CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 4

Lecture 21: Recurrences

[Recurrence relations; Linear homogeneous recurrences with constant coefficients; Linear nonhomogeneous recurrences]

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Notation table

Symbol / Notation	Meaning
$\triangle(r)$	Characteristic polynomial of linear homogeneous recurrence, with solution of form $a_n = r^n$, where r = constant ($r \neq 0$).

Recurrence relations

- Recurrence relation: rule for determining subsequent terms of sequence from preceding terms of sequence, based on prespecified one or more initial terms of sequence.
  - Property:: Sequence → solution of recurrence relation, if terms of sequence satisfying recurrence relation.
  - Applicability: analyze complexity of algorithms of —
    (i) divide-and-conquer paradigm (recursively dividing problem into fixed number of non-overlapping subproblems, until simple enough to be solved directly),
    (ii) dynamic programming paradigm (recursively breaking down problem into simpler overlapping subproblems, and computing solutions of subproblems to solve overall problem).

Recurrence relations

- Recurrence relation types: linear homogeneous recurrence, linear nonhomogeneous recurrence etc.
  - Property:: **Linear homogeneous recurrence** of k-th order (or degree k) with constant coefficients: recurrence equation of form — $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}) = \underbrace{c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}}_{k \text{ terms}}$ where $c_1, c_2, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$.
  - Property:: Homogeneity: every monomial at R.H.S. expressed as multiples of individual terms of sequence $\{a_n\}$.
  - Property:: Constant coefficients: $c_1, c_2, \ldots, c_k$ not depending on n.

Recurrence relations

- Recurrence relation types:
  - Property:: Linearity: R.H.S. expressed as sum of monomial(s), in which each monomial being function of n of degree at most 1.
  - Property:: Degree (or order) of recurrence: k, as a_n expressed by preceding k terms of sequence $\{a_n\}$.
  - Property:: Solution to given recurrence $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$: sequence $a_0, a_1, \ldots, a_{n-k}, a_{n-k-1}, \ldots, a_{n-2}, a_{n-1}, a_n$: uniquely determined based on (i) given recurrence, and (ii) its k boundary conditions i.e., initial values of 1st k elements of sequence.

Recurrence relations

- Recurrence relation types:
  - Property:: **Linear nonhomogeneous recurrence** of degree k with constant coefficients: recurrence equation of form — $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k \in \mathbb{R}$, $c_k \neq 0$, $F(n)$ = function depending only on n (and not identically zero).
  - Property:: Nonhomogeneity: at least one monomial at R.H.S. not expressed as multiple of term(s) of sequence.
  - Property:: Associated homogeneous recurrence of linear non--homogeneous recurrence: $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$.

Recurrence relations

- Recurrence relation types:
  - Property:: Solution to given recurrence $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \ldots + c_k \cdot a_{n-k} + F(n)$: sequence $\left(a_0^{(p)} + a_0^{(h)}\right), \left(a_1^{(p)} + a_1^{(h)}\right), \ldots, \left(a_{n-k}^{(p)} + a_{n-k}^{(h)}\right), \left(a_{n-k-1}^{(p)} + a_{n-k-1}^{(h)}\right), \ldots, \left(a_{n-1}^{(p)} + a_{n-1}^{(h)}\right), \left(a_n^{(p)} + a_n^{(h)}\right)$, denoted by $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $\left\{a_n^{(h)}\right\}$ = solution of associated homogeneous recurrence, $\left\{a_n^{(p)}\right\}$ = particular solution of recurrence with nonhomogeneous term.

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution: two key ideas for solving — (A) solution of form $a_n = r^n$, where r = constant ($r \neq 0$); (B) solution of form linear combination of two solutions of linear homogeneous recurrence.
  - Property:: For recurrence equation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$, if $a_n = r^n$ ($r \neq 0$) to be solution of that recurrence, then $r^n = c_1 \cdot r^{n-1} + c_2 \cdot r^{n-2} + \cdots + c_k \cdot r^{n-k}$, and conversely. Simplifying above equation: $r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} - \cdots - c_k = 0$.
  - Property:: Characteristic polynomial of recurrence: $\triangle(r)$, where $\triangle(r) = r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} - \cdots - c_k$.

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property:: Characteristic equation of recurrence: $\triangle(r) = 0$, where $\triangle(r) = r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} - \cdots - c_k$.
  - Property:: Characteristic root(s) of characteristic equation of recurrence: roots of $\triangle(r) = 0$.
  - Property:: If both s_n (where, $s_n = c_1 \cdot s_{n-1} + c_2 \cdot s_{n-2} + \cdots + c_k \cdot s_{n-k}$) and t_n (where, $t_n = c_1 \cdot t_{n-1} + c_2 \cdot t_{n-2} + \cdots + c_k \cdot t_{n-k}$) to become solutions of recurrence equation $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$, then $b_1 \cdot s_n + b_2 \cdot t_n$ ($b_1, b_2 \in \mathbb{R}$) also to become solution of same recurrence.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property (contd.)::
    Reasoning: $b_1 \cdot s_n + b_2 \cdot t_n = b_1 \cdot (c_1 \cdot s_{n-1} + c_2 \cdot s_{n-2} + \cdots + c_k \cdot s_{n-k}) + b_2 \cdot (c_1 \cdot t_{n-1} + c_2 \cdot t_{n-2} + \cdots + c_k \cdot t_{n-k})$
    $= c_1 \cdot (b_1 \cdot s_{n-1} + b_2 \cdot t_{n-1}) + c_2 \cdot (b_1 \cdot s_{n-2} + b_2 \cdot t_{n-2}) + \cdots + c_k \cdot (b_1 \cdot s_{n-k} + b_2 \cdot t_{n-k})$.
    Each of k terms to be linear combination of corresponding terms of source solutions s_n and t_n.

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property:: ⟨Theorem (Order 1)⟩: For linear homogeneous recurrence equation of degree $k = 1$ with constant coefficient, $a_n = c_1 \cdot a_{n-1}$ (constant $c_1 \in \mathbb{R}$, $c_1 \neq 0$), sequence $\{a_n\}$ to become solution of that recurrence if and only if $a_n = \lambda \cdot r^n$, where $\lambda =$ constant, $n \in \mathbb{N}$, as well as r obtained from recurrence's characteristic equation $r - c_1 = 0$, $r \neq 0$, based on boundary condition $a_0 = \lambda = C_0$ (say).

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property:: ⟨Theorem (Order 2, distinct roots)⟩: For linear homogeneous recurrence of degree $k = 2$ with constant coefficients, $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ (constants $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$), sequence $\{a_n\}$ to become solution of that recurrence iff $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$, where λ_1, λ_2 considered as constants, $n \in \mathbb{N}$, as well as r_1 and r_2 as two distinct roots of recurrence's characteristic equation $r^2 - c_1 \cdot r - c_2 = 0$, $r \neq 0$.

  Proof: Two cases to prove — (1) $\{a_n\}$ when $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$ to become solution of given recurrence; (2) for given recurrence, if $\{a_n\}$ to be considered as solution, then $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  Proof of ⟨Theorem (Order 2, distinct roots)⟩ contd.
  Case-1: If $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$, then $\{a_n\}$ to become solution.
  Given: $r_1, r_2 =$ two distinct roots of $r^2 - c_1 \cdot r - c_2 = 0$.
  So, $(r_1)^2 = c_1 \cdot r_1 + c_2$, and $(r_2)^2 = c_1 \cdot r_2 + c_2$.
  Then, R.H.S. of given recurrence $= c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$
  $= c_1 \cdot (\lambda_1 \cdot (r_1)^{n-1} + \lambda_2 \cdot (r_2)^{n-1}) + c_2 \cdot (\lambda_1 \cdot (r_1)^{n-2} + \lambda_2 \cdot (r_2)^{n-2})$
  $= \lambda_1 \cdot (r_1)^{n-2} \cdot (c_1 \cdot r_1 + c_2) + \lambda_2 \cdot (r_2)^{n-2} \cdot (c_1 \cdot r_2 + c_2)$.
  $= \lambda_1 \cdot (r_1)^{n-2} \cdot (r_1)^2 + \lambda_2 \cdot (r_2)^{n-2} \cdot (r_2)^2 = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n = a_n$.
  So, $\{a_n\} =$ solution, if $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  Proof of ⟨Theorem (Order 2, distinct roots)⟩ contd-2.
  Case-2: Every solution $\{a_n\}$ to be of form $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$ for every $n \in \mathbb{N}$. [Use of principle of mathematical induction.]
  Let $n_0 = 0$, and $T = \{n \mid$ solution $\{a_n\}$ fulfilling required form$\}$.
  (Basis step) Assume $\{a_0\}$, $\{a_1\}$ satisfied for $n = 0{,}1$, with $a_0 = C_0$, $a_1 = C_1$.
  So, C_0 and C_1 expressed as: $C_0 = \lambda_1 + \lambda_2$, $C_1 = \lambda_1 \cdot r_1 + \lambda_2 \cdot r_2$.
  Given $r_1 \neq r_2$, solving λ_1 and λ_2 by simplifying above two equations:
  $\lambda_1 = \frac{C_1 - C_0 \cdot r_2}{r_1 - r_2}$, $\lambda_2 = \frac{C_0 \cdot r_2 - C_1}{r_1 - r_2}$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  Proof of ⟨Theorem (Order 2, distinct roots)⟩ contd-3.
  Then, $\{a_n\}$ with $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$ and above values of λ_1, λ_2, to be solution satisfying two initial conditions. So, $0, 1 \in T$.
  (Inductive step) Choosing any $n \geq 2$, and assuming that $\{a_0\}$, $\{a_1\}, \ldots, \{a_{n-1}\}$ fulfilling required form (strong inductive step), then given recurrence $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$
  $= c_1 \cdot (\lambda_1 \cdot (r_1)^{n-1} + \lambda_2 \cdot (r_2)^{n-1}) + c_2 \cdot (\lambda_1 \cdot (r_1)^{n-2} + \lambda_2 \cdot (r_2)^{n-2})$
  $= \lambda_1 \cdot (r_1)^{n-2} \cdot (c_1 \cdot r_1 + c_2) + \lambda_2 \cdot (r_2)^{n-2} \cdot (c_1 \cdot r_2 + c_2)$.
  $= \lambda_1 \cdot (r_1)^{n-2} \cdot (r_1)^2 + \lambda_2 \cdot (r_2)^{n-2} \cdot (r_2)^2 = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:

  Proof of ⟨Theorem (Order 2, distinct roots)⟩ contd-4.

  Then, $\{a_n\}$ with form $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n$ to be solution of given recurrence. So, $n \in \mathcal{T}$.

  By Strong Mathematical Induction, conclusion: $\mathcal{T} = \mathbb{N}$. ■

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property:: ⟨Theorem (Order 2, same roots)⟩: For linear homogeneous recurrence of degree $k = 2$ with constant coefficients, $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2}$ (constants $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$), sequence $\{a_n\}$ to become solution of that recurrence iff $a_n = \lambda_1 \cdot (r_0)^n + \lambda_2 \cdot n \cdot (r_0)^n$, where λ_1, λ_2 considered as constants, $n \in \mathbb{N}$, as well as r_0 as only root of recurrence's characteristic equation $r^2 - c_1 \cdot r - c_2 = 0, r \neq 0$.

Linear homogeneous recurrence relations

- Linear homogeneous recurrence (of degree k) solution:
  - Property:: ⟨Theorem (Order k)⟩: For linear homogeneous recurrence of degree $k > 2$ with constant coefficients, $a_n = c_1 \cdot a_{n-1} + c_2 \cdot a_{n-2} + \cdots + c_k \cdot a_{n-k}$ (constants $c_1, c_2, \dots, c_k \in \mathbb{R}$, $c_k \neq 0$), sequence $\{a_n\}$ to become solution of that recurrence iff $a_n = \lambda_1 \cdot (r_1)^n + \lambda_2 \cdot (r_2)^n + \cdots + \lambda_k \cdot (r_k)^n$, where $\lambda_1, \lambda_2, \dots, \lambda_k$ considered as constants, $n \in \mathbb{N}$, as well as $r_1, r_2, \dots, r_k$ as k distinct roots of recurrence's characteristic equation $r^k - c_1 \cdot r^{k-1} - c_2 \cdot r^{k-2} - \cdots - c_k = 0, r \neq 0$.

Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-1:: Given recurrence $a_n = a_{n-1} + 2 \cdot a_{n-2}$, $a_0 = 2$, $a_1 = 7$.
    Case of second-order (i.e. degree $k = 2$) linear homogeneous recurrence with constant coefficients $c_1 = 1$, $c_2 = 2$.
    Characteristic equation: $r^2 - r - 2 = 0$, with roots $\overbrace{r_1 = 2, r_2 = -1}^{\text{distinct}}$.
    So, according to theorem, sequence $\{a_n\}$, where $a_n = \lambda_1 \cdot 2^n + \lambda_2 \cdot (-1)^n$, for constants λ_1, λ_2, to be solution of given recurrence equation.
    Applying initial conditions (i.e., $n = 0$, $n = 1$) to find λ_1, λ_2:
    $a_0 = 2 = \lambda_1 + \lambda_2$; $a_1 = 7 = 2 \cdot \lambda_1 - \lambda_2$. Solving, $\lambda_1 = 3$, $\lambda_2 = -1$.
    So, solution of given recurrence: $\{a_n\}$ where $a_n = 3 \cdot 2^n - (-1)^n$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-2:: Fibonacci recurrence $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$.
    Case of second-order (i.e. degree $k = 2$) linear homogeneous recurrence with constant coefficients $c_1 = 1$, $c_2 = 1$.
    Characteristic equation: $r^2 - r - 1 = 0$, with distinct roots $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$.
    So, according to theorem, sequence $\{f_n\}$, where $f_n = \lambda_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$, for constants λ_1, λ_2, to be solution of Fibonacci recurrence equation. That solution helped to obtain its terms.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-2 contd.::
    Applying initial conditions to find λ_1, λ_2:
    $f_0 = 0 = \lambda_1 + \lambda_2$ $f_1 = 1 = \lambda_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + \lambda_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)$.
    Solving, $\lambda_1 = \frac{1}{\sqrt{5}}$, $\lambda_2 = -\frac{1}{\sqrt{5}}$.
    So, solution of Fibonacci recurrence to produce its terms: $\{f_n\}$, where $f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-3:: Given recurrence equation $a_n = 6 \cdot a_{n-1} - 9 \cdot a_{n-2}$, with initial conditions $a_0 = 1$, $a_1 = 6$.
    Case of second-order (i.e. degree $k = 2$) linear homogeneous recurrence with constant coefficients $c_1 = 1$, $c_2 = 2$.
    Characteristic equation: $r^2 - 6 \cdot r + 9 = 0$, with same root $r_0 = 3$.
    So, according to theorem, sequence $\{a_n\}$ when $a_n = \lambda_1 \cdot 3^n + \lambda_2 \cdot n \cdot 3^n$ (for constants λ_1, λ_2) to be solution of given recurrence equation.
    Applying initial conditions to find λ_1, λ_2:
    $a_0 = 1 = \lambda_1$; $a_1 = 6 = 3 \cdot \lambda_1 + 3 \cdot \lambda_2 = 3 + 3 \cdot \lambda_2$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-3 contd.::
    Solving, $\lambda_1 = 1$, $\lambda_2 = 1$.
    So, solution of given recurrence: $\{a_n\}$, where $a_n = 3^n + n \cdot 3^n$.
  - Example-4:: Given recurrence $a_n = 6 \cdot a_{n-1} - 11 \cdot a_{n-2} + 6 \cdot a_{n-3}$, with initial conditions $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.
    Case of third-order (i.e. degree $k = 3$) linear homogeneous recurrence with constant coefficients $c_1 = 6$, $c_2 = -11$, $c_3 = 6$.
    Characteristic equation: $r^3 - 6 \cdot r^2 + 11 \cdot r - 6 = 0$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-4 contd.::
    Discriminant $= \frac{4 \cdot \left((-6)^2 - 3 \cdot 1 \cdot 11\right)^3 - \left(2 \cdot (-6)^3 - 9 \cdot 1 \cdot (-6) \cdot 11 + 27 \cdot 1^2 \cdot (-6)\right)^2}{27 \cdot 1^2} = \frac{108 - 0}{27} = 4$
    as, for cubic equation $a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0$,
    discriminant $= \frac{4 \cdot \left(b^2 - 3 \cdot a \cdot c\right)^3 - \left(2 \cdot b^3 - 9 \cdot a \cdot b \cdot c + 27 \cdot a^2 \cdot d\right)^2}{27 \cdot a^2}$.
    Positive discriminant indicating three distinct roots.
    Further, $r^3 - 6 \cdot r^2 + 11 \cdot r - 6 = (r-1) \cdot (r-2) \cdot (r-3)$, indicating distinct roots $r_1 = 1$, $r_2 = 2$, $r_3 = 3$.

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Linear homogeneous recurrence relations

- Linear homogeneous recurrence examples:
  - Example-4 contd-2.::
    So, according to theorem, sequence $\{a_n\}$ when $a_n = \lambda_1 \cdot 1^n + \lambda_2 \cdot 2^n + \lambda_3 \cdot 3^n$ (for constants $\lambda_1, \lambda_2, \lambda_3$) to be solution of given recurrence equation.
    Applying initial conditions to find $\lambda_1, \lambda_2, \lambda_3$: $a_0 = 2 = \lambda_1 + \lambda_2 + \lambda_3$;
    $a_1 = 5 = \lambda_1 + 2 \cdot \lambda_2 + 3 \cdot \lambda_3$; $a_2 = 15 = \lambda_1 + 4 \cdot \lambda_2 + 9 \cdot \lambda_3$.
    Solving, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$.
    So, solution of given recurrence: $\{a_n\}$, where $a_n = 1 - 2^n + 2 \cdot 3^n$.

Linear nonhomogeneous recurrence relations

- Linear nonhomogeneous recurrence examples:
  - Example-1:: Given recurrence equation $a_n = 3 \cdot a_{n-1} + 2 \cdot n$, to find all solutions.
    Case of first-order (i.e. degree $k = 1$) linear nonhomogeneous recurrence with constant coefficient $c_1 = 3$, and $F(n) = 2 \cdot n$.
    Associated homogeneous recurrence: $a_n = 3 \cdot a_{n-1}$, case of linear homogeneous recurrence with constant coefficient $c_1 = 3$, $k = 1$.
    So, according to theorem, sequence $\left\{a_n^{(h)}\right\}$ when $a_n^{(h)} = \lambda \cdot 3^n$, for constant λ, to be solution of associated homogeneous recurrence equation.

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Linear nonhomogeneous recurrence relations

- Linear nonhomogeneous recurrence examples:
  - Example-1 contd.
    Next to find particular solution of given nonhomogeneous recurrence.
    $F(n) = 2 \cdot n$ being linear polynomial (i.e. degree 1), reasonable trial solution → linear function in n, like $c \cdot n + d$, $c, d =$ constants.
    To verify whether particular solution of form: $a_n = c \cdot n + d$.
    Replacing in given recurrence: $c \cdot n + d = 3 \cdot (c \cdot (n-1) + d) + 2 \cdot n$.
    So, $(2 + 2 \cdot c) \cdot n + (2 \cdot d - 3 \cdot c) = 0$, of form $c' \cdot n + d'$.
    Further, $c \cdot n + d$ to be considered as particular solution, if —
    (i) $(2 + 2 \cdot c) = 0$; (ii) $(2 \cdot d - 3 \cdot c) = 0$.

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Linear nonhomogeneous recurrence relations

- Linear nonhomogeneous recurrence examples:
  - Example-1 contd-2.
    For $(2 + 2 \cdot c) = 0$: $c = -1$.
    For $(2 \cdot d - 3 \cdot c) = 0$ and $c = -1$: $d = -3/2$.
    So, particular solution of given recurrence: $a_n^{(p)} = -n - 3/2$.
    So, according to theorem, all solutions of given recurrence to be of form: sequence $\{a_n\}$, where $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \lambda \cdot 3^n$, for constant λ.
    Value of λ to be further calculated based on boundary conditions, if given. In this case, no boundary condition specified.

Summary

- Focus: Recurrences.
- Recurrence relations.
- Linear homogeneous recurrence.
- Solution of linear homogeneous recurrence of k-th order with constant coefficients, and related theorems.
- Linear nonhomogeneous recurrence.
- Solution of linear nonhomogeneous recurrence of degree k with constant coefficients, and related theorems.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, Schaum's Outline of Theory and Problems of Discrete Mathematics, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Recurrence relations:: [Ros19]:527-533.
- Linear homogeneous recurrence:: [Ros19]:540-547.
- Solution of linear homogeneous recurrence of k-th order with constant coefficients:: [Ros19]:541-547.
- Linear nonhomogeneous recurrence:: [Ros19]:547-550.
- Solution of linear nonhomogeneous recurrence of degree k with constant coefficients:: [Ros19]:547-550.