1 contd.::

    $\mathcal{G}'$ obtained from $\mathcal{G}$, by removing vertices h, j, and k and all edges incident with these vertices, viz. $\{c,h\}$, $\{h,g\}$, $\{b,j\}$, $\{j,c\}$, $\{i,k\}$, $\{k,g\}$.

    $K_5 \xrightarrow{\text{elementary subdivisions}} \mathcal{G}'$: (1) removing $\{c,g\}$, adding d, adding $\{c,d\}$, $\{d,g\}$; (2) removing $\{d,g\}$, adding vertex e, adding $\{d,e\}$, $\{e,g\}$; (3) removing $\{e,g\}$, adding f, adding $\{e,f\}$, $\{f,g\}$.

    So, K_5 and $\mathcal{G}'$ to be homeomorphic.

    As K_5 nonplanar, so $\mathcal{G}'$ also nonplanar, and hence $\mathcal{G}$ also nonplanar.

Graph planarity

- Kuratowski's theorem:
  - Property:: Crossing number of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: minimum number of crossings on drawing $\mathcal{G}$ in plane, such that no three edges permitted to cross at same point.
  - Property:: Crossing number of $K_{3,3}$: 1.
  - Property:: For m, n to be even positive integers, crossing number of $K_{m,n}$ to be less than or equal to $\frac{m \cdot n \cdot (m-2) \cdot (n-2)}{16}$.

Graph planarity

- Kuratowski's theorem:
  - Property:: Thickness of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: smallest number of planar subgraphs of $\mathcal{G}$ such that $\mathcal{G}$ = union of these subgraphs.
  - Property:: Thickness of $K_{3,3}$: 2.
  - Property:: Thickness of K_n: at least $\left\lfloor \frac{(n+7)}{6} \right\rfloor$, where $n \in \mathbb{Z}^+$.
  - Property:: For connected simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{E}| = e$ edges and $|\mathbf{V}| = v \geq 3$ vertices, then thickness of $\mathcal{G}$ to be at least $\left\lceil \frac{e}{3 \cdot v - 6} \right\rceil$.
  - Property:: For connected simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{E}| = e$ edges, $|\mathbf{V}| = v \geq 3$ vertices and no circuits of length 3, then thickness of $\mathcal{G}$ to be at least $\left\lceil \frac{e}{2 \cdot v - 4} \right\rceil$.

Summary

- Focus: Planarity of graph (contd.).
- Corollaries of Euler's Formula, with proofs.
- Planarity of K_5 and $K_{3,3}$, revisited for proofs using corollaries of Euler's Formula.
- Kuratowski's theorem, with proof.
- Crossing number of graph.
- Thickness of graph.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.