CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 17: Counting

[Combination; r-combinations theorem; Bijective, double counting; Combinations with repetitions; r-combinations with repetitions theorem]

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Combinations

- Combination: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, one *r-combination* = any unordered selection of r elements from n elements of $\mathbf{S}$. [Note: empty sets excluded.]
  - Property:: Each r-combination = subset of r distinct elements of $\mathbf{S}$.
  - Property:: ⟨Theorem⟩: Number of r-combinations = Binomial coefficient = $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!} = C(n,r)$ or C_r^n or $C_{n,r}$ or ${}_nC_r$ or "n-choose-r". [Proof similar to Binomial theorem proof.] [Alternative proofs: r-permutation theorem, generating function.]
  - Property:: Number of r-combinations = number of subsets (having cardinality r) of $\mathbf{S}$.

Combinations

- Combination identities:
  - Property:: ⟨Corollary⟩: $C(n,r) = C(n, n-r)$, where $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leq n$.

  Proof: Using algebraic manipulations. From previous theorem, $C(n,r) = \frac{n!}{r! \cdot (n-r)!}$.
  Then, $C(n, n-r) = \frac{n!}{(n-r)! \cdot (n-(n-r))!} = \frac{n!}{r! \cdot (n-r)!} = C(n,r)$. ■

  Alternate proof: [Using bijective proof (one form of *combinatorial proof*)]
  Let set $\mathbf{S}$ having n elements, and any set $\mathbf{A} \subseteq \mathbf{S}$ having r elements. Then, set $\bar{\mathbf{A}} = \mathbf{S} \backslash \mathbf{A}$ (where, $\bar{\mathbf{A}} \subseteq \mathbf{S}$) to have $(n-r)$ elements. Clearly, some mapping possible between $\mathbf{A}$ and $\bar{\mathbf{A}}$.

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Combinations

- Combination identities:

  Alternate proof contd.

  Let $\mathcal{P}$ be set of r-element subsets (like, $\mathbf{A}$) of $\mathbf{S}$, so that $|\mathcal{P}| = C(n, r)$.

  Let $\mathcal{Q}$ be set of $(n - r)$-element subsets (like, $\overline{\mathbf{A}}$) of $\mathbf{S}$, so that $|\mathcal{Q}| = C(n, n - r)$.

  Defining function $f: \mathcal{P} \to \mathcal{Q}$, such that $f(\mathbf{X}) = \overline{\mathbf{X}}$, to map any subset $\mathbf{X}$ of $\mathbf{S}$ (where, $|\mathbf{X}| = r$) to its complement $\overline{\mathbf{X}}$ (where, $|\overline{\mathbf{X}}| = n - r$).

  Plan: to show bijection of $f: \mathcal{P} \to \mathcal{Q}$; then, for two finite sets, $|\mathcal{P}| = |\mathcal{Q}|$.

  Again, bijection of $f: \mathcal{P} \to \mathcal{Q}$ = (injection of $f: \mathcal{P} \to \mathcal{Q}$) $\wedge$ (surjection of $f: \mathcal{P} \to \mathcal{Q}$).

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Combinations

- Combination identities:

  Alternate proof contd-2.

  Case: Injection of $f: \mathcal{P} \to \mathcal{Q}$ — Let $f(\mathbf{X}) = f(\mathbf{Y})$, for arbitrarily chosen subsets $\mathbf{X}, \mathbf{Y} \in \mathcal{P}$. Then, $\overline{\mathbf{X}} = \overline{\mathbf{Y}}$. Taking complements on both sides, $\overline{\overline{\mathbf{X}}} = \overline{\overline{\mathbf{Y}}}$, i.e. $\mathbf{X} = \mathbf{Y}$, establishing injection.

  Case: Surjection of $f: \mathcal{P} \to \mathcal{Q}$ — Let arbitrary $\mathbf{B} \in \mathcal{Q}$, $\mathbf{B} \subseteq \mathbf{S}$. Then, $\mathbf{B} = \overline{\overline{\mathbf{B}}} = f(\overline{\mathbf{B}})$, where $\overline{\mathbf{B}} \in \mathcal{P}$. Also, as $|\mathbf{B}| = (n - r)$ and $\mathbf{B} \cup \overline{\mathbf{B}} = \mathbf{S}$, so $|\overline{\mathbf{B}}| = r$, establishing surjection.

  Combinedly, $f: \mathcal{P} \to \mathcal{Q}$ to be bijective, and hence $|\mathcal{P}| = |\mathcal{Q}|$.

  Then, $C(n, r) = C(n, n - r)$. ■

Combinations

- Combination identities:

  Another alternate proof: [Using double counting (another form of *combinatorial proof*)]

  Let any set $\mathbf{S}$ having n elements.

  Then, number of subsets of $\mathbf{S}$ with r elements = number of ways of choosing r elements from n elements of $\mathbf{S}$ to form subsets = $C(n, r)$.

  However, determining each r-element subset $\mathbf{A} \subseteq \mathbf{S}$ also possible → by specifying $(n - r)$ elements not to be included in $\mathbf{A}$.

  For every $\mathbf{A} \subseteq \mathbf{S}$, such $(n - r)$ elements then becoming members of $\overline{\mathbf{A}}$, where $\overline{\mathbf{A}} \subseteq \mathbf{S}$.

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Combinations

- Combination identities:

  Another alternate proof contd.

  Then, number of subsets of $\mathbf{S}$ with $(n-r)$ elements = number of ways of choosing $(n-r)$ elements from n elements of $\mathbf{S}$ to form subsets = $C(n, n-r)$. So,

  $C(n, r)$ = number of r-element subsets of $\mathbf{S}$

  = number of complements of r-element subsets of $\mathbf{S}$

  = number of $(n-r)$-element subsets of $\mathbf{S} = C(n, n-r)$. ■

  - Demo:: If $\mathbf{S} = \{1,2,3,4\}$, any subset $\mathbf{A} = \{1,3\} = \{2,4\} = \bar{\mathbf{A}} = \mathbf{S} \setminus \mathbf{A}$.

    $\boldsymbol{\mathcal{P}} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$.

Combinations

- Combinations examples:
  - Example-1:: To determine how many different committees of three students formed from group of four students.

    Let $\mathbf{S}$ be set of students, denoted by last digit of roll no. $\{1, 2, 3, 4\}$.

    Number of (unordered) subsets with 3 from 4 students of $\mathbf{S}$ = number of 3-combinations from 4 students of $\mathbf{S} = C(4,3) = \binom{4}{3} = \frac{4!}{3! \cdot 1!} = 4$.

    [Note: Unordered subsets of $\mathbf{S}$ with 3 elements: {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}.]

Combinations

- Combinations examples:
  - Example-2:: To determine how many ways to select 47 cards from standard deck of 52 cards.

    No order restriction while dealing 47 cards from deck of 52 cards.

    So, number of different ways to select 47 cards from deck of 52 cards $= C(52,47) = \frac{52!}{47! \cdot 5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2598960$.

Combinations

- Combinations examples:
  - Example-3:: To count number of bit strings of length n to contain exactly r number of bit value 1s.
    For creating any bit string of length n, consider n positions of bits $b_1, b_2, \ldots, b_n$ in string to be filled.
    Choosing r number of positions from n positions to be filled by 1s $= C(n,r)$. Remaining $(n-r)$ positions to be filled by 0s.
    $\therefore$ Number of required bit strings $= C(n,r)$.

Combinations with repetition

- Combination with repetition: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, one r-*combination with repetitions* $=$ any unordered selection of r not necessarily distinct elements from n elements of $\mathbf{S}$.
  - Property:: Each r-combination with repetitions $=$ multiset of r elements taken from $\mathbf{S}$.
  - Property:: r-combination with repetitions: also "n-multichoose-r", r-multicombination.
  - Property:: Number of r-combinations with repetitions $=$ number of multisets (having cardinality r) of elements taken from $\mathbf{S}$.

Combinations with repetition

- Combination with repetition:
  - Property:: ⟨Theorem⟩: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, number of r-combinations with repetitions $=$ $\left(\binom{n}{r}\right) = \frac{(n+r-1)!}{r! \cdot (n-1)!} = C(n+r-1, r) = \bar{C}(n,r) = C(n+r-1, n-1)$.

  Proof: Each r-combination with repetition from n distinct elements (of $\mathbf{S}$) to be represented by **list** of $(n-1)$ '|' (bar symbol) and r '*' (star symbol), with total of $(\boldsymbol{n-1+r} = n+r-1)$ slots in list.
  Nature of **list**: $(n-1)$ '|' (bars) to separate n distinct element-groups, where repetition (= same element to occur possibly more than once, and kept together in that element-group) allowed.

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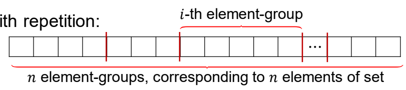
Combinations with repetition

- Combination with repetition:

  Proof contd.



Further, for $1 \le i \le n$, i-th element-group in list to contain 1 '*' (star) for each time i-th element of set to appear in combination; if such element not present in combination, no '*' in i-th element-group.

So, among r slots for r '*' (stars) in list of $(n + r - 1)$ slots, different counts of slots among each of n element-groups (corresponding to their counts of appearances), which totalled to r.

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Combinations with repetition

- Combination with repetition:

  Proof contd-2.

So, each list of r '*' (stars) → r-combination with repetition from n elements of $\mathbf{S}$ → choice of r slots to place r '*' (stars) from $(n + r - 1)$ slots of list.

∴ No. of such lists $= \frac{(n+r-1)\cdot(n+r-2)\cdot\cdots\cdot n}{r\cdot(r-1)\cdot\cdots\cdot 1} = \frac{(n+r-1)!}{r!\cdot(n-1)!} = C(n + r - 1, r)$.

So, number of r-combinations with repetitions from $\mathbf{S}$

$= C(n + r - 1, r)$. ■

Combinations with repetition

- Combination with repetition examples:
  - Example-1:: To determine how many ways to take 4 pieces of fruit from a bowl containing apples (A), oranges (O), and mangoes (M), with no regard to selection order (i.e. only type of fruit important and not individual piece), and considering enough (or at least 4 pieces of each type) of fruit available in bowl.

    All selections: [A: One apple; O: one orange; M: one mango]

AAAA	AAAO	AAAM	AAOO	AAMM	AAOM
AOOO	AMMM	AOOM	AOMM	OOOO	OOOM
OOMM	OMMM	MMMM	[listing all selections in same order: first A, then O, then M]		

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Combinations with repetition

- Combination with repetition examples:
  - Example-1 contd.

    Separating types of fruit by '|' (bar symbol): [first A, then O, then M]

AAAA||	AAA|O|	AAA||M	AA|OO|	AA||MM	AA|O|M
A|OOO|	A||MMM	A|OO|M	A|O|MM	|OOOO|	|OOO|M
|OO|MM	|O|MMM	||MMMM			

Actual fruit notations not needed, to be replaced by '*' (star symbol):

****||	***|*|	***||*	**|**|	**||**	**|*|*
*|***|	*||***	*|**|*	*|*|**	|****|	|***|*
|**|**	|*|***	||****			

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Combinations with repetition

- Combination with repetition examples:
  - Example-1 contd-2.

    Here, $(n-1=)$ 2 '|' symbols required to separate $(n=)$ 3 fruit types.
    Again, $(r=)$ 4 '*' symbols to express "taking 4 fruit pieces from bowl".
    So, total $(n-1+r=$ 2+4 $=)$ 6 slots, with decision of where to place — (i) either $(n-1=)$ 2 '|' symbols, (ii) or $(r=)$ 4 '*' symbols.
    Case (i): number of ways to place 2 '|' symbols in 6 slots $= C(6,2)$.
    Case (ii): number of ways to place 4 '*' symbols in 6 slots $= C(6,4)$.
    So, number of $(r=4)$-combinations with repetitions $= C(n+r-1,r) = C(6,4) = 15 = C(6,2) = C(n+r-1,n-1)$.

Combinations with repetition

- Combination with repetition examples:
  - Example-2:: Continuing example-1, where selecting 5 pieces of fruit from set $\{\text{apple}, \text{orange}, \text{mango}\}$.

    Here, $(n-1=)$ 2 '|' symbols required to separate $(n=)$ 3 fruit types.
    Again, $(r=)$ 5 '*' symbols to express "taking 5 fruit pieces from bowl."
    So, number of $(r=5)$-combinations with repetitions $= C(n+r-1,r) = C(7,5) = 21 = C(7,2) = C(n+r-1,n-1)$.

Combinations with repetition

- Combination with repetition examples:
  - Example-3:: To determine number of solutions of equation: $x_1 + x_2 + x_3 = 11$, where $x_1, x_2, x_3 \in \mathbb{N}$.
    Given problem → selecting 11 count of 'unknowns' from set $\{\text{unknown}_{x_1}, \text{unknown}_{x_2}, \text{unknown}_{x_3}\}$, no regard to selection order.
    Here, $(n-1=)$ 2 '|' symbols → to separate $(n=)$ 3 'unknown' types.
    Again, $(r=)$ 11 '*' symbols → "selecting 11 'unknowns' to form solution."
    So, number of solutions = number of $(r=11)$-combinations with repetitions $= C(n+r-1, r) = C(13,11) = C(13,2) = 78$.

Summary

- Focus: Combinatorics.
- Combination.
- r-combinations theorem, corollaries, and properties.
- Two forms of combinatorial proof — bijective proof, double counting proof.
- Combinations examples.
- r-combinations with repetitions theorem, and properties.
- Combinations with repetitions examples.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Mot08] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, PHI, Second edition, 2008.
3. [Lip07] Seymour Lipschutz and Marc Lars Lipson, Schaum's Outline of Theory and Problems of Discrete Mathematics, Third edition, McGraw-Hill Education, 2007.
4. https://www.cs.sfu.ca/~ggbaker/zju/math/perm-comb.html.
5. https://www.cs.sfu.ca/~ggbaker/zju/math/perm-comb-more.html.
6. https://en.wikipedia.org/wiki/Glossary_of_mathematical_symbols.

Further Reading

- Combination:: [Ros19]:428,431-434.
- r-combinations theorem:: [Ros19]:431-432.
- Combinatorial proof:: [Ros19]:433.
- Bijective proof:: [Ros19]:433-434.
- Double counting proof:: [Ros19]:433-434.
- Combinations with repetitions:: [Ros19]:445,446-450.
- r-combinations with repetitions theorem:: [Ros19]:448.