CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 1

Lecture 01: Propositional Logic

[Mathematical logic; Proposition; Connectives; Conjunction; Disjunction; Negation; Exclusive disjunction; Implication; Bi-implication]

Dr. Suddhasil De

Discrete mathematics

- Discrete mathematics: part of mathematics devoted to study discrete objects.
  - Discrete → study of distinct or non-connected elements/objects.
- Discrete structures: representations for discrete objects.
  - Examples: sets, collections, combinations, relations, graphs etc.
- Importance of 'discrete mathematics and graph theory' study: programming logic, software development, formal specification and verification, networking, system development etc.
- Foundation of discrete mathematics: mathematical logic.

Mathematical logic

- Mathematical logic: (commonly) formal study of correct reasoning;
  (rigorously) mathematical properties (particularly, expressive and deductive ability) of formal system of logic.
  - Focus → how to deduce conclusion, starting from premise(s), by solely following rules of proof system, with being affected by topic/content of context.
  - Categories of mathematical logic: propositional logic, first-order logic, second-order logic, infinitary logic, intuitionistic logic etc.

Propositional logic

- Propositional logic (also called propositional calculus): area of mathematical logic dealing with propositions.
  - Applications of propositional logic:
    a. Translating imprecise/ambiguous English sentences into precise propositions.
    b. Specifying both hardware and software systems into precise and unambiguous propositions.
    c. Expressing custom and complicated search query in of large collections of information by unambiguous propositions.
    d. Designing logic circuits for computer hardware by propositions.

Proposition

- Proposition: declarative factual statement (formed following syntactic rule, with meaning as per semantic interpretation) having truth value of either TRUE or FALSE, but not both.
  - Examples::
    "Ice floats in water." → Proposition, value = TRUE.
    "2 + 2 = 4" → Proposition, value = TRUE.
    "India is in Europe." → Proposition, value = FALSE.
    "2 + 2 = 5" → Proposition, value = FALSE.
    "Do your homework." → Not Proposition.
    "Where are you going?" → Not Proposition.
    "x + 1 = 2" → Not Proposition

Proposition

- Two types: (i) primitive; (ii) compound (or composite).

A. Primitive (or atomic) proposition: proposition not able to break down into simpler subpropositions of respective truth values.
  - Examples:: 4 propositions in last slide.

B. Compound proposition: composite proposition, consisting of subpropositions (with respective truth values) and connectives to combine them.
  - Examples::
    "I am smart AND I revise before class." → Compound proposition.
    "I am smart OR I revise before class." → Compound proposition.

Proposition

B. Compound proposition:
- Truth value of compound proposition ← completely determined by truth values of subpropositions, and manner of connecting them.
- Connective (also called operator): representation of logical operation, to be performed on truth value(s) of one or more subpropositions.
  Example:: AND (∧), OR (∨), NOT (¬), IMPLIES (→) etc.

Propositional variable & Truth function

- Propositional variables (also called sentential variables, or statement variables): use of letters to denote variables for representing propositions.
  - Conventional letters as propositional variables: $p, q, r, s, \ldots$
- Truth function: function in logic that accepting only truth values as input and producing a unique truth value as output.

Logical operation

- Logical operation:
  - 3 basic "truth-functional" logical operations:
    conjunction (due to AND connective; operator notation: ∧);
    disjunction (due to OR connective, operator notation: ∨);
    negation (due to NOT connective; operator notation: ¬, ‾, ~, ′, !).
  - Additional logical operations:
    exclusive disjunction (equivalent to XOR connective; operator notations: ⊕, $\underline{\vee}$, $\dot{\vee}$);
  - implication (equivalent to IF…THEN…; operator notation: →);
  - bi-implication (equivalent to …IF AND ONLY IF…; operator: ↔).

Logical operation: Conjunction

- Conjunction of given propositions: combining truth values of any two or more given propositions (when connected by AND connective in any compound proposition), using operator $\wedge$.
  - $p \wedge q$ [read: "p and q"]: conjunction of p and q, to form compound proposition "p AND q", for given propositions p, q, having truth value depending only on p, q.
  - Definition of truth value of $p \wedge q$: if both p and q to be TRUE, then $p \wedge q$ to become TRUE; otherwise $p \wedge q$ to become FALSE.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical operation: Conjunction

- Conjunction:
  - Example-1::
    p = "Ice floats in water." → p's value = TRUE.
    q = "2 + 2 = 4" → q's value = TRUE.
    $p \wedge q$ = "Ice floats in water AND 2 + 2 = 4." → value of $p \wedge q$ = TRUE.
  - Example-2::
    p = "Ice floats in water." → p's value = TRUE.
    q = "2 + 2 = 5" → q's value = FALSE.
    $p \wedge q$ = "Ice floats in water AND 2 + 2 = 5." → value of $p \wedge q$ = FALSE.

Logical operation: Conjunction

- Conjunction:
  - Example-3::
    p = "India is in Europe." → p's value = FALSE.
    q = "2 + 2 = 4" → q's value = TRUE.
    $p \wedge q$ = "India is in Europe AND 2 + 2 = 4." → value of $p \wedge q$ = FALSE.
  - Example-4::
    p = "India is in Europe." → p's value = FALSE.
    q = "2 + 2 = 5" → q's value = FALSE.
    $p \wedge q$ = "India is in Europe AND 2 + 2 = 5." → value of $p \wedge q$ = FALSE.

Logical operation: Disjunction

- Disjunction of given propositions: combining truth values of any two or more given propositions (when connected by *inclusive*-OR connective in any compound proposition), using operator ∨.
  - $p \vee q$ [read: "p or q"]: disjunction of p and q, to form compound proposition "p OR q", for given propositions p, q, having truth value depending only on p, q.
  - Definition of truth value of $p \vee q$:
    if both p and q to be FALSE,
    then $p \vee q$ to become FALSE;
    otherwise $p \vee q$ to become TRUE.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical operation: Disjunction

- Disjunction:
  - Example-1::
    p = "Ice floats in water." → p's value = TRUE.
    q = "2 + 2 = 4" → q's value = TRUE.
    $p \vee q$ = "Ice floats in water OR 2 + 2 = 4." → value of $p \vee q$ = TRUE.
  - Example-2::
    p = "Ice floats in water." → p's value = TRUE.
    q = "2 + 2 = 5" → q's value = FALSE.
    $p \vee q$ = "Ice floats in water OR 2 + 2 = 5." → value of $p \vee q$ = TRUE.

Logical operation: Disjunction

- Disjunction:
  - Example-3::
    p = "India is in Europe." → p's value = FALSE.
    q = "2 + 2 = 4" → q's value = TRUE.
    $p \vee q$ = "India is in Europe OR 2 + 2 = 4." → value of $p \vee q$ = TRUE.
  - Example-4::
    p = "India is in Europe." → p's value = FALSE.
    q = "2 + 2 = 5" → q's value = FALSE.
    $p \vee q$ = "India is in Europe OR 2 + 2 = 5." → value of $p \vee q$ = FALSE.

Logical operation: Negation

- Negation of given proposition: modifying truth value of any given proposition (by — (i) preceding with "It is not true that" or "It is false that" or "It is not the case that" connectives, or (ii) possibly inserting "negate" meaning inside given proposition, to form compound proposition), using operators '$\neg$', '′' or '‾'.
  - $\neg p$ [read: "not p"]: negation of p, to form compound proposition "$\neg p$", for given proposition p, having truth value depending only on p.
  - Definition of truth value of $\neg p$ (or p' or $\bar{p}$):
    if p to be TRUE, then $\neg p$ to become FALSE;
    and, if p to be FALSE, then $\neg p$ to become TRUE.

Truth table

p	$\neg p$
T	F
F	T

Logical operation: Negation

- Negation:
  - Example-1::
    p = "Ice floats in water." → p's value = TRUE.
    $\neg p$ = "**It is false that** ice floats in water." → value of $\neg p$ = FALSE.
  - Example-2::
    p = "Ice floats in water." → p's value = TRUE.
    $\neg p$ = "Ice **does not** float in water." → value of $\neg p$ = FALSE.
  - Example-3::
    p = "2 + 2 = 5" → p's value = FALSE.
    $\neg p$ = "2 + 2 ≠ 5." → value of $\neg p$ = TRUE.

Logical operation: Negation

- Negation:
  - Other used notations (apart from $\neg p$): $\overline{p}$, $\sim p$, p', $!p$, etc.

Logical operation: Exclusive disjunction

- Exclusive disjunction of given propositions: combining truth values of any two or more given propositions (when connected by *exclusive*-OR connective in form (i) "Either…or…", (ii) "Either…or…, but not both" in any compound proposition), by operator ⊕.
  - $p \oplus q$ [read: "p xor q"]: exclusive disjunction of p and q, for "EITHER p OR q", of given propositions p, q.
  - Definition of truth value of $p \oplus q$: if exactly one of p and q to be TRUE, then $p \oplus q$ to become TRUE; otherwise $p \oplus q$ to become FALSE.

Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical operation: Exclusive disjunction

- Exclusive disjunction:
  - Example-1::

    p = "At 8:30AM on Monday, I attend lecture of CS34110."
    → p's value = TRUE.
    q = "At 8:30AM on Monday, I attend lecture of CS34105."
    → q's value = TRUE.
    $p \oplus q$ = "At 8:30AM on Monday, I attend lecture of **EITHER** CS34110 **OR** CS34105, **IS FALSE**."
    → value of $p \oplus q$ = FALSE.

Logical operation: Exclusive disjunction

- Exclusive disjunction:
  - Example-2::

    p = "At 8:30AM on Monday, I attend lecture of CS34110."
    → p's value = TRUE.
    q = "**It is not the case that** at 8:30AM on Monday, I attend lecture of CS34105."
    → q's value = FALSE.
    $p \oplus q$ = "At 8:30AM on Monday, I attend lecture of **EITHER** CS34110 **OR** CS34105."
    → value of $p \oplus q$ = TRUE.

Logical operation: Exclusive disjunction

- Exclusive disjunction:
  - Example-3::

    p = "At 8:30AM on Monday, I **DO NOT** attend lecture of CS34110."

    → p's value = FALSE.

    q = "At 8:30AM on Monday, I attend lecture of CS34105."

    → q's value = TRUE.

    $p \oplus q$ = "At 8:30AM on Monday, I attend lecture of **EITHER** CS34110 **OR** CS34105."

    → value of $p \oplus q$ = TRUE.

Logical operation: Exclusive disjunction

- Exclusive disjunction:
  - Example-4::

    p = "At 8:30AM on Monday, I **DO NOT** attend lecture of CS34110."

    → p's value = FALSE.

    q = "At 8:30AM on Monday, I **DO NOT** attend lecture of CS34105."

    → q's value = FALSE.

    $p \oplus q$ = "At 8:30AM on Monday, I attend lecture of **EITHER** CS34110 **OR** CS34105, **IS FALSE**."

    → value of $p \oplus q$ = FALSE.

Logical operation: Implication

- Implication of given propositions: conditionally combining truth values of any two given propositions p and q (when connected by condition in any compound proposition), by operator '→'.
  - Other names given to implication:

"p implies q"	"q whenever p"	"p is sufficient for q"
"if p, then q"	"q follows from p"	"a sufficient condition for q is p"
"if p, q"	"q provided that p"	"p only if q"
"q if p"	"q unless $\neg p$"	"a necessary condition for p is q"
"q when p"		"q is necessary for p"

Logical operation: Implication

- Implication:
  - $p \rightarrow q$ [read: "p implies q"]: "IF p THEN q", of given propositions p, q.
  - p: hypothesis or **antecedent** or premise;
    q: conclusion or **consequent**.
  - Definition of truth value of $p \rightarrow q$:
    if p to be TRUE and q to be FALSE,
    then $p \rightarrow q$ to become FALSE;
    otherwise $p \rightarrow q$ to become TRUE.
  - Note: when p to be FALSE, $p \rightarrow q$ to be TRUE regardless of truth value of q.

Truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical operation: Implication

- Implication:
  - Example-1:: <Politician's pledge>
    p = "I win election." → p's value = TRUE.
    q = "I will lower taxes." → q's value = TRUE.
    $p \rightarrow q$ = "**IF** I win election, **THEN** I will lower taxes."
    → value of $p \rightarrow q$ = TRUE.
    [think as obligation or contract fulfilled]
  - Example-2::
    p's value = TRUE; q's value = FALSE.
    Value of $p \rightarrow q$ = FALSE. [obligation not fulfilled]

Logical operation: Implication

- Implication:
  - Example-3::
    p's value = FALSE; q's value = TRUE. [by some influence]
    Value of $p \rightarrow q$ = TRUE.
  - Example-4::
    p's value = FALSE; q's value = FALSE.
    Value of $p \rightarrow q$ = TRUE. [no obligation]
  - Difference from 'if p then S' construct in programming languages:
    S: set of executable instructions/statements, instead of proposition;
    If p's value = FALSE, then S not to be executed.

Logical operation: Bi-implication

- Bi-implication of given propositions: biconditionally combining truth values of any two given propositions p and q (when connected by conditions in any compound proposition), using operator '↔'.
  - Other names given to bi-implication:

"p if and only if q"	"p is necessary and sufficient for q"
"p iff q"	"if p then q, and conversely"
"p exactly when q"	

Logical operation: Bi-implication

- Bi-implication:
  - $p \leftrightarrow q$ [read: "p if and only if q"]: "p IF AND ONLY IF q", of given propositions p, q.
  - Definition of truth value of $p \leftrightarrow q$:
    if p and q to have same truth values, then $p \leftrightarrow q$ to become TRUE;
    otherwise $p \leftrightarrow q$ to become FALSE.

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

  - Property: $p \leftrightarrow q$ to become TRUE when both $p \rightarrow q$ **and** $q \rightarrow p$ to be TRUE, and $p \leftrightarrow q$ to become FALSE otherwise.

➲ Note: Truth table of $p \leftrightarrow q$ opposite of $p \oplus q$.

Logical operation: Bi-implication

- Bi-implication:
  - Example-1::
    p = "I can board airplane." → p's value = TRUE.
    q = "I get boarding pass." → q's value = TRUE.
    $p \leftrightarrow q$ = "I can board airplane, **IF AND ONLY IF** I get boarding pass."
    → value of $p \leftrightarrow q$ = TRUE. [think as both-way obligation fulfilled]
    "IF I can board airplane, THEN I get boarding pass."
    → value of $p \rightarrow q$ = TRUE.
    "IF I get boarding pass, THEN I can board airplane."
    →value of $q \rightarrow p$ = TRUE.

Logical operation: Bi-implication

- Bi-implication:
  - Example-2::

    p's value = TRUE; q's value = FALSE.

    Value of $p \leftrightarrow q$ = FALSE. [obligation not fulfilled]

    "IF I can board airplane, THEN I do not get boarding pass."

    → value of $p \rightarrow q$ = FALSE.

    "IF I do not get boarding pass, THEN I can board airplane."

    → value of $q \rightarrow p$ = TRUE.

Logical operation: Bi-implication

- Bi-implication:
  - Example-3::

    p's value = FALSE; q's value = TRUE.

    Value of $p \leftrightarrow q$ = FALSE. [obligation not fulfilled]

    "IF I can not board airplane, THEN I get boarding pass."

    → value of $p \rightarrow q$ = TRUE.

    "IF I get boarding pass, THEN I can not board airplane."

    → value of $q \rightarrow p$ = FALSE.

Logical operation: Bi-implication

- Bi-implication:
  - Example-4::

    p's value = FALSE; q's value = FALSE.

    Value of $p \leftrightarrow q$ = TRUE. [no obligation]

    "IF I can not board airplane, THEN I do not get boarding pass."

    → value of $p \rightarrow q$ = TRUE.

    "IF I do not get boarding pass, THEN I can not board airplane."

    → value of $q \rightarrow p$ = TRUE.

Proposition and connectives: Nutshell

Sentence	Variable, Operator	Precedence
Simple proposition	p	
Another simple proposition	q	
Negation	$\neg p$	①
Conjunction	$p \wedge q$	②
Disjunction	$p \vee q$	③
Implication	$p \rightarrow q$	④
Bi-implication	$p \leftrightarrow q$	⑤

Summary

- Focus: Propositional logic.
- Discrete mathematics, discrete structures, and importance.
- Mathematical logic, propositional logic, and applicability.
- Proposition definition, with examples.
- Primitive and compound propositions.
- Propositional variables definitions.
- Connectives and logical operations of propositional variables.
- Conjunction, (inclusive) disjunction, negation, and exclusive disjunction definitions, truth tables, with examples.
- Implication and bi-implication definitions and truth tables.

Summary

- Examples of implication and bi-implication.
- Precedence of connectives.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Proposition:: [Ros21]:2-3.
- Logical operation:conjunction:: [Ros21]:4.
- Logical operation:disjunction:: [Ros21]:4-5.
- Logical operation:negation:: [Ros21]:3-4.
- Logical operation:exclusive disjunction:: [Ros21]:5-6.
- Logical operation:implication:: [Ros21]:6-9.
- Logical operation:bi-implication:: [Ros21]:10.

Lecture Exercises: Problem 1 [Ref: Gate 2024, Q.12, p.9 (Set2)]

Let p and q be the following propositions:

p: Fail grade can be given.

q: Student scores more than 50% marks.

Consider the statement: "*Fail grade cannot be given when student scores more than 50% marks*." Which one of following is CORRECT representation of above statement in propositional logic?

(A) $q \rightarrow \neg p$
(B) $q \rightarrow p$
(C) $p \rightarrow q$
(D) $\neg p \rightarrow q$

Lecture Exercises: Problem 1 Ans

- Given:: p: "Fail grade can be given."
  q: "Student scores more than 50% marks."
- $\therefore$ "Fail grade cannot be given": $\neg p$.
- Given:: (Fail grade cannot be given) when (Student scores more than 50% marks)
- $\therefore$ Given proposition: $q \rightarrow \neg p$.

Lecture Exercises: Problem 2 [Ref: Gate 2017, Q.11, p.5 (Set2)]

Let p, q, r denote the statements "It is raining", "It is cold", and "It is pleasant", respectively. Then the statement "It is not raining and it is pleasant, and it is not pleasant only if it is raining and it is cold" is represented by ______.

Lecture Exercises: Problem 2 Ans

- $p =$ "It is raining". $q =$ "It is cold". $r =$ "It is pleasant".
- $\therefore$ "It is NOT raining" = $\neg p$.
- $\therefore$ "It is NOT raining AND It is pleasant" = $(\neg p \land r)$.
- Again, "It is NOT pleasant" = $\neg r$.
- Also, "It is raining AND It is cold" = $(p \land q)$.
- $\therefore$ "It is NOT pleasant ONLY IF It is raining AND It is cold" = $\neg r \rightarrow (p \land q)$.
- $\therefore$ Combining, "((It is NOT raining) AND It is pleasant), AND ((It is NOT pleasant) ONLY IF (It is raining AND It is cold))" = $(\neg p \land r) \land (\neg r \rightarrow (p \land q))$.