MTH310/520: Submission 6

Time: 15 Minutes, Marks: 5

April 6, 2024

Name and Roll No:

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1. (5 points) Given a graph G, let $S_1, \dots, S_k$ and T be subsets of $V(G)$ such that each S_i has odd size. These sets form a generalized cover of G if every edge of G has one endpoint in T or both endpoints in some S_i. The weight of a generalized cover is $|T| + \sum_i \lfloor |S_i|/2 \rfloor$. Let $\beta^*(G)$ be the minimum weight of a generalized cover. Prove that $\alpha'(G) = \beta^*(G)$.

Solution. By Tutte-Berge formula, $\alpha'(G) = \frac{1}{2}\min_T\{n - d(T)\}; d(T) = o(G \setminus T) - |T|$. Choose a maximal T. Observe that no component is of even order in $G \setminus T$ as otherwise we could bring one of the endpoint of a matched edge in that component to T. Let $S_1, \dots, S_k$ be the components in $G \setminus T$. Therefore, $k = o(G \setminus T) = d(T) + |T|$ since $d(T) = o(G \setminus T) - |T|$.

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\begin{aligned}
\beta^*(G) &= \sum_{i=1}^{k} \frac{1}{2}(|S_i| - 1) + |T| \\
&= \frac{1}{2}(n - |T| - k) + |T| \\
&= \frac{n + |T| - d(T) - |T|}{2} \\
&= \frac{n - d(T)}{2} \\
&= \alpha'(G)
\end{aligned}
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