CS34110 Discrete Mathematics and Graph Theory

UNIT – III, Module – 1

Lecture 26: Graphs & Trees

[Special graphs; Bipartite graphs; Adjacency matrix, power; Adjacency list; Incidence, degree matrices; Graph isomorphism; Graph invariants]

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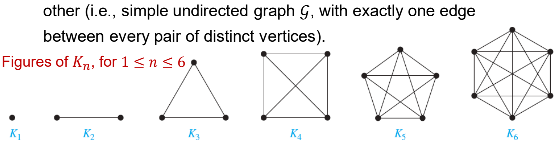
Notation table

Symbol / Notation	Meaning
K_n	Complete undirected graph of n vertices.
C_n	Cycle graph of n vertices.
W_n	Wheel graph of $n+1$ vertices.
Q_n	n-dimensional hypercube graph (also called n-cube graph).
$K_{m,n}$	Complete bipartite graph with m-vertex and n-vertex partitions.
$Adj_{\mathcal{G}}$	Adjacency matrix of simple graph $\mathcal{G}$.
$Adj_{\mathcal{G}}{}^{r}$	r-th power adjacency matrix of simple graph $\mathcal{G}$.
$I_{\mathcal{G}}$	Incidence matrix of simple graph $\mathcal{G}$.
$\tilde{I}_{\mathcal{G}}$	Oriented incidence matrix of simple graph $\mathcal{G}$.
$D_{\mathcal{G}}$	Degree matrix of simple graph $\mathcal{G}$.
$\mathcal{L}_{\mathcal{G}}$	Laplacian matrix of simple graph $\mathcal{G}$.

Special graphs

- Special graphs:
  - Property:: K_n $(n > 0)$: complete undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices, where every distinct vertex pair adjacent to each other (i.e., simple undirected graph $\mathcal{G}$, with exactly one edge between every pair of distinct vertices).

Figures of K_n, for $1 \le n \le 6$



  - Property:: K_n → applicable to unlabeled graph.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

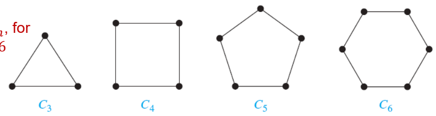
Special graphs

- Special graphs:
  - Property:: Degree of every vertex in K_n $(n > 0)$: $(n - 1)$.
  - Property:: ⟨Theorem⟩: Number of edges in K_n $(n > 0)$: $\binom{n}{2} = \frac{1}{2} \cdot n \cdot (n - 1)$.

Special graphs

- Special graphs:
  - Property:: C_n $(n \geq 3)$: cycle graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, being simple undirected graph with $|\mathbf{V}| = n$ vertices, vertex set $\mathbf{V} = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$, and edge set $\mathbf{E} = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

Figures of C_n, for $3 \leq n \leq 6$

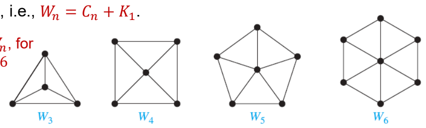


[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Special graphs

- Special graphs:
  - Property:: W_n $(n \geq 3)$: wheel graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, being simple undirected graph with $|\mathbf{V}| = n + 1$, obtained by adding new vertex to C_n and connecting this new vertex to each of n vertices in C_n by new edges, i.e., $W_n = C_n + K_1$.

Figures of W_n, for $3 \leq n \leq 6$

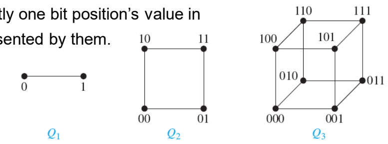


[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Special graphs

- Special graphs:
  - Property:: Q_n $(n > 0)$: n-dimensional hypercube graph (also called, n-cube graph) $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, being simple undirected graph with vertices representing 2^n bit strings of length n and adjacent vertices to differ by exactly one bit position's value in bit strings represented by them.

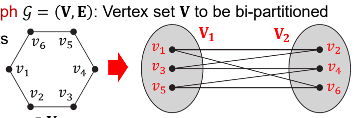
Figures of Q_n, for $1 \leq n \leq 3$



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Special graphs

- Bipartite graph: possibility of bi-partitioning of vertex set of graph.
  - Property:: Bipartite graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: Vertex set $\mathbf{V}$ to be bi-partitioned into two disjoint subsets $\mathbf{V_1}$ and $\mathbf{V_2}$, such that every edge in $\mathcal{G}$ to join vertex $u \in \mathbf{V_1}$ with vertex $v \in \mathbf{V_2}$.



  - Property:: Bipartite graph: also called bigraph, bicolorable graph, pair graph.
  - Property:: Partite sets: $\mathbf{V_1}$ and $\mathbf{V_2}$ in bipartite graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$.

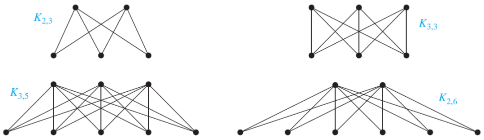
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Special graphs

- Bipartite graph:
  - Property:: Bipartite graph to have — (i) no self-loop, (ii) parallel edges between vertices to be represented by single edge.

Special graphs

- Bipartite graph:
  - Property:: $K_{m,n}$ $(m, n > 0)$: complete bipartite graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, with bi-partitioned $\mathbf{V} = \mathbf{V_1} \cup \mathbf{V_2}$, $|\mathbf{V_1}| = m$, $|\mathbf{V_2}| = n$, and $\mathbf{V_1} \cap \mathbf{V_2} = \emptyset$, such that edge present between every vertex of $\mathbf{V_1}$ to every vertex of $\mathbf{V_2}$.



[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Special graphs

- Bipartite graph:
  - Property:: $K_{1,n}$ $(n > 0)$: also called star.
  - Property:: $K_{1,3}$: also called claw, cherry etc.
  - Property:: ⟨Theorem⟩: Number of edges in $K_{m,n}$ $(m, n > 0) = (m \cdot n)$.

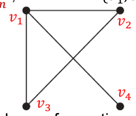
Graph representations

- Graph representation: convenient and simplified representation of graphs for carrying out graph algorithms; common approaches —
  - adjacency matrix;
  - adjacency list;
  - incidence matrix;
  - degree matrix.

Graph representations

- Graph representation:
  - Property:: Adjacency matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $n \times n$ zero-one matrix $Adj_{\mathcal{G}} = (a_{ij})_{1 \le i,j \le n}$, where $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, $|\mathbf{V}| = n$, such that

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in \mathbf{E} \\ 0, & \text{otherwise.} \end{cases}$$



$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

  - Property:: Adjacency matrix of graph based on ordering chosen for vertices in its vertex set.
  - Property:: ⟨Theorem⟩: $n!$ different adjacency matrices for graph with n vertices.

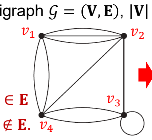
[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph representations

- Graph representation:
  - Property:: For simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$, symmetric adjacency matrix $Adj_{\mathcal{G}} = (a_{ij})_{1 \le i,j \le n} = (a_{ji})_{1 \le i,j \le n}$, $a_{ii} = 0$ (no self-loop).
  - Property:: Adjacency matrix of multigraph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, $|\mathbf{V}| = n$, [also applicable to pseudograph]:

$Adj_{\mathcal{G}} = (a'_{ij})_{1 \le i,j \le n}$, where

$$a'_{ij} = \begin{cases} \text{multiplicity}(v_i, v_j), & \text{if } \{v_i, v_j\} \in \mathbf{E} \\ 0, & \text{if } \{v_i, v_j\} \notin \mathbf{E}. \end{cases}$$



$$\Rightarrow \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph representations

- Graph representation:
  - Property:: Adjacency matrix: also called connection matrix.
  - Property:: For disconnected undirected graph $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$, possibility of two components $\mathcal{g}'_1 = (\mathbf{v}'_1, \mathbf{e}'_1)$, $\mathcal{g}'_2 = (\mathbf{v}'_2, \mathbf{e}'_2)$, where $\mathcal{g}'_1 \cup \mathcal{g}'_2 = \mathcal{G}'$, $\mathbf{v}'_1 \cap \mathbf{v}'_2 = \emptyset$, if and only if possibility to partition $Adj_{\mathcal{G}'}$ as:

$$Adj_{\mathcal{G}'} = \left[\begin{array}{c|c} Adj_{\mathcal{g}'_1} & [0] \\ \hline [0] & Adj_{\mathcal{g}'_2} \end{array}\right], \text{ where } [0] = \text{zero matrix.}$$

Graph representations

- Graph representation:
  - Property:: Power of adjacency matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$:

    $Adj_{\mathcal{G}}{}^2 = Adj_{\mathcal{G}} \cdot Adj_{\mathcal{G}} = \left({}^2a_{ij}\right)_{1 \le i,j \le n}$, where

    if $i \ne j$, ${}^2a_{ij}$ = number of 1's in product of row i and column j

    = number of 1's in product of row i and row j (due to symmetry)

    = number of positions in both row i and row j to have 1's

    = number of vertices adjacent to both i-th vertex v_i and j-th vertex v_j

    = number of different paths between v_i and v_j, where $len(\text{path } P)=2$,

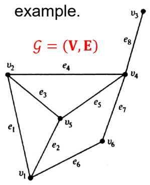
    if $i = j$, ${}^2a_{ij}$ = number of 1's in row i (or column i)

    = degree of vertex v_i, $\deg(v_i)$ (provided no self-loops in $\mathcal{G}$).

Graph representations

- Graph representation:
  - Property:: Power of adjacency matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$:

    example.



$$Adj_{\mathcal{G}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$Adj_{\mathcal{G}}{}^2 = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 1 & 2 & 1 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 2 & 2 \end{bmatrix}$$

$\deg(v_4) = 4$

Two paths between v_5 and v_2 where $len(\text{path})=2$

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph representations

- Graph representation:
  - Property:: Power of adjacency matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$:

    $Adj_{\mathcal{G}}{}^3 = Adj_{\mathcal{G}}{}^2 \cdot Adj_{\mathcal{G}} = \left({}^3a_{ij}\right)_{1 \le i,j \le n}$, where

    if $i \ne j$, ${}^3a_{ij}$ = product of row i of $Adj_{\mathcal{G}}{}^2$ and column/row j of $Adj_{\mathcal{G}}$

    $= \sum_{k=1}^{n}\left({}^2a_{ik} \cdot a_{kj}\right) = \sum_{k=1}^{n}$(number of all different paths of length 3 from v_i to v_j via v_k)

    = number of different paths between v_i and v_j, where $len(\text{path } P)=3$,

    if $i = j$, ${}^3a_{ij}$ = twice of number of different circuits of length 3 (i.e., triangles) passing through v_i

Graph representations

- Graph representation:
  - Property:: ⟨Theorem⟩: For any undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to directed graphs], with adjacency matrix $Adj_{\mathcal{G}}$ for $\mathbf{V} = \{v_1, v_2, \ldots, v_n\}$, $|\mathbf{V}| = n$, $1 \le i, j \le n$, number of different paths of length r ($r \in \mathbb{Z}^+$) from v_i to v_j (i.e., count of different edge sequences of r edges between them) to be obtained from (i, j)-th entry, i.e., ${}^{r}a_{ij} = \sum_{k=1}^{n} \left({}^{r-1}a_{ik} \cdot a_{kj}\right)$ of ${Adj_{\mathcal{G}}}^r$, where ${Adj_{\mathcal{G}}}^r = {Adj_{\mathcal{G}}}^{r-1} \cdot Adj_{\mathcal{G}}$, ${Adj_{\mathcal{G}}}^r = \left({}^{r}a_{ij}\right)_{1 \le i,j \le n}$, ${Adj_{\mathcal{G}}}^{r-1} = \left({}^{r-1}a_{ij}\right)_{1 \le i,j \le n}$, $Adj_{\mathcal{G}} = \left(a_{ij}\right)_{1 \le i,j \le n}$.

Proof: Proof by mathematical induction.[Ros19:723-724;Deo74:161]

Graph representations

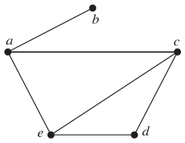
- Graph representation:
  - Property:: ⟨Corollary⟩: For any connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, with adjacency matrix $Adj_{\mathcal{G}}$ for $\mathbf{V} = \{v_1, v_2, \ldots, v_n\}$, $|\mathbf{V}| = n$, $1 \le i, j \le n$, distance between vertices v_i and v_j, $d(v_i, v_j) = \ell$ (for $i \ne j$), if and only if ℓ being smallest integer for which ${}^{\ell}a_{ij} \ne 0$ in ${Adj_{\mathcal{G}}}^{\ell}$.
  - Property:: ⟨Corollary⟩: If any undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$, with adjacency matrix $Adj_{\mathcal{G}}$ for $\mathbf{V} = \{v_1, v_2, \ldots, v_n\}$, $|\mathbf{V}| = n$, and $\mathcal{A} = Adj_{\mathcal{G}} + {Adj_{\mathcal{G}}}^2 + {Adj_{\mathcal{G}}}^3 + \cdots + {Adj_{\mathcal{G}}}^{n-1}$, where $\mathcal{A} = \left(\alpha_{ij}\right)_{1 \le i,j \le n}$, $\alpha_{ij} \in \mathbb{Z}$, then $\mathcal{G}$ to become disconnected if and only if $\exists\, \alpha_{ij} \left(\alpha_{ij} = 0\right)$.

Graph representations

- Graph representation:
  - Benefit of adjacency matrix: efficient for simple dense graph (i.e., graph with many edges, such as graph with more than half of all possible edges), due to very few numerical comparisons to decide presence of edges.
  - Shortcomings of adjacency matrix: special sparse matrix related algorithms required for adjacency matrix of simple sparse graph.

Graph representations

- Graph representation:
  - Property:: Adjacency list of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: tabular view of vertices and their corresponding list (like, array) of adjacent vertices.



Adjacency List	
Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

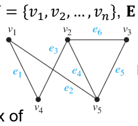
  - Benefit of adjacency list: efficient for simple sparse graph.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph representations

- Graph representation:
  - Property:: Incidence matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ [also applicable to multigraph, pseudograph]: $n \times m$ zero-one matrix $I_{\mathcal{G}} = (b_{ij})_{1 \le i \le n, 1 \le j \le m}$, where $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, $\mathbf{E} = \{e_1, e_2, \dots, e_m\}$, $|\mathbf{V}| = n$, $|\mathbf{E}| = m$, such that

$$b_{ij} = \begin{cases} 1, & \text{if } e_j \text{ incident with } v_i \\ 0, & \text{otherwise.} \end{cases}$$



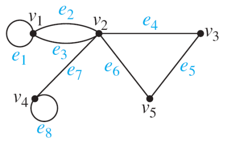
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

  - Property:: For incidence matrix of multigraph, pseudograph, $\mathbf{E}$ to contain parallel edges, self-loops also.

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph representations

- Graph representation:
  - Property:: Incidence matrix of pseudograph: example.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

[Ref: Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.]

Graph representations

- Graph representation:
  - Property:: Oriented incidence matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $n \times m$ matrix $\hat{\mathcal{I}}_{\mathcal{G}} = (\mathbf{b}_j')_{1 \le j \le m}$, where $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, $\mathbf{E} = \{e_1, e_2, \dots, e_m\}$, $|\mathbf{V}| = n$, $|\mathbf{E}| = m$, and $\mathbf{b}_j'$ to be obtained from j-th column of $\mathcal{I}_{\mathcal{G}}$ by arbitrarily replacing anyone of two 1's in that column by -1.

Graph representations

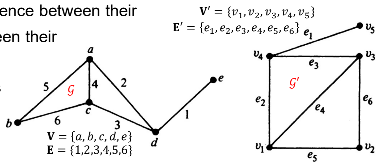
- Graph representation:
  - Property:: Degree matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $n \times n$ diagonal matrix $\mathcal{D}_{\mathcal{G}} = (\deg_{ij})_{1 \le i,j \le n}$, where $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, $|\mathbf{V}| = n$, and $\deg_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$
  - Property:: Laplacian matrix of simple graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$: $n \times n$ matrix $\mathcal{L}_{\mathcal{G}} = \mathcal{D}_{\mathcal{G}} - \mathcal{A}dj_{\mathcal{G}} = (L_{ij})_{1 \le i,j \le n}$, where $\mathbf{V} = \{v_1, v_2, \dots, v_n\}$, $|\mathbf{V}| = n$, and $L_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } i \ne j \text{, and } v_i \text{ adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}$

Graph representations

- Graph representation:
  - Property:: ⟨Theorem⟩: For simple connected undirected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = n$ vertices, $\hat{\mathcal{I}}_{\mathcal{G}} \cdot (\hat{\mathcal{I}}_{\mathcal{G}}^{\mathrm{T}}) = \mathcal{D}_{\mathcal{G}} - \mathcal{A}dj_{\mathcal{G}} = \mathcal{L}_{\mathcal{G}}$.
    [Above equation indicating relation between adjacency and incidence matrices.]

Graph

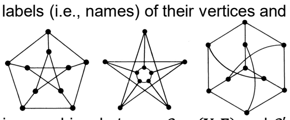
- Special graphs:
  - Property:: Graph isomorphism: $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$ to become equivalent (and called isomorphic to each other) on establishing one-to-one correspondence between their vertices and between their edges preserving incidence relations (for simple graphs $\mathcal{G}, \mathcal{G}'$).



[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph

- Special graphs:
  - Property:: Isomorphic graphs to be same, perhaps drawn differently, with different labels (i.e., names) of their vertices and edges.



  - Condition for isomorphism between $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and $\mathcal{G}' = (\mathbf{V}', \mathbf{E}')$:
    $f\colon \mathbf{V} \to \mathbf{V}'$, with $\forall e \in \mathbf{E}\left((e = \{u, v\}) \to \left(\exists e' \in \mathbf{E}'(e' = \{f(u), f(v)\})\right)\right)$,
    where, $u, v \in \mathbf{V}$, $f(u), f(v) \in \mathbf{V}'$.

[Ref: Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.]

Graph

- Special graphs:
  - Property:: Graph invariants of isomorphic simple graphs —
    i. same number of vertices, due to one-to-one correspondence between vertex sets of isomorphic simple graphs;
    ii. same number of edges, again due to one-to-one correspondence between edges implied from one-to-one correspondence between vertices of isomorphic simple graphs;
    iii. same degrees of vertices in isomorphic simple graphs;
    iv. existence of circuit of same length (say, $k > 2$) in isomorphic simple graphs;

(contd. to next slide)

Graph

- Special graphs:
  - Property:: Graph invariants of isomorphic simple graphs (contd.) —
    v. existence of path traversing through all vertices of isomorphic simple graphs, so that corresponding vertices in two graphs to have same degree.
  - Property:: Determining graph isomorphism based on —
    (a) $Adj_{\mathcal{G}}$, $Adj_{\mathcal{G}'}$,
    (b) $Adj_{\mathcal{G}}{}^{r}$, $Adj_{\mathcal{G}'}{}^{r}$, to justify satisfying these graph invariants.

Summary

- Focus: Special graphs, graph representation, isomorphism.
- Special graphs K_n, C_n, W_n, Q_n, with examples.
- Bipartite graphs and $K_{m,n}$, with examples.
- Graph representations.
- Adjacency matrix of simple graphs, multigraphs, disconnected graphs, with examples.
- Power of adjacency matrix of simple graph, and its properties and related theorems, with examples.
- Adjacency list of simple graph, with examples.

Summary

- Incidence matrix of simple graphs, multigraphs, with examples.
- Oriented incidence matrix of simple undirected graphs.
- Degree matrix of simple graphs.
- Graph isomorphism.
- Isomorphic graphs, with examples.
- Graph invariants of isomorphic simple graphs.

References

1. [Ros19] Kenneth H. Rosen, *Discrete Mathematics and its Applications*, Eighth edition, McGraw-Hill Education, 2019.
2. [Lip07] Seymour Lipschutz and Marc Lars Lipson, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, Third edition, McGraw-Hill Education, 2007.
3. [Wes01] Douglas Brent West, *Introduction to Graph Theory*, Second edition, Prentice-Hall, 2001.
4. [Deo74] Narsingh Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice-Hall, 1974.
5. [Har69] Frank Harary, *Graph Theory*, Addison-Wesley, 1969.