1) $L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{(x_i)^{(\alpha-1)}(1-x_i)^{(\beta-1)}}{\Gamma(\alpha)\,\Gamma(\beta)} \times \Gamma(\alpha+\beta)$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\,\Gamma\beta}\right)^n \times \prod_{i=1}^{n}\left((x_i)^{\alpha-1}(1-x_i)^{(\beta-1)}\right)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)^n \left[\frac{1}{\prod x_i(1-x_i)}\right]\left[\prod(x_i)^\alpha \left(\prod(1-x_i)^\beta\right)\right]$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)^n \left(\frac{1}{\prod x_i(1-x_i)}\right)\left(\prod x_i^\alpha (1-x_i)^\beta\right)$$

$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\right)^n \left(\frac{1}{\prod x_i(1-x_i)}\right)\left(e^{\alpha\sum \ln(x_i) + \beta \sum \ln(1-x_i)}\right)$$

Comparing with exp family of distribution

$$L(\underset{\sim}{\theta}) = [C(\theta)]^n \left(\prod h(x_i)\right) e^{\sum_{j=1}^{k} w_j(\theta)\left[\sum_{i=1}^{n} T_j(x_i)\right]}$$

$\Rightarrow C(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$; $h(x_i) = x_i(1-x_i)$

$\boxed{K=2}$ $w_1(\theta) = \alpha$; $w_2(\theta) = \beta$

$T_1(x_i) = \ln(x_i)$

$T_2(x_2) = \ln(1-x_i)$

b) Find sufficient statistic(s) for $\theta = (\alpha, \beta)$.

From factorization Theorem; a statistic $T(X_1, \ldots, X_n)$ is sufficient if the joint density can be factored as

$$f(x_1, x_2, \ldots x_n | \theta) = g(T(x_1, \ldots, x_n), \theta)\, h(x_1, \ldots, x_n)$$

For $X_1, \ldots, X_n \in$ iid Beta(α, β), the JPDF is

$$\prod_{i=1}^{n} \frac{x_i^{\alpha-1}(1-x_i)^{\beta-1}}{B(\alpha,\beta)} \Rightarrow \frac{1}{B(\alpha,\beta)} \prod x_i^{\alpha-1}(1-x_i)^{\beta-1}$$

Can be written as:

$$e^{\left((\alpha-1) \sum \log x_i + (\beta-1) \sum \log(1-x_i) - n \log B(\alpha,\beta) \right)}$$

$\therefore$ Sufficient statistic for (α, β) are:

$$T_1 = \sum_{i=1}^{n} \log x_i \quad \& \quad T_2 = \sum_{i=1}^{n} \log(1-x_i)$$

This can also come from part (a)!

1(c) Find minimal sufficient statistic for $\theta = (\alpha, \beta)$

Let X, Y be two samples, By Lehman-Scheffe Th^m

$\frac{f_\theta(\underset{\sim}{X})}{f_\theta(\underset{\sim}{Y})}$: depends on $\underset{\sim}{\theta}$ iff $\underset{\sim}{X} = \underset{\sim}{Y}$.

$$\frac{f_X(x)}{f_Y(y)} = \frac{\prod_{i=1}^{n} \dfrac{e^{(\alpha-1)\ln x_i + (\beta-1)\ln(1-x_i)}}{\cancel{B(\alpha,\beta)}}}{\prod_{i=1}^{n} \dfrac{e^{(\alpha-1)\ln y_i + (\beta-1)\ln(1-y_i)}}{\cancel{B(\alpha,\beta)}}}$$

$$= \frac{e^{(\alpha-1)\sum \ln x_i + (\beta-1)\sum_i \ln(1-x_i)}}{e^{(\alpha-1)\sum \ln y_i + (\beta-1)\sum \ln(1-y_i)}}$$

$$= e^{(\alpha-1)\sum_1^n(\log x_i - \log y_i) + (\beta-1)\sum_1^n \log(1-x_i) - \log(1-y_i))}.$$

For MSS, $\sum \ln x_i = \sum \ln y_i$ [or equivalently $\prod x_i = \prod y_i$]

& $\sum \ln(1-x_i) = \sum \ln(1-y_i)$ [or equivalently $\prod(1-x_i) = \prod(1-y_i)$]

$\therefore T(X) = (\sum \ln x_i, \sum \ln(1-x_i))$ or $(\prod x_i, \prod(1-x_i))$ is a MSS