Rubric Q2 Mid-sem

Q2 $Y_1, Y_2 \cdots Y_n$ are iid samples from the pdf $f_\theta(y_i)$

$$f_\theta(y_i) = \frac{1}{2\beta^3} y_i^2 e^{-y_i/\beta} \quad y_i > 0,\ \beta > 0$$

a) MoM estimator for β

$$m_1 = \frac{1}{n} \Sigma Y_i$$

$$E[X] = \int_0^\infty y f(y) \cdot dy$$

$$\Rightarrow \int y f(y) \cdot dy = \int_0^\infty y \cdot \frac{1}{2\beta_3} y_\theta^2 e^{-y/\beta} \cdot dy$$

$$\Rightarrow \frac{1}{2\beta^3} \int_0^\infty y^3 \cdot e^{-y/\beta} \cdot dy$$

+0.5

applying integration by parts $\int u \cdot v\,dy = u \int v\,dy - \int u' \int v \cdot dy \cdot dy$

$$\Rightarrow \frac{1}{2\beta^3} \left[y^3 \int e^{-y/\beta} \cdot dy - \int \left(3y^2 \int e^{-y/\beta} \cdot dy\right) \cdot dy \right]$$

$$\Rightarrow \frac{1}{2\beta^3} \left[-\beta y^3 e^{-y/\beta} - \underbrace{\int -3y^2 \beta e^{-y/\beta} \cdot dy}_{\text{again applying by parts}} \right]$$

$$\Rightarrow \frac{1}{2\beta^3} \left[-\beta y^3 e^{-y/\beta} + 3\beta \int y^2 e^{-y/\beta} \cdot dy \right]$$

$$\Rightarrow \frac{1}{2\beta^3} \left[-\beta y^3 e^{-y/\beta} + 3\beta \left[y^2 \int e^{-y/\beta} \cdot dy - \int 2y \left(\int e^{-y/\beta} \cdot dy\right) \cdot dy \right] \right]$$

$$\Rightarrow \frac{1}{2\beta^3} \left[-\beta y^3 e^{-y/\beta} + 3\beta \left[-\beta y^2 e^{-y/\beta} + \underbrace{2\beta \int y e^{-y/\beta} \cdot dy}_{} \right] \right]$$

again appying by parts,

$$\Rightarrow \frac{1}{2\beta^3}\left[-\beta y^3 e^{-y/\beta} - 3\beta^2 y^2 e^{-y/\beta} + 6\beta^2\left[y\int e^{-y/\beta}\,dy - \iint e^{-y/\beta}\,dy\,dy\right]\right]$$

$$\Rightarrow \frac{1}{2\beta^3}\left[-\beta y^3 e^{-y/\beta} - 3\beta^2 y^2 e^{-y/\beta} - 6\beta^3 y e^{-y/\beta} - 6\beta^4 e^{-y/\beta}\right]$$

$$\Rightarrow \frac{-1}{2\beta^2}\left[y^3 e^{-y/\beta} + 3\beta y^2 e^{-y/\beta} + 6\beta^2 y e^{-y/\beta} + 6\beta^3 e^{-y/\beta}\right]\Big|_0^\infty$$

$$\Rightarrow 0 - \left(\frac{-1}{2\beta^2}\right)\left[6\beta^3\right]$$

+1 depending on correctness of integration steps.

$$\Rightarrow 3\beta$$

$\therefore E[y] = 3\beta$

$m_1 = E[y]$

$$\frac{\sum y_i}{n} = 3\beta$$

+0.5

$$\Rightarrow \boxed{\hat{\beta} = \frac{\sum y_i}{3n}} \to \text{mom estimate}$$

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b) MLE estimator for β

+0.5

$$L(\theta) = \prod_{i=1}^{n} f_\theta(y_i)$$

$$= \prod_{i=1}^{n} \frac{1}{2\beta^3} y_i^2 e^{-y_i/\beta} \qquad y_i > 0, \ \beta > 0$$

$$= \left(\frac{1}{2\beta^3}\right)^n \left(\prod_{i=1}^{n} y_i\right)^2 e^{-\sum y_i/\beta}$$

+0.5

$$\log L(\theta) = \log\left[\left(\frac{1}{2\beta^3}\right)^n \left(\prod_{i=1}^{n} y_i\right)^2 e^{-\sum y_i/\beta}\right]$$

$$= -n\log(2\beta^3) + 2\log\left(\prod_{i=1}^{n} y_i\right) - \sum y_i/\beta$$

+1

$$\frac{d \log L(\theta)}{d\theta} = \frac{-n}{2\beta^3} \times 6\beta^2 + 0 + \frac{\sum y_i}{\beta^2}$$

$$\frac{d \log L(\theta)}{d\theta} = 0$$

$$\frac{6\beta^2 n}{2\beta^3} = \frac{\sum y_i}{\beta^2}$$

$$\Rightarrow \boxed{\hat{\beta} = \frac{\sum y_i}{3n}}$$

check $\frac{d^2L}{d\theta^2}$ at $\hat{\beta} = \frac{\Sigma y_i}{3n}$ __/__/____

$$\frac{d^2L}{d\theta^2} = \frac{3n}{\beta^2} + 0 - \frac{2\Sigma y_i}{\beta^3}$$

$$\left.\frac{d^2L}{d\theta^2}\right|_{\hat{\beta} = \frac{\Sigma y_i}{3n}}$$

$$\Rightarrow \frac{27n^3}{(\Sigma y_i)^2} - \frac{2\cancel{\Sigma y_i}}{(\Sigma y_i)^{\cancel{3}}} \times 27n^3$$

$$\Rightarrow \frac{-27n^3}{(\Sigma y_i)^2}$$

we can see that $(\Sigma y_i)^2 > 0$ as $y_i > 0$

$$\therefore, \left.\frac{d^2L}{d\theta^2}\right|_{\hat{\beta}} < 0$$

since 2^{nd} order derivative is $-ve$, $\hat{\beta} = \frac{\Sigma y_i}{3n}$

is the maxima.

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c) $Y \sim f_\theta(y) = \frac{1}{2\beta^2} y_i^2 e^{-y_i/\beta}$ $y_i > 0, \beta > 0$

let $Z = cy + d$

$\Rightarrow y = \frac{z-d}{c} \Rightarrow \frac{dy}{dz} = \frac{1}{c}$

+0.5 By transformation theorem distribution/pdf of Z is given by

$$f(z) = \left|\frac{dy}{dz}\right| \cdot f(y)$$

$$f(z) = \frac{1}{c} \cdot \frac{1}{2\beta^2} y_i^2 e^{-y_i/\beta}$$

+0.5

$$= \frac{1}{c} \cdot \frac{1}{2\beta^3}\left(\frac{z-d}{c}\right)^2 e^{-\left(\frac{z-d}{\beta c}\right)}$$

$$= \frac{1}{c} \cdot \frac{1}{2\beta^3}\left(\frac{z-d}{c}\right)^2 e^{-\frac{1}{\beta}\left(\frac{z-d}{c}\right)}$$

$$= \frac{1}{c} \cdot f_y\left(\frac{z-d}{c}\right)$$

+0.5 Thus, Y belongs to the loc-scale family