CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 2

Lecture 06: Predicate logic

[Quantification nesting; Nested universal, existential quantifiers; Negation of nested quantifiers; Predicate logic rules of inference]

Dr. Suddhasil De

Quantification nesting

- Quantification nesting: first-order logical expression, where one quantifier within scope of another quantifier.
  - Property:: Quantification nesting to resemble nested loops: to find truth value of nested quantifier of variables x and y, looping through values for x, and for each x again looping through values for y.
  - Property:: Quantification nesting depending on order of quantifiers, unless all universal quantifiers or all existential quantifiers.

Quantification nesting

- Quantification nesting:
  - Property:: Ordering property-1: order of solely nested universal quantifiers in statement, without other quantifiers, allowed to be changed without changing meaning of quantified statement, i.e., $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$.
  - Property:: Ordering property-2: order of solely nested existential quantifiers in statement, without other quantifiers, allowed to be changed without changing meaning of quantified statement, i.e., $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$.

Quantification nesting

- Quantification nesting:
  - Property:: Ordering property-3: order of mixed nested universal and existential quantifiers (appearing in some mixed form) in statement NOT allowed to be changed (and strictly to be adhered to) without changing meaning of quantified statement, i.e., universal quantifier followed by existential quantifier in quantified statement NOT logically equivalent to reverse order of existential quantifier followed by universal quantifier in same statement for same domain of discourse, and vice versa, i.e., $\exists x \forall y P(x,y) \not\equiv \forall y \exists x P(x,y)$, and $\forall x \exists y P(x,y) \not\equiv \exists y \forall x P(x,y)$.

Quantification nesting

- Quantification nesting:
  - Property:: Ordering property-4: existential quantifier followed by universal quantifier when adjudged TRUE in statement, implying universal quantifier followed by existential quantifier to also become TRUE in same statement for same domain of discourse, in which 'existential quantifier independent of universal quantifier' in 1st part, but 'existential quantifier NOT independent of universal quantifier' in 2nd part, i.e., ($\exists x \forall y P(x,y)$ adjudged TRUE) → ($\forall y \exists x P(x,y)$ to become TRUE).

Quantification nesting

- Connection between quantification nesting and looping:
  - Determining truth value of quantification nesting → nested looping and searching through domains of discourse successively.
  - Determining truth values of $\forall x \forall y P(x,y)$, $\forall x \exists y P(x,y)$, $\exists x \forall y P(x,y)$, $\exists x \exists y P(x,y)$ in m-element domain of discourse $\mathfrak{D}_x$ (for x) and n-element domain of discourse $\mathfrak{D}_y$ (for y):: looping through all m values of x, and for each value of x looping through all n values for y, to find truth values of $P(x,y)$ in each scenario.
  - Time to stop looping → based on nature of quantification and truth value. [shown in next slide's table]

Quantification nesting

- Loop stopping in quantification nesting:

Quantifications of Two Variables

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y. [x, y: till end]	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$ [x: till end]	For every x there is a y for which $P(x, y)$ is true. [y: once find]	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$ [x: once find]	There is an x for which $P(x, y)$ is true for every y. [y: till end]	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true. [x, y: once find]	$P(x, y)$ is false for every pair x, y.

[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Quantification nesting

- Quantification nesting examples:
  - Example-1:: Given, $\forall x \forall y(x + y = y + x)$ in domain of discourse $\mathbb{R}$.
    Denoting $P(x, y)$: $(x + y = y + x)$,
    $\forall x \forall y P(x, y)$ representing commutativity property of addition for all real numbers x and y, i.e., commutative law for addition of real numbers.
  - Example-2:: Given, $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$ in domain of discourse $\mathbb{R}$.
    Denoting $Q(x, y, z)$: $(x + (y + z) = (x + y) + z)$,
    $\forall x \forall y \forall z Q(x, y, z)$ representing associativity of addition for all real numbers x, y and z, i.e., associative law for addition of real numbers.

Quantification nesting

- Quantification nesting examples:
  - Example-3:: Given, $\forall x \exists y(x + y = 0)$ in domain of discourse $\mathbb{R}$.
    Denoting $S(x, y)$: $(x + y = 0)$.
    Case1: $\forall x \exists y S(x, y)$ representing that "for every real number x another real number y present and their sum to become 0," i.e., inverse law for addition of real numbers, in which $x = -y$, for every x.
    $\therefore$ Truth value of $\forall x \exists y S(x, y)$: TRUE.
    In opposite order case2: $\exists y \forall x S(x, y)$ representing "there present a real number y such that for every real number x, their sum to become 0," which being FALSE, as except $y = -x$, no other number present.

Quantification nesting

- Quantification nesting examples:
  - Example-4:: To express "The sum of two positive integers is always positive."
    Rewriting, "For every two integers, if these integers are both positive, then the sum of these integers is positive."
    Rewriting with variables, "For every two integers x and y, if x and y are both positive, then $x + y$ is also positive." Given, $\forall x \forall y(x + y > 0)$.
    Denoting $P(x, y)$: $\Big(\big((x > 0) \wedge (y > 0)\big) \rightarrow (x + y > 0)\Big)$,
    logical expression: $\forall x \forall y P(x, y)$, in domain of discourse $\mathbb{Z}$ for both x and y.

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Quantification nesting

- Quantification nesting examples:
  - Example-4 (contd.)::
    With restricted quantifiers, previous logical expression to become:
    $\forall x > 0\ \forall y > 0\ (x + y > 0)$, in domain of discourse $\mathbb{Z}$ for both x and y.
    Also possible to update domain of discourse in logical expression.
    Equivalent logical expression: $\forall x \forall y(x + y > 0)$, in domain of discourse $\mathbb{Z}^+$ for both x and y.

Quantification nesting

- Quantification nesting examples:
  - Example-5:: To express "Every real number except zero has a multiplicative inverse." [Multiplicative inverse = product to become multiplicative identity]
    Rewriting with variables, "For real number x, if $x \neq 0$, then there exists a real number y such that $x \cdot y = 1$."
    Logical expression: $\forall x\big((x \neq 0) \rightarrow \exists y(x \cdot y = 1)\big)$,
    in domain of discourse $\mathbb{R}$ for both x and y.

Quantification nesting

- Quantification nesting examples:
  - Example-6:: To express definition of limit: "For every real number $\epsilon > 0$, there exists a real number $\delta > 0$, such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$." ← definition of $\lim_{x \to a} f(x) = L$.
    Logical expression with variables in restricted quantifiers:
    $\forall \epsilon > 0\ \exists \delta > 0\ \forall x\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)$, in domain of discourse $\mathbb{R}$ for ϵ, δ and x.
    Also possible to update domain of discourse in above logical expression: $\forall \epsilon \exists \delta \forall x\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)$, in domain of discourse $\mathbb{R}^+$ for both ϵ and δ, and $\mathbb{R}$ for x.

Negations of quantification nesting

- Negation of nested quantifiers in any domain of discourse: negating by successively applying rules (according to quantifier ordering) for negating statements involving single quantifier.

Negations of quantification nesting

- Negation of nested quantifiers examples:
  - Example-1:: Given, $\forall x \exists y (x \cdot y = 1)$ in domain of discourse $\mathfrak{D}$.
    Negating, $\neg(\forall x \exists y (x \cdot y = 1))$
    $\equiv \exists x\ \neg \exists y (x \cdot y = 1)$ De Morgan's law (II.2a)
    $\equiv \exists x \forall y \neg (x \cdot y = 1)$ De Morgan's law (II.2b)
    $\equiv \exists x \forall y (x \cdot y \neq 1)$ alternate representation
    $\therefore$ Negated form of "$\forall x \exists y (x \cdot y = 1)$":: $\exists x \forall y (x \cdot y \neq 1)$.

Negations of quantification nesting

- Negation of nested quantifiers examples:
  - Example-2:: [Negation of Example-6, slide no.13] "$\lim_{x \to a} f(x)$ does not exist where $f(x)$ is a real-valued function of a real variable x and a belongs to the domain of f." [$\mathfrak{D} = \mathbb{R}$]

    To say "$\lim_{x \to a} f(x)$ does not exist" same as "$\lim_{x \to a} f(x) \neq L$ for real number L." This to be expressed in negated form (of Example-6)::

    $\neg\big(\forall \epsilon > 0\, \exists \delta > 0\, \forall x\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)\big)$

    $\equiv \exists \epsilon > 0\, \neg\big(\exists \delta > 0\, \forall x\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)\big)$ (II.2a)

    $\equiv \exists \epsilon > 0\, \forall \delta > 0\, \neg\big(\forall x\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)\big)$ (II.2b)

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Negations of quantification nesting

- Negation of nested quantifiers examples:
  - Example-2 (contd.)::

    $\equiv \exists \epsilon > 0\, \forall \delta > 0\, \exists x\, \neg\big((0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)\big)$ (II.2a)

    $\equiv \exists \epsilon > 0\, \forall \delta > 0\, \exists x (0 < |x - a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)$ (I.12b)

    $\equiv \exists \epsilon > 0\, \forall \delta > 0\, \exists x (0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)$

    alternate representation

    Quantifying "$\lim_{x \to a} f(x)$ does not exist" for all real numbers::

    $\forall L\, \exists \epsilon > 0\, \forall \delta > 0\, \exists x (0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)$.

    $\therefore$ Negation:: $\forall L\, \exists \epsilon > 0\, \forall \delta > 0\, \exists x (0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)$.

Rules of Inference for predicate logic

- Rules of Inference: rules for 'universal instantiation', 'universal generalization', 'existential instantiation', 'existential generalization', 'universal transitivity' among key forms.
  - Property:: For universal rules: choice of arbitrary element from domain of discourse, with no control of choosing of that element and no assumptions about that element (except being member of domain of discourse).
  - Property:: For existential rules: choice of specific (and NOT arbitrary) element from domain of discourse.

Rules of Inference for predicate logic (w. argument form)

Universal instantiation (U.I.):	(II.7) $\forall x P(x) \vdash P(c)$, where $c \in$ domain of x	$\forall x P(x)$ $\therefore P(c)$
Universal generalization (U.G.):	(II.8) $P(c)$ for arbitrary c $\vdash \forall x P(x)$, where $c \in$ domain of x	$P(c)$ for arbitrary c $\therefore \forall x P(x)$
Existential instantiation (E.I.):	(II.9) $\exists x P(x) \vdash P(c)$ for some c, where $c \in$ domain of x	$\exists x P(x)$ $\therefore P(c)$ for some c
Existential generalization (E.G.):	(II.10) $P(c)$ for some c $\vdash \exists x P(x)$, where $c \in$ domain of x	$P(c)$ for some c $\therefore \exists x P(x)$

Rules of Inference for predicate logic (w. argument form)

Universal transitivity (U.T.):	(II.11) $\forall x\big(P(x) \to Q(x)\big)$, $\forall x\big(Q(x) \to R(x)\big)$ $\vdash \forall x\big(P(x) \to R(x)\big)$, for any domain of x	$\forall x\big(P(x) \to Q(x)\big)$ $\forall x\big(Q(x) \to R(x)\big)$ $\therefore \forall x\big(P(x) \to R(x)\big)$
Universal modus ponens (U.M.P.)	(II.12) $\forall x\big(P(x) \to Q(x)\big)$, $P(c)$ for particular c $\vdash Q(c)$, where $c \in$ domain of x	$\forall x\big(P(x) \to Q(x)\big)$ $P(c)$ for particular c $\therefore Q(c)$
Universal modus tollens (U.M.T.)	(II.13) $\forall x\big(P(x) \to Q(x)\big)$, $\neg Q(c)$ for particular c $\vdash \neg P(c)$, where $c \in$ domain of x	$\forall x\big(P(x) \to Q(x)\big)$ $\neg Q(c)$ for particular c $\therefore \neg P(c)$

Rules of Inference for predicate logic (w. argument form)

	(II.14) $\forall x(P(x) \to (Q(x) \wedge S(x)))$, $\forall x\big(P(x) \wedge R(x)\big)$ $\vdash \forall x\big(R(x) \wedge S(x)\big)$, for any domain of x	$\forall x(P(x) \to (Q(x) \wedge S(x)))$ $\forall x\big(P(x) \wedge R(x)\big)$ $\therefore \forall x\big(R(x) \wedge S(x)\big)$
	(II.15) $\forall x\big(P(x) \vee Q(x)\big)$, $\forall x\big((\neg P(x) \wedge Q(x)\big)$ $\vdash \forall x\big(\neg R(x) \to P(x)\big)$, for any domain of x	$\forall x\big(P(x) \vee Q(x)\big)$ $\forall x\big((\neg P(x) \wedge Q(x)) \to R(x)\big)$ $\therefore \forall x\big(\neg R(x) \to P(x)\big)$
	(II.16) $\forall x\big(P(x) \vee Q(x)\big)$, $\forall x\big(\neg Q(x) \vee S(x)\big)$, $\forall x\big(R(x) \to \neg S(x)\big)$, $\exists x \neg P(x) \vdash \exists x \neg R(x)$, for any domain of x	$\forall x\big(P(x) \vee Q(x)\big)$ $\forall x\big(\neg Q(x) \vee S(x)\big)$ $\forall x\big(R(x) \to \neg S(x)\big)$ $\exists x \neg P(x)$ $\therefore \exists x \neg R(x)$

Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-1:: Consider premises "Everyone in this CSE class has registered in discrete mathematics course," "ABCD is a student in this CSE class," and conclusion "ABCD has registered in discrete mathematics course."

    Let $P(x)$ be "x is a student in this CSE class," and $Q(x)$ be "x has registered in discrete mathematics course."

    Then, given premises to become: $\forall x(P(x) \rightarrow Q(x))$, and $P(\text{ABCD})$.

    Also, given conclusion to become: $Q(\text{ABCD})$.

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Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-1 (contd.)::

    So, constructing valid argument form to get conclusion —

① $\forall x(P(x) \rightarrow Q(x))$	premise
② $P(\text{ABCD}) \rightarrow Q(\text{ABCD})$	Universal Instantiation of ① by (II.7)
③ $P(\text{ABCD})$	premise
④ $Q(\text{ABCD})$	modus ponens of ② and ③ by (I.20a)

Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-2:: Consider premises "A student in this class has not read the textbook," "Everyone in this class has passed the course," and conclusion "Someone who has passed the course has not read the textbook."

    Let $P(x)$ be "x is in this class," $Q(x)$ be "x has read the textbook," and $R(x)$ be "x has passed the course."

    Then, given premises to be: $\exists x(P(x) \wedge \neg Q(x))$, and $\forall x(P(x) \rightarrow R(x))$.

    Given that conclusion to be: $\exists x(R(x) \wedge \neg Q(x))$.

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Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-2 (contd.)::

    So, constructing valid argument form to get conclusion —

① $\exists x\big(P(x) \land \neg Q(x)\big)$	premise
② $P(a) \land \neg Q(a)$	Existential Instantiation of ① by (II.9)
③ $P(a)$	Simplification of ② by (I.25)
④ $\forall x\big(P(x) \to R(x)\big)$	premise
⑤ $P(a) \to R(a)$	Universal Instantiation of ④ by (II.7)
⑥ $R(a)$	Modus ponens of ③ and ⑤ by (I.20a)
⑦ $\neg Q(a)$	also, Simplification of ② by (I.25)
⑧ $R(a) \land \neg Q(a)$	Conjunction of ⑥ and ⑦ by (I.26)
⑨ $\exists x\big(R(x) \land \neg Q(x)\big)$	Existential generalization of ⑧ by (II.10)

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Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-2 (contd-2.)::

    So, alternate construction of valid argument form for conclusion —

① $\exists x\big(P(x) \land \neg Q(x)\big)$	premise
② $P(a) \land \neg Q(a)$	Existential Instantiation of ① by (II.9)
③ $P(a)$	Simplification of ② by (I.25)
④ $\forall x\big(P(x) \to R(x)\big)$	premise
⑤ $R(a)$	Univ. modus ponens of ③ and ④ by (II.12)
⑥ $\neg Q(a)$	also, Simplification of ② by (I.25)
⑦ $R(a) \land \neg Q(a)$	Conjunction of ⑥ and ⑦ by (I.26)
⑧ $\exists x\big(R(x) \land \neg Q(x)\big)$	Existential generalization of ⑧ by (II.10)

Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-3:: Consider premise "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n," and conclusion "$100^2 < 2^{100}$."

    Let $P(n)$ be "$n > 4$," $Q(n)$ be "$n^2 < 2^n$," and $R(n = 100)$ be "$100^2 < 2^{100}$," with domain of discourse $= \mathbb{Z}^+$.

    Then, given premises to be: $\forall n\big(P(n) \to Q(n)\big)$, $P(100)$; and conclusion to be: $Q(100)$.

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Rules of Inference for predicate logic

- Rules of Inference examples:
  - Example-3 (contd.)::

    So, constructing valid argument form to get conclusion —

    ① $\forall x(P(n) \rightarrow Q(n))$ premise

    ② $P(100)$ premise; TRUE as $100 > 4$

    ③ $Q(100)$ Universal modus ponens of ① and ② by (II.12)

Summary

- Focus: Predicate logic (contd.).
- Quantification nesting properties, with examples.
- Nested quantification and looping resemblance.
- Negation of nested quantifiers, with examples.
- Rules of inference for predicate logic, with examples.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Quantification nesting:: [Ros21]:60-61,62-67.
- Nested quantification and looping:: [Ros21]:61-62.
- Negation of nested quantifiers:: [Ros21]:67-68.
- Rules of inference for predicate logic:: [Ros21]:79-82.

Lecture Exercises: Problem 1 [Ref: Gate 2025, Q.48, p.48 (Set-1)]

Which of the following predicate logic formulae/formula is/are CORRECT representation(s) of the statement: "*Everyone has exactly one mother*"?

The meanings of the predicates used are:

$mother(y,x)$: y is the mother of x

$noteq(x,y)$: x and y are not equal

A. $\forall x \exists y \exists z(mother(y,x) \wedge \neg mother(z,x))$

B. $\forall x \exists y[mother(y,x) \wedge \forall z(noteq(z,y) \rightarrow \neg mother(z,x))]$

C. $\forall x \forall y[mother(y,x) \rightarrow \exists z(mother(z,x) \wedge \neg noteq(z,y))]$

D. $\forall x \exists y[mother(y,x) \wedge \neg \exists z(noteq(z,y) \wedge mother(z,x))]$.

Lecture Exercises: Problem 1 Ans

- As per premise, given predicate being interpreted by domain of discourse, say $\mathfrak{D}$, as set of all persons; $x, y, z \in \mathfrak{D}$.
- Given predicate: $P(\cdot)$ = Everyone has exactly one mother
  ≡ For each person, there is another person who is his/her mother, and all other persons are not his/her mother
  ≡ For each person x, there is another person y who is his/her mother, and every other person z is not his/her mother
- That is, $P(x,y,z) = \forall x \exists y \left(mother(y,x) \wedge \forall z\big((z \neq y) \rightarrow \neg mother(z,x)\big)\right)$,
  i.e., $P(x,y,z) = \forall x \exists y \left(mother(y,x) \wedge \forall z\big(noteq(z,y) \rightarrow \neg mother(z,x)\big)\right)$.
- One correct option: B.

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Lecture Exercises: Problem 1 Ans (contd.)

- Further, equivalent of uniqueness nested predicate possible.
- That is, $\forall z(noteq(z,y) \rightarrow \neg mother(z,x))$.
  $\equiv \neg\exists z \neg(noteq(z,y) \rightarrow \neg mother(z,x))$ De Morgan's law (II.2b)
  $\equiv \neg\exists z(noteq(z,y) \wedge mother(z,x))$ Implication rule (I.12b)
- So, $P(x,y,z) = \forall x \exists y \left(mother(y,x) \wedge \neg\exists z(noteq(z,y) \wedge mother(z,x))\right)$.
- Another correct option: D.

Lecture Exercises: Problem 2 [Ref: Gate 2023, Q.26, p.24]

Geetha has a conjecture about integers, which is of the form

$$\forall x(P(x) \rightarrow \exists y Q(x,y))$$

where P is a statement about integers, and Q is a statement about pairs of integers. Which of the following (one or more) option(s) would *imply* Geetha's conjecture?

(a) $\exists x(P(x) \wedge \forall y Q(x,y))$.
(b) $\forall x \forall y Q(x,y)$.
(c) $\exists y \forall x(P(x) \rightarrow Q(x,y))$.
(d) $\exists x(P(x) \wedge \exists y Q(x,y))$.

Lecture Exercises: Problem 2 Ans

(A) Given premises: $\forall x(P(x) \rightarrow \exists y Q(x,y))$, and $\exists x(P(x) \wedge \forall y Q(x,y))$.

- Conclusion to get: $\left(\exists x(P(x) \wedge \forall y Q(x,y))\right) \rightarrow \left(\forall x(P(x) \rightarrow \exists y Q(x,y))\right)$.
- Not possible: (i) existential quantification of x in antecedent not to be converted to universal quantification of x in consequent; (ii) as per implication rule (I.13b) on propositions, conjunction in antecedent leads to negation of implication.
- So, $\left(\exists x(P(x) \wedge \forall y Q(x,y))\right) \nrightarrow \left(\forall x(P(x) \rightarrow \exists y Q(x,y))\right)$.

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Lecture Exercises: Problem 2 Ans (contd.)

(B) Given premises: $\forall x(P(x) \rightarrow \exists y Q(x,y))$, and $\forall x \forall y Q(x,y)$.

- Conclusion to get: $(\forall x \forall y Q(x,y)) \rightarrow (\forall x(P(x) \rightarrow \exists y Q(x,y)))$.

① $(\forall x \forall y Q(x,y)) \rightarrow (\forall x(P(x) \rightarrow \exists y Q(x,y)))$ Conclusion as premise

② $(\forall x \forall y Q(x,y)) \rightarrow (\forall x \exists y(P(x) \rightarrow Q(x,y)))$ Null quantification of consequent of ① by (II.5b)

③ $(\forall x \forall y Q(x,y)) \rightarrow (\forall x \exists y(\neg P(x) \vee Q(x,y)))$ Implication rule (I.12a) on consequent of ②

④ $(\forall x \forall y Q(x,y)) \rightarrow \left(\forall x \left(\neg \forall y(P(x) \wedge \neg Q(x,y))\right)\right)$ De Morgan's law (II.2a) on consequent of ③

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Lecture Exercises: Problem 2 Ans (contd.-2)

(B) contd.

- Both antecedent and consequent now quantified by universal quantifier
- When $Q(x,y)$ to be TRUE, antecedent to become TRUE.
- Then, $\neg Q(x,y)$ to become FALSE, resulting in $(P(x) \wedge \neg Q(x,y))$ also to become FALSE.
- So, $(\neg \forall y(P(x) \wedge \neg Q(x,y)))$ to become TRUE, and thus consequent to become TRUE. So, implication in ④ (and so in ①) to become TRUE.
- When $Q(x,y)$ to be FALSE, following similar argument, implication in ① to be TRUE. So, $(\forall x \forall y Q(x,y)) \rightarrow (\forall x(P(x) \rightarrow \exists y Q(x,y)))$ a tautology.

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Lecture Exercises: Problem 2 Ans (contd.-3)

(C) Given premises: $\forall x(P(x) \rightarrow \exists y Q(x,y))$, and $\exists y \forall x(P(x) \rightarrow Q(x,y))$.

- Domain of discourse: $\mathbb{Z}$.
- Conclusion to get: $(\exists y \forall x(P(x) \rightarrow Q(x,y))) \rightarrow (\forall x(P(x) \rightarrow \exists y Q(x,y)))$.

① $\exists y \forall x(P(x) \rightarrow Q(x,y))$ premise; TRUE for $x, y \in \mathbb{Z}$; choice of y independent of x

② $(\exists y \forall x(P(x) \rightarrow Q(x,y))) \rightarrow (\forall x \exists y(P(x) \rightarrow Q(x,y)))$ Quantification nesting order property-4; choice of y not independent of x

③ $(\exists y \forall x(P(x) \rightarrow Q(x,y))) \rightarrow (\forall x(P(x) \rightarrow \exists y Q(x,y)))$ Null quantification of consequent of ② by (II.5b)

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Lecture Exercises: Problem 2 Ans (contd.-4)

(D) Given premises: $\forall x\big(P(x) \rightarrow \exists y Q(x,y)\big)$, and $\exists x\big(P(x) \wedge \exists y Q(x,y)\big)$.

- Conclusion to get: $\Big(\exists x\big(P(x) \wedge \exists y Q(x,y)\big)\Big) \rightarrow \Big(\forall x\big(P(x) \rightarrow \exists y Q(x,y)\big)\Big)$.
- Not possible: (i) existential quantification of x in antecedent not to be converted to universal quantification of x in consequent; (ii) as per implication rule (I.13b) on propositions, conjunction in antecedent leads to negation of implication.
- So, $\Big(\exists x\big(P(x) \wedge \exists y Q(x,y)\big)\Big) \nrightarrow \Big(\forall x\big(P(x) \rightarrow \exists y Q(x,y)\big)\Big)$.