CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 2

Lecture 33: Graph/Tree Connectivity

[Euler graph, trail, circuit; Königsberg graph; Euler graph conditions; Hamilton graph, path, circuit; Icosian game solution; Dirac's, Ore's theorem]

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Euler graph

- Euler graph [also Eulerian graph]: connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, containing some closed trail $\mathfrak{I}$ listing all edges of $\mathbf{E}$.
  - Property:: Euler trail (also called Euler line): closed trail $\mathfrak{I}$ containing all edges of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
  - Property:: Euler circuit: circuit $\mathcal{C}$ containing all edges of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
  - Property:: Euler graph → always connected, except for any isolated vertices as its components.
  - Applicability:: Euler graph → necessary and sufficient conditions for existence of trails and circuits in multigraph.

Euler graph

- Euler graph:
  - Property:: ⟨Theorem⟩: A given graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be Euler graph if and only if $\mathcal{G}$ having at most one nontrivial component and all vertices of $\mathcal{G}$ having even degree.
    [Alternate statement: A given connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be Euler graph if and only if all vertices of $\mathcal{G}$ having even degree.]

  Proof: Two cases to prove — (A) Euler graph → both conditions to hold; (B) Both conditions satisfied → Euler graph.
  (A) Euler graph → both conditions to hold. [Necessity]
  Let Euler graph $\mathcal{G}$ to have Euler trail $\mathfrak{I}$, a closed trail.

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Euler graph

- Euler graph:

  Proof of ⟨Theorem⟩ contd.:

  Observation: in tracing $\mathfrak{J}$, every time $\mathfrak{J}$ to meet each vertex $v \in \mathbf{V}$, $\mathfrak{J}$ to enter v through edge $e = \{u, v\}$ ($e \in \mathbf{E}$, $u \in \mathbf{V}$), and $\mathfrak{J}$ to exit v through edge $e' = \{v, u'\}$ ($e' \in \mathbf{E}$, $u' \in \mathbf{V}$), i.e., $\mathfrak{J}$ to encounter two "new" edges e, e' ($e \neq e'$) incident on each v.

  Above observation true not only of all intermediate vertices of $\mathfrak{J}$, but also of terminal vertex, because terminal vertex "exited" at beginning and "entered" at end of $\mathfrak{J}$.

  So, every vertex of $\mathcal{G}$ to have even degree.

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Euler graph

- Euler graph:

  Proof of ⟨Theorem⟩ contd-2.:

  Also, for $\mathfrak{J}$ containing all edges of $\mathcal{G}$, two distinct edges to be included in $\mathfrak{J}$ only when such edges lying in same component of $\mathcal{G}$.

  Consequently, at most one nontrivial component in $\mathcal{G}$.

  Hence, Euler graph → both conditions to hold.

  (B) Both conditions satisfied → Euler graph. [Sufficiency]

  Let all vertices of $\mathcal{G}$ to be of even degree.

  Now to construct a walk, with tracing to start at arbitrary vertex $v \in \mathbf{V}$ and continuing through edges of $\mathcal{G}$ with no edge repetition.

(contd. to next slide)

Euler graph

- Euler graph:

  Proof of ⟨Theorem⟩ contd-3.:

  Let this walk be $\mathfrak{J}$, in which tracing to continue as far as possible.

  Since every vertex of even degree, "exit" from every "entered" vertex possible while tracing, i.e., $\mathfrak{J}$ not ending at any vertex but v.

  And since v also of even degree, eventually $\mathfrak{J}$ to reach v at end of tracing. Let $\mathbf{e} = \{e \in \mathbf{E} \mid e \in \mathfrak{J}\}$, so that ($\mathbf{e} \subseteq \mathbf{E}$).

  If this closed walk (i.e., trail) $\mathfrak{J}$ included all edges of $\mathcal{G}$ (i.e., $\mathbf{e} = \mathbf{E}$) then $\mathcal{G}$ to become Euler graph, and $\mathfrak{J}$ to be called Euler trail.

  If not (i.e., $\mathbf{e} \subset \mathbf{E}$), then trail formation to be repeated in $\mathcal{G}$.

(contd. to next slide)

Euler graph

- Euler graph:

  Proof of ⟨Theorem⟩ contd-4.:

  For repeatedly tracing all edges of $\mathcal{G}$, let $\mathscr{g} \subset \mathcal{G}$ be formed, where $\mathscr{g} = (\mathbf{v}, \mathbf{e}')$, such that $\mathbf{e}' = \mathbf{E} \setminus \mathbf{e}$, $\mathbf{v} \subseteq \mathbf{V}$.

  Since all vertices of both $\mathcal{G}$ and $\mathfrak{I}$ to be even degree, degrees of vertices of $\mathscr{g}$ to be also even. [As per Handshaking theorem]

  Further as $\mathcal{G}$ connected, $\mathscr{g}$ to connect to $\mathfrak{I}$ at least at vertex $u \in \mathbf{V} \cap \mathbf{v}$.

  Now, another walk to be constructed in $\mathscr{g}$, with tracing to start at u and continuing through edges of $\mathscr{g}$ with no edge repetition.

  Let this walk be $\mathfrak{I}'$, with tracing to continue as far as possible in $\mathscr{g}$.

(contd. to next slide)

Euler graph

- Euler graph:

  Proof of ⟨Theorem⟩ contd-5.:

  Since all vertices of even degree in $\mathscr{g}$, "exit" from every "entered" vertex possible in $\mathfrak{I}'$, and eventually $\mathfrak{I}'$ to reach u at end of tracing.

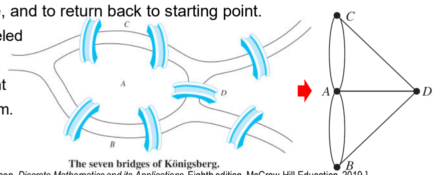
  But, possibility to combine $\mathfrak{I}'$ with $\mathfrak{I}$ to form new trail $\mathfrak{I}''$, starting and ending at v with more edges than $\mathfrak{I}$.

  This process to be repeated until a trail obtained traversing all edges of $\mathcal{G}$. Then, $\mathcal{G}$ to become Euler graph.

  Hence, both conditions satisfied → Euler graph. ∎

Euler graph

- Euler graph:
  - Example:: Königsberg bridge problem [seven bridges of Königsberg]: to begin at any of four land areas, to travel across each bridge exactly once, and to return back to starting point.
    - Euler modeled multigraph to represent that problem.



The seven bridges of Königsberg.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Euler graph

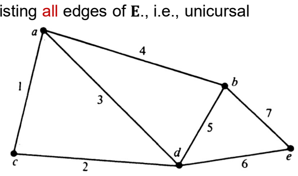
- Euler graph:
  - Example:: Königsberg bridge problem (contd.): problem ≡ "closed trail" traversing all seven bridges.
    As per previous theorem, Euler trail possible when both — at most one nontrivial component and all vertices of even degree, satisfied.
    In multigraph of Königsberg bridge problem (commonly called Königsberg graph), not all its vertices of even degree, viz.
    $\deg(A) = 5$, $\deg(B) = 3$, $\deg(C) = 3$, $\deg(D) = 3$.
    So, Königsberg graph ≠ Euler graph.
    Answer to Königsberg bridge problem: NOT POSSIBLE.

Euler graph

- Euler graph:
  - Property:: ⟨Theorem⟩: A connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to contain Euler path (but not Euler circuit), if and only if $\mathcal{G}$ having exactly two vertices of odd degree.
  - Property:: ⟨Theorem⟩: A connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to be considered as Euler graph, if and only if possibility of decomposing $\mathcal{G}$ into circuits.

Euler graph

- Euler graph:
  - Property:: Unicursal graph: connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, containing some open trail $\mathfrak{J}$ listing all edges of $\mathbf{E}$., i.e., unicursal line.
  - Property:: Unicursal line (also called open Euler line): open trail $\mathfrak{J}$ containing all edges of connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.



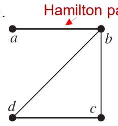
Unicursal line = {1,c,2,d,3,a,4,b,5,d,6,e,7,b}

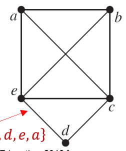
[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Hamilton graph

- Hamilton graph [also Hamiltonian graph]: simple undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, with a spanning circuit.
  - Property:: Hamilton path (or Hamiltonian path): spanning path of simple undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
  - Property:: Hamilton cycle (or Hamilton circuit, or Hamiltonian cycle, or Hamiltonian circuit): spanning circuit of simple undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.

Hamilton path$_1$: $\{a, b, c, d\}$





Hamilton circuit$_1$: $\{a, b, c, d, e, a\}$

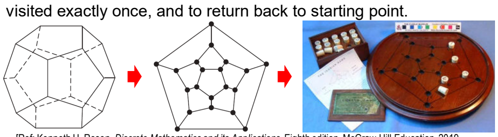
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Hamilton graph

- Hamilton graph:
  - Property:: Hamilton circuit length: n (or, n edges), for graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$ vertices.
  - Property:: Hamilton path: $C - e$, where C = Hamilton circuit in simple undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $e \in C$, i.e., removing any edge from Hamilton circuit resulting in Hamilton path.
  - Property:: Hamilton path length: $n - 1$ (or, $n - 1$ edges), for graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$ vertices.

Hamilton graph

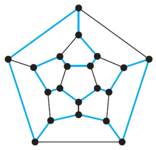
- Hamilton graph:
  - Example:: Icosian game [also "a voyage round the world" puzzle; traveler's dodecahedron]: to begin at any vertex of dodecahedron, to travel along its edges exactly once such that every vertex also visited exactly once, and to return back to starting point.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
https://www.geogebra.org/m/u3xggkcj]

Hamilton graph

- Hamilton graph:
  - Example:: Icosian game (contd.):
    - Solution → Hamilton circuit.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Hamilton graph

- Hamilton graph:
  - Applicability:: Hamilton graph → necessary and sufficient conditions for existence of spanning paths, spanning circuits in simple graph.
  - Property:: Necessary and sufficient condition for connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to have Hamilton circuit: still unsolved.

Hamilton graph

- Hamilton graph:
  - Property:: If graph $\mathcal{G}$ to contain at least one pendant vertex (of degree one), then $\mathcal{G}$ not to have Hamilton circuit.

    [Reason: For presence of Hamilton circuit, each vertex to be incident with two edges, and its both incident edges to be part of Hamilton circuit.

    Once Hamilton circuit traversed through vertex of degree more than two, then all remaining incident edges of this vertex to be removed from consideration.

    Further, Hamilton circuit not containing smaller circuit within it.]

Hamilton graph

- Hamilton graph:
  - Property:: ⟨Dirac's Theorem⟩: For simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$ $(n \geq 3)$, if minimum degree $\delta(\mathcal{G}) = {}^{n}/_{2}$ (i.e., $\deg(v) \geq {}^{n}/_{2}$, $v \in \mathbf{V}$), then $\mathcal{G}$ to contain Hamilton circuit.
  - Property:: ⟨Ore's Theorem⟩: For simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$ $(n \geq 3)$, if $\deg(u) + \deg(v) \geq n$, $\forall u, v \in \mathbf{V}$, $e = \{u, v\} \in \mathbf{E}$, then $\mathcal{G}$ to contain Hamilton circuit.
  - Property:: Dirac's theorem and Ore's theorem → only sufficient conditions for existence of Hamilton circuit in graph.

Summary

- Focus: Euler graph, Hamilton graph.
- Euler graph, Euler trail, Euler circuit, Euler path, with properties and examples.
- Königsberg bridge problem, with illustrations.
- Königsberg graph, and solution to Königsberg bridge problem.
- Euler graph conditions, and related theorems with proofs.
- Hamilton graph, Hamilton path, Hamilton circuit with properties and examples.
- Icosian game problem, with illustrations.
- Solution to Icosian game problem.

Summary

- Hamilton graph conditions, and related theorems with proofs.
- Dirac's theorem, Ore's theorem, with proofs.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.