CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 2

Lecture 31: Graph/Tree Connectivity

[Edge cut, edge connectivity λ; k-edge-connected graph; κ, λ, δ inequality; Connectivity invariants; Cut set; Connectivity theorems]

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Notation table

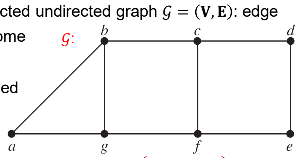
Symbol / Notation	Meaning
$\lambda(\cdot)$	Minimum cardinality of edge cut of graph. E.g., $\lambda(\mathcal{G})$: for graph $\mathcal{G}$.

Graph connectivity

- Graph connectivity: determining connectivity invariants of given graph, based on cut vertex, cut edge, block etc.
  - Property:: k-connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: if $\kappa(\mathcal{G}) \geq k$, where $\kappa(\mathcal{G})$ = vertex connectivity of $\mathcal{G}$.
  - Property:: ⟨Theorem⟩: If graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be k-connected ($k \geq 2$), then every set of k vertices of $\mathcal{G}$ to lie on some circuit.

Graph connectivity

- Graph connectivity:
  - Property:: Edge cut of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: edge subset $\mathbf{e} \subseteq \mathbf{E}$, if $\mathcal{G} - \mathbf{e}$ to become disconnected or trivial.
  - Property:: Edge cut: also called disconnecting set.
  - Property:: Edge cut applicable to connected graph only.

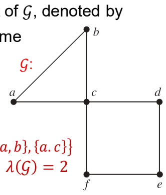


Edge cut: $\{\{b, c\}, \{g, f\}\}$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph connectivity

- Graph connectivity:
  - Property:: Edge connectivity of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| > 1$: minimum cardinality of edge cut of $\mathcal{G}$, denoted by $\lambda(\mathcal{G})$, i.e., $\lambda(\mathcal{G}) = \min\{|\mathbf{e}| \mid \mathbf{e} \subseteq \mathbf{E},\ \mathcal{G} - \mathbf{e}$ to become disconnected$\}$.
  - Property:: Edge connectivity: another form of connectivity invariant.
  - Property:: Edge connectivity: also called line connectivity.



Edge cut: $\{\{a, b\}, \{a. c\}\}$
$\lambda(\mathcal{G}) = 2$

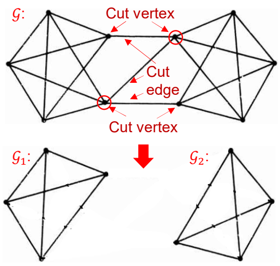
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph connectivity

- Graph connectivity:
  - Property:: $\lambda(\mathcal{G}') = 0$, if $\mathcal{G}'$ disconnected graph or $\mathcal{G}'$ trivial graph.
  - Property:: $\lambda(\mathcal{G}) = 1$, for connected graph $\mathcal{G}$ with cut edge.
  - Property:: $\lambda(K_n) = n - 1$, for complete graph K_n $(n > 0)$.
  - Property:: If $\mathcal{G}$ not complete graph, $\lambda(\mathcal{G}) \leq n - 2$.
  - Property:: For any graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$, $0 \leq \lambda(\mathcal{G}) \leq n - 1$.
  - Property:: k-edge-connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: if $\lambda(\mathcal{G}) \geq k$.

Graph connectivity

- Graph connectivity:
  - Example-1:: For the adjacent graph $\mathcal{G}$, two connected components $\mathcal{G}_1$ and $\mathcal{G}_2$ formed after removing 3 marked cut edges and two encircled cut vertices. So, $\kappa(\mathcal{G}) = 2$, $\lambda(\mathcal{G}) = 3$, $\delta(\mathcal{G}) = 4$.



[Ref: Frank Harary, *Graph Theory*, Addison-Wesley, Reading, MA, USA, 1969.]

Graph connectivity

- Graph connectivity:
  - Property:: ⟨Theorem⟩: For any graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $\kappa(\mathcal{G}) \le \lambda(\mathcal{G}) \le \delta(\mathcal{G})$, where $\delta(\mathcal{G}) = \min\{\deg(v) \mid v \in \mathbf{V}\}$.

  Proof: (A) $\lambda(\mathcal{G}) \le \delta(\mathcal{G})$.

  For any graph $\mathcal{G}$ with no edges, $\lambda(\mathcal{G}) = \delta(\mathcal{G}) = 0$.

  For disconnected graph $\mathcal{G}$ with no isolated vertex, $\lambda(\mathcal{G}) = 0$, $\delta(\mathcal{G}) > 0$.

  In case of disconnected $\mathcal{G}$ with isolated vertex, $\lambda(\mathcal{G}) = \delta(\mathcal{G}) = 0$.

  For connected graph $\mathcal{G}$ having some vertex v such that $\deg(v) = \delta(\mathcal{G}) > 0$, removing all edges incident with v resulting in disconnected $\mathcal{G}$, i.e., $\lambda(\mathcal{G}) = 0$. Combining all such cases, $\lambda(\mathcal{G}) \le \delta(\mathcal{G})$.

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Graph connectivity

- Graph connectivity:

  Proof of ⟨Theorem⟩ contd.:

  (B) $\kappa(\mathcal{G}) \le \lambda(\mathcal{G})$.

  For any graph $\mathcal{G}$ with no edges, $\kappa(\mathcal{G}) = \lambda(\mathcal{G}) = 0$.

  For any disconnected graph $\mathcal{G}$, $\kappa(\mathcal{G}) = \lambda(\mathcal{G}) = 0$.

  For connected graph $\mathcal{G}$ having some cut edge e, $\lambda(\mathcal{G}) = 1$. In this case, two possibilities — (i) vertex v incident with e to become cut vertex of $\mathcal{G}$, or (ii) $\mathcal{G}$ actually K_2. For both possibilities, $\kappa(\mathcal{G}) = 1$.

  In another case, connected $\mathcal{G}$ having $\lambda(\mathcal{G}) > 2$, i.e., $\mathcal{G}$ having more than 2 cut edges.

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Graph connectivity

- Graph connectivity:
  Proof of ⟨Theorem⟩ contd-2.:
  Then, removal of $\lambda(\mathcal{G}) - 1$ of these cut edges from $\mathcal{G}$ resulting in another graph $\mathcal{G}'$ with one cut edge remaining, say $e = \{u, v\}$.
  So, removal of either u or v from $\mathcal{G}'$ resulting in disconnected graph.
  On the other hand, for each of these $\lambda(\mathcal{G}) - 1$ cut edges of $\mathcal{G}$, selecting one incident vertex w (where $w \neq u$, $w \neq v$) and removing w also to remove one of $\lambda(\mathcal{G}) - 1$ cut edges of $\mathcal{G}$ and quite possibly some more edges of $\mathcal{G}$, resulting in another graph $\mathcal{G}''$.
  If $\mathcal{G}''$ to become disconnected, then $\kappa(\mathcal{G}) < \lambda(\mathcal{G})$.

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Graph connectivity

- Graph connectivity:
  Proof of ⟨Theorem⟩ contd-3.:
  If $\mathcal{G}''$ still connected, one cut edge still remaining, say $e = \{u, v\}$.
  So, removal of either u or v from $\mathcal{G}''$ resulting in disconnected graph.
  Combining all cases together, $\kappa(\mathcal{G}) \leq \lambda(\mathcal{G})$. ■

Graph connectivity

- Graph connectivity:
  - Property:: ⟨Theorem⟩: For graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$, if $\delta(\mathcal{G}) \geq \lfloor {}^{n}/_{2} \rfloor$ (indicating dense graph), then $\lambda(\mathcal{G}) = \delta(\mathcal{G})$.
  - Property:: ⟨Theorem⟩: Among all undirected graphs with $|\mathbf{V}| = p$ vertices and $|\mathbf{E}| = q$ edges, i.e., among all undirected (p, q) graphs, maximum vertex connectivity and maximum edge connectivity to be given by —

$$\kappa_{\max}(p,q) = \lambda_{\max}(p,q) = \begin{cases} 0, & q < p-1 \\ \lfloor 2 \cdot {}^{q}/_{p} \rfloor, & q \geq p-1. \end{cases}$$

Graph connectivity

- Cut set: set of edges of given graph, such that their removal to produce subgraph with more components than original given graph, but no proper subset of that set of edges to have same property.
  - Property:: Cut set in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: set of edges $\mathbf{e}$ $(\mathbf{e} \subseteq \mathbf{E})$, such that $\big(\omega(\mathcal{G} - \mathbf{e}) > \omega(\mathcal{G})\big) \wedge \big(\omega(\mathcal{G} - \mathbf{e}') \ngtr \omega(\mathcal{G})\big)$, where $\mathcal{G} - \mathbf{e} =$ deletion of all edges in $\mathbf{e}$ one by one from $\mathcal{G}$, $\mathcal{G} - \mathbf{e}' =$ deletion of all edges in $\mathbf{e}'$ one by one from $\mathcal{G}$, $(\mathbf{e}' \subset \mathbf{e})$, and $\omega(\cdot) =$ number of components of given graph.

Graph connectivity

- Cut set:.
  - Applicability:: Cut set applicable to any form of graph, whether connected or disconnected.
  - Property:: Cut set: also called edge cut set, minimal cut-set, proper cut-set, simple cut set, cocycle.
  - Property:: For connected graph, cut set = edge cut.
    [Note: Edge cut not defined for disconnected graph.]
  - Property:: For connected graph $\mathcal{G}$, with $\lambda(\mathcal{G}) = 1$, cut set = cut edge.

Summary

- Focus: Graph connectedness and connectivity (contd.).
- k-connected graph theorems.
- Edge cut of graph, with examples.
- Edge connectivity of graph, with properties.
- k-edge-connected graph, with properties.
- κ, λ, δ inequality, other connectivity invariants.
- Cut set, with properties.
- Graph connectivity related theorems, with proofs.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.