Midsem

MTH-310/520: Graph Theory
Winter 2024

Time: 120 mins

Instructions:

- There are **five questions** with a total of 25 **points**.
- The questions are *not* arranged by difficulty.
- Questions marked (**520**) are for students registered for the 520 course, and questions marked (**310**) are for students registered for the 310 course. If no number is given, then the question is common for both groups of students.

Problem 1 (5 points)**.** Prove by induction that if $d_1, \ldots, d_n$ are non-negative integers, and $\sum_i d_i$ is even, then there is an n-vertex graph with vertex degrees $d_1, \ldots, d_n$.

Solution: Proceed by induction on n. For $n = 1, d_1$ should be even. Then the corresponding graph is of a single vertex and multiple loops. For $n = 2$, the graph is a set of parallel edges if $d_1 = d_2$, otherwise one of them has loops. Suppose the statement is true for $n = k$. For $n = k + 1$, we remove the integer d_k. Suppose d_k is even. Then $\sum_{i=1}^{n-1} d_i$ is even and by inductive hypothesis there exists a graph G' whose degrees $d_1, \ldots, d_{n-1}$. We take a vertex v_n and add $d_n/2$ loops and $G' \cup \{v_n\}$ is the desired graph. Now suppose d_n is odd. Observe that $\sum_{i=1}^{n-1} d_i$ is odd and hence strictly positive. We remove one from each of the integers $d_1, \ldots, d_l$, where l is an integer at most d_n and add edges from v_n to each of $d_1, \ldots, d_{n-1}$. Since $\sum_{i=1}^{n-1} d_i$ is now even, by inductive hypothesis there is a graph G' with degrees $d_1, \ldots, d_{n-1}$. Finally if $d_n > l$ then we add loops on v_n and thus $G' \cup \{v_n\}$ is the desired graph.

Problem 2 (5 points)**.** Prove that every connected graph has an orientation with at most one vertex having odd out-degree [*Hint: Use Eulerian tours*]

Solution: Suppose the graph is Eulerian. Then we can take an Eulerian tour and any Eulerian tour can be oriented in such a way that at most one vertex has odd degree as follows: for a vertex v, we orient the incident edges outwards, and for all neighbours of v, orient the incident edges inwards. Since degrees of all the vertices are even, the above claim satisfies. Suppose the graph is not Eulerian. Then number of odd degree vertices are even. We form pairs of vertices add dummy edges between each such pairs. Thus we obtain an Eulerian graph. We orient the edges in the above manner. Now we start deleting the dummy edges added to the graph. Let (u, v) is such a pair of vertices. If both of their outdegrees were even, we take an u-v path and flip the orientation of the edges. This will yield both u and v having odd out degrees. On other hand, if one of u and v had odd out degree then removing the edge between them leaves one of them having odd out degree. We repeat this process and obtain the desired graph.

Problem 3 (5 points)**.**

(**310**) Prove that if n is odd, there is a tournament such that all players are kings.

(**520**) Let $T = (V, A)$ be a tournament. For a vertex $v \in V$, v let $D(v) = \{u \in V : (v, u) \in A\} \cup \{v\}$. Prove that there is a set S, such that $|S| \leq \lceil \log_2 n \rceil$, $\cup_{v \in S} D(v) = V$.

Solution (310): Put the vertices on a circle and let $v_1, \ldots, v_n$ be the cyclic order of the vertices. We orient the edges $(v_i \to v_{i+j}), 1 \leq i \leq n, 1 \leq j \leq \lfloor n/2 \rfloor$ with j modulo n. For any pair (v_k, v_l) of vertices, if $l \leq k + \lfloor n/2 \rfloor$ then v_k can be reached from v_k in just one step. Otherwise $v_{k+\lfloor n/2 \rfloor}$ can be reached in one step from v_k and v_l can be reached from $v_{k+\lfloor n/2 \rfloor}$ in one step since $k + \lfloor n/2 \rfloor + \lfloor n/2 \rfloor = k$ modulo n.

Solution (520): Observe that $\sum_{v \in V} D(v) = \binom{n}{2}$. This implies $\frac{1}{n} \sum_{v \in V} D(v) = n - 1/2$. Therefore there exists at least one vertex v such that $|D(v)| \leq n - 1/2$. We remove all the vertices in $D(v)$ and repeat the procedure. Let $C(n)$ denote the count on total number of such vertices in a graph with n vertices. Thus we obtain the following recurrence: $C(n) \leq C(n - 1/2) + 1$ with $C(1) = 1$ and $C(0) = 0$. This yields $C(n) \leq \lceil \log_2 n \rceil$.

Problem 4 (5 points)**.**

(**310**) Let T and T' be two spanning trees of a graph G. Let $e \in E(T)) \setminus E(T')$. Show that there is an edge $e' \in E(T') \setminus E(T)$ such that

(a) $T_1 = (T \setminus \{e\}) \cup \{e'\}$ is a spanning tree of G, and

(b) $T_2 = (T' \setminus \{e'\} \cup \{e\}$ is a spanning tree of G.

(**520**) Let G be a connected graph and let S be a set of edges of $G = (V, E)$. Prove that the following are equivalent: (i) S is a spanning tree in G, (ii) S meets every *bond* of G, and is (edge-)minimal with this property, where a set of edges F of E is a bond if $E \setminus F$ induces two connected subgraphs.

Solution (310): From the book of Douglas B. West, Page 69-70

Solution (520): Suppose for the sake of contradiction B is a bond in G but $S \cap B = \phi$. Then $E \setminus B$ disconnects the graph into two connected components say C_1 and C_2. However that would mean there is no other path between any $u \in C_1$ and $v \in C_2$, which further contradicts the fact that S is a spanning tree in G. On other hand Suppose S is a tree of G such that there exists $v \in G$ such that $v \notin S$. This implies $E(S)$ does not contain all the edges incident to v. Let $(u, v) \in E$. Observe that exists a bond B whose removal yields two connected components C_1 and C_2 such that $u \in C_1$ and $v \in C_2$. Clearly at least one of the edges incident on v belongs to B. But since $v \notin S$, S does not meet B.

Problem 5 (5 points)**.** Prove that every tree has at most one perfect matching.

Solution: Proceed by induction on n. For $n = 1$ we have no perfect matching and for $n = 2$ the tree T itself is a perfect matching. Suppose the statement is true for $n = k$. For $n = k+1$, we consider removing one leaf node v and it's parent u of T and obtain T'. By inductive hypothesis T' has at most one perfect matching. If T' has no perfect matching, we note that T also has no perfect matching because if u was not matched with v, then v remains unmatched in T. Suppose T' has one perfect matching M. Then $M \cup (u, v)$ is a perfect matching of T.