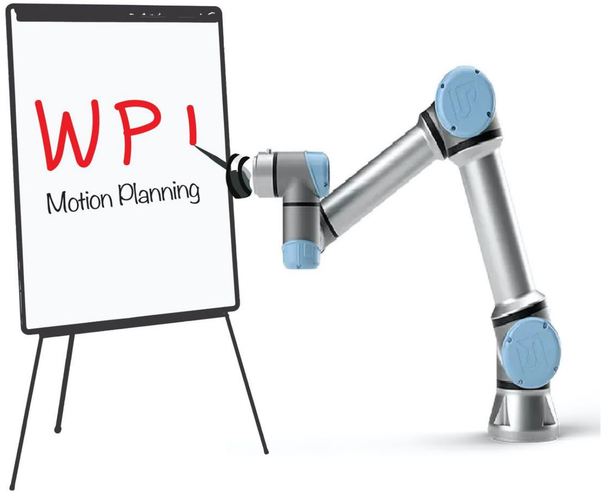
RBE550
Motion Planning
Theoretical Issues



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Disclaimer and Acknowledgments

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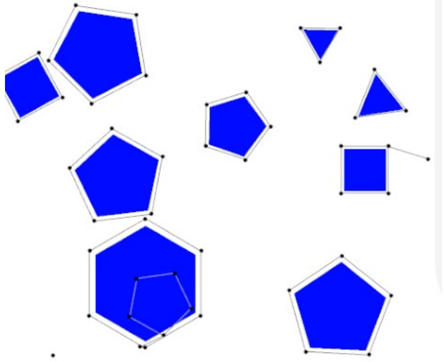
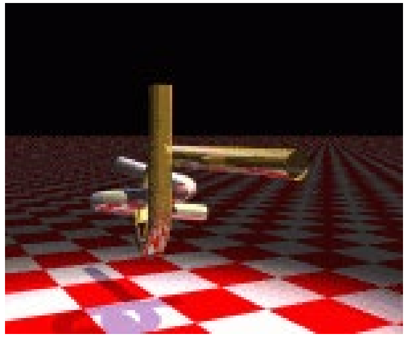
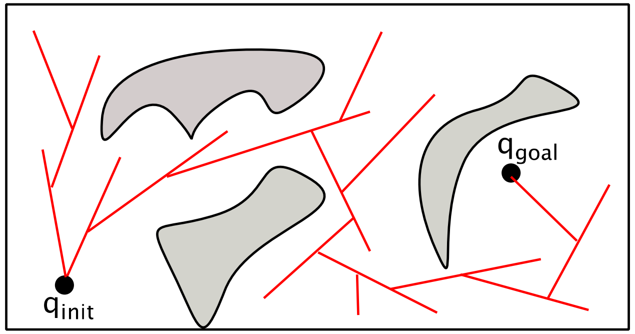
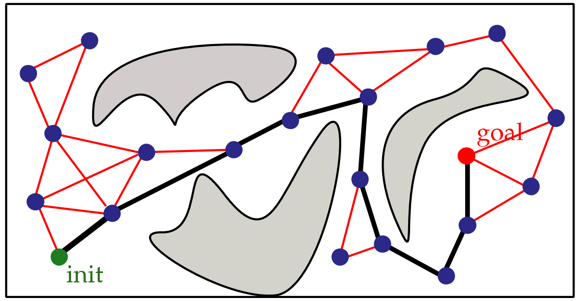
Last time

- Multi-query and Single-query planners
- RRT and variants
- PRM and variants







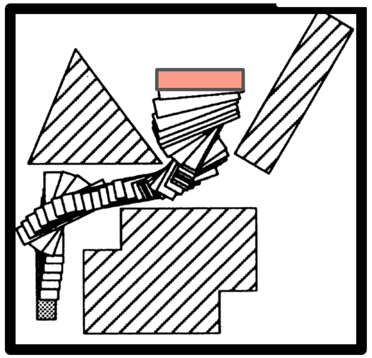


Overview

- Probabilistic Completeness
- Analysis of PRM
- Characterization of space
    - ϵ−good
    - β−lookout
    - $(\epsilon, \alpha, \beta)$−expansive
    - Theoretical results

Sampling-based planners

- How to specify motion for rigid bodies?
- C-space gives us a point robot



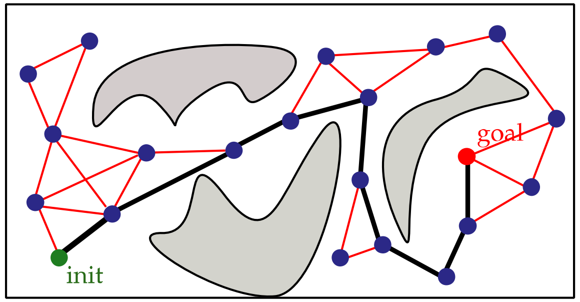
- **Sampling-based technique:**

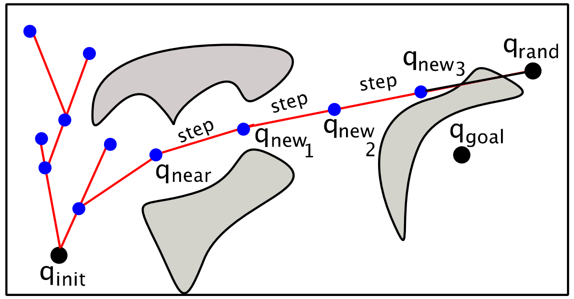
  Advantages

  - Computationally efficient
  - Solves high-dimensional problems
  - Easy to implement

  Disadvantages

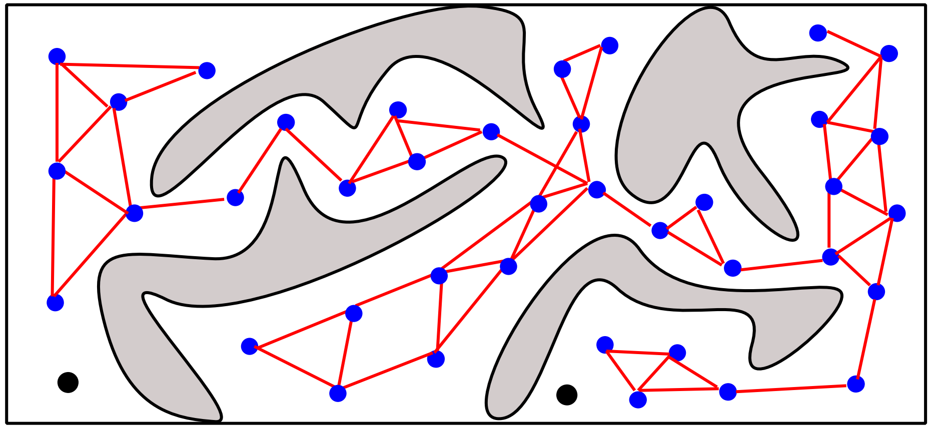
  - Not complete:
    - cannot report when no solution exists
  - Probabilistically complete:
    - If a path exists, the probability of not finding it $\rightarrow 0$ as number of samples $\rightarrow \infty$





Probabilistic RoadMap

- PRM-based planners aim to construct a roadmap that captures the whole connectivity of the configuration space
- Steps: sampling configurations and connecting k nearest



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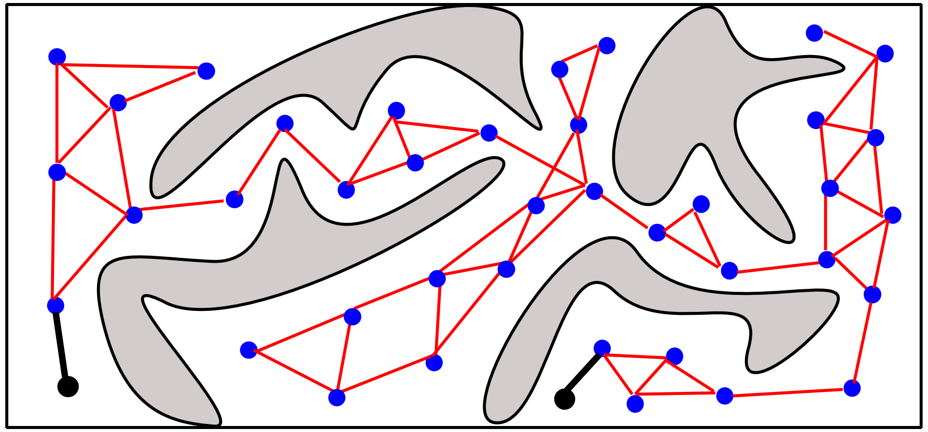


Analysis of Probabilistic RoadMap

- **Probabilistic completeness definition:**
  - Suppose $a, b \in Q_{free}$ can be connected by a path in Q_{free}. A planner is probabilistically complete if

$$\lim_{N\to\infty} \underbrace{\Pr\big((a,b)\ \ Failure\big)} = 0$$

Probability that PRM fails to answer query (a, b) with N samples

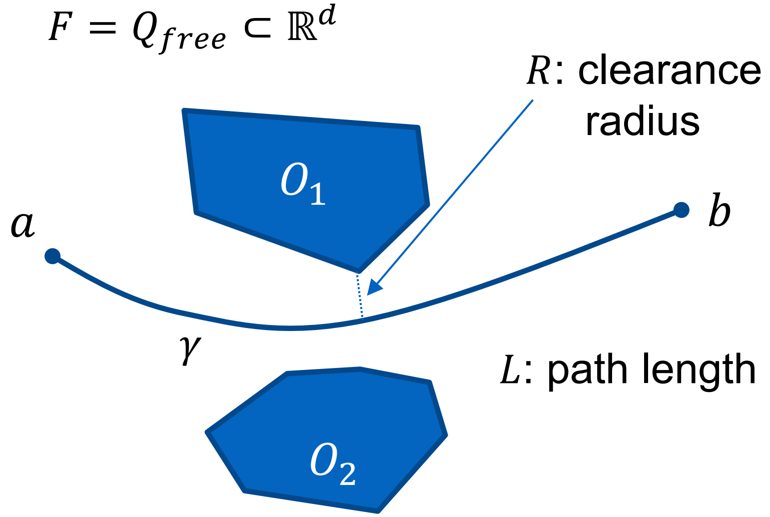
- For analysis, assume:
  - random sampling with uniform distribution in a bounded space
  - $Q_{free} \subset \mathbb{R}^d$

Random Sampling Scheme

- How do various parameters affect performance?
- Can properties of the C-space be related to parameters of the scheme?
- Are there properties of the C-space that make the scheme even more efficient?
- To answer these questions, we consider two notions:
  - Path clearance
  - ϵ –goodness

Path Clearance: Minimum Distance

- $a, b \in F$ can be connected by a path $\gamma: [0, L] \to F$
- L is the length of γ
- R is the minimum distance of γ from O, aka: path clearance radius



- **Theorem:** the probability of failing to connect a and b by a PRM with N samples is

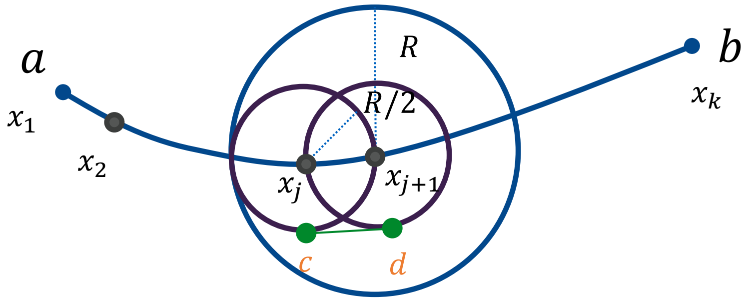
$$\Pr\big((a, b), Failure\big) \le \left\lceil \frac{2L}{R} \right\rceil \left(1 - \alpha\, R^d\right)^N \le \left\lceil \frac{2L}{R} \right\rceil e^{(-\alpha\, R^d\, N)}$$

where

- $\alpha = \dfrac{\mathrm{Vol}\big(B_1(.)\big)}{2^d\ \mathrm{Vol}\,(F)}$
- $\mathrm{Vol(F)}$ is the volume of region (set) F, and $B_1(.)$ is a ball of radius 1.

Path Clearance: Minimum Distance

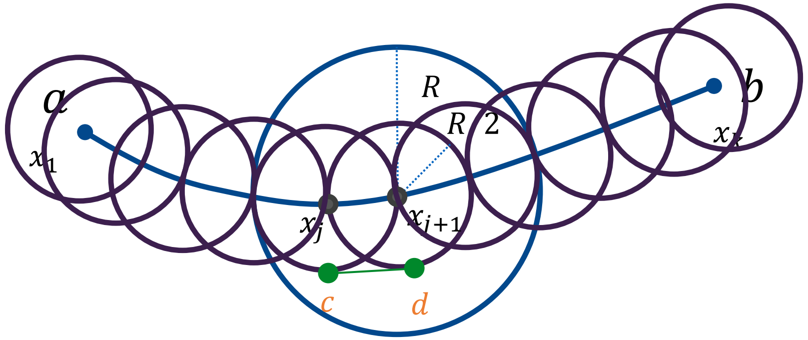
Proof



- Idea: Cover γ with large balls that overlap to a certain extent
- Let $k = \left\lceil \frac{L}{R/2} \right\rceil$
- Choose $x_0 = a, \ldots, x_k = b$ on γ s.t. $|x_j - x_{j+1}| \leq \frac{R}{2}$
- $B_r(x)$: the ball (sphere) with radius r centered at x
- Then:
  - $B_{\frac{R}{2}}(x_j) \subseteq B_R(x_{j+1})$
  - Line $\overline{cd} \subset Q_{free}$ if $c \in B_{R/2}(x_j)$ and $d \in B_{\frac{R}{2}}(x_{j+1})$

Path Clearance: Minimum Distance

Proof (cont.)



- PRM succeeds in answering query (a, b) if there is a sample in every $B_{\frac{R}{2}}(x_i)$ for all $i = 0, \dots, k$

- Let I_i indicate that $B_{\frac{R}{2}}(x_i)$ contains no sample point

- Then, $\Pr\big((a, b)\ Failure\big) \;\leq\; \Pr(I_0 \vee I_1 \vee \cdots \vee I_k) \;\leq\; \Pr(I_0) + \cdots + \Pr(I_k)$

(Union Bound)

- 1st point not in $B_{\frac{R}{2}}(x_i)$: $\Pr(I_i) = \left(1 - \dfrac{\mathrm{vol}\left(B_{\frac{R}{2}}(x_i)\right)}{\mathrm{vol}(F)}\right)$

- N points not in $B_{\frac{R}{2}}(x_i)$: $\Pr(I_i) = \left(1 - \dfrac{\mathrm{vol}\left(B_{\frac{R}{2}}(x_i)\right)}{\mathrm{vol}(F)}\right)^N$ (Independence of Samples)

Path Clearance: Minimum Distance

Proof (cont.)

- $\Pr\big((a,b)\ Failure\big) \le \Pr(I_0) + \cdots + \Pr(I_k)$

$$= k\left(1 - \frac{\mathrm{vol}\left(B_{\frac{R}{2}}(x_i)\right)}{\mathrm{vol}(F)}\right)^N$$

$vol(B_R^d(.)) = vol(B_1^d(.))R^d$, where d is the dimension of the space

$$= \left\lceil \frac{L}{R/2} \right\rceil \left(1 - \frac{\left(\frac{R}{2}\right)^d \mathrm{vol}\big(B_1(\,.\,)\big)}{\mathrm{vol}(F)}\right)^N$$

$$= \left\lceil \frac{2L}{R} \right\rceil \left(1 - \frac{R^d\ \mathrm{vol}\big(B_1(\,.\,)\big)}{2^d\ \ \mathrm{vol}(F)}\right)^N$$

$(1-x) \le e^{-x}$

$$= \left\lceil \frac{2L}{R} \right\rceil \left(1 - R^d\ \alpha\right)^N \le \left\lceil \frac{2L}{R} \right\rceil e^{-\,\alpha\, R^d\, N}$$

Random Sampling Scheme

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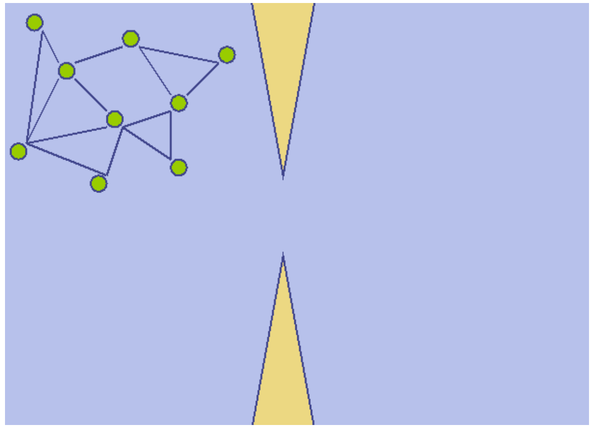
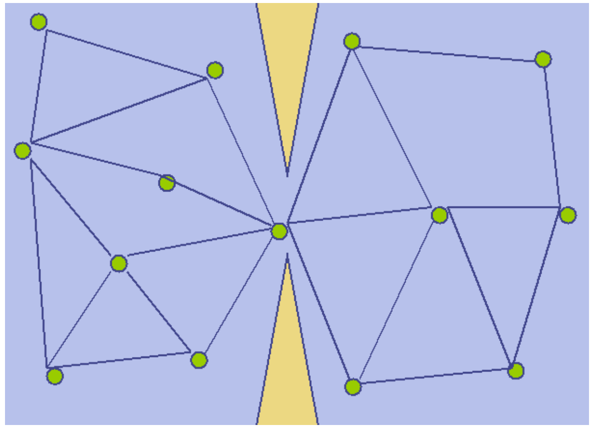
Coverage



VS

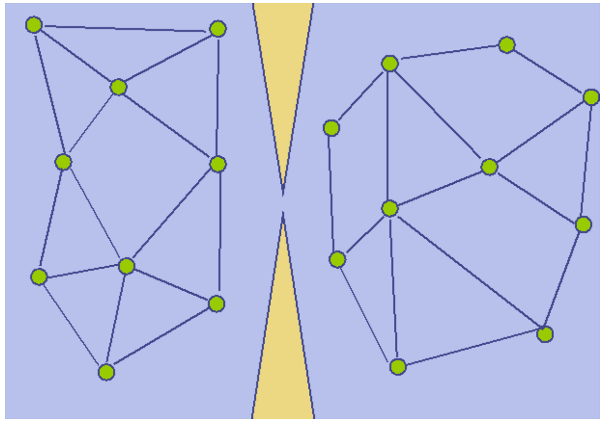


Bad

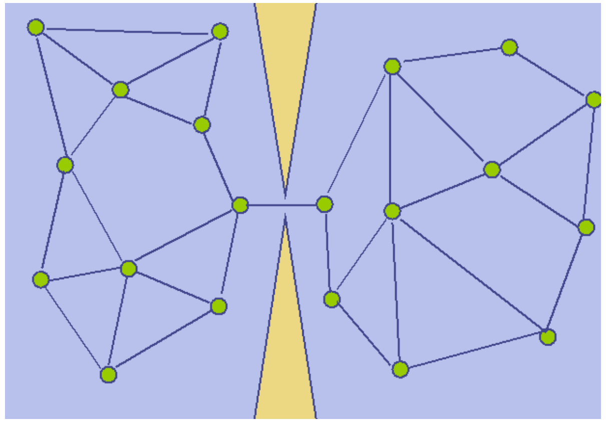
Good

almost any point of the configuration space can be connected
by a straight line segment to some sampled node

Connectivity



VS



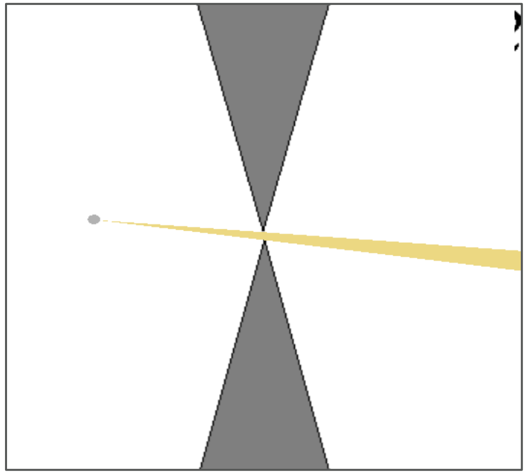
Bad

Good

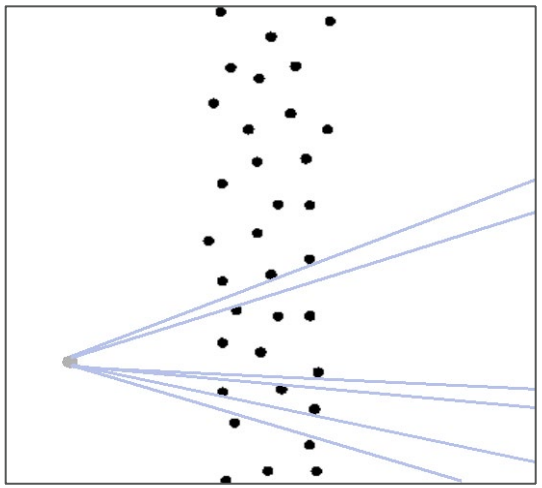
1-1 correspondence between the connected components of the roadmap and those of F

Narrow Passages

- Connectivity is difficult to capture when there are narrow passages
  - A narrow passage is difficult to define!



difficult

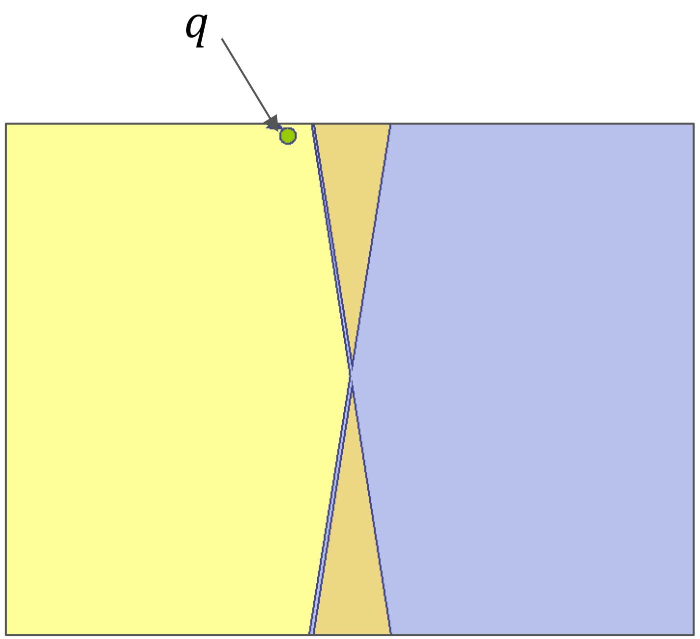


easy

- How to characterize coverage/connectivity (expansiveness)?

Visibility

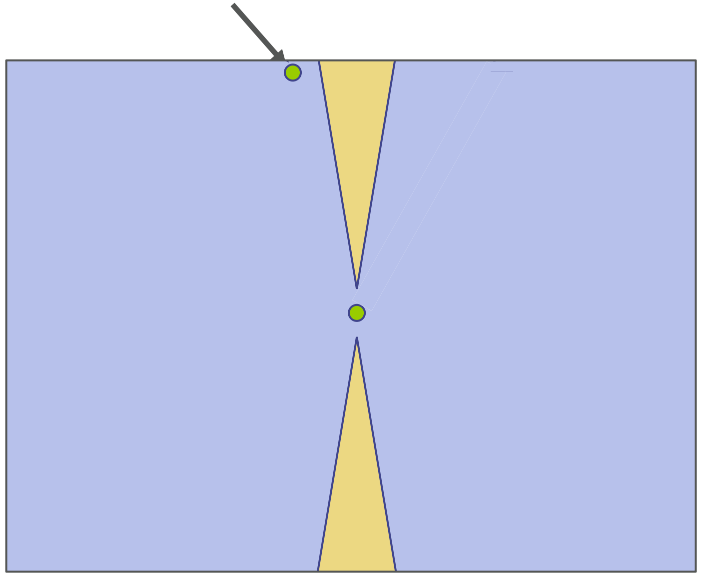
- **Visibility:**
  - All the configurations in free space that can be seen by a free configuration q



ϵ–good

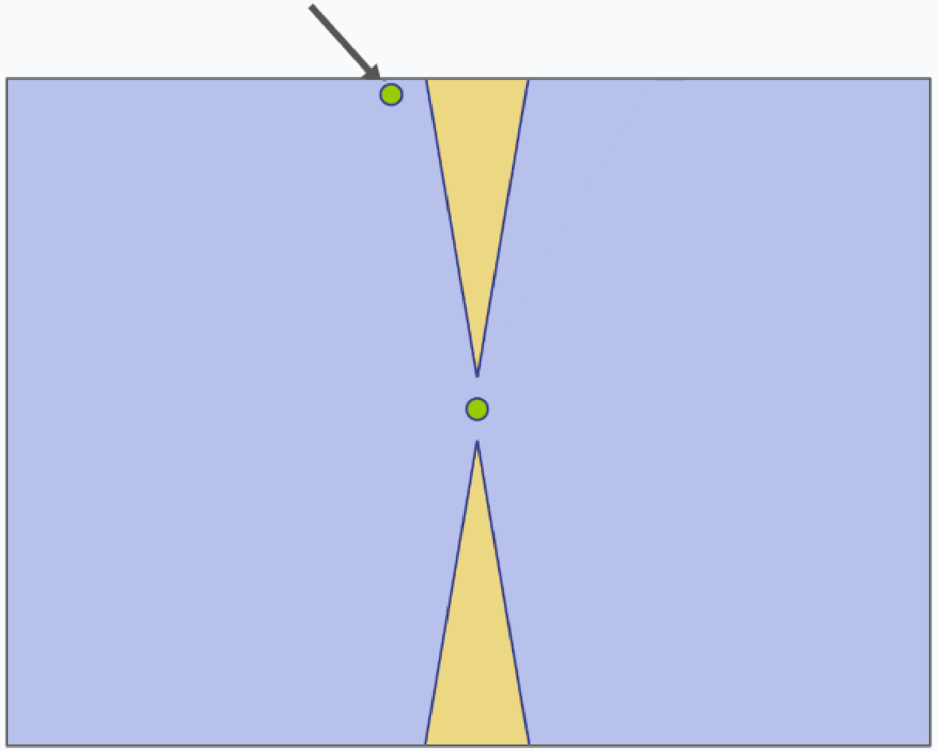
- **ϵ–good:**
  - Every free configuration “sees” at least an ϵ fraction of the free space

$\epsilon \in (0,1]$



What is the fraction ε that this configuration can see?

0



0.1

0.4

0.5

1.0

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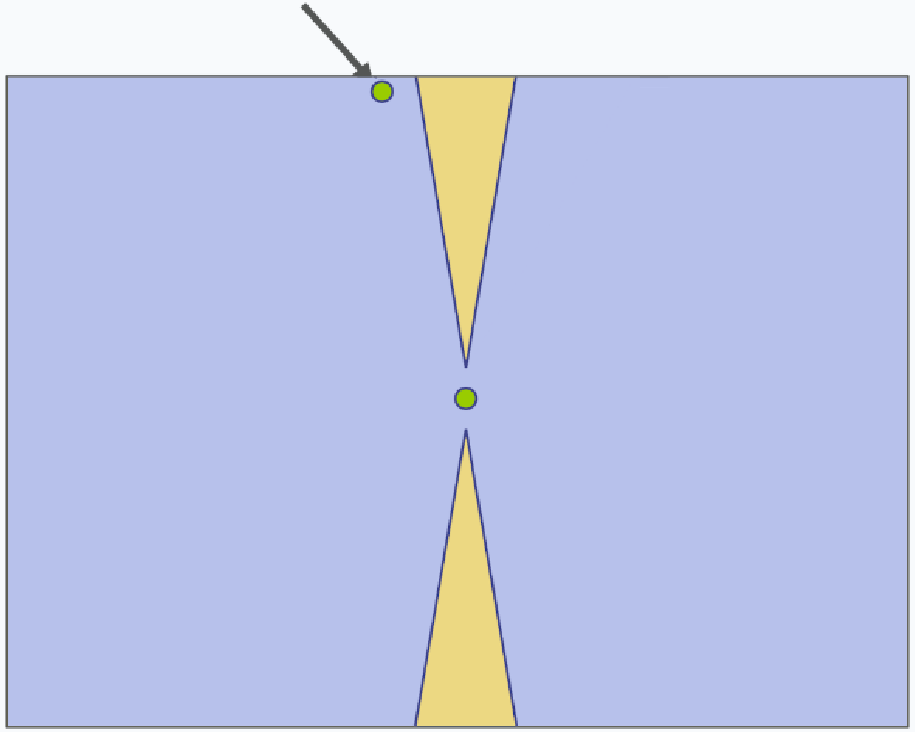
What is the fraction ε that this configuration can see?

0



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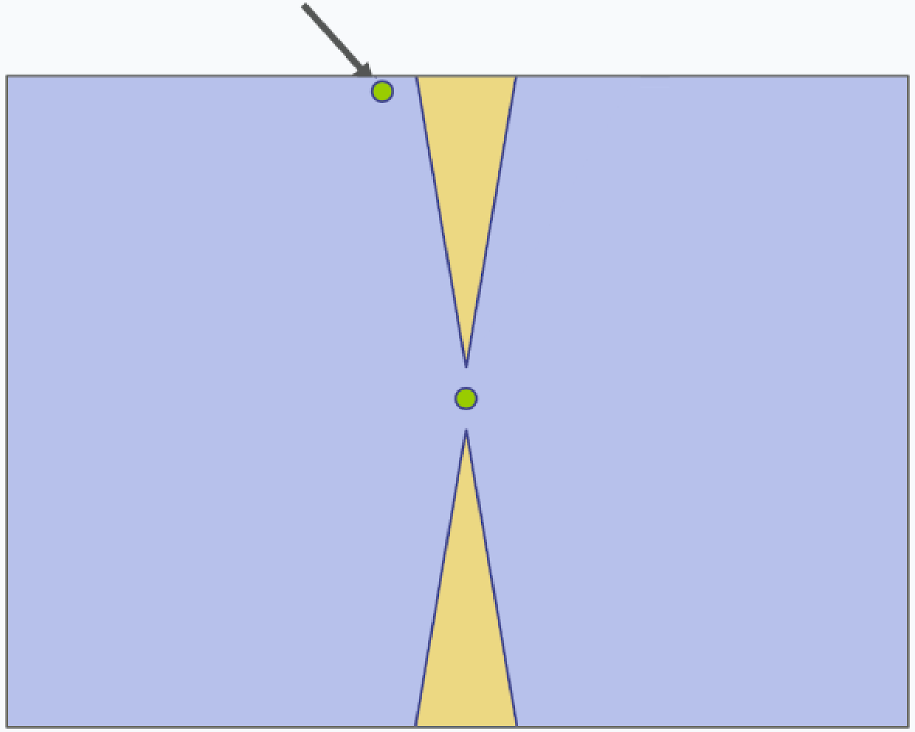
✓✓ 0



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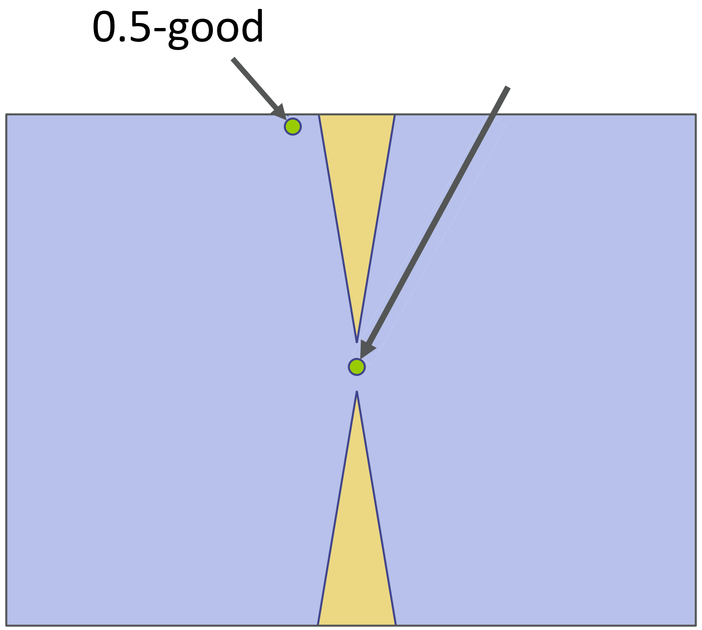
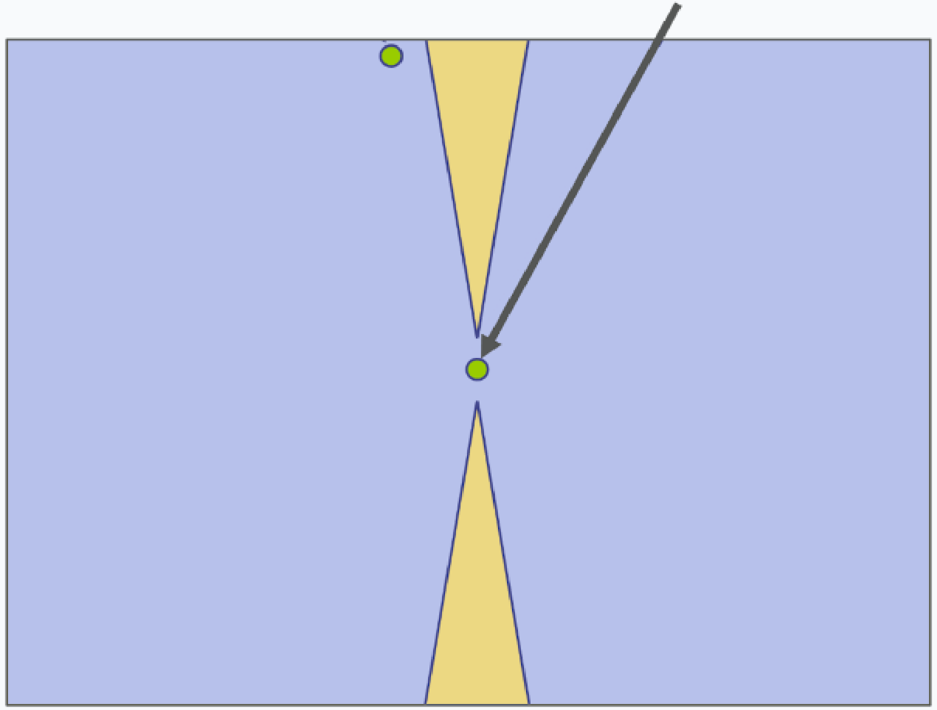


How about this point?

What is the fraction ε that this other configuration can see?

0



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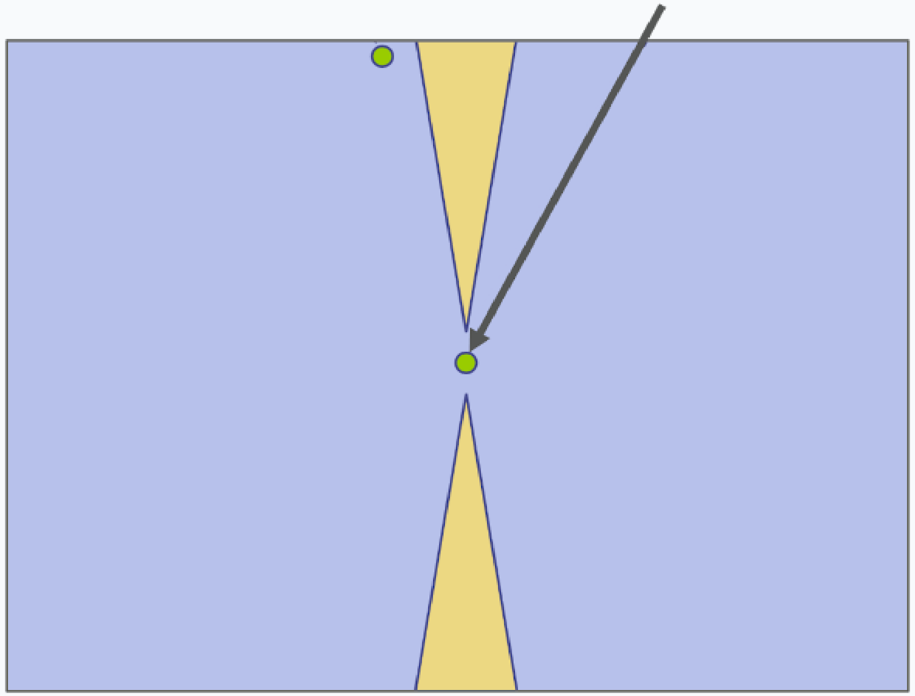
What is the fraction ε that this other configuration can see?

0



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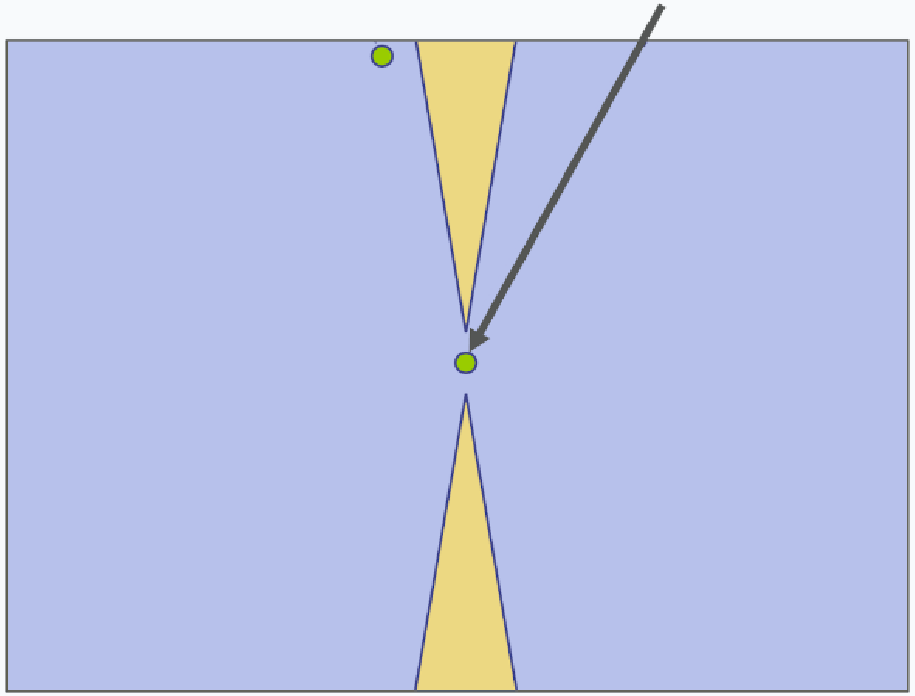
What is the fraction ε that this other configuration can see?

0



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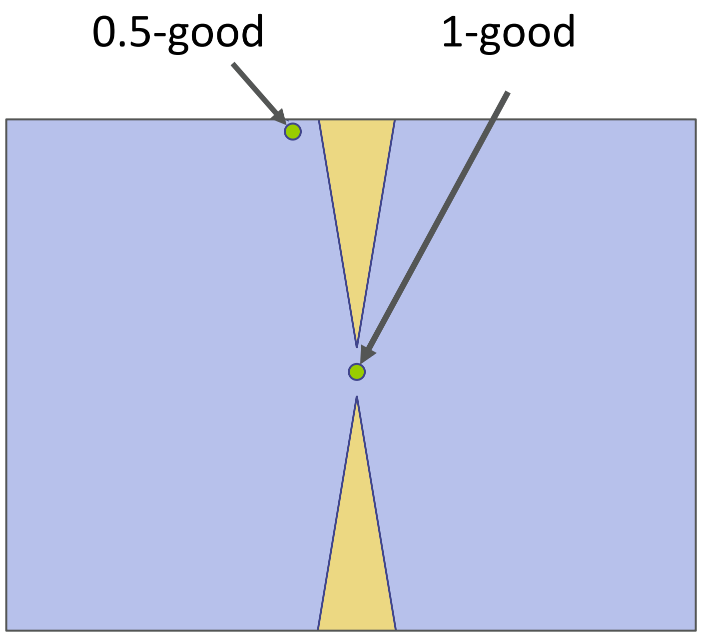
ϵ−good

- **ϵ−good:**
  - Every free configuration “sees” at least an ϵ fraction of the free space
  - A space F is ϵ -good if every configuration can see at least an ϵ fraction of the free space

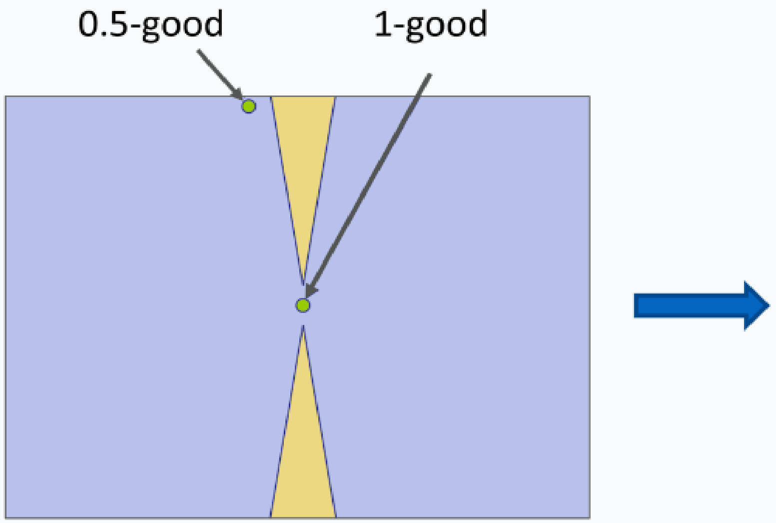
$\epsilon \in (0,1]$



$\Rightarrow$ F is

How about the whole space?

What is ε-goodness of the free space F?



How about the whole space?

F is

- 0.1
- 0.2
- 0.5
- 1.0

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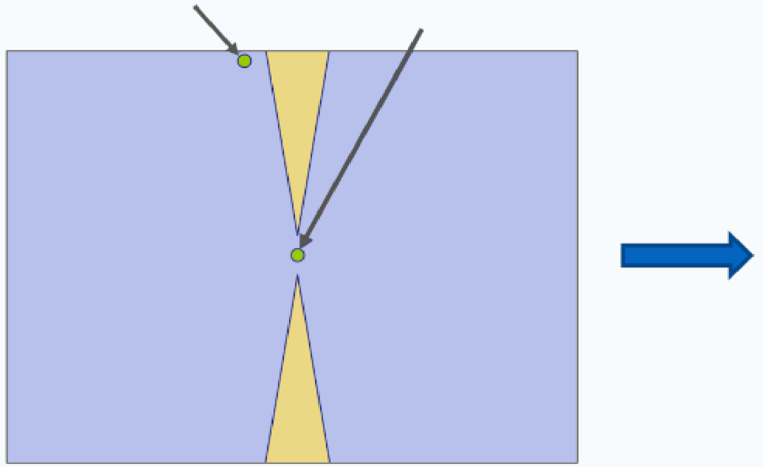
What is ε-goodness of the free space F?



F is

How about the whole space?

0.1	0%
0.2	0%
0.5	0%
1.0	0%

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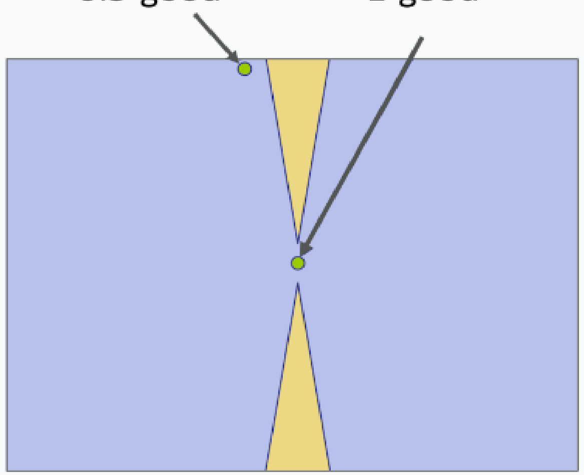
What is ε-goodness of the free space F?



How about the whole space?

F is

0.1 — 0%

0.2 — 0%

0.5 — 0%

1.0 — 0%

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ϵ −good

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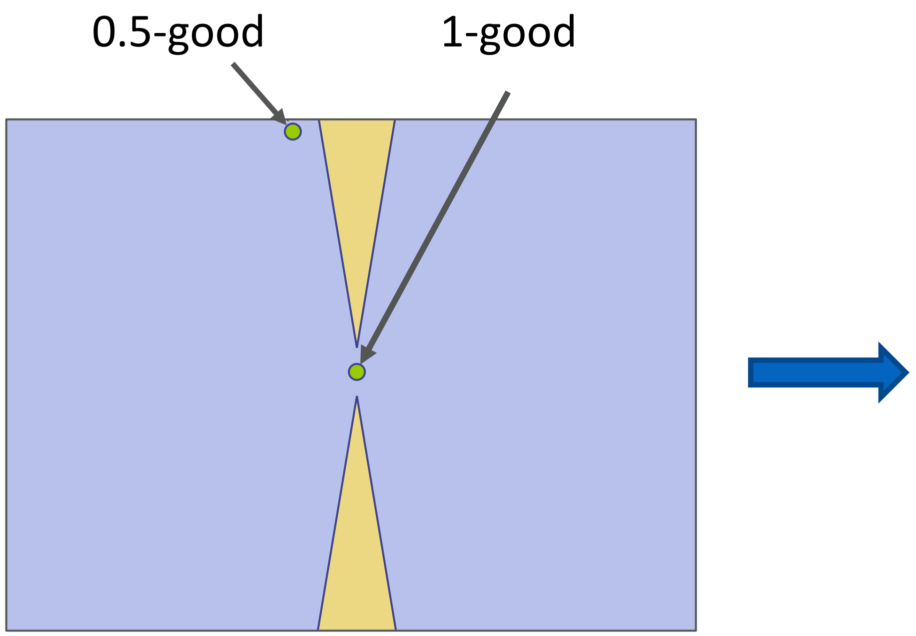
$$\epsilon \in (0,1]$$



F is 0.5-good

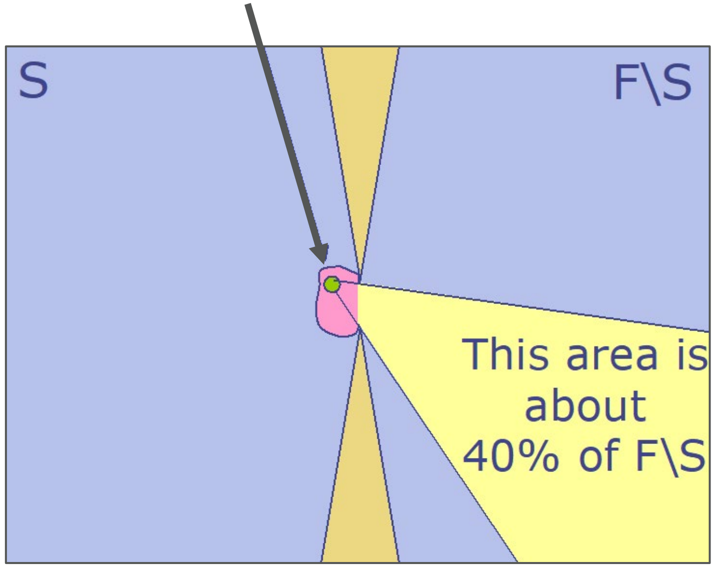
How about the whole Free space?

β –Lookout

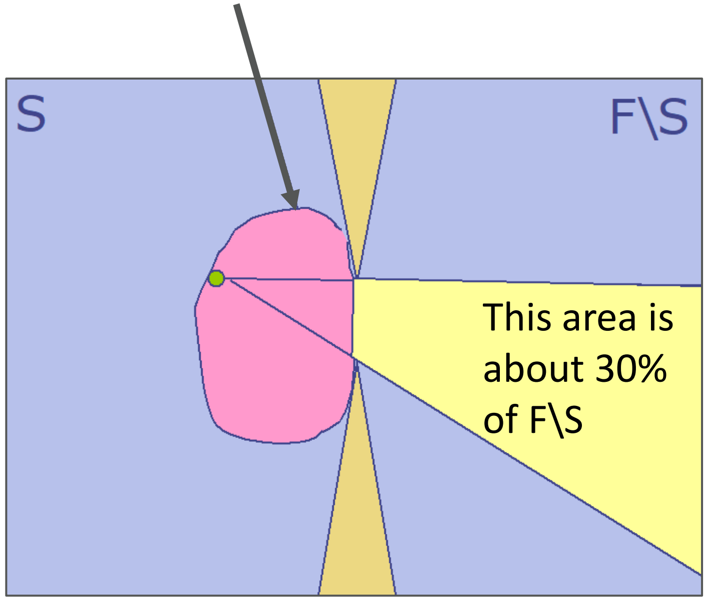
- **β–lookout of a subspace S:**
  - Subset of points in S that can see at least β fraction of $F \setminus S$.

$\beta \in (0,1]$

0.4-lookout of S



0.3-lookout of S

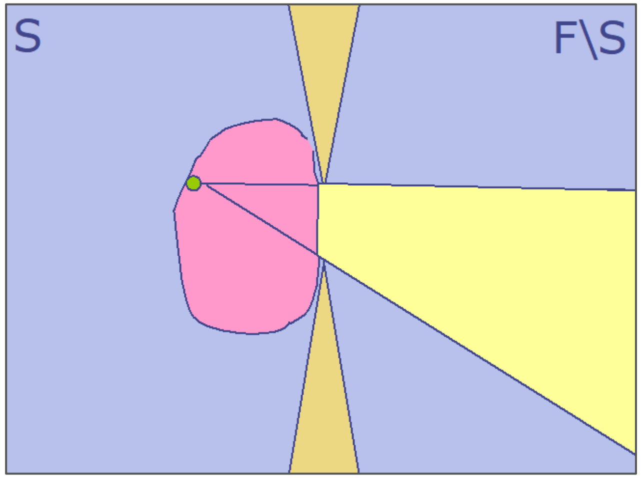


This is a different space!

Expansiveness

- **$(\epsilon, \alpha, \beta)$–expansive:**
  - The free space F is $(\epsilon, \alpha, \beta)$-expansive if
    - Free space F is ϵ-good
    - For each subspace S of F, its β-lookout is at least α fraction of S

$$\epsilon, \alpha, \beta, \in (0,1]$$



F is ϵ-good $\rightarrow$ $\epsilon = 0.5$

β-lookout $\rightarrow$ $\beta = 0.3$

$$\alpha = \frac{\mathrm{Vol}(\beta \text{ lookout})}{\mathrm{Vol}(S)} \rightarrow \alpha = 0.2$$

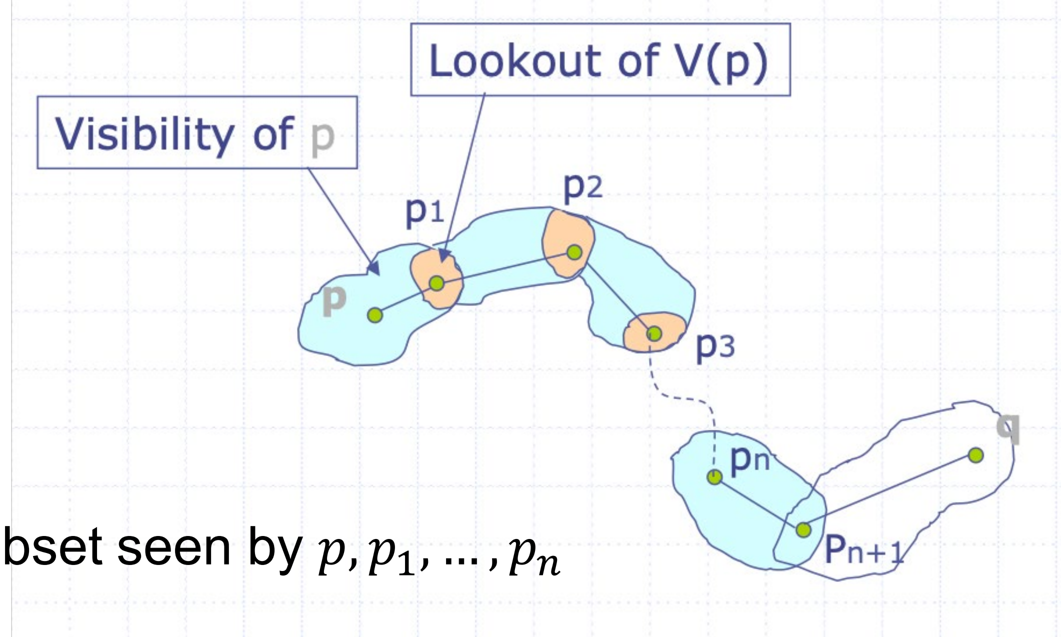
F is $(\epsilon, \alpha, \beta)$-expansive, where

$\epsilon = 0.5, \quad \alpha = 0.2, \quad \beta = 0.3$

Linking Sequences

- **Linking sequence**

Definition 2 *The linking sequence of a point $p \in \mathcal{F}$ is a sequence of points $p_0 = p, p_1, p_2, \ldots$ and a sequence of sets $V_0 = \mathcal{V}(p_0), V_1, V_2, \ldots \subseteq \mathcal{F}$ such that for all $i \geq 1$, $p_i \in$ LOOKOUT(V_{i-1}) and $V_i = V_{i-1} \cup \mathcal{V}(p_i)$.*



- Linking sequence of length t
- p_{n+1} is chosen from the lookout of the subset seen by $p, p_1, \ldots, p_n$

Proof sketch

1. Given a sequence of points we have defined a visibility set F, and unvisible F'

2. The probability of sampling inside the β-Lookout is α.

3. If we sample inside β-lookout the unvisible F' will decrease my β-percent

Uniform sampling

Lemma 1 *Suppose that a set M of n milestones is chosen independently and uniformly at random from the free space $\mathcal{F}$. Let $s = 1/\alpha\epsilon$. Given any milestone $p \in M$, there exists a linking sequence in M of length t for p with probability at least $1 - se^{-(n-t-1)/s}$.*

- **Theorem 1** : A roadmap of $\dfrac{16\ln(1/\gamma)}{\varepsilon\alpha} + \dfrac{6}{\beta}$ uniformly-sampled milestones has the correct connectivity with probability at least $1-\gamma$.

Expansiveness

- **Theorem:**

  Probability of achieving good connectivity increases exponentially with the number of sampled nodes in an expansive space.

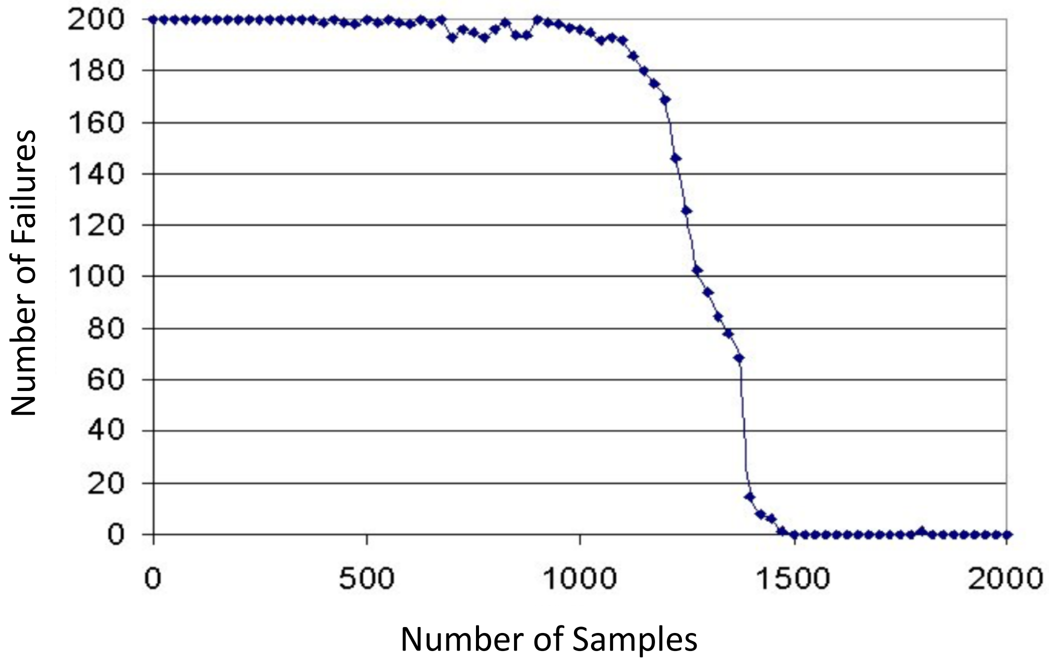
- As $(\epsilon, \alpha, \beta)$ decreases then the number of milestones needs to increase to maintain good connectivity.

- **Theorem:**

  The probability of achieving good coverage, increases exponentially with the number of sampled nodes in an expansive space.

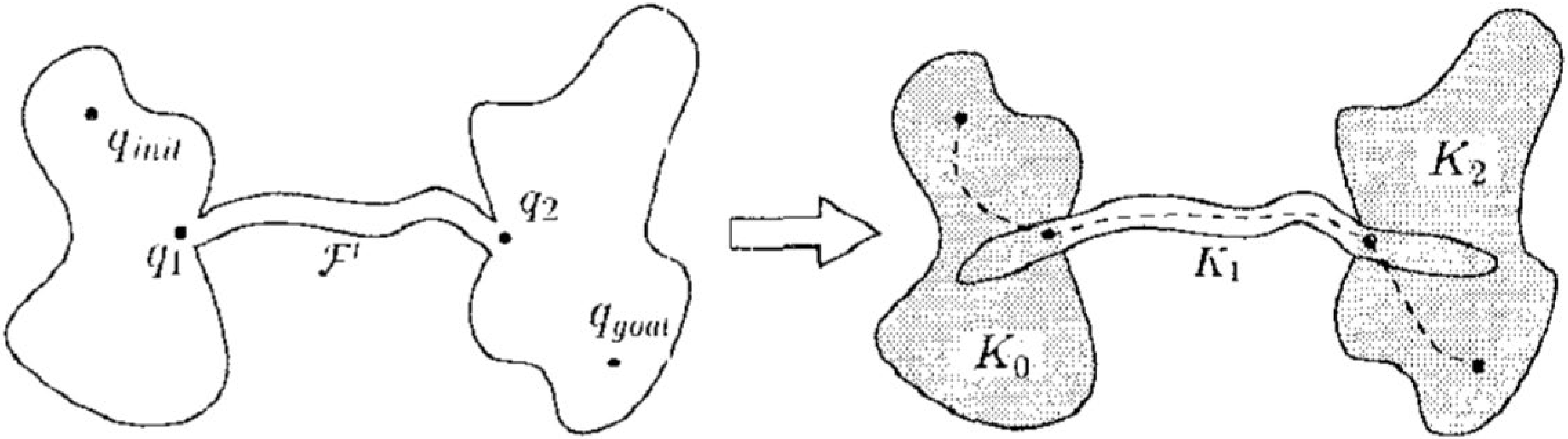
Probabilistic Completeness

- In an expansive space, the probability that a PRM planner fails to find a path when one exists goes to 0 exponentially in the number of sampled nodes.



Expansiveness Decomposition

- Expansive decomposition, by inserting q1, q2 we create three spaces



Summary

- **Main result:**
  - If a C-space is expansive, then a roadmap can be constructed efficiently with good connectivity and coverage
  - Placing samples in narrow passages, improves the e,a,b values
- **Limitation in implementation**
  - No theoretical guidance about the stopping time
  - Calculating expansiveness is often as hard as solving the actual problem
  - A planner stops when either a path is found or maximum number of steps have been taken