(4) $X_i \overset{iid}{\sim} N(\theta, \sigma^2)$; θ, σ^2 unknown.

LRT for $H_0: \theta \le \theta_0$ v/s $H_1: \theta > \theta_0$ is

$$T = \frac{\sqrt{n}(\bar{x} - \theta_0)}{s} \ge c_1$$

To find the value of c_1.

$$\beta(\theta) = P_\theta(X \in RR) \quad \text{for } \theta \in \Theta_0$$

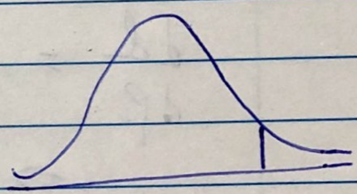
$$= P_\theta\left(\sqrt{n}\left(\frac{\bar{x} - \theta_0}{s}\right) \ge c_1\right) \quad \text{for } \theta \in \Theta_0$$

Here, $\sqrt{n}\frac{(\bar{x} - \theta_0)}{s} \sim t(n-1)$. Thus we don't need to do any manipulations. We can re-write above as

$$\beta(\theta) = P_\theta(T \ge c_1)$$

Now, $\beta(\theta) = \alpha = 0.05$ (given)

Thus, $\alpha = P_\theta(T \ge c_1)$



Thus $c_1 = t_{\alpha, (n-1)}$

Or $c_1 = t_{0.05}(n-1)$

We can find $t_{0.05}(n-1)$ from t-table or using statistical s/w.

Thus we reject H_0 if $\sqrt{n}\frac{(\bar{x} - \theta_0)}{s} \ge t_{\alpha = 0.05}(n-1)$

(5) $X_i \overset{iid}{\sim}$ Bernoulli(p).

$T = \sum X_i \sim$ Binomial (n, p) ; n known.

$H_0 : p \leq p_0$ v/s $H_1 : p > p_0$.

Likelihood of T will be

$$L(p) = \binom{n}{t} p^t (1-p)^{n-t}.$$

To perform LRT we need to max. $L(p)$ under restricted & unrestricted parameter space of p.

For unrestricted it is equivalent to finding MLE w/o any restriction. Thus,

$$l(p) = \log L(p) = c + t \log p + (n-t) \log(1-p)$$

$$\frac{dl}{dp} = \frac{t}{p} + \frac{(n-t)(-1)}{1-p} = 0$$

$$\Rightarrow \hat{p}_{MLE} = t/n.$$

$$\left.\frac{d^2 l}{dp^2}\right|_{\hat{p}} = \frac{-t}{\hat{p}^2} - \frac{(n-t)}{(1-\hat{p})^2} < 0.$$

Thus $\hat{p}_{MLE} = \frac{t}{n} = \frac{\sum x_i}{n} = \bar{x}$.

So for the denominator of LRT statistic we have

$$L(\hat{p}_{MLE}) = \binom{n}{t} \left(\frac{t}{n}\right)^t \left(1 - \frac{t}{n}\right)^{n-t}$$

Now, for restricted MLE, we need to max $L(p)$.
$\{p \le p_0\}$

Clearly, there are 2 possibilities.

Case I When $\frac{t}{n} \le p_0$.

then by previous solution (explanation).

$$\hat{\hat{p}}_{MLE} = t/n$$

if not then

Case II When $\frac{t}{n} > p_0$.

In that case, as MLE is attained at $\hat{p} = \frac{t}{n}$. But $\frac{t}{n} > p_0$ and the max. value of parameter p can be p_0 so we will take max. possible value of the parameter 'p' under null.

Thus $\hat{\hat{p}}_{MLE} = p_0$.

Correspondingly, we will have two cases of $\lambda(x)$.

Case I. When $t/n \le p_0$

$$\lambda(x) = \frac{L(\hat{\hat{p}})}{L(\hat{p})} = 1.$$

Case II when $t/n > p_0$

$$\lambda(x) = \frac{L(\hat{\hat{p}})}{L(\hat{p})}$$

$$= \frac{\binom{n}{t} p_0^{t} (1-p_0)^{n-t}}{\binom{n}{t} \left(\frac{t}{n}\right)^t \left(1-\frac{t}{n}\right)^{n-t}}$$

$$= \frac{(p_0)^t (1-p_0)^{n-t}}{\left(\frac{t}{n}\right)^t \left(1-\frac{t}{n}\right)^{n-t}}$$

Thus by LRT we reject H_0 if

$$\{\lambda(x) \leq c\}$$

$\Rightarrow$ reject H_0 if $\frac{(p_0)^t (1-p_0)^{n-t}}{(t/n)^t (1-t/n)^{n-t}} \leq c$.

We need to simplify this expression. for that we can verify that $\lambda(t)$ here is a decreasing function of t when $t/n > p_0$.

Thus we can re-write above as reject H_0 if $\lambda(t) \leq c$

or $\quad t \geq c$.

or $\quad \Sigma x_i \geq c$.

(Side note: to verify it's a decreasing f^n in t; take derivati[ve]
w.r.t.

i.e. $\ln \lambda(t) = t \ln p_0 + (n-t) \ln(1-p_0) - t \ln \frac{t}{n} - (n-t)\ln\left(1-\frac{t}{n}\right)$

$$\frac{\partial \ell}{\partial t} = \ln p_0 - \ln(1-p_0) - \ln\frac{t}{n} \cancel{- \frac{t(1/n)}{t/n}} + \ln\left(1-\frac{t}{n}\right) \cancel{+ \frac{(n-t)(1/n)}{(1-t/n)}}$$

$$= \ln p_0 - \ln(1-p_0) - \ln\frac{t}{n} + \ln\left(1-\frac{t}{n}\right)$$

~~Now~~ $= \ln\left[\frac{p_0}{1-p_0} \cdot \frac{(1-t/n)}{t/n}\right]$

Now, $\left(\frac{p_0}{1-p_0}\right)\left(\frac{1-t/n}{t/n}\right) < 1$

$\Leftrightarrow \quad p_0 < t/n$.

Thus, $\frac{\partial \ell}{\partial t} < 0$.

or $\lambda(t)$ is a decreasing f^n. in t.).

1. $X_i \overset{iid}{\sim} N(\theta, \sigma^2)$; σ^2 known.

$H_0: \theta \leq \theta_0$ V/s $H_1: \theta > \theta_0$.

To perform LRT we need to max. $L(\theta)$ w.r.t θ under restricted & unrestricted parameter space. Here the unknown parameter is only one, θ. The likelihood will be,

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{1}{2} \frac{(x_i-\theta)^2}{\sigma^2}\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2} \frac{\Sigma(x_i-\theta)^2}{\sigma^2}\right]$$

To get the unrestricted $\max_{-\infty<\theta<\infty} L(\theta)$, we have to find MLE of $L(\theta)$ or ~~log~~ max $\log L(\theta)$ w.r.t. θ.

Recall, $\hat{\theta}_{MLE} = \bar{X}$

(In exams you have to find it).

Now, for the LRT statistic the denominator will be,

$$L(\hat{\theta}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2} \frac{\Sigma(x_i-\bar{x})^2}{\sigma^2}\right].$$

For the restricted parameter space, we need to

$$\max_{\theta \le \theta_0} L(\theta).$$

Now, we have two cases.

Case I. When $\bar{X} \le \theta_0$

then using the same steps as for unrestricted

$$\hat{\hat{\theta}}_{MLE} = \bar{X}.$$

And, correspondingly

$$L(\hat{\hat{\theta}}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{-1}{2\sigma^2} \Sigma (x_i - \bar{x})^2\right]$$

& $\lambda(x) = \frac{L(\hat{\hat{\theta}})}{L(\hat{\theta})} = 1.$

Case II. When $\bar{X} > \theta_0$.

Here, as we have seen that $L(\theta)$ is max. at $\hat{\theta} = \bar{X}$. But as per the restriction $\theta \le \theta_0 < \bar{X}$. Thus $\bar{X}$

can't be attained. Thus the best we can do is reach θ_0. Or we take the max. possible value of θ i.e. θ_0.

So, $\hat{\hat{\theta}}_{MLE} = \theta_0$

& $$\lambda(x) = \frac{L(\hat{\hat{\theta}})}{L(\hat{\theta})} = \frac{\exp\left[-\frac{1}{2\sigma^2} \Sigma (x_i - \theta_0)^2\right]}{\exp\left[-\frac{1}{2\sigma^2} \Sigma (x_i - \bar{x})^2\right]}$$

By LRT reject H_0 if $\lambda(x) \le C$.

$$\Rightarrow \exp\left[-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2 + \frac{1}{2\sigma^2}\sum(x_i-\bar{x})^2\right] \le C.$$

$$\Rightarrow \sum(x_i-\bar{x})^2 - \sum(x_i-\theta_0)^2 \le C.$$

$$\Rightarrow n\frac{(\bar{x}-\theta_0)^2}{\sigma^2} \ge C.$$

$$\Rightarrow \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \ge C \qquad (\text{as } \bar{X} > \theta_0)$$

Thus, reject H_0 if $\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} \ge C$.

(For finding value of C refer to LRT II slides).

$C = z_\alpha$.