CS34110 Discrete Mathematics and Graph Theory

UNIT – I, Module – 2

Lecture 05: Predicate logic

[Quantification with restricted domains; Bound, free variables; Logical equivalence; Negations; Predicates in practice; Laws for quantifiers]

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Quantification: Restricted domain of discourse

- Quantification with restricted domain of discourse: notation specifying condition for quantified variable to be satisfied to restrict domain of discourse of quantifier.
  - Property:: Restriction in domain of discourse to be included after quantifier.
  - Property:: Restriction of universal quantification = universal quantification of conditional statement (i.e., implication statement).
  - Property:: Restriction of existential quantification = existential quantification of conjunction of statements.

Quantification: Restricted domain of discourse

- Quantification with restricted domain of discourse examples:
  - Example-1:: consider $P(x)$: "$x^2 > 0$", domain of discourse $= \mathbb{R}$, restriction: negative numbers of $\mathbb{R}$.
    $P(x) \equiv$ T, for every $x \in \mathbb{R}$ and $x < 0$, as $x < 0$ fulfills $x^2 > 0$.
    That is, $\forall x < 0\ (x^2 > 0)$ to be TRUE in domain of discourse $\mathbb{R}$.
    In above sentence, condition $x < 0$ in universal quantification to express given restriction.
    Also, $(x < 0) \rightarrow (x^2 > 0)$, i.e., IF $(x < 0)$, THEN $(x^2 > 0)$.
    So, $\forall x < 0\ (x^2 > 0) \equiv \forall x\ ((x < 0) \rightarrow (x^2 > 0))$.

Quantification: Restricted domain of discourse

- Quantification with restricted domain of discourse examples:
  - Example-2:: interpretation of $\forall x < 0\ (x^2 > 0)$ in $\mathbb{R}$.
    $\forall x < 0\ (x^2 > 0)$ stating "for every real number x with $x < 0$, $x^2 > 0$";
    i.e., "Square of negative real numbers are positive."
    As above statement TRUE, so given domain of discourse with restriction to become model of given predicate.
  - Example-3:: interpretation of $\exists z > 0\ (z^2 = 2)$ in $\mathbb{R}$.
    $\exists z > 0\ (z^2 = 2)$ stating "there is a positive real number, which is square root of 2" → TRUE, so given domain be model of predicate.
    Also, $\exists z > 0\ (z^2 = 2) \equiv \exists z((z > 0) \wedge (z^2 = 2))$.

Variable binding

- Variable binding: binding of all propositional variables in propositional function, by quantifying or setting to particular value to turn into proposition.
  - Property:: Variable binding: by combination of universal quantifiers, existential quantifiers, and value assignments.
  - Property:: Bound variable: when any form of quantification applied on variable of propositional function.
  - Property:: Free variable: occurrence of variable — (i) not bound by quantifier, or (ii) set by value assignment, in propositional function.

Variable binding

- Variable binding:
  - Property:: Free variable → if outside scope of all quantifiers in formula specifying it.
  - ➲ Note: in common usage of variables in predicate logic, same letter allowed to represent variables bound by different quantifiers, but with non-overlapping scopes of quantification.

Variable binding

- Variable binding examples:
  - Example-1:: consider $\exists x(x + y = 1)$.
    Variable x bound by existential quantification $\exists x$.
    Variable y free, as not bound by any quantifier or no value assigned.
  - Example-2:: consider $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, disjuncts of $\exists$ & $\forall$.
    Variable x bound in both disjuncts separately by $\exists x$ and $\forall x$.
    Binding of x separated by non-overlapping scope of quantification, i.e., $\exists x$ binding in $P(x) \wedge Q(x)$, $\forall x$ binding in $R(x)$, no overlap between them.

Logical equivalence in predicate logic

- Logical equivalence in predicate logic: given statements S and T (involving predicates and quantifiers) to be logically equivalent (notation: '$S \equiv T$'), if S and T having same truth values, irrespective of predicates substituted in S and T, and irrespective of domain of discourse used for variables in propositional functions of S and T.

Logical equivalence in predicate logic

- Logical equivalence examples:
  - Example-1:: to show $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$ in same domain of discourse — two-part proof.
    (1) To show if $\forall x(P(x) \wedge Q(x))$ TRUE, then $\forall x P(x) \wedge \forall x Q(x)$ TRUE.

$\forall x(P(x) \wedge Q(x)) \equiv$ T	premise, in any domain of discourse $\mathfrak{D}$.
So, $P(a) \wedge Q(a) \equiv$ T	for every $a \in \mathfrak{D}$.
So, $P(a) \equiv$ T and $Q(a) \equiv$ T	for every $a \in \mathfrak{D}$.
So, $\forall x P(x) \equiv$ T and $\forall x Q(x) \equiv$ T	in $\mathfrak{D}$.
So, $\forall x P(x) \wedge \forall x Q(x) \equiv$ T	in $\mathfrak{D}$.

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Logical equivalence in predicate logic

- Logical equivalence examples:
  - Example-1 contd.

    (2) To show if $\forall x P(x) \land \forall x Q(x)$ TRUE, then $\forall x(P(x) \land Q(x))$ TRUE.

$\forall x P(x) \land \forall x Q(x) \equiv$ T	premise, in $\mathfrak{D}$.
So, $\forall x P(x) \equiv$ T and $\forall x Q(x) \equiv$ T	in $\mathfrak{D}$.
So, $P(a) \equiv$ T and $Q(a) \equiv$ T	for every $a \in \mathfrak{D}$.
So, $P(a) \land Q(a) \equiv$ T	for every $a \in \mathfrak{D}$.
So, $\forall x(P(x) \land Q(x)) \equiv$ T	in $\mathfrak{D}$.

    In conclusion, $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$.

Negations of quantification

- Negation of universal quantification $\forall x P(x)$ in any domain of discourse: $\exists x \neg P(x)$ in same domain of discourse.
  - Property:: Negation of $\forall x_1 \forall x_2 \ldots \forall x_n\, P(x_1, x_2, \ldots, x_n)$ in any product set of n domains of discourse: $\exists x_1 \exists x_2 \ldots \exists x_n\, \neg P(x_1, x_2, \ldots, x_n)$ in same domains of discourse.
- Negation of existential quantification $\exists x P(x)$ in any domain of discourse: $\forall x \neg P(x)$ in same domain of discourse.
  - Property:: Negation of $\exists x_1 \exists x_2 \ldots \exists x_n\, P(x_1, x_2, \ldots, x_n)$ in any product set of n domains of discourse: $\forall x_1 \forall x_2 \ldots \forall x_n\, \neg P(x_1, x_2, \ldots, x_n)$ in same domains of discourse.

Negations of quantification

- Negations of quantification:
  - Property:: also known by De Morgan's laws for quantifiers.

De Morgan's Laws for Quantifiers

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

  - Property:: $\neg \forall x P(x) \equiv \exists x \neg P(x)$ in same domain of discourse;
  - Property:: $\neg \exists x P(x) \equiv \forall x \neg P(x)$ in same domain of discourse.

[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Negations of quantification

- Negations of quantification examples:
  - Example-1:: $\neg\forall x P(x) \equiv \exists x \neg P(x)$ in same domain of discourse.

    $\neg\forall x P(x) \equiv$ T iff $\forall x P(x) \equiv$ F in any domain of discourse $\mathfrak{D}$.
    Also, $\forall x P(x) \equiv$ F iff $P(a) \equiv$ F for some $a \in \mathfrak{D}$.
    Then, $P(a) \equiv$ F iff $\neg P(a) \equiv$ T for some $a \in \mathfrak{D}$.
    Next, $\neg P(a) \equiv$ T iff $\exists x \neg P(x) \equiv$ T for some $a \in \mathfrak{D}$.
    So, $\neg\forall x P(x) \equiv$ T iff $\exists x \neg P(x) \equiv$ T in $\mathfrak{D}$.

  Thus, to conclude, $\neg\forall x P(x) \equiv \exists x \neg P(x)$.

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Negations of quantification

- Negations of quantification examples:
  - Example-2:: $\neg\exists x Q(x) \equiv \forall x \neg Q(x)$ in same domain of discourse.

    $\neg\exists x Q(x) \equiv$ T iff $\exists x Q(x) \equiv$ F in any domain of discourse $\mathfrak{D}$.
    Also, $\exists x Q(x) \equiv$ F iff $Q(a) \equiv$ F for every $a \in \mathfrak{D}$.
    Then, $Q(a) \equiv$ F iff $\neg Q(a) \equiv$ T for every $a \in \mathfrak{D}$.
    Next, $\neg Q(a) \equiv$ T iff $\forall x \neg Q(x) \equiv$ T for every $a \in \mathfrak{D}$.
    So, $\neg\exists x Q(x) \equiv$ T iff $\forall x \neg Q(x) \equiv$ T in $\mathfrak{D}$.

  Thus, to conclude, $\neg\exists x Q(x) \equiv \forall x \neg Q(x)$.

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Negations of quantification

- Negations of quantification examples:
  - Example-3:: Negation of statement: "There is an honest politician".
    Let $H(x)$ denoting "x is honest".
    So, statement: "There is an honest politician", represented by $\exists x H(x)$, where domain of discourse = all politicians.
    Now, $\neg\exists x H(x) \equiv \forall x \neg H(x)$ expressing "Every politician is dishonest".
  - Example-4:: Negation of $\forall x(x^2 > x)$ in some domain of discourse.
    $\neg\forall x(x^2 > x) \equiv \exists x \neg(x^2 > x)$, and $\neg(x^2 > x)$ rewritten as $(x^2 \leq x)$.
    So, $\neg\forall x(x^2 > x) \equiv \exists x(x^2 \leq x)$.

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Negations of quantification

- Negations of quantification examples:
  - Example-5:: to show $\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$ in same domain of discourse $\mathfrak{D}$.

    $\neg\forall x(P(x) \to Q(x)) \equiv \exists x\neg(P(x) \to Q(x))$ De Morgan's law in $\mathfrak{D}$

    Also, $\neg(P(x) \to Q(x)) \equiv P(x) \wedge \neg Q(x)$ Implication rule (I.12b)

    for every $x = a \in \mathfrak{D}$, as each instantiation to result into proposition.

    Then, $\exists x\neg(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$ by substitution

    Thus, to conclude, $\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$ in $\mathfrak{D}$.

Predicate logic in practice

- English sentences into logical expressions using predicates and quantifiers (case – single quantifier):
  - Example-1:: Given: "Every student in this class has learnt calculus."

    Rewriting, $\mathfrak{S}$ as, "For every student in this class, that student has learnt calculus."

    Introducing variable x in $\mathfrak{S}$, "For every student x in this class, x has learnt calculus."

    Let $C(x)$ be "x has learnt calculus."

    So, $\mathfrak{S}$ represented by, $\forall x C(x)$, where domain of discourse = all students in class.

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Predicate logic in practice

- English sentences into logical expressions (single quantifier):
  - Example-1-alternate (with different domain of discourse)::

    Let, $\mathfrak{S}'$ be, "For every person x, if person x is a student in this class, then x has learnt calculus.", with all people as domain of discourse.

    Let $C(x)$ be, "x has learnt calculus.";

    $P(x)$ be, "x is a student in this class."

    So, $\mathfrak{S}'$ represented by, $\forall x(P(x) \to C(x))$, where domain of discourse = all people.

Predicate logic in practice

- English sentences into logical expressions (single quantifier):
  - Example-2:: Given: "Some student in this class has visited Himalayas."
    Let, $\mathfrak{Y}$, "There is a student x in this class having property that x has visited Himalayas.", with all students in class as domain of discourse.
    Let, $\mathfrak{Y}'$ be, "There is a person x having properties that x is a student in this class and x has visited Himalayas.", with all people as domain of discourse.
    Let $H(x)$ be, "x has visited Himalayas."; $P(x)$ be, "x is a student in this class."

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Predicate logic in practice

- English sentences into logical expressions (single quantifier):
  - Example-2 contd.
    So, $\mathfrak{Y}$ represented by, $\exists x H(x)$, where domain of discourse = all students in class.
    And, $\mathfrak{Y}'$ represented by, $\exists x (P(x) \wedge H(x))$, where domain of discourse = all people.

Predicate logic in practice

- English sentences into logical expressions (double quantifier):
  - Example-1:: Given: "Every student in this class has learnt calculus."
    Let, $\mathfrak{I}$ be, "For every student x in this class, x has learnt calculus.", with all students in class as domain of discourse.
    And, $\mathfrak{I}'$ be, "For every person x, if person x is a student in this class, then x has learnt calculus.", with all people as domain of discourse.
    Let $B(x, y)$ be, "x has learnt subject y."
    So, $\mathfrak{I}'$ become, $\forall x (P(x) \rightarrow B(x, \text{calculus}))$, where domain of discourse = all people.
    And, $\mathfrak{I}$, $\forall x B(x, \text{calculus})$, domain of discourse = all students in class.

Predicate logic in practice

- English sentences into logical expressions (double quantifier):
  - Example-2:: Given: "Every student in this class has visited either Himalayas or Nilgiris."
    Let, $\mathfrak{U}$ be, "For every person x, if x is a student in this class, then x has the property that x has visited Himalayas or x has visited Nilgiris.", with all people as domain of discourse, and assuming inclusive OR.
    Let $W(x, y)$ be, "x has visited location y."
    So, $\mathfrak{U}$ become, $\forall x\big(P(x) \rightarrow (W(x, \text{Himalayas}) \vee W(x, \text{Nilgiris}))\big)$, where domain of discourse = all people.

Predicate logic in practice

- English sentences into logical expressions (multiple quantifier):
  - Example-1:: Given: "If a user is active, at least one network link will be available."
    Let, $\mathfrak{N}$ be, "If a user u is active, network link ℓ will be in state available.", with 'all users', 'all network links', and 'all possible states for any network link' as domains of discourse.
    Let, $\mathfrak{N}'$ be, "There is a person u having property that if u is a user and if u is active, then network link ℓ will be in state available.", with 'all people', 'all network links', and 'all possible states for any network link' as domains of discourse.

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Predicate logic in practice

- English sentences into logical expressions (multiple quantifier):
  - Example-1 contd.
    Let $A(u)$ be, "u is active."; $R(\ell, y)$ be, "ℓ is in state y."; $P(u)$ be, "u is a user."
    So, $\mathfrak{N}$ become, $\exists u A(u) \rightarrow \exists \ell S(\ell, \text{available})$, where domains of discourse = 'all users' (for u), 'all network links' (for ℓ).
    And, $\mathfrak{N}'$ become, $\exists u(P(u) \wedge A(u)) \rightarrow \exists \ell S(\ell, \text{available})$, where domains of discourse = 'all people' (for u), 'all network links' (for ℓ).

Laws for quantifiers (for same domains)

Distributive laws:	(II.1a) $\forall x(P(x) \land Q(x)) \equiv \forall xP(x) \land \forall xQ(x)$	(II.1b) $\exists x(P(x) \lor Q(x)) \equiv \exists xP(x) \lor \exists xQ(x)$
De Morgan's laws:	(II.2a) $\neg\forall xP(x) \equiv \exists x\neg P(x)$	(II.2b) $\neg\exists xP(x) \equiv \forall x\neg P(x)$
Null quantification laws:	(II.3a) $P(x) \land \forall yQ(y) \equiv \forall y(P(x) \land Q(y))$ (II.3b) $\forall yQ(y) \land P(x) \equiv \forall y(Q(y) \land P(x))$ both for nonempty domain of y	(II.3c) $P(x) \land \exists yQ(y) \equiv \exists y(P(x) \land Q(y))$ (II.3d) $\exists yQ(y) \land P(x) \equiv \exists y(Q(y) \land P(x))$
	(II.4a) $P(x) \lor \forall yQ(y) \equiv \forall y(P(x) \lor Q(y))$ (II.4b) $\forall yQ(y) \lor P(x) \equiv \forall y(Q(y) \lor P(x))$	(II.4c) $P(x) \lor \exists yQ(y) \equiv \exists y(P(x) \lor Q(y))$ (II.4d) $\exists yQ(y) \lor P(x) \equiv \exists y(Q(y) \lor P(x))$ both for nonempty domain of y

Laws for quantifiers (for same domains)

Null quantification laws (contd.):	(II.5a) $\forall y(P(x) \to Q(y)) \equiv P(x) \to \forall yQ(y)$	(II.5b) $\exists y(P(x) \to Q(y)) \equiv P(x) \to \exists yQ(y)$, for nonempty domain of y
	(II.6a) $\forall y(Q(y) \to P(x)) \equiv \exists yQ(y) \to P(x)$	(II.6b) $\exists y(Q(y) \to P(x)) \equiv \forall yQ(y) \to P(x)$ for nonempty domain of y

Summary

- Focus: Predicate logic (contd.).
- Quantification with restricted domain of discourse, with examples.
- Bound and free variables definitions, with examples.
- Logical equivalence of predicate definition, with examples.
- Negations of universal and existential quantifications definitions, with examples.
- Predicates with single quantifier, double quantifier and multiple quantifier examples.
- Laws for quantifiers (for same domains).

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Quantification with restricted domain:: [Ros21]:48.
- Bound and free variables:: [Ros21]:48-49.
- Logical equivalence of predicate:: [Ros21]:49.
- Negation of quantification:: [Ros21]:49-51.
- Predicates with single/double/multiple quantifiers:: [Ros21]:51-53.
- Laws for quantifiers (for same domains):: [Ros21]:49,51,59.

Lecture Exercises: Problem 1 [Ref: Gate 2020, Q.39, p.14]

Which one of the following predicate formulae is NOT logically valid? Note that W is a predicate formula without any free occurrence of x.

(a) $\forall x(P(x) \vee W) \equiv \forall x P(x) \vee W$.
(b) $\exists x(P(x) \wedge W) \equiv \exists x P(x) \wedge W$.
(c) $\forall x(P(x) \rightarrow W) \equiv \forall x P(x) \rightarrow W$.
(d) $\exists x(P(x) \rightarrow W) \equiv \forall x P(x) \rightarrow W$.

Lecture Exercises: Problem 1 Ans

- Logical validity of given predicate formulae: in each formula, L.H.S. logically equivalent to R.H.S.

(C) Given L.H.S.: ① $\forall x(P(x) \rightarrow W)$ premise

② $\forall x(\neg P(x) \vee W)$ Implication rule (I.12a) on ①

③ $\forall x(\neg P(x)) \vee W$ Null quantification of ② by (II.4b), as no free occurrence of x in W

④ $\neg(\exists x P(x)) \vee W$ De Morgan's law (II.2b) on ③

⑤ $\exists x P(x) \rightarrow W$ Implication rule (I.12a) on ④

Given R.H.S.: $\forall x P(x) \rightarrow W$.

- So, L.H.S. NOT logically equivalent to R.H.S.
- Check for other options (A), (B), (D) similarly.