END SEMESTER EXAM
BIO525 COURSE

Introduction to Bioinspired Algorithms and Optimal Control (BAOC)

May 4, 2025
Solution outline and marks distribution

<u>50 Marks EXAM Questions:</u>

Q 1-14 (1 MARK).
1 mark only for correct answer
0 for no or incorrect answer
If spelling mistake is done in fill in the blanks, 0.5 is deducted.

1. (1 mark) Major disadvantage in global best approach in Particle Swarm Optimization (PSO) is:
   a) Slow Convergence
   b) No Convergence
   c) Premature Convergence
   d) Overfitting

2. (1 mark) Which of the following is true about error reduction in Newton's method or Gradient descent method?
   a) Newton's Method reduces error linearly
   b) Gradient Descent always outperforms Newton's Method
   c) Newton's Method reduces error quadratically near the optimum
   d) Both methods reduce error at the same rate

3. (1 mark) What happens if the learning rate in gradient descent is too large?
   a) The algorithm converges faster
   b) The algorithm may overshoot and diverge
   c) The gradient becomes zero
   d) The function always reaches the global minimum

4. (1 mark) In optimal control problems with free terminal time and no terminal cost, Pontryagin's Maximum Principle includes the following condition at the terminal time:
   a) The state must satisfy a boundary condition at t_f
   b) The adjoint equation must be zero at t_f
   c) The Hamiltonian must be zero at t_f
   d) The control input must satisfy a condition at t_0

5. (1 mark) In PSO, When the cognitive component is higher than the social component, the swarm tends to:
   a) Converge faster to the global optimum
   b) Explore the global best solution more
   c) Focus more on individual exploration
   d) Avoid local optima more efficiently

(1 mark) During ant bridge formation, how is the bridge structure maintained?
   a) Ants secrete sticky substances
   b) Ants physically link their bodies together
   c) Ants weave leaves and twigs into bridges
   d) Ants build mud structures

7. (1 mark) Suppose you are climbing the stairs with n steps. Each time you can either climb 1 or 2 steps. Let dp[i] represent the number of distinct ways to reach step i. What is the correct recurrence relation to find the number of ways to reach step i. Choose the correct option -
   a) dp[i]=dp[i−1]+dp[i−2]
   b) dp[i]=dp[i−1]×dp[i−2]
   c) dp[i]=dp[i−1]−dp[i−2]
   d) dp[i]=dp[i−2]

8. (1 mark) Minimize the cost functional: $J = \int_0^1 (x^2 + u^2)\ dt$
   Subject to the dynamic constraint:

$$\frac{dx}{dt} = u$$
$$x(0) = 1$$

   Using Pontryagin's Maximum Principle, find the optimal control $u(t)$.
   a) $u(t) = -x(t)$
   b) $u(t) = -\lambda(t)$
   c) $u(t) = -\frac{1}{2}\lambda(t)$
   d) $u(t) = \lambda(t)^2$

9. (1 mark) In Bang-Bang control, the optimal control strategy typically involves:
   a) Gradually adjusting the control input based on the state trajectory
   b) Using a continuous range of control values to minimize cost
   c) Switching abruptly between the maximum and minimum allowable control values
   d) Holding the control input constant throughout the time horizon

10. (1 mark) The ____**TERMINAL**______ cost function.

11. (1 mark) In genetic algorithms, the process of copying one or more of the best individuals directly into the next generation is called ________**ELITISM**__.

12. (1 mark) The ______**PHEROMONE**____ intensity on a path in ACO increases when more ants use that path, reinforcing it as a preferred route.

13. (1 mark) True/False: The Free Lunch Theorem suggests that there is a universal optimization algorithm that outperforms all others for every problem.

14. (1 mark) True / False: In random optimization methods, the algorithm explores the solution space by generating random candidates without utilizing any gradient or prior knowledge of the objective function.

3 MARKS QUESTIONS

15. (3 mark) True/False: If f(x,y) has a maximum or minimum subject to constraint g(x,y)=0, then the gradients of f and g must be perpendicular (**1 mark)- for correct options**.

    Support your answer with relevant equation(s) (2 mark)- for correct explanation. If reason is not properly explained, 1 mark is given.
    Answer: False
    They are parallel, not perpendicular.
    Using Lagrange Multipliers:

$L = f + \lambda g$ Lagrange Multiplier
For optimum-
$\nabla f(x, y) = -\lambda \nabla g(x, y)$

16. (3 mark) A plant stem grows between two fixed points, minimizing energy spent in bending and stretching. The energy cost functional is:

$$J[y] = \int_0^1 (y'^2 + y^2)\, dt$$

Where:
$y(t)$ represents the vertical displacement of the stem at position $t \in [0,1]$,
$y(0) = 0$ (stem starts at ground) $y(1) = 1$ (stem reaches fixed height)

Using Euler-Lagrange equation, find the optimum shape $y(t)$ of the plant stem that minimizes the total energy.

Answer: Euler Lagrange equation (2 marks if all steps are written correctly including EL equation till general solution)

L = $(y'^2 + y^2)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$2y'' - 2y = 0 \quad \Rightarrow \quad y'' - y = 0$$

$r^2 - 1 = 0,\ r = \pm 1$

$$y(t) = Ae^t + Be^{-t}$$

Applying Boundary conditions: (+1 if boundary condition is implemented correctly)

$$y(0) = A + B = 0$$
$$y(1) = Ae + Be^{-1} = 1$$

$$A = \frac{1}{e - e^{-1}}, \quad B = -A$$

$$y(t) = \frac{1}{e - e^{-1}}\left(e^t - e^{-t}\right)$$

This is the equation for minimum energy shape of the plant stem.

17. (3marks) In the forward-backward sweep method used in optimal control, the state equation is solved backward in time, and the adjoint equation is solved forward in time. True / False (1 mark- only for correct choice). Give justification (2 mark for proper reason explained for both state and adjoint equations).
Answer: False
In forward backward sweep method:
- State equation for $x(t)$ is solved forward in time with initial condition:
$$\dot{x}(t) = f(x(t), u(t), t),\ x(0) = x_0$$
- Adjoint equation for $\lambda(t)$ is solved backward in time:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}$$

With boundary condition given at final time t = T (terminal condition), hence solved backward.

18. (3 marks) Consider the optimal control problem-

- System Dynamics: $\dot{x}(t) = u(t)$
- Cost functional: $J = \int_0^T \left(x(t)^2 + \frac{1}{2}u(t)^2\right) dt$
- Assume the value function is: $V(x,t) = A(t)x^2$, where A(t) is a time-dependent coefficient.

a) Write the Hamilton-Jacobi Bellman (HJB) equation. Mention the principle that underlies the HJB formulation. (1.5 mark)
b) Using the value function form, what is the optimal control law u*(t) in terms of A(t). (1.5 mark)

Answer: Principle of optimality: "Any optimal policy has the property that, whatever the current state and decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the current decision." (1 mark)

HJB (0.5 mark):

$$0 = \min_u \left[x(t)^2 + \frac{1}{2}u(t)^2 + V_x(x,t) \cdot u\right] + V_t(x,t)$$

For optimal control law u*(t)-

$$\frac{d}{du}\left[x^2 + \frac{1}{2}u^2 + V_x . u\right] = 0 \qquad \text{(0.5 mark)}$$

$$Using, V_x = 2A(t)x(t)$$

u* + 2 A(t) x(t) = 0

Answer: $u^*(t) = -2A(t)x(t)$ (1 mark with all steps)

7 MARKS QUESTIONS

19. (7 marks) Given a function f(x) in n-dimensional space with N particles-
    a) Write the algorithm for classical Particle Swarm Optimization (PSO). The output should include the global best position and function value at the global best when the termination condition is reached. You may choose to maximize or minimize the function, but make sure you mention it. The termination condition is checked with either a maximum iteration limit or a convergence condition. Provide the update equations for velocity and position, explain each part and each parameter clearly. (4 marks)
    b) Pictorially represent with proper labels, one PSO update step for a particle pi at position xi and velocity v_i. Clearly show the learning coefficient and resulting updated velocity and position. (3 marks). (4 marks).

Answer: Classical PSO algorithm (4 marks):
Other way in which all steps are explained are also given marks. If any steps are missing, appropriate deductions are done (1/2 marks).
Initialization

- For each particle i = 1 to N:

- For each dimension d in 1 to n:
    - Randomly initialize position, $x_i \in R^n$
    - Randomly initialize velocity, $v_i \in R^n$
    - Evaluate: For each fitness $f(x_i)$
    - Set personal best $p_{best,i} = x_i$

- Set global best g_{best} *to the best among all* $p_{best,i}$
- Set parameters: cognitive coefficient c_1, social coefficient c_2, *and* maximum iterations. Tmax and convergence threshold ϵ.

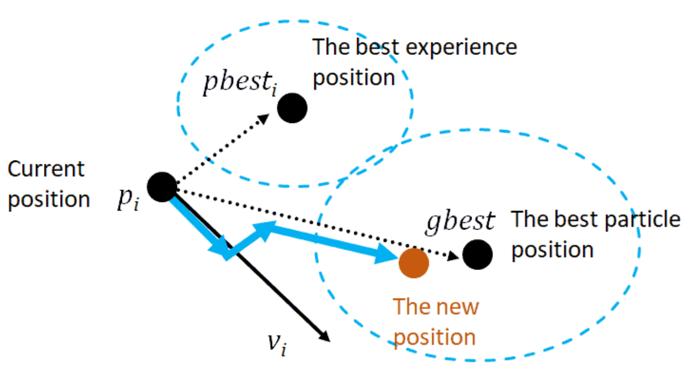
Loop (for t = 1 to Tmax):

- For each particle i in 1 to N:
- For each dimension d in 1 to n:
- - Generate two random numbers r1, r2 ∈ [0, 1]
- - **Update velocity**:
- v_i[d] = v_i[d]
- + c1 * r1 * (pbest_i[d] - x_i[d])
- + c2 * r2 * (gbest[d] - x_i[d])
- - Restrict velocities, such that |vij |<vmax.
- - **Update position**:
- x_i[d] = x_i[d] + v_i[d]
-

For minimization-
- Evaluate fitness f(x_i)
- If f(x_i) < f(pbest_i):
- Update pbest_i = x_i
- If f(pbest_i) < f(gbest):
- Update gbest = pbest_i

Termination condition:
- Stop if maximum iterations Tmax is reached, or $|g^{t+1} - g^t| < \epsilon$ (i.e convergence condition met).

Output
- Return gbest and f(gbest) as the best solution found



2 marks for drawing + 1 marks for correct explanation (if other pictorial representation is made and appropriately explained, marks are given)

Particle pi, xi (Current position):
This is where the particle is currently located in the search space.

v_i (Velocity vector):
This black arrow shows the direction and speed the particle wants to move, based on its current knowledge.
$pbest_i$ (Personal best position):
This is the best position that this particle has found so far — like its best memory.
gbest (Global best position):
This is the best position found by any particle in the whole swarm — like the best team solution

Orange dot – New position:
This is where the particle moves next, influenced by both its own experience ($pbest_i$) and the swarm's best experience (gbest).

Particle is trying to balance between: - Exploring new areas (based on current movement direction) and Returning to good spots (personal and global bests).

20. (7 marks) Let the chromosomes A (0101), B (1010), D (1001), and E (1110) represent binary strings encoding integer values in the range [0,15]. Each chromosome's fitness is calculated using the function:

$$f(x) = x^2$$

, where x=decimal equivalent of the binary chromosome
Note: Indexing should be considered starting from 0. (THIS HAD TO BE TAKEN CARE OF)

a) Convert each chromosome to decimal value and evaluate the fitness. (1.5 mark)
b) Perform two tournaments of size 3 (randomly pick 3 chromosomes each time) and select the winners (those with highest fitness). Ensure 2 different parents are selected. (1.5 marks)
c) Apply single-point crossover between the two parents after position 2. Write the resulting two offspring. (1.5 marks)
d) Apply 1-bit mutation at position 3 to Offspring 1 and at position 2 to Offspring 2. Write the mutated chromosomes and their fitness values. (1.5 marks)
e) Explain briefly the biological inspiration behind genetic algorithm. (1 mark)

Answer

a) A (0101)-5, B (1010)- 10, D (1001)- 9, E = 1110- 14

Fitness- 25, 100, 81, 196 respectively (1.5 marks if correctly decimal values and fitness are evaluated. If notation change is done, 0.5 is deducted)

b) Two tournaments of size 3 (1.5 mark for correct tournaments and winner selection – 2 separate .)
T1: A,B,D (winner B)- 1010 [Parent 1]
T2: BDE (winner E)- 1110 [Parent 2]

c) Offspring (Single-Point Crossover after position 2) (1.5 mark for correct crossover (if index (starting from 0 as asked in the question) is chosen correctly for crossover and step shown), otherwise marks deducted)
Crossover point after position (index) 2:
101|0
111|0 1010 (offspring 1) and 1110 (offspring 2)

d) 1- bit mutation (1.5 marks for correct index used as asked, mutation and fitness evaluation, otherwise marks deducted.)
1010 Mutate at index 3 (1011) –decimal =11, fitness-121
1110 Mutate at index 2 (1100) – decimal = 12, fitness- 144

e) Genetic Algorithms are inspired by the process of nature selection and biological evolution. They mimic how population evolves over generations using mechanisms such as selection, crossover and mutation, like how traits are passed and adapted in nature to improve fitness or survival. (1 mark for correct explanation, 0.5 deducted if not properly explained).

10 MARKS QUESTION

21. Cells regulate their energy usage to balance growth and maintenance. Suppose a cell invests a fraction $\alpha(t)$ of its energy into growth, while the remaining fraction $1 - \alpha(t)$ is used for cellular maintenance and function.

The dynamics of cellular growth follow the equation:

$$\dot{x}(t) = k\,\alpha(t)\,x(t) \qquad x(0) = x_0 > 0 \text{ and } 0 \leq t \leq T$$

where:

- $x\,(t)$ represents the biomass
- $\alpha(t)$ is the control variable ($0 \leq \alpha \leq 1$),
- $k > 0$ is the growth rate,
- T is the total time available

The cell aims to maximize energy to maintenance over time:

$$J = \int_0^T (1 - \alpha(t))x(t)dt$$

Questions:

a. Write the Hamiltonian for this problem. (1 marks)
b. Write PMP equations and identify the optimal control strategy $\alpha^*(t)$. What type of control is this? Show pictorially by marking a switching time t* (4 marks).
c. Find switching time t^* in terms of T *and* k where the cell transitions from full growth to full maintenance. Write the answer for k = 1 (2.5 marks)
d. How does increasing k affect the switching time and control strategy? (1.5 marks)
e. Briefly explain what the control strategy suggests about how a cell should allocate energy over time for maximum maintenance benefit. (1 mark)

a) Hamiltonian- (1 mark for correct Hamiltonian, and adjoint label)

$$\mathcal{H} = (1 - \alpha(t))x(t) + \lambda(t) \cdot (k\alpha(t)x(t))$$

$\lambda(t)$: Adjoint function

b) PMP equations: (2 marks- deduction done if not properly explained or steps missing)
State equation:

$$\dot{x}(t) = k\alpha(t)x(t)$$

Adjoint condition:

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x} = -[(1 - \alpha(t)) + \lambda(t)k\alpha(t)]$$

Transversality condition:

$$\lambda(T) = 0$$

Optimality condition:

$$\frac{\partial \mathcal{H}}{\partial \alpha} = -x(t) + \lambda(t)kx(t) = x(t)(\lambda(t)k - 1)$$

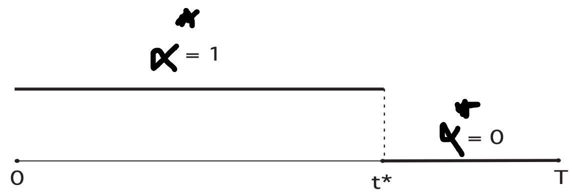
This is a bang bang control (1 mark for proper explanation))-

If $\lambda(t)k - 1 > 0$, then $\alpha^*(t) = 1$ (full growth)

If $\lambda(t)k - 1 < 0$, then $\alpha^*(t) = 0$ (full maintenance)

If $\lambda(t)k = 1$, switching point

1 mark for diagram-



c) Switching time
(2 marks for t* derivation in terms of k)

For $t > t^*$, $\alpha(t) = 0$:

Solving adjoint equation and using transversality condition:

$$\lambda(t) = T - t \quad \lambda(t^*)k = 1 \Rightarrow k(T - t^*) = 1$$

$$t^* = T - \frac{1}{k}$$

0.5 mark

For k = 1: t* = T – 1

d) Higher k (faster growth rate) ⇒ the cell spends more time in growth phase Switching to maintenance happens later. The control strategy remains bang-bang, but the switch occurs later. (1.5 marks if correctly explained for switching time and control strategy)

e) The optimal control strategy suggests that the cell should allocate all its energy to growth initially (i.e., set α(t)=1 for t<t*) to rapidly increase its biomass. Once it reaches a certain switching time t*, the cell should switch entirely to maintenance (i.e., set α(t)=0) for the remainder of the time t* ≤ t ≤ T. (1 mark)