MTH310/520: Submission 5

Time: 15 Minutes, Marks: 5

April 5, 2024

Name and Roll No:

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1. (5 points) Let there be n bus drivers, n morning routes with durations $x_1, \ldots, x_n$, and n afternoon routes with durations $y_1, \ldots, y_n$. A driver is paid overtime when the morning route and afternoon route exceed total time t. The objective is to assign one morning run and one afternoon run to each driver to minimize the total amount of overtime. Express this as a weighted matching problem. Prove that giving the i-th longest morning route and i-th shortest afternoon route to the same driver, for each i, yields an optimal solution.

Solution. We form a bipartite graph $G = (M \cup A, E)$ where M represents set of morning routes and A represents set of afternoon routes. Add an edge (m_i, a_j) with weight $\min\{0, x_i + y_j - t\}$. We can find a min-cost perfect matching to find an optimal solution with minimum overall overtime.

Assigning a driver to a route is having a permutation $\sigma : [n] \to [n]$. Observe that either of the following holds:

i) $i < j$ and $\sigma(i) > \sigma(j)$

ii) $i > j$ and $\sigma(i) < \sigma(j)$

Suppose for any $i < j$

i) $W_1 = \max\{0, x_i + y_{\sigma(i)} - t\} + \max\{0, x_j + y_{\sigma(j)-t}\}$

ii) $W_2 = \max\{0, x_i + y_{\sigma(j)} - t\} + \max\{0, x_j + y_{\sigma(i)-t}\}$

Since $\sigma(i) > \sigma(j), y_\sigma(i) > y_\sigma(j)$ and therefore we have
$x_i + y_{\sigma(i)} - t \geq x_i + y_{\sigma(j)} - t \geq x_j + y_{\sigma(j)} - t$, and
$x_i + y_{\sigma(i)} - t \geq x_j + y_{\sigma(i)} - t \geq x_j + y_{\sigma(j)} - t$
Since the quantities are non-negative, we have $W_1 \geq W_2$. This proves the first claim which further implies that for any pair i, j in the bipartite graph formed above there the edges cross. Symmetrically this holds when $i > j$ and $\sigma(i) < \sigma(j)$. Finally, it is easy to prove by induction that the only way to satisfy the same is by following the given algorithm.

Rubric: 2 marks for reducing to a matching problem. 3 marks to prove the correctness of the algorithm.