CS34110 Discrete Mathematics and Graph Theory

UNIT – IV, Module – 1

Lecture 35: Graph Planarity

[Planar, nonplanar graphs; Region, degree of region; Elementary subdivision; Homeomorphism; Kuratowski's two graphs; Euler's Formula]

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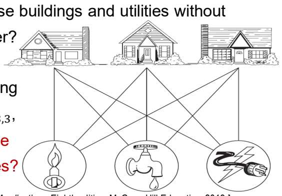
Notation table

Symbol / Notation	Meaning
R	Region in planar representation of graph.
$\deg(R)$	Degree of region R in planar representation of graph.
K_5	Kuratowski's first graph.
$K_{3,3}$	Kuratowski's second graph.

Graph planarity

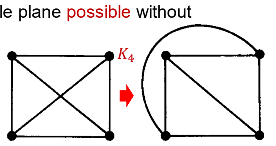
- Motivation for graph planarity: to find answers to some real-life problems. E.g., buildings-and-utilities problem.
  - Property:: Possibility to join these buildings and utilities without connections crossing each other? Rephrased problem: Capturing given scenario as graph, resulting into complete bipartite graph $K_{3,3}$, possibility to draw $K_{3,3}$ in a plane without crossing any of its edges?



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Planar graph: graph or multigraph, where its drawing (i.e., geometric representation) in single plane possible without crossing any of its edges.
  - Property:: Possibility of planar graph to be embedded (i.e., drawn) in plane surface.
  - Property:: Crossing of edges: intersection of lines or arcs representing edges at some point other than their common vertices.

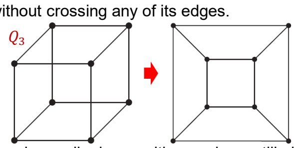


Redrawing K_4 in a plane → Planar representation of K_4

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Planar graph:
  - Property:: Planar representation of graph: embedding of graph in single plane without crossing any of its edges.



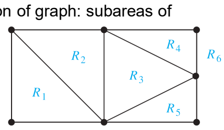
Planar representation of Q_3

  - Note: planar graph usually drawn with crossings; still planar, due to possibility to draw it in different way without crossings.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

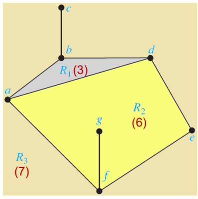
- Planar graph:
  - Property:: Regions in planar representation of graph: subareas of plane generated while drawing planar representation of graph (i.e., embedding graph in plane); also called faces.
  - Property:: Both bounded and unbounded regions possible. In fig. R_6 → unbounded.
  - Property:: Unbounded region: also called exterior face.
  - Property:: Nonplanar graph: graph not possible to be drawn on plane without crossover between its edges.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Planar graph:
  - Property:: Degree of region R in planar representation of graph: denoted by $\deg(R)$, where $\deg(R)$ = number of edges on boundary of R, such that an edge to contribute two to $\deg(R)$ when that edge to be traced out twice while tracing boundary of R.



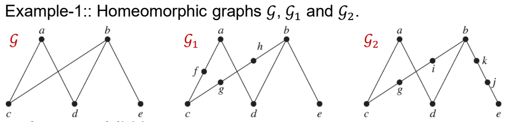
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph planarity

- Planar graph:
  - Property:: Elementary subdivision of planar graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ obtained after performing following operations in sequence — (i) removing existing edge $\{u, v\}$ such that $\mathbf{E} \setminus \{u, v\}$, i.e., $\mathcal{G} - \{u, v\}$; (ii) adding new vertex w, such that $\mathbf{V}' = \mathbf{V} \cup \{w\}$; and (iii) adding new edges such that $\mathbf{E}' = \mathbf{E} \setminus \{u, v\} \cup \{\{u, w\}, \{w, v\}\}$.
  - Property:: Homeomorphism: Homeomorphic graphs $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$ of planar graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, if $\mathcal{G}_1$ and $\mathcal{G}_2$ obtained from $\mathcal{G}$ by a sequence of elementary subdivisions.

Graph planarity

- Planar graph:
  - Example-1:: Homeomorphic graphs $\mathcal{G}$, $\mathcal{G}_1$ and $\mathcal{G}_2$.



$\mathcal{G} \xrightarrow{\text{elementary subdivision}} \mathcal{G}$: empty sequence of elementary subdivisions.

$\mathcal{G} \xrightarrow{\text{elementary subdivisions}} \mathcal{G}_1$: (1) removing $\{a, c\}$, adding f, adding $\{a, f\}$, $\{f, c\}$; (2) removing $\{b, c\}$, adding g, adding $\{b, g\}$, $\{g, c\}$; (3) removing $\{b, g\}$, adding h, adding $\{b, h\}$, $\{h, g\}$.

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[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

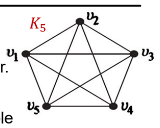
Graph planarity

- Planar graph:
  - Example-1 contd.::
    $\mathcal{G} \xrightarrow{\text{elementary subdivisions}} \mathcal{G}_2$: (1) removing $\{b, c\}$, adding g, adding $\{b, g\}$, $\{g, c\}$; (2) removing $\{b, g\}$, adding i, adding $\{b, i\}$, $\{i, g\}$; (3) removing $\{b, e\}$, adding j, adding $\{b, j\}$, $\{j, e\}$; (4) removing $\{b, j\}$, adding k, adding $\{b, k\}$, $\{k, j\}$.

Graph planarity

- Planar graph:
  - Benefits:: (i) planar graphs for modeling of electronic circuits to print on single board with no wires crossing;
    (ii) planar subgraphs (after vertex partition) for multi-layered circuit print, with insulated wires at crossings, and then objective to draw with fewest possible crossings;
    (iii) planar graphs in design of road networks to connect a group of cities by highways without using underpasses or overpasses.

Graph planarity



- Nonplanarity of graph:
  - Property:: ⟨Theorem: Planarity of K_5⟩: K_5 nonplanar. [K_5: called Kuratowski's first graph]

  Proof: [Based on Jordan curve theorem: "every simple curve divides plane into two regions;" Special case: every simple polygon divides plane into two regions — interior (consisting of points inside curve), and exterior (consisting of points outside curve)]
  Idea: drawing $K_5 = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}$, in sequence, starting from simple pentagon, and successively adding edge(s) (incident on existing vertex) at each stage.

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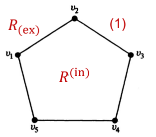
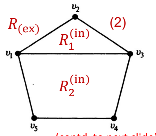
Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of K_5⟩ contd.:

  In figure-(1), simple pentagon to connect all vertices of K_5 by one edge, forming one circuit. As per Jordan curve theorem, number of regions $= 2$, interior region $R^{(\text{in})}$ and exterior region $R^{(\text{ex})}$. Adding edge $\{v_1, v_3\}$ to figure-(1) resulting in figure-(2), whose drawing possible through inside or outside pentagon, without intersecting edges of figure-(1). Let $\{v_1, v_3\}$ be drawn through inside.





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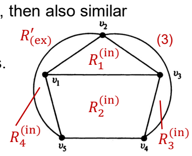
Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of K_5⟩ contd-2.:

  In case when $\{v_1, v_3\}$ be drawn through outside, then also similar arguments possible to prove theorem.
  $\{v_1, v_3\}$ splitting $R^{(\text{in})}$ into $R_1^{(\text{in})}$ and $R_2^{(\text{in})}$ regions.
  In figure-(2), adding edges $\{v_2, v_4\}$ and $\{v_2, v_5\}$ resulting in figure-(3), whose drawing through inside of pentagon of figure-(2) not possible without intersecting edges. So, these two edges drawn through outside of pentagon of figure-(2), splitting $R^{(\text{ex})}$ into $R_1^{(\text{ex})}$,$R_3^{(\text{in})}$,$R_4^{(\text{in})}$



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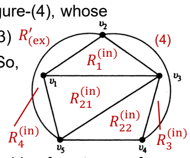
Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of K_5⟩ contd-3.:

  In figure-(3), adding edge $\{v_3, v_5\}$ resulting in figure-(4), whose drawing through outside of pentagon of figure-(3) not possible without intersecting edge $\{v_2, v_4\}$. So, this edge drawn through inside of pentagon of figure-(3), splitting $R_2^{(\text{in})}$ into $R_{21}^{(\text{in})}$, $R_{22}^{(\text{in})}$ regions.
  At this point, adding edge $\{v_1, v_4\}$ in figure-(4) resulting in figure-(5), whose drawing through inside of pentagon of figure-(4) not possible without intersecting edge $\{v_1, v_4\}$.



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Graph planarity

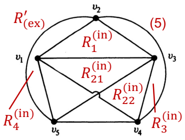
- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of K_5⟩ contd-4.:

  But drawing of $\{v_1, v_4\}$ through outside of pentagon of figure-(4) also not possible without intersecting edge $\{v_2, v_5\}$.

  Thus, at least one edge of K_5 found, with no possibility to be placed inside or outside pentagon without crossover.

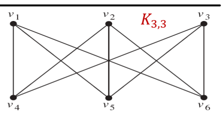
  Hence, K_5 not possible to be embedded in plane. ■



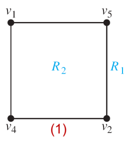
  - Property:: K_5 (or, Kuratowski's first graph) = nonplanar graph with smallest number of vertices (only 5 vertices).

Graph planarity

- Nonplanarity of graph:
  - Property:: ⟨Theorem: Planarity of $K_{3,3}$⟩: $K_{3,3}$ nonplanar. [$K_{3,3}$: also called Kuratowski's second graph]



  Proof: In any planar representation of $K_{3,3}$, vertices v_1 and v_2 must be connected to both v_4 and v_5 through four edges $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$, shown in figure-(1). These four edges form 'closed curve' square, and as per Jordan curve theorem, number of regions $= 2$, exterior region R_1 and interior region R_2.



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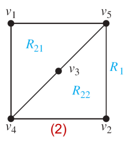
Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of $K_{3,3}$⟩ contd.:

  Adding vertex v_3 to figure-(1) resulting in figure-(2), whose drawing possible through inside or outside of square in either R_1 or R_2.

  Let v_3 be drawn through R_2.



(2)

  In case if v_3 drawn through R_1, then also similar arguments possible to prove theorem.

  In figure-(2), v_3 must be connected to both v_4 and v_5 through edges $\{v_3, v_4\}$, $\{v_3, v_5\}$, splitting R_2 into R_{21} and R_{22} regions.

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Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of $K_{3,3}$⟩ contd-2.:

  At this point, adding vertex v_6 in R_{21} region of figure-(2) not possible without crossing, due to drawing of edge $\{v_2, v_6\}$ from v_6 not possible without intersecting any of existing edges $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$.

  But, adding v_6 in R_{22} region of figure-(2) also not possible without crossing, due to drawing of edge $\{v_1, v_6\}$ without intersecting any of existing edges $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$ not possible.

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Graph planarity

- Nonplanarity of graph:

  Proof of ⟨Theorem: Planarity of $K_{3,3}$⟩ contd-3.:

  Further, drawing of v_6 in R_1 region of figure-(2) also not possible without crossing, due to drawing of edge $\{v_3, v_6\}$ not possible without intersecting any of existing edges $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$.

  Thus, at least one vertex of $K_{3,3}$ found, with no possibility to be placed inside or outside square of figure-(2) without crossover.

  Hence, $K_{3,3}$ not possible to be embedded in plane. ∎

  - Note:: Answer to buildings-and-utilities problem: not possible to connect three houses and three utilities in plane without a crossing.

Graph planarity

- Nonplanarity of graph:
  - Property:: $K_{3,3}$ (or, Kuratowski's second graph) = nonplanar graph with smallest number of edges (only 9 edges).

Graph planarity

- Euler's formula:
  - Property:: ⟨Theorem: Euler's Formula⟩: If $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ be connected planar simple graph with $|\mathbf{E}| = e$ edges and $|\mathbf{V}| = v$ vertices, and if r be number of regions in planar representation of $\mathcal{G}$, then $r = e - v + 2$.

  Proof: [By principle of mathematical induction]

  Idea: constructing sequence of planar subgraphs inductively as — $\mathcal{G}_1$, $\mathcal{G}_2, \ldots, \mathcal{G}_e = \mathcal{G}$, successively adding one edge (incident on existing vertex) at each stage; such construction possible, $\mathcal{G}$ being connected.

  For any arbitrary integer n $(1 \le n \le e)$, r_n, e_n, v_n to represent count of regions, edges, vertices of planar representation of $\mathcal{G}_n$ respectively.

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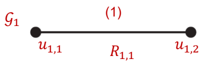
Graph planarity



- Euler's formula:

  Proof of ⟨Euler's Formula⟩ contd:

  (*Basis step*) Arbitrarily picking one edge of $\mathcal{G}$ from $\mathbf{E}$ to obtain $\mathcal{G}_1$. Then, for planar $\mathcal{G}_1$, $r_1 = e_1 - v_1 + 2$, because $e_1 = 1$ (single edge), $v_1 = 2$ (two vertices $u_{1,1}$ and $u_{1,2}$), and $r_1 = 1$ (single region $R_{1,1}$), as evident in figure-(1).

  (*Inductive step*) Inductive hypothesis: premise that Euler's formula to become true for planar subgraph $\mathcal{G}_k$ $(1 < k < n)$, i.e., $r_k = e_k - v_k + 2$.

  Let arbitrary edge $\{u_{k+1,i}, u_{k+1,j}\}$ of $\mathcal{G}$, picked from $\mathbf{E}$, be added to $\mathcal{G}_k$ to construct $\mathcal{G}_{k+1}$. Two possible cases to handle for $\mathcal{G}_{k+1}$.

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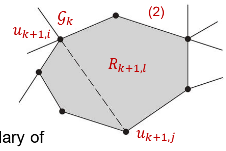
Graph planarity



- Euler's formula:

  Proof of ⟨Euler's Formula⟩ contd-2:

  (A) Both vertex $u_{k+1,i}$ and vertex $u_{k+1,j}$ already included in $\mathcal{G}_k$ in figure-(2).

  Then, $u_{k+1,i}$ and $u_{k+1,j}$ must be on boundary of common region $R_{k+1,l}$ of $\mathcal{G}_k$, else impossible to add edge $\{u_{k+1,i}, u_{k+1,j}\}$ to $\mathcal{G}_k$ without crossing other edges of $\mathcal{G}_k$ (thereby satisfying $\mathcal{G}_{k+1}$ planar), as shown in figure-(2).

  Then, addition of $\{u_{k+1,i}, u_{k+1,j}\}$ splitting $R_{k+1,l}$ into two regions.

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Graph planarity

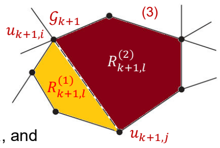
- Euler's formula:

  Proof of ⟨Euler's Formula⟩ contd-3:

  Consequently, two regions $R^{(1)}_{k+1,l}$ and $R^{(2)}_{k+1,l}$ generated, shown in figure-(3).

  Then, in $\mathcal{G}_{k+1}$, $r_{k+1} = r_k + 1$, $e_{k+1} = e_k + 1$, and $v_{k+1} = v_k$.

  So, $r_{k+1} = r_k + 1 = e_k - v_k + 2 + 1 = (e_k + 1) - v_{k+1} + 2 = e_{k+1} - v_{k+1} + 2$. Because, each side of formula '$r_k = e_k - v_k + 2$' in $\mathcal{G}_k$ to increase by exactly one, thereby formula '$r_{k+1} = e_{k+1} - v_{k+1} + 2$' still true for $\mathcal{G}_{k+1}$.



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Graph planarity

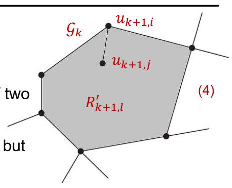
- Euler's formula:

  Proof of ⟨Euler's Formula⟩ contd-4:

  (B) For edge $\{u_{k+1,i}, u_{k+1,j}\}$, any one of two vertices of that edge not present in $\mathcal{G}_k$.

  Let vertex $u_{k+1,i}$ already included in $\mathcal{G}_k$, but vertex $u_{k+1,j}$ not in $\mathcal{G}_k$ in figure-(4).

  Adding $\{u_{k+1,i}, u_{k+1,j}\}$ to $\mathcal{G}_k$ not producing any new region in $\mathcal{G}_{k+1}$, because $u_{k+1,j}$ must be within region $R'_{k+1,l}$ of $\mathcal{G}_k$ with $u_{k+1,i}$ on its boundary, else impossible to add that edge to $\mathcal{G}_k$ without crossing existing edges (satisfying $\mathcal{G}_{k+1}$ planar) in figure-(4).



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Graph planarity

- Euler's formula:

  Proof of ⟨Euler's Formula⟩ contd-5:

  Then, in $\mathcal{G}_{k+1}$, $r_{k+1} = r_k$, $e_{k+1} = e_k + 1$, and $v_{k+1} = v_k + 1$.

  So, $r_{k+1} = r_k = e_k - v_k + 2 = (e_k + 1) - (v_k + 1) + 2 = e_{k+1} - v_{k+1} + 2$.

  Because, each side of formula '$r_k = e_k - v_k + 2$' in $\mathcal{G}_k$ to remain same, and increase in e_{k+1} and v_{k+1} to cancel each other, thereby formula '$r_{k+1} = e_{k+1} - v_{k+1} + 2$' still true for $\mathcal{G}_{k+1}$.

  Hence, Euler's formula to be true for planar subgraph $\mathcal{G}_n$ $(1 \le n \le e)$.

  As $\mathcal{G}_e = \mathcal{G}$, by Mathematical Induction, $r = e - v + 2$. ■

Summary

- Focus: Planarity of graph.
- Motivation for graph planarity.
- Planar graph, with properties, examples and benefits.
- Planar representation of graph.
- Regions in planar graph, with properties and examples.
- Nonplanar graph.
- Graph nonplanarity theorems on Kuratowski's first graph and second graph, with proofs.
- Euler's Formula on graph planarity, with proof.
- Degree of region in planar graph.

References

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