CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 2

Lecture 04: Predicate Logic

[Predicate logic; Predicate; Quantification; Interpretation; Universal, existential, uniqueness quantifiers; Domain of discourse]

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Predicate logic

- Predicate logic: mathematical logic for reasoning with predicates.
  - Also called first-order logic, quantificational logic, first-order predicate calculus.
  - Why called "first-order": propositional variables as arguments of predicates, and quantification over propositional variables.
  - Difference from propositional logic: in predicate logic, no definite truth value in general till instantiation, unlike propositional logic, in which always definite truth value.
  - Difference from higher-order logic: predicates themselves as arguments of other predicates, and quantification over predicates.

Predicate logic

- Predicate logic:
  - Two important aspect: predicate, quantifier.

Predicate

- Predicate: statement of $P(x_1, x_2, \ldots, x_n)$, where $P =$ propositional function involving n propositional variables $x_1, x_2, \ldots, x_n$, with truth value of predicate being value of P at n-tuple $\langle x_1, x_2, \ldots, x_n \rangle$.
  - Predicate with n-tuple: n-place predicate or n-ary predicate.
  - Propositional function: also called open sentence or condition.
  - $P(x_1, x_2, \ldots, x_n)$ with values assigned to variables $x_1, x_2, \ldots, x_n$ → proposition with certain truth value.
  - Truth set of $P(x_1, x_2, \ldots, x_n)$: $\{(a_1, a_2, \ldots, a_n) \mid (a_1, a_2, \ldots, a_n) \in \prod_{i=}^{n} A_i$, $P(a_1, a_2, \ldots, a_n)$ become TRUE$\}$, i.e., set of all elements of Cartesian product $\prod_{i=}^{n} A_i$ of n $a_i \in A_i$ values, with $P(a_1, a_2, \ldots, a_n)$ to be TRUE.

Predicate

- Predicate:
  - Truth set of $P(x)$ to contain —
    (i) all $a \in A$,
    (ii) some $a \in A$,
    (iii) no $a \in A$.

Predicate

- Predicate examples:
  - Example-1:: let $P(x)$ denoting statement "$x > 3$".
    Instantiating x by 4, $P(4) \equiv 4 > 3$. So, truth value of $P(4) =$ T.
    Instantiating x by 2, $P(2) \equiv 2 > 3$. So, truth value of $P(2) =$ F.
  - Example-2:: let $Q(x, y)$ denoting statement "$x = y + 3$".
    Instantiating $x = 1, y = 2$, $Q(1,2) \equiv 1 = 2 + 3$.
    So, truth value $Q(1,2) =$ F.
    Instantiating $x = 3, y = 0$, $Q(3,0) \equiv 3 = 0 + 3$.
    So, truth value $Q(3,0) =$ T.

Predicate

- Predicate examples:
  - Example-3:: let $R(x, y, z)$ denoting statement "$x + y = z$".
    Instantiating $x = 1, y = 2, z = 3$, $R(1,2,3) \equiv 1 + 2 = 3$.
    So, truth value $R(1,2,3) =$ T.
    Instantiating $x = 0, y = 0, z = 1$, $R(0,0,1) \equiv 0 + 0 = 1$.
    So, truth value $R(0,0,1) =$ F.

Precondition and Postcondition

- (i) Precondition and (ii) Postcondition: predicate statements to describe — (i) valid input, and
  (ii) condition to be satisfied by program output after execution.
  - Purpose: in order to show computer programs always producing desired output for given valid input.

Precondition and Postcondition

- Precondition and postcondition examples:
  - Example-1:: program fragment to interchange values x and y —
    temp := x $x := y$ $y :=$ temp
    Precondition: predicate, to express x and y with particular values, $P(x, y) \equiv x = a$ and $y = b$, with a and b being values of x and y before program execution.
    Postcondition: predicate (corresponding to given precondition), to express swapped values of x and y, $Q(x, y) \equiv x = b$ and $y = a$, with b and a being values of x and y after program execution.

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Precondition and Postcondition

- Precondition and postcondition examples:
  - Example-1(contd.):: verifying correctness of program.
    To verify correctness of program fragment using these predicates.
    Suppose $P(x, y)$ holds, i.e., "$x = a$ and $y = b$" ≡ T. So, $x = a$, $y = b$.
    After 1st step (i.e., assigning value of x to variable temp),
    $x = a$, temp $= a$, $y = b$.
    After 2nd step (i.e., assigning value of y to x), $x = b$, temp $= a$, $y = b$.
    Finally, after 3rd step, $x = b$, temp $= a$, $y = a$.
    So, after program fragment execution, $Q(x, y)$ holds.

Quantification

- Quantification: operation to specify count of elements in domain of discourse satisfying propositional function in predicate.
  - Quantification: extent to which predicate to become TRUE over range of elements.
  - Propositional function $\xrightarrow[\text{Quantification}]{}$ Proposition.
  - Domain of discourse (or universe of discourse, or universe, or domain): set of entities over which quantifiers of variables of interest span for instantiation of first-order formula.
  - Variable $\xrightarrow[\text{Quantifier}]{}$ Quantified variable.

Quantification

- Quantification:
  - Scope of quantification: part of logical expression involving one or more propositional function(s) to apply quantification.
  - Model (or interpretation) of predicate: elements of domain of discourse, with all such elements instantiating given predicate to satisfy with truth value TRUE. So, model → interpretable.
  - Co-model of predicate: elements of domain of discourse, with all such elements instantiating given predicate to satisfy with truth value FALSE.

Quantification

- Quantification:
  - Three types of quantification in predicate logic:
    (i) universal quantification – predicate TRUE for every element in specified domain of discourse;
    (ii) existential quantification: predicate TRUE for one or more element in specified domain of discourse;
    (ii) uniqueness quantification: predicate TRUE for only one unique element in specified domain of discourse.

Quantification: Universal quantification

- Universal quantification of $P(x)$: statement "$\forall x\ P(x)$" (read: "$P(x)$ for all values of x in specified domain of discourse", or "for all x $P(x)$", or "for every x $P(x)$"), to produce TRUE as truth value of $P(x)$ for every element x in non-empty domain of discourse.
  - Universal quantification = proposition asserting $P(x)$ TRUE for all values of x in domain of discourse.
  - '∀' → universal quantifier.
  - ➲ Note: truth values of universal quantification of $P(x)$ to change with change of domain of discourse.
  - ➲ Note: no domain of discourse → universal quantification not defined.

Quantification: Universal quantification

- Universal quantification:
  - Truth set of $\forall x\ P(x)$: exactly same to domain of discourse.
  - For empty domain of discourse, $\forall x\ P(x)$ to become TRUE, as no element x producing truth value of $P(x)$ as FALSE.
  - Counter-example to $\forall x\ P(x)$: any element in domain of discourse for which truth value of $P(x)$ to become FALSE.
  - Universal quantification expressed as: "for all", "for every", "all of", "for each", "for arbitrary".
  - ➲ Note: "any" to be avoided due to ambiguous interpretation — "any" meaning "every" or "some".

Quantification: Universal quantification

- Universal quantification:
  - For finite domain of discourse, universal quantification statements also expressed using conjunction of propositional logic: $\forall x\, P(x)$ same as "$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$", where $x_1, x_2, \dots, x_n$ to be elements of domain of discourse for positive integer n.
    Reason: conjunction to be TRUE, if and only if all $P(x_1)$, $P(x_2)$,…, $P(x_n)$ become TRUE.

Quantification: Universal quantification

- Universal quantification examples:
  - Example-1:: consider $P(x) =$ "$x + 1 > x$", domain of discourse = set of all real numbers $= \mathbb{R}$.
    $P(x) \equiv$ T, for every $x \in \mathbb{R}$.
    That is, $\forall x\, P(x)$ to be TRUE in domain of discourse $\mathbb{R}$.
  - Example-2:: consider $Q(x) =$ "$x^2 > 0$", domain of discourse $= \mathbb{Z}$.
    $Q(x) \equiv$ F, for $x = 0$, which becoming counter-example for universal quantification.
    That is, $\forall x\, Q(x)$ to be FALSE in domain of discourse $\mathbb{Z}$.

Quantification: Universal quantification

- Universal quantification examples:
  - Example-3:: consider $\forall x\, N(x)$ to be TRUE, where $N(x) =$ "computer x is connected to campus network", and domain of discourse = set of all computers on campus.
    $\forall x\, N(x) \equiv$ "for every computer x on campus, computer x is connected to campus network".
    Corresponding English statement: "Every computer on campus is connected to campus network."

Quantification: Universal quantification

- Universal quantification examples:
  - Next example to show importance of domain of discourse.
  - Example-4:: consider $\forall x\ (x^2 \geq x)$, and two separate domains of discourse, viz. $\mathbb{R}$, $\mathbb{Z}$. To determine truth value of quantification.
    For given predicate to be TRUE, $x^2 - x \geq 0$, that is, $x(x-1) \geq 0$.
    Three cases — (i) positive case: $x > 0$ and $(x-1) > 0$, which means $x \geq 1$; (ii) negative case: $x < 0$ and $(x-1) < 0$, which means $x < 0$; (iii) zeroth case: $x = 0$. So, allowed range : $x < 0$, or $x = 0$, or $x \geq 1$.
    For $\mathbb{R}$, inequality not to hold if $0 < x < 1$. So, $\forall x\ (x^2 \geq x) \equiv$ FALSE.
    For $\mathbb{Z}$, inequality to always hold. So, $\forall x\ (x^2 \geq x) \equiv$ TRUE.

Quantification: Universal quantification

- Universal quantification examples:
  - Next two examples to show effect of finite domain of discourse.
  - Example-5:: consider $P(x) =$ "$x^2 < 10$", domain of discourse $\mathfrak{D} =$ positive integers not exceeding 3.
    So, $P(1) \equiv$ T, $P(2) \equiv$ T, $P(3) \equiv$ T. Their conjunction $\equiv$ T.
    That is, $\forall x\ P(x)$ to be TRUE in domain of discourse $\mathfrak{D}$.
  - Example-6:: consider $P(x) =$ "$x^2 < 10$", domain of discourse $\mathfrak{D}' =$ positive integers not exceeding 4.
    So, $P(1) \equiv$ T, $P(2) \equiv$ T, $P(3) \equiv$ T, $P(4) \equiv$ F. Their conjunction $\equiv$ F.
    That is, $\forall x\ P(x)$ to be FALSE in domain of discourse $\mathfrak{D}'$.

Quantification: Existential quantification

- Existential quantification of $P(x)$: statement "$\exists x\ P(x)$" (read: "There exists an element x in specified domain of discourse such that $P(x)$", or "There is an x such that $P(x)$", or "for some x $P(x)$", or "for at least one x $P(x)$"), to produce TRUE as truth value of $P(x)$ for that element x in non-empty domain of discourse.
  - Existential quantification = proposition asserting $P(x)$ TRUE for at least one value of x in domain of discourse.
  - '$\exists$' → existential quantifier.
  - ➲ Note: truth values of existential quantification of $P(x)$ to change with change of domain of discourse.

Quantification: Existential quantification

- Existential quantification:
  - For empty domain of discourse, $\exists x\, P(x)$ to become FALSE, as no element x able to produce truth value of $P(x)$ as TRUE.
  - For finite domain of discourse, universal quantification statements also expressed using disjunction of propositional logic: $\exists x\, P(x)$ same as "$P(x_1) \lor P(x_2) \lor \cdots \lor P(x_n)$", where $x_1, x_2, \ldots, x_n$ to be elements of domain of discourse for positive integer n.
    Reason: disjunction to be TRUE, if and only if at least one of $P(x_1)$, $P(x_2)$,…, $P(x_n)$ become TRUE.

➲ Note: no domain of discourse → existential quantification undefined.

Quantification: Existential quantification

- Existential quantification:
  - Truth set of $\exists x\, P(x)$: nonempty subset of domain of discourse.

Quantification: Existential quantification

- Existential quantification examples:
  - Example-1:: consider $P(x) =$ "$x > 3$", domain of discourse $= \mathbb{R}$.
    $P(x) \equiv$ T, for some real number $x \in \mathbb{R}$, like $x = 3.1$, $x = 3.01$ etc.
    That is, $\exists x\, P(x)$ to be TRUE in domain of discourse $\mathbb{R}$.
  - Example-2:: let $Q(x)$ denoting "$x = x + 1$", domain of discourse $= \mathbb{R}$.
    $Q(x) \equiv$ F, for every real number $x \in \mathbb{R}$.
    That is, $\exists x\, Q(x)$ to be FALSE in domain of discourse $\mathbb{R}$.

Quantification: Existential quantification

- Existential quantification examples :
  - Next two examples to show effect of finite domain of discourse.
  - Example-3:: consider $Q(x) =$ "$x^2 > 10$", domain of discourse $\mathfrak{D}' =$ positive integers not exceeding 4.
    So, $Q(1) \equiv$ F, $Q(2) \equiv$ F, $Q(3) \equiv$ F, $Q(4) \equiv$ T. Their disjunction $\equiv$ T.
    That is, $\exists x\, Q(x)$ to be TRUE in domain of discourse $\mathfrak{D}'$.
  - Example-4:: consider $Q(x) =$ "$x^2 > 10$", domain of discourse $\mathfrak{D} =$ positive integers not exceeding 3.
    So, $Q(1) \equiv$ F, $Q(2) \equiv$ F, $Q(3) \equiv$ F. Their disjunction $\equiv$ F.
    That is, $\exists x\, Q(x)$ to be FALSE in domain of discourse $\mathfrak{D}$.

Quantification: Universal vs. Existential quantifications

- Comparison between universal and existential quantifications:

Quantifiers

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

  - Precedence: '$\forall$' and '$\exists$' to have higher precedence than all logical operators from propositional logic.

[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Quantification: Uniqueness quantification

- Uniqueness quantification of $P(x)$: statement "$\exists!\, x\, P(x)$" or "$\exists_1 x\, P(x)$" (read: "There exists a unique element x in specified domain of discourse such that $P(x)$", or "there is exactly one x such that $P(x)$", or "there is one and only one x such that $P(x)$", to produce TRUE as truth value of $P(x)$ for that element x in non-empty domain of discourse.
  - '$\exists!$' or '$\exists_1$' → uniqueness quantifier.
  - ➲ Note: Possibility of expressing uniqueness through universal or existential quantification and propositional logic. So, uniqueness quantification commonly avoided.

Quantification: Uniqueness quantification

- Uniqueness quantification:
  - $\exists! x P(x) \equiv \exists x \left(P(x) \land \neg \left(\exists y \left(P(y) \land (y \neq x)\right)\right)\right)$
    $\equiv \exists x \left(P(x) \land \neg \left(\exists y \neg \left(P(y) \rightarrow (y = x)\right)\right)\right)$ after (I.12b)
    $\equiv \exists x \left(P(x) \land \forall y \left(P(y) \rightarrow (y = x)\right)\right)$
    $\equiv \exists x \forall y \left(P(y) \leftrightarrow (y = x)\right)$.

Quantification: Uniqueness quantification

- Uniqueness quantification examples:
  - Example-1:: consider $P(x) =$ "$x - 1 = 0$", domain of discourse $= \mathbb{R}$.
    $P(x) \equiv$ T, for only real number $x = 1$.
    That is, $\exists! x\, P(x)$ to be TRUE in domain of discourse $\mathbb{R}$.

Quantification & looping-searching

- Connection between quantification and looping:
  - Determining truth value of quantification → looping and searching through domain of discourse.
  - Showing $\forall x\, P(x)$ to be TRUE in n-element domain of discourse $\mathfrak{D}$:: looping through all n values of x (till end) to see whether $P(x)$ to be always true. [Same steps to show $\exists x\, P(x)$ to be FALSE in $\mathfrak{D}$.]
  - While looping, if any value of x resulting in $P(x)$ to be FALSE → $\forall x\, P(x)$ to be FALSE in n-element $\mathfrak{D}$.
  - In case of $\exists x\, P(x)$ to be TRUE in $\mathfrak{D}$ ← if any value of x resulting in $P(x)$ to be TRUE.

Summary

- Focus: Predicate logic.
- Predicate logic, and difference with propositional logic.
- Predicate definition, with examples.
- Precondition and postcondition, with examples.
- Quantification, domain of discourse and model/interpretation of predicate.
- Unary quantification in predicate logic.
- Universal quantification definitions, truth set, with examples.
- Existential quantification definitions, truth set, with examples.
- Comparison between universal and existential quantifiers.

Summary

- Uniqueness quantification definitions, with examples.
- Connection between quantification and looping.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Predicate logic:: [Ros21]:40.
- Predicate:: [Ros21]:40-43.
- Precondition, postcondition:: [Ros21]:43.
- Quantification:: [Ros21]:43.
- Domain of discourse:: [Ros21]:44.
- Universal quantification:: [Ros21]:44-45.
- Existential quantification:: [Ros21]:45-46.
- Uniqueness quantification:: [Ros21]:46.
- Quantification and looping:: [Ros21]:47.

Lecture Exercises: Problem 1 [Ref: Gate 2018, Q.28, p.11 (Set-3)]

Consider the first-order logic sentence:

$$\varphi \equiv \exists s \exists t \exists u \forall v \forall w \forall x \forall y\ \psi(s,t,u,v,w,x,y)$$

where $\psi(s,t,u,v,w,x,y)$ is a quantifier-free first-order logic formula using only predicate symbols, and possibly equality, but no function symbols. Suppose φ has a model with a universe containing 7 elements. Which one of the following statements is necessarily true?

(a) There exists at least one model of φ with universe of size less than or equal to 3.

(b) There exists no model of φ with universe of size less than or equal to 3.

(c) There exists no model of φ with universe of size greater than 7.

(d) Every model of φ has a universe of size equal to 7.

Lecture Exercises: Problem 1 Ans

- As per premise, predicate φ being interpreted by domain of discourse, say $\mathfrak{D}$, having 7 elements, say $\mathfrak{D} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$.
- That is, ψ to be satisfied based on one assignment of elements of $\mathfrak{D}$ to s, t, u, v, w, x, y.
- Let $\mathfrak{D}'$ be another carefully-chosen domain of discourse having $\mathfrak{D}' = \{a_i, a_j, a_k\}$, where $\mathfrak{D}'$ may or may not have any relation to $\mathfrak{D}$.
- Since $\mathfrak{D}'$ carefully chosen, and because ψ being quantifier-free, function-free with predicate symbols, and possibly equality, then ψ to be satisfied again based on one assignment of elements of $\mathfrak{D}'$ to s, t, u, v, w, x, y. (with possibly repetition). So, φ to be interpreted by $\mathfrak{D}'$ of 3 elements.

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Lecture Exercises: Problem 1 Ans (contd.)

- Continuing in this fashion, another $\mathfrak{D}''$ can be carefully-chosen domain of discourse, such that $\mathfrak{D}'' = \{b_1, b_2\}$, where $\mathfrak{D}''$ may or may not have any relation to $\mathfrak{D}$ and $\mathfrak{D}'$.
- Then also, ψ to be shown satisfied based on one assignment of elements of $\mathfrak{D}''$ to s, t, u, v, w, x, y. (with possibly repetition).
  So, φ to be interpreted by $\mathfrak{D}''$ of less than 3 elements.

- Answer: There exists at least one model of φ with universe of size less than or equal to 3.