Q4) Rayleigh distribution

$$f_\theta(x_i) = \frac{2}{\theta} x_i e^{-x_i^2/\theta} \quad ; x_i > 0, \theta > 0.$$

Sol^n: observe that $f_\theta(x_i)$ is of the exponential family

$$f_\theta(x_i) = \underbrace{x_i}_{h(x)} \cdot \underbrace{\frac{2}{\theta}}_{c(\theta)} \cdot e^{\left(\underbrace{-\frac{1}{\theta}}_{w(\theta)} \cdot \underbrace{x_i^2}_{t(x)}\right)}.$$

Also, observe $\theta \in (0,\infty)$ which is an open subset of $\mathbb{R}$.
Using known result that $T(\underset{\sim}{X}) = (\Sigma t_1(x), \dots \Sigma t_k(x))$ is a complete and minimal sufficient statistic, we have that

$T(\underset{\sim}{X}) = \Sigma X_i^2$ is complete and MSS for θ.

~~2 marks for~~
1 mark each for complete and sufficient (either by def^n, or using result).

Now observe:

$$\frac{1}{n}\Sigma X_i^2 = \frac{1}{n} T(\underset{\sim}{X})$$

ie. $\hat{\theta}(T(\underset{\sim}{X})) = \frac{1}{n}\Sigma X_i^2$

1 mark for correct estimator

To show unbiasedness:

0.5 for formula

$$E\left[\hat{\theta}(T(\underset{\sim}{X}))\right] = \frac{1}{n}\Sigma E[X_i^2] = E[X^2] \text{ since } X \sim f_\theta(x).$$

[$\because$ sample is iid].

$$E[X^2] = \int_0^\infty x^2 \cdot \frac{2}{\theta} x \, e^{-x^2/\theta} dx.$$

0.5 for integration

$$= \int_0^\infty \frac{2}{\theta} x^3 \cdot e^{-x^2/\theta} dx$$

$\left(\text{let } t = \frac{x^2}{\theta} \Rightarrow \frac{2x}{\theta} = \frac{dt}{dx} \text{ and } \theta > 0, \text{ so}\right)$

$$= \int_0^\infty \theta t \cdot e^{-t} dt = \theta \int_0^\infty t \cdot e^{-t} dt$$

Method 1: observe $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

Method 2: observe that this is the expectation of a exp(1) random variable.

Method 3: integrate by parts.

$\Rightarrow$ we get $\int_0^\infty t e^{-t} dt = 1.$

$\therefore \hat{\theta}(T(\underset{\sim}{X}))$ is unbiased.

Using Lehmann-Scheffe theorem, we have that $\hat{\theta}(T(\underset{\sim}{X}))$ is the UMVUE.

$\rightarrow$ 1 mark for identifying Lehmann-Scheffe.