CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 16: Counting

[Binomial coefficient, binomial expression; Binomial theorem; Binomial identities; Pascal's identity theorem, Pascal's triangle]

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Binomial coefficients

- Binomial coefficient: coefficient of any term after expansion of powers of binomial expressions (or simply, binomials).
  - Property:: Binomial: polynomial expressed as sum of two monomials.
  - Property:: Monomial: polynomial expressed by only one term with nonzero coefficient.
  - Property:: Coefficient: generally, multiplicative factor involved in any of all terms of polynomial, or series, or expression.
  - Property:: Types of coefficient (in general case): (i) number (dimensionless), also called numerical factor; (ii) constant (with units of measurement), also called constant multiplier.

Binomial coefficients

- Binomial expression's expansion:
  - Property:: Binomial expression: of general form $(a+b)^n$.
  - Property:: Expansion of binomial: using combinatorial reasoning.
  - Example-1:: $(a+b)^3 = (a+b)\cdot(a+b)\cdot(a+b)$ (1)

    $= (a\cdot a + a\cdot b + b\cdot a + b\cdot b)\cdot(a+b)$

    $= a{\cdot}a{\cdot}a + a{\cdot}a{\cdot}b + a{\cdot}b{\cdot}a + a{\cdot}b{\cdot}b + b{\cdot}a{\cdot}a + b{\cdot}a{\cdot}b + b{\cdot}b{\cdot}a + b{\cdot}b{\cdot}b$.

    To obtain any (blue) term, say $a{\cdot}a{\cdot}a$, a MUST be chosen from every multiplier $(a+b)$, i.e., each choice of which possible independent of previous choice and each choice possible in only 1 way, so as per product rule, no. of ways = $1{\cdot}1{\cdot}1$ = 1 = coefficient value.

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Binomial coefficients

- Binomial expression's expansion:
  - Example-1 (contd.):: $(a+b)^3 = (a+b)\cdot(a+b)\cdot(a+b)$ (1)

    $= a{\cdot}a{\cdot}a + a{\cdot}a{\cdot}b + a{\cdot}b{\cdot}a + a{\cdot}b{\cdot}b + b{\cdot}a{\cdot}a + b{\cdot}a{\cdot}b + b{\cdot}b{\cdot}a + b{\cdot}b{\cdot}b$.

    For $a^2{\cdot}b$, three (red) terms in expansion, where each term to start either by a or by b only; their ways also independent. Reason:

    If a chosen 1st, then after choosing a, for next choice, another a or b to be independently chosen from remaining 2 multipliers in 2 ways, and then last choice also independent. So, by product rule, no. of addends=1·2·1=2.

    If b chosen 1st, by product rule, no. of addends=1·1·1=1.

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Binomial coefficients

- Binomial expression's expansion:
  - Example-1 (contd-2.):: $(a+b)^3 = (a+b)\cdot(a+b)\cdot(a+b)$ (1)

    $= a{\cdot}a{\cdot}a + a{\cdot}a{\cdot}b + a{\cdot}b{\cdot}a + a{\cdot}b{\cdot}b + b{\cdot}a{\cdot}a + b{\cdot}a{\cdot}b + b{\cdot}b{\cdot}a + b{\cdot}b{\cdot}b$.

    Then for $a^2{\cdot}b$, as per sum rule, no. of addends = no. of ways = 2+1 = 3 = coefficient value.

    Alternative explanation:: number of ways to select 1st a from three $(a+b)$ multipliers, leaving rest multipliers for choice of 2nd a and one b.

    Same explanation for $b^2{\cdot}a$.

Binomial coefficients

- Binomial expression's expansion:
  - Example-2:: $(a+b)^4 = (a+b)\cdot(a+b)\cdot(a+b)\cdot(a+b)$ (2)

    $= (a{\cdot}a{\cdot}a{\cdot}a + b{\cdot}b{\cdot}b{\cdot}b) + (a{\cdot}a{\cdot}a{\cdot}b + a{\cdot}a{\cdot}b{\cdot}a + a{\cdot}b{\cdot}a{\cdot}a + b{\cdot}a{\cdot}a{\cdot}a) +$
    $(b{\cdot}b{\cdot}b{\cdot}a + b{\cdot}b{\cdot}a{\cdot}b + b{\cdot}a{\cdot}b{\cdot}b + a{\cdot}b{\cdot}b{\cdot}b) + (a_1{\cdot}a_2{\cdot}b_3{\cdot}b_4[+a_1{\cdot}a_2{\cdot}b_4{\cdot}b_3] +$
    $a_1{\cdot}a_3{\cdot}b_2{\cdot}b_4[+a_1{\cdot}a_3{\cdot}b_4{\cdot}b_2] + a_1{\cdot}a_4{\cdot}b_2{\cdot}b_3[+a_1{\cdot}a_4{\cdot}b_3{\cdot}b_2] + \cdots)$.

    For brown terms of form $a^2{\cdot}b^2$, to select 1st a from four $(a+b)$ multipliers in 4 ways; for each of these 4 ways, to select 2nd a from remaining $(a+b)$ multipliers in 3 ways independently.

    So, by product rule, no. of ways = $4\cdot 3$, where 2 ways be equivalent.

    So, by division rule, distinct no. of summands = $\frac{4}{2}\cdot\frac{3}{1}$ = 6 = coefficient.

Binomial theorem

- Binomial theorem: expanding nonnegative power of generalized binomial expression as sum of terms involving binomial coefficients.
  - Property:: ⟨Binomial theorem⟩: For variables (& constants) x and y,
    $(x+y)^n = \binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}\cdot y + \cdots + \binom{n}{n-1}\cdot x\cdot y^{n-1} + \binom{n}{n}\cdot y^n$
    $= \sum_{r=0}^{n}\binom{n}{r}\cdot x^{n-r}\cdot y^r = \sum_{r=0}^{n}\binom{n}{r}\cdot x^r\cdot y^{n-r}$
    where, $n\in\mathbb{N}$, and "n-choose-r" denoted by $\binom{n}{r}$ = number of ways of selecting r elements from set of n elements, $r\in\mathbb{N}$, $r\le n$.

Binomial theorem

- Binomial theorem:
  Proof of ⟨Binomial theorem⟩: [Using counting principles].
  Expanded product to contain summands of form: $x^{n-r}\cdot y^r$, for $r = 0,1,2,3,\ldots,n$. Each summand = one/more addends summed together.
  Number of addends resulting in summand of form $x^{n-r}\cdot y^r$ (for some r) = distinct ways to select x for $(n-r)$ times from n no. of $(x+y)$ multipliers, leaving rest r no. of $(x+y)$ for choice of r times of y.
  ∴ Coefficient of $x^{n-r}\cdot y^r$ (as per product rule & division rule): $\frac{n}{r}\cdot\frac{(n-1)}{(r-1)}\cdot \ldots \cdot \frac{(n-(r-1))}{1} = \frac{n\cdot(n-1)\cdot\cdots\cdot(n-(r-1))}{r\cdot(r-1)\cdot\cdots\cdot 1} = \frac{n!}{r!\cdot(n-r)!}$ denoted by $\binom{n}{r}$.
  Also, same coefficient $\binom{n}{r}$ for $x^r\cdot y^{n-r}$. ■

Binomial theorem

- Binomial theorem examples:
  - Example-1:: Expansion of $(x+y)^4$. From binomial theorem,
    $(x+y)^4 = \sum_{j=0}^{4}\binom{4}{j}\cdot x^{4-r}\cdot y^r = \sum_{j=0}^{4}\binom{4}{r}\cdot x^r\cdot y^{4-r}$
    $= \binom{4}{0}\cdot x^4 + \binom{4}{1}\cdot x^3\cdot y + \binom{4}{2}\cdot x^2\cdot y^2 + \binom{4}{3}\cdot x\cdot y^3 + \binom{4}{4}\cdot y^4$
    $= 1\cdot x^4 + 4\cdot x^3\cdot y + 6\cdot x^2\cdot y^2 + 4\cdot x\cdot y^3 + 1\cdot y^4$.
  - Example-2:: Coefficient of $x^{12}\cdot y^{13}$ in $(x+y)^{25}$. From binomial theorem, given term's coefficient (from 1st expression of summands)
    $=\binom{25}{13} = \frac{25!}{13!\cdot 12!} = 5200300$.
    From 2nd expression of summands in binomial theorem, given term's coefficient $=\binom{25}{12} = \frac{25!}{12!\cdot 13!} = 5200300$.

Binomial theorem

- Binomial theorem examples:
  - Example-3:: Coefficient of $x^{12} \cdot y^{13}$ in expansion of $(2 \cdot x - 3 \cdot y)^{25}$.
    Let $u = 2 \cdot x$, $v = -3 \cdot y$; so given binomial to become $(u + v)^{25}$, and required term to become $2^{12} \cdot u^{12} \cdot (-1)^{13} \cdot 3^{13} \cdot v^{13}$.
    From 1st expression of summands in binomial theorem, given term's coefficient $= \binom{25}{13} \cdot 2^{12} \cdot (-1)^{13} \cdot 3^{13} = -\frac{25!}{13! \cdot 12!} \cdot 2^{12} \cdot 3^{13}$.
    From 2nd expression of summands in binomial theorem, given term's coefficient $= \binom{25}{12} \cdot 2^{12} \cdot (-1)^{13} \cdot 3^{13} = -\frac{25!}{13! \cdot 12!} \cdot 2^{12} \cdot 3^{13}$.

Binomial identities

- Binomial identity: formula expressing products of binomial factors as sum over terms, each including a binomial coefficient.
  - Property:: ⟨Corollary⟩: $\sum_{r=0}^{n} \binom{n}{r} = 2^n$, where $n \in \mathbb{N}$.
  Proof of above corollary to obtain straight from binomial theorem on assigning $x = y = 1$, and simplifying.
  - Property:: ⟨Corollary⟩: $\sum_{r=0}^{n} \left((-1)^r \cdot \binom{n}{r}\right) = 0$, where $n \in \mathbb{Z}^+$.
  - Property:: ⟨Corollary⟩: $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$, where $n \in \mathbb{Z}^+$.
  - Property:: ⟨Corollary⟩: $\sum_{r=0}^{n} \left(2^r \cdot \binom{n}{r}\right) = 3^n$, where $n \in \mathbb{N}$.

Binomial identities

- Binomial identity:
  - Property:: ⟨Pascal's identity theorem⟩: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$, where $n, r \in \mathbb{Z}^+$, $r \leq n$.
  Proof: Using counting principles. Let $\mathbf{T}$ be any set, where $|\mathbf{T}| = n + 1$. Let any arbitrary element $a \in \mathbf{T}$, and let $\mathbf{S} = \mathbf{T} \setminus \{a\}$, where $|\mathbf{S}| = n$. Then, number of subsets of $\mathbf{T}$ containing r elements $= \binom{n+1}{r}$, $r \leq n$. Such subset formation of $\mathbf{T}$ containing r elements possible in two procedures — either (i) each subset to contain a, together with $r - 1$ elements taken from n elements of $\mathbf{S}$, or (ii) each subset to contain r elements taken from n elements of $\mathbf{S}$, and not containing a.

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Binomial identities

- Binomial identity:
  - Proof of ⟨Pascal's identity theorem⟩ contd.

    In first case, number of ways to form subsets of $r-1$ elements from $\mathbf{S} = \binom{n}{r-1}$. ∴ Number of subsets of $\mathbf{T}$ (of r elements) with $a = \binom{n}{r-1}$.

    In second case, number of ways to form subsets of r elements from $\mathbf{S} = \binom{n}{r}$. ∴ Number of subsets of $\mathbf{T}$ (of r elements) without $a = \binom{n}{r}$.

    Then, according to sum rule,

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}. \qquad \blacksquare$$

Binomial identities

- Binomial identity:
  - ⟨Pascal's identity theorem⟩ Alternate proof: based on algebraic manipulations. To derive L.H.S., starting with R.H.S. of given equation. Use of expression of binomial coefficients.

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)! \cdot (n-r+1)!} + \frac{n!}{r! \cdot (n-r)!} = \frac{n! \cdot r}{r! \cdot (n-r+1)!} + \frac{n! \cdot (n-r+1)}{r! \cdot (n-r+1)!} =$$

$$= \frac{n! \cdot (r+n-r+1)}{r! \cdot (n-r+1)!} = \frac{(n+1)!}{r! \cdot (n+1-r)!} = \binom{n+1}{r}. \qquad \blacksquare$$

Binomial identities

- Binomial identity:
  - Property:: ⟨Theorem⟩: $\binom{n}{r} = \binom{n}{n-r}$, where $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, $r \leq n$.

  Proof: Based on algebraic manipulations. To derive L.H.S., starting with R.H.S. of given equation. Use of expression of binomial coefficients.

$$\binom{n}{n-r} = \frac{n!}{(n-r)! \cdot (n-(n-r))!} = \frac{n!}{(n-r)! \cdot r!} = \binom{n}{r}. \qquad \blacksquare$$

Binomial identities

- Binomial identity examples:
  - Example-1:: Pascal's triangle.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}
$$

By Pascal's identity: $\binom{6}{4}+\binom{6}{5}=\binom{7}{5}$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Extended binomial coefficients

- Extended binomial coefficient: for $u \in \mathbb{R}$, $r \in \mathbb{N}$, extended binomial coefficient, $\binom{u}{r} = \begin{cases} \frac{u\cdot(u-1)\cdot \cdots \cdot(u-(r-1))}{r!}, & r > 0 \\ 1, & r = 0 \end{cases}$

$$= \begin{cases} \frac{u!}{r!\cdot(u-r)!}, & r > 0 \\ 1, & r = 0\,. \end{cases}$$

  - Property:: Negative extended binomial coefficient: for $u \in \mathbb{R}^+$, $r \in \mathbb{N}$, $\binom{-u}{r} = \begin{cases} \frac{(-1)^r\cdot(u+r-1)!}{r!\cdot(u-1)!}, & r > 0 \\ 1, & r = 0 \end{cases}$, where $\frac{(-1)^r\cdot(u+r-1)!}{r!\cdot(u-1)!} = (-1)^r \cdot \binom{u+r-1}{r}$.

Extended binomial theorem

- Extended binomial theorem: extension of binomial theorem to $\mathbb{R}$ from $\mathbb{Z}^+$.
  - Property:: ⟨Extended binomial theorem⟩: For variable (& constant) $x \in \mathbb{R}$, where $|x| < 1$, and for $u \in \mathbb{R}$, $(1+x)^u = \sum_{k=0}^{\infty}\binom{u}{k}\cdot x^k$.

Summary

- Focus: Binomial coefficients and identities.
- Binomial coefficients, and binomial expression expansion.
- Binomial theorem, with proof and examples.
- Binomial identities, with proofs.
- Pascal's identity theorem, with proof.
- Extended binomial coefficients.
- Extended binomial theorem.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip17] Seymour Lipschutz, Marc L. Lipson, Varsha H. Patil, *Discrete Mathematics (Schaum's Outlines)*, Revised Third edition, McGraw-Hill Education, 2017.

Further Reading

- Binomial coefficients:: [Ros21]:437.
- Binomial expression:: [Ros21]:437.
- Binomial theorem:: [Ros21]:437-438.
- Binomial identities:: [Ros21]:439-440,442-443.
- Pascal's identity theorem:: [Ros21]:440.
- Pascal's triangle:: [Ros21]:440-441.