

CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 1

Lecture 02: Logic

[Propositional logic truth table; Converse, inverse, contrapositive; Tautology, contradiction; Logical equivalence; Laws of algebra of propositions]

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Propositional logic

- **Propositional logic:** nature.
 - Atomic propositions and compound propositions of different forms.
 - Fundamental property of compound proposition: its truth value completely determined by truth values of its subpropositions, together with connectives used to connect them.
 - Tautology (compound proposition always TRUE), contradiction (compound proposition always FALSE), contingency (neither tautology nor contradiction).
 - Equivalence: when two compound propositions always having same truth values, regardless of truth values of propositional variables.

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Compound proposition: Truth table formation

- Truth table construction of any compound proposition: required for finding truth value of any compound proposition.
- **Truth table:** mathematical tabular representation displaying all possible truth values of any proposition (particularly, compound proposition).
- Two different approaches to construct truth table of compound proposition.

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Compound proposition: Truth table formation

A) Separate columns approach::

- Formation steps:

- ① use of separate column to find truth value of each compound expression occurring in compound proposition; ② first columns for propositional variables, with rows for all possible combinations of their truth values (i.e., for n variables, 2^n rows required); ③ column for each compound expression in "elementary" stages, each truth value determined from previous stages by definitions of connectives as per precedence; ④ truth value of compound proposition (for each combination of truth values of variables) in final column of truth table.

Compound proposition: Truth table formation

A) Separate columns approach::

- Example:: $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Compound proposition: Truth table formation

B) Sub-columns approach::

- Formation steps:

- ① apart from columns for propositional variables, single final column for compound proposition, with multiple sub-columns corresponding to every variable and every connective in that compound proposition;
- ② rows with all possible combinations of truth values of variables (i.e., for n variables, 2^n rows required), plus one more row labeled "step" to record step count; ③ filled-up sub-columns for variables, indicated as step count 1; ④ truth values entered into remaining sub-columns for connectives, with step count, as per **precedence**.

Compound proposition: Truth table formation

B) Sub-columns approach::

- Example:: $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$(p \vee \neg q) \rightarrow (p \wedge q)$							
		(p	v	\neg	q)	\rightarrow	(p	\wedge	q)
T	T	T	T	F	T	T	T	T	T
T	F	T	T	T	F	F	T	F	F
F	T	F	F	F	T	T	F	F	T
F	F	F	T	T	F	F	F	F	F

Step no. ① ④ ② ① ⑤ ① ③ ①

Compound proposition: Fundamental property

- Fundamental property of compound proposition: its truth value completely determined by truth values of its subpropositions, together with connectives used to connect them.
- $P(p, q, \dots)$: denoting proposition constructed from variables p, q, \dots having truth values TRUE (T) or FALSE (F), connected through connectives.
- Truth value of $P(p, q, \dots)$: based on logical operations of truth values of p, q, \dots and their connectives.

Compound proposition: Conditional statement

- Implication.
- Bi-implication.
- Converse of implication.
- Inverse of implication.
- Contrapositive of implication.

Compound proposition: Converse of implication

- Converse of implication $p \rightarrow q$: conditional compound proposition

$q \rightarrow p$ of given propositions p, q .

Truth table	p	q	$p \rightarrow q$	$q \rightarrow p$
	T	T	T	T
	T	F	F	T
	F	T	T	F
	F	F	T	T

if p to be FALSE and q to be TRUE,

then $q \rightarrow p$ to become FALSE;

otherwise $q \rightarrow p$ to become TRUE.

- Note: when p to be TRUE, $q \rightarrow p$ to be TRUE regardless of truth value of q .
- Note: alternately, when q to be FALSE, $q \rightarrow p$ to be TRUE regardless of truth value of p .

Compound proposition: Converse of implication

- Converse of implication:

Converse of implication \rightarrow not equivalent to given implication.

Example::

Given: "The home team wins whenever it is raining."

[" q WHENEVER p "]

= "IF it is raining, THEN the home team wins." ["if p , then q "]

$\rightarrow p$ = "It is raining"; q = "The home team wins."

$\therefore (q \rightarrow p)$ = "IF the home team wins, THEN it is raining."

Compound proposition: Inverse of implication

- Inverse of implication $p \rightarrow q$: conditional compound proposition

$\neg p \rightarrow \neg q$ of given propositions p, q .

Truth table	p	q	$p \rightarrow q$	$\neg p \rightarrow \neg q$
	T	T	T	T
	T	F	F	T
	F	T	T	F
	F	F	T	T

if p to be FALSE and q to be TRUE,

then $\neg p \rightarrow \neg q$ to become FALSE;

otherwise $\neg p \rightarrow \neg q$ to become TRUE.

- Note: same truth table as converse of implication.
- Inverse of implication \rightarrow not equivalent to given implication.
- Inverse of implication \rightarrow logically equivalent converse of implication of same given implication.

Compound proposition: Inverse of implication

- Inverse of implication:

Example::

Given: "The home team wins whenever it is raining."

$[q \text{ WHENEVER } p]$

= "IF it is raining, THEN the home team wins." ["if p , then q "]

$\rightarrow p = \text{"It is raining.}"; q = \text{"The home team wins.}''$

$\therefore (\neg p \rightarrow \neg q) = \text{"IF it is NOT raining, THEN the home team does NOT win.}''$

[Note: "IF it is NOT raining, THEN the home team does NOT win." \equiv "If the home team wins, THEN it is raining."]

Compound proposition: Contrapositive of implication

- Contrapositive of implication $p \rightarrow q$: conditional compound proposition $\neg q \rightarrow \neg p$ of given propositions p, q .

Definition of truth value of $\neg q \rightarrow \neg p$:

Truth table	p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
if p to be TRUE and q to be FALSE;	T	T	T	T
then $\neg q \rightarrow \neg p$ to become FALSE;	T	F	F	F
otherwise $\neg q \rightarrow \neg p$ to become TRUE.	F	T	T	T
Note: same truth table as given implication.	F	F	T	T

- Contrapositive of implication \rightarrow logically equivalent to given implication.

Compound proposition: Contrapositive of implication

- Contrapositive of implication:

Example::

Given: "The home team wins whenever it is raining."

$[q \text{ WHENEVER } p]$

= "IF it is raining, THEN the home team wins." ["if p , then q "]

$\rightarrow p = \text{"It is raining.}"; q = \text{"The home team wins.}''$

$\therefore (\neg q \rightarrow \neg p) = \text{"IF the home team does NOT win, THEN it is NOT raining.}''$

Compound proposition: Tautology

- Tautology:** compound proposition with TRUE as truth value for any truth values of its propositional variables (i.e., compound proposition with only T in final column of its truth table).
- Note: negation of tautology \rightarrow contradiction.
- Example:: any proposition p .
Then, $(p \vee \neg p) =$ Tautology.
- (Theorem) "Principle of Substitution": If $P(p, q, \dots)$ is a tautology, then $P(P_1, P_2, \dots)$ is also a tautology for any propositions $P_1(p, q, \dots), P_2(p, q, \dots), \dots$

Truth table	p	$\neg p$	$p \vee \neg p$
	T	F	T
	F	T	T

Compound proposition: Contradiction

- Contradiction** (also called absurdity): compound proposition with FALSE as truth value for any truth values of its propositional variables (i.e., compound proposition with only F in final column of its truth table).
- Note: negation of contradiction \rightarrow tautology.
- Example:: any proposition p .
Then, $(p \wedge \neg p) =$ Contradiction.

Truth table	p	$\neg p$	$p \wedge \neg p$
	T	F	F
	F	T	F

Compound proposition: Equivalence

- Logical equivalence** (notation: ' \equiv '): compound propositions $P(p, q, \dots)$ and $Q(p, q, \dots)$ to be **logically equivalent** (or **equivalent**) or **equal**, i.e., $P(p, q, \dots) \equiv Q(p, q, \dots)$, if P and Q having same truth values in all possible cases (i.e., identical truth tables).
 - Example-1:: propositions $P(p, q) = \neg(p \wedge q)$, $Q(p, q) = (\neg p \vee \neg q)$.
So, $P \equiv Q$.
 - Other notations for logical equivalence: ' \Leftrightarrow '.
- ☞ Note: symbols ' \equiv ' and ' \Leftrightarrow ': not logical connectives of propositions.

Compound proposition: Equivalence

- Logical equivalence:

Property:: For any propositions $P(p, q, r)$ and $Q(p, q, r)$, $P \equiv Q$, whenever $P \leftrightarrow Q$ to be tautology.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Same truth values of P and Q

Example-1 (contd.):

p	q	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

} Tautology

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Compound proposition: Equivalence

- Logical equivalence:
- Example-2 [distributive law of disjunction over conjunction]:

propositions	p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
$P(p, q, r) =$	T	T	T	T	T	T	T	T
$p \vee (q \wedge r),$	T	T	F	F	T	T	T	T
$Q(p, q, r) =$	T	F	T	F	T	T	T	T
$(p \vee q) \wedge (p \vee r),$	F	T	T	T	T	T	T	T
As all possible truth values of	F	T	F	F	F	T	F	F
P and Q agreeing, so, $P \equiv Q$.	F	F	F	F	F	F	F	F

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Compound proposition: Equivalence

- Logical equivalence:
- Example-3 [conditional-disjunction equivalence]: propositions

$P(p, q) = p \rightarrow q,$	p	q	$p \rightarrow q$	p	q	$(\neg p \vee q)$
$Q(p, q) = (\neg p \vee q).$	T	T	T	T	T	T
So, $P \equiv Q$.	T	F	F	T	F	F
	F	T	T	F	T	T
	F	F	T	F	F	T

Property:: Identity: logically equivalent propositions.

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Compound proposition: Equivalence

- Logical equivalence → **Laws of algebra of propositions.**
- Proof of each law based on proving equivalence either by truth table of given propositions, or by showing their bi-implication as tautology.

Laws of algebra of propositions

Idempotent laws:	(I.1a) $p \vee p \equiv p$	(I.1b) $p \wedge p \equiv p$
Identity laws:	(I.2a) $p \vee F \equiv p$	(I.2b) $p \wedge T \equiv p$
Domination laws:	(I.3a) $p \vee T \equiv T$	(I.3b) $p \wedge F \equiv F$
Associative laws:	(I.4a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(I.4b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	(I.5a) $p \vee q \equiv q \vee p$	(I.5b) $p \wedge q \equiv q \wedge p$
Distributive laws:	(I.6a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(I.6b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Involution law:	(I.7) $\neg\neg p \equiv p$ (also called Double negation law)	
Negation laws:	(I.8a) $p \vee \neg p \equiv T$	(I.8b) $p \wedge \neg p \equiv F$
Complement laws:	(I.9a) $\neg T \equiv F$	(I.9b) $\neg F \equiv T$
De Morgan's laws:	(I.10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(I.10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Laws of algebra of propositions

Absorption laws:	(I.11a) $p \vee (p \wedge q) \equiv p$	(I.11b) $p \wedge (p \vee q) \equiv p$
Implication rules:	(I.12a) $p \rightarrow q \equiv \neg p \vee q$	(I.12b) $\neg(p \rightarrow q) \equiv p \wedge \neg q$
	(I.13a) $p \vee q \equiv \neg p \rightarrow q$	(I.13b) $p \wedge q \equiv \neg(p \rightarrow \neg q)$
	(I.14) $p \rightarrow q \equiv \neg q \rightarrow \neg p$	
	(I.15a) $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$	(I.15b) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
	(I.16a) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	(I.16b) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
Bi-implication rules:	(I.17a) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$	(I.17b) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
	(I.18) $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	
	(I.19) $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	

Laws of algebra of propositions

Exclusive
disjunction rules:

(I.20) $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$	
(I.21a) $p \oplus q \equiv \neg(p \leftrightarrow q)$	(I.21b) $p \leftrightarrow q \equiv \neg(p \oplus q)$
(I.22) $p \oplus q \equiv \neg((p \wedge q) \vee (\neg p \wedge \neg q))$	

Laws of algebra of propositions

- Generalization of Laws of algebra of propositions:
 - Associative** law for disjunction: for propositions $p_1, p_2, p_3, \dots, p_{n-1}, p_n$,
 $\dots (p_1 \vee p_2) \vee p_3 \dots \vee p_n \equiv p_1 \vee (p_2 \vee (p_3 \vee \dots (p_{n-1} \vee p_n) \dots))$
 - Associative** law for conjunction: for propositions $p_1, p_2, p_3, \dots, p_{n-1}, p_n$,
 $\dots (p_1 \wedge p_2) \wedge p_3 \dots \wedge p_n \equiv p_1 \wedge (p_2 \wedge (p_3 \wedge \dots (p_{n-1} \wedge p_n) \dots))$
- In same manner, generalization of **distributive** laws for disjunction and conjunction.
- De Morgan's laws:** for propositions $p_1, p_2, p_3, \dots, p_{n-1}, p_n$,

$$\begin{aligned}\neg(p_1 \vee p_2 \vee \dots \vee p_n) &\equiv (\neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n), \neg(\bigvee_{j=1}^n p_j) \equiv \bigwedge_{j=1}^n \neg p_j \\ \neg(p_1 \wedge p_2 \wedge \dots \wedge p_n) &\equiv (\neg p_1 \vee \neg p_2 \vee \dots \vee \neg p_n), \neg(\bigwedge_{j=1}^n p_j) \equiv \bigvee_{j=1}^n \neg p_j\end{aligned}$$

Laws of algebra of propositions

- Example-1::
- For **negation** of "You have an iPhone **and** an iPad": $p \wedge q$,
 where p : "You have an iPhone", q : "You have an iPad".
 By one of De Morgan's laws in (I.10b), $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
 Here, $\neg p$: "It is false that you have an iPhone" = "You don't have an iPhone".
 Similarly, $\neg q$: "It is false that you have an iPad" = "You don't have an iPad".
 So, $\neg p \vee \neg q$ = "You **don't** have an iPhone **or** an iPad".

Laws of algebra of propositions

- Example-2::

To prove $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ to be logically equivalent —

$$\begin{aligned} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) \text{ by De Morgan's law in (I.10a)} \\ &\equiv \neg p \wedge (\neg(\neg p) \vee \neg q) \text{ by De Morgan's law in (I.10b)} \\ &\equiv \neg p \wedge (p \vee \neg q) \text{ by double negation law in (I.7)} \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \text{ by distributive law in (I.6b)} \\ &\equiv F \vee (\neg p \wedge \neg q) \text{ by negation law in (I.8b)} \\ &\equiv (\neg p \wedge \neg q) \vee F \text{ by commutative law in (I.5a)} \\ &\equiv \neg p \wedge \neg q \text{ by identity law in (I.2a)} \end{aligned}$$

Compound proposition: Operator '|'

- Logical operator '|': (called Sheffer stroke, named after H. M. Sheffer) to represent **NAND** logical operation, with FALSE as truth value for TRUE truth values of its propositional variables.

Definition of truth value of $p | q \equiv p \text{ NAND } q$:

Truth table	p	q	$p q$
if both p and q to be TRUE, then $p q$ to become FALSE; otherwise $p q$ to become TRUE.	T	T	F
	T	F	T
	F	T	T
	F	F	T

Compound proposition: Operator '↓'

- Logical operator '↓': (called Peirce arrow, named after Charles Sanders Peirce) to represent **NOR** logical operation, with TRUE as truth value for FALSE truth values of its propositional variables.

Definition of truth value of $p \downarrow q \equiv p \text{ NOR } q$:

Truth table	p	q	$p \downarrow q$
if both p and q to be FALSE, then $p \downarrow q$ to become TRUE; otherwise $p \downarrow q$ to become FALSE.	T	T	F
	T	F	F
	F	T	F
	F	F	T

Summary

- Focus: Propositional logic (contd.).
 - Propositional logic nature.
 - Truth table construction for compound propositions - separate columns approach, and sub-columns approach, with examples.
 - Fundamental property of compound propositions.
 - Definitions and truth tables of converse, inverse, and contrapositive of implication, with examples.
 - Definitions and truth tables of tautology, contradiction, and logical equivalence, with examples.

Summary

- Principle of substitution.
 - Laws of algebra of propositions, and generalizations of laws, with examples.
 - Logical operators Sheffer stroke and Peirce arrow definitions, truth tables.

References

- [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
 - [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
 - [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
 - [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
 - https://www.cs.odu.edu/~toida/nerzic/content/logic/prop_logic/identities/entities.html.

Further Reading

- Truth tables construction of compound propositions:: [Ros21]:11.
- Conditional statements in compound propositions:: [Ros21]:6-9.
- Converse, inverse, and contrapositive of implication:: [Ros21]:9.
- Tautology and contradiction:: [Ros21]:26-27.
- Logical equivalence:: [Ros21]:27-32.
- Laws of algebra of propositions:: [Ros21]:29-30.

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Lecture Exercises: Problem 1 [Ref: Gate 2019, Q.6, p.2 (Set2)]

Which one of the following is NOT a valid identity?

- ($x \oplus y$) $\oplus z = x \oplus (y \oplus z)$.
- $(x + y) \oplus z = x \oplus (y + z)$.
- $x \oplus y = x + y$, if $xy = 0$.
- $x \oplus y = (xy + x'y')'$.

Note: $x' = \neg x$.

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Lecture Exercises: Problem 1 Ans

- Is $(x \oplus y) \oplus z \equiv x \oplus (y \oplus z)$?
- Is $(x \vee y) \oplus z \equiv x \oplus (y \vee z)$?
- Is $x \oplus y \equiv x \vee y$, if $x \wedge y \equiv F$?
- Is $x \oplus y \equiv \neg((x \wedge y) \vee (\neg x \wedge \neg y))$?

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Lecture Exercises: Problem 2 [Ref: Gate 2018, Q.4, p.2 (Set3)]

Let \oplus and \odot denote the Exclusive OR and Exclusive NOR operations, respectively. Which one of the following is NOT CORRECT?

- (a) $\bar{P} \oplus \bar{Q} = P \odot Q$.
- (b) $\bar{P} \oplus Q = P \odot Q$.
- (c) $\bar{P} \oplus \bar{Q} = P \oplus Q$.
- (d) $(P \oplus \bar{P}) \oplus Q = (P \odot \bar{P}) \odot \bar{Q}$.

Note: $\bar{P} = P' = \neg P$.

Lecture Exercises: Problem 2 Ans

- Is $\bar{P} \oplus Q \equiv P \odot Q$?
- Is $\bar{P} \oplus Q \equiv P \odot Q$?
- Is $\bar{P} \oplus \bar{Q} \equiv P \oplus Q$?
- Is $(P \oplus \bar{P}) \oplus Q \equiv (P \odot \bar{P}) \odot \bar{Q}$?
