CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 3

Lecture 19: Relations

[Binary relation; Homogeneous, heterogeneous relations; Reflexivity, symmetry, antisymmetry, transitivity; Binary relation operations]

Dr. Suddhasil De

Notation table

Symbol / Notation	Meaning
$\mathcal{R}$	Relation.
$\not\mathcal{R}$	No relation.
$\mathcal{R}^{-1}$	Inverse of relation $\mathcal{R}$.
$\bar{\mathcal{R}}$	Complementary of relation $\mathcal{R}$.
$\boldsymbol{\Delta}$	Diagonal relation.
$\mathscr{P}$-relation	Relation fulfilling property $\mathscr{P}$.

Relations

- Binary relation (also commonly referred as *relation*): For sets $\mathbf{A}$ and $\mathbf{B}$, binary relation from $\mathbf{A}$ to $\mathbf{B}$ = subset of $\mathbf{A} \times \mathbf{B}$.
  - Property:: Binary relation from $\mathbf{A}$ to $\mathbf{B}$ = set $\mathcal{R}$ of ordered pairs (i.e., ordered 2-tuples) as elements, i.e., $\mathcal{R} = \{(a, b) \mid (a \in \mathbf{A}) \wedge (b \in \mathbf{B})\}$.
  - Property:: $a\mathcal{R}b \equiv (a, b) \in \mathcal{R} \equiv$ "a is related to b by $\mathcal{R}$" $\equiv$ "a is $\mathcal{R}$-related to b"; Domain of $\mathcal{R} = \mathbf{A}$; Range of $\mathcal{R} = \mathbf{B}$.
    No relation: $a\not\mathcal{R}b \equiv \neg(a\mathcal{R}b) \equiv \neg\big((a, b) \in \mathcal{R}\big) \equiv (a, b) \notin \mathcal{R}$.
  - Applicability:: to model wide variety of concepts in mathematics — (i) "greater than", " equal to", "divides" relations in arithmetic, (ii) "is adjacent to" relation in graph theory etc.

Relations

- Binary relation:
  - Property:: Relation on set $\mathbf{A}$ = *Homogeneous* relation on $\mathbf{A}$ = Binary relation from $\mathbf{A}$ to $\mathbf{A}$ = subset of $\mathbf{A} \times \mathbf{A}$.
  - Property:: Universal relation $= \mathbf{A} \times \mathbf{A}$. Empty relation $= \emptyset$.
  - Property:: ⟨Theorem⟩: "2^{n^2} distinct homogeneous relations possible on a set $\mathbf{A}$ of n elements."

  Proof: As $|\mathbf{A}| = n$, so $|\mathbf{A} \times \mathbf{A}| = n^2$. Again, relation on $\mathbf{A} \subseteq \mathbf{A} \times \mathbf{A}$.
  $\therefore$ Number of relations on $\mathbf{A}$ = numbers of subsets of $\mathbf{A} \times \mathbf{A} = 2^{n^2}$. ■
  - Property:: *Heterogeneous* relation: binary relation from $\mathbf{A}$ to $\mathbf{B}$, with possibly $\mathbf{A}$ and $\mathbf{B}$ containing distinct elements (i.e., $\mathbf{A} \cap \mathbf{B} \neq \emptyset$).

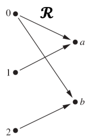
Relations

- Binary relation:
  - Property:: Inverse binary relation of 'relation $\mathcal{R}$ from $\mathbf{A}$ to $\mathbf{B}$': denoted as $\mathcal{R}^{-1}$; $\mathcal{R}^{-1} = \{(b,a) \mid (a,b) \in \mathcal{R}\} = \{(b,a) \mid (a \in \mathbf{A}) \wedge (b \in \mathbf{B})\}$.
  - Property:: Complementary binary relation of 'relation $\mathcal{R}$ from $\mathbf{A}$ to $\mathbf{B}$': denoted as $\overline{\mathcal{R}}$; $\overline{\mathcal{R}} = \{(a,b) \mid (a,b) \notin \mathcal{R}\} = \{(a,b) \mid (a \in \overline{\mathbf{A}}) \vee (b \in \overline{\mathbf{B}})\}$.
  - Property:: Diagonal relation on set $\mathbf{A}$: denoted as Δ; $\Delta = \{(a,a) \mid a \in \mathbf{A}\}$; subset of $\mathbf{A} \times \mathbf{A}$.

Relations

- Binary relation examples:
  - Example-1:: For sets $\mathbf{A} = \{0, 1, 2\}$, $\mathbf{B} = \{a, b\}$, their Cartesian products $\mathbf{A} \times \mathbf{B}$ to form relation $\mathcal{R}$ from $\mathbf{A}$ to $\mathbf{B}$: $\mathbf{A} \times \mathbf{B} = \mathcal{R} = \{(0,a), (0,b), (1,a), (2,b)\}$.
  So, $0\mathcal{R}a$, but $1\mathcal{R}b$.
  - Example-2:: For set $\mathbf{A} = \{1,2,3,4\}$, to find ordered pairs in relation $\mathcal{R} = \{(a,b) \mid a, b \in \mathbf{A}, a$ divides $b\}$.
  So, $\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$.
  Above elements of satisfying "a divides b", and $a, b \in \mathbf{A}$.

$\mathcal{R}$	a	b
0	×	×
1	×	
2		×



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Relations

- Binary relation examples:
  - Example-3:: For following relations (on infinite set), where $a, b \in \mathbb{Z}$:
    $\mathcal{R}_1 = \{(a,b) \mid a \leq b\}$. $\mathcal{R}_2 = \{(a,b) \mid a > b\}$. $\mathcal{R}_3 = \{(a,b) \mid a = b\}$.
    $\mathcal{R}_4 = \{(a,b) \mid a = b$ or $a = -b\}$. $\mathcal{R}_5 = \{(a,b) \mid a = b + 1\}$.
    $\mathcal{R}_6 = \{(a,b) \mid a + b \leq 3\}$,
    to find relations containing each of pairs:
    (1,1), (1,2), (2,1), (1,−1), and (2,2).
    (1,1): member of $\mathcal{R}_1$, $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_6$.
    (1,2): member of $\mathcal{R}_1$, $\mathcal{R}_6$. (2,1): member of $\mathcal{R}_2$, $\mathcal{R}_5$, $\mathcal{R}_6$.
    (1,−1): member of $\mathcal{R}_2$, $\mathcal{R}_4$, $\mathcal{R}_6$. (2,2): member of $\mathcal{R}_1$, $\mathcal{R}_3$, $\mathcal{R}_4$.

Relations

- Binary relation *types*: based on property $\mathscr{P}$ being satisfied or not.
  - Property:: $\mathscr{P}$-relation: any binary relation fulfilling property $\mathscr{P}$.
  - Property:: Typical types of $\mathscr{P}$: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, transitivity etc.

Relations

- Binary relation *types*:
  - Reflexivity property:: Reflexive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $a\mathcal{R}a$ for every element $a \in \mathbf{A}$, i.e., $\mathcal{R} = \{(a,a) \mid \forall a \in \mathbf{A}\}$, i.e.,
    $\forall a{\in}\mathbf{A}\,\big((a,a) \in \mathcal{R}\big) \equiv \forall a\Big((a \in \mathbf{A}) \rightarrow \big((a,a) \in \mathcal{R}\big)\Big)$.
    In words, reflexive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ to relate every element of $\mathbf{A}$ to itself by $\mathcal{R}$.
  - Irreflexivity property:: Irreflexive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $a\not\mathcal{R}a$ for every element $a \in \mathbf{A}$, i.e., $\forall a{\in}\mathbf{A}\,\big((a,a) \notin \mathcal{R}\big)$.
    In words, irreflexive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ not to relate any element of $\mathbf{A}$ to itself by $\mathcal{R}$.

Relations

- Binary relation *types*:
  - Property:: Relation $\mathcal{R}$ on finite set $\mathbf{A}$ neither reflexive nor irreflexive: $\left(\exists a \in \mathbf{A}\left((a, a) \notin \mathcal{R}\right)\right) \wedge \left(\exists b \in \mathbf{A}\left((b, b) \in \mathcal{R}\right)\right)$.
  - Property:: Binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ irreflexive, if and only if binary relation $\left((\mathbf{A} \times \mathbf{A}) \setminus \mathcal{R}\right)$ reflexive.

Relations

- Binary relation *types*:
  - Symmetry property:: Symmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $b\mathcal{R}a$ whenever $a\mathcal{R}b$ for every element $a, b \in \mathbf{A}$., i.e.,
  $$\forall a \in \mathbf{A} \forall b \in \mathbf{A}\left(\left((a, b) \in \mathcal{R}\right) \rightarrow \left((b, a) \in \mathcal{R}\right)\right).$$
  In words, symmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ to relate element b of $\mathbf{A}$ to element a of $\mathbf{A}$ by $\mathcal{R}$, whenever a related to b by $\mathcal{R}$.
  - Property:: $\mathcal{R}$ on set $\mathbf{A}$ not symmetric: $\exists a \in \mathbf{A} \exists b \in \mathbf{A}\left(\left((a, b) \in \mathcal{R}\right) \wedge \left((b, a) \notin \mathcal{R}\right)\right)$.

Relations

- Binary relation *types*:
  - Asymmetry property:: Asymmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $b\mathcal{R}a$ whenever $a\mathcal{R}b$ for every element $a, b \in \mathbf{A}$, i.e.,
  $$\forall a \in \mathbf{A} \forall b \in \mathbf{A}\left(\left((a, b) \in \mathcal{R}\right) \rightarrow \left((b, a) \notin \mathcal{R}\right)\right)$$
  $$\equiv \forall a \in \mathbf{A} \forall b \in \mathbf{A} \neg\left(\left((a, b) \in \mathcal{R}\right) \wedge \left((b, a) \in \mathcal{R}\right)\right).$$
  In words, asymmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ to not relate every element b of $\mathbf{A}$ to every other element a of $\mathbf{A}$ by $\mathcal{R}$, whenever a related to b by $\mathcal{R}$.

Relations

- Binary relation *types*:
  - Property:: $\mathcal{R}$ on set $\mathbf{A}$ neither symmetric nor asymmetric:

$$\left(\exists a{\in}\mathbf{A}\exists b{\in}\mathbf{A}\left(\left((a,b)\in\mathcal{R}\right)\wedge\left((b,a)\notin\mathcal{R}\right)\right)\right)\wedge$$
$$\left(\exists c{\in}\mathbf{A}\exists d{\in}\mathbf{A}\left(\left((c,d)\in\mathcal{R}\right)\wedge\left((d,c)\in\mathcal{R}\right)\right)\right).$$

Relations

- Binary relation *types*:
  - Antisymmetry property:: Antisymmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $a = b$ whenever $a\mathcal{R}b$ and $b\mathcal{R}a$ for every element $a, b \in \mathbf{A}$, i.e.,
    $$\forall a{\in}\mathbf{A}\forall b{\in}\mathbf{A}\left(\left(\left((a,b)\in\mathcal{R}\right)\wedge\left((b,a)\in\mathcal{R}\right)\right)\rightarrow(a=b)\right).$$
    In words, antisymmetric binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ to not relate any pairs of distinct elements of $\mathbf{A}$ to each other by $\mathcal{R}$.
  - Property:: $\mathcal{R}$ on set $\mathbf{A}$ not antisymmetric:
    $$\exists a{\in}\mathbf{A}\exists b{\in}\mathbf{A}\left((a\neq b)\wedge\left((a,b)\in\mathcal{R}\right)\wedge\left((b,a)\in\mathcal{R}\right)\right).$$

Relations

- Binary relation *types*:
  - Property:: Possibility of relation $\mathcal{R}$ to hold both antisymmetric and symmetric properties. E.g., (i) $\mathcal{R} = \emptyset$, $\mathbf{A} = \{a\}$ (vacuously satisfied); (ii) $\mathcal{R} = \{(1,1),(2,2)\}$, $\mathbf{A} = \mathbb{Z}$.
  - Property:: Possibility of relation $\mathcal{R}$ to not satisfy combinedly symmetric, asymmetric and antisymmetric properties (i.e., neither symmetric nor asymmetric nor antisymmetric).
    E.g., $\mathcal{R} = \{(a,b),(b,a),(a,c)\}$, $\mathbf{A} = \{a,b,c\}$ (where, $a \neq b \neq c$); reasons: $(c,a) \notin \mathcal{R}$, $(b,a) \in \mathcal{R}$, and $a \neq b$.

Relations

- Binary relation *types*:
  - Transitivity property:: Transitive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $a\mathcal{R}c$ whenever $a\mathcal{R}b$ and $b\mathcal{R}c$ for every element $a, b, c \in \mathbf{A}$, i.e.,
    $\forall a \in \mathbf{A} \forall b \in \mathbf{A} \forall c \in \mathbf{A} \left(\left(\left((a,b) \in \mathcal{R}\right) \wedge \left((b,c) \in \mathcal{R}\right)\right) \rightarrow \left((a,c) \in \mathcal{R}\right)\right)$.
    In words, transitive binary relation $\mathcal{R}$ on finite set $\mathbf{A}$ to relate element a of $\mathbf{A}$ to element c of $\mathbf{A}$ by $\mathcal{R}$, whenever a related to element b of $\mathbf{A}$ by $\mathcal{R}$ and b related to c by $\mathcal{R}$.
  - Property:: $\mathcal{R}$ on set $\mathbf{A}$ not transitive:
    $\exists a \in \mathbf{A} \exists b \in \mathbf{A} \exists c \in \mathbf{A} \left(\left((a,b) \in \mathcal{R}\right) \wedge \left((b,c) \in \mathcal{R}\right) \wedge \left((a,c) \notin \mathcal{R}\right)\right)$.

Relations

- Binary relation *types*:
  - Property:: ⟨Theorem⟩: "$2^{n \cdot (n-1)}$ reflexive relations possible on a set of n elements."

  Proof: [Using basic counting principle] Constituents of job: 2 tasks.
  Let $\mathbf{A} =$ arbitrary set of n elements, $\mathcal{R} =$ some arbitrary relation.
  Then, $\mathcal{R}$ on $\mathbf{A} =$ subset of $\mathbf{A} \times \mathbf{A}$, i.e., $\mathcal{R}$ to specify whether each of n^2 ordered pairs in $\mathbf{A} \times \mathbf{A}$ to belong to $\mathcal{R}$.
  Task-1: $\mathcal{R}$ reflexive. Then, $\mathcal{R}$ to contain all n ordered pairs (a, a) for $a \in \mathbf{A}$. So, 1 subset of $\mathbf{A} \times \mathbf{A}$ holding all (a, a) ordered pairs of reflexive relation formed based on ordered pairing of same element.

(contd. to next slide)

Relations

- Binary relation *types*:

  Proof contd.:
  For any $a, b \in \mathbf{A}$, considering ordered pairs of form (a, b), where $a \neq b$, as per product rule, number of such ordered pairs $= n \cdot (n-1)$.
  However, each of $n \cdot (n-1)$ ordered pairs may or may not be in $\mathcal{R}$.
  Task-2: selecting $n \cdot (n-1)$ ordered pairs.
  Constituent: $n \cdot (n-1)$ sub-tasks.
  Sub-task T_i $(i = 1, 2, \ldots, n \cdot (n-1))$: choice of adding or not i-th ordered pair of distinct elements in $\mathcal{R}$.
  Single procedure to perform sub-task T_i.

(contd. to next slide)

Relations

- Binary relation *types*:

  Proof contd-2.:

  Sub-task T_i performing regardless of how $T_1, T_2, \ldots, T_{i-1}$ performed.
  Then, according to product rule, number of subsets of $\mathbf{A} \times \mathbf{A}$ for all (a, b) ordered pairs in reflexive relation formed based on ordered pairing of distinct elements $= \underbrace{2 \cdot 2 \cdot 2 \cdot \cdots \cdot 2}_{n\cdot(n-1)\text{factors}} = 2^{n\cdot(n-1)}$.
  So, task-2 providing $2^{n\cdot(n-1)}$ relations $\mathcal{R}$.
  Task-2 performing regardless of how task-1 performed.
  So, again applying product rule, number of $\mathcal{R} = 1 \cdot 2^{n\cdot(n-1)}$. ■

Relations

- Binary relation *types* examples:
  - Example-1:: Given relations on $\{1,2,3,4\}$, to find reflexive relations:

    $\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$.
    $\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\}$.
    $\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$.
    $\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$.
    $\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$.
    $\mathcal{R}_3$ → reflexive: as all $a\mathcal{R}_3 a$ present, i.e. $\forall a\big((a,a) \in \mathcal{R}_3\big)$; here, $(1,1),(2,2),(3,3),(4,4) \in \mathcal{R}_3$.
    $\mathcal{R}_5$ → reflexive: as $(1,1),(2,2),(3,3),(4,4) \in \mathcal{R}_5$.

(contd. to next slide)

Relations

- Binary relation *types* examples:
  - Example-1 contd.

    For other given relations: $(3,3) \notin \mathcal{R}_1$; $(2,2),(3,3),(4,4) \notin \mathcal{R}_2$; $(1,1),(2,2),(3,3),(4,4) \notin \mathcal{R}_4$.
    So, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_4$ → not reflexive.
  - Example-2:: To find reflexive nature of "divides" ('|') relation on $\mathbb{Z}^+$.

    Considering $\mathcal{R}$ as '|', $\mathcal{R} = \{(a,b) \mid a, b \in \mathbb{Z}^+, a|b\}$, where $a|b =$ a divides b.
    As "a divides a", so $(a,a) \in \mathcal{R}$, i.e., $a\mathcal{R}a$.
    So, '|' → reflexive.

Relations

- Binary relation *types* examples:
  - Example-3:: Given relations on $\{1,2,3,4\}$, to find symmetric relations:
    $\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$.
    $\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\}$.
    $\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$.
    $\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$.
    $\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$.
    $\mathcal{R}_2$ → symmetric: as for each such case whenever $a\mathcal{R}_2 b$ present, then $b\mathcal{R}_2 a$ also present; here, $(1,2),(2,1) \in \mathcal{R}_2$, and no other case in $\mathcal{R}_2$.

(contd. to next slide)

Relations

- Binary relation *types* examples:
  - Example-3 contd.
    $\mathcal{R}_3$ → symmetric: as, $(1,2),(2,1),(1,4),(4,1) \in \mathcal{R}_3$.
    For other given relations: $(4,1) \in \mathcal{R}_1$, but $(1,4) \notin \mathcal{R}_1$;
    $(2,1),(3,1),(3,2),(4,1),(4,2),(4,3) \in \mathcal{R}_4$, but
    $(1,2),(1,3),(2,3),(1,4),(2,4),(3,4) \notin \mathcal{R}_4$;
    $(1,2),(1,3),(1,4),(2,3),(2,4),(3,4) \in \mathcal{R}_5$, but
    $(2,1),(3,1),(4,1),(3,2),(4,2),(4,3) \notin \mathcal{R}_5$.
    So, $\mathcal{R}_1$, $\mathcal{R}_4$, $\mathcal{R}_5$ → not symmetric.

Relations

- Binary relation *types* examples:
  - Example-4:: To find symmetric nature of "equality" ('=') relation on $\mathbb{Z}$.
    Considering $\mathcal{R}$ as '=', $\mathcal{R} = \{(a,b) \mid a,b \in \mathbb{Z}, a \text{ equals } b\}$.
    As "a equals b" → "b equals a", so $\left(\left((a,b) \in \mathcal{R}\right) \rightarrow \left((b,a) \in \mathcal{R}\right)\right)$,
    i.e., $b\mathcal{R}a$ whenever $a\mathcal{R}b$.
    So, '=' → symmetric.

Relations

- Binary relation *types* examples:
  - Example-5:: Given relations on $\{1,2,3,4\}$, to find antisymmetric relations:

    $\mathcal{R}_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$.
    $\mathcal{R}_2 = \{(1,1), (1,2), (2,1)\}$.
    $\mathcal{R}_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$.
    $\mathcal{R}_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$.
    $\mathcal{R}_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$.
    For determining relations satisfying antisymmetric property, to start with relations not holding symmetric property.

(contd. to next slide)

Relations

- Binary relation *types* examples:
  - Example-5 contd.

    $\mathcal{R}_4$ → antisymmetric: as no such case possible, whenever $a\mathcal{R}_4b$ and $b\mathcal{R}_4a$ present, then $a = b$; here,
    $(2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \in \mathcal{R}_4$, but
    $(1,2), (1,3), (2,3), (1,4), (2,4), (3,4) \notin \mathcal{R}_4$, and so no equality scenario.
    $\mathcal{R}_5$ → antisymmetric: as, similar to $\mathcal{R}_4$, no such case possible; here,
    $(1,2), (1,3), (1,4), (2,3), (2,4), (3,4) \in \mathcal{R}_5$, but
    $(2,1), (3,1), (4,1), (3,2), (4,2), (4,3) \notin \mathcal{R}_5$, and so no equality comparison scenario.

(contd. to next slide)

Relations

- Binary relation *types* examples:
  - Example-5 contd-2.

    For other given relations: $(1,2) \in \mathcal{R}_1$ and $(2,1) \in \mathcal{R}_1$, but $1 \neq 2$;
    Same reasoning of $\mathcal{R}_1$ also for $\mathcal{R}_2$; and $\mathcal{R}_3$.
    So, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ → not antisymmetric.

Relations

- Binary relation *types* examples:
  - Example-6:: To find antisymmetric nature of "less than or equal to" ('≤') relation on $\mathbb{Z}$.
    Considering $\mathcal{R}$ as '≤', $\mathcal{R} = \{(a,b) \mid a,b \in \mathbb{Z}, a$ less or equal to $b\}$.
    As "a less or equal to b" and "b less or equal to a" → "a equals b", so
    $\left(\left(\left((a,b) \in \mathcal{R}\right) \wedge \left((b,a) \in \mathcal{R}\right)\right) \rightarrow (a = b)\right)$,
    i.e., when $a\mathcal{R}b$ and $b\mathcal{R}a$, then $a = b$.
    So, '≤' → antisymmetric.

Relations

- Binary relation *types* examples:
  - Example-7:: Given relations on $\{1,2,3,4\}$, to find transitive relations:
    $\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$.
    $\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\}$.
    $\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$.
    $\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$.
    $\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$.
    Here, $\mathcal{R}_4$ → transitive: as for each such case, whenever $a\mathcal{R}_4b$ and $b\mathcal{R}_4c$ present, then $a\mathcal{R}_4c$ present; here, $(3,2),(2,1),(3,1) \in \mathcal{R}_4$, $(4,2),(2,1),(4,1) \in \mathcal{R}_4$, $(4,3),(3,2),(4,2) \in \mathcal{R}_4$, $(4,3),(3,1),(4,1) \in \mathcal{R}_4$.

(contd. to next slide)

Relations

- Binary relation *types* examples:
  - Example-7 contd.
    $\mathcal{R}_5$ → transitive: as $(1,2),(2,3),(1,3) \in \mathcal{R}_5$, $(1,2),(2,4),(1,4) \in \mathcal{R}_5$, $(1,3),(3,4),(1,4) \in \mathcal{R}_5$, $(2,3),(3,4),(2,4) \in \mathcal{R}_5$, to satisfy all possible transitive conditions in $\mathcal{R}_5$.
    For other given relations: $(3,4) \in \mathcal{R}_1$ and $(4,1) \in \mathcal{R}_1$, but $(3,1) \notin \mathcal{R}_1$,
    $(4,1) \in \mathcal{R}_1$ and $(1,2) \in \mathcal{R}_1$, but $(4,2) \notin \mathcal{R}_1$;
    $(2,1) \in \mathcal{R}_2$ and $(1,2) \in \mathcal{R}_2$, but $(2,2) \notin \mathcal{R}_2$;
    $(4,1) \in \mathcal{R}_3$ and $(1,2) \in \mathcal{R}_3$, but $(4,2) \notin \mathcal{R}_3$.
    So, $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ → not transitive.

Relations

- Binary relation *types* examples:
  - Example-8:: To find transitive nature of "divides" ('|') relation on $\mathbb{Z}^+$. Considering $\mathcal{R}$ as '|', $\mathcal{R} = \{(a,b) \mid a,b \in \mathbb{Z}^+, a|b\}$, where $a|b =$ a divides b.
    For any $a,b,c \in \mathbb{Z}^+$, let "a divides b". Then, $b = k \cdot a$, for some $k \in \mathbb{Z}^+$.
    Again let "b divides c". Then, $c = l \cdot b$, for some $l \in \mathbb{Z}^+$.
    $\therefore c = l \cdot k \cdot a = (l \cdot k) \cdot a$ where $(l \cdot k) \in \mathbb{Z}^+$, i.e., "$a$ divides c". So, $\Big(\big(((a,b) \in \mathcal{R}) \wedge ((b,c) \in \mathcal{R})\big) \rightarrow ((a,c) \in \mathcal{R})\Big)$,
    i.e., when $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$.
    So, '|' → transitive.

Relations

- Binary relation *operations*: similar to operations on sets.
  - Union property:: Union of two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on finite sets $\mathbf{A}$ and $\mathbf{B}$: $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(a,b) \mid ((a,b) \in \mathcal{R}_1) \vee ((a,b) \in \mathcal{R}_2)\}$.
  - Intersection property:: Intersection of two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on finite sets $\mathbf{A}$ and $\mathbf{B}$: $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(a,b) \mid ((a,b) \in \mathcal{R}_1) \wedge ((a,b) \in \mathcal{R}_2)\}$.
  - Subtraction property:: Subtraction of two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on finite sets $\mathbf{A}$ and $\mathbf{B}$: $\mathcal{R}_1 \setminus \mathcal{R}_2 = \{(a,b) \mid ((a,b) \in \mathcal{R}_1) \wedge ((a,b) \notin \mathcal{R}_2)\}$.

Relations

- Binary relation *operations*:
  - Property:: Symmetric difference of two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on finite sets $\mathbf{A}$ and $\mathbf{B}$: $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}_1 \ominus \mathcal{R}_2 = \mathcal{R}_1 \Delta \mathcal{R}_2 = \{(a,b) \mid ((a,b) \in \mathcal{R}_1) \oplus ((a,b) \in \mathcal{R}_2)\}$.
  - Composition property:: Composition (or composite) of binary relations $\mathcal{R}$ (from finite sets $\mathbf{A}$ to $\mathbf{B}$) and $\mathcal{S}$ (from finite sets $\mathbf{B}$ to $\mathbf{C}$): $\mathcal{S} \circ \mathcal{R} = \{(a,c) \mid a \in \mathbf{A}, c \in \mathbf{C}, \exists b \in \mathbf{B}((a,b) \in \mathcal{R}) \wedge ((b,c) \in \mathcal{S})\}$.
  - Power property:: Powers of binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $\mathcal{R}^{n+1} = \mathcal{R}^n \circ \mathcal{R}$, provided $\mathcal{R}^1 = \mathcal{R}$, where, $n \in \mathbb{N} \setminus \{0\}$.
  - Property:: $\mathcal{R}^2 = \mathcal{R} \circ \mathcal{R}$.

Relations

- Binary relation *operations*:
  - Property:: ⟨Theorem⟩: Binary relation $\boldsymbol{R}$ on finite set $\mathbf{A}$ to become transitive if and only if $\mathcal{R}^n \subseteq \boldsymbol{R}$ for $n \in \mathbb{N} \backslash \{0\}$.

Summary

- Focus: Relations.
- Binary relation, and properties.
- Homogeneous and heterogeneous relations, and properties.
- Binary relation types, and $\mathscr{P}$-relation.
- Reflexive and irreflexive binary relations, with examples.
- Symmetric and asymmetric binary relations, with examples.
- Antisymmetric binary relation, with examples.
- Transitive binary relation, with examples.
- Union of two binary relations.
- Intersection of two binary relations.

Summary

- Subtraction of two binary relations.
- Symmetric difference of two binary relations.
- Composition of two binary relations.
- Powers of binary relation.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Mot08] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, PHI, Second edition, 2008.
3. [Lip07] Seymour Lipschutz and Marc Lars Lipson, Schaum's Outline of Theory and Problems of Discrete Mathematics, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Binary relations:: [Ros19]:599-601.
- Homogeneous relations:: [Ros19]:601-602.
- Binary relation types:: [Ros19]:602-605.
- Reflexive binary relations:: [Ros19]:602-603,605.
- Symmetric binary relations:: [Ros19]:603-604.
- Antisymmetric binary relations:: [Ros19]:603-604.
- Transitive binary relations:: [Ros19]:604-605.
- Union, intersection of binary relations:: [Ros19]:606.
- Composite of binary relations:: [Ros19]:606-608.