CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 2

Lecture 32: Graph/Tree Connectivity

[Graphic sequence; Degree sequence, theorems; Graph realization problem; Havel-Hakimi theorem, algorithm steps]

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Notation table

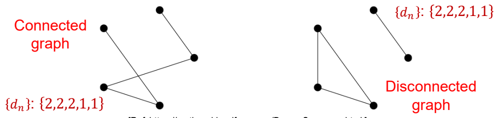
Symbol / Notation	Meaning
$\{d_n\}$	Degree sequence of simple graph of n vertices. Also graphic sequence, if graph realization possible from $\{d_n\}$.

Graphic sequence

- Graphic sequence: solution to problem of finding some or all graphs represented by given degree sequence as nonincreasing sequence of nonnegative integer terms.
  - Property:: Graphic sequence: also called graphical sequence.
  - Property:: Graphic sequence → Degree sequence.

Graphic sequence

- Degree sequence: list of vertex degrees of all vertices of given graph in form of sequence.
  - Property:: Degree sequence of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ ($|\mathbf{V}| = n$): sequence $\{d_n\}$, where $d = \deg(v)$, $v \in \mathbf{V}$, $d \geq 0$; usually listed in monotonic nonincreasing order — $d_1 \geq d_2 \geq \cdots \geq d_n$.



[Ref: https://mathworld.wolfram.com/DegreeSequence.html.]

Graphic sequence

- Degree sequence:
  - Property:: ⟨Theorem⟩: For given finite nonnegative sequence $\{d_n\}$, $\{d_n\}$ to be considered as degree sequence of some graph (with self-loops and parallel edges), if and only if $\sum_{i=1}^{n} d_i$ to become even.

  Proof: Two cases to prove — (A) Degree sequence → Even degree-sum; (B) Even degree-sum → Degree sequence.

  (A) Degree sequence → Even degree-sum. [Necessity]
  Let $\{d_n\}$ representing degree sequence of graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$.
  Then, d_i ($1 \leq i \leq n$) representing vertex degree of some vertex in $\mathcal{G}$.
  So, $\sum_{i=1}^{n} d_i$ to become even, as per Handshaking theorem.

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Graphic sequence

- Degree sequence:

  Proof of ⟨Theorem⟩ contd.:

  (B) Even degree-sum → Degree sequence. [Sufficiency]
  Based on principle of mathematical induction.
  (*Basis step*) $n = 1$. Even nonnegative sum to represent vertex degree of some graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $|\mathbf{V}| = 1$, $|\mathbf{E}| =$ half of sum.
  So, $\{d_1\}$ to become degree sequence.
  (*Inductive step*) Inductive hypothesis: premise that even sum of $n - 1$ terms in $\{d_{n-1}\}$ sequence to become degree sequence of some graph.

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Graphic sequence

- Degree sequence:
  Proof of ⟨Theorem⟩ contd-2.:
  Premise: in sequence $\{d_n\}$, $\sum_{i=1}^{n} d_i = \left(\sum_{i=1}^{n-1} d_i\right) + d_n$ to be even.
  Two possible nature of d_n leading to two cases.
  Case-1: term d_n to be odd.
  Then, $\left(\sum_{i=1}^{n-1} d_i\right)$ also to be odd, so that $\sum_{i=1}^{n} d_i$ to remain even.
  So, sequence $\{d_{n-1}\}$ no longer degree sequence.
  To get back degree sequence from sequence $\{d_{n-1}\}$, let any
  $k \in \{1, \dots, n-1\}$ be chosen from $\{d_{n-1}\}$ so that term d_k be odd.

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Graphic sequence

- Degree sequence:
  Proof of ⟨Theorem⟩ contd-3.:
  Then, new sequence $\{d'_{n-1}\}$ to be defined so that $\left(\sum_{i=1}^{n-1} d'_i\right)$ become even. For sequence $\{d'_{n-1}\}$, to define every term:

$$d'_i = \begin{cases} d_i\,, & i \neq k \\ d_i - 1, & i = k. \end{cases}$$

  Thus, odd term d_k converted to even d'_k, resulting in even $\left(\sum_{i=1}^{n-1} d'_i\right)$.
  Then, based on sequence $\{d'_{n-1}\}$, some graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$, where
  $|\mathbf{V}'| = n - 1$, $\mathbf{V}' = \{v_1, \dots, v_{n-1}\}$, $|\mathbf{E}'| = {}^{1}/_{2} \cdot \left(\sum_{i=1}^{n-1} d'_i\right)$, to be formed,
  such that terms in $\{d'_{n-1}\}$ representing degrees of $v_1, \dots, v_{n-1}$.

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Graphic sequence

- Degree sequence:
  Proof of ⟨Theorem⟩ contd-4.:
  Now, new vertex v_n to be added to $\mathcal{G}'$, with edge $e = \{v_k, v_n\}$.
  This edge e to justify decrement $d_k - 1$ while defining d'_k.
  Graph $\mathcal{G}'$ then having $\mathbf{V}' = \mathbf{V}' \cup \{v_n\}$. In modified $\mathcal{G}'$, to accommodate remaining degrees of vertex v_n (i.e., $d_n - 1$, of even value), ${}^{d_n - 1}/_{2}$ number of self loops of v_n added to $\mathcal{G}'$. Then, $|\mathbf{E}'| = {}^{1}/_{2} \cdot \left(\sum_{i=1}^{n} d_i\right)$.
  So, $\{d_n\}$ to become degree sequence.
  Case-2: term d_n to be even. Then, $\left(\sum_{i=1}^{n-1} d_i\right)$ also to be even. So,
  $\left(\sum_{i=1}^{n-1} d_i\right) - 2$ also even.

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Graphic sequence

- Degree sequence:

  Proof of ⟨Theorem⟩ contd-5.:

  Degree-sum decremented by 2 to accommodate new edges later.
  Then new sequence $\{d''_{n-1}\}$ to be defined to accommodate degree reduction, in which let any $k \in \{1, \ldots, n-1\}$, $l \in \{1, \ldots, n-1\}$, $k \neq l$ be chosen from $\{d_{n-1}\}$ so that term d_k, d_l be even, and to define:

$$d''_i = \begin{cases} d_i, & i \neq k, i \neq l \\ d_i - 1, & i = k \\ d_i - 1, & i = l. \end{cases}$$

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Graphic sequence

- Degree sequence:

  Proof of ⟨Theorem⟩ contd-6.:

  Then, based on sequence $\{d''_{n-1}\}$, some graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$, where $|\mathbf{V}'| = n - 1$, $\mathbf{V}' = \{v_1, \ldots, v_{n-1}\}$, $|\mathbf{E}'| = {}^1/_2 \cdot (\sum_{i=1}^{n-1} d''_i)$, to be formed, representing degrees of $v_1, \ldots, v_{n-1}$.
  Now, new vertex v_n to be added to $\mathcal{G}'$, with two edges $e_1 = \{v_k, v_n\}$, $e_2 = \{v_l, v_n\}$. Both edges e_1, e_2 to justify decrement $d_k - 1$, $d_l - 1$ while defining d''_k, d''_l respectively.
  Graph $\mathcal{G}'$ then having $\mathbf{V}' = \mathbf{V}' \cup \{v_n\}$.

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Graphic sequence

- Degree sequence:

  Proof of ⟨Theorem⟩ contd-7.:

  In modified graph $\mathcal{G}'$, to accommodate remaining degrees of vertex v_n (i.e., $d_n - 2$, of even value), ${}^{d_n-2}/_2$ number of self loops of v_n added to $\mathcal{G}'$. Then, $|\mathbf{E}'| = {}^1/_2 \cdot (\sum_{i=1}^{n} d_i)$.
  So, $\{d_n\}$ to become degree sequence.
  Combinedly, even vertex degree-sum → degree sequence. ■

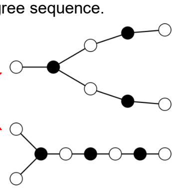
Graphic sequence

- Degree sequence:
  - Property:: ⟨Corollary⟩: Number of odd-valued terms in any degree sequence to be always even.
    [Based on previous theorem: number of vertices of odd degree in undirected graph to be always even.]
  - Property:: For r-regular graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, where $|\mathbf{V}| = n$, its degree sequence $\{d_n\}$ to contain only multiple (here, n) copies of single integer term r.

Graphic sequence

- Degree sequence:
  - Property:: Possibility for two topologically (structurally) distinct graphs to have same degree sequence.

$\{d_n\}$: $\{3,2,2,2,2,1,1,1\}$



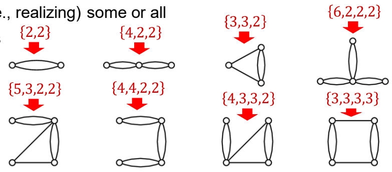
[Ref: https://en.wikipedia.org/wiki/Degree_(graph_theory).]

Graphic sequence

- Degree sequence:
  - Property:: k-connected degree sequence: if some k-connected graph construction possible corresponding to degree sequence.

Graphic sequence

- Graphic sequence:
  - Property:: Graphic sequence: sequence $\{d_n\}$ (usually listed in monotonic nonincreasing order of nonnegative terms), such that constructing (i.e., realizing) some or all possible graphs considering $\{d_n\}$ as degree sequence.



[Ref: J Li et al., Group Connectivity, Strongly $\mathbb{Z}_m$-Connectivity, and Edge Disjoint Spanning Trees, SIAM Journal on Discrete Mathematics 31(3):1909-1922, 2017.]

Graphic sequence

- Graph realization: graph having degree sequence $\{d_n\}$ "realize" $\{d_n\}$.
  - Property:: Graph realization problem: for given finite sequence $\{d_n\}$ of natural numbers, problem of constructing some graph $\mathcal{G}$, such that $\mathcal{G}$ to satisfy given degree sequence $\{d_n\}$.
  - Property:: Graphic: sequence $\{d_n\}$ to be called graphic, if some graph realizing $\{d_n\}$.
  - Property:: Solution to graph realization problem:
    (i) Havel-Hakimi theorem (applicable to simple graphs);
    (ii) Erdős-Gallai theorem.

Graphic sequence

- Graph realization:
  - Property:: ⟨Havel−Hakimi Theorem⟩: For $n > 1$, sequence $\{d_n\}$ to become graphic, if and only if sequence $\{d'_{n-1}\}$ to be graphic, where $\{d'_{n-1}\}$ to be obtained from $\{d_n\}$ by deleting largest term $\hbar$ of $\{d_n\}$ and decrementing 1 from $\hbar$ number of next largest terms of $\{d_n\}$; sequence $\{d_1\}$, where $d_1 = 0$, to be only $n = 1$ (i.e., single-term) graphic sequence.

Proof: For $n = 1$, theorem statement to become trivial.
For $n > 1$, two cases to prove — (A) $\{d_n\}$ graphic → $\{d'_{n-1}\}$ graphic; (B) $\{d'_{n-1}\}$ graphic → $\{d_n\}$ graphic.

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Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd.:

  (A) $\{d_n\}$ graphic → $\{d'_{n-1}\}$ graphic. [Necessity]

  Premise that simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ realizing $\{d_n\}$, and $\hat{\hbar}$ to be largest term of $\{d_n\}$.

  Let $u \in \mathbf{V}$, such that $\deg(u) = \hat{\hbar}$, and neighborhood of u, $N(u) = \{v \in \mathbf{V} \mid v \neq u,\ \exists e \in \mathbf{E}\ (e = \{u, v\})\}$.

  Also, let $\mathbf{v}$ = set of vertices of $\mathcal{G}$ having degrees $d_{l_1}, d_{l_2}, \ldots, d_{l_{\hat{\hbar}}}$, i.e., $\hat{\hbar}$ number of next largest terms in $\{d_n\}$, where $\mathbf{v} \subseteq \mathbf{V}$.

  Two possible cases, based on relation between $N(u)$ and $\mathbf{v}$.

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Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd-2.:

  Case-1: $N(u) = \mathbf{v}$.

  Then deleting u from $\mathcal{G}$ resulting into $\mathcal{G}' = \mathcal{G} - u$, where $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$, such that $\mathbf{V}' = \mathbf{V} \setminus \{u\}$ and $\mathbf{E}' = \mathbf{E} \setminus \{e = \{u, v\} \in \mathbf{E} \mid v \in \mathbf{v}, \mathbf{v} \subseteq \mathbf{V}'\}$.

  Correspondingly, degrees of vertices in $\mathbf{v}$ also decremented by 1, i.e., $d_{l_1} - 1, d_{l_2} - 1, \ldots, d_{l_{\hat{\hbar}}} - 1$. Also, $d_{\hat{\hbar}} = 0$. So, $\{d'_{n-1}\}$ obtained.

  Thus, simple graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ realizing $\{d'_{n-1}\}$.

  Case-2: $N(u) \neq \mathbf{v}$.

  So, at least one vertex $q \in \mathbf{V}$, such that $\big(q \in N(u)\big) \wedge (q \notin \mathbf{v})$.

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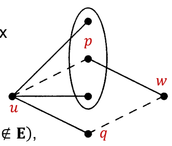
Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd-3.:

  Target: to increase $|N(u) \cap \mathbf{v}|$, preserving vertex degrees to satisfy given degree sequence.

  Then, after choosing $p, w \in \mathbf{V}$, such that $p \in \mathbf{v}$, and $p \notin N(u)$ (due to $\{u, p\} \notin \mathbf{E}$) as well as $\big(w \notin N(u)\big) \wedge (w \notin \mathbf{v})$ and $(\{p, w\} \in \mathbf{E}) \wedge (\{q, w\} \notin \mathbf{E})$, next step: to modify $\mathcal{G}$ to generate $\mathcal{G}^* = (\mathbf{V}, \mathbf{E}^*)$ such that $(\mathbf{E} \neq \mathbf{E}^*) \wedge (|\mathbf{E}| = |\mathbf{E}^*|)$ and $\left\{d_n^{\mathcal{G}}\right\} = \left\{d_n^{\mathcal{G}^*}\right\} = \{d_n\}$, i.e., same degrees.

  [Ref: Douglas B West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.]

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Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd-4.:

  One possibility for $\mathcal{G} \rightarrow \mathcal{G}^*$ to increase $|N(u) \cap \mathbf{v}|$: removing edges $\{u, q\}$, $\{p, w\}$, i.e., $\mathbf{E} = \mathbf{E} \setminus \{\{u, q\}, \{p, w\}\}$, and then adding edges $\{u, p\}$, $\{q, w\}$, i.e., $\mathbf{E} = \mathbf{E} \cup \{\{u, p\}, \{q, w\}\}$.

  Thus, in one attempt, modified graph $\mathcal{G}^*$ generated preserving edge count and degree sequence and increasing $|N(u) \cap \mathbf{v}|$ by 1.

  In case $\mathbf{v} = \emptyset$, increase of $|N(u) \cap \mathbf{v}|$ allowed at most $\hbar$ times.

  After modification complete, $N(u) = \mathbf{v}$, $\deg(u) = \hbar$, and $\mathcal{G}^* = (\mathbf{V}, \mathbf{E}^*)$ realizing $\{d_n\}$.

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Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd-5.:

  Now, following arguments of *Case-1*, $\mathcal{G}'' = \mathcal{G}^* - u$ realizing $\{d'_{n-1}\}$.

  Hence, in either case, $\{d_n\}$ graphic → $\{d'_{n-1}\}$ graphic.

  (B) $\{d'_{n-1}\}$ graphic → $\{d_n\}$ graphic. [Sufficiency]

  Let $\{d_n\}$: $\hbar, d_{l_1}, d_{l_2}, \dots, d_{l_\hbar}, \dots, d_n$ (in nonincreasing order), for some $\hbar \in \mathbb{Z}^+$, $\hbar < n$.

  Premise that simple graph $\breve{\mathcal{G}} = (\breve{\mathbf{V}}, \breve{\mathbf{E}})$ realizing $\{d'_{n-1}\}$, where $\{d'_{n-1}\}$: $d'_1, d'_2, \dots, d'_\hbar, \dots, d'_{n-1}$.

  Now, goal to add new vertex u to $\breve{\mathcal{G}}$ to generate $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.

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Graphic sequence

- Graph realization:

  Proof of ⟨Havel−Hakimi Theorem⟩ contd-6.:

  Addition of u to $\breve{\mathcal{G}}$ also requiring addition of $\hbar$ edges to connect u to $\breve{\mathcal{G}}$, such that $\mathbf{V} = \breve{\mathbf{V}} \cup \{u\}$, $\mathbf{E} = \breve{\mathbf{E}} \cup \{\{u, v\} \mid v \in \breve{\mathbf{V}}, \deg(v) = d_{l_i} - 1, 1 \le i \le \hbar\}$, resulting in $\deg(u) = \hbar$ in $\mathcal{G}$.

  Then, $\{d_n\}$ and $\{d'_{n-1}\}$ to become related as: $\{d'_{n-1}\} = \{d_n\}$ after deleting largest term $\hbar$ of $\{d_n\}$ and decrementing 1 from $\hbar$ number of next largest terms of $\{d_n\}$.

  And, $\mathcal{G}$ realizing $\{d_n\}$, as degrees of vertices in $\mathbf{V}$ matching $\{d_n\}$.

  Hence, $\{d'_{n-1}\}$ graphic → $\{d_n\}$ graphic. ∎

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: ⟨Graph realizing degree sequence steps⟩:

    Input: Degree sequence $\{d_n\}$ of natural numbers.

    Output: $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ ($|\mathbf{V}| = n$), $Adj_{\mathcal{G}} = (a_{ij})_{1 \le i,j \le n}$, or FALSE.

    Step-1: Initialize empty $\mathcal{G}$, i.e., $\mathbf{V} = \mathbf{E} = \emptyset$, $Adj_{\mathcal{G}} \leftarrow n \times n$ zero matrix.

    Step-2: Find largest degree in $\{d_n\}$, and record in d_i, with position i.

    Step-3: Update $\{d_n\}$ by setting 0 in i-th term, and set $\mathbf{V} \leftarrow \mathbf{V} \cup \{v_i\}$.

    Step-4: Find d_i number of terms of largest degrees in $\{d_n\}$, record term positions $\{b_1, \dots, b_{d_i}\}$, and decrement their values by 1.

    Step-5: If any negative terms in $\{d_n\}$, return FALSE and terminate.

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Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm contd.::

    Step-6: For each $j = b_1, \dots, b_{d_i}$, set $\mathbf{V} \leftarrow \mathbf{V} \cup \{v_j\}$, $\mathbf{E} \leftarrow \mathbf{E} \cup \{v_j, v_i\}$, $a_{ij} \leftarrow a_{ij} + 1$, and $a_{ji} \leftarrow a_{ji} + 1$.

    Step-7: Repeat steps 2-6, until terms in $\{d_n\}$ to become all zeros.

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1.

    Given: Degree sequence $\{d_n\}$.

$$\{d_n\}: \{4,3,3,3,3\}.$$

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd.).

Initialize $\mathbf{V}$, $\mathbf{E}$ to empty; term-index array j to zero; $Adj_{\mathcal{G}}$ to zero matrix.

$\mathbf{E} = \{\}$.

Iterations: until only zero terms in $\{d_n\}$.

d_i: $\perp$; i: $\perp$. $\{d_n\}$: $\{4,3,3,3,3\}$.

$\mathbf{V} = \{\}$.

j:

$$Adj = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd-2.).

Iteration $it = 1$

$\mathbf{E} = \{\{v_2, v_1\}, \{v_3, v_1\}, \{v_4, v_1\}, \{v_5, v_1\}\}$.

d_i: 4; i: 1. $\{d_n\}$: $\{\cancel{4}0, \cancel{3}2, \cancel{3}2, \cancel{3}2, \cancel{3}2\}$.

$\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}$.

j:

2	3	4	5	

$$Adj = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd-3.).

Iteration $it = 2$

$\mathbf{E} = \{\{v_2, v_1\}, \{v_3, v_1\}, \{v_4, v_1\}, \{v_5, v_1\}, \{v_3, v_2\}, \{v_4, v_2\}\}$.

d_i: 2; i: 2. $\{d_n\}$: $\{\cancel{4}0, \cancel{32}0, \cancel{32}1, \cancel{32}1, \cancel{3}2\}$.

$\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}$.

j:

3	4			

$$Adj = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd-4.).

    Iteration $it = 3$

    $\mathbf{E} = \{\{v_2, v_1\}, \{v_3, v_1\}, \{v_4, v_1\}, \{v_5, v_1\}, \{v_3, v_2\}, \{v_4, v_2\}, \{v_3, v_5\}, \{v_4, v_5\}\}.$

    $d_i: 2;\ i: 5.$ $\{d_n\}: \{\cancel{4}0, \cancel{32}0, \cancel{321}0, \cancel{321}0, \cancel{32}0\}.$

    $\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}.$

    j:

3	4			

$$Adj = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Graphic sequence

- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd-5.).

    Iteration stop after $it = 3$, as $\{d_n\}$ becoming all zeros.

    $\mathbf{E} = \{\{v_2, v_1\}, \{v_3, v_1\}, \{v_4, v_1\}, \{v_5, v_1\}, \{v_3, v_2\}, \{v_4, v_2\}, \{v_3, v_5\}, \{v_4, v_5\}\}.$

    $d_i: 2;\ i: 5.$ $\{d_n\}: \{\cancel{4}\mathbf{0}, \cancel{32}\mathbf{0}, \cancel{321}\mathbf{0}, \cancel{321}\mathbf{0}, \cancel{32}\mathbf{0}\}.$

    $\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}.$

    j:

$$Adj = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Graphic sequence

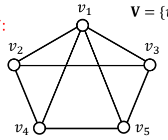
- Graph realization:
  - Havel-Hakimi Algorithm:: Example-1 (contd-6.).

    Graph $\mathcal{G}$, its adjacency matrix $Adj_{\mathcal{G}}$. $\{d_n\}: \{\cancel{4}0, \cancel{32}0, \cancel{321}0, \cancel{321}0, \cancel{32}0\}.$

    $\mathbf{E} = \{\{v_2, v_1\}, \{v_3, v_1\}, \{v_4, v_1\}, \{v_5, v_1\}, \{v_3, v_2\}, \{v_4, v_2\}, \{v_3, v_5\}, \{v_4, v_5\}\}.$

    $\mathbf{V} = \{v_1, v_2, v_3, v_4, v_5\}.$

$\mathcal{G}$:



$$Adj_{\mathcal{G}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Summary

- Focus: Graph realization.
- Graphic sequence.
- Degree sequence, with properties and examples.
- Degree sequence related theorems, with proofs.
- Graphic sequence properties and examples.
- Graph realization.
- Havel-Hakimi theorem, with proof, corresponding algorithm and examples.

References

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