CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 1

Lecture 03: Propositional Logic

[Duality; Satisfiability; Assertion; Valid argument; Valid argument normal form; Rules of inference; Fallacy]

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Duality in propositional logic

- Dual of given compound proposition s containing only logical operators ∨, ∧, and ¬: compound proposition s^* obtained by replacing — (i) each ∨ by ∧, (ii) each ∧ by ∨, (iii) each T by F, and (iv) each F by T.
  - Nature: For any compound proposition s, $(s^*)^* = s$.
  - Nature: $s^* = s$, when s consisting of only one propositional variable.
  - Example-1:: given: $p \vee \neg q$; dual: $p \wedge \neg q$.
  - Example-2:: given: $(p \vee \neg q) \wedge (q \vee \mathrm{T})$; dual: $(p \wedge \neg q) \vee (q \wedge \mathrm{F})$.
  - Example-3:: given: $p \vee \mathrm{F}$; dual: $p \wedge \mathrm{T}$; also, $p \vee \mathrm{F} \equiv p \wedge \mathrm{T} \equiv p$; so, $s^* = s$.

Satisfiability in propositional logic

- Satisfiability of compound proposition: in case when its truth value to become TRUE for any/all assignment of truth values to its variables (i.e., when that compound proposition to become either a tautology or a contingency).
  - Unsatisfiability of compound proposition: when its truth value to become FALSE for all assignments of truth values to its variables (i.e., when that compound proposition to become a contradiction).
  - Unsatisfiable proposition: if and only if proposition's negation to become TRUE for all assignments of truth values to its variables (i.e., unsatisfiable proposition → proposition's negation ≡ tautology).

Satisfiability in propositional logic

- Satisfiability:
  - To show satisfiability → to find a particular assignment of truth values for compound proposition to become TRUE.
    Truth value assignment = solution of specific satisfiability problem.
  - To show unsatisfiability → to show that every assignment of truth values to variables of compound proposition making it FALSE.

Satisfiability in propositional logic

- Satisfiability examples:
  - Example-1:: To determine satisfiability of given proposition —
    $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ given
    $\equiv ((p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)) \wedge ((p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r))$
    For 1st term of given proposition: $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ to be TRUE, 3 conjunctive terms (of 1st term) to be TRUE.
    Consider one possible assignment: $p = \mathrm{T}$, $q = \mathrm{T}$, $r = \mathrm{T}$.
    $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ 1st term of given proposition
    $\equiv (\mathrm{T} \vee \mathrm{F}) \wedge (\mathrm{T} \vee \mathrm{F}) \wedge (\mathrm{T} \vee \mathrm{F}) \equiv \mathrm{T}$ assignment: $p = \mathrm{T}$, $q = \mathrm{T}$, $r = \mathrm{T}$
    For any other assignments, truth value of 1st term not become TRUE.

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Satisfiability in propositional logic

- Satisfiability examples:
  - Example-1 (contd.).
    For 2nd term of given proposition: $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be TRUE, its 2 conjunctive terms to be TRUE. With same assignment,
    $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ 2nd term of given proposition
    $\equiv (\mathrm{T} \vee \mathrm{T} \vee \mathrm{T}) \wedge (\mathrm{F} \vee \mathrm{F} \vee \mathrm{F}) \equiv \mathrm{F}$ for same assignment of p, q, r
    2nd term to be TRUE only possible when at least one of p, q, r to be TRUE and at least one to be FALSE (not: $p = \mathrm{T}$, $q = \mathrm{T}$, $r = \mathrm{T}$).
    So, given proposition (i.e., ANDing of 1st and 2nd terms) to be FALSE.
    So, given proposition not tautology, i.e., unsatisfiable.

Assertion

- Assertion: precise, unambiguous, mathematical statement with claim of — (i) truth value being TRUE or FALSE (e.g., that some relation between mathematical objects holds; that some object has some property; etc.), or (ii) statement being well-formed formula with truth value being TRUE or FALSE on instantiation.
  - Classification of assertions: definite assertions, indefinite assertions.
  - Definite assertions: also called "propositions" or "statements."
  - Claim of definite assertion: TRUE or FALSE.
  - Indefinite assertions: also sometimes called "predicates" or "open sentences."

Argument in propositional logic

- Argument in propositional logic: assertion that given sequence of statements (here, propositions) $p_1, p_2, \dots, p_n$, yielding (i.e., having consequence) another proposition q; denoted by
  $p_1, p_2, \dots, p_n \vdash q$, ('$\vdash$' → sequent notation).
  - Argument: sequence of statements, starting with at least one *premise*, then leading to one/more *subconclusions* (to act as premises for succeeding statements) along with possibly some more premises, and ending with one *conclusion*.
  - Hypotheses: all statements above horizontal bar.

Premises, plus Subconclusions
Premise
Statement1
Statement2
.
.
.
∴ Conclusion

Argument in propositional logic

- Argument:
  - Valid argument (also logical argument): argument $p_1, p_2, \dots, p_n \vdash q$ to be valid if q to become TRUE whenever all $p_1, p_2, \dots, p_n$ to be TRUE.
    [Alt-1] argument to be valid if conclusion of argument following from truth of premises of argument.
    [Alt-2]: argument to be valid if and only if impossibility for all premises to be TRUE and conclusion to be FALSE.
  - ➲ Note: To show valid argument → conjunction of all premises as antecedent, implies, conclusion as consequent, to be tautology.
  - ➲ Note: FALSE q for al least one FALSE p possible in valid argument.

Argument in propositional logic

- Argument:
  - Argument form: argument, when expressed by sequence of compound propositions (of propositional variables) to represent premises and conclusion.
  - Valid argument form: argument form to be valid, irrespective of propositions substituted for propositional variables in its premises, if premises to be all TRUE then conclusion to be also TRUE.
    i.e., (Conjunction of all propositional variables of Premises) → (propositional variables of Conclusion) ≡ Tautology.

Argument in propositional logic

- Valid argument examples:
  - Example-1:: Given propositions "If you have a current password, then you can log onto the campus network," and "You have a current password."
    Rewriting, "you have a current password" → "you can log onto the campus network." Also, another premise, "you have a current password." ∴ "you can log onto the campus network."
    Structure-wise::
    Premise1: "you have a current password" → "you can log onto the campus network"
    Premise2: "you have a current password"
    ∴ Conclusion: "you can log onto the campus network"

Argument in propositional logic

- Valid argument examples:
  - Example-2:: To show example-1 as valid argument.
    In turn, to show ((("you have a current password" → "you can log onto the campus network") ∧ ("you have a current password")) → "you can log onto the campus network") ≡ Tautology.
    Changing to argument form, to show $(((p \to q) \land p) \to q) \equiv$ Tautology.
    Next truth table for showing above relation.

p	q	$p \to q$	$(p \to q) \land p$	$((p \to q) \land p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rules of Inference for propositional logic

- Rules of Inference: rules (in relatively simpler argument form) justifying logical step from premise(s) to conclusion.
  - Clause: expressing compound proposition as — (i) disjunction of propositional variables, or (ii) negations of such variables.
  - Benefit: (1) to be used as building blocks to construct more complicated and valid argument forms;
    (2) to reduce tedious tasks of validating through truth table of 2^n rows for n propositional variables.

Rules of Inference for propositional logic (w. tautology)

Modus ponens: (or law of detachment)	(I.20a) $p, p \rightarrow q \vdash q$ (Read: From p and $p \rightarrow q$, infer q)	$((p \land (p \rightarrow q)) \rightarrow q) \equiv$ T
	(I.20b) $p \rightarrow q, p \vdash q$	$(((p \rightarrow q) \land p) \rightarrow q) \equiv$ T
Modus tollens:	(I.21a) $\neg q, p \rightarrow q \vdash \neg p$	$((\neg q \land (p \rightarrow q)) \rightarrow \neg p) \equiv$ T
	(I.21b) $p \rightarrow q, \neg q \vdash \neg p$	$(((p \rightarrow q) \land \neg q) \rightarrow \neg p) \equiv$ T
Hypothetical syllogism:	(I.22) $p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r$	$(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)) \equiv$ T
Disjunctive syllogism:	(I.23) $p \lor q,\ \neg p \vdash q$	$(((p \lor q) \land \neg p) \rightarrow q) \equiv$ T
Addition:	(I.24) $p \vdash p \lor q$	$(p \rightarrow (p \lor q)) \equiv$ T

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Rules of Inference for propositional logic (w. tautology)

Simplification:	(I.25) $p \land q \vdash p$	$((p \land q) \rightarrow p) \equiv$ T
Conjunction:	(I.26) $p,\ q \vdash p \land q$	$((p \land q) \rightarrow (p \land q)) \equiv$ T
Absorption:	(I.27) $p \rightarrow q \vdash p \rightarrow (p \lor q)$	$((p \rightarrow q) \rightarrow (p \rightarrow (p \lor q))) \equiv$ T
Constructive dilemma:	(I.28) $(p \rightarrow q) \land (r \rightarrow s),\ (p \lor r) \vdash q \lor s$	$((((p \rightarrow q) \land (r \rightarrow s)) \land (p \lor r)) \rightarrow (q \lor s)) \equiv$ T
Resolution:	(I.29) $p \lor q, \neg p \lor r \vdash q \lor r$ where $q \lor r$ called resolvent	$(((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)) \equiv$ T

Rules of Inference for propositional logic (w. tautology)

- Soundness of 'Modus ponens':: proof of modus ponens being tautology —

$\big(p \land (p \to q)\big) \to q \equiv \big(p \land (\neg p \lor q)\big) \to q$ by implication rule (I.12a)
$\equiv \big((p \land \neg p) \lor (p \land q)\big) \to q$ by distributive law in (I.6b)
$\equiv \neg\big((p \land \neg p) \lor (p \land q)\big) \lor q$ by implication rule (I.12a)
$\equiv \neg\big(\mathrm{F} \lor (p \land q)\big) \lor q$ by negation law (I.8b)
$\equiv \neg(p \land q) \lor q$ by commutative & identity laws (I.5a), (I.2a)
$\equiv (\neg p \lor \neg q) \lor q$ by De Morgan's law (I.10b)
$\equiv \neg p \lor (\neg q \lor q)$ by associative law (I.4a)
$\equiv \neg p \lor \mathrm{T} \equiv \mathrm{T}$ by negation & domination laws (I.8a), (I.3a)

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-1:: [valid argument with no truth in conclusion]

    Let two propositions be — p: "$\sqrt{2} > \frac{3}{2}$," and q: "$\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$."

    Also let another premise: "If $\sqrt{2} > \frac{3}{2}$, then $\left(\sqrt{2}\right)^2 > \left(\frac{3}{2}\right)^2$," i.e., $p \to q$.

    So, valid argument form based on modus ponens in (I.20a) —

$$\begin{array}{ll} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

    However, premise p being FALSE, q to be NOT TRUE (i.e., no truth in conclusion).

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-2:: Given valid argument —

    "It is below freezing now. Therefore, it is below freezing or raining now."

    Let p be "It is below freezing now," and q be "It is raining now."

    So, argument form —

$$\begin{array}{ll} & p \quad \text{premise} \\ \hline \therefore & p \lor q \end{array}$$

    Above argument form used 'law of addition' inference rule in (I.24).

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-3:: Consider premise "It is below freezing and raining now."; and conclusion "It is below freezing now."
    Let p be "It is below freezing now," and q be "It is raining now."
    So, valid argument form to get conclusion —

$p \land q$	premise
$\therefore p$	based on law of simplification in (I.25)

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-4:: Given premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to conclusion "We will be home by sunset."
    Let p be "It is sunny this afternoon," q be "It is colder than yesterday," r be "We will go swimming," s be "We will take a canoe trip," and t be "We will be home by sunset."
    So, all given premises: $\neg p \land q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$.

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Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-4 contd.
    So, constructing valid argument form to establish conclusion —

① $\neg p \land q$	premise
② $\neg p$	simplification of ① by (I.25)
③ $r \rightarrow p$	premise
④ $\neg r$	modus tollens of ② and ③ by (I.21a)
⑤ $\neg r \rightarrow s$	premise
⑥ s	modus ponens of ④ and ⑤ by (I.20a)
⑦ $s \rightarrow t$	premise
⑧ t	modus ponens of ⑥ and ⑦ by (I.20a)

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-5:: Given premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed."

    Let p be "You send me an e-mail message," q be "I will finish writing the program," r be "I will go to sleep early," and s be "I will wake up feeling refreshed."

    So, all given premises: $p \rightarrow q$, $\neg p \rightarrow r$, $r \rightarrow s$.

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Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-5 contd.

    So, finding conclusion through valid argument form —

①	$p \rightarrow q$	premise
②	$\neg q \rightarrow \neg p$	contrapositive of ①; ② equivalent to ①
③	$\neg p \rightarrow r$	premise
④	$\neg q \rightarrow r$	hypothetical syllogism of ② and ③ by (I.22)
⑤	$r \rightarrow s$	premise
⑥	$\neg q \rightarrow s$	hypothetical syllogism of ④ and ⑤ by (I.22)

So, conclusion obtained: $\neg q \rightarrow s$, i.e., "If I do not finish writing the program, then I will wake up feeling refreshed."

Rules of Inference for propositional logic

- Rules of Inference examples:
  - Example-6:: Given premises "$(p \wedge q) \vee r$" and "$r \rightarrow s$"; to infer conclusion "$p \vee s$".

    Constructing valid argument form —

①	$(p \wedge q) \vee r$	premise
②	$(p \vee r) \wedge (q \vee r)$	distributive law of ① by (I.6a)
③	$p \vee r$	clause from ②; via simplification of ② by (I.25)
④	$r \rightarrow s$	premise
⑤	$\neg r \vee s$	equivalent clause from ④
⑥	$p \vee s$	resolution of ③ and ⑤ by (I.29)

Fallacy in propositional logic

- Fallacy: common forms of incorrect reasoning based on contingency (rather than tautology), leading to invalid arguments.
  - Different forms of fallacy: (i) fallacy of affirming conclusion; (ii) fallacy of denying hypothesis.

Fallacy in propositional logic

- Fallacy examples:
  - Example-1:: [fallacy affirming conclusion]
    Given premises: "If you solve every problem in this book, then you will learn discrete mathematics," "You learnt discrete mathematics," and conclusion: "You solved every problem in this book."
    To check validity of above argument, let p be "You solved every problem in this book," q be "You learnt discrete mathematics."
    So, argument form of given —

$$\begin{array}{l} p \to q \\ \underline{q} \\ \therefore p \end{array}$$

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Fallacy in propositional logic

- Fallacy examples:
  - Example-1 contd.
    Considering both $p \to q$ and q to be TRUE, no conclusion possible to be drawn about p, as possibility of p to be both TRUE and FALSE to produce same effect in $p \to q$, as per implication truth table.
    So, case of incorrect argument using fallacy of affirming conclusion.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Fallacy in propositional logic

- Fallacy examples:
  - Example-2:: [fallacy denying hypothesis]
    Given premises: "If you solve every problem in this book, then you will learn discrete mathematics," "You did not solve every problem in this book," and conclusion: "You did not learn discrete mathematics."
    So, argument form of given — $p \rightarrow q$

    $\neg p$

    $\therefore \neg q$

    Considering premises to be TRUE, no conclusion possibly be drawn about q, as per implication truth table. So, incorrect argument.

Summary

- Focus: Propositional logic (contd.).
- Duality in propositional logic, with examples.
- Satisfiability in propositional logic, with example.
- Assertion in mathematical logic.
- Argument in propositional logic, definition and struture.
- Valid argument and valid argument form definitions and examples.
- Rules of inference for propositional logic, with examples.
- Fallacy in propositional logic, with examples.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Duality in propositional logic:: [Ros21]:30.
- Satisfiability in propositional logic:: [Ros21]:33-37.
- Argument:: [Ros21]:73-74.
- Valid argument and valid argument form:: [Ros21]:73-74.
- Rules of inference:: [Ros21]:74-78.
- Fallacy:: [Ros21]:79.