

UNIT – III, Module – 1

Lecture 27: Graphs & Trees

[Tree, forest; Minimally connected graph; Path uniqueness, edge set cardinality in tree; Distance, center, radius, diameter; Cayley's theorem]

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Notation table

Symbol / Notation	Meaning
$\mathcal{T} = (V, E)$	Tree, with V set of vertices and E set of edges.
P, C	Representations of path and circuit in undirected tree.
$d(u, v)$	Distance between vertices u, v in undirected tree \mathcal{T} .
$center(\mathcal{T})$	Center of undirected tree \mathcal{T} .
$rad(\mathcal{T})$	Radius of undirected tree \mathcal{T} .
$diam(\mathcal{T})$	Diameter of undirected tree \mathcal{T} .

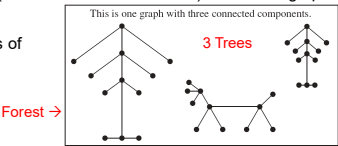
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2

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Tree

- Tree: connected undirected graph without circuits, i.e., acyclic connected undirected graph. [Rigorous definition in slide 16.]
- Property:: Tree \rightarrow simple graph (i.e., no self loop, no parallel edges).
- Property:: **Forest**: any (connected or disconnected) undirected graph without circuits.
- Property:: Components of forest \rightarrow trees.



[Ref: Kenneth H. Rosen, Discrete Mathematics and its Applications, Eighth edition, McGraw-Hill Education, 2019.]

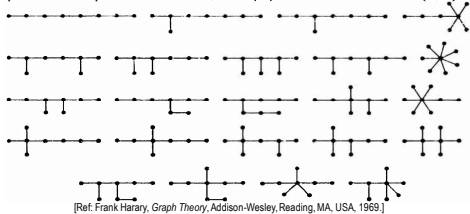
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3

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Tree

- Tree:
 - Special example:: all 23 trees, with $|V| = 8$ vertices in $\mathcal{T} = (V, E)$.



[Ref: Frank Harary, Graph Theory, Addison-Wesley, Reading, MA, USA, 1969.]

Tree

- Tree:
 - Property:: **Tree** \rightarrow minimally connected graph.
 [Minimally connected graph: if removal of any one edge from connected graph disconnecting it, i.e., connected graph with minimum number of edges and no circuits.]

Tree

- Tree fundamentals:
 - Property:: (**Theorem**): One and only one path (i.e., unique path) to be present between every pair of vertices in tree $\mathcal{T} = (V, E)$.

Proof: From definition of tree, \mathcal{T} to be taken as connected graph.

So, for arbitrary pair of vertices $u, v \in V$, at least one path to exist between u and v , i.e., $\exists P(u \in P) \wedge (v \in P)$, where P = path in \mathcal{T} .

To prove: $(\mathcal{T} \text{ to be tree}) \rightarrow (\text{one and only one path between } u \text{ and } v)$.

Proof by contraposition.

Let, $\neg r = \neg(\text{one and only one path between } u \text{ and } v) = \text{more than one path (say, two paths } P_1, P_2 \text{ in } \mathcal{T} \text{ between } u \text{ and } v)$.

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Tree

- Tree fundamentals:
Proof (contd.):
So, $\exists P_1(u \in P_1) \wedge (v \in P_1)$ and $\exists P_2(u \in P_2) \wedge (v \in P_2)$.
Then, union of $P_1, P_2 = P_1 \cup P_2$ also to be present in \mathcal{T} .
In $P_1 \cup P_2$, if traversal started from vertex u through vertices and edges in P_1 , vertex v reached, and still another path P_2 present to return back to u by traversing through vertices and edges in P_2 .
So, $P_1 \cup P_2 = \{u, \text{vertex-edge sequence from } u \text{ up to } v \text{ in } P_1, v, \text{vertex-edge sequence from } v \text{ up to } u \text{ in } P_2, u\}$, resulting in closed trail in \mathcal{T} with no vertex repetition.

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Tree

- Tree fundamentals:
Proof (contd-2.):
 $\therefore \mathcal{T}$ to contain at least one circuit.
So, \mathcal{T} not possible to become tree $= \neg(\mathcal{T} \text{ to be tree}) = \neg q$ (say).
Thus, $\neg r \rightarrow \neg q$, and so $q \rightarrow r$ (by definition of contrapositive). ■

Tree

- Tree fundamentals:
 - Property:: (**Theorem**): If one and only one path (i.e., unique path) between every pair of vertices in graph $\mathcal{G} = (V, E)$, then \mathcal{G} to become tree.
- Proof: Given, unique path between any two vertices of graph \mathcal{G} , i.e., for arbitrary pair of vertices $u, v \in V$, one and only one path to exist between u and v , i.e., $\exists P(u \in P) \wedge (v \in P)$, where $P = \text{path in } \mathcal{G}$.
Then, \mathcal{G} must be connected graph.
To prove: (One and only one path between u and v) \rightarrow (no circuit between u and v), for $|V| \geq 2$.

(contd. to next slide)

Tree

- Tree fundamentals:
Proof (contd.):
Proof by contraposition.
Let, $\neg r = \neg(\text{no circuit between } u \text{ and } v) = \text{circuit } C \text{ existing between } u \text{ and } v$.
Let $C = \{u, \text{vertex sequence from } u \text{ up to } v, v, \text{vertex sequence from } v \text{ up to } u, u\}$ (closed trail with no vertex repeat).
Then, C to be possibly split into two separate vertex sequences, viz.
 $\{u, \text{vertex sequence from } u \text{ up to } v, v\} = P_1$ (say), and
 $\{v, \text{vertex sequence from } v \text{ up to } u, u\} = P_2$ (say).
(contd. to next slide)

Tree

- Tree fundamentals:
Proof (contd-2.):
Clearly, $(u \in P_1) \wedge (v \in P_1)$ and $(u \in P_2) \wedge (v \in P_2)$.
Thus, both $P_1, P_2 =$ open trails between u and v with no vertex repeat.
 \therefore Two paths P_1, P_2 in G between u and $v = \neg(\text{one and only one path between } u \text{ and } v) = \neg q$ (say).
Thus, $\neg r \rightarrow \neg q$, and so $q \rightarrow r$ (by definition of contrapositive).
Accordingly, G to become acyclic connected graph.
Hence G to become tree. ■

Tree

- Tree fundamentals:
 - Property:: (Theorem): A tree $T = (V, E)$ with $|V| = n$ ($n \in \mathbb{Z}^+$) vertices to contain $|E| = n - 1$ edges.
- Proof: Proof by mathematical induction.
(Basis step) When $n = 1$, isolated vertex in T with no edge $= n - 1$ edge. Then, theorem to become true for $n = 1$.
(Inductive step)
Inductive hypothesis: premise that theorem to become true for every tree with fewer than n vertices (including $n - 1$ vertices).
Then, in $T = (V, E)$ with $|V| = n$, let $e = \{u, v\}$, $e \in E$, $u, v \in V$.
(contd. to next slide)

Tree

- Tree fundamentals:

Proof (contd.):

According to previous theorem, due to one and one path between u and v in \mathcal{T} , except e , no other path present.

So, deletion of e from \mathcal{T} resulting

in \mathcal{T} getting disconnect, i.e., $\mathcal{T} - e$

to become disconnected graph,

with exactly two partitions \mathcal{T}_1 and \mathcal{T}_2 , due to acyclic nature of \mathcal{T} , resulting in two components, viz. $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T} - e$, $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T} - e$.

Also, no circuit in \mathcal{T}_1 and \mathcal{T}_2 as no circuit in \mathcal{T} .

[Ref: Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall, 1974.] (contd. to next slide)

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13

Tree

- Tree fundamentals:

Proof (contd-2.):

So, $\mathcal{T}_1 = (V_1, E_1)$ and $\mathcal{T}_2 = (V_2, E_2)$ to become tree.

By induction hypothesis, theorem to hold for both \mathcal{T}_1 and \mathcal{T}_2 due to containing fewer than n vertices each, and therefore, each to contain one less edge than number of vertices in it.

Considering $|V_1| = n_1$, $|V_2| = n_2$, then $n_1 + n_2 = n$, $|E_1| = n_1 - 1$,

$|E_2| = n_2 - 1$.

Therefore, number of edges in $\mathcal{T} - e = n_1 - 1 + n_2 - 1 = n - 2$.

So, adding back e to $\mathcal{T} - e$ to get back given \mathcal{T} , with $|E| = n - 1$. ■

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14

Tree

- Tree fundamentals:

• Property: (Theorem): For any undirected graph $\mathcal{T} = (V, E)$ with $|V| = n$ ($n \in \mathbb{Z}^+$) vertices, satisfying any two of following conditions also implying third condition, and establishing \mathcal{T} as tree.

(i) \mathcal{T} to be connected;

(ii) \mathcal{T} to have no circuits;

(iii) \mathcal{T} to contain $|E| = n - 1$ edges.

Proof: Proof based on previous theorems.

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15

Tree

- Tree fundamentals:
 - (Five different but equivalent definitions):: Tree: undirected graph $\mathcal{T} = (V, E)$ with $|V| = n$ ($n \in \mathbb{Z}^+$) vertices holding any of following conditions —
 - (i) \mathcal{T} to be connected and circuitless (or cycle-free), or
 - (ii) \mathcal{T} to be connected and to contain $|E| = n - 1$ edges, or
 - (iii) \mathcal{T} to be circuitless and to contain $|E| = n - 1$ edges, or
 - (iv) exactly one unique path between every pair of vertices in \mathcal{T} , or
 - (v) \mathcal{T} to be minimally connected graph.

Tree

- Tree fundamentals:
 - Property:: (Theorem): A connected undirected graph $\mathcal{T} = (V, E)$ to become tree, if and only if adding a new edge e between any two vertices $u, v \in V$ in \mathcal{T} resulting in exactly one circuit in \mathcal{T} .

Tree

- Tree fundamentals:
 - Property:: (Theorem): In any tree $\mathcal{T} = (V, E)$ with two or more vertices (i.e., $|V| = n, n \geq 2$) to contain at least two pendant vertices.
- Proof: Proof by contradiction.
- From previous theorem, \mathcal{T} with $|V| = n$ ($n \geq 2$) vertices, where $V = \{v_1, \dots, v_n\}$, to contain $|E| = n - 1$ edges.
- Then, from handshaking theorem, sum of degrees of all vertices in $\mathcal{T} = \sum_{i=1}^n \deg(v_i) = 2 \cdot |E| = 2 \cdot (n - 1)$.
- For connected \mathcal{T} , no vertex v_i possible with zero degree.

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Tree

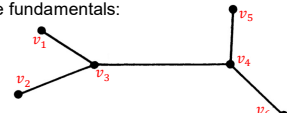
- Tree fundamentals:
Proof (contd.):
Premise: less than two vertices of degree one (i.e., less than two pendant vertices) present in \mathcal{T} .
Consider v to be only pendant vertex in \mathcal{T} .
Then, sum of degrees of all vertices in \mathcal{T}
 $= \deg(v) + \sum_{v_i \in V \setminus \{v\}} \deg(v_i) = 1 + 2 \cdot (n - 1)$, contradicting
handshaking theorem.
Hence, at least two pendant vertices present in \mathcal{T} . ■

Tree

- Tree fundamentals:
 - Property: Distance between any pair of vertices u, v in undirected tree $\mathcal{T} = (V, E)$: $d(u, v)$ = length of **unique path** between u, v (i.e., number of edges in that path) in \mathcal{T} .
 $d(v_1, v_2) = 1$
 $d(v_1, v_3) = 2$
 \vdots
 - Property: Center of undirected tree $\mathcal{T} = (V, E)$:
 $center(\mathcal{T}) = \{u \in V \mid E(u) = \min\{E(v) \mid \forall v \in V\}\}$,
where $E(v) = \max\{d(v, v_i) \mid v_i \in V, i = 1, 2, \dots, |V|\}$.
 $E(v_1) = E(v_3) = E(v_4) = 2$
 $E(v_2) = 1$
 $center(\mathcal{T}) = \{v_2\}$
 - Property: Two centers possible in some tree.

[Ref: Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall, 1974.]

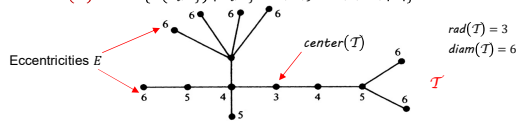
Tree

- Tree fundamentals:

 $E(v_1) = 3$
 $E(v_2) = 3$
 $E(v_3) = 2$
 $E(v_4) = 2$
 $E(v_5) = 3$
 $E(v_6) = 3$
 $center(\mathcal{T}) = \{v_3, v_4\}$
- Property: (Theorem): One or two in every undirected tree $\mathcal{T} = (V, E)$, i.e., $|center(\mathcal{T})| = 1$ or $|center(\mathcal{T})| = 2$.
- Property: (Corollary): For any undirected tree $\mathcal{T} = (V, E)$ with two centers, i.e., $|center(\mathcal{T})| = 2$, then two centers to be neighbors in \mathcal{T} .

[Ref: Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall, 1974.]

Tree

- Tree fundamentals:
 - Property:: **Radius** of undirected tree $T = (V, E)$:
 $rad(T) = E(v)$, where $v \in center(T)$.
 - Property:: **Diameter** of undirected tree $T = (V, E)$:
 $diam(T) = \max\{d(v_i, v_j) \mid v_i, v_j \in V, i, j = 1, 2, \dots, |V|\}$.



[Ref: Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science, Prentice-Hall, 1974.]

Tree

- Tree fundamentals:
 - Property:: **Labeled undirected tree**: with unique name or label assigned to each vertex.
 - Property:: **Unlabeled undirected tree**: no assigned vertex distinctions.
 - Property:: (**Cayley's Theorem**): Number of labeled trees to be equal to n^{n-2} , where each labeled tree $T = (V, E)$ to have two or more vertices (i.e., $|V| = n, n \geq 2$),.
[10 different proofs of Cayley's Theorem available.]

Summary

- Focus: Tree fundamentals.
- Tree, forest.
- Minimally connected graph, and other properties of trees.
- Path uniqueness in tree, and related theorems.
- Edge set cardinality in tree, and related theorems.
- Rigorous definition of tree.
- Distance between vertices, eccentricity of vertex in tree.
- Center, radius, diameter of tree.
- Labeled, unlabeled tree.
- Cayley's theorem.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.

2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.

4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.

5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.
