MTH310/520: Submission 2

February 9, 2024

1. (5 points) Let C and D be cycles in a graph G. Prove that $C\Delta D$ decomposes into cycles. Here $C\Delta D$ indicates symmetric differences between two sets, i.e., $C\Delta D = (C \setminus D) \cup (D \setminus C)$.

   Solution. Suppose C and D are two disjoint cycles. Then the result trivially holds. Therefore we assume $C \cap D \neq \phi$. We claim that for any maximal path P that lies in C but not D, there exists another path P' with the same endpoints that lies in D but not C and thus $P' \cup P$ is a cycle. Let $C = \{v_1, e_1, \ldots, v_i, e_i, \ldots, v_j, e_j, \ldots, v_k, e_k, v_1\}$ such that $P = \{v_i, e_i, \ldots, e_{j-1}, v_j\}$ be a maximal path with $P \subseteq C \setminus D$. Therefore there exists another path $P' = \{v_i, e'_i, \ldots, e'_{j-1}, v_j\}$ such that $P' \subseteq D \setminus C$. Observe that $P \cup P' = \{v_i, e_i, \ldots, e_{j-1}, v_j, e'_{j-1}, \ldots, e'_1, v_1\}$ is a cycle.

   Rubric: 2 marks for a correct proof idea, 3 marks for precise writing.