CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 2

Lecture 18: Counting

[Permutation; r-permutations theorem; r-permu- -tations with repetitions theorem; n-permutations with indistinguishable elements theorem]

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Permutations

- Permutation: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{Z}^+$, where $r \leq n$, one r-permutation $=$ any ordered arrangement of r elements from n elements of $\mathbf{S}$. [Note: empty sets excluded.]
  - Property:: Permutation: any linearly-ordered arrangement (or sequence or list) of distinct objects from set.
  - Property:: Each r-permutation $=$ ordering of r distinct elements of $\mathbf{S}$ $=$ (partial) permutation of n elements of $\mathbf{S}$ taken r at a time.
  - Property:: For $r = 0$, 0-permutation $=$ exactly one list possible, with no elements in it, i.e., 0-permutation $=$ empty sequence or list.

Permutations

- Permutation:
  - Property:: ⟨Theorem⟩: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{Z}^+$, where $r \leq n$, number of r-permutations $= n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-(r-1)) = \frac{n!}{(n-r)!} = P(n,r)$ or $P_{n,r}$ or ${}_nP_r$ or $(n)_r$.

 Proof: Target: count of ways of choosing r elements from n elements.
 Constituents: r tasks, each task to select each of r elements.
 Task T_i $(i = 1,2,\ldots,r)$: choice of one of $(n-(i-1))$ elements from $\mathbf{S}$ for i-th element of r-permutation, independent of how $T_1, T_2, \ldots, T_{i-1}$ performed, except that elements of $\mathbf{S}$ already chosen by previous tasks not to be used again.

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Permutations

- Permutation:

  Proof (contd.):

  T_1: from n distinct elements, number of ways to choose 1st element of r-permutation $= n$.

  T_2: from remaining $(n-1)$ distinct elements, number of ways to pick 2nd element of r-permutation $= n-1$.

  Continuing in this manner,

  T_r: from remaining $(n-(r-1))$ distinct elements, number of ways to choose r-th element of r-permutation $= (n-(r-1)) = n-r+1$.

  Accordingly, product rule to be applied.

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Permutations

- Permutation:

  Proof (contd-2.):

  Then, as per product rule, number of different ways to choose r elements from n elements = number of r-permutations = $P(n,r)$
  $= \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-(r-1))}_{r \text{ factors}} = \frac{n!}{(n-r)!}$, on simplifying. ■

  - Property:: ⟨Corollary⟩: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and $r = n$, number of n-permutations $= \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! = P(n,n)$
    $\equiv \sigma: \mathbf{S} \to \mathbf{S}$ = bijective function σ from $\mathbf{S}$ to itself.
    [Also called, reordering.]

Permutations

- Permutation:
  - Property:: ⟨Corollary⟩: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and $r = 0$, number of 0-permutations $= \frac{n!}{(n-0)!} = 1 = P(n,0)$.
  - Property:: ⟨Theorem Revisited⟩: Number of r-combinations in selecting r distinct elements from set $\mathbf{S}$ of n distinct elements = $C(n,r) = \frac{n!}{r! \cdot (n-r)!}$.

  Proof: [Using division rule and r-permutation theorem]

  $P(n,r)$ number of r-permutations of $\mathbf{S}$ to be obtained by forming $C(n,r)$ number of r-combinations of $\mathbf{S}$, and then ordering elements in each r-combination.

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Permutations

- Permutation:
  Proof (contd.):
  So, number of ways of forming r-combinations = $P(n,r)$, ignoring order, as no effect of order of elements of $\boldsymbol{S}$ in r-combination.
  Also, $P(r,r)$ number of ways to order r elements in any r-combination of n elements.
  So, each r-combination in $C(n,r)$ corresponding to exactly $P(r,r)$ number of r-permutations.
  Then, equivalent factor $d = P(r,r)$.
  Accordingly, division rule to become applicable.

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Permutations

- Permutation:
  Proof (contd-2.):
  Then, as per division rule, number of r-combinations $= \frac{P(n,r)}{P(r,r)}$
  $= \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r! \cdot (n-r)!}$, on simplifying. ■

Permutations

- Permutations examples:
  - Example-1:: To determine how many ways to select 3 students from group of 5 students to stand in line for picture.
    Let $\boldsymbol{S}$ be set of students, denoted by last digit of roll no. $\{1, 2, 3, 4, 5\}$.
    To lineup 3 selected students → need for order of selected students.
    Then, by product rule, number of (ordered) lineup of 3 from 5 students $=$ number of 3-permutations $= P(5,3) = 5 \cdot 4 \cdot 3 = \frac{5!}{2!} = 60$.
  - Example-2:: Example-1, where all 5 students to stand in line.
    By product rule, number of (ordered) lineup of all 5 students $=$ number of 5-permutations $= P(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \frac{5!}{0!} = 5! = 120$.

Permutations

- Permutations examples:
  - Example-3:: To determine how many ways to award gold, silver, and bronze medals in marathon race of 8 participants.
    To arrange 3 winners → need for order from 8 racers.
    Then, by product rule, number of (ordered) awardees of 3 from 8 racers = number of 3-permutations = $P(8,3) = 336$.

Permutations with repetition

- Permutation with repetition: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, one r-permutation with repetition = any ordered arrangement of r (not necessarily distinct) elements from n elements of $\mathbf{S}$.
  - Property:: ⟨Theorem⟩: For set $\mathbf{S}$ of n distinct elements ($n \in \mathbb{Z}^+$) and any $r \in \mathbb{N}$, where $r \leq n$, number of r-permutations with repetition $= n^r$.

  Proof: Target — choosing r elements with repetitions from n elements.
  Constituents: r tasks, each task to select each of r elements.

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Permutations with repetition

- Permutation with repetition:
  Proof (contd.):
  Task T_i $(i = 1,2,\ldots,r)$: choice of one of n elements (because for each choice all n elements to be available due to repetition) from $\mathbf{S}$ for i-th element of r-permutation, independent of how $T_1, T_2, \ldots, T_{i-1}$ performed.
  Then, as per product rule, number of different ways to choose r elements from n elements with repetitions
  = number of r-permutations with repetitions
  $= \underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{r \text{ factors}} = n^r$. ■

Permutations with repetition

- Permutation with repetition examples:
  - Example-1: to count number of strings of length r formed from uppercase letters of English alphabet.
    Given, $n = 26$, repetition allowed in r-permutations.
    So, number of r-permutations with repetitions = 26^r.

Permutations with indistinguishable elements

- Permutation with indistinguishable elements: For finite multiset $\mathbf{M}$ of n elements with n_1 indistinguishable elements of type 1, n_2 indistinguishable elements of type 2,…, and n_k indistinguishable elements of type k, and $n = n_1 + n_2 + \cdots + n_k$, $(n, n_1, n_2, \ldots, n_k \in \mathbb{Z}^+)$, one n-permutation with indistinguishable elements = any ordered arrangement of all elements from n elements of $\mathbf{M}$.
  - Property:: ⟨Generalized Theorem⟩: For multiset $\mathbf{M}$ of n $(= n_1 + n_2 + \cdots + n_k)$ elements corresponding to each type of k indistinguishable element-types $(n, n_1, n_2, \ldots, n_k \in \mathbb{Z}^+)$, number of n-permutations with indistinguishable elements = $\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$.

Permutations with indistinguishable elements

- Permutation with indistinguishable elements:
  Proof of ⟨Generalized Theorem⟩: Target: n-permutation with indistinguishable elements of k element-types in $\mathbf{M}$.
  Constituents: k tasks, each to place one of $n_1, n_2, \ldots, n_k$ elements of respective k element-types from $\mathbf{M}$ among n positions in sequence.
  Task T_i $(i = 1,2, \ldots, k)$: choice of n_i elements (i.e., elements of i-th element-type from $\mathbf{M}$) to be placed among available $(n - n_1 - n_2 - \cdots - n_{i-1})$ positions of n-permutation ($\because$ remaining positions of n-permutation already filled by tasks $T_1, T_2, \ldots, T_{i-1}$), independent of how $T_1, T_2, \ldots, T_{i-1}$ performed.

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Permutations with indistinguishable elements

- Permutation with indistinguishable elements:

  Proof of ⟨Generalized Theorem⟩ (contd.):

  T_1: number of ways to place n_1 elements among n positions $= n_1$-combination $= C(n, n_1)$. [$\because$ elements taken from multiset, so elements no longer distinct; for any particular element type, combination of count of that type to be considered for available positions in sequence.]

  T_2: number of ways to place n_2 elements among remaining $(n - n_1)$ positions $= C(n - n_1, n_2)$.

  Continuing in this manner,

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Permutations with indistinguishable elements

- Permutation with indistinguishable elements:

  Proof of ⟨Generalized Theorem⟩ (contd-2.):

  T_k: number of ways to place n_k elements among remaining $(n - n_1 - n_2 - \cdots - n_{k-1})$ positions $= C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$.

  Then, as per product rule, number of different ways to place $n_1, n_2, \ldots, n_k$ elements of k element-types among n positions $=$ number of n-permutations with indistinguishable elements

  $= \underbrace{C(n, n_1) \cdot C(n - n_1, n_2) \cdot \cdots \cdot C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)}_{k \text{ factors}}$

  $= \frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdot \cdots \cdot \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k! \cdot (n-n_1-n_2-\cdots-n_{k-1}-n_k)!}$.

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Permutations with indistinguishable elements

- Permutation with indistinguishable elements:

  Proof of ⟨Generalized Theorem⟩ (contd-3.):

  Simplifying, no. of n-permutations with indistinguishable elements

  $= \frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdot (n-n_1-n_2)!} \cdot \cdots \cdot \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k! \cdot 0!} = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$

  $= \frac{\left(\sum_{i=1}^{k} n_i\right)!}{\prod_{i=1}^{k} (n_i!)}$. ■

  - Property:: ⟨Special Theorem⟩: For multiset $\mathbf{M}$ of n elements, with identical k elements of one element-type (where $n, k \in \mathbb{Z}^+, n, k > 1$) and rest $(n - k)$ distinct elements, number of n-permutations with identical elements $= \frac{n!}{k!}$.

Permutations with indistinguishable elements

- Permutations with indistinguishable elements examples:
  - Example-1:: To find how many ways to permute letters of word HAPPY.
    Given: multiset $\mathbf{M} = \{1 \cdot \text{H}, 1 \cdot \text{A}, 2 \cdot \text{P}, 1 \cdot \text{Y}\}$, $n = 5$.
    With element 'P' multiplicity = 2, number of ways to place 2 'P's in 5 positions $= C(5,2)$.
    For remaining 3 positions, number of ways to permute remaining 3 distinct elements $= P(3,3)$.
    By product rule: $C(5,2) \cdot P(3,3) = \frac{5!}{2! \cdot 3!} \cdot \frac{3!}{0!} = 60$.
    Alternate sol^n: As per previous special theorem, $n = 5$, $k = 2$ (two P).
    So, number of permutations (with identical elements) $= \frac{5!}{2!} = 60$.

Permutations with indistinguishable elements

- Permutations with indistinguishable elements examples:
  - Example-2:: To determine how many ways to reorder letters of word SUCCESS.
    Given: multiset $\mathbf{M} = \{3 \cdot \text{S}, 1 \cdot \text{U}, 2 \cdot \text{C}, 1 \cdot \text{E}\}$, $n = 7$.
    Number of ways to place 3 'S's in 7 positions $= C(7,3)$.
    Number of ways to place 1 'U' in 4 positions $= C(4,1)$.
    Number of ways to place 2 'C's in 3 positions $= C(3,2)$.
    Number of ways to place 1 'E' in last position $= C(1,1)$.
    By product rule: $C(7,3) \cdot C(4,1) \cdot C(3,2) \cdot C(1,1) = \frac{7!}{3! \cdot 4!} \cdot \frac{4!}{1! \cdot 3!} \cdot \frac{3!}{2! \cdot 1!} \cdot \frac{1!}{1! \cdot 0!}$
    $= 7 \cdot 5 \cdot 4 \cdot 3 = 420$.

Summary

- Focus: Combinatorics contd.
- Permutation.
- r-permutations theorem, corollaries, and properties.
- Permutations examples.
- r-permutations with repetitions theorem, and properties.
- Permutations with repetitions examples.
- n-permutations with indistinguishable elements theorems.
- Permutations with indistinguishable elements examples.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip17] Seymour Lipschutz, Marc L. Lipson, Varsha H. Patil, *Discrete Mathematics (Schaum's Outlines)*, Revised Third edition, McGraw-Hill Education, 2017.
5. https://www.cs.sfu.ca/~ggbaker/zju/math/perm-comb.html.
6. https://www.cs.sfu.ca/~ggbaker/zju/math/perm-comb-more.html.

Further Reading

- Permutation:: [Ros21]:428-431.
- r-permutations theorem:: [Ros21]:429-430.
- Permutation with repetition:: [Ros21]:445-446.
- r-permutations with repetitions theorem:: [Ros21]:446.450.
- Permutation with indistinguishable elements:: [Ros21]:450-451.
- n-permutations with indistinguishable elements theorem:: [Ros21]:450-451.