CS34110   Discrete Mathematics and Graph Theory

UNIT – II, Module – 1

Lecture 11: Discrete Structures

[Function composition; Partial function;
Sequence, subsequence; Recurrence, closed
formula; Forward, backward substitution; Series]

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Recap:: Discrete structures

- Discrete structures examples:
  - (i) sets (collections of objects) and multisets;
    (ii) combinations (built from sets; unordered collections of objects to
    be used in counting);
    (iii) relations (sets of ordered pairs representing relationships
    between objects);
    (iv) graphs (sets of vertices and edges connecting vertices);
    (v) sequences (ordered lists of elements, as well as special type of
    functions expressing relationships among elements)
    (vi) matrices.

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Notation table

| Symbol / Notation	Meaning
ι	Identity function. E.g. Identity function on S, $\iota_S : S \rightarrow S$
$\circ$	Composition of functions. E.g., $f \circ g$ denoting composition of f and g
$\lfloor\ \rfloor$	Floor function. E.g., function $\lfloor x \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ from $\mathbb{R}$ to $\mathbb{Z}$
$\lceil\ \rceil$	Ceiling function. E.g., function $\lceil x \rceil : \mathbb{R} \rightarrow \mathbb{Z}$ from $\mathbb{R}$ to $\mathbb{Z}$

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Function

- Function:
  - Composition property:: For function $g: \mathbf{A} \to \mathbf{B}$ and function $f: \mathbf{B} \to \mathbf{C}$, composition of functions f and g, denoted by $(f \circ g): \mathbf{A} \to \mathbf{C}$ — assignment "$(f \circ g)(a) = f(g(a))$" for all $a \in \mathbf{A}$, where domain of $f \circ g$ = domain of g = $\mathbf{A}$, and range of $f \circ g$ = images of "range of g with respect to f".
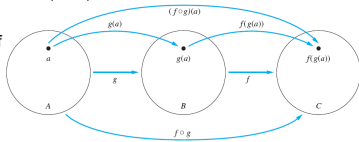
[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Function

- Function:
  - Property:: Nature of composition of functions —

$(f \circ g): \mathbf{A} \to \mathbf{C}$	$g: \mathbf{A} \to \mathbf{B}$	$f: \mathbf{B} \to \mathbf{C}$
If $(f \circ g)$ be one-to-one (or injective)	Then, g must be one-to-one	But, f may or mayn't be one-to-one.
If $(f \circ g)$ be onto (or surjective)	Then, g may or mayn't be onto	But, f must be onto.
If $(f \circ g)$ be one-to-one correspondence (or bijective)	Then, g must be onto	And, f must also be one-to-one.

Function

- Function:
  - Property:: For any two functions, composition of those functions not commutative, even for their same domain and codomain.
  - Property:: For bijective function $f: \mathbf{A} \to \mathbf{B}$, composition of f and f^{-1} — $f \circ f^{-1} = \iota_{\mathbf{B}}: \mathbf{B} \to \mathbf{B}$, where $f^{-1}: \mathbf{B} \to \mathbf{A}$, due to $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x$, where $f^{-1}(x) = y$ when $f(y) = x$, for $x \in \mathbf{B}$, $y \in \mathbf{A}$.
  - Property:: For bijective function $f: \mathbf{A} \to \mathbf{B}$, composition of f^{-1} and f — $f^{-1} \circ f = \iota_{\mathbf{A}}: \mathbf{A} \to \mathbf{A}$, due to $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$, where $f^{-1}(y) = x$ when $f(x) = y$, for $x \in \mathbf{A}$, $y \in \mathbf{B}$.

Function

- Function:
  - Property:: Floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ — $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$, or equivalently $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$, for $x \in \mathbb{R}$, $n \in \mathbb{Z}$, i.e., $x = n + \varepsilon$, where $0 \le \varepsilon < 1$, $\varepsilon \in \mathbb{R}$.
  - Property:: Ceiling function $\lceil x \rceil : \mathbb{R} \to \mathbb{Z}$ — $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$, or equivalently $\lceil x \rceil = n$ if and only if $x \le n < x + 1$, for $x \in \mathbb{R}$, $n \in \mathbb{Z}$, i.e., $x = n - \varepsilon$, where $0 \le \varepsilon < 1$, $\varepsilon \in \mathbb{R}$.
  - Property:: $\lfloor x + n \rfloor = \lfloor x \rfloor + n$. $\lceil x + n \rceil = \lceil x \rceil + n$.
  - Property:: $\lfloor -x \rfloor = -\lceil x \rceil$. $\lceil -x \rceil = -\lfloor x \rfloor$.
  - Property:: $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$.

Function

- Function:
  - Property:: For all $x \in \mathbb{R}$, $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2 \cdot x \rfloor$.

Function

- Function examples:
  - Example-13:: For functions $f : \mathbb{Z} \to \mathbb{Z}$, $g : \mathbb{Z} \to \mathbb{Z}$, such that $f(x) = 2 \cdot x + 3$, $g(x) = 3 \cdot x + 2$, $x \in \mathbb{Z}$, to obtain their composition of functions — $(f \circ g)(x) = f\big(g(x)\big) = f(3 \cdot x + 2) = 2 \cdot (3 \cdot x + 2) + 3 = 6 \cdot x + 7$, for all $x \in \mathbb{Z}$; domain of $f \circ g$ = domain of $g = \mathbb{Z}$; range of $f \circ g$ = image of "range of g" with respect to $f = \mathbb{Z}$. $\therefore f \circ g : \mathbb{Z} \to \mathbb{Z}$, $(f \circ g)(x) = 6 \cdot x + 7$.
    $(g \circ f)(x) = g\big(f(x)\big) = g(2 \cdot x + 3) = 3 \cdot (2 \cdot x + 3) + 2 = 6 \cdot x + 11$, for all $x \in \mathbb{Z}$; domain of $g \circ f$ = domain of $f = \mathbb{Z}$; range of $g \circ f$ = image of "range of f" with respect to $g = \mathbb{Z}$.
    $\therefore g \circ f : \mathbb{Z} \to \mathbb{Z}$, $(g \circ f)(x) = 6 \cdot x + 11$.

Function

- Function examples:
  - Example-14:: For function $f: \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$, where $f(x) = x^2$, $x \in \mathbb{R}$, and function $g: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$, with $g(x) = \sqrt{x}$, where $\sqrt{x}$ = nonnegative square root of x, $x \in \mathbb{R}^+ \cup \{0\}$, to obtain composition $(f \circ g)(x)$ —
    Domain of $f \circ g$ = domain of $g = \mathbb{R}^+ \cup \{0\}$.
    $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$, for all $x \in \mathbb{R}^+ \cup \{0\}$, due to each x being nonnegative real number.
    Range of $f \circ g$ = image of "range of g" with respect to $f = \mathbb{R}^+ \cup \{0\}$.
    $\therefore f \circ g: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$, with $(f \circ g)(x) = x$, for all $x \in \mathbb{R}^+ \cup \{0\}$.

Partial function

- Partial function f from set **A** to set **B** — $f: \mathbf{A} \rightarrow \mathbf{B}$ = assignment "$f(a) = b$" of unique element $b \in \mathbf{B}$ corresponding to each element $a \in$ subset of **A**. In other words, mapping not defined for all elements in domain of partial function.
  - Property:: Subset of **A**: "domain of definition" of f.
  - Property:: **B**: codomain of f.
  - Property:: Partial function $f: \mathbf{A} \rightarrow \mathbf{B}$ undefined for others — $f(a)$ undefined for all other elements $a \notin$ "domain of definition" subset of **A**.
  - Property:: Total function $f: \mathbf{A} \rightarrow \mathbf{B}$ — when "domain of definition" of f = **A**.

Partial function

- Partial function examples:
  - Example-1:: For function $f: \mathbb{Z} \rightarrow \mathbb{R}$, with $f(x) = \sqrt{x}$, to check partial —
    Domain of definition of f = set of nonnegative integers = $\mathbb{Z}^+ \cup \{0\}$.
    But domain of $f = \mathbb{Z}$.
    So, given $f: \mathbb{Z} \rightarrow \mathbb{R}$ to become partial function.

Recap:: Discrete structures

- Discrete structures examples:
  - (i) sets (collections of objects) and multisets;
    (ii) combinations (built from sets; unordered collections of objects to be used in counting);
    (iii) relations (sets of ordered pairs representing relationships between objects);
    (iv) graphs (sets of vertices and edges connecting vertices);
    (v) sequences (ordered lists of elements, as well as special type of functions expressing relationships among elements)
    (vi) matrices.

Sequence

- Sequence — function from nonnegative set of integers (usually $\mathbb{N}$ or $\mathbb{Z}^{+}$) to set S (i.e., $f : \mathbb{N} \rightarrow S$); notation: $\{a_n\}$.
  - Property:: a_n: term of sequence S; also, image of integer n.
  - Property:: Arithmetic progression: sequence of form —
    $\{a_n\} = \{b + n \cdot d\}: \ b, (b + d), (b + 2 \cdot d), \dots, (b + n \cdot d), \dots$
    where initial term $= b$, common difference $= d$, $b, d \in \mathbb{R}$.
  - Property:: Geometric progression: sequence of form —
    $\{a_n\} = \{c \cdot r^n\}: \ c, (c \cdot r), (c \cdot r^2), \dots, (c \cdot r^n), \dots$
    where initial term $= c$, common ratio $= r$, $c, r \in \mathbb{R}, r \neq 0$.

Sequence

- Sequence:
  - Property:: Strictly increasing (and strictly decreasing) sequence: each term larger (i.e., increasing order) than term preceding it, and each term smaller (i.e., decreasing order) than term preceding it.
  - Property:: Subsequence: For sequence $\{a_n\}$ of real numbers (of terms $a_1, a_2, \dots, a_n$), sequence of terms $a_{i_1}, a_{i_1}, \dots, a_{i_m}$ only (where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$), $i_1, i_2, \dots, i_m, m, n \in \mathbb{N} \backslash \{0\}$.
  - Property:: [In words] Subsequence = sequence obtained from given original sequence by including some terms of original sequence in their original order, and perhaps not including other terms.

Sequence

- Sequence:
  - Property:: Sequence $\{a_n\}$ = solution of **recurrence** (specified by recurrence equation), expressing a_n in terms of one or more of previous terms $a_0, a_1, \dots, a_{n-1}$ of that sequence for all integers $n \geq n_0$, where $n_0 =$ nonnegative integer.
    i.e., recurrence $\xrightarrow{\text{recursively define}}$ sequence.
    - Property:: Initial conditions for recursively defined sequence $\{a_n\}$: terms preceding first term of recurrence.
  - Property:: Source of sequence examples: Online Encyclopedia of Integer Sequences (OEIS).

Sequence

- Sequence:
  - Property:: Closed formula of recursively defined sequence $\{a_n\}$: closed-form solution of recurrence, whereby expressing a_n in terms of explicit formula.
  - Property:: Summation of terms of sequence $\{a_n\}$: series, in which summation of m-th term to n-th term of $\{a_n\} = \sum_{j=m}^{n} a_j = \sum_{m \leq j \leq n} a_j = (a_m + a_{m+1} + \cdots + a_n)$, $(n \geq m)$, where $j =$ index of summation (choice of letter j being arbitrary, be replaced by any other letter of choice, such as i or k), $m =$ lower limit and $n =$ upper limit of summation.

Sequence

- Sequence:
  - Property:: Addition properties equally applicable to summation —
    $\sum_{j=1}^{n}(\alpha \cdot a_j + \beta \cdot b_j) = \alpha \cdot \left(\sum_{j=1}^{n} a_j\right) + \beta \cdot \left(\sum_{j=1}^{n} b_j\right)$.
  - Property:: Geometric series: sum of terms of geometric progression
    $$= \sum_{j=0}^{n}(a \cdot r^j) = \begin{cases} \dfrac{a \cdot r^{n+1} - a}{r - 1}, & r \neq 1 \\ (n+1) \cdot a, & r = 1 \end{cases}$$
    for $a, r \in \mathbb{R}, r \neq 0$.
  - Property:: Summation of terms of function and set —
    $\sum_{x \in A} f(x) =$ sum of function values $f(x)$, for all $x \in A$.

Sequence

- Sequence:
  - Property:: Infinite geometric series: sum of infinite number of terms of geometric progression
  $$= \sum_{j=0}^{\infty}\left(a \cdot r^j\right) = \frac{a}{1-r}, \qquad |r| < 1$$
  for $a, r \in \mathbb{R}, r \neq 0$.
  - Property:: Series with even powers of r — $\sum_{j=0}^{\infty}\left(a \cdot r^{2 \cdot j}\right) = \frac{a}{1-r^2}$.
  - Property:: Series with odd powers of r — $\sum_{j=0}^{\infty}\left(a \cdot r^{2 \cdot j+1}\right) = \frac{a \cdot r}{1-r^2}$.
  - Property:: Series starting at m-th term — $\sum_{j=m}^{\infty}\left(a \cdot r^j\right) = \frac{a \cdot r^m}{1-r}$.

Sequence

- Sequence:
  - Property:: Product of terms of sequence $\{a_n\}$: product of m-th term up to n-th term of $\{a_n\} = \prod_{j=m}^{n} a_j = \prod_{m \le j \le n} a_j = (a_m \cdot a_{m+1} \cdot \cdots \cdot a_n)$, $(n \ge m)$, where j = index of product, m = lower limit and n = upper limit of product.

Sequence

- Sequence examples:
  - Example-1:: Fibonacci sequence $\{f_n\}$, with initial conditions: $f_0 = 0$, $f_1 = 1$, and recurrence: $f_n = f_{n-1} + f_{n-2}$, for $n \ge 2$.
  - Example-2:: For sequence $\{a_n\}$ of integers, where $a_n = n!$ (value of factorial function at $n \in \mathbb{Z}^+$) — Closed formula
  $a_n = n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n \cdot (n-1)! = n \cdot a_{n-1}$.
  So, recurrence: $a_n = n \cdot a_{n-1}$, for $n \ge 2$,
  with initial condition: $a_1 = 1$ due to $1!$.

Sequence

- Sequence examples:
  - Example-3:: Given sequence $\{a_n\}$, obtained by recurrence: $a_n = 2 \cdot a_{n-1} - a_{n-2}$, for $n \geq 2$, with initial conditions: $a_0 = 0$, $a_1 = 3$, to verify $a_n = 3 \cdot n$ (for $n \in \mathbb{Z}^+$) being its closed formula —
    For $n \geq 2$, $2 \cdot a_{n-1} - a_{n-2} = 2 \cdot \big(3 \cdot (n-1)\big) - 3 \cdot (n-2) = 3 \cdot n =$
    $= a_n$, obtained by use of $a_n = 3 \cdot n$ to recurrence.
    So, closed formula $a_n = 3 \cdot n$ (for $n \in \mathbb{Z}^+$) for given recurrence.

Sequence

- Sequence examples:
  - Example-4:: For sequence $\{a_n\}$, obtained by recurrence: $a_n = a_{n-1} + 3$, for $n \geq 2$, with initial condition: $a_1 = 2$ —
    To deduce closed formula, based on initial term through n-th term.
    $a_2 = a_1 + 3 = 2 + 3$.
    $a_3 = a_2 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2$.   Forward substitution
    $a_4 = a_3 + 3 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$.
    $\vdots$
    $a_n = a_{n-1} + 3 = \big(2 + 3 \cdot (n-2)\big) + 3 = 2 + 3 \cdot (n-1)$.
    So, closed formula: $a_n = 2 + 3 \cdot (n-1)$ [independent of recurrence].

Sequence

- Sequence examples:
  - Example-4 (contd.):: [alternative solution].
    To deduce closed formula, based on n-th term being successively worked out towards preceding term until reaching initial condition.
    $a_n = a_{n-1} + 3 = (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$ Backward substitution
    $\quad = (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$
    $\quad \vdots$
    $\quad = a_2 + 3 \cdot (n-2) = (a_1 + 3) + 3 \cdot (n-2)$
    $\quad = a_1 + 3 \cdot (n-1) = 2 + 3 \cdot (n-1)$.
    So, closed formula: $a_n = 2 + 3 \cdot (n-1)$.

Sequence

- Sequence examples:
  - Example-5:: For sequence $\{a_n\}$ of integers, where $a_n = n^2$, sum of first 5 terms of sequence in series —
    $\sum_{j=1}^{5} j^2 = \frac{5 \cdot 6 \cdot 11}{6} = 55$, as $\sum_{j=1}^{n} j^2 = \frac{n \cdot (n+1) \cdot (2 \cdot n+1)}{6}$.
  - Example-6:: For following double summation —
    $\sum_{i=1}^{4} \sum_{j=1}^{3} (i \cdot j) = \sum_{i=1}^{4} (i + i \cdot 2 + i \cdot 3) = \sum_{i=1}^{4} (6 \cdot i) = 6 \cdot \sum_{i=1}^{4} i$.
    $= 6 \cdot \frac{4 \cdot 5}{2} = 60$, as $\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2}$.
  - Example-7:: For given summation $\sum_{s \in \{0,2,4\}} s$ —
    $\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6$.

Recap:: Discrete structures

- Discrete structures examples:
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Summary

- Focus: Function (contd.), sequence as discrete structures.
- Composition of functions, with examples.
- Floor, ceiling functions, and related theorems.
- Partial function.
- Sequence, and its representation, with examples.
- Sequence of arithmetic, geometric progressions; examples.
- Sequence from recurrence relation, with examples.
- Closed formula of recursively defined sequence.
- Summation of sequence terms as series, with examples.
- Finite and infinite geometric series.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

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Further Reading

- Identity function:: [Ros21]:153.
- Inverse function:: [Ros21]:153-155.
- Composition of functions:: [Ros21]:155-156.
- Floor, ceiling functions:: [Ros21]:157-160.
- Partial function:: [Ros21]:161.
- Sequence:: [Ros21]:165-166,170-172.
- Sequence from recurrence relation:: [Ros21]:167-169.
- Finite series:: [Ros21]:172-176.
- Infinite series:: [Ros21]:176-177.

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Lecture Exercises: Problem 1 [Ref: Gate 2025,Q.17,p.17(Set1)]

$g(\cdot)$ is a function from A to B, $f(\cdot)$ is a function from B to C, and their composition defined as $f(g(\cdot))$ is a mapping from A to C. If $f(\cdot)$ and $f(g(\cdot))$ are onto (surjective) functions, which ONE of the following is TRUE about the function $g(\cdot)$?

(a) $g(\cdot)$ must be an onto (surjective) function.
(b) $g(\cdot)$ must be a one-to-one (injective) function.
(c) $g(\cdot)$ must be a bijective function, i.e., both one-to-one and onto.
(d) $g(\cdot)$ is not required to be a one-to-one or onto function.

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