

CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 1

Tree

- Tree: connected undirected graph without circuits, i.e., acyclic connected undirected graph. [Rigorous definition in slide 16.]
- Property:: Tree \rightarrow simple graph (i.e., no self loop, no parallel edges).
- Property:: **Forest**: any (connected or disconnected) undirected graph without circuits.
- Property:: Components of forest \rightarrow trees.

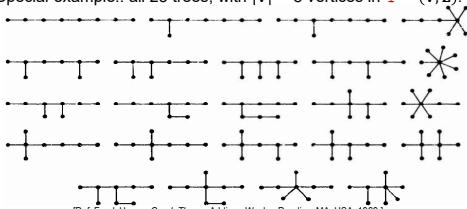
Forest \rightarrow

This is one graph with three connected components.
3 Trees

[Ref: Kenneth H. Rosen, Discrete Mathematics and its Applications, Eighth edition, McGraw-Hill Education, 2019.]

Tree

- Tree:
 - Special example:: all 23 trees, with $|V| = 8$ vertices in $\mathcal{T} = (V, E)$.



[Ref: Frank Harary, *Graph Theory*, Addison-Wesley, Reading, MA, USA, 1969.]

Tree

- Tree:
 - Property: Tree \rightarrow minimally connected graph.
[Minimally connected graph: if removal of any one edge from connected graph disconnecting it, i.e., connected graph with minimum number of edges and no circuits.]

Tree

- Tree fundamentals:
 - Property:: (**Theorem**): One and only one path (i.e., unique path) to be present between every pair of vertices in tree $T = (V, E)$.

Proof: From definition of tree, T to be taken as connected graph.

So, for arbitrary pair of vertices $u, v \in V$, at least one path to exist between u and v , i.e., $\exists P(u \in P) \wedge (v \in P)$, where $P = \text{path in } T$.

To prove: $(T \text{ is a tree}) \rightarrow (\text{there is exactly one path between } u \text{ and } v)$.

Proof by contraposition.

Let, $\neg r = \neg(\text{one and only one path between } u \text{ and } v) = \text{more than one path (say, two paths } P_1, P_2 \text{ in } \mathcal{T}) \text{ between } u \text{ and } v.$

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Tree

- Tree fundamentals:

Proof (contd.):

So, $\exists P_1(u \in P_1) \wedge (v \in P_1)$ and $\exists P_2(u \in P_2) \wedge (v \in P_2)$.

Then, union of P_1 , $P_2 = P_1 \cup P_2$ also to be preset in \mathcal{T} .
 In $P_1 \cup P_2$, if traversal started from vertex u through vertices and edges in P_1 , vertex v reached, and still another path P_2 present to return back to u by traversing through vertices and edges in P_2 .

So, $P_1 \sqcup P_2 = \{u\}$, vertex-edge sequence from v_1 up to v_2 in

So, $P_1 \cup P_2 = \{u, \text{vertex-edge sequence from } u \text{ up to } v \text{ in } P_1, v, \text{vertex-edge sequence from } v \text{ up to } u \text{ in } P_2, u\}$, resulting in closed trail in \mathcal{T} with no vertex repetition.

(contd. to next slide)

Tree

- Tree fundamentals:

Proof (contd-2.):

$\therefore \mathcal{T}$ to contain at least one circuit.

So, T not possible to become tree = $\neg(T \text{ to be tree}) = \neg q$ (say).

Thus, $\neg r \rightarrow \neg q$, and so $q \rightarrow r$ (by definition of contrapositive)

Graph Theory

Tree

- Tree fundamentals:

- Property:: (**Theorem**): If one and only one path (i.e., unique path) between every pair of vertices in graph $G = (V, E)$, then G to become tree.

Proof: Given, unique path between any two vertices of graph \mathcal{G} , i.e., for arbitrary pair of vertices $u, v \in V$, one and only one path to exist between u and v , i.e., $\exists P(u \in P) \wedge (v \in P)$, where $P = \text{path in } \mathcal{G}$.

Then, G must be connected graph.

To prove: (One and only one path between u and v) \rightarrow (no circuit between u and v), for $|V| \geq 2$. (contd. to next slide)

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Tree

- Tree fundamentals:

Proof (contd.):
 Proof by contraposition.
 Let, $\neg r = \neg(\text{no circuit between } u \text{ and } v) = \text{circuit } C \text{ existing between } u \text{ and } v.$
 Let $C = \{u, \text{vertex sequence from } u \text{ up to } v, v, \text{vertex sequence from } v \text{ up to } u, u\}$ (closed trail with no vertex repeat).
 Then, C to be possibly split into two separate vertex sequences, viz.
 $\{u, \text{vertex sequence from } u \text{ up to } v, v\} = P_1$ (say), and
 $\{v, \text{vertex sequence from } v \text{ up to } u, u\} = P_2$ (say).

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- Tree fundamentals:

Proof (contd-2.):
 Clearly, $(u \in P_1) \wedge (v \in P_1)$ and $(u \in P_2) \wedge (v \in P_2)$.
 Thus, both P_1, P_2 = open trails between u and v with no vertex repeat.
 \therefore Two paths P_1, P_2 in G between u and $v = \neg(\text{one and only one path between } u \text{ and } v) = \neg q$ (say).
 Thus, $\neg r \rightarrow \neg q$, and so $q \rightarrow r$ (by definition of contrapositive).
 Accordingly, G to become acyclic connected graph.
 Hence G to become tree. ■

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- Tree fundamentals:

Property: (**Theorem**): A tree $T = (V, E)$ with $|V| = n$ ($n \in \mathbb{Z}^+$) vertices to contain $|E| = n - 1$ edges.

Proof: Proof by mathematical induction.

(Basis step) When $n = 1$, isolated vertex in T with no edge = $n - 1$ edge. Then, theorem to become true for $n = 1$.

(Inductive step)
 Inductive hypothesis: premise that theorem to become true for every tree with fewer than n vertices (including $n - 1$ vertices).
 Then, in $T = (V, E)$ with $|V| = n$, let $e = \{u, v\}$, $e \in E$, $u, v \in V$.

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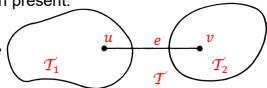
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- Tree fundamentals:

Proof (contd.): According to previous theorem, due to one and one path between u and v in T , except e , no other path present.

So, deletion of e from \mathcal{T} resulting in \mathcal{T} getting disconnect, i.e., $\mathcal{T} - e$ to become disconnected graph,



with exactly two partitions \mathcal{T}_1 and \mathcal{T}_2 , due to acyclic nature of \mathcal{T} , resulting in two components, viz. $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T} - e$, $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T} - e$. Also, no circuit in \mathcal{T}_1 and \mathcal{T}_2 as no circuit in \mathcal{T} .

Also, no circuit in T_1 and T_2 as no circuit in T .
 Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*

[Ref: Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall, 1974.] (contd. to next slide)

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- Tree fundamentals:

Proof (contd-2.):

So, $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ to become tree.
 By induction hypothesis, theorem to hold for both T_1 and T_2 due to containing fewer than n vertices each, and therefore, each to contain one less edge than number of vertices in it.

Considering $|V_1| = n_1$, $|V_2| = n_2$, then $n_1 + n_2 = n$, $|E_1| = n_1 - 1$, $|E_2| = n_2 - 1$.

Therefore, number of edges in $\mathcal{T} - e = n_1 - 1 + n_2 - 1 = n - 2$.
 So, adding back e to $\mathcal{T} - e$ to get back given \mathcal{T} , with $|E| = n - 1$.

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- Tree fundamentals:

- Property:: (**Theorem**): For any undirected graph $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ ($n \in \mathbb{Z}^+$) vertices, satisfying any two of following conditions also implying third condition, and establishing \mathcal{T} as tree.
 - (1) \mathcal{T} is connected.
 - (2) \mathcal{T} has no cycles.
 - (3) \mathcal{T} has $n - 1$ edges.

- (i) \mathcal{T} to be connected;
 - (ii) \mathcal{T} to have no circuits;
 - (iii) \mathcal{T} to contain $|E| = n - 1$ edges.

Proof: Proof based on previous theorems.

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- Tree fundamentals:
 - (Five different but equivalent definitions):: **Tree**: undirected graph $\mathcal{T} = (V, E)$ with $|V| = n$ ($n \in \mathbb{Z}^+$) vertices holding any of following conditions —
 - \mathcal{T} to be connected and circuitless (or cycle-free), or
 - \mathcal{T} to be connected and to contain $|E| = n - 1$ edges, or
 - \mathcal{T} to be circuitless and to contain $|E| = n - 1$ edges, or
 - exactly one unique path between every pair of vertices in \mathcal{T} , or
 - \mathcal{T} to be minimally connected graph.

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- Tree fundamentals:
 - Property: (**Theorem**): A connected undirected graph $T = (V, E)$ to become tree, if and only if adding a new edge e between any two vertices $u, v \in V$ in T resulting in exactly one circuit in T .

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- Tree fundamentals:
 - Property:: (**Theorem**): In any tree $T = (V, E)$ with two or more vertices (i.e., $|V| = n$, $n \geq 2$) to contain at least two pendant vertices.

Proof: Proof by contradiction.

From previous theorem, \mathcal{T} with $|V| = n$ ($n \geq 2$) vertices, where $V = \{v_1, \dots, v_n\}$, to contain $|E| = n - 1$ edges.

Then, from handshaking theorem, sum of degrees of all vertices in \mathcal{T} = $\sum_{i=1}^n \deg(v_i) = 2 \cdot |\mathbf{E}| = 2 \cdot (n - 1)$.

For connected T , no vertex v_i possible with zero degree.

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- Tree fundamentals:

Proof (contd.):

Premise: less than two vertices of degree one (i.e., less than two pendant vertices) present in T .
 Consider v to be only pendent vertex in T .
 Then, sum of degrees of all vertices in T
 $= \deg(v) + \sum_{v_i \in V \setminus \{v\}} \deg(v_i) = 1 + 2 \cdot (n - 1)$, contradicting
 handshaking theorem.
 Hence, at least two pendant vertices present in T . ■

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- Tree fundamentals:

Property:: **Distance** between any pair of vertices u, v in undirected tree $T = (V, E)$: $d(u, v) =$ length of **unique path** between u, v (i.e., number of edges in that path) in T .

Property:: **Center** of undirected tree $T = (V, E)$:
 $\text{center}(T) = \{u \in V \mid d(u, v) = \min\{d(v, u) \mid \forall v \in V\}\}$, where $d(v) = \max\{d(v, v_i) \mid v_i \in V, i = 1, 2, \dots, |V|\}$.

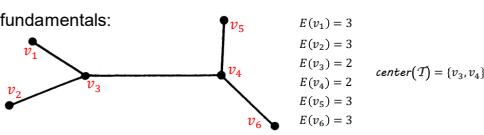
Property:: Two centers possible in some tree.

[Ref: Narasimha Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

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Tree

- Tree fundamentals:



$E(v_1) = 1$
 $E(v_2) = 1$
 $E(v_3) = 1$
 $E(v_4) = 2$
 $E(v_5) = 2$
 $E(v_6) = 2$
 $\text{center}(T) = \{v_3, v_4\}$

Property:: (**Theorem**): One or two in every undirected tree $T = (V, E)$, i.e., $|\text{center}(T)| = 1$ or $|\text{center}(T)| = 2$.
 Property:: (**Corollary**): For any undirected tree $T = (V, E)$ with two centers, i.e., $|\text{center}(T)| = 2$, then two centers to be neighbors in T .

[Ref: Narasimha Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

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- Tree fundamentals:
 - Property: **Radius** of undirected tree $T = (V, E)$:
 $\text{rad}(T) = E(v)$, where $v \in \text{center}(T)$.
 - Property: **Diameter** of undirected tree $T = (V, E)$:
 $\text{diam}(T) = \max\{d(v_i, v_j) \mid v_i, v_j \in V, i, j = 1, 2, \dots, |V|\}$.

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

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- Tree fundamentals:
 - Property: **Labeled undirected tree**: with unique name or label assigned to each vertex.
 - Property: **Unlabeled undirected tree**: no assigned vertex distinctions.
 - Property: (**Cayley's Theorem**): Number of labeled trees to be equal to n^{n-2} , where each labeled tree $T = (V, E)$ to have two or more vertices (i.e., $|V| = n, n \geq 2$).
[10 different proofs of Cayley's Theorem available.]

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Summary

- Focus: Tree fundamentals.
- Tree, forest.
- Minimally connected graph, and other properties of trees.
- Path uniqueness in tree, and related theorems.
- Edge set cardinality in tree, and related theorems.
- Rigorous definition of tree.
- Distance between vertices, eccentricity of vertex in tree.
- Center, radius, diameter of tree.
- Labeled, unlabeled tree.
- Cayley's theorem.

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References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.