MTH 377/577 Convex Optimization
Quiz 1

Instructions: Answer all questions. Time= 1 hr 10 minutes.

1. Let A be an $m \times n$ matrix, $b \in R^m$.

   (a) Show that if the system of inequalities $yA \geq 0$, $yb < 0$ has a solution then the feasible set of non-negative solutions for $Ax = b$ is empty. (3)
   Ans. Suppose $\exists y$ such that $yA \geq 0$, $yb < 0$ and the set $F = \{x \geq 0 | Ax = b\}$ is non-empty. For any $x \in F$, $Ax = b \implies yAx = yb$. However $yA \geq 0$ and $yb < 0$. Therefore either y does not exist or F is empty.

   (b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$. Use Farkas Lemma to find out if the system $Ax = b$ has a non-negative solution. Show all the steps. (2)
   Ans.(indicative) Solution to $yA \geq 0$, $yb < 0$ exists therefore $Ax = b$ does not have a non-negative solution. For example: $y = (1, -1, 2)$. Show all the steps on your own as demonstrated in earlier problem sets.

2. State whether each of the following statements are true or false. Provide a brief mathematical explanation/counterexample to support your answer.

   (a) A polyhedron is always convex. (2)
   Ans. True. A polyhedron is the intersection of finite number of halfspaces and hyperplanes. $P = \{x \in R^n | Ax \leq b\}$. Consider $x_1, x_2 \in P$, and any $\theta \in [0, 1]$. $\theta x_1 \leq \theta b$ and $(1 - \theta)x_2 \leq (1 - \theta)b$. Adding, we get $\theta x_1 + (1 - \theta)x_2 \leq b$ therefore $\theta x_1 + (1 - \theta)x_2 \in P$.

(b) Let $A = \{x, y, z\}$ be a set of vectors in R^2. Is the conic hull of A always convex? Is the convex hull of A a cone? (3)
Ans. (a) True. (b) False- convex hull is not necessarily a cone.

(Refer Page 4 onwards for 2(b) sol^n) →

3. Let $S = \{(x_1, x_2, x_3) \in R^3 | -10 \leq x_1, x_2, x_3 \leq 10\}$. Do the following functions have a maxima/minima? Provide an explanation in support of your answer. (2+2+1)

   (a) Let $f : S \to R$ where $f(x) = x_1^2 + x_2^2 + x_3^2$; for all $x = (x_1, x_2, x_3)$ in S. **Ans.** f is continuous and S is compact. By Weierstrass theorem, maxima and minima exist.

   (b) Let $g : S \to R$ such that $g(x) = \begin{cases} f(x) + 2, & x_1, x_2, x_3 \geq 0 \\ 0, & \text{otherwise} \end{cases}$
   Ans g is discontinuous at $x = 0$. Weirstrass theorem cannot be applies. We however can observe that $x < 0$ and $x = 10$ generate the minimum and maximum values of the function.

   (c) Does your answer change for f and g if

   $$S = \{(x_1, x_2, x_3) \in R^3 | -10 < x_1, x_2, x_3 < 10\}$$

   Ans. Yes, S is no longer closed, therefore it is not compact. We cannot apply weierstrass theorem to f. For g now supremum exists at $x = 10$ but it is no longer in the feasible set, therefore not maxima.

4. Consider the following sets in R^n:

   $$A = \{(x_1, x_2, \ldots, x_n) \in R^n | x_i \geq 2\}$$

   $$B = \{(x_1, x_2, \ldots, x_n) \in R^n | \sum_{i=1,2,\ldots,n} x_i^2 \leq 2\}$$

   $$C = \{(x_1, x_2, \ldots, x_n) \in R^n | (x_1 - 2)^2 + \sum_{j=2,\ldots,n} x_j^2 < 1\}$$

   (a) Does a separating hyperplane exist that separates A from B? What about between A and $B \cup C$? Why/why not? Use separating hyperplane theorems in your explanation. (3)
   Ans. A, B are disjoint. Therefore, we must claim that a strictly separating hyperplane will exist using the hyperplane theorems (must contain some kind of mathematical derivation in order to prove that A and B are disjoint sets, diagrams might be included as

well.). $B \cup C$ is non-convex (show diagram/brief proof) therefore, we cannot claim using the theorems that a separating hyperplane will exist between A and $B \cup C$(1.5+1.5).

(b) Does a separating hyperplane exist that separates A from C? If yes, explain how you can find such a hyperplane. If no, provide a reason/counterexample. Use separating hyperplane theorems in your explanation. (2)
Ans. A and C are not disjoint even if they might be convex and non-empty hence, according to the hyperplane theorem, since the intersection is not a null set, no separating hyperplane would exist. Construction (indicative): take $a \in A$, $c \in C$ such that these minimze $d(x, y)$ for all $x \in A$ and $y \in C$ and show the shaded intersecting region. (1+1).

Q2 (b) **conic hull :** set of all conic combinations of points in set C

$$\text{cone}(C) = \left\{ \sum_{i}^{k} \theta_i x_i \mid x_i \in C,\ \theta_i \geq 0,\ i = 1 \text{ to } k \right\}$$

set $A = \{x, y, z\}$

conic hull of set A = $\text{cone}(A) = \{\theta_1 x + \theta_2 y + \theta_3 z \mid \theta_1, \theta_2, \theta_3 \geq 0\}$

for convexity let $\alpha \in [0,1]$ $\vartheta, \omega \in \text{cone}(A)$

$\vartheta = \lambda_1 x + \lambda_2 y + \lambda_3 z$, $\lambda_1, \lambda_2, \lambda_3 \geq 0$

$\omega = \mu_1 x + \mu_2 y + \mu_3 z$, $\mu_1, \mu_2, \mu_3 \geq 0$

$$\begin{aligned}\alpha\vartheta + (1-\alpha)\omega &= \alpha(\lambda_1 x + \lambda_2 y + \lambda_3 z) + (1-\alpha)(\mu_1 x + \mu_2 y + \mu_3 z) \\ &= (\alpha\lambda_1 + (1-\alpha)\mu_1)x + (\alpha\lambda_2 + (1-\alpha)\mu_2)y \\ &\quad + (\alpha\lambda_3 + (1-\alpha)\mu_3)z\end{aligned}$$

since $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 \geq 0$

$\alpha \in [0,1]$

$\Rightarrow$ $\alpha \cdot \lambda_i + (1-\alpha)\mu_i \geq 0$

$\therefore$ $\alpha\vartheta + (1-\alpha)\omega \in \text{cone}(A)$

$\therefore$ conic hull of set A is convex

convex hull : convex hull of set $C \subseteq \mathbb{R}^n$ is set of all convex combinations of elements in C

$$\text{conv}(C) = \left\{ \sum_{i}^{k} \theta_i x_i \mid x_i \in C,\ \theta_i \geq 0\ \forall i,\ \sum_{i \leq k} \theta_i = 1 \right\}$$

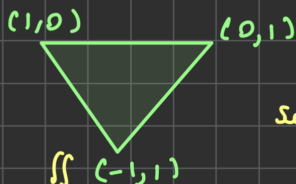
cone : set C is called cone if for every $x \in C$, $\theta \geq 0$ $\theta x \in C$.

No, convex hull of set A is not necessarily a cone

as a cone requires that if $v \in A$ then $\theta v \in A$, $\theta \geq 0$

But, convex hull do not necessarily satisfy this property, as they allow convex combinations of set A.

counter eg :



set $A = \{(1,0), (0,1), (-1,-1)\} \in \mathbb{R}^2$

convex hull of set A

it is not a cone as $(1,0) \in A$

but $2(1,0) = (2,0) \notin A$