CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 12: Counting

[Countable set; Countably infinite set; Cantor-Schröder-Bernstein theorem; Countability of $\mathbb{Z}$; Uncountable set; Uncountability of $(0,1)$ and $\mathbb{R}$]

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Notation table

Symbol / Notation	Meaning
$\curlywedge$	Caret insertion point, to insert word later in written English sentence

Counting

- Counting: problem concerned with enumeration of objects with specific properties in field of 'Combinatorics.'
  - Set counting: application of counting principles for deriving properties of finite and infinite sets.

Set counting

- Countable set $\mathbf{S}$: (i) if $\mathbf{S}$ = finite set, or (ii) if $\mathbf{S}$ = infinite set, and in $\mathbf{S}$, possibility to arrange its elements as sequence (i.e., presence of some bijective function $f: \mathbb{N} \to \mathbf{S}$ to list elements indexed by positive integers taken from "$\mathbb{N} \setminus \{0\}$").
  - Property:: In countable set $\mathbf{S}$, $|\mathbf{S}| \leq \aleph_0$.
  - Property:: Countably infinite set $\mathbf{S}$: if and only if (i) countable set $\mathbf{S} \neq$ finite set, (ii) $|\mathbf{S}| = \aleph_0$, (iii) bijection $f: \mathbb{N} \to \mathbf{S}$ resulting in terms of sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n)$ and so on.

Set counting

- Countable set:
  - Property:: ⟨Theorem⟩: For countable sets $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \cup \mathbf{B}$ also to become countable, i.e., union of countable sets to become countable.
  - Property:: ⟨Theorem⟩: Subset of countable set also countable, i.e., if $\mathbf{B} \subseteq \mathbf{A}$ for countable set $\mathbf{A}$, then $\mathbf{B}$ also to become countable.
  - Property:: ⟨Cantor−Schröder−Bernstein theorem⟩: If sets $\mathbf{A}$ and $\mathbf{B}$ satisfy $|\mathbf{A}| \leq |\mathbf{B}|$ and $|\mathbf{B}| \leq |\mathbf{A}|$, then $|\mathbf{A}| = |\mathbf{B}|$,

    [Proof strategy: $((\mathbf{A} \subseteq \mathbf{B}) \wedge (\mathbf{B} \subseteq \mathbf{A})) \to (\mathbf{A} = \mathbf{B})$]

    i.e., if injective functions $f: \mathbf{A} \to \mathbf{B}$ and $g: \mathbf{B} \to \mathbf{A}$ present, then bijection between $\mathbf{A}$ and $\mathbf{B}$ also present.

Set counting

- Countable set theorems and proofs:
  - Proof of ⟨Theorem⟩: Union of countable sets to become countable.

  Proof: Given: $\mathbf{A}$ and $\mathbf{B}$ to be countable sets. Three cases to prove.

  Assumptions, without loss of generality —

  (i) $\mathbf{A}$ and $\mathbf{B}$ to be disjoint; if not, then $\mathbf{B}$ to be updated as $\mathbf{B} = \mathbf{B} \setminus \mathbf{A}$, as $\mathbf{A} \cap (\mathbf{B} \setminus \mathbf{A}) = \emptyset$, and $\mathbf{A} \cup (\mathbf{B} \setminus \mathbf{A}) = \mathbf{A} \cup \mathbf{B}$;

  (ii) for different nature of countability (i.e., one finite, while other countably infinite), $\mathbf{B}$ to be finite set.

  Case-1: $\mathbf{A}$ and $\mathbf{B}$ both finite sets. So, their union $\mathbf{A} \cup \mathbf{B}$ also finite set. So, $\mathbf{A} \cup \mathbf{B}$ to become countable.

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Set counting

- Countable set theorems and proofs:

  Proof (contd.): Case-2: $\mathbf{A}$ to be countably infinite set and $\mathbf{B}$ finite set.

  Then, elements of $\mathbf{A}$ to be listed in infinite sequence $a_1, a_2, \dots, a_n, \dots$, and so on. Also, elements of $\mathbf{B}$ to be listed in sequence $b_1, b_2, \dots, b_m$, for some $m, n \in \mathbb{N}\setminus\{0\}$.

  For $\mathbf{A} \cup \mathbf{B}$, a new sequence to list elements of $\mathbf{A} \cup \mathbf{B}$ as:

  $$b_1, b_2, \dots, b_m, a_1, a_2, \dots, a_n, \dots.$$

  For above sequence, bijection $f: \mathbb{N} \rightarrow (\mathbf{A} \cup \mathbf{B})$ possible, and so $\mathbf{A} \cup \mathbf{B}$ to become countable.

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Set counting

- Countable set theorems and proofs:

  Proof (contd-2.): Case-3: $\mathbf{A}$ and $\mathbf{B}$ both countably infinite sets.

  Then, elements of $\mathbf{A}$ and $\mathbf{B}$ to be respectively listed in infinite sequences $a_1, a_2, \dots, a_n, \dots$, and $b_1, b_2, \dots, b_n, \dots$.

  For $\mathbf{A} \cup \mathbf{B}$, another new sequence to list elements of $\mathbf{A} \cup \mathbf{B}$ as:

  $$a_1, b_1, a_2, b_2, a_3, b_3, \dots, a_n, b_n, \dots.$$

  For above sequence, another bijection $g: \mathbb{N} \rightarrow (\mathbf{A} \cup \mathbf{B})$ possible, and so $\mathbf{A} \cup \mathbf{B}$ to become countable.

  Thus, proof to show $\mathbf{A} \cup \mathbf{B}$ countable in all three cases complete. ■

Set counting

- Countable set theorems and proofs:
  - Property:: ⟨Theorem⟩: Union of a countable number of countable sets to become countable.

  Proof: Given: multiple $\mathbf{A}_i$ (where, $1 \le i \le n$, $n \in \mathbb{N}$); each countable set.

  To find: countability of $\mathcal{A}$, where $\mathcal{A} = \bigcup_{i=1}^{n} \mathbf{A}_i$ and n possibly infinite.

  As $\mathbf{A}_i$ countable, possibility of listing of elements of $\mathbf{A}_i$ in a sequence: $a_{i1}, a_{i2}, a_{i3}, \cdots$.

  Let a new sequence to be created by listing all elements of $\mathcal{A}$ as per following fashion: first listing all terms a_{ij} with $i+j=2$, then all terms a_{ij} with $i+j=3$, then all terms a_{ij} with $i+j=4$, and so on.

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Set counting

- Countable set theorems and proofs:

  Proof (contd.):

  Then, new sequence: $\underbrace{a_{ij_1}, a_{ij_2}, \cdots}_{i+j=2}, \underbrace{a_{ij_1}, a_{ij_2}, \cdots}_{i+j=3}, \underbrace{a_{ij_1}, a_{ij_2}, \cdots}_{i+j=4}, \cdots$.

  For above sequence, a bijection $f: \mathbb{N} \to \mathcal{A}$ possible, and so $\mathcal{A}$ to become countable. ■

Set counting

- Set counting examples:
  - Example-1:: to decide countability of "set of odd positive integers" — For given case, to exhibit presence of one-to-one correspondence (i.e., bijection) between $\mathbb{N}\backslash\{0\}$ and "set of odd positive integers," and thus, inferring "set of odd positive integers" as countable set.

    Considering function $f: \mathbb{N} \to$"Set of odd positive integers", with $f(n) = 2 \cdot n - 1$. Bijection of $f \equiv$ injection of $f \wedge$ surjection of f.

    Injection of f: Let $f(n) = f(m)$, for arbitrarily-chosen $n, m \in \mathbb{N}\backslash\{0\}$.

    Then, $2 \cdot n - 1 = 2 \cdot m - 1$.

    So, $n = m$, establishing injection of f.

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Set counting

- Set counting examples:
  - Example-1 (contd.)::

    Surjection of f: Let $a =$ odd positive integer.

    Then, $a =$ (even positive number succeeding a) $- 1 = 2 \cdot x - 1 = f(x)$, where $x \in \mathbb{N}\backslash\{0\}$.

    So, $a = f(x)$, showing surjection of f.

    Combinedly, injection and surjection of f shown in following.

Set of odd positive integers:	1	3	5	7	9	11	13	15	17	19	21	23	…
(Injection + Surjection)	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{N}\backslash\{0\}$:	1	2	3	4	5	6	7	8	9	10	11	12	…

Set counting

- Set counting examples:
  - Example-2:: to decide countability of "set of all integers" —
    For given case, to exhibit bijection between positive integers $\mathbb{N}\backslash\{0\}$ and "set of all integers" $\mathbb{Z}$, and thus, inferring $\mathbb{Z}$ as countable set.
    Considering function $f: \mathbb{N} \to \mathbb{Z}$, with $f(1) = 0$, and
    $$f(n) = \begin{cases} {}^{n}/_{2}, & n = \text{even} \\ {}^{-(n-1)}/_{2}, & n = \text{odd}, n > 1 \end{cases}$$
    Injection of f: Let $f(n) = f(m)$, where arbitrary $n, m \in \mathbb{N}\backslash\{0\}$.
    Then, either $\frac{n}{2} = \frac{m}{2}$, or $\frac{-(n-1)}{2} = \frac{-(m-1)}{2}$.
    In both cases, $n = m$, establishing injection of f.

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Set counting

- Set counting examples:
  - Example-2 (contd.)::
    Surjection of f: Let $i =$ any arbitrarily-chosen integer, i.e. $i \in \mathbb{Z}$.
    Considering $i > 0$, $i = f(x) = \frac{x}{2}$. $\therefore x = 2 \cdot i > 0$, and so $x \in \mathbb{N}\backslash\{0\}$.
    For $i < 0$, $i = f(x) = \frac{-(x-1)}{2}$. $\therefore x = (-2) \cdot i + 1 > 0$, and so $x \in \mathbb{N}\backslash\{0\}$.
    When $i = f(x) = 0$, $x = 1$. So, $x \in \mathbb{N}\backslash\{0\}$, showing surjection of f.
    Combinedly, injection and surjection of f shown in following.

Set of all integers:	0	1	–1	2	–2	3	–3	4	–4	5	–5	6	–6	…
⟨Injection + Surjection⟩	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{N}\backslash\{0\}$:	1	2	3	4	5	6	7	8	9	10	11	12	13	…

Set counting

- Set counting examples:
  - Example-3:: to decide countability of "set of all rational numbers in $[0, 1]$" —
    "Set of all rational numbers in $[0, 1]$" = $(\mathbb{Q} \cap [0, 1]) = \{{}^{p}/_{q} \mid p \in \mathbb{N}, q \in \mathbb{Z}^{+}, {}^{p}/_{q} \leq 1\}$, where $[0, 1]$ = set of all real numbers between 0 and 1, including 0 and 1, i.e., closed interval of real numbers between 0 and 1.
    For given case, to exhibit bijection between positive integers $\mathbb{N}\backslash\{0\}$ and $(\mathbb{Q} \cap [0, 1])$, and thus, inferring $(\mathbb{Q} \cap [0, 1])$ as countable set.
    Now, to prepare sequence all numbers in $(\mathbb{Q} \cap [0, 1])$, starting with $p = 0$, $q = 1$ and increment q by 1, and for each such q $(0 \leq p \leq q)$, to add rational number ${}^{p}/_{q}$ to list, if not already present.

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Set counting

- Set counting examples:
  - Example-3 (contd.)::
    Final sequence: $0, 1, {}^1/_2, {}^1/_3, {}^2/_3, {}^1/_4, {}^3/_4, {}^1/_5, {}^2/_5, {}^3/_5, {}^4/_5, {}^1/_6, {}^5/_6, \cdots$.
    Clearly, after completion of listing, enumerating this sequence also possible, i.e., bijection function $f: \mathbb{N} \to (\mathbb{Q} \cap [0, 1])$ to exist, as shown in following.

$\mathbb{Q} \cap [0, 1]$:	0	1	${}^1/_2$	${}^1/_3$	${}^2/_3$	${}^1/_4$	${}^3/_4$	${}^1/_5$	${}^2/_5$	${}^3/_5$	${}^4/_5$	${}^1/_6$	${}^5/_6$	…
(Bijection)	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{N}\backslash\{0\}$:	1	2	3	4	5	6	7	8	9	10	11	12	13	…

So, set of all rational numbers in $[0, 1] = (\mathbb{Q} \cap [0, 1])$ to become countably infinite, i.e., countable.

Set counting

- Set counting examples:
  - Example-4:: to show $|(0,1)| = |(0,1]|$, i.e., cardinality of open unit interval and mixed (i.e., open-close) unit interval to be same.
    Plan: in order to show equality, to exhibit one-to-one correspondence between sets in L.H.S. and R.H.S., i.e. to show their respective injections first, and then to apply Cantor-Schröder-Bernstein theorem.
    Case-1 – injective function from $(0,1)$ to $(0,1]$: as $(0,1) \subset (0,1]$, with $f: (0,1) \to (0,1]$, choosing $f(x) = x$, for arbitrary $a, b \in (0,1)$, if $a \neq b$, then $f(a) = a \neq b = f(b)$, and if $a = b$, $f(a) = a = b = f(b)$.
    Thus, injection of f concluded.

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Set counting

- Set counting examples:
  - Example-4 (contd.)::
    Case-2 – injective function from $(0,1]$ to $(0,1)$: as $(0, {}^1/_2] \subset (0,1)$, with $g: (0,1] \to (0, {}^1/_2]$, choosing $g(x) = {}^x/_2$, for arbitrary $c, d \in (0,1]$, if $c \neq d$, then $g(c) = {}^c/_2 \neq {}^d/_2 = g(d)$, and if $c = d$, $g(c) = {}^c/_2 = {}^d/_2 = g(d)$. Thus, injection of g concluded, which inferring presence of injective function from $(0,1]$ to $(0,1)$.
    Using above two intermediate results and applying Cantor-Schröder-Bernstein theorem, $|(0,1)| = |(0,1]|$.

Set counting

- Uncountable set $\mathbf{S}'$: if infinite set $\mathbf{S}' \neq$ countable set, i.e., no bijective function $f: \mathbb{N} \rightarrow \mathbf{S}'$ possible, and so no listing of elements of $\mathbf{S}'$ based on indexing by positive integers from "$\mathbb{N} \backslash \{0\}$".
  - Property:: ⟨Theorem⟩: Superset of uncountable set also uncountable, i.e., if uncountable set $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{B}$ also to become uncountable.
  - Property:: Set of real numbers $\mathbb{R}$ = uncountable set.
  - Property:: Cardinality of Continuum $= |\mathbb{R}| = \mathfrak{c}$ (Fraktur 'c') $= \aleph_1$.
  - Property:: ⟨Theorem⟩: $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, i.e., $\aleph_0 < 2^{\aleph_0}$.
  - Property:: ⟨Theorem⟩: $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, and so, $\mathfrak{c} = 2^{\aleph_0}$.
  - Property:: $\mathfrak{c} = \aleph_1 = 2^{\aleph_0} > \aleph_0$.

Set counting

- Uncountable set:
  - Property:: ⟨Cantor's theorem⟩: For any (finite/infinite) set $\mathbf{S}$, $|\mathbf{S}| < |\mathcal{P}(\mathbf{S})|$.
  - Property:: ⟨Continuum hypothesis⟩: No cardinal number between $\aleph_0$ and $\mathfrak{c}$ (i.e., $\aleph_1$), i.e., no finite/infinite set $\mathbf{X}$ such that: $\aleph_0 < |\mathbf{X}| < \aleph_1 = \mathfrak{c}$.

Set counting

- Set counting examples:
  - Example-5:: to decide uncountability of "set of all real numbers" — [Proof by contradiction]
    Let $\neg p \equiv$ "$\mathbb{R}$ be countable" (premise). So, $(0, 1) \subseteq \mathbb{R}$ also countable, as "subset of countable sets also countable." So $q \equiv$ "$f: \mathbb{N} \rightarrow (0, 1)$ bijective." Then, $q \equiv$ TRUE, and so possibility of $f(i)$ to enumerate any $r_i \in (0, 1)$, $i \in \mathbb{N} \backslash \{0\}$, with decimal representation $r_i = 0.d_{i1}d_{i2}d_{i3}d_{i4}\ldots$.

    $r_1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$ $\quad$ $r_2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$

    $r_3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$ $\quad$ $\ldots\ldots$

    where, $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $j \in \mathbb{N}$.

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Set counting

- Set counting examples:
  - Example-5 (contd.)::

    Now, to form a 'new' arbitrary real number $r \in (0, 1)$ stage-wise (while building sequence) having decimal representation $r = 0.d'_1 d'_2 d'_3 d'_4 \ldots$, where each d'_i given by,

    digit $d'_i = \begin{cases} 1, & \text{if } d_{ii} \neq 1 \\ 2, & \text{if } d_{ii} = 1. \end{cases}$ Purpose: $d'_i \neq d_{ii}$, i.e., not to make equal.

$r_1 = 0.d_{11}d_{12}d_{13}d_{14} \ldots$
$r_2 = 0.d_{21}d_{22}d_{23}d_{24} \ldots$
$r_3 = 0.d_{31}d_{32}d_{33}d_{34} \ldots$
$r_4 = 0.d_{41}d_{42}d_{43}d_{44} \ldots$
$\vdots \quad \vdots$

    Continuing in this manner, r finally formed once all real numbers in $(0, 1)$ enumerated, and r not present in that ordering, as its creation kept r distinct at every stage of enumeration.

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Set counting

- Set counting examples:
  - Example-5 (contd-2.)::

    So, r not enumerated, i.e., $r \neq f(i)$, $\forall i \in \mathbb{N} \backslash \{0\}$. Thus, $f: \mathbb{N} \rightarrow (0, 1)$ not surjective. So, $f: \mathbb{N} \rightarrow (0, 1)$ not bijective, i.e., $\neg q \equiv$ TRUE. Contradiction.

    Then, $(0, 1)$ uncountable, and so based on "superset of uncountable set also uncountable," its superset $\mathbb{R}$ also to be uncountable.

    Note: creating new r allowed to be arbitrary, with only restriction of $d'_i \neq 9$. [Reason: real number with terminated decimal expansion also having second decimal expansion ending with infinite sequence of 9s, because $0.5 = 0.4999\ldots$.]

Summary

- Focus: Set countability.
- Set counting.
- Countable set and countably infinite set, with examples.
- Countable set related theorems and properties.
- Countability of set of integers.
- Cantor-Schröder-Bernstein theorem, with examples.
- Uncountable set, and related theorems and properties.
- Uncountability of set of real numbers.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
5. https://www.math.cmu.edu/~wgunther/127m12/notes/CSB.pdf.

Further Reading

- Countable set:: [Ros19]:180-183.
- Countably infinite set:: [Ros19]:181-183.
- Countability of set of integers:: [Ros19]:182.
- Schröder-Bernstein theorem:: [Ros19]:184-185.
- Uncountable set:: [Ros19]:183-184.

Lecture Exercises: Problem 1 [Ref: Gate 2021,Q.11,p.16(Set2)]

Consider the following sets, where $n \geq 2$:

S_1: Set of all $n \times n$ matrices with entries from the set $\{a, b, c\}$

S_2: Set of all functions from the set $\{0,1,2,\cdots,n^2-1\}$ to the set $\{0,1,2\}$

Which of the following choice(s) is/are correct?

(a) There does not exist a bijection from S_1 to S_2.

(b) There exists a surjection from S_1 to S_2.

(c) There exists a bijection from S_1 to S_2.

(d) There does not exist an injection from S_1 to S_2.

Lecture Exercises: Problem 2 [Ref: Gate 2018, Set-3, Q.27, p.8]

Let $\mathbb{N}$ be the set of natural numbers. Consider the following sets.

$\mathbf{P}$: set of Rational numbers (positive and negative)
$\mathbf{Q}$: set of functions from $\{0,1\}$ to $\mathbb{N}$
$\mathbf{R}$: set of functions from $\mathbb{N}$ to $\{0,1\}$
$\mathbf{S}$: set of finite subsets of $\mathbb{N}$.

Which of the sets above are countable?

(a) $\mathbf{Q}$ and $\mathbf{S}$ only.
(b) $\mathbf{P}$ and $\mathbf{S}$ only.
(c) $\mathbf{P}$ and $\mathbf{R}$ only.
(d) $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{S}$ only.

Lecture Exercises: Problem 2 Ans

- Countable sets = Finite sets + Countably infinite sets.
- Countability of $\mathbf{P}$:: [based on countability theorems]
  Using proof method for theorem, "set of all rational numbers in $[0,1]$ to be countable," proof possible for "set of all rational numbers in $[n, n+1]$ to be countable, for any arbitrary $n \in \mathbb{Z}$," i.e., $(\mathbb{Q} \cap [n, n+1])$ countable.
  Then, $\forall n \in \mathbb{Z}\ (\mathbb{Q} \cap [n, n+1])$ to become countable, based on universal generalization rule.
  Based on theorem "union of countable sets to be countable," and corollary "countable union of countable sets to be countable," $\bigcup_{n \in \mathbb{Z}} (\mathbb{Q} \cap [n, n+1]) = \mathbb{Q}$ to become countable.

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Lecture Exercises: Problem 2 Ans (contd.)

- Countability of $\mathbf{Q}$:: [based on countability definition]
  Given, $\mathbf{Q} = \{f \mid f: \{0,1\} \to \mathbb{N}\}$.
  For each $f \in \mathbf{Q}$, range $= \{f(0), f(1)\}$, such that $f(0), f(1) \in \mathbb{N}$.
  Consequently, each range to contain two ordered pairs: $(f(0), f(1))$ and $(f(1), f(0))$. Then,
  $\mathbf{Q}$ to become set of all such ordered pairs from infinite count of functions.
  An infinite sequence created as follows for all such ordered pairs of $\mathbf{Q}$:
  $$(f_1(0), f_1(1)), (f_1(1), f_1(0)), (f_2(0), f_2(1)), (f_2(1), f_2(0)), \cdots,$$
  $$(f_n(0), f_n(1)), (f_n(1), f_n(0)), \cdots \infty$$
  Then, some bijective function $g: \mathbb{N} \to \mathbf{Q}$ possible based on above.
  So, $\mathbf{Q}$ to become countably infinite, i.e., countable.

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Lecture Exercises: Problem 2 Ans (contd-2.)

- Countability of $\mathbf{R}$:: [based on product rule and uncountability theorems]
  Given, $\mathbf{R} = \{f \mid f: \mathbb{N} \to \{0,1\}\}$, i.e., each f in $\mathbf{R}$ to map to either 0 or 1.
  Number of ways of performing such mapping for each f: 2.
  For any two $f, g \in \mathbf{R}$, mapping of f independent of mapping of g.
  Then, as per product rule, number of mappings of all functions = (number of mapping of 1st function) · (number of mapping of 2nd function) · (number of mapping of 3rd function) $\cdots = \underbrace{2 \cdot 2 \cdot 2 \cdot \cdots}_{\aleph_0 \text{ terms}} = 2^{\aleph_0} = \mathfrak{c}$,
  where $\aleph_0$ = cardinality of $\mathbb{N}$, $\mathfrak{c} = \aleph_1$ = cardinality of $\mathbb{R}$. Then, $|\mathbf{R}| = |\mathbb{R}|$.
  Based on theorem "set of all real numbers uncountable," $\mathbf{R}$ to become uncountable.

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Lecture Exercises: Problem 2 Ans (contd-3.)

- Countability of $\mathbf{S}$:: [based on set, countability theorems]
  Given, $\mathbf{S} = \{\boldsymbol{s} \mid \boldsymbol{s} \subseteq \mathbb{N}, \boldsymbol{s} \text{ finite}\}$, i.e., finite count of natural numbers in $\boldsymbol{s}$.
  Then, a largest finite element of $\boldsymbol{s}$ also possible, say $n \in \mathbb{N}$.
  So $\boldsymbol{s}$ to be considered subset of set $\mathfrak{N} = \{0,1,2,\cdots,n\}$, where $|\mathfrak{N}| = n+1$.
  Let $\boldsymbol{s}_n$ be set of all subsets of $\mathfrak{N}$ with largest element n in it, i.e., $\boldsymbol{s}_n = \mathcal{P}(\mathfrak{N})$, where $\mathcal{P}(\mathfrak{N})$ = power set of $\mathfrak{N}$, and $|\boldsymbol{s}_n| = 2^{n+1}$.
  As $n \in \mathbb{N}$ and n finite, so $\boldsymbol{s}_n$ to become countable (based on definition of countability).
  Then, $\mathbf{S} = \bigcup_{n \in \mathbb{N}} \boldsymbol{s}_n$, i.e., $\mathbf{S}$ to become union of countable number of countable sets. ∴ $\mathbf{S}$ also countable, based on theorem 'countable union of countable sets to become countable.'