CS34110 Discrete Mathematics and Graph Theory

UNIT – II, Module – 1

Lecture 10: Discrete Structures

[Multiset; Multiset operations; Function; Injection, surjection, bijection; Increasing, decreasing functions; Identity, inverse functions]

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Recap:: Discrete structures

- Discrete structures examples:
  - (i) sets (collections of objects) and multisets;
    (ii) combinations (built from sets; unordered collections of objects to be used in counting);
    (iii) relations (sets of ordered pairs representing relationships between objects);
    (iv) graphs (sets of vertices and edges connecting vertices);
    (v) sequences (ordered lists of elements, as well as special type of functions expressing relationships among elements)
    (vi) matrices.

Multiset

- Multiset (short for Multiple-membership *set*; also called bag, or mset): unordered collection of multiple instances of distinct members, i.e., possibility of any element to occur as member more than once.
  - Property:: Multiset $\mathbf{A} = \{m_1 \cdot a_1, m_2 \cdot a_2, \dots, m_r \cdot a_r\}$, where element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on.
    Alternate representation:: $\mathbf{A} = \{{a_1}^{m(a_1)}, {a_2}^{m(a_2)}, \dots, {a_r}^{m(a_r)}\}$, where $m =$ multiplicity function.
  - Property:: Multiplicities of members of multiset: m_i $(i = 1,2, \dots, r)$, of elements a_i, each m_i being nonnegative integer.

Multiset

- Multiset:
  - Property:: Cardinality of multiset $\mathbf{M}$: $|\mathbf{M}| = \sum_{i=1}^{r} m_i = \sum_{i=1}^{r} m(a_i)$
    $= \sum_{x \in \mathcal{U}} m_{\mathbf{M}}(x)$
  - Inclusion property:: Multiset $\mathbf{M}$ to be subset of multiset $\mathbf{N}$:
    $\mathbf{M} \subseteq \mathbf{N}$ ($\equiv \mathbf{N} \supseteq \mathbf{M}$), if and only if $\forall x \left((x \in \mathcal{U}) \wedge \left(m_{\mathbf{M}}(x) \leq m_{\mathbf{N}}(x) \right) \right)$.
  - Union property:: Union of multisets $\mathbf{M}$ and $\mathbf{N}$:
    $$\mathbf{M} \cup \mathbf{N} = \left\{ x \in \mathcal{U} \;\middle|\; m_{\mathbf{M} \cup \mathbf{N}}(x) = \max\left(m_{\mathbf{M}}(x), m_{\mathbf{N}}(x) \right) \right\}.$$
  - Intersection property:: Intersection of multisets $\mathbf{M}$ and $\mathbf{N}$:
    $$\mathbf{M} \cap \mathbf{N} = \left\{ x \in \mathcal{U} \;\middle|\; m_{\mathbf{M} \cap \mathbf{N}}(x) = \min\left(m_{\mathbf{M}}(x), m_{\mathbf{N}}(x) \right) \right\}.$$

Multiset

- Multiset:
  - Subtraction property:: Difference between multisets $\mathbf{M}$ and $\mathbf{N}$:
    $$\mathbf{M} \setminus \mathbf{N} = \mathbf{M} - \mathbf{N} = \left\{ x \in \mathcal{U} \;\middle|\; m_{\mathbf{M} \setminus \mathbf{N}}(x) = \max\left(\left(m_{\mathbf{M}}(x) - m_{\mathbf{N}}(x) \right), 0 \right) \right\}.$$
  - Addition property:: Sum of multisets $\mathbf{M}$ and $\mathbf{N}$:
    $$\mathbf{M} + \mathbf{N} = \{ x \in \mathcal{U} \mid m_{\mathbf{M}+\mathbf{N}}(x) = \left(m_{\mathbf{M}}(x) + m_{\mathbf{N}}(x) \right) \}.$$

Multiset

- Multiset examples:
  - Example-1:: For multisets $\mathbf{P} = \{4 \cdot a, 1 \cdot b, 3 \cdot c\}$, $\mathbf{Q} = \{3 \cdot a, 4 \cdot b, 2 \cdot d\}$:
    $\mathbf{P} \cup \mathbf{Q} = \left\{ \max\left(m_{\mathbf{P}}(a), m_{\mathbf{Q}}(a) \right) \cdot a, \max\left(m_{\mathbf{P}}(b), m_{\mathbf{Q}}(b) \right) \cdot b, \right.$
    $\left. \max\left(m_{\mathbf{P}}(c), m_{\mathbf{Q}}(c) \right) \cdot c, \max\left(m_{\mathbf{P}}(d), m_{\mathbf{Q}}(d) \right) \cdot d \right\}$
    $= \{\max(4,3) \cdot a, \max(1,4) \cdot b, \max(3,0) \cdot c, \max(0,2) \cdot d\}$
    $= \{4 \cdot a, 4 \cdot b, 3 \cdot c, 2 \cdot d\} = \{a^{(4)}, b^{(4)}, c^{(3)}, d^{(2)}\}$.
    $\mathbf{P} \cap \mathbf{Q} = \left\{ \min\left(m_{\mathbf{P}}(a), m_{\mathbf{Q}}(a) \right) \cdot a, \min\left(m_{\mathbf{P}}(b), m_{\mathbf{Q}}(b) \right) \cdot b, \right.$
    $\left. \min\left(m_{\mathbf{P}}(c), m_{\mathbf{Q}}(c) \right) \cdot c, \min\left(m_{\mathbf{P}}(d), m_{\mathbf{Q}}(d) \right) \cdot d \right\}$
    $= \{\min(4,3) \cdot a, \min(1,4) \cdot b, \min(3,0) \cdot c, \min(0,2) \cdot d\} = \{3 \cdot a, 1 \cdot b\}$.

Multiset

- Multiset examples:
  - Example-1 (contd.)::

    $\mathbf{P} \setminus \mathbf{Q} = \Big\{\max\Big(\big(m_{\mathbf{P}}(a) - m_{\mathbf{Q}}(a)\big), 0\Big) \cdot a, \max\Big(\big(m_{\mathbf{P}}(b) - m_{\mathbf{Q}}(b)\big), 0\Big) \cdot b, \max\Big(\big(m_{\mathbf{P}}(c) - m_{\mathbf{Q}}(c)\big), 0\Big) \cdot c, \max\Big(\big(m_{\mathbf{P}}(d) - m_{\mathbf{Q}}(d)\big), 0\Big) \cdot d\Big\}$

    $= \{\max(1,0) \cdot a, \max(-3,0) \cdot b, \max(3,0) \cdot c, \max(-2,0) \cdot d\}$

    $= \{1 \cdot a, 3 \cdot c\}$.

    $\mathbf{P} + \mathbf{Q} = \Big\{\big(m_{\mathbf{P}}(a), +m_{\mathbf{Q}}(a)\big) \cdot a, \big(m_{\mathbf{P}}(b) + m_{\mathbf{Q}}(b)\big) \cdot b, \big(m_{\mathbf{P}}(c) + m_{\mathbf{Q}}(c)\big) \cdot c, \big(m_{\mathbf{P}}(d) + m_{\mathbf{Q}}(d)\big) \cdot d\Big\} = \{7 \cdot a, 5 \cdot b, 3 \cdot c, 2 \cdot d\}$.

Recap:: Discrete structures

- Discrete structures examples:
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    (vi) matrices.

Function

- Function f from nonempty set $\mathbf{A}$ to nonempty set $\mathbf{B}$: $f: \mathbf{A} \to \mathbf{B}$ = "f maps $\mathbf{A}$ to $\mathbf{B}$" = assignment "$f(a) = b$" of exactly one element $b \in \mathbf{B}$ corresponding to each element $a \in \mathbf{A}$.
  - [Alternate] $f: \mathbf{A} \to \mathbf{B}$ = relation from $\mathbf{A}$ to $\mathbf{B}$ containing one, and only one, ordered pair (a, b), for every element $a \in \mathbf{A}$, with a as first element and $b \in \mathbf{B}$ as lest element, leading to assignment $f(a) = b$.
  - Property:: $\mathbf{A}$: domain of f.
  - Property:: $\mathbf{B}$: codomain of f.
  - Property:: b: image of a.
  - Property:: a: preimage of b.

Function

- Function:
  - Property:: Range of f — set of all images of elements of $\mathbf{A}$.
  - Property:: Range: also called image. Range of $f \subseteq$ codomain $\mathbf{B}$.
  - Property:: one-to-one (or injective) function f — $\forall a \forall b\big((f(a) = f(b)) \rightarrow (a = b)\big) \equiv \forall a \forall b\big((a \neq b) \rightarrow (f(a) \neq f(b))\big)$, where domain of discourse = domain of f.
  - Property:: onto (or surjective) function f — $\forall y \exists x(f(x) = y)$, where domain for x = domain of f, domain for y = codomain of f.
  - Property:: one-to-one correspondence (or bijective) function f — if both one-to-one and onto.

Function

- Function:
  - Property:: Real-valued function — function $f: \mathbf{A} \rightarrow \mathbb{R}$ to be real-valued, if codomain of f = set of real numbers = $\mathbb{R}$.
  - Property:: Integer-valued function — function $f: \mathbf{A} \rightarrow \mathbb{Z}$ to be integer-valued, if codomain of f = set of integers = $\mathbb{Z}$.
  - Property:: Sum of real-valued functions $f_1: \mathbf{A} \rightarrow \mathbb{R}$, $f_2: \mathbf{A} \rightarrow \mathbb{R}$ — real-valued sum function $f_1 + f_2: \mathbf{A} \rightarrow \mathbb{R}$, where $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, for all $x \in \mathbf{A}$.

Function

- Function:
  - Property:: Product of real-valued functions $f_1: \mathbf{A} \rightarrow \mathbb{R}$, $f_2: \mathbf{A} \rightarrow \mathbb{R}$ — real-valued product function $f_1 \cdot f_2: \mathbf{A} \rightarrow \mathbb{R}$, where $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$, for all $x \in \mathbf{A}$.
  - Property:: Sum of integer-valued functions $g_1: \mathbf{A} \rightarrow \mathbb{Z}$, $g_2: \mathbf{A} \rightarrow \mathbb{Z}$ — integer-valued sum function $g_1 + g_2: \mathbf{A} \rightarrow \mathbb{Z}$, where $(g_1 + g_2)(x) = g_1(x) + g_2(x)$, for all $x \in \mathbf{A}$.
  - Property:: Product of integer-valued functions $g_1: \mathbf{A} \rightarrow \mathbb{Z}$, $g_2: \mathbf{A} \rightarrow \mathbb{Z}$ — integer-valued product function $g_1 \cdot g_2: \mathbf{A} \rightarrow \mathbb{Z}$, where $(g_1 \cdot g_2)(x) = g_1(x) \cdot g_2(x)$, for all $x \in \mathbf{A}$.

Function

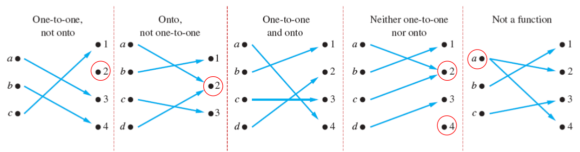
- Function:
  - Property:: Increasing function $f: \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B} \subseteq \mathbb{R}$ —
    $\forall a \forall b\big((a < b) \to (f(a) \le f(b))\big)$, $a, b \in \mathbf{A}$,
    where domain of discourse = domain of $f = \mathbf{A}$.
  - Property:: Strictly increasing function $f: \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B} \subseteq \mathbb{R}$ —
    $\forall a \forall b\big((a < b) \to (f(a) < f(b))\big)$, $a, b \in \mathbf{A}$,
    where domain of discourse = domain of $f = \mathbf{A}$.
  - Property:: Decreasing function $f: \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B} \subseteq \mathbb{R}$ —
    $\forall a \forall b\big((a < b) \to (f(a) \ge f(b))\big)$, $a, b \in \mathbf{A}$,
    where domain of discourse = domain of $f = \mathbf{A}$.

Function

- Function:
  - Property:: Strictly decreasing function $f: \mathbf{A} \to \mathbf{B}$, where $\mathbf{A}, \mathbf{B} \subseteq \mathbb{R}$ —
    $\forall a \forall b\big((a < b) \to (f(a) > f(b))\big)$, $a, b \in \mathbf{A}$,
    where domain of discourse = domain of $f = \mathbf{A}$.
  - Property:: One-to-one (or injective) function: either (i) strictly increasing function, or (ii) strictly decreasing function.
  - Property:: Not one-to-one (or NOT injective) function:
    either (i) increasing function, but not strictly increasing function,
    or (ii) decreasing function, but not strictly decreasing function.
  - Property:: Onto (or surjective) function: equal range and codomain.

Function

- Function:



[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Function

- Function examples:
  - Example-1:: For function g assigning grades {A+, A, B, C, D, F} to set of students $\{\text{stud}_1, \text{stud}_2, \text{stud}_3, \dots, \text{stud}_n\}$ of discrete mathematics course, with no D grades being assigned to any of n students,
    domain of g = set $\{\text{stud}_1, \text{stud}_2, \text{stud}_3, \dots, \text{stud}_n\}$,
    codomain of g = set {A+, A, B, C, D, F},
    range of g = set {A+, A, B, C, F}.

Function

- Function examples:
  - Example-2:: For function f extracting last two bits of bit string of length 2 or greater course,
    domain of f = set of all bit strings of length 2 or greater,
    codomain of f = set $\{00, 01, 10, 11\}$,
    range of f = set $\{00, 01, 10, 11\}$.
  - Example-3:: For $f: \mathbb{Z} \to \mathbb{Z}$ assigning squares of integer, i.e. $f(x) = x^2$,
    domain of f = set of all integers $\mathbb{Z}$,
    codomain of f = set of all integers $\mathbb{Z}$,
    range of f = set of all perfect square integers = $\{0, 1, 4, 9, \dots\}$.

Function

- Function examples:
  - Example-4:: For functions $f_1: \mathbb{R} \to \mathbb{R}$, $f_2: \mathbb{R} \to \mathbb{R}$, such that $f_1(x) = x^2$, $f_2(x) = x - x^2$, their sum and product to be —
    $(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$, for all $x \in \mathbb{R}$; and
    $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = x^2 \cdot (x - x^2) = x^3 - x^4$, for all $x \in \mathbb{R}$.
  - Example-5:: For function $f: \mathbb{Z}^+ \to \mathbb{Z}$, $f(x) = x^2$, to show its one-to-one nature, then in turn, to show $f(x)$ as strictly increasing —
    For any $x, y \in \mathbb{Z}^+$ and assuming $x < y$, to obtain $x^2 < x \cdot y$, and $x \cdot y < y^2$. Combining, $f(x) = x^2 < x \cdot y < y^2 = f(y)$, i.e., $f(x) \neq f(y)$ if $x \neq y$. Thus, $\forall x \forall y \big((x \neq y) \to (f(x) \neq f(y))\big)$.

Function

- Function examples:
  - Example-6:: For function $f: \mathbb{Z} \rightarrow \mathbb{Z}$, where $f(x) = x^2$ —
    Considering $x = 1$ and $y = -1$, where $x \neq y$, $f(x) = 1 = f(y)$.
    So, $\exists x \exists y \neg \big((x \neq y) \rightarrow (f(x) \neq f(y))\big)$, i.e., $\neg \Big(\forall x \forall y \big((x \neq y) \rightarrow (f(x) \neq f(y))\big)\Big)$.
    So, $f: \mathbb{Z} \rightarrow \mathbb{Z}$ not one-to-one.
  - Example-7:: For function $g: \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = x^2$ not one-to-one, for same reason as above.
  - Example-8:: For functions $g_1: \mathbb{R}^+ \rightarrow \mathbb{R}$, $g_1(x) = x^2$, and $g_2: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $g_2(x) = x^2$ both one-to-one, for same reason as given in Example-5.

Function

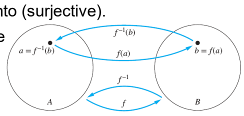
- Function examples:
  - Example-9:: For function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x + 1$, to show one-to-one (alternately) leading to show unique codomain-domain mapping —
    For any $x, y \in \mathbb{R}$, and assuming $f(x) = f(y)$, resulting $x + 1 = y + 1$, i.e., $x = y$. So, $\forall x \forall y \big((f(x) = f(y)) \rightarrow (x = y)\big)$.
    So, $f: \mathbb{R} \rightarrow \mathbb{R}$ one-to-one.
  - Example-10:: For function $g: \mathbb{Z} \rightarrow \mathbb{Z}$, $g(x) = x + 1$, to show onto, thus to show entire codomain-domain mapping (i.e. codomain=range) —
    For arbitrary $y \in \mathbb{Z}$, and assuming $g(x) = y$, resulting in $x + 1 = y$, i.e., $x = y - 1$. So, $x \in \mathbb{Z}$. So, $\forall y \exists x (f(x) = y)$. So, $g: \mathbb{Z} \rightarrow \mathbb{Z}$ onto.

Function

- Function examples:
  - Example-11:: for function $g: \mathbb{Z} \rightarrow \mathbb{Z}$, $g(x) = x^2$ —
    Considering $y = -1$, leading to $x^2 = -1$, resulting in $x \notin \mathbb{Z}$.
    So, $\exists y \neg \big(\exists x (g(x) = y)\big)$, i.e. $\neg \big(\forall y \exists x (g(x) = y)\big)$.
    So, $g: \mathbb{Z} \rightarrow \mathbb{Z}$ not onto.

Function

- Function:
  - Property:: Identity function on $\mathbf{A}$, $\iota_{\mathbf{A}}: \mathbf{A} \to \mathbf{A}$ — assignment "$\iota_{\mathbf{A}}(a) = a$", for all $a \in \mathbf{A}$, i.e., identity function on set assigning each element to itself.
  - Property:: $\iota_{\mathbf{A}}: \mathbf{A} \to \mathbf{A}$ of nature bijective (one-to-one correspondence), i.e. both one-to-one (injective) and onto (surjective).
  - Property:: Inverse function of bijective function $f: \mathbf{A} \to \mathbf{B}$ — assignment "$f^{-1}(b) = a$" only when $f(a) = b$, where $a \in \mathbf{A}$, $b \in \mathbf{B}$.



[Ref: Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.]

Function

- Function examples:
  - Example-12:: For function $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 1$, shown one-to-one and onto (in Example-9,10 of previous lecture), to obtain inverse — For any $y \in \mathbb{Z}$, and considering $y = f(x) = x + 1$, resulting in $x = y - 1$, i.e. $f^{-1}(y) = x = y - 1$. So, $x \in \mathbb{Z}$.
    So, $f: \mathbb{Z} \to \mathbb{Z}$ invertible, where $f^{-1}(y) = y - 1$.

Summary

- Focus: Multiset, function as discrete structures.
- Multiset, and its representation.
- Multiset inclusion, union and intersection, with examples.
- Cardinality of multiset.
- Multiset operations, with examples.
- Function, its representation and related terminologies.
- Injective, surjective, bijective functions, with examples.
- Real-valued, integer-valued functions.
- Increasing, decreasing functions.
- Identity function, inverse function.

References

1. [Ros21] Kenneth H. Rosen, Kamala Krithivasan, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2021.
2. [Ross12] Kenneth A. Ross, Charles R. B. Wright, *Discrete Mathematics*, Fifth edition, Pearson Education, 2012.
3. [Mot15] Joe L. Mott, Abraham Kandel, Theodore P. Baker, *Discrete Mathematics for Computer Scientists and Mathematicians*, Second edition, Pearson Education, 2015.
4. [Lip07] Seymour Lipschutz, Marc L. Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.

Further Reading

- Multiset:: [Ros21]:143-144.
- Function:: [Ros21]:147-149.
- Injective, surjective, bijective functions:: [Ros21]:150-153.
- Real-valued, integer-valued functions:: [Ros21]:149-150.
- Increasing, decreasing functions:: [Ros21]:151.

Lecture Exercises: Problem 1 [Ref: Gate 2024, Set-1, Q.32, p.20]

Let $\mathbf{A}$ and $\mathbf{B}$ be non-empty finite sets such that there exist one-to-one and onto functions — (i) from $\mathbf{A}$ to $\mathbf{B}$, and (ii) from $(\mathbf{A} \times \mathbf{A})$ to $(\mathbf{A} \cup \mathbf{B})$. The number of possible values of $|\mathbf{A}|$ is ___________.

Lecture Exercises: Problem 1 Ans

- Given: $|\mathbf{A}| \neq 0$, $|\mathbf{B}| \neq 0$, bijection $f\colon \mathbf{A} \rightarrow \mathbf{B}$, bijection $g\colon (\mathbf{A} \times \mathbf{A}) \rightarrow (\mathbf{A} \cup \mathbf{B})$.
- For arbitrary $n \in \mathbb{Z}^+$, let "$|\mathbf{A}| = n$" be TRUE. Then, "$|\mathbf{B}| = n$" becomes TRUE, due to existence of bijection $f\colon \mathbf{A} \rightarrow \mathbf{B}$.
- Also, "$|\mathbf{A} \times \mathbf{A}| = |\mathbf{A}| \cdot |\mathbf{A}| = n^2$" becomes TRUE. Then, "$|\mathbf{A} \cup \mathbf{B}| = n^2$" becomes TRUE, due to existence of bijection $g\colon (\mathbf{A} \times \mathbf{A}) \rightarrow (\mathbf{A} \cup \mathbf{B})$.
- Following principle of inclusion-exclusion, $|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$. So, $|\mathbf{A} \cup \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \equiv n^2 \leq 2 \cdot n \equiv n^2 - 2 \cdot n \leq 0 \equiv n \cdot (n-2) \leq 0$.
- For the last simplified inequality to hold, value of n to be within closed interval $[0,2]$. But, $n \neq 0$, as per premise.
- So, $n \in [1,2]$. Only two possible values of $|\mathbf{A}|$.