Q: Let X be an observation from Pdf:

$$f_\theta(x) = \left(\frac{\theta}{2}\right)^{|x|}(1-\theta)^{1-|x|} \quad \left\{ \begin{array}{l} x = -1, 0, 1 \\ 0 \le \theta \le 1 \end{array} \right\}$$

Is X a complete sufficient statistic.

Notes ↓

Definition of complete sufficient statistic

→ Statistic that is both complete and sufficient !!

Soln:

(I) Checking for Sufficiency: [1 mark]

Approach 1: Since here we've 1 observation X and $T(X) = X$, observation itself, Therefore it is sufficient statistic since it'd obviously contain all info about θ. [1 mark]

OR

Approach 3: Suff defn

→ writing $T(X) = X$ correctly
→ using corr form $f_\theta(T(X))$
→ proving sufficient.

Approach 2: factorization theorem.

$$f_\theta(x) = g(t|\theta)\, h(x)$$ [0.5 mark defn]

$$= \underbrace{\left(\theta\right)^{|x|} (1-\theta)^{1-|x|}}_{g(t|\theta)} \times \underbrace{\left(\frac{1}{2}\right)^{|x|}}_{h(x)}$$ [0.5 mark]

II) Checking for Completeness: [1.5 mark]

→ Statistic $T = T(X)$ is said to be complete if $E_\theta[g(t)] = 0 \;\forall\, \theta \in \Theta \Rightarrow g(t) = 0$ a.e

0.25 mark

→ Suppose $E_\theta[g(t)] = 0 \;\forall\, \theta \in [0,1]$

where $T(X) = X$

$x = -1, 0, 1 \;;\; \theta \in [0,1]$

0.25 mark

$$\sum_{t \in \{-1,0,1\}} g(t)\, P_\theta(X = x) = 0 \;\forall\, \theta \in [0,1]$$

$$g(-1)\frac{\theta}{2} + g(0)(1-\theta) + g(1)\frac{\theta}{2} = 0 \;\forall\, \theta \in [0,1]$$

$$\frac{\theta}{2}[g(-1)+g(1)]+(1-\theta)g(0)=0 \quad \forall \theta\in[0,1]$$

0.25 mark

$$\frac{\theta}{2}[g(-1)+g(1)-2g(0)]+g(0)=0 \quad \forall \theta\in[0,1]$$

for above to hold $\forall \theta \in [0,1]$

0.25 mark

$g(0)=0$ and $g(-1)+g(1)-2g(0)=0$

putting $g(0)=0$ in $g(-1)+g(1)-2g(0)=0$

0.25 mark

$$\Rightarrow g(-1)+g(1)=0$$

i.e., $\exists$ a non trivial solution [$g(-1)$ & $g(1)$ are not necessarily zero $\forall \theta\in[0,1]$]

$\therefore E_\theta[g(t)]=0 \not\Rightarrow g(t)=0 \quad \forall \theta\in[0,1]$ (does not imply)

0.25 mark

$\therefore$ Not Complete!!