CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 1

Lecture 29: Graphs & Trees

[Directed graph, multigraph, in-, out-degrees; Digraph walk, trail, path, circuit, component; Rooted tree, subtree; m-ary, binary tree]

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Notation table

Symbol / Notation	Meaning
$\mathcal{G} = (\mathbf{V}, \mathbf{E}, \Psi)$	Directed graph (or digraph); vertex set $\mathbf{V}$, edge set $\mathbf{E}$, mapping Ψ.
$e = (u, v)$	Directed edge (or arc) of digraph, starting from u and ending in v.
$\deg^-(v)$	In-degree (or in-valence) of vertex v in digraph $\mathcal{G}$.
$\deg^+(v)$	Out-degree (or out-valence) of vertex v in digraph $\mathcal{G}$.
$\hat{\mathcal{G}}$	Underlying undirected graph of digraph $\mathcal{G}$.
$\mathcal{T} = (\mathbf{V}, \mathbf{E})$	Rooted tree, with vertex set $\mathbf{V}$, edge set $\mathbf{E}$, root r as one vertex.
$\mathcal{T}$	Rooted subtree of rooted tree $\mathcal{T}$.
$level(v)$	Level of vertex v in rooted tree $\mathcal{T}$.
h	Height of rooted tree $\mathcal{T}$.
m	At most count of child vertices in m-ary tree $\mathcal{T}$.

Digraph

- Directed graph: graph, represented as $\mathcal{G} = (\mathbf{V}, \mathbf{E}, \Psi)$, consisting of nonempty finite set of vertices $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, set of arcs $\mathbf{E} = \{e_1, e_2, \dots\}$, and mapping Ψ to map each arc $e_k \in \mathbf{E}$ onto some ordered pair of vertices (v_i, v_j), $v_i, v_j \in \mathbf{V}$, so that arc e_k to start at v_i and to end at v_j, and $(v_i, v_j) \neq (v_j, v_i)$.
  - Property:: Directed graph $\mathcal{G}$: also called digraph.
  - Property:: Vertex (in $\mathcal{G}$): also called node, point.
  - Property:: Arc (in $\mathcal{G}$): also called directed edge.
  - Property:: Arrow in arc: to depict arc e_k direction from v_i to v_j.

Digraph

- Directed graph:
  - Property:: Self directed loop in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: arc e starting and ending at same vertex, i.e., $e = (v, v)$, $e \in \mathbf{E}$, $v \in \mathbf{V}$.
  - Property:: Multiple directed edges in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: two distinct arcs $e_1, e_2 \in \mathbf{E}$, where $e_1 \neq e_2$, starting and ending at same vertex pair, i.e., $e_1 = (u, v)$, $e_2 = (u, v)$, $e_1 \neq e_2$, $e_1, e_2 \in \mathbf{E}$, $u, v \in \mathbf{V}$.
  - Property:: Possibility of multiple arcs connecting vertices u and v in both directions in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $e_1 = (u, v)$, $e_2 = (v, u)$, $e_1 \neq e_2$, $e_1, e_2 \in \mathbf{E}$, $u, v \in \mathbf{V}$.

Digraph

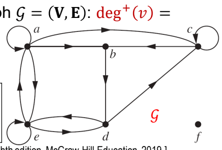
- Directed graph:
  - Property:: Simple directed graph: finite digraph, with each arc starting and ending at two distinct vertices (i.e., no self directed loop) and no two arcs starting and ending at same pair of vertices (i.e., no multiple directed edges between same vertex pair).
  - Property:: Directed multigraph: digraph, with multiple directed edges between same vertex pair;
    multiplicity (of multiple directed edges between specified pair of vertices) = count of multiple distinct directed edges for same pair of vertices.

Digraph

- Digraph fundamentals:
  - Property:: Adjacent vertices in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: vertex u to be adjacent to v (or, v to be adjacent from u) on presence of arc $e = (u, v)$, $u, v \in \mathbf{V}$, $u \neq v$, $e \in \mathbf{E}$.
    u: initial vertex of (u, v).
    v: terminal or end vertex of (u, v).

Digraph

- Digraph fundamentals:
  - Property:: In-degree of vertex in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\deg^-(v) =$ number of arcs with vertex v ($v \in \mathbf{V}$) as terminal vertex. (Also called in-valence.)
  - Property:: Out-degree of vertex in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\deg^+(v) =$ number of arcs with vertex v ($v \in \mathbf{V}$) as initial vertex. (Also called out-valence.)



$\deg^+(a) = 4$	$\deg^-(b) = 2$	$\deg^+(d) = 2$	$\deg^-(e) = 3$
$\deg^-(a) = 2$	$\deg^+(c) = 2$	$\deg^-(d) = 2$	$\deg^+(f) = 0$
$\deg^+(b) = 1$	$\deg^-(c) = 3$	$\deg^+(e) = 3$	$\deg^-(f) = 0$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Digraph

- Digraph fundamentals:
  - Property:: Any self directed loop $e = (v, v)$, $e \in \mathbf{E}$, $v \in \mathbf{V}$ in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ to contribute value 1 to both $\deg^+(v)$ and $\deg^-(v)$.

Digraph

- Digraph fundamentals:
  - Property:: ⟨Theorem⟩: In digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $\mathbf{V} = \{v_1, \ldots, v_n\}$, $\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |\mathbf{E}|$, i.e., sum of in-degrees of all vertices same as sum of out-degrees, also same as number of arcs.

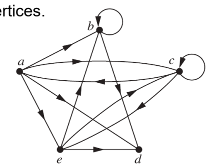
Proof: Each arc $e \in \mathbf{E}$ to have exactly one head and one tail, thereby contributing 1 in-degree $\deg^-$ and 1 out-degree $\deg^+$ in $\mathcal{G}$ respectively. So, sum of all in-degrees $= \sum_{i=1}^{n} \deg^-(v_i) =$ counting all arc heads, and sum of all out-degrees $= \sum_{i=1}^{n} \deg^+(v_i) =$ counting all arc tails. So, total number of arcs $= |\mathbf{E}|$, and all contributing 1 to in-degrees and out-degrees. Hence, $\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |\mathbf{E}|$. ■

Digraph

- Digraph fundamentals:
  - Property:: Underlying undirected graph of digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: undirected graph $\hat{\mathcal{G}}$ resulting from ignoring directions of arcs in $\mathbf{E}$.

Digraph

- Digraph representation:
  - Property:: Adjacency list of any digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: tabular view of initial vertices and their corresponding list (like, array) of terminal vertices.



Adjacency List

Initial Vertex	*Terminal Vertices*
a	*b, c, d, e*
b	*b, d*
c	*a, c, e*
d	
e	*b, c, d*

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Digraph

- Digraph connectedness: vertex connectedness within digraph.
  - Property:: Directed walk of digraph — finite alternating sequence of vertices and arcs in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, beginning and ending with vertices, such that each arc to start from and terminate into vertices preceding and following it, respectively, i.e., $W = \{v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n\}$, where arc $e_i = (v_i, v_{i+1})$, $e_i \in \mathbf{E}$, vertices $v_i, v_{i+1} \in \mathbf{V}$, $i, j = 1, \ldots, (n-1)$.
  - Property:: Vertex repetition, arc repetition **allowed** in directed walk of digraph.

Digraph

- Digraph connectedness:
  - Property:: Directed trail of digraph — finite alternating sequence of vertices and arcs in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, beginning and ending with vertices (called directed trail's terminal vertices), such that — (i) each arc to start from and terminate into vertices preceding and following it, respectively, and (ii) no arc to appear (i.e. covered or traversed) more than once, i.e., $\mathfrak{I} = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where (i) arc $e_i = (v_i, v_{i+1})$, $e_i \in \mathbf{E}$, vertices $v_i, v_{i+1} \in \mathbf{V}$, $i, j = 1, \dots, (n-1)$, (ii) $\forall e_i, e_j \in \mathfrak{I}\ (e_i \neq e_j)$.
  - Property:: Only vertex repetition allowed in directed trail of digraph.

Digraph

- Digraph connectedness:
  - Property:: Closed directed trail — directed trail to begin and end at same terminal vertex.
  - Property:: Open directed trail — directed trail to begin and end at separate, distinct terminal vertices.

Digraph

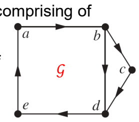
- Digraph connectedness:
  - Property:: Directed path of digraph — open directed trail (i.e. distinct terminal vertices) in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, in which no vertex to appear more than once, i.e., $P = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where (i) arc $e_i = (v_i, v_{i+1})$, $e_i \in \mathbf{E}$, vertices $v_i, v_{i+1} \in \mathbf{V}$, $i, j = 1, \dots, (n-1)$ (ii) $\forall e_i, e_j \in P\ (e_i \neq e_j)$, (iii) $(v_1 \neq v_n)$, (iv) $\forall v_i, v_j \in P\ (v_i \neq v_j)$.
  - Property:: Directed path of digraph $\mathcal{G}$ = Open directed trail of $\mathcal{G}$ without vertex repetition.

Digraph

- Digraph connectedness:
  - Property:: Directed circuit of digraph — closed directed trail (i.e. same terminal vertex) in digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, in which no vertex to appear more than once, i.e., $C = \{v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_{n-1}, v_n\}$, where (i) arc $e_i = (v_i, v_{i+1})$, $e_i \in \mathbf{E}$, vertices $v_i, v_{i+1} \in \mathbf{V}$, $i = 1, \dots, (n-1)$, (ii) $\forall e_i, e_j \left(e_i \neq e_j\right)$, where $e_i, e_j \in C$, (iii) $(v_1 = v_n)$, (iv) $\forall v_i, v_j \left(\left((v_i \neq v_1) \vee (v_j \neq v_n)\right) \rightarrow (v_i \neq v_j)\right)$, where $v_i, v_j \in C$.
  - Property:: Directed circuit of digraph $\mathcal{G}$ = Closed directed trail of $\mathcal{G}$ without vertex repetition.

Digraph

- Digraph connectedness:
  - Property:: Strongly connected digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: in $\mathcal{G}$, presence of sequence of arcs from any vertex to any other vertex, i.e., for every vertex pair $u, v \in \mathbf{V}$, presence of directed path from u to v and directed path from v to u, where each directed path comprising of sequence of arcs.

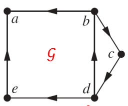
    

    In other words, digraph $\mathcal{G}$ to be strongly connected, if $\forall u,v \in \mathbf{V} \left(\exists P\left((u \in P) \wedge (v \in P)\right)\right)$, where P = directed path in $\mathcal{G}$.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Digraph

- Digraph connectedness:
  - Property:: Weakly connected digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $\mathcal{G}$ not strongly connected, but presence of path from any vertex to any other vertex in $\hat{\mathcal{G}}$ = underlying undirected graph of $\mathcal{G}$.

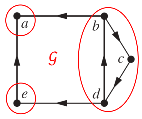
    $u = a, v = b$

    In other words, digraph $\mathcal{G}$ to be weakly connected, if $\exists u,v \in \mathbf{V} \left(\forall P\left((u \notin P) \vee (v \notin P)\right)\right) \wedge \forall u,v \in \mathbf{V} \left(\exists \hat{P} \left((u \in \hat{P}) \wedge (v \in \hat{P})\right)\right)$, where P = directed path in $\mathcal{G}$, and $\hat{P}$ = undirected path of $\hat{\mathcal{G}}$.
  - Property:: Any strongly connected digraph also weakly connected.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Digraph

- Digraph connectedness:
  - Property:: Strong component of any digraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: strongly connected subgraph $\mathcal{g} = (\mathbf{v}, \mathbf{e})$, where $\mathcal{g} \subset \mathcal{G}$, provided $\mathcal{g}$ not contained in any larger strongly connected subgraphs of $\mathcal{G}$.
  - Property:: Component of $\mathcal{G}$ = maximal strongly connected subgraph of $\mathcal{G}$.
  - Property:: Strong component: also called strongly connected component.

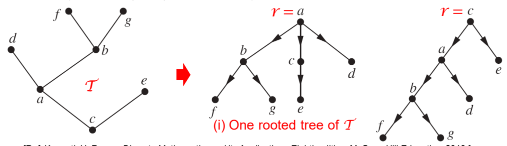


3 strong components of $\mathcal{G}$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

- Rooted tree: tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, of digraph nature, in which one vertex to be designated as root and every edge to become arc directed away from root.
  - Property:: Root (in $\mathcal{T}$): denoted by r.



(i) One rooted tree of $\mathcal{T}$

(ii) Another rooted tree of $\mathcal{T}$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

- Rooted tree:
  - Property:: Parent vertex of any vertex $v \in \mathbf{V}$ in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: unique vertex $u \in \mathbf{V}$, such that arc $e = (u, v)$, $e \in \mathbf{E}$, $(v, u) \notin \mathbf{E}$.
  - Property:: Child vertex of any vertex $v \in \mathbf{V}$ in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: unique vertex $u \in \mathbf{V}$, such that arc $e = (v, u)$, $e \in \mathbf{E}$, $(u, v) \notin \mathbf{E}$.
  - Property:: Sibling vertices in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, $\mathbf{V} = \{v_1, v_2, \ldots, v_n\}$: subset of vertices $\mathbf{v} \subset \mathbf{V}$, where vertices $v_i, v_j \in \mathbf{v}$ $(i, j = 1, \ldots, n)$, such that unique vertex $v_k \in \mathbf{V} \backslash \mathbf{v}$ $(k = 1, \ldots, n)$ as parent, with arcs $e = (v_k, v_i)$, $e' = (v_k, v_j)$, $e, e' \in \mathbf{E}$, $(v_i, v_k) \notin \mathbf{E}$, $(v_j, v_k) \notin \mathbf{E}$.

Rooted tree

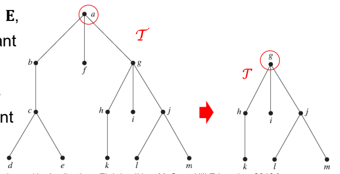
- Rooted tree:
  - Property:: Ancestor vertices of any 'non-root' vertex $v \in \mathbf{V}$ in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: set of vertices (being subset of $\mathbf{V}$) in directed path from root r to v, excluding v itself and including r, i.e., parent of v, its parent's parent, and so on, until r.
  - Property:: Descendant vertices of any vertex $v \in \mathbf{V}$ in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: set of vertices (being subset of $\mathbf{V}$), each having v as its ancestor.

Rooted tree

- Rooted tree:
  - Property:: Leaf vertex in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: any vertex $v \in \mathbf{V}$ having no child vertex.
  - Property:: Internal vertex in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: any vertex $v \in \mathbf{V}$ having at least one child vertex.
  - Property:: Ordered rooted tree: applicable typically to any rooted tree, where child vertices of each internal vertex to be ordered from left to right.

Rooted tree

- Rooted tree:
  - Property:: Rooted subtree of rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: for any vertex $v \in \mathbf{V}$, tree $\mathcal{T} = (\mathbf{v}, \mathbf{e})$ to be subtree of $\mathcal{T}$, with root $r = v$, iff
    (i) $\mathcal{T} \subset \mathcal{T}$, $\mathbf{v} \subseteq \mathbf{V}$, and $\mathbf{e} \subseteq \mathbf{E}$,
    (ii) $\mathbf{v}$ to contain descendant vertices of v,
    (iii) $\mathbf{e}$ to contain all edges incident to descendant vertices of v.



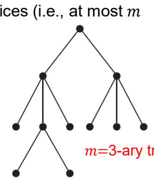
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

- Rooted tree:
  - Property:: Level of vertex in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: for any vertex $v \in \mathbf{V}$, $level(v)$ = length of unique path from root of $\mathcal{T}$ to v.
  - Property:: Level of root vertex in rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: $level(\text{root}) = 0$.
  - Property:: Height of rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: $h = \max\{level(v) \mid \forall v \in \mathbf{V}\}$, i.e., length of longest path from root to any vertex in $\mathcal{T}$.

Rooted tree

- Rooted tree:
  - Property:: m-ary tree: rooted tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, where every internal vertex $v \in \mathbf{V}$ having no more than m child vertices (i.e., at most m child vertices).
  - Property:: Binary tree: m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, where $m = 2$.

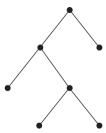


m=3-ary tree

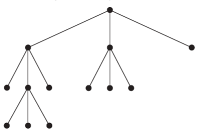
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

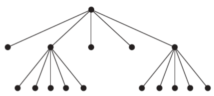
- Rooted tree:
  - Property:: Full m-ary tree: m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, where every internal vertex $v \in \mathbf{V}$ having exactly m child vertices.



Full binary tree



Full 3-ary tree



Full 5-ary tree

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

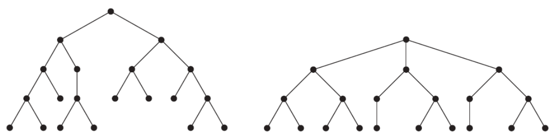
- Rooted tree:
  - Property:: Complete m-ary tree: full m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$, in which every leaf vertex having exactly same level.
  - Property:: Left child vertex of interval vertex $v \in \mathbf{V}$ in binary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: first child vertex of v, when its out-degree $\deg^+(v) = 2$.
  - Property:: Right child vertex of interval vertex $v \in \mathbf{V}$ in binary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: second child vertex of v, when its out-degree $\deg^+(v) = 2$.

Rooted tree

- Rooted tree:
  - Property:: Left subtree of interval vertex $v \in \mathbf{V}$ in binary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: tree $T = (\mathbf{v}, \mathbf{e})$, $T \xrightarrow[\text{subtree}]{} \mathcal{T}$, with root $r = u$, where $u =$ left child of v such that arc $e = (v, u)$, $e \in \mathbf{E}$, $(u, v) \notin \mathbf{E}$.
  - Property:: Right subtree of interval vertex $v \in \mathbf{V}$ in binary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: tree $T = (\mathbf{v}, \mathbf{e})$, $T \xrightarrow[\text{subtree}]{} \mathcal{T}$, with root $r = u'$, where $u' =$ right child of v such that arc $e' = (v, u')$, $e' \in \mathbf{E}$, $(u', v) \notin \mathbf{E}$.

Rooted tree

- Rooted tree:
  - Property:: Balanced m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$: rooted m-ary tree $\mathcal{T}$ having all leaf vertices at levels h or $h-1$, where $h =$ height of $\mathcal{T}$.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Rooted tree

- Rooted tree:
  - Property:: ⟨Theorem⟩: A full m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ with —
    (i) I number of internal vertices in $\mathcal{T}$, to satisfy number of vertices $|\mathbf{V}| = n = m \cdot I + 1$, and number of leaf vertices $l = (m-1) \cdot I + 1$;
    (ii) $|\mathbf{V}| = n$ vertices in $\mathcal{T}$, to satisfy number of internal vertices $I = {}^{n-1}/_{m}$, and number of leaf vertices $l = {}^{(m-1)\cdot n+1}/_{m}$;
    (iii) l number of leaf vertices in $\mathcal{T}$, to satisfy number of vertices $|\mathbf{V}| = n = {}^{m \cdot l - 1}/_{m-1}$, and number of internal vertices $I = {}^{l-1}/_{m-1}$.

Rooted tree

- Rooted tree:
  - Property:: ⟨Theorem⟩: At most m^h leaf vertices possible in any m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ of height h.
  - Property:: ⟨Corollary⟩: For number of leaf vertices $= l$ in any m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ of height h, then $h \geq \lceil \log_m l \rceil$.
  - Property:: ⟨Corollary⟩: For number of leaf vertices $= l$ in any full and balanced m-ary tree $\mathcal{T} = (\mathbf{V}, \mathbf{E})$ of height h, then $h = \lceil \log_m l \rceil$.

Summary

- Focus: Directed graphs, rooted trees.
- Digraph, vertex, arc, with properties.
- Self directed loop, multiple directed edges.
- Simple directed graph, directed multigraph.
- Vertex in-degree, out-degree, with examples.
- Underlying undirected graph of digraph.
- Digraph related theorems, with proofs.
- Digraph representation, with examples.
- Directed walk, directed trail, directed path, directed circuit, with properties and examples.

Summary

- Strongly connected, weakly connected digraphs, with properties and examples.
- Strong component of digraph, with examples.
- Rooted tree, with examples.
- Parent vertex, child vertex, sibling vertices in rooted trees.
- Ancestor, descendant vertices in rooted trees.
- Leaf vertex, internal vertex in rooted trees.
- Ordered rooted tree.
- Rooted subtree, with examples.
- Level of vertex in rooted tree, height of rooted tree.

Summary

- m-ary tree, full m-ary tree, complete m-ary tree, balanced m-ary tree, with examples.
- Binary tree.
- Rooted tree related theorems, with proofs.

References

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