CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 14: Counting

[Sum rule; Applicability of sum rule to set theory; Subtraction rule, its applicability to set theory; Division rule, its applicability to set theory]

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Sum rule

- Sum rule [generalized case]: If any task T of job can be done either in one of n_1 ways, or in one of n_2 ways,..., or in one of n_m ways, where any n_i ways of doing T not same as any n_j ways (for all pairs i and j, $1 \leq i < j \leq m$), then $(n_1 + n_2 + \cdots + n_m)$ total number of ways to do task T of job.
  - Sum rule [two-procedure case]: If any task can be done either in one of n_1 ways or in one of n_2 ways, where none of set of n_1 ways same as any of set of n_2 ways, then $(n_1 + n_2)$ total number of ways to do given task of job.
  - Resembling exclusive disjunction of procedures of task.

Sum rule

- Sum rule examples:
  - Example-1:: case of choosing a representative to institute committee from 32 department faculty members or 275 UG students.
    Task of job = choosing representative.
    Possible manners of performing (i.e., procedures for task): 2.
    Procedure-1: Number of ways to choose a representative from 32 department faculty members = 32.
    Procedure-2: Number of ways to choose a representative from 275 UG students = 275.

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Sum rule

- Sum rule examples:
  - Example-1 (contd.)::
    Ways of performing Procedure-1 not same as ways of performing Procedure-2, due to none common from both department faculty members and UG students.
    Then, according to sum rule, number of possible ways to choose this representative = 32 + 275 = 307.

Sum rule

- Sum rule examples:
  - Example-2:: case of choosing a major project from one of the three lists containing 23, 15, and 19 possible projects, with no project in common in these three lists.
    Task of job = picking project; possible manners of performing: 3.
    Procedure-1: Number of ways to pick a project from List-A = 23.
    Procedure-2: Number of ways to pick a project from List-B = 15.
    Procedure-3: Number of ways to pick a project from List-C = 19.
    As three procedures pairwise not same, then according to sum rule, number of possible ways to pick project = 23 + 15 + 19 = 57.

Sum rule

- Sum rule examples:
  - Example-3:: case of supported variable names in programming language BASIC (Beginners' All-purpose Symbolic Instruction Code), where name = string of one or two alphanumeric characters, starting with letter, not distinguishing uppercase and lowercase letters, and excluding five reserved strings of two characters; alphanumeric character = either one of 26 English letters or one of 10 digits.
    Task of job = counting variables; possible manners of performing: 2.
    Procedure-1: Number of variables of one-character long = number of ways to form one-character long variables = v_1 (say).

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Sum rule

- Sum rule examples:
  - Example-3 (contd.)::
    Procedure-2: Number of variables of two-character long excluding five reserved strings = number of ways to form two-character long variables excluding five reserved strings = v_2 (say).
    Now, for Procedure-1, v_1 = 26, due to one-character variable name to must add one of 26 English letters.
    For Procedure-2, two-character variable name to begin with one of 26 English letters, and to end with one of (26+10=) 36 alphanumeric characters.

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Sum rule

- Sum rule examples:
  - Example-3 (contd-2.)::
    Since, selection of characters in two-character variable name, to be performed in sequence and independently, applying product rule, number of ways to form two-character long variables $= 26 \cdot 36 = 936$. Excluding five reserved strings, $v_2 = 936 - 5 = 931$.
    Ways of performing Procedure-1 independent of ways of performing Procedure-2.
    Then, according to sum rule, number of possible ways to form variable names supported in BASIC = $v_1 + v_2$ = 26 + 931 = 957.

Sum rule

- Sum rule applied to set theory: for m pairwise-disjoint finite sets $A_1, A_2, \ldots, A_m$, choosing an element in union $(A_1 \cup A_2 \cup \cdots \cup A_m)$ done by either choosing an element from A_1 (done in $|A_1|$ ways), or an element from A_2 (done in $|A_2|$ ways),..., or an element from A_m (done in $|A_m|$ ways), but not selecting from more than one set, resulting in (as per sum rule), where $1 \leq i, j \leq m$:

$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$, as $A_i \cap A_j = \emptyset$, for all i, j,

i.e., number of elements in union of $A_1, A_2, \ldots, A_m$ = sum of number of elements in each of m sets.

Subtraction rule

- Subtraction rule [generalized case]: If any task T of job be done either in one of n_1 ways, or in one of n_2 ways,..., or in one of n_m ways, where multiple common ways possible among any of $n_1, n_2, \cdots, n_m$ ways of doing T, then number of ways to do task T:

$$(n_1 + n_2 + \cdots + n_m) - \left(\sum_{1 \le i < j \le m} n_{ij}\right) + \left(\sum_{1 \le i < j < k \le m} n_{ijk}\right) - \cdots + \left((-1)^{m+1} \cdot n_{12\ldots m}\right)$$

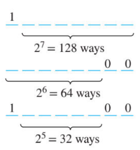
$$= \sum_{k=1}^{m} \left((-1)^{k+1} \cdot \left(\sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} n_{i_1 i_2 \ldots i_k}\right)\right)$$

Subtraction rule

- Subtraction rule:
  - Subtraction rule [two-procedure case]: If any task be done either in one of n_1 ways or in one of n_2 ways, where n_1 and n_2 having some common ways, then number of ways to do given task $= n_1 + n_2 -$ (number of ways common between n_1 and n_2) $= (n_1 + n_2 - n_{12})$.
  - Also called principle of inclusion-exclusion (particularly in relation to set theory).
  - Resembling inclusive disjunction (or simply, disjunction) of procedures of task.

Subtraction rule

- Subtraction rule examples:
  - Example-1:: case of counting bit strings of length eight either starting with bit $(1)_2$ or ending with two-bit combination $(00)_2$. Task of job = counting bit strings; possible manners of performing: 2 procedures.
    Procedure-1: Number of ways to form bit strings of length eight starting with bit $(1)_2$ = 128.
    Procedure-2: Number of ways to form bit strings of length eight ending with two-bit combination $(00)_2$ = 64.



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[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Subtraction rule

- Subtraction rule examples:
  - Example-1 (contd.)::
    Procedure-1 and Procedure-2 share common ways to form strings. Number of common ways to form bit strings of length eight, with both starting with $(1)_2$ and ending with $(00)_2$ = 32.
    These three counts of number of ways based on 'product rule'.
    Then, according to subtraction rule, number of bit strings of length eight starting with $(1)_2$ or end with $(00)_2$ = number of ways to form bit string of length eight starting with $(1)_2$ or end with $(00)_2$ = 128 + 64 − 32 = 160.

Subtraction rule

- Subtraction rule on set theory [Principle of inclusion-exclusion]: for m non-disjoint finite sets $A_1, A_2, \dots, A_m$, choosing an element in union $(A_1 \cup A_2 \cup \cdots \cup A_m)$ done by choosing an element from either A_1 (in $|A_1|$ ways), or A_2 (in $|A_2|$ ways),..., or A_m (in $|A_m|$ ways), while successively countering over-generous inclusion and exclusion, resulting in (as per subtraction rule), where $1 \leq i, j \leq m$:

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{k=1}^{m} \left((-1)^{k+1} \cdot \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \right) \right)$$

Division rule

- Division rule [generalized case]: If any task T of job be done using single procedure in n ways, where for each of n ways, there are exactly d equivalent ways of doing T, then there are $({}^{n}/_{d})$ distinct ways to do task T of job.
  - Division rule [based on function]: For function $f: \mathbf{A} \rightarrow \mathbf{B}$ with finite sets $\mathbf{A}$ and $\mathbf{B}$, and for every value $y \in \mathbf{B}$, exactly d values of $x \in \mathbf{A}$ present such that $f(x) = y$ (i.e., f to become d-to-one), then $|\mathbf{B}| = {}^{|\mathbf{A}|}/_{d}$.
  - Division rule applied to set theory: for n pairwise disjoint finite sets $A_1, A_2, \dots, A_n$, each with exactly d elements, i.e., $|A_1| = |A_2| = \cdots = |A_n|$, if $\mathbf{A} = (A_1 \cup A_2 \cup \cdots \cup A_m)$, then $n = {}^{|\mathbf{A}|}/_{d}$.

Division rule

- Division rule examples:
  - Example-1:: case of different seating of four persons around circular table with four seats, where two seating positions of each person to become equivalent, if same left neighbor **and** same right neighbor.
    Task of job = counting distinct seating arrangements.
    Procedure for task: arbitrarily selecting any seat at table, and labeling it as seat-1; assigning rest of seats in numerical order, clockwise around table; choosing one of 4 persons for seat-1; choosing one of remaining 3 persons for seat-2; choosing one of remaining 2 persons for seat-3; assigning last person for seat-4.

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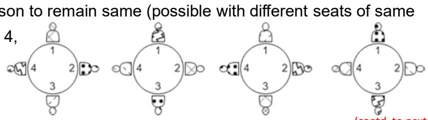
Division rule

- Division rule examples:
  - Example-1 (contd.)::
    Number of ways to perform Procedure = number of ways to order four people on four seats without any distinctions $= 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
    Within 24 ways, number of ways in which left **and** right neighbors of every person to remain same (possible with different seats of same person) = 4, shown in adjacent diagram.



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Division rule

- Division rule examples:
  - Example-1 (contd-2.)::
    Then, according to division rule, $d = 4$, and number of distinct seating arrangements = ${}^{24}/_{4}$ = 6.

Summary

- Focus: Basic counting principles (contd.).
- Sum rule definition for generalized case.
- Sum rule definition in two-procedure counting.
- Practical examples to demonstrate applicability of sum rule.
- Applicability of sum rule to set-theoretic problems.
- Subtraction rule definition for generalized case.
- Subtraction rule definition in two-procedure counting.
- Practical examples to demonstrate applicability of subtraction rule.
- Applicability of subtraction rule to set-theoretic problems.

Summary

- Division rule definition for generalized case.
- Division rule definition based on function.
- Practical examples to demonstrate applicability of division rule.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip17] Seymour Lipschutz, Marc L. Lipson, Varsha H. Patil, *Discrete Mathematics (Schaum's Outlines)*, Revised Third edition, McGraw-Hill Education, 2017.
5. https://brilliant.org/wiki/rule-of-sum-and-rule-of-product-problem-solving/.

Further Reading

- Sum rule definition for generalized case:: [Ros21]:410.
- Sum rule definition in two-procedure counting:: [Ros21]:409.
- Practical examples to demonstrate applicability of sum rule:: [Ros21]:409-410,411-412.
- Applicability of sum rule to set-theoretic problems:: [Ros21]:410-411.
- Subtraction rule definition in two-procedure counting:: [Ros21]:413.
- Practical examples to demonstrate applicability of subtraction rule:: [Ros21]:412-414.
- Applicability of subtraction rule to set-theoretic problems:: [Ros21]:413.
- Division rule definition for generalized case:: [Ros21]:414.

Further Reading

- Division rule definition based on function:: [Ros21]:414.
- Practical examples to demonstrate applicability of division rule:: [Ros21]:414-415.

Lecture Exercises: Problem 1 [Ref: Gate 2017, Q.47, p.21 (Set-1)]

The number of integers between 1 and 500 (both inclusive) that are divisible by 3 or 5 or 7 is _______.