CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 1

Lecture 24: Graphs & Trees

[Vertex neighborhood, degree, independent set; Regular graph; Subgraph; Union, intersection, ring sum; Complement; Decomposition, deletion, fusion]

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Notation table

Symbol / Notation	Meaning
$N(v)$	Neighborhood of vertex v in graph.
$\deg(v)$	Degree (or valency) of vertex v in graph.
$\deg(\mathcal{G})$	Degree of all vertices in graph $\mathcal{G}$.
$\delta(\mathcal{G})$	Minimum degree among vertices in graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
$\triangle(\mathcal{G})$	Maximum degree among vertices in graph $\mathcal{G}$.
$\beta(\mathcal{G})$	Independence number of graph $\mathcal{G}$.
K_n	Complete graph of n vertices.
$g \subset \mathcal{G}$	Graph g to be subgraph of graph $\mathcal{G}$.
$\mathcal{G}_1 \cup \mathcal{G}_2$	Union of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.
$\mathcal{G}_1 \cap \mathcal{G}_2$	Intersection of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.
$\mathcal{G}_1 \oplus \mathcal{G}_2$	Ring sum of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.
$\mathcal{G} \oplus g, \mathcal{G} \setminus g$	Complement of subgraph g w.r.t. supergraph $\mathcal{G}$.

Notation table (contd.)

Symbol / Notation	Meaning
$\mathcal{G} - v$	Vertex deletion of vertex $v \in \mathbf{V}$ from $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.
$\mathcal{G} - e$	Edge deletion of edge $e \in \mathbf{E}$ from $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.

Graph

- Graph fundamentals:
  - Property:: Adjacent (or neighbor) vertices in undirected $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: distinct vertices u and v (i.e., $u \neq v$) connected by $e = \{u, v\}$, $u, v \in \mathbf{V}$, $e \in \mathbf{E}$.
  - Property:: Adjacent (or neighbor) edges in undirected $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: nonparallel edges e_1 and e_2 incident on common vertex u, i.e. $e_1 = \{u, v_1\}$, $e_2 = \{u, v_2\}$, $e_1, e_2 \in \mathbf{E}$, $u, v_1, v_2 \in \mathbf{V}$.
  - Property:: Neighborhood of vertex in undirected $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: (denoted as $N(v)$) set of all neighbor vertices of vertex v, other than v ($v \in \mathbf{V}$), of $\mathcal{G}$, i.e., $N(v) = \{u \in \mathbf{V} \mid v \neq u,\ \exists e \in \mathbf{E}\ (e = \{u, v\})\}$.

Graph

- Graph fundamentals:
  - Property:: Degree (also called valency) of vertex in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\deg(v)$ = number of edges incident with vertex v ($v \in \mathbf{V}$) of $\mathcal{G}$, with self loops counted twice. Degree of graph $\mathcal{G}$: $\deg(\mathcal{G})$.
  - Property:: Degree of isolated vertex = zero degree.
  - Property:: Pendant vertex (also end vertex): vertex of degree one.
  - Property:: Regular graph: undirected graph with all vertices of equal degree r.
  - Property:: n-regular graph: undirected graph with all vertices of same degree n.

Graph

- Graph fundamentals:
  - Property:: Even graph: undirected graph with all vertices of even degree.
  - Property:: Two adjacent edges in series: if their common vertex (on which both edges incident on) having degree two.
  - Property:: Degree sequence of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: sequence of degrees of vertices of $\mathcal{G}$ in nonincreasing order, i.e., $\ldots, \deg(v_i), \deg(v_j), \ldots$, where $\deg(v_i) \geq \deg(v_j)$, $v_i, v_j \in \mathbf{V}$, $1 \leq i, j \leq |\mathbf{V}|$.

Graph

- Graph fundamentals:
  - Property:: Minimum degree among vertices in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\delta(\mathcal{G}) = \min\{\deg(v) \mid v \in \mathbf{V}\}$.
  - Property:: Maximum degree among vertices in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\triangle(\mathcal{G}) = \max\{\deg(v) \mid v \in \mathbf{V}\}$.
  - Property:: For regular graph $\mathcal{G}$, $\delta(\mathcal{G}) = \triangle(\mathcal{G}) = r = \deg(\mathcal{G})$.

Graph

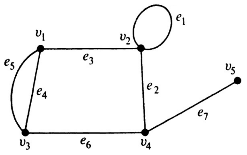
- Graph fundamentals:
  - Property:: Independent set of vertices (or simply, independent set) in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: set of non-adjacent vertices of $\mathcal{G}$; also called internally stable set, or coclique or anticlique.
  - Property:: Isolated vertex = independent set.
  - Property:: Maximal independent set of vertices (or simply, maximal independent set) in undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: independent set of $\mathcal{G}$ to which adding at least one more vertex of $\mathcal{G}$ destroyed its independence property; also called maximal internally stable set.
  - Property:: Multiple maximal independent sets of graph $\mathcal{G}$ possible.

Graph

- Graph fundamentals:
  - Property:: Independence number (or coefficient of internal stability) of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: denoted by $\beta(\mathcal{G})$, where $\beta(\mathcal{G}) =$ number of vertices in largest independent set of $\mathcal{G}$.

Graph

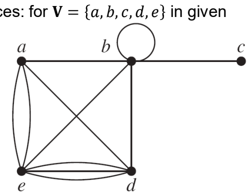
- Graph fundamentals examples:
  - Example-1:: Degrees of vertices: for $\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}$ in given graph with $\mathbf{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, degree of each $v \in \mathbf{V}$:
    $\deg(v_1) = 3$; $\deg(v_2) = 4$;
    $\deg(v_3) = 3$; $\deg(v_4) = 3$; $\deg(v_5) = 1$.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph

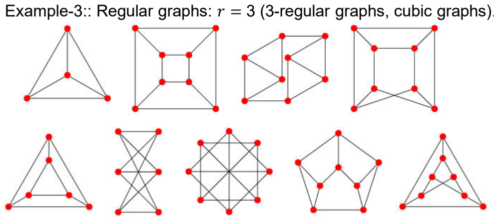
- Graph fundamentals examples:
  Example-2:: Neighborhoods of vertices: for $\mathbf{V} = \{a, b, c, d, e\}$ in given graph, neighborhood of each $v \in \mathbf{V}$:
  $N(a) = \{b, d, e\}$; $N(b) = \{a, b, c, d, e\}$;
  $N(c) = \{b\}$; $N(d) = \{a, b, e\}$;
  $N(e) = \{a, b, d\}$.
  Also, $\deg(a) = 4$; $\deg(b) = 6$;
  $\deg(c) = 1$; $\deg(d) = 5$;
  $\deg(e) = 6$.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph

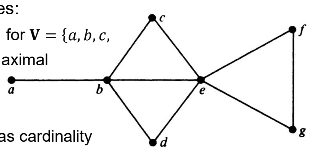
- Graph fundamentals examples:
  - Example-3:: Regular graphs: $r = 3$ (3-regular graphs, cubic graphs).



[Ref: https://mathworld.wolfram.com/CubicGraph.html.]

Graph

- Graph fundamentals examples:
  - Example-4:: Independent set: for $\mathbf{V} = \{a, b, c, d, e, f, g\}$ in given graph, all maximal independent sets: $\{a, c, d, f\}$; $\{a, c, d, g\}$; $\{b, f\}$; $\{b, g\}$; $\{a, e\}$.
    Further, β(given graph) = 4, as cardinality of largest independent sets (either $\{a, c, d, f\}$ or $\{a, c, d, g\}$) to be 4.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph

- Graph fundamental theorems:
  - Property:: ⟨Handshaking Theorem⟩: In undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V} = \{v_1, \dots, v_n\}$, $\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |\mathbf{E}|$, i.e., sum of degrees of all vertices same as twice number of edges.

  Proof: Each edge $e \in \mathbf{E}$ contributing two degrees in $\mathcal{G}$.
  Total number of edges = $|\mathbf{E}|$.
  So, all edges in $\mathbf{E}$ contributing total $(2 \cdot |\mathbf{E}|)$ degrees in $\mathcal{G}$. ①
  Again, sum of degrees of $v_1, \dots, v_n$ in $\mathcal{G}$, i.e., $\sum_{i=1}^{n} \deg(v_i)$ also contributing same value. ②
  Combining ① and ②, $\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |\mathbf{E}|$. ■

Graph

- Graph fundamental theorems:
  - Property:: ⟨Theorem⟩: Number of vertices of odd degree in undirected graph to be always even.

  Proof: From previous theorem, $\sum_{i=1}^{n} \deg(v_i) = 2 \cdot |\mathbf{E}|$ = even number.
  Considering vertices v_j with odd degrees and vertices v_k with even degrees separately, $\sum_{i=1}^{n} \deg(v_i) = \sum_{\text{odd}} \deg(v_j) + \sum_{\text{even}} \deg(v_k)$.
  L.H.S. being even, and second expression on R.H.S. being even (due to sum of even numbers), $\sum_{\text{odd}} \deg(v_j)$ = even number.
  Each $\deg(v_j)$ being odd, total number of terms in $\sum_{\text{odd}} \deg(v_j)$ = even, i.e. number of vertices of odd degree to be always even. ■

Graph

- Graph fundamental theorems:
  - Property:: ⟨Theorem⟩: In simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $|\mathbf{V}| = n$, maximum value of $|\mathbf{E}| = \frac{1}{2} \cdot n \cdot (n-1)$, i.e., maximum number of edges in simple graph with n vertices to be $\frac{1}{2} \cdot n \cdot (n-1)$.
  - Property:: ⟨Alternate theorem⟩: In complete graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $|\mathbf{V}| = n$, $|\mathbf{E}| = \frac{1}{2} \cdot n \cdot (n-1)$. [Notation of n-vertex complete graph: K_n]

  Proof: Simple n-vertex graph with maximum edges = Complete graph.
  For any vertex $v \in \mathbf{V}$ in $\mathcal{G}$ or K_n, $\max(\deg(v)) = (n-1)$.
  So, maximum sum of degrees in $\mathcal{G}$ or K_n: $n \cdot (n-1) = 2 \cdot |\mathbf{E}|$.
  Then, maximum value of $|\mathbf{E}|$ in $\mathcal{G}$ (or $|\mathbf{E}|$ in K_n): $\frac{1}{2} \cdot n \cdot (n-1)$. ■

Graph containment

- Graph containment: one graph being contained in (or, part of) another graph.
  - Property:: Subgraph: undirected graph $g = (\mathbf{v}, \mathbf{e})$ to be subgraph of another undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, if and only if —
    (i) $\mathbf{v} \subseteq \mathbf{V}$, and
    (ii) $(\mathbf{e} \subseteq \mathbf{E}) \wedge \big((\{u,v\} \in \mathbf{e}) \rightarrow ((\{u,v\} \in \mathbf{E}) \wedge (u, v \in \mathbf{V}))\big)$.
  - Property:: Notation of subgraph: $g \subset \mathcal{G}$. [Ref: Deo74.]
  - Property:: Every graph to be its own subgraph.
  - Property:: Subgraph of subgraph of $\mathcal{G}$ = subgraph of $\mathcal{G}$.
  - Property:: Single vertex in $\mathcal{G}$ = subgraph of $\mathcal{G}$.

Graph containment

- Graph containment:
  - Property:: Single edge in $\mathcal{G}$ (with its end vertices) = subgraph of $\mathcal{G}$.
  - Property:: $(\mathcal{G} - v) \subset \mathcal{G}$, where undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and $v \in \mathbf{V}$, obtained by deleting v from $\mathcal{G}$ and deleting all corresponding edges in $\mathcal{G}$ containing v.
  - Property:: $(\mathcal{G} - e) \subset \mathcal{G}$, where undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and $e \in \mathbf{E}$, obtained by deleting e from $\mathcal{G}$.
  - Property:: Spanning subgraph of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: graph $g = (\mathbf{V}, \mathbf{e})$ with same vertex set of $\mathcal{G}$, so that $(g \subset \mathcal{G})$, i.e., subgraph of $\mathcal{G}$ containing every vertex of $\mathcal{G}$.

Graph containment

- Graph containment:
  - Property:: Edge-disjoint subgraph: undirected graphs $\mathcal{g} = (\mathbf{v}, \mathbf{e})$ and $\mathcal{g}' = (\mathbf{v}', \mathbf{e}')$ to be edge-disjoint subgraphs of undirected graph $\mathcal{G}$, if —
    (i) $(\mathcal{g} \subset \mathcal{G}) \wedge (\mathcal{g}' \subset \mathcal{G})$, and
    (ii) $\mathbf{e} \cap \mathbf{e}' = \emptyset$.
  - Property:: Possibility of $\mathbf{v} \cap \mathbf{v}' \neq \emptyset$ between edge-disjoint graphs $\mathcal{g}$ and $\mathcal{g}'$.

Graph containment

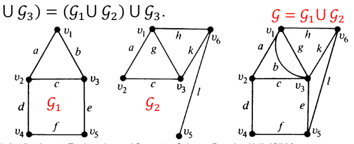
- Graph containment:
  - Property:: Vertex-disjoint subgraph: undirected graphs $\mathcal{g} = (\mathbf{v}, \mathbf{e})$ and $\mathcal{g}' = (\mathbf{v}', \mathbf{e}')$ to be vertex-disjoint subgraphs of undirected graph $\mathcal{G}$, if —
    (i) $(\mathcal{g} \subset \mathcal{G}) \wedge (\mathcal{g}' \subset \mathcal{G})$, and
    (ii) $\mathbf{v} \cap \mathbf{v}' = \emptyset$.
  - Property:: Vertex-disjoint graphs $\mathcal{g}$ and $\mathcal{g}'$ to also satisfy $\mathbf{e} \cap \mathbf{e}' = \emptyset$.

Graph containment

- Graph containment:
  - Property:: Induced subgraph: undirected graph $\mathcal{g} = (\mathbf{v}, \mathbf{e})$ to be induced subgraph of undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, induced by $\mathbf{v}$ of $\mathcal{g}$, if —
    (i) $\mathbf{v} \subseteq \mathbf{V}$ and $\mathbf{e} \subseteq \mathbf{E}$,
    (ii) $\forall_{u,v \in \mathbf{v}} \left(\exists_{e \in \mathbf{E}} \left(\left((e = \{u, v\}) \wedge (\{u, v\} \notin \mathbf{e}) \right) \to (e \in \mathbf{e}) \right) \right)$.

Graph operations

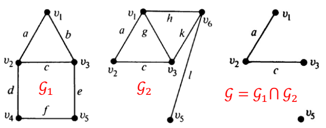
- Graph operation: Property of combined graph, to be derived from properties of combining graphs.
  - Union property:: Union of two graphs $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$: $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$, such that $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$ and $\mathbf{E} = \mathbf{E}_1 \cup \mathbf{E}_2$.
  - Property:: $\mathcal{G} = \mathcal{G}_1 \cup (\mathcal{G}_2 \cup \mathcal{G}_3) = (\mathcal{G}_1 \cup \mathcal{G}_2) \cup \mathcal{G}_3$.
  - Property:: $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$
    $= \mathcal{G}_2 \cup \mathcal{G}_1$.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Intersection property:: Intersection of two graphs $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$: $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$, such that $\mathbf{V} = \mathbf{V}_1 \cap \mathbf{V}_2$ and $\mathbf{E} = \mathbf{E}_1 \cap \mathbf{E}_2$.
  - Property:: Intersection of two graphs = graph consisting only of those vertices and edges present in both given graphs.



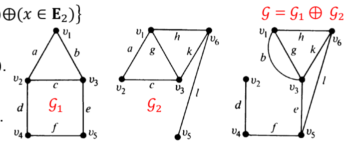
[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Property:: $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2 = \mathcal{G}_2 \cap \mathcal{G}_1$.

Graph operations

- Graph operation:
  - Ring sum property:: Ring sum [also called, disjoint union or vertex-disjoint union] of two graphs $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$ and $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$: $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$, such that $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$ and $\mathbf{E} = \mathbf{E}_1 \,\Delta\, \mathbf{E}_2 = \mathbf{E}_1 \oplus \mathbf{E}_2 = \{x \mid (x \in \mathbf{E}_1) \oplus (x \in \mathbf{E}_2)\}$ $= (\mathbf{E}_1 \setminus \mathbf{E}_2) \cup (\mathbf{E}_2 \setminus \mathbf{E}_1)$ $= (\mathbf{E}_1 \cup \mathbf{E}_2) \setminus (\mathbf{E}_1 \cap \mathbf{E}_2)$.
  - Property:: $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ $= \mathcal{G}_2 \oplus \mathcal{G}_1$.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Property:: For edge-disjoint undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ —
    (i) $\mathcal{G}_1 \cup \mathcal{G}_2 = \mathcal{G}_1 \oplus \mathcal{G}_2$.
    (ii) $\mathcal{G}_1 \cap \mathcal{G}_2$ = null graph.
  - Property:: For vertex-disjoint undirected graphs $\mathcal{G}_1'$ and $\mathcal{G}_2'$ —
    (i) $\mathcal{G}_1' \cap \mathcal{G}_2'$ = null graph.
  - Property:: For any undirected graph $\mathcal{G}$ —
    (i) $\mathcal{G} \cup \mathcal{G} = \mathcal{G} \cap \mathcal{G} = \mathcal{G}$.
    (ii) $\mathcal{G} \oplus \mathcal{G}$ = null graph.

Graph operations

- Graph operation:
  - Property:: Complement of subgraph $\mathscr{g} = (\mathbf{v}, \mathbf{e})$, where $\mathscr{g} \subseteq \mathcal{G} = (\mathbf{V}, \mathbf{E})$: denoted by $(\mathcal{G} \oplus \mathscr{g})$ or $(\mathcal{G} \setminus \mathscr{g})$, where $(\mathcal{G} \oplus \mathscr{g}) = (\mathbf{v}, \bar{\mathbf{e}})$, $(\mathcal{G} \oplus \mathscr{g}) \subseteq \mathcal{G}$, such that $\bar{\mathbf{e}} = \mathbf{E} \setminus \mathbf{e} = \{e \in \mathbf{E} \mid e \notin \mathbf{e}\}$.
  - Property:: Complement of subgraph of given graph = another subgraph of given graph, remaining after all edges in specified subgraph being removed from given graph.

Graph operations

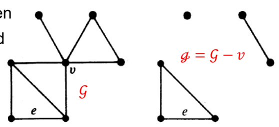
- Graph operation:
  - Decomposition property:: Decomposition of graph $\mathcal{G}$ into two subgraphs $g_1 \subseteq \mathcal{G}$ and $g_2 \subseteq \mathcal{G}$: possible, if —
    (i) $g_1 \cup g_2 = \mathcal{G}$, and
    (ii) $g_1 \cap g_2 =$ null graph.
  - Property:: Every edge of $\mathcal{G}$ to be present after decomposition either in g_1 or in g_2, but not in both. However, possibility of common vertices (of $\mathcal{G}$) in g_1 and g_2.

Graph operations

- Graph operation:
  - Property:: ⟨Theorem⟩: A graph containing m edges $\{e_1 . e_2, \ldots, e_m\}$ to be decomposed in total $2^{m-1} - 1$ different ways into pairs of subgraphs g_1, g_2.

Graph operations

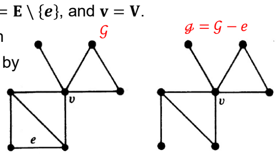
- Graph operation:
  - Property:: Vertex deletion of $v \in \mathbf{V}$ from $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: subgraph $g = \mathcal{G} - v$, where $g = (\mathbf{v}, \mathbf{e})$, such that $\mathbf{v} = \mathbf{V} \setminus \{v\}$, and $\mathbf{e} = \{e \in \mathbf{E} \,|\, (e = \{u_i, u_j\}) \wedge (u_i, u_j \in \mathbf{v})\} = \mathbf{E} \setminus \{\{v, u\} \in \mathbf{E} \,|\, u \in \mathbf{V}\}$.
  - Property:: Vertex deletion of given vertex from given graph obtained by removing said vertex and removing all edges incident on that vertex from said graph.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Property:: Edge deletion of $e \in \mathbf{E}$ from $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: subgraph $g = \mathcal{G} - e$, where $g = (\mathbf{v}, \mathbf{e})$, such that $\mathbf{e} = \mathbf{E} \setminus \{e\}$, and $\mathbf{v} = \mathbf{V}$.
  - Property:: Edge deletion of given edge from given graph obtained by removing said edge only from said graph (and not removing end vertices of that edge).

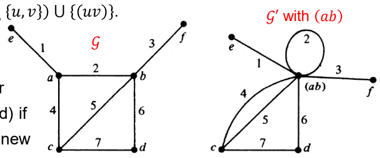


  - Property:: $\mathcal{G} - e = \mathcal{G} \oplus e$.

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Property:: Fusion of $u, v \in \mathbf{V}$ in $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: modified graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E})$ such that $\mathbf{V}' = (\mathbf{V} \setminus \{u, v\}) \cup \{(uv)\}$.
  - Property:: Two vertices in graph said to be fused (or merged or identified) if replaced by single new vertex with their every incident edge to be incident on new vertex.



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph operations

- Graph operation:
  - Property:: Fusion of two vertices in graph not to alter number of edges, but to reduce number of vertices by one.

Summary

- Focus: Graph fundamentals and operations.
- Graph neighborhood and vertex/edge adjacency.
- Degree of vertex in graph, with examples.
- Neighborhood, and independent sets, with examples.
- Regular graph, with examples.
- Graph related theorems, with proofs.
- Graph containment and subgraph, with properties.
- Edge-disjoint subgraphs, vertex-disjoint subgraphs.
- Graph operations.
- Union, intersection and ring sum of two graphs.

Summary

- Properties of graph operations.
- Complement of subgraph, and properties.
- Decomposition of graphs into two subgraphs.
- Vertex deletion, edge deletion from graph, with examples.
- Fusion of two vertices in graph, with examples.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.