

UNIT – II, Module – 4

Lecture 22: Recurrence

[Divide-and-conquer recurrence; Master theorem; Generating function; Generating function for r -combinations]

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Notation table

Symbol / Notation	Meaning
$G(x), F(x)$	Generating function notations for given sequence.

Divide-and-conquer recurrences

- Divide-and-conquer recurrence: For given problem P of size n ($n \in \mathbb{N}$ or \mathbb{Z}^+), $f(n)$ operations needed to solve P , where S subproblems of P obtained by dividing P , each S 's size $= n/b$, with conquering by $g(n)$ extra operations to combine solutions of S subproblems into solution of given problem, i.e., $f(n) = S \cdot f(n/b) + g(n)$.
- Divide-and-conquer paradigm: dividing given problem into one/more instances of same problem of smaller size, until solutions of smaller problems found quickly, and then conquering (i.e., finding solution of) given problem, possibly by combining solutions of smaller problems, and perhaps with some additional operations.

Divide-and-conquer recurrences

- Divide-and-conquer recurrence:
 - Applicability: Algorithm design strategy to solve diverse problems —
 - (i) binary search comparisons in sequence of size n : $f(n) = f(n/2) + 2$ (n even positive integer);
 - (ii) maximum, minimum of sequence of size n : $f(n) = 2 \cdot f(n/2) + 2$ (n even positive integer);
 - (iii) merge sort comparisons in sequence of size n : $f(n) = 2 \cdot f(n/2) + n$ (n even positive integer);
 - (iv) fast $n \times n$ matrix multiplication (Volker Strassen, 1969): $f(n) = 7 \cdot f(n/2) + 15 \cdot n^2/4$ (n even positive integer); and many more.

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence:
 - Property: For divide-and-conquer strategy to solve problem \mathcal{P} , operation count/(complexity) recurrence: $f(n) \rightarrow$ increasing function.
 - Property: (**Lemma**): For given divide-and-conquer recurrence $f(n) = S \cdot f(n/b) + g(n)$, where $n \in \mathbb{N}$ or \mathbb{Z}^+ , $S, b, k \in \mathbb{Z}^+$, $S \geq 1$, $b > 1$, $n = b^k$, then $f(n) = S^k \cdot f(1) + \sum_{i=0}^{k-1} (S^i \cdot g(n/b^i))$.
 - Proof:** Successive $(k-1)$ substitutions of f , given $n=b^k$, i.e., $n/b^k=1$.

$$f(n) = S \cdot f(n/b) + g(n) = S^2 \cdot f(n/b^2) + S \cdot g(n/b) + g(n) = \dots$$

$$= S^k \cdot f(n/b^k) + \sum_{i=0}^{k-1} (S^i \cdot g(n/b^i)) = S^k \cdot f(1) + \sum_{i=0}^{k-1} (S^i \cdot g(n/b^i)). \blacksquare$$

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence:
 - Property: (**Theorem**): Considering f as increasing function, satisfying recurrence: $f(n) = S \cdot f(n/b) + c$, where $n \in \mathbb{N}$ or \mathbb{Z}^+ , $S, b \in \mathbb{Z}^+$, $b > 1$, $c \in \mathbb{R}^+$ —
 - (i) whenever ' n divisible by b ' and $S \geq 1$,

$$f(n) \in \begin{cases} O(n^{\log_b S}), & \text{if } S > 1 \\ O(\log_b n), & \text{if } S = 1. \end{cases}$$
 - (ii) When $n = b^k$ ($k \in \mathbb{Z}^+$) and $S \neq 1$, $f(n) = C_1 \cdot n^{\log_b S} + C_2$, where $C_1 = f(1) + c/(S-1)$ and $C_2 = -c/(S-1)$.

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence:
- Property:: (**Master theorem**): Let increasing function f satisfying recurrence: $f(n) = S \cdot f(n/b) + c \cdot n^d$, where $n \in \mathbb{N}$ or \mathbb{Z}^+ , $S, b \in \mathbb{Z}^+$, $S \geq 1$, $b > 1$, $c, d \in \mathbb{R}$, $c > 0$, $d \geq 0$, $n = b^k$ ($k \in \mathbb{Z}^+$). Then —

$$f(n) \in \begin{cases} O(n^d), & \text{if } S < b^d \\ O(n^d \cdot \log_b n), & \text{if } S = b^d \\ O(n^{\log_b S}), & \text{if } S > b^d. \end{cases}$$

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence examples:
- Example-1:: Let divide-and-conquer recurrence $f(n) = 5 \cdot f(n/2) + 3$ and $f(1) = 7$.
 - (i) Find $f(2^k)$, where $k \in \mathbb{Z}^+$. (ii) Also estimate $f(n)$, if f is increasing function.
 - (i) From **theorem**, with $S = 5 \neq 1$, $b = 2 > 1$, and $c = 3$, if $n = 2^k$, then $f(n)$ to become —

$$f(n) = (f(1) + c/(S-1)) \cdot n^{\log_b S} - c/(S-1) = (7 + 3/4) \cdot (2^k)^{\log_2 5} - 3/4 = 31/4 \cdot (2^{\log_2 5})^k - 3/4 = 31/4 \cdot 5^k - 3/4. \quad (\text{contd. to next slide})$$

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence examples:
- Example-1 contd.:
 - (ii) From **theorem**, for given increasing nature of $f(n)$, with $S = 5 > 1$, $b = 2 > 1$, and $n = 2^k$ (i.e., n divisible by b), $f(n)$ to become —

$$f(n) \in O(n^{\log_b S}) = O(n^{\log_2 5}).$$
 - Alternately, from **Master theorem**, with $d = 0$ (as $n^d = 1$ in 2^{nd} addend of given recurrence satisfied by f), $c = 3 > 0$, $b = 2 > 1$, $S = 5 > b^d$ (as $b^d = 1$), and n divisible by b , then $f(n)$ to become —

$$f(n) \in O(n^{\log_b S}) = O(n^{\log_2 5}).$$

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Divide-and-conquer recurrences

- Divide-and-conquer recurrence examples:
 - Example-2:: Big- O estimate for number of comparisons used by binary search algorithm.
Number of comparisons required to perform binary search on sequence of size n : $f(n) = f(n/2) + 2$ (where, n = even positive integer).
From **theorem**, with $S = 1$, $b = 2 > 1$, n divisible by b , and $c = 2$, then $f(n)$ to become — $f(n) \in O(\log_2 n)$.

Divide-and-conquer recurrences

- Divide-and-conquer recurrence examples:
 - Example-3:: Big- O estimate for number of operations (combinedly multiplications and additions) required to multiply two $n \times n$ matrices using fast matrix multiplication (Volker Strassen,1969).
Number of operations (combinedly multiplications and additions) required to perform matrix multiplication of two $n \times n$ matrices:
 $f(n) = 7 \cdot f(n/2) + 15 \cdot n^2/4$ (where, n = even positive integer).
From **Master theorem**, with $d = 2$ (as n^2 present in 2nd addend of given recurrence satisfied by f), $c = 15/4 > 0$, $b = 2 > 1$, $S = 7 > b^d$ (as $b^d = 4$), and n divisible by b , then $f(n) \in O(n^{\log_b S}) = O(n^{\log_2 7})$.

Generating functions

- Generating function: For given sequence $a_0, a_1, \dots, a_k, \dots$ of real numbers, (ordinary) **generating function** for $\{a_k\}$ expressed as infinite series $G(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_k \cdot x^k + \dots$
 $= \sum_{k=0}^{\infty} (a_k \cdot x^k)$.
 - Property:: **Generating**: series to generate terms of sequence.
 - Property:: Purpose of ' x ': mechanism to locate term a_k from summand involving x^k .
 - Property:: **Ordinary**: simply 'power of x ' used to locate any term a_k of sequence.

Generating functions

- Generating function:
 - Property: x not to be assigned any numerical value, for infinitely-many nonzero coefficient in sequence.
 - Property: Generating function for finite sequence a_0, a_1, \dots, a_n of real numbers: $G(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \dots + a_n \cdot x^n + 0 + 0 + \dots = \sum_{k=0}^n (a_k \cdot x^k)$.

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Generating functions

- Generating function:
 - Property: (**Theorem**): For two generating functions $G(x) = \sum_{k=0}^{\infty} (a_k \cdot x^k)$, and $F(x) = \sum_{k=0}^{\infty} (b_k \cdot x^k)$,
 $G(x) + F(x) = \sum_{k=0}^{\infty} ((a_k + b_k) \cdot x^k)$, and
 $G(x) \cdot F(x) = \sum_{k=0}^{\infty} \left(\left(\sum_{j=0}^k (a_j \cdot b_{k-j}) \right) \cdot x^k \right)$.
 Note-1: Above theorem valid for power series converging in some interval.
 Note-2: For non-converging series, statements in above theorem to be considered as definitions (instead of theorems) of addition and multiplication of generating functions.

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Generating functions

- Generating function:
 - Property: (**Theorem Revisited**): For set S of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, number of r -combinations = $C(n, r) = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$.
- Proof:** Using product rule, generating function and binomial theorem.
- From n , no (i.e., zero) elements selected = $C(n, 0) = 1$ way.
- From n , 1 element selected = $C(n, 1)$ ways.
- From n , 2 elements selected = $C(n, 2)$ ways.
- \vdots \vdots \vdots
- From n , n elements selected = $C(n, n)$ ways.

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Generating functions

- Generating function:
Proof contd.:
So, sequence of terms expressing different r -combinations: $1, C(n, 1), C(n, 2), \dots, C(n, n)$.
Corresponding generating function: $\mathcal{F}(x) = \sum_{r=0}^n (C(n, r) \cdot x^r)$. ①
On other hand, each of n elements in S involved in "to be chosen or NOT chosen for r -combinations," i.e., each element to contribute $(1 + x)$ to $\mathcal{F}(x)$, independent of choosing prior elements.
So, applying product rule, $\mathcal{F}(x) = (1 + x)^n$. ②
Combining ① and ②, $(1 + x)^n = \sum_{r=0}^n (C(n, r) \cdot x^r)$. ③ (contd. to next slide)

Generating functions

- Generating function:
Proof contd-2:
Again, according to binomial theorem, $(1 + x)^n = \sum_{r=0}^n \left(\binom{n}{r} \cdot x^r \right)$ ④
where $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$.
Combining ③ and ④, $\sum_{r=0}^n (C(n, r) \cdot x^r) = \sum_{r=0}^n \left(\binom{n}{r} \cdot x^r \right)$.
Series equality (in above) implying individual summand-wise equality.
So, $C(n, r) = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$. ■

Generating functions

- Generating function examples:
 - Example-1:: Generating function for sequence $1, 1, 1, 1, 1, 1$.
Let $\bar{G}(x)$ be generating function for sequence $1, 1, 1, 1, 1, 1$. Then,
 $\bar{G}(x) = 1 + x + x^2 + x^3 + x^4 + x^5 = \frac{(x^6-1)}{(x-1)}$, when $x \neq 1$ (as per finite geometric series).
Because 'powers of x ' only place holders for terms of sequence in generating function, no need to worry that $\bar{G}(1)$ undefined.

Generating functions

- Generating function examples:
 - Example-2:: Generating function for sequence 1, 1, 1, ...
Let $G(x)$ be generating function for sequence 1, 1, 1, ... Then,
 $G(x) = 1 + x + x^2 + \dots \infty = \frac{1}{(1-x)}$, when $|x| < 1$ (as per infinite geometric series).
Again, because 'powers of x ' only place holders for terms of sequence in generating function, no need to worry for bound of x .

Generating functions

- Generating function examples:
 - Example-3:: Let $f(x) = \frac{1}{(1-x)^2}$. To find coefficients a_0, a_1, a_2, \dots in expansion of $f(x) = \sum_{k=0}^{\infty} (a_k \cdot x^k)$ form.
As per infinite geometric series, let generating function $g(x) = \frac{1}{(1-x)} = 1 + x + x^2 + \dots \infty, (|x| < 1)$.
Then, from **theorem**, given generating function $f(x) = \frac{1}{(1-x)^2} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)} = g(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\left(\sum_{j=0}^k (1 \cdot 1) \right) \cdot x^k \right) = \sum_{k=0}^{\infty} ((k+1) \cdot x^k) = 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 + \dots \infty$.

Generating functions

- Generating function examples:
 - Example-4:: Generating function for $(1+x)^{-n}$, where $n \in \mathbb{Z}^+$, using extended binomial theorem.
By **extended binomial theorem**, generating function $G(x)$ become:
 $G(x) = \sum_{k=0}^{\infty} \binom{-n}{k} \cdot x^k = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{(n+k-1)!}{k! \cdot (n-1)!} \cdot x^k$
 $= \sum_{k=0}^{\infty} (-1)^k \cdot C(n+k-1, k) \cdot x^k$.

Summary

- Focus: Recurrences (contd.).
- Divide-and-conquer recurrence relations, and applicability.
- Solution of divide-and-conquer recurrence.
- Master theorem for generalized divide-and-conquer recurrence, and special cases.
- Generating functions of sequences and series, with examples.
- Operations between two generating functions.
- r -combinations theorem proof using generating functions.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Mot08] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, PHI, Second edition, 2008.
3. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Divide-and-conquer recurrence:: [Ros19]:553-556.
- Master theorem, and special cases:: [Ros19]:556-559.
- Generating functions:: [Ros19]:563-568.
- r -combinations proof using generating function:: [Ros19]:570-571.

Lecture Exercises: Problem 1 [Ref: Gate 2022, Q.36, p.24]

Which one of the following is the closed form for the generating function of the sequence $\{a_n\}_{n \geq 0}$ defined below?

$$a_n = \begin{cases} n+1, & n \text{ is odd} \\ 1, & \text{otherwise} \end{cases}$$

- (a) $\frac{x \cdot (1+x^2)}{(1-x^2)^2} + \frac{1}{1-x}$
 (b) $\frac{x \cdot (3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$
 (c) $\frac{2 \cdot x}{(1-x^2)^2} + \frac{1}{1-x}$
 (d) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

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Lecture Exercises: Problem 1 Ans

- Given sequence rewritten as: $a_0 = 1, a_1 = 2, a_2 = 1, a_3 = 4, a_4 = 1, a_5 = 6, a_6 = 1, a_7 = 8, a_8 = 1, a_9 = 10, \dots \infty$
- Corresponding closed-form of generating function = infinite series

$$G(x) = 1 + 2 \cdot x + 1 \cdot x^2 + 4 \cdot x^3 + 1 \cdot x^4 + 6 \cdot x^5 + 1 \cdot x^6 + 8 \cdot x^7 + 1 \cdot x^8 + 10 \cdot x^9 + \dots \infty = (1 + 1 \cdot x^2 + 1 \cdot x^4 + 1 \cdot x^6 + 1 \cdot x^8 + \dots \infty) +$$

$$(2 \cdot x + 4 \cdot x^3 + 6 \cdot x^5 + 8 \cdot x^7 + 10 \cdot x^9 + \dots \infty). \quad (1)$$
- 1st term after simplification: $T_1 = \sum_{k=0}^{\infty} (1 \cdot x^{2 \cdot k}) = \frac{1}{1-x^2}$, as per formula

$$\sum_{j=0}^{\infty} (a \cdot r^{2 \cdot j}) = \frac{a}{1-r^2}, \text{ which after further simplification:}$$

$$T_1 = \frac{1+(x-x)}{1-x^2} = \frac{1+x}{(1+x) \cdot (1-x)} = \frac{x}{1-x^2} = \frac{1}{1-x} - \frac{x}{1-x^2}. \quad (2) \text{ (contd. to next slide)}$$

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Lecture Exercises: Problem 1 Ans (contd.)

- 2nd term combined A.P. and G.P. in series, where coefficients forming A.P., and terms of 'x' forming G.P. Now, plan for removing A.P. from 2nd term: $T_2 = \sum_{k=0}^{\infty} ((2 \cdot k + 2) \cdot x^{2 \cdot k+1})$.
 So, $x^2 \cdot T_2 = x^2 \cdot \sum_{k=0}^{\infty} ((2 \cdot k + 2) \cdot x^{2 \cdot k+1}) = \sum_{k=0}^{\infty} ((2 \cdot k + 2) \cdot x^{2 \cdot k+3})$.
- Then, $T_2 - x^2 \cdot T_2 = 2 \cdot x + (\sum_{k=1}^{\infty} ((2 \cdot k + 2) \cdot x^{2 \cdot k+1}) - \sum_{k=0}^{\infty} ((2 \cdot k + 2) \cdot x^{2 \cdot k+3})) = 2 \cdot x + \sum_{k=1}^{\infty} ((2 \cdot k + 2 - 2 \cdot (k-1) - 2) \cdot x^{2 \cdot k+1})$

$$= 2 \cdot x + \sum_{k=1}^{\infty} (2 \cdot x^{2 \cdot k+1}) = \sum_{k=0}^{\infty} (2 \cdot x^{2 \cdot k+1}) = 2 \cdot x \cdot \sum_{k=0}^{\infty} (1 \cdot x^{2 \cdot k})$$

$$= \frac{2 \cdot x}{1-x^2}, \text{ as per formula } \sum_{j=0}^{\infty} (a \cdot r^{2 \cdot j}) = \frac{a}{1-r^2}.$$
- So, $T_2 \cdot (1 - x^2) = \frac{2 \cdot x}{1-x^2}$. Then, $T_2 = \frac{2 \cdot x}{(1-x^2)^2}. \quad (3) \text{ (contd. to next slide)}$

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Lecture Exercises: Problem 1 Ans (contd.-2)

- Combining Eq. ② and Eq. ③ in Eq. ①:

$$G(x) = \frac{1}{1-x} - \frac{x}{1-x^2} + \frac{2x}{(1-x^2)^2} = \frac{1}{1-x} + \frac{2x-x(1-x^2)}{(1-x^2)^2} = \frac{1}{1-x} + \frac{x+x^3}{(1-x^2)^2}$$
$$= \frac{1}{1-x} + \frac{x(1+x^2)}{(1-x^2)^2}.$$

- Correct answer: (A)
