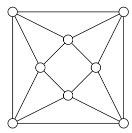
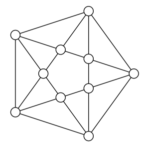
Endsem, MTH-310/520, Time: 180 mins

Problem 1 (5 points)**.** A graph G is called k*-critical* if its chromatic number $\chi(G)$ is k but any subgraph G' of G has chromatic number $\chi(G') < \chi(G)$.

1. Show that a k-chromatic graph must have a k-critical sub-graph.
2. For each G below, compute its chromatic number $\chi(G)$ and give a $\chi(G)$-critical subgraph. Justify your answer.

Solution:

1. Let $G_1 = G \setminus \{v_1\}$, where $v_1 \in V$. If $\chi(G_1) = \chi(G)$, then we repeat this process with $v_2, \ldots$ until we obtain a subgraph G_ℓ where $\chi(G_\ell - v) < \chi(G)$ for each $v \in G_\ell$. Then, G_ℓ is a k-critical subgraph.
2. $\chi(G) = 4$ for both graphs. Since $\Delta(G) = 4$, we need at most 4 colors. On other hand $\omega(G) = 3$, hence $3 \leq \chi(G) \leq 4$. We need three colors to color each triangles adjacent to outer face, and since the triangles are connected by edges inside, we need one more color.

Problem 2 (5 points)**.**

1. Prove or disprove: If u and v are the only odd-degree vertices in a graph G, then there is a u-v path in G.
2. Prove that $G = (V, E)$ is a tree if and only if it is connected, and $|V| = |E| + 1$.

Solution:

1. By the handshake lemma, a graph has an even number of odd degree vertices. Hence, each component must have an even number of odd degree vertices. Hence, the two odd degree vertices are in a connected component, which implies there is a path between them.
2. We prove by induction. We know a tree has a leaf. Removing it yields a smaller tree. Inductively, the equality holds, and adding this leaf node back yields the desired bound.

Problem 3 (5 points)**.**

1. For M, a matching in a bipartite graph $G = (X \uplus Y, E)$, if $S \subseteq X$ is matched in M, then show $\exists$ a *maximum* matching in which all nodes in S are matched.
2. (**520**) Two people play a game on a graph G by alternately selecting distinct vertices $v_1, v_2, \ldots$ forming a path. The last player able to select a vertex wins. Prove that the second player has a winning strategy if G has a perfect matching, and the first player has a winning strategy if G has no perfect matching. (Hint: For the second part, the first player should start by picking a vertex omitted by some maximum matching.)
3. (**320**) Design an algorithm that adds the fewest number of edges to a graph to make it Eulerian.

Solution:

1. Following the algorithm of finding augmenting paths, if S is a set of covered vertices, the vertices remain in S remain covered.

2. If there exists a perfect matching M, the first player chooses a vertex v and the second player can always choose the other endpoint u, where $(u, v) \in M$. If there is no perfect matching, the first player can start the game by choosing a vertex v not saturated by a maximum matching. In return the second player chooses a vertex u from which an M-alternating path begins. Since M is maximum matching, the game ends in the last move by the first player.

3. There are an even number of odd degree vertices by the handshake lemma. Thus, add an edge cover of these vertices by putting a matching on these vertices.

Problem 4 (5 points)**.**

1. Use the max-flow/min-cut theorem to prove the following version of Menger's theorem: Let $S, T \subseteq V$ of a digraph $D = (V, A)$. The maximum number of $S - T$ vertex disjoint paths is equal to the minimum size of an $S - T$ disconnecting set, i.e., a subset of vertices K, the removal of which separates S and T.

2. **(320)** Consider the following network on 4 vertices $\{s, t, u, v\}$ with edges $(s, u), (s, v), (u, t), (v, t)$ and (u, v). For any $n \in \mathbb{N}$, set capacities on the arcs so that running the Ford-Fulkerson algorithm on this network results in more than n augmentation steps.

3. **(520)** Show that $S \cap T, S \cup T$ are both min-cuts if S and T are min-cuts.

Solution:

1. Replace each vertex by a directed arc and apply MFMC.

2. Set the capacity of all arcs except (u, v) to be n, and set the capacity of (u, v) to be 1. At each step if the algorithm chooses the st path including the edge uv, we require n iterations.

Problem 5 (5 points)**.**

1. Let G be a maximal planar graph. Show that G^* is 2-edge connected and 3-regular.

2. Let L be a set of lines in the plane so that no pair of lines is parallel. We construct a graph whose vertices are the intersection points of the lines, and two vertices are adjacent if they lie consecutively on one of the lines in L. Determine the chromatic number of this graph by proving upper and lower bounds on the chromatic number.

Solution:

1. Each edge in the dual belongs to a cycle as the faces around a vertex form a cycle in the graph. Each vertex has degree 3 since each face has 3 sides.

2. $\chi(G) = 3$ since removing the leftmost vertex at each step shows that the graph is 2-degenerate. It is at least 3 since the graph contains triangles.