$X \sim N(0, \sigma^2)$

1) unbiased estimator of σ^2

$\rightarrow$ when mean is not known:

$$V = \frac{1}{n} \sum (X_i - \bar{X})^2 \quad \text{is biased.}$$

<u>Proof</u>: $$E[V] = \frac{1}{n} \sum_{i=1}^{n} \underbrace{E[(X_i - \bar{X})^2]}$$

$$\downarrow$$

$$E\left[(X_i - \mu)^2 + (\bar{X} - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu)\right]$$

$$= E\left[\frac{1}{n} \sum (X_i - \mu)^2 + \frac{1}{n} \sum (\bar{X} - \mu)^2 - \frac{2}{n}(\bar{X} - \mu) \sum (X_i - \mu)\right]$$

$$= E\left[\frac{1}{n} \sum (X_i - \mu)^2 + \frac{1}{n} \sum (\bar{X} - \mu)^2 - 2(\bar{X} - \mu)(\bar{X} - \mu)\right]$$

$$= E\left[\frac{1}{n} \sum (X_i - \mu)^2 - (\bar{X} - \mu)^2\right]$$

$$= Var[x] - Var[\bar{x}]$$

$$= \sigma^2 - \frac{\sigma^2}{n}. \qquad \therefore \text{Biased} \quad \blacksquare$$

→ However, when mean is known

$$V = \frac{1}{n} \sum (x-\mu)^2$$

$$= Var[x].$$

a) Here, mean is known:

$$\text{estimator} = \frac{1}{n} \sum (x-\mu)^2$$

$$= x_1^2 \qquad (\text{in this case}).$$

b) $$L(\sigma|\underline{x}) = f(x|\sigma)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$\log L(\sigma|\underline{x}) = \frac{-x^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{x^2}{\sigma^3} - \frac{1}{\cancel{2}}\left(\frac{1}{\cancel{2}\pi\sigma^2} \cancel{4\pi\sigma}\right)$$

$$- x^2 - \sigma^2 = 0$$

$$\overline{\qquad \sigma^3 \qquad}$$

$$\Rightarrow \quad x^2 = \sigma^2$$

$$\hat{\sigma}^2 = x^2 \quad \rightarrow \quad \underline{\underline{MLE}}$$

c) <u>In method of moments:</u>

$$\frac{1}{n} \sum X_i \quad — \quad E[X]$$

$$\frac{1}{n} \sum X_i^2 \quad — \quad E[X^2]$$

in a single sample:

$$E[X^2] - \cancel{(E(X))}^2 = var[X]$$

$$= \sigma^2$$

MOM estimator of $\hat{\sigma}^2 = X_1^2$