CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 1

Lecture 25: Graphs & Trees

[Walk, trail, path, circuit; Component of graph, graph partitioning; Rank, nullity; Distance, eccentricity, center, radius, diameter]

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Notation table

Symbol / Notation	Meaning
W	Representation of walk in undirected graph.
$\mathfrak{I}$	Representation of trail in undirected graph.
P	Representation of path in undirected graph.
$len(P)$	Length of path P of undirected graph.
$wt(P)$	Weight of path P of weighted undirected graph.
C	Representation of circuit in undirected graph.
$\omega(\mathcal{G})$	Number of components of disconnected graph $\mathcal{G}$.
$rank(\mathcal{G})$	Rank of undirected graph $\mathcal{G}$.
$\mu(\mathcal{G})$	Nullity of undirected graph $\mathcal{G}$.
$d(u,v)$	Distance between vertices u, v in connected undirected graph $\mathcal{G}$.
$E(v)$	Eccentricity of vertex v in connected undirected graph $\mathcal{G}$.
$center(\mathcal{G})$	Center of connected undirected graph $\mathcal{G}$.

Notation table (contd.)

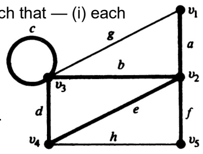
Symbol / Notation	Meaning
$rad(\mathcal{G})$	Radius of connected undirected graph $\mathcal{G}$.
$diam(\mathcal{G})$	Diameter of connected undirected graph $\mathcal{G}$.

Graph connectedness

- Graph connectedness: vertex connectedness within graph.
  - Property:: Walk of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: finite alternating sequence of vertices and edges in $\mathcal{G}$, beginning and ending with vertices (called walk's terminal vertices), such that each edge incident with vertices preceding and following it.
  - Property:: Walk representation: $W = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where $e_i = \{v_i, v_{i+1}\}$, $e_i \in \mathbf{E}$, $v_i, v_{i+1} \in \mathbf{V}$, $i = 1, \dots, (n-1)$.
  - Property:: Walk of graph $\mathcal{G}$ = subgraph of $\mathcal{G}$, i.e., $W \subset \mathcal{G}$.
  - Property:: Vertex repetition, edge repetition **allowed** in walk of graph.
  - Property:: Significance of walk: expressing connectedness of graph.

Graph connectedness

- Graph connectedness:
  - Property:: Trail of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: finite alternating sequence of vertices and edges in $\mathcal{G}$, beginning and ending with vertices (called trail's terminal vertices), such that — (i) each edge incident with vertices preceding and following it, and (ii) no edge to appear (i.e., covered or traversed) more than once.
  - Property:: Trail of graph $\mathcal{G}$ = subgraph of $\mathcal{G}$.
  - Property:: No edge repetition.
    Vertex repetition possible.



Trail = $\{v_1, a, v_2, b, v_3, c, v_3, d, v_4, e, v_2, f, v_5\}$

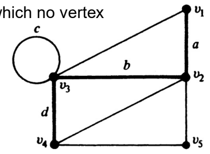
[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph connectedness

- Graph connectedness:
  - Property:: Trail representation: $\mathfrak{I} = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where (i) $e_i = \{v_i, v_{i+1}\}$, $e_i \in \mathbf{E}$, $v_i, v_{i+1} \in \mathbf{V}$, $i = 1, \dots, (n-1)$, (ii) $\forall e_i, e_j \left(e_i \neq e_j\right)$, where $e_i, e_j \in \mathfrak{I}$, $i, j = 1, \dots, (n-1)$.
  - Property:: Trail of graph $\mathcal{G}$ = Walk of $\mathcal{G}$ without edge repetition.
  - Property:: Vertex repetition allowed in trail of graph.
  - Property:: Closed trail: trail to begin and end at same terminal vertex.
  - Property:: Open trail: trail to begin and end at separate, distinct terminal vertices.

Graph connectedness

- Graph connectedness:
  - Property:: Path of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: open trail (i.e. distinct terminal vertices) in undirected graph, in which no vertex to appear more than once.
  - Property:: Path of graph $\mathcal{G}$ = Open trail of $\mathcal{G}$ without vertex repetition.
  - Property:: Path: also called simple path, elementary path (indicating no edge repeat).



Path $= \{v_1, a, v_2, b, v_3, d, v_4\}$

Path $\neq \{v_1, a, v_2, b, v_3, c, v_3, d, v_4, e, v_2, f, v_5\}$

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph connectedness

- Graph connectedness:
  - Property:: Path representation: $P = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where —
    (i) $e_i = \{v_i, v_{i+1}\}$, $e_i \in \mathbf{E}$, $v_i, v_{i+1} \in \mathbf{V}$, $i = 1, \dots, (n-1)$,
    (ii) $\forall e_i, e_j \left(e_i \neq e_j\right)$, where $e_i, e_j \in P$, $i, j = 1, \dots, (n-1)$,
    (iii) $(v_1 \neq v_n)$,
    (iv) $\forall v_i, v_j \left(v_i \neq v_j\right)$, where $v_i, v_j \in P$, $i, j = 1, \dots, (n-1)$.
  - Property:: Self-loop allowed to be included in trail, but not in path.
  - Property:: Maximal path of undirected graph $\mathcal{G}$: path P in $\mathcal{G}$ not contained in longer path of $\mathcal{G}$.

Graph connectedness

- Graph connectedness:
  - Property:: Length of path P of undirected graph $\mathcal{G}$: $len(P)$ = number of edges in P.
  - Property:: Every edge (except self loop) of graph = path of length 1.
  - Property:: Weight of path P of weighted undirected graph $\mathcal{G}$: $wt(P)$ = sum of weights of edges in P.
  - Property:: Spanning path of undirected graph $\mathcal{G}$: single path of $\mathcal{G}$ covering every vertex of $\mathcal{G}$.

Graph connectedness

- Graph connectedness:
  - Property:: Shortest path between any two given vertices $u, v \in \mathbf{V}$ of weighted undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: path of minimum weight between u and v as terminal vertices.

Graph connectedness

- Graph connectedness:
  - Property:: Circuit of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: closed trail (i.e. same terminal vertex) in $\mathcal{G}$, in which no vertex to appear more than once.
  - Property:: Circuit of graph $\mathcal{G}$ = Closed trail of $\mathcal{G}$ without vertex repetition.
  - Property:: Circuit: also called cycle, simple circuit, elementary cycle (indicating no edge repeat), circular path, polygon, loop (IMPORTANT:: every self-loop → loop/circuit/cycle, but every loop NOT self-loop).

Graph connectedness

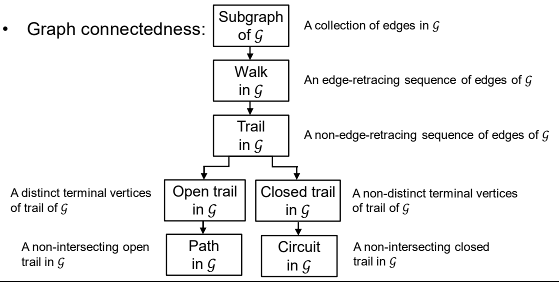
- Graph connectedness:
  - Property:: Circuit representation: $C = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where (i) $e_i = \{v_i, v_{i+1}\}$, $e_i \in \mathbf{E}$, $v_i, v_{i+1} \in \mathbf{V}$, $i = 1, \dots, (n-1)$,
    (ii) $\forall e_i, e_j \, (e_i \neq e_j)$, where $e_i, e_j \in C$, $i, j = 1, \dots, (n-1)$,
    (iii) $(v_1 = v_n)$,
    (iv) $\forall v_i, v_j \left(\left((v_i \neq v_1) \vee (v_j \neq v_n) \right) \rightarrow (v_i \neq v_j) \right)$, where $v_i, v_j \in C$, $i, j = 1, \dots, (n-1)$.
  - Property:: Spanning circuit of undirected graph $\mathcal{G}$: single circuit of $\mathcal{G}$ covering every vertex of $\mathcal{G}$.

Graph connectedness

- Graph connectedness:
  - Property:: Path, circuit said to pass through vertices or to traverse edges, as per their respective sequence of vertices and edges.
  - Property:: [Re-definition] Connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: graph, with at least one path between every pair of its vertices.
  - Property:: [Re-definition] Disconnected undirected graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$: graph, in which no path present between at least one pair of its vertices.
  - Property:: Null graph of more than one vertex (i.e., $|\mathbf{V}'| > 1$) = Disconnected graph.

Graph connectedness

- Graph connectedness:



Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: A path to be present between every pair of distinct vertices of any connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.

<u>Proof</u>: Let $u, v \in \mathbf{V}$ be two distinct arbitrary vertices in $\mathcal{G}$.

One trail $P = \{u = v_0, v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_{j-1}, v_j, \dots, v_{n-1}, v_n = v\}$ at least present of minimum length between u and v, for connected graph $\mathcal{G}$.

To prove: "trail P to be actually a path."

Assume, "vertex repetition present in trail P," where $v_i = v_j$ ($i < j$, $0 \le i, j \le n$).

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Graph connectedness

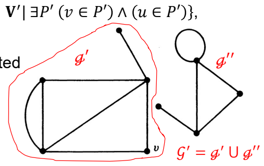
- Graph connectedness:
  Proof (contd.):
  Then, another trail P' possible to be constructed from P by deleting edges corresponding to vertex sequence: $v_i, \dots, v_{j-1}$.
  In P', at least one edge of P to get removed, i.e., $len(P') < len(P)$.
  Then, another trail P' possible between u and v with no vertex repetition and minimum length.
  Thus, always a path possible between arbitrary u, v with no vertex repetition and minimum length. ■

Graph connectedness

- Graph connectedness:
  - Property:: Component of any undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: connected subgraph $\mathscr{g} = (\mathbf{v}, \mathbf{e})$, where $\mathscr{g} \subset \mathcal{G}$, provided $\mathscr{g}$ not contained in any larger connected subgraph of $\mathcal{G}$.
  - Property:: Component of $\mathcal{G}$ = maximal, connected subgraph of $\mathcal{G}$.
  - Property:: Component: also called connected component.
  - Property:: Component of connected undirected graph $\mathcal{G}$ = connected undirected graph $\mathcal{G}$ itself.

Graph connectedness

- Graph connectedness:
  - Property:: Component of disconnected undirected graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$: connected graph $\mathscr{g}' = (\mathbf{v}', \mathbf{e}')$, where $(\mathscr{g}' \subset \mathcal{G}') \land (\mathbf{v}' \subset \mathbf{V}')$, provided that if $v \in \mathbf{v}'$, then $\mathbf{v}'$ to contain $\{u \in \mathbf{V}' \mid \exists P' \, (v \in P') \land (u \in P')\}$, where P' = path in $\mathcal{G}'$.
  - Property:: Partitioning of disconnected graph $\mathcal{G}'$ = set of components of $\mathcal{G}'$.
  - Property:: Number of components of disconnected graph $\mathcal{G}' = \omega(\mathcal{G}')$.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: Graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ to be disconnected if and only if $\mathbf{V}'$ to be partitioned into two nonempty, disjoint subsets $\mathbf{V_1}$ and $\mathbf{V_2}$ such that no edge in $\mathcal{G}'$ possible with one end vertex in subset $\mathbf{V_1}$ and other end vertex in subset $\mathbf{V_2}$. In other words,

    In $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$, with partitioning of $\mathbf{V}'$ into $\mathbf{V_1}$ and $\mathbf{V_2}$, where $(\mathbf{V}' = \mathbf{V_1} \cup \mathbf{V_2}) \wedge (\mathbf{V_1} \cap \mathbf{V_2} = \emptyset)$, such that $(u \in \mathbf{V_1}) \wedge (v \in \mathbf{V_2}) \rightarrow (\{u, v\} \notin \mathbf{E}')$, then $\mathcal{G}'$ said to be disconnected, and conversely.

  Proof: Two cases to prove — (A) Partitioning → Disconnectedness; (B) Disconnectedness → Partitioning.

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Graph connectedness

- Graph connectedness:

  Proof (contd.):

  (A) Partitioning → Disconnectedness. Let partitioning of $\mathbf{V}'$ into $\mathbf{V_1}$ and $\mathbf{V_2}$ possible. For any arbitrary vertices a and b of $\mathcal{G}'$, such that $a \in \mathbf{V_1}$, $b \in \mathbf{V_2}$, considering every path P' in $\mathcal{G}'$, $\neg\big((a \in P') \wedge (b \in P')\big)$.
  Reason: if any P' to contain both a and b, then some edge $e \in P'$ also required for P' to exist, where $e = \{c, c'\}$ and $(c \in \mathbf{V_1}) \wedge (c' \in \mathbf{V_2})$; not possible. So, no path present in $\mathcal{G}'$ containing at least one edge whose one end vertex in $\mathbf{V_1}$ and other end vertex in $\mathbf{V_2}$.
  Hence, partitioning in $\mathcal{G}' \rightarrow \mathcal{G}'$ disconnected.

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Graph connectedness

- Graph connectedness:

  Proof (contd-2.):

  (B) Disconnectedness → Partitioning. Consider any arbitrary vertex a of disconnected $\mathcal{G}'$. Then, $\mathbf{V_1} = \{v \in \mathbf{V}' \mid \exists P' \text{ path in } \mathcal{G}', (a \in P') \wedge (v \in P')\}$, i.e. set of all vertices that are joined by paths to a.
  As $\mathcal{G}'$ disconnected, $\mathbf{V_1} \subset \mathbf{V}'$, i.e. $\mathbf{V_1}$ not including all vertices of $\mathcal{G}'$.
  Then, $\mathbf{V_2} = \mathbf{V}' \setminus \mathbf{V_1}$, i.e. remaining vertices to form $\mathbf{V_2}$, and $\mathbf{V_2} \neq \emptyset$.
  So, for every path P' in $\mathcal{G}', \neg\big((v \in P') \wedge (u \in P')\big)$, where $u \in \mathbf{V_1}$ and $v \in \mathbf{V_2}$. $\therefore \forall e = \{c, c'\}, \big((c \in \mathbf{V_1}) \wedge (c' \in \mathbf{V_2})\big) \rightarrow (e \notin \mathbf{E}')$.
  So, $\mathbf{V}'$ partioned into $\mathbf{V_1}$ and $\mathbf{V_2}$. ■

Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: If a graph (connected or disconnected) to have exactly two vertices of odd degree, mandatorily a path joining these two vertices also to be present.

  Proof: Based on earlier theorem "Number of vertices of odd degree in undirected graph to be always even", which not only applicable to 'connected graph', but also to for every component of disconnected graph and therefore to 'disconnected graph'.
  So, 'connected graph' or 'component of disconnected graph' having two vertices of odd degree must have a path between them. ■

Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: For undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices, $|\mathbf{E}| = m$ edges and k components, following to be satisfied —
    (i) $\mathcal{G}$ connected when $k = 1$;
    (ii) $n \geq k$, i.e., $n - k \geq 0$;
    (iii) $m \geq n - k$, i.e., $m - n + k \geq 0$.

Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: A simple graph with n vertices and k components to have at most $(n-k)\cdot(n-k+1)/2$ edges.

  Proof: Let graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to contain n vertices and k components.
  Designating number of vertices in each of k components as: n_1, n_2, …, n_k, where $n_i \geq 1$ or $n_i - 1 \geq 0$ ($i = 1,2,\ldots,k$), then summation $\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k = n$.
  Consequently, $\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$.
  Squaring both sides, $\left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2\cdot n\cdot k$. Here, L.H.S. to contain nonnegative cross terms of multiple n_is.

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Graph connectedness

- Graph connectedness:
  - Property:: Proof of ⟨Theorem⟩ contd.
    Denoting sum of all nonnegative cross terms of multiple n_is from L.H.S. of last equation as **N**, where $\mathbf{N} \geq 0$, and then simplifying
    $\sum_{i=1}^{k}(n_i{}^2) - \sum_{i=1}^{k}(2 \cdot n_i) + k + \mathbf{N} = n^2 + k^2 - 2 \cdot n \cdot k$.
    So, $\sum_{i=1}^{k}(n_i{}^2) = n^2 + k^2 - 2 \cdot n \cdot k + 2 \cdot n - k - \mathbf{N}$.
    $\leq n^2 + k^2 - 2 \cdot n \cdot k + 2 \cdot n - k = n^2 - (k-1) \cdot (2 \cdot n - k)$. ①
    Given $\mathcal{G}$ to be simple, so each of its k components also simple.
    From earlier theorem, maximum number of edges in i-th component (being simple connected graph) $= \frac{1}{2} \cdot n_i \cdot (n_i - 1)$. ② (contd. to next slide)

Graph connectedness

- Graph connectedness:
  - Property:: Proof of ⟨Theorem⟩ contd.
    Summing Eq. ② for all component of $\mathcal{G}$, maximum number of edges in $\mathcal{G} = \sum_{i=1}^{k}\left(\frac{1}{2} \cdot n_i \cdot (n_i - 1)\right) = \frac{1}{2} \cdot \left(\sum_{i=1}^{k}(n_i{}^2)\right) - \frac{1}{2} \cdot \left(\sum_{i=1}^{k} n_i\right)$
    $= \frac{1}{2} \cdot \left(\sum_{i=1}^{k}(n_i{}^2)\right) - \frac{n}{2} \leq \frac{1}{2} \cdot \left(n^2 - (k-1) \cdot (2 \cdot n - k)\right) - \frac{n}{2}$
    (based on Eq. ①)
    $= \frac{1}{2} \cdot (n^2 + k^2 - 2 \cdot n \cdot k + 2 \cdot n - k - n)$
    $= \frac{1}{2} \cdot (n^2 + k^2 - 2 \cdot n \cdot k + n - k) = \frac{1}{2} \cdot (n-k) \cdot (n-k+1)$. ■

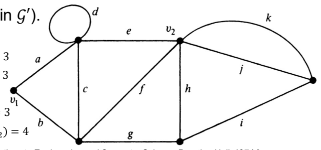
Graph connectedness

- Graph connectedness:
  - Property:: Rank of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices and k components: $rank(\mathcal{G}) = n - k$.
  - Property:: Nullity of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices, $|\mathbf{E}| = m$ edges and k components: $\mu(\mathcal{G}) = m - n + k$.
  - Property:: Nullity: also called cyclomatic number.
  - Property:: ⟨Theorem⟩: For connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices and $|\mathbf{E}| = m$ edges, $rank(\mathcal{G}) = n - 1$, and $\mu(\mathcal{G}) = m - n + 1$.

Graph connectedness

- Graph connectedness:
  - Property:: Distance between any pair of vertices u, v in connected undirected graph $\mathcal{G}$ (or in component $\mathcal{G}'$ of undirected graph $\mathcal{G}$): $d(u,v)$ = length of shortest path (i.e., number of edges in shortest path) between u, v in $\mathcal{G}$ (or in $\mathcal{G}'$).

Path$_1(v_1, v_2) = \{v_1, a, \mathrm{v}, e, v_2\}$; $d_1(v_1, v_2) = 2$
Path$_2(v_1, v_2) = \{v_1, a, \mathrm{v}, c, \mathrm{v}, f, v_2\}$; $d_2(v_1, v_2) = 3$
Path$_3(v_1, v_2) = \{v_1, b, \mathrm{v}, c, \mathrm{v}, e, v_2\}$; $d_3(v_1, v_2) = 3$
Path$_4(v_1, v_2) = \{v_1, b, \mathrm{v}, f, v_2\}$; $d_4(v_1, v_2) = 2$
Path$_5(v_1, v_2) = \{v_1, b, \mathrm{v}, g, \mathrm{v}, h, v_2\}$; $d_5(v_1, v_2) = 3$
Path$_6(v_1, v_2) = \{v_1, b, \mathrm{v}, g, \mathrm{v}, i, \mathrm{v}, k, v_2\}$; $d_6(v_1, v_2) = 4$



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph connectedness

- Graph connectedness:
  - Property:: ⟨Theorem⟩: $d(u,v) \geq 0$, between vertices u, v in graph $\mathcal{G}$.
  - Property:: ⟨Theorem⟩: If $d(u,v) = 0$, then $u = v$, and conversely.
  - Property:: $d(u,v) = d(v,u)$.
  - Property:: ⟨Theorem⟩: $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$, for any intermediate vertex v_k in path between v_i, v_j.

  Proof: $d(v_i, v_j)$ = length of shortest path between v_i, v_j.
  So, not possible that this path to become longer than another path between v_i, v_j, passing through specified intermediate vertex v_k.
  Then, $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$. ■

Graph connectedness

- Graph connectedness:
  - Property:: Eccentricity of vertex v in connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to component $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ of undirected graph $\mathcal{G}$]: distance from v to another vertex, farthest from v in $\mathcal{G}$; i.e., $E(v) = \max_{v_i \in \mathbf{V}} d(v, v_i) = \max\{d(v, v_i) \mid v_i \in \mathbf{V},\ i = 1,2,\cdots,|\mathbf{V}|\}$.
  - Property:: Eccentricity: also called associated number, separation.
  - Property:: Center of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to component $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ of undirected graph $\mathcal{G}$]: vertex with minimum eccentricity in $\mathcal{G}$; i.e., $\mathit{center}(\mathcal{G}) = \{u \in \mathbf{V} \mid E(u) = \min\{E(v) \mid \forall v \in \mathbf{V}\}\} = \left\{u \in \mathbf{V} \mid E(u) = \min_{v \in \mathbf{V}} E(v)\right\}$, where $\mathit{center}(\mathcal{G}) \subseteq \mathbf{V}$.

Graph connectedness

- Graph connectedness:
  - Property:: Radius of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to component $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ of undirected graph $\mathcal{G}$]: distance from $center(\mathcal{G})$ to farthest vertex in $\mathcal{G}$ = eccentricity of vertices in $center(\mathcal{G})$; i.e., $rad(\mathcal{G}) = E(v)$, where $v \in center(\mathcal{G})$.
  - Property:: Diameter of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to component $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ of undirected graph $\mathcal{G}$]: maximum distance between any two vertices in $\mathcal{G}$ = length of longest path in $\mathcal{G}$, i.e., $diam(\mathcal{G}) = \max\{d(v_i, v_j) \mid v_i, v_j \in \mathbf{V},\ i, j = 1,2, \cdots, |\mathbf{V}|\}$ $= \max_{v_i, v_j \in \mathbf{V}} d(v_i, v_j)$.

Summary

- Focus: Graph connectedness.
- Graph connectedness.
- Walk, trail of undirected graph, with examples.
- Closed trail, open trail.
- Path, circuit of undirected graph, with examples.
- Graph connectedness theorems.
- Component of undirected graph, with examples.
- Partitioning of disconnected graph.
- Graph partitioning theorems.

Summary

- Distance between vertices, eccentricity of vertex in connected graph.
- Center, radius, diameter of connected undirected graph.

References

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