Q1: Let $X_1 \ldots X_n$ be a random sample from distbn from Normal $N(0, \theta)$, $\theta > 0$. Find the MSE for $\frac{1}{n}\sum X_i^2$.

$$\underset{RV}{X_i} \sim N(0, \theta) \quad (\theta > 0)$$

→ we are given estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

MSE ≔ Mean Squared Error $= E[(\underbrace{\hat{\theta}}_{\text{pred}} - \underbrace{\theta}_{\text{actual.}})^2]$

we can interpret $MSE = E[(\hat{\theta} - \theta)^2] = Var(\hat{\theta})$

∴ MSE of $\underset{\text{some estimator}}{W(X)} = E_\theta[(W - \theta)^2]$

$$= Var_\theta[W] + (E_\theta[W] - \theta)^2$$

$\left(∵ Bias_\theta(W) = E_\theta[W] - \theta\right)$ $= Var_\theta[W] + (Bias_\theta(W))^2$

now here $Bias_\theta(\hat{\theta}) = E\left[\frac{1}{n}\sum X_i^2\right] - \theta$

$$= \frac{1}{n} E[\sum X_i^2] - \theta$$

$$= \frac{1}{n}\left(\sum var(X_i) + (\cancel{E(X_i)}^{0})^2\right) - \theta = \frac{1}{n}(n\theta) - \theta = \theta - \theta = 0$$

now

$$\mathrm{Var}_\theta(\hat{\theta}) = \mathrm{Var}\left[\frac{1}{n}\sum X_i^2\right] = \frac{1}{n^2}\sum \underbrace{\mathrm{Var}(X_i^2)}_{??}$$

Distribution of X_i^2 ?

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

where $S^2 = \frac{1}{n}\sum(X_i - \mu)^2$

$X_i \sim N(\mu, \sigma^2)$

Using above since $X_i \sim N(0, \theta)$, i.e.,

$$\frac{nS^2}{\theta} \sim \chi^2_n \qquad \text{where } S^2 = \frac{1}{n}\sum X_i^2$$

Using the fact that $E[\chi^2_{(n)}] = n$

$$E[S^2] = \frac{\theta}{n} E[\chi^2_n] = \theta \qquad [\because \mathrm{Var}[\chi^2_n] = 2n]$$

$$\mathrm{Var}(S^2) = \mathrm{Var}\left[\frac{\theta}{n}\chi_n^2\right] = \frac{\theta^2}{n^{\cancel{2}}} \times 2\cancel{n} = \frac{2\theta^2}{n}$$

$$\therefore \mathrm{MSE} = \mathrm{Var}\left[\frac{1}{n}\sum X_i^2\right] = \mathrm{Var}(S^2) = \frac{2\theta^2}{n}$$

Q29 Let $X_1 \cdots X_n$ form random sample from Uniform distribution on interval $(-\theta, \theta)$. What is unbiased esti for of θ.

Unbiased Estimator $\Rightarrow$ Bias $= 0$ i.e

$$E_\theta(W) = \theta$$
$\hookrightarrow$ estimator

→ If we have a uniform distbn $X \sim U(a, b)$

so, $f_X(x) = \dfrac{1}{b-a} \quad x \in [a, b]$

$E[X] = a+b/2 \qquad var(X) = b-a^2/12$

- You can approach finding estimators by guessing some statistics that carries max info about your parameters you're estimating (Sufficient statistics!!)

Sufficient statistic $T(X)$ in this case

$$f_X(X_1 \cdots X_n \mid \theta) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \frac{1}{2\theta} I(-\theta < x_i < \theta)$$

$$= \left(\frac{1}{2\theta}\right)^n \prod_{i=1}^{n} I(|x_i| < \theta) \times 1$$

suff stat $= \max_{i=1 \text{ to } n} |x_i|$

denote it by M.

$M = \max |X_i| \rightarrow$ Suff Stat

$E[M] = \sum x f_M(x)$

[CHANGES HERE!!]

$\Rightarrow (|X_1| < x ; |X_2| < x \ldots |X_n| < x)$

$F_M(x) = P(\max_{i=1,\ldots n} |X_i| < x) = \prod \dfrac{x - 0}{\theta - 0}$ $\quad x_i \in [-\theta, \theta]$

$\because |X_i| \sim U(0, \theta)$

$= \left(\dfrac{x}{\theta}\right)^n \qquad = \left(\dfrac{x}{\theta}\right)^n$

$$f_M(x) = n\left(\frac{x}{\theta}\right)^{n-1} \times \frac{1}{\theta^n} = n x^{n-1} \theta^{-n}$$

$\therefore E[M] = \int x f_M(x) \qquad = \dfrac{n}{n+1}\theta$

So, if we have $W = \dfrac{n+1}{n} M$, we'd get

$E[W] = \dfrac{n+1}{n} \times \dfrac{n}{n+1} \times \theta = \theta$ ✓

$\therefore$ Unbiased Estimator $= \dfrac{n+1}{n} \max_{i=1 \text{ to } n} |X_i|$

Ques 36 $X_1 \dots X_n \sim \text{Poisson}(\lambda)$; $\lambda > 0$.

i) verify $\bar{X}$ and $s^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$ unbiased est. of λ

$$E[\bar{X}] = \frac{1}{n}\sum E[X_i] = \lambda \quad \checkmark \text{ (Yes)}$$

$$E[s^2] = \frac{1}{n-1}\sum E(X_i - \bar{X})^2$$

$$= \frac{1}{n-1}\sum E[X_i^2 + (\bar{X})^2 - 2\bar{X}X_i]$$

$$= \frac{1}{n-1}\sum E(X_i^2 + \cancel{(\bar{X})^2} - \cancel{2}n(\bar{X})^2) = \frac{1}{n-1}\sum E(X_i^2 - n\bar{X}^2)$$

$[X_i \sim \text{Poisson}(\lambda) \quad E(X_i) = \lambda \quad E(X_i^2) = \lambda + \lambda^2]$

$\sum X_i \sim \text{Poisson}(n\lambda) \quad E(\sum X_i) = n\lambda \quad E((\sum X_i)^2) = n\lambda + n^2\lambda^2]$

$$= \frac{1}{n-1}\left[\sum E[X_i^2] - n\,E\left(\frac{\sum X_i}{n}\right)^2\right] \quad \text{(Yes!)}$$

$$= \frac{1}{n-1}\left(n(\lambda + \lambda^2) - \frac{1}{n}(n\lambda + n^2\lambda^2)\right) = \lambda$$

b) Which one of $\bar{X}$ & s^2 is better & why?

Logic: Lower MSE | better estimator

∵ MSE = Bias & variance & ∵ both here have bias zero so here variance should be lower for the better estimator!

$$V(\bar{X}) = \frac{1}{n^2}\sum Var(X_i) = \frac{\cancel{n}\lambda}{n^2} = \frac{\lambda}{n}$$

$$V(S^2) = \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right]$$

where $\sigma^2 = \lambda$ & $\mu_4 = E((X-\lambda)^4)$

$$= \lambda(1+3\lambda)$$

Ques 48 Let $X_1 \dots X_n$ be random sample from bernoulli distribution with parameter 'p'. Find UMVUE of p

Defn UMVUE :

→ Estimator W^* is UMVUE of $\tau(\theta)$ if

1) $E_\theta[W^*] = \tau(\theta) \quad \forall \theta \in \Theta$

2) $Var_\theta[W^*] \leq Var_\theta(W) \ \forall \theta \in \Theta$

↳ any other estimator.

→ To find UMVUE, two approaches

① Rao Blackwell theorem :

Let W be any unbiased estimator of $\tau(\theta)$ & Let T be sufficient stat of θ. Define $\phi(T) = E[W|T]$

Then, ① $E[\phi(T)] = \tau(\theta)$

② $Var(\phi(T)) \leq Var(W)$

$X_1 \dots X_n \sim$ Bernoulli (p).

Suff stat : $f(X_1 \dots X_n) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$

$$= \underbrace{p^{\Sigma x_i} (1-p)^{1-\Sigma x_i}}_{g(t|\theta)} \times 1$$

$T = \Sigma x_i$

Step 2: Prove it to be Complete.

Step 3: Now here $\tau(\theta) = p$! . You'll use the complete sufficient to find $\phi(T)$

$\phi(T) = \bar{X}$ $\quad\quad E(\phi(T)) = p$

$\therefore$ By Lehman Scheffe thm, It is UMVUE!