1. Let $X_1, \ldots, X_n$ be i.i.d. with the following pdf

$$f_\theta(x_i) = \frac{e^{-x_i}}{e^{-\theta} - e^{-b}}, \quad \theta < x_i < b, \ b \text{ known.}$$

Answer the following questions

(a) (2 points) Apart from the data itself, find sufficient statistic(s) for θ.

(b) (2 points) Find minimal sufficient statistic(s) for θ.

(a) $f_\theta(x_i) = \dfrac{e^{-x_i}}{e^{-\theta} - e^{-b}}, \quad \theta < x_i < b$

we can define Indicator function for x_i

$$I_\theta(x_i) = \begin{cases} 1, & \theta < x_i < b \\ 0, & \text{else} \end{cases}$$

so we can write

$$f_\theta(x_i) = \frac{e^{-x_i}}{e^{-\theta} - e^{-b}} I_\theta(x_i)$$

now

$$f_\theta(x) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \frac{e^{-x_i}}{e^{-\theta} - e^{-b}} I_\theta(x_i)$$

$$= e^{-\Sigma x_i} \times (e^{-\theta} - e^{-b})^{-n} \times \prod_{i=1}^{n} I_\theta(x_i)$$

$$= e^{-\Sigma x_i} \times (e^{-\theta} - e^{-b})^{-n} \times \prod_{i=1}^{n} I_\theta(x_i > \theta) \cdot \prod_{i=1}^{n} I_\theta(x_i < b)$$

→ Using order statistics

$$= e^{-\sum x_i}\left(e^{-\theta}-e^{-b}\right)^{-n} \times I(x_{(1)}>\theta)\, I(x_{(n)}<b)$$

$$= \underbrace{\left(e^{-\theta}-e^{-b}\right)^{-n} \times I(x_{(1)}>\theta)}_{g(t, \theta)} \times \underbrace{e^{-\sum x_i} \times I(x_{(n)}<b)}_{h(x)}$$

$\therefore T(X) = X_{(1)}$ is the sufficient statistic.

(b) for Minimally sufficient statistic $T(X)$, It holds $\forall\, x, y \in \mathcal{X}$ when,

$$\frac{f_X(x|\theta)}{f_Y(y|\theta)} \text{ is independent of } \theta \iff T(X) = T(y)$$

$\therefore$ Let's consider $\dfrac{f_\theta(x|\theta)}{f_\theta(y|\theta)}$

$$= \frac{e^{-\sum x_i}\cancel{(e^{-\theta} - e^{-b})^{-n}} \times I(x_{(1)} > \theta)\, I(x_{(n)} < b)}{e^{-\sum y_i}\cancel{(e^{-\theta} - e^{-b})^{-n}} \times I(y_{(1)} > \theta)\, I(y_{(n)} < b)}$$

$$= \frac{e^{\sum y_i - \sum x_i}\, I(x_{(1)} > \theta)\, I(x_{(n)} < b)}{I(y_{(1)} > \theta)\, I(y_{(n)} < b}$$

this expression is free / independent of θ iff $x_{(1)} = y_{(1)}$ as it is very evident above

$\therefore$ $T(X) = X_{(1)}$ is the minimal sufficient statistic here.