CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 3

Lecture 20: Relations

[Closures of relations; Reflexive, symmetric, transitive closures; Equivalence relations; Equivalence classes; N-ary relations]

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Notation table

Symbol / Notation	Meaning
$\mathscr{P}(\boldsymbol{\mathcal{R}})$	Closure of relation $\boldsymbol{\mathcal{R}}$ with respect to property $\mathscr{P}$.
$\mathit{Reflexive}(\boldsymbol{\mathcal{R}})$	Reflexive closure of binary relation $\boldsymbol{\mathcal{R}}$.
$\mathit{Symmetric}(\boldsymbol{\mathcal{R}})$	Symmetric closure of binary relation $\boldsymbol{\mathcal{R}}$.
$\mathit{Transitive}(\boldsymbol{\mathcal{R}})$, $\boldsymbol{\mathcal{R}}^\star$	Transitive closure of binary relation $\boldsymbol{\mathcal{R}}$.
$\sim_{\boldsymbol{\mathcal{R}}}$	Equivalence relation $\boldsymbol{\mathcal{R}}$ (other notations: '$\sim$', '$\equiv_{\boldsymbol{\mathcal{R}}}$').
$\nsim_{\boldsymbol{\mathcal{R}}}$	Non-equivalence relation $\boldsymbol{\mathcal{R}}$ (other notations: '$\nsim$', '$\not\equiv_{\boldsymbol{\mathcal{R}}}$').
$[a]_{\boldsymbol{\mathcal{R}}}$	Equivalence class of some element a in a set by relation $\boldsymbol{\mathcal{R}}$.
$\mathbf{A}/\sim_{\boldsymbol{\mathcal{R}}}$	Set of disjoint nonempty partitions (subsets) of set $\mathbf{A}$ by $\boldsymbol{\mathcal{R}}$.
$S_{\mathfrak{C}}$	Selection operator of n-ary $\boldsymbol{\mathcal{R}}$ satisfying condition $\mathfrak{C}$.
$P_{i_1,i_2,\ldots,i_m}$	Projection operator of n-ary $\boldsymbol{\mathcal{R}}$ ($1 \leq i_1 < i_2 < \cdots < i_m \leq n$ ($m \in \mathbb{Z}^+$))
$J_p(\boldsymbol{\mathcal{R}}, \boldsymbol{S})$	Join operator of m-ary $\boldsymbol{\mathcal{R}}$ with n-ary $\boldsymbol{S}$ (where, $p \leq n$ and $p \leq m$).

Closures of relations

- Closure of relation: For binary relation $\boldsymbol{\mathcal{R}}$ on finite set $\mathbf{A}$, closure of $\boldsymbol{\mathcal{R}}$ with respect to property $\mathscr{P}$ (if $\mathscr{P}$ existing) = smallest relation $\boldsymbol{S}$ on $\mathbf{A}$ with $\mathscr{P}$, containing $\boldsymbol{\mathcal{R}}$.
  - Property:: Notation for closure of $\boldsymbol{\mathcal{R}}$ with respect to $\mathscr{P}$: $\mathscr{P}(\boldsymbol{\mathcal{R}})$.
  - Property:: $\mathscr{P}(\boldsymbol{\mathcal{R}})$ = subset of every subset of $\mathbf{A} \times \mathbf{A}$ containing $\boldsymbol{\mathcal{R}}$ with property $\mathscr{P}$. Then, $\mathscr{P}(\boldsymbol{\mathcal{R}}) = \bigcap_{\boldsymbol{S} \subseteq \mathbf{A} \times \mathbf{A}} \{\boldsymbol{S} \mid (\boldsymbol{S} = \mathscr{P}\text{–relation}) \wedge (\boldsymbol{\mathcal{R}} \subseteq \boldsymbol{S})\}$.

Closures of relations

- Closure of relation:
  - Property:: ⟨Theorem⟩: For binary $\mathscr{P}$-relation $\mathcal{R}$ on finite set $\mathbf{A}$, if any $\mathscr{P}$-relation $\boldsymbol{S}$ exists, where $\boldsymbol{S}$ = subset of every $\mathscr{P}$-relation containing $\mathcal{R}$, then $\boldsymbol{S}$ to be unique.

  Proof: ['Proof by contradiction']

  Let arbitrary distinct relations $\boldsymbol{S}$ and $\boldsymbol{T}$ present, both with property $\mathscr{P}$, i.e., $\boldsymbol{S} = \mathscr{P}$−relation, as well as, $\boldsymbol{T} = \mathscr{P}$−relation.

  Further, let $\boldsymbol{S} \subseteq$ every $\mathscr{P}$−relation containing $\mathcal{R}$, and then $\boldsymbol{T} \subseteq$ every $\mathscr{P}$−relation containing $\mathcal{R}$.

  Then, $\boldsymbol{S} \subseteq \boldsymbol{T}$ and $\boldsymbol{T} \subseteq \boldsymbol{S}$. So, $\boldsymbol{S} = \boldsymbol{T}$. A contradiction. ■

Closures of relations

- Closure of relation:
  - Reflexive closure property:: Reflexive closure of binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $\mathit{Reflexive}(\mathcal{R}) = \bigcap_{\boldsymbol{S}\subseteq\mathbf{A}\times\mathbf{A}}\{\boldsymbol{S} \mid (\forall a{\in}\mathbf{A}((a,a) \in \boldsymbol{S})) \wedge (\mathcal{R} \subseteq \boldsymbol{S})\}$.
  In other words, reflexive closure relation — (i) to contain $\mathcal{R}$, (ii) to satisfy reflexivity property, and (iii) being contained within every reflexive relation that also containing $\mathcal{R}$.
  - Procedure to form reflexive closure:: given $\mathcal{R}$ on $\mathbf{A}$, steps required — adding to $\mathcal{R}$ all pairs of form (a,a), if not already in $\mathcal{R}$, where $a \in \mathbf{A}$.
  - Property:: $\mathit{Reflexive}(\mathcal{R}) = \mathcal{R} \cup \Delta_{\mathbf{A}}$, where $\mathcal{R}$ on set $\mathbf{A}$, and $\Delta_{\mathbf{A}} = \{(a,a) \mid \forall a \in \mathbf{A}\}$ = diagonal relation on $\mathbf{A}$.

Closures of relations

- Closure of relation:
  - Symmetric closure property:: Symmetric closure of binary relation $\mathcal{R}$ on finite set $\mathbf{A}$: $\mathit{Symmetric}(\mathcal{R}) = \bigcap_{\boldsymbol{S}\subseteq\mathbf{A}\times\mathbf{A}}\{\boldsymbol{S} \mid (\forall a \forall b{\in}\mathbf{A}(((a,b) \in \boldsymbol{S}) \rightarrow ((b,a) \in \boldsymbol{S}))) \wedge (\mathcal{R} \subseteq \boldsymbol{S})\}$.
  In other words, symmetric closure relation — (i) to contain $\mathcal{R}$, (ii) to satisfy symmetry property, and (iii) being contained within every symmetric relation that also containing $\mathcal{R}$.
  - Procedure to form symmetric closure:: given $\mathcal{R}$ on $\mathbf{A}$, steps required — adding to $\mathcal{R}$ all ordered pairs of form (b,a), if not already in $\mathcal{R}$, but corresponding (a,b) already in $\mathcal{R}$, where $a,b \in \mathbf{A}$.

Closures of relations

- Closure of relation:
  - Property:: $\mathcal{Symmetric}(\boldsymbol{\mathcal{R}}) = \boldsymbol{\mathcal{R}} \cup \boldsymbol{\mathcal{R}}^{-1}$, where $\boldsymbol{\mathcal{R}}$ on set $\mathbf{A}$, and $\boldsymbol{\mathcal{R}}^{-1} = \{(b, a) \mid \forall (a, b) \in \boldsymbol{\mathcal{R}},\ a, b \in \mathbf{A}\}$ = inverse relation of $\boldsymbol{\mathcal{R}}$.

Closures of relations

- Closure of relation:
  - Transitive closure property:: Transitive closure of binary relation $\boldsymbol{\mathcal{R}}$ on finite set $\mathbf{A}$: $\mathcal{Transitive}(\boldsymbol{\mathcal{R}}) = \bigcap_{\boldsymbol{S} \subseteq \mathbf{A} \times \mathbf{A}} \left\{ \boldsymbol{S} \mid \left(\forall a \forall b \forall c \in \mathbf{A} \left(\left(\left((a, b) \in \boldsymbol{S} \right) \wedge \left((b, c) \in \boldsymbol{S} \right) \right) \rightarrow \left((a, c) \in \boldsymbol{S} \right) \right) \right) \wedge (\boldsymbol{\mathcal{R}} \subseteq \boldsymbol{S}) \right\}$.

    In other words, transitive closure relation — (i) to contain $\boldsymbol{\mathcal{R}}$, (ii) to satisfy transitivity property, and (iii) being contained within every transitive relation that also containing $\boldsymbol{\mathcal{R}}$.

Closures of relations

- Closure of relation:
  - Procedure to form transitive closure: given $\boldsymbol{\mathcal{R}}$ on $\mathbf{A}$, steps required — (i) adding to $\boldsymbol{\mathcal{R}}$ all new ordered pairs of form (a, c), if not already in $\boldsymbol{\mathcal{R}}$, but corresponding (a, b) and (b, c) already in $\boldsymbol{\mathcal{R}}$, where $a, b, c \in \mathbf{A}$; (ii) repeating step-(i) until no new ordered pairs found.
  - Property:: $\mathcal{Transitive}(\boldsymbol{\mathcal{R}}) = \bigcup_{i=1}^{m} \boldsymbol{\mathcal{R}}^{i} = \boldsymbol{\mathcal{R}} \cup \boldsymbol{\mathcal{R}}^{2} \cup \ldots \cup \boldsymbol{\mathcal{R}}^{m}$, where $|\mathbf{A}| = m$, and $\boldsymbol{\mathcal{R}}^{m} = \boldsymbol{\mathcal{R}}^{m-1} \circ \boldsymbol{\mathcal{R}}$.
  - Property:: For infinite set, $\mathcal{Transitive}(\boldsymbol{\mathcal{R}}) = \boldsymbol{\mathcal{R}}^{*} = \bigcup_{i=1}^{\infty} \boldsymbol{\mathcal{R}}^{i}$.

Closures of relations

- Closure of relation examples:
  - Example-1:: To find reflexive closure of $\mathcal{R} = \{(a, b) \mid a, b \in \mathbb{Z}, a < b\}$.
    For given $\mathcal{R}$, its reflexive closure $= \mathcal{Reflexive}(\mathcal{R}) = \mathcal{R} \cup \Delta_{\mathbf{A}} =$
    $\{(a, b) \mid a, b \in \mathbb{Z}, a < b\} \cup \{(a, a) \mid \forall a \in \mathbb{Z}\} = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$,
    where diagonal relation $\Delta_{\mathbf{A}} = \{(a, a) \mid \forall a \in \mathbb{Z}\}$.
    So, $\mathcal{Reflexive}(\mathcal{R}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.

Closures of relations

- Closure of relation examples:
  - Example-2:: To find symmetric closure of $\mathcal{R} = \{(a, b) \mid a, b \in \mathbb{Z}^+, a > b\}$.
    For given $\mathcal{R}$, $\mathcal{R}^{-1} = \{(b, a) \mid a, b \in \mathbb{Z}^+, (a, b) \in \mathcal{R}\}$.
    So, symmetric closure of $\mathcal{R} = \mathcal{Symmetric}(\mathcal{R}) = \mathcal{R} \cup \mathcal{R}^{-1} =$
    $\{(a, b) \mid a, b \in \mathbb{Z}^+, a > b\} \cup \{(b, a) \mid a, b \in \mathbb{Z}^+, (a, b) \in \mathcal{R}\} =$
    $\{(a, b) \mid a, b \in \mathbb{Z}^+, a \neq b\}$.
    Inequality → justified, as both (a, b) and (b, a) present, where $a > b$, implying $a \neq b$.
    So, $\mathcal{Symmetric}(\mathcal{R}) = \{(a, b) \mid a, b \in \mathbb{Z}^+, a \neq b\}$.

Closures of relations

- Closure of relation examples:
  - Example-3:: To find transitive closure of $\mathcal{R} = \{(1,2), (2,3), (3,3)\}$ on set $\mathbf{A} = \{1, 2, 3\}$.
    For given $\mathcal{R}$, $\mathcal{R}^2 = \mathcal{R} \circ \mathcal{R} = \{(a, c) \mid a, b, c \in \mathbf{A}, \exists b\big((a, b) \in \mathcal{R}\big) \wedge \big((b, c) \in \mathcal{R}\big)\} = \{(1,3), (2,3), (3,3)\}$.
    Similarly, $\mathcal{R}^3 = \mathcal{R}^2 \circ \mathcal{R} = \{(1,3), (2,3), (3,3)\}$.
    So, transitive closure of $\mathcal{R} = \mathcal{Transitive}(\mathcal{R}) = \mathcal{R} \cup \mathcal{R}^2 \cup \mathcal{R}^3 =$
    $\{(1,2), (1,3), (2,3), (3,3)\}$.

Equivalence relations

- Equivalence relation: For binary relation $\mathcal{R}$ on finite set $\mathbf{A}$, $\mathcal{R}$ to become equivalence relation, if $\mathcal{R}$ satisfying reflexivity, symmetry, and transitivity properties.
  - Property:: Notation of equivalence relation $\mathcal{R}$: '$\sim_{\mathcal{R}}$', '$\sim$', '$\equiv_{\mathcal{R}}$'.
  - Property:: Notation of non-equivalence relation $\mathcal{R}$: '$\nsim_{\mathcal{R}}$', '$\nsim$', '$\not\equiv_{\mathcal{R}}$'.
  - Property:: Equivalence relation $\mathcal{R}$ on $\mathbf{A}$: $\forall a{\in}\mathbf{A}\forall b{\in}\mathbf{A}\forall c{\in}\mathbf{A}$ $\Big((a{\sim_{\mathcal{R}}}a) \wedge \big((a{\sim_{\mathcal{R}}}b) \leftrightarrow (b{\sim_{\mathcal{R}}}a)\big) \wedge \Big(\big((a{\sim_{\mathcal{R}}}b) \wedge (b{\sim_{\mathcal{R}}}c)\big) \rightarrow (a{\sim_{\mathcal{R}}}c)\Big)\Big)$, where, $(a{\sim_{\mathcal{R}}}b)$= arbitrary elements $a{\in}\mathbf{A}$ and $b{\in}\mathbf{A}$ related by $\mathcal{R}$.
  - Property:: $(a{\sim_{\mathcal{R}}}a)$: every element of $\mathbf{A}$ to be equivalent to itself by $\mathcal{R}$.

Equivalence relations

- Equivalence relation:
  - Property:: $(a{\sim_{\mathcal{R}}}b) \leftrightarrow (b{\sim_{\mathcal{R}}}a)$: arbitrary elements a and b in $\mathbf{A}$ equivalent to each other by $\mathcal{R}$.
  - Property:: $\big((a{\sim_{\mathcal{R}}}b) \wedge (b{\sim_{\mathcal{R}}}c)\big) \rightarrow (a{\sim_{\mathcal{R}}}c)$: if arbitrary elements a, b equivalent to each other and b, c equivalent to each other by $\mathcal{R}$, then a, c also equivalent to each other by $\mathcal{R}$.

Equivalence classes of relations

- Equivalence class of relation: For binary equivalence relation $\mathcal{R}$ on finite set $\mathbf{A}$, equivalence class of element a (where, $a \in \mathbf{A}$) = set of all elements of $\mathbf{A}$ related to a by $\mathcal{R}$.
  - Property:: Notation of equivalence class of a by $\mathcal{R}$: $[a]_{\mathcal{R}}$, $[a]$.
  - Property:: $[a]_{\mathcal{R}} = \{x \mid \forall x \in \mathbf{A}\ (x{\sim_{\mathcal{R}}}a)\} = \big\{x \mid (\mathcal{R} = \text{equivalence relation}) \wedge \big((a, x) \in \mathcal{R}\big),\ \ a, x \in \mathbf{A}\big\}$.
  - Property:: Representative of equivalence class $[a]_{\mathcal{R}}$: any $b \in \mathbf{A}$, if $b \in [a]_{\mathcal{R}}$.

Equivalence classes of relations

- Equivalence class of relation:
  - Property:: ⟨Theorem⟩: For binary equivalence relation $\mathcal{R}$ on finite set $\mathbf{A}$, statements (i), (ii) and (iii) to be equivalent for arbitrary elements $a, b \in \mathbf{A}$ — (i) $a\mathcal{R}b$ (or, $a \sim_{\mathcal{R}} b$). (ii) $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$. (iii) $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset$.

  Proof: To successively show: (i)→(ii), (ii)→(iii), and (iii)→(i).

  (A) (i)→(ii): Given binary equivalence relation $\mathcal{R}$, $a\mathcal{R}b$, $[a]_{\mathcal{R}}$, $[b]_{\mathcal{R}}$.
  Antecedent $a\mathcal{R}b \equiv$ T (premise). Then, $b\mathcal{R}a \equiv$ T (as $\mathcal{R}$ symmetric).
  Let arbitrary $c \in [a]_{\mathcal{R}}$. Then, $a\mathcal{R}c \equiv$ T (definition of equivalence class).
  So, $b\mathcal{R}c \equiv$ T (as $\mathcal{R}$ transitive). Then, $c \in [b]_{\mathcal{R}}$ (equivalence class).
  So, all elements of $[a]_{\mathcal{R}}$ in $[b]_{\mathcal{R}}$, i.e., $[a]_{\mathcal{R}} \subseteq [b]_{\mathcal{R}}$.

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Equivalence classes of relations

- Equivalence class of relation:

  Proof contd.:

  (A) (i)→(ii) contd.: $a\mathcal{R}b \equiv$ T (premise). Again, let arbitrary $d \in [b]_{\mathcal{R}}$.
  Then, $b\mathcal{R}d \equiv$ T (definition of equivalence class).
  So, $a\mathcal{R}d \equiv$ T (as $\mathcal{R}$ transitive). Then, $d \in [a]_{\mathcal{R}}$ (equivalence class).
  So, all elements of $[b]_{\mathcal{R}}$ in $[a]_{\mathcal{R}}$, i.e., $[b]_{\mathcal{R}} \subseteq [a]_{\mathcal{R}}$.
  Combining $[a]_{\mathcal{R}} \subseteq [b]_{\mathcal{R}}$ and $[b]_{\mathcal{R}} \subseteq [a]_{\mathcal{R}}$, $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$ (consequent).
  (B) (ii)→(iii): Given equivalence relation $\mathcal{R}$, $[a]_{\mathcal{R}}$, $[b]_{\mathcal{R}}$, $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$.
  Both $[a]_{\mathcal{R}} \neq \emptyset$, $[b]_{\mathcal{R}} \neq \emptyset$ (as $\mathcal{R}$ reflexive; so, $a \in [a]_{\mathcal{R}}$ and $b \in [b]_{\mathcal{R}}$).
  Antecedent $[a]_{\mathcal{R}} = [b]_{\mathcal{R}} \equiv$ T (premise).

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Equivalence classes of relations

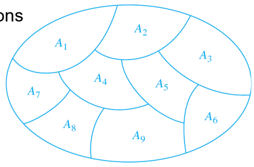
- Equivalence class of relation:

  Proof contd-2.:

  (B) (ii)→(iii) contd.: Then, $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset$ (consequent).
  (C) (iii)→(i): Given equivalence relation $\mathcal{R}$, $[a]_{\mathcal{R}}$, $[b]_{\mathcal{R}}$, $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset$.
  Antecedent $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset \equiv$ T (premise).
  Then, some common arbitrary element c possible, such that $c \in [a]_{\mathcal{R}}$ and $c \in [b]_{\mathcal{R}}$., i.e., $a\mathcal{R}c \equiv$ T and $b\mathcal{R}c \equiv$ T.
  So, $c\mathcal{R}a \equiv$ T and $c\mathcal{R}b \equiv$ T (as $\mathcal{R}$ symmetric).
  Finally, $a\mathcal{R}b \equiv$ T (as $\mathcal{R}$ transitive) (consequent).
  Combining (i)→(ii), (ii)→(iii), and (iii)→(i), (i)≡(ii)≡(iii). ■

Equivalence classes of relations

- Equivalence relation partitioning: For binary equivalence relation $\mathcal{R}$ on finite set $\mathbf{A}$, partitioning of $\mathbf{A}$ by $\mathcal{R}$ = splitting of $\mathbf{A}$ into disjoint nonempty subsets $\mathbf{A}_i$, such that $\bigcup_{i \in \mathbf{I_A}} \mathbf{A}_i = \mathbf{A}$, where $\mathbf{I_A}$ = index set created from partitions of $\mathbf{A}$ by $\mathcal{R}$, and each $\mathbf{A}_i$ being equivalence class formed by $\mathcal{R}$.
  - Property:: $\mathbf{A}_i = [a]_\mathcal{R} = [b]_\mathcal{R} = [c]_\mathcal{R} = \cdots = [x]_\mathcal{R}$, if $a \sim_\mathcal{R} b \sim_\mathcal{R} c \sim_\mathcal{R} \dots \sim_\mathcal{R} x$, $a, b, c, \dots, x \in \mathbf{A}$.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Equivalence classes of relations

- Equivalence relation partitioning:
  - Property:: Equivalence classes of $\mathcal{R}$ on $\mathbf{A} \xrightarrow[\text{partition}]{} \mathbf{A}/\sim_\mathcal{R}$, where $\mathbf{A}/\sim_\mathcal{R} = \{[a]_\mathcal{R} \mid a \in \mathbf{A}\}$ = set of disjoint nonempty subsets of $\mathbf{A}$ after partitioning of $\mathbf{A}$ = quotient set of $\mathbf{A}$ by $\mathcal{R}$.

Equivalence classes of relations

- Equivalence class of relation examples:
  - Example-1:: To find equivalence classes of equivalence relation $\mathcal{R} = \{(a,b) \mid (a = b) \vee (a = -b), \ \ a, b \in \mathbb{Z}\}$.
    Given set: $\mathbb{Z}$. Given relation: $\mathcal{R}$, equivalence.
    For any $x \in \mathbb{Z}$, $[x]_\mathcal{R} = \{x, -x\}$, as $x\mathcal{R}(-x)$ and $(-x)\mathcal{R}x$.
    Number of equivalence classes: infinite.
    Some classes: $[0]_\mathcal{R} = \{0\}$, $[1]_\mathcal{R} = [-1]_\mathcal{R} = \{1, -1\}$, $\cdots$, $[100]_\mathcal{R} = [-100]_\mathcal{R} = \{100, -100\}$, $\cdots$.

Equivalence classes of relations

- Equivalence class of relation examples:
  - Example-1:: To find equivalence classes of equivalence relation $\mathcal{R} = \{(a,b) \mid (a=b) \vee (a=-b),\ \ a,b \in \mathbb{Z}\}$.
    Given set: $\mathbb{Z}$. Given relation: $\mathcal{R}$, equivalence.
    For any $x \in \mathbb{Z}$, $[x]_{\mathcal{R}} = \{x, -x\}$, as $x\mathcal{R}(-x)$ and $(-x)\mathcal{R}x$.
    Number of equivalence classes: infinite.
    Some classes: $[0]_{\mathcal{R}} = \{0\}$, $[1]_{\mathcal{R}} = [-1]_{\mathcal{R}} = \{1,-1\}$, $\cdots$, $[100]_{\mathcal{R}} = [-100]_{\mathcal{R}} = \{100,-100\}$, $\cdots$.

Equivalence classes of relations

- Equivalence class of relation examples:
  - Example-2:: To find equivalence classes of integers 0, 1, 2 and 3 for equivalence relation $\mathcal{R} = \big\{(a,b) \mid \big(a \equiv b \pmod 4\big) \equiv (a - b = k \cdot m),\ a,b,k \in \mathbb{Z}\big\}$.
    Given set: $\mathbb{Z}$. Given relation: $\mathcal{R}$, equivalence.
    $[0]_{\mathcal{R}} = \{a \mid a \equiv 0 \pmod 4,\ a \in \mathbb{Z}\} = \{\cdots,-8,-4,0,4,8,\cdots\}$, as $0\mathcal{R}0$, $0\mathcal{R}4$, $0\mathcal{R}(-4)$, $0\mathcal{R}8$, $0\mathcal{R}(-8)$,$\cdots$ — set of all integers divisible by 4.
    $[1]_{\mathcal{R}} = \{a \mid a \equiv 1 \pmod 4,\ a \in \mathbb{Z}\} = \{\cdots,-7,-3,1,5,9,\cdots\}$, as $1\mathcal{R}1$, $1\mathcal{R}5$, $1\mathcal{R}(-3)$, $1\mathcal{R}9$, $1\mathcal{R}(-7)$,$\cdots$ — set of all integers with absolute remainder 1 when divided by 4.

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Equivalence classes of relations

- Equivalence class of relation examples:
  - Example-2 contd.::
    $[2]_{\mathcal{R}} = \{a \mid a \equiv 2 \pmod 4,\ a \in \mathbb{Z}\} = \{\cdots,-6,-2,2,6,10,\cdots\}$, as $2\mathcal{R}2$, $2\mathcal{R}6$, $2\mathcal{R}(-2)$, $2\mathcal{R}10$, $2\mathcal{R}(-6)$,$\cdots$ — set of all integers with absolute remainder 2 when divided by 4.
    $[3]_{\mathcal{R}} = \{a \mid a \equiv 3 \pmod 4,\ a \in \mathbb{Z}\} = \{\cdots,-5,-1,3,7,11,\cdots\}$, as $3\mathcal{R}3$, $3\mathcal{R}7$, $3\mathcal{R}(-1)$, $3\mathcal{R}11$, $3\mathcal{R}(-5)$,$\cdots$ — set of all integers with absolute remainder 3 when divided by 4.
    Note: Congruence classes modulo 4: equivalence classes $[0]_{\mathcal{R}}$, $[1]_{\mathcal{R}}$, $[2]_{\mathcal{R}}$, $[3]_{\mathcal{R}}$, $[4]_{\mathcal{R}} = [0]_{\mathcal{R}}$, $[5]_{\mathcal{R}} = [1]_{\mathcal{R}}$, $\cdots$.

N-ary relations

- N-ary relation: For finite sets $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ (where $n \in \mathbb{N}$, $n > 2$), n-ary relation $\mathcal{R}$ on $\mathbf{A}_i$ (i=1,2,⋯,n) = subset of $\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_n$.
  - Property:: n-ary relation $\mathcal{R}$ on $\mathbf{A}_i$ (i=1,2,⋯,n) = set $\mathcal{R}$ of ordered pairs (i.e. ordered n-tuples) as elements, i.e., $\mathcal{R} = \{(a_1, a_2, \cdots, a_n) \mid \wedge_{i=1}^{n}(a_i \in \mathbf{A}_i)\}$.
  - Property:: Domain of $\mathcal{R} = \mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$.
  - Property:: Degree of $\mathcal{R} = n$.
  - Applicability:: to model relationships involving more than two sets — (i) computer databases, to answer queries on data in databases, (ii) “comparison", “betweenness", etc. relations in arithmetic.

N-ary relations

- N-ary relation:
  - Property:: Ternary relation $\mathcal{R}$ on $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$: set $\mathcal{R}$ of ordered 3-tuples i.e., $\mathcal{R} = \{(a_1, a_2, a_3) \mid (a_1 \in \mathbf{A}_1) \wedge (a_2 \in \mathbf{A}_2) \wedge (a_3 \in \mathbf{A}_3)\}$.

N-ary relations

- N-ary relation examples:
  - Example-1:: For relation $\mathcal{R} = \{(a, b, c) \mid a, b, c \in \mathbb{N}, a < b < c\}$,
    $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$.
    Degree of $\mathcal{R}$: 3, i.e., ternary relation.
    Domains of $\mathcal{R}$: $\mathbb{N}$ as all 3 domains.
    Membership of $\mathcal{R}$: $(1,2,3) \in \mathcal{R}$, $(2,4,3) \notin \mathcal{R}$.

N-ary relations

- N-ary relation operations:
  - Selection property:: Selection operator $S_{\mathfrak{C}}$ of n-ary relation $\boldsymbol{\mathcal{R}}$ on finite sets $\mathbf{A}_i$ (i=1,2,⋯,n), where elements in $\boldsymbol{\mathcal{R}}$ satisfying condition $\mathfrak{C}$: $S_{\mathfrak{C}}$: $\boldsymbol{\mathcal{R}}$→$\boldsymbol{\mathcal{R}}_{\mathfrak{C}}$ in which $S_{\mathfrak{C}}$ to map $\boldsymbol{\mathcal{R}}$ to 'all n-tuples from $\boldsymbol{\mathcal{R}}$ satisfying $\mathfrak{C}$'.
  - Projection property:: Projection operator $P_{i_1,i_2,\ldots,i_m}$ of n-ary relation $\boldsymbol{\mathcal{R}}$ on finite sets $\mathbf{A}_i$ (i=1,2,⋯,n), where $1 \le i_1 < i_2 < \cdots < i_m \le n$ ($m \in \mathbb{Z}^+$): $P_{i_1,i_2,\ldots,i_m}$: $\boldsymbol{\mathcal{R}}$→$\boldsymbol{\mathcal{R}}_P$ in which $P_{i_1,i_2,\ldots,i_m}$ to map 'all n-tuples from $\boldsymbol{\mathcal{R}}$' to 'corresponding m-tuples projection of $\boldsymbol{\mathcal{R}}$' (where $1 \le m \le n$).
    In other words, $P_{i_1,i_2,\ldots,i_m}$ to delete $n-m$ components of each n-tuple, leaving i_1th, i_2th, … , i_mth components to form m-tuple.

N-ary relations

- N-ary relation *operations*:
  - Join property:: Join operation of relation $\boldsymbol{\mathcal{R}}$ having degree m on finite sets $\mathbf{A}_i$ (i=1,2,⋯,m), with relation $\boldsymbol{S}$ having degree n on finite sets $\mathbf{B}_j$ (j=1,2,⋯,n): $J_p(\boldsymbol{\mathcal{R}},\boldsymbol{S})$ = new relation of degree $m+n-p$ (where, $p \le n$ and $p \le m$), in which $\{(a_1,a_2,\cdots,a_{m-p},c_1,c_2,\cdots,c_p,b_1,b_2,\cdots,b_{n-p}) \mid ((a_1,a_2,\cdots,a_{m-p},c_1,c_2,\cdots,c_p) \in \boldsymbol{\mathcal{R}}) \wedge ((c_1,c_2,\cdots,c_p,b_1,b_2,\cdots,b_{n-p}) \in \boldsymbol{S}) \wedge (c_1,c_2,\cdots,c_p$ in $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{S}$ to agree$)\}$.
    In other words, $J_p(\boldsymbol{\mathcal{R}},\boldsymbol{S})$ to combine all m-tuples of $\boldsymbol{\mathcal{R}}$ with all n-tuples of $\boldsymbol{S}$, where last p components of m-tuples to agree with first p components of n-tuples.

Summary

- Focus: Relations (contd.).
- Closure of relation.
- Reflexive closure of binary relation, and its properties, with examples.
- Symmetric closure of binary relation, and its properties, with examples.
- Transitive closure of binary relation, and its properties, with examples.
- Equivalence binary relation, and its properties.
- Equivalence class of binary relation.

Summary

- Equivalence relation partitioning.
- N-ary relation, and properties.
- Selection, projection and join operations of n-ary relation.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Mot08] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, PHI, Second edition, 2008.
3. [Lip07] Seymour Lipschutz and Marc Lars Lipson, Schaum's Outline of Theory and Problems of Discrete Mathematics, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Closure of relation:: [Ros19]:628-638.
- Reflexive closure:: [Ros19]:628.
- Symmetric closure:: [Ros19]:629.
- Transitive closure:: [Ros19]:630-633.
- Equivalence relation:: [Ros19]:638-641.
- Equivalence class of relation:: [Ros19]:641-643.
- Equivalence relation partitioning:: [Ros19]:643-646.
- N-ary relation and its operations:: [Ros19]:611-616.