3. Let $X \sim$ Exponential (λ), $\lambda > 0$. Consider $\theta = 1/\lambda$

(a) (1.5 points) Consider $T(X) = X^2$, find bias of $T(X)$.

(b) (2 points) Find MSE of $T(X)$.

(c) (2 points) Can we find Cramer-Rao lower bound. If yes, find it. If not, why not.

(a) Statistic $T(X) = X^2$

$X \sim$ Exponential(λ) $(\lambda > 0)$

$\therefore f_X(x) = \lambda e^{-\lambda x}$ $(x \geq 0)$ [put $\lambda = \frac{1}{\theta}$]

$= \frac{1}{\theta} e^{-x/\theta}$

[0.25 Marks]

Now Bias $= E[T(X)] - \theta$ → Parameter we're estimating

$\therefore$ Bias $= E[X^2] - \theta$

[0.25 marks]

Note: ① Bias formula and correct identification of parameter we're estimating, both have to be correct to fetch 0.25 marks above.

→ Calculating $E(X^2)$ → 0.75 marks

(Next page has 3 approaches).

(I) Approach-1 :

we know $Var(X) = E[X^2] - (E[X])^2$ — 0.25 Marks

for $X \sim Exp(\lambda)$ $E[X] = \frac{1}{\lambda}$ $Var(X) = \frac{1}{\lambda^2}$

$\therefore X \sim Exp\left(\frac{1}{\theta}\right)$ or $Exp(\lambda) \Rightarrow E[X] = \theta$, $Var[X] = \theta^2$ — 0.25 Marks

so, $E[X^2] = \theta^2 + \theta^2 = 2\theta^2$ — 0.25 Marks

Approach-2 :

$$E[X^2] = \int_0^\infty x^2 f_X(x)\, dx$$

$$= \int_0^\infty x^2 \frac{1}{\theta} e^{-x/\theta}\, dx$$ — 0.25 Marks

constant

Use integration by Parts

$$\int uv\, dx = u\int v\, dx - \int\left(\frac{\partial u}{\partial x}\int v\, dx\right)dx$$

Here $u = x^2$ $v = e^{-x/\theta}$

$$= \frac{1}{\theta}\left(-x^2\theta e^{-x/\theta}\Big|_0^\infty - \int_0^\infty 2x\theta e^{-x/\theta}\, dx\right)$$

$$= \frac{1}{\theta}\left(-x^2\theta e^{-x/\theta}\Big|_0^\infty - 2\left(-x\theta e^{-x/\theta}\Big|_0^\infty + \int_0^\infty \theta e^{-x/\theta}\right)\right)$$ — 0.25 Marks

Solve it

$E[X^2] = 2\theta^2$ — 0.25 Marks

Approach-3 :

we know MGF of exp distb

$M_X(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}$ $(X \sim exp(\lambda))$ — 0.25 Marks

we know, $E[X^n] = M_X^{(n)}(t)\Big|_{at\ t=0}$ — n^{th} derivative w.r.t t — 0.25 Marks

$M_X'(t) = \frac{-\lambda}{(\lambda - t)^2}$ $M_X''(t) = \frac{2\lambda}{(\lambda - t)^3}$

$\therefore M_X''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = 2\theta^2$ — 0.25 Marks

So, Bias $= 2\theta^2 - \theta$ [0.25 Marks] (for final correct expression)

(b) (2 points) Find MSE of $T(X)$.

Using $MSE_\theta(W) = Var_\theta(W) + Bias_\theta^2(W)$ 0.25 Marks

for estimator 'W(X)' estimating param 'θ'.

Note: If formula is fully correct → 0.25 marks
else → 0 !!

We already know $Bias_\theta(X^2)$ from Q3 (a)

So, we'll calculate $Var_\theta(X^2)$.

$Var_\theta(X^2) = E[X^4] - E[X^2]$ 0.25 marks

Approach-1: (MGF)

MGF of $X = \frac{\lambda}{\lambda - t}$ 0.25

$E[X^n] = M_X^{(n)}(t)\Big|_{at\ t=0}$ 0.25

So,

$M_X(t) = \frac{\lambda}{\lambda - t}$ $M_X'(t) = \frac{-\lambda}{(\lambda-t)^2}$

$M_X''(t) = \frac{2\lambda}{(\lambda-t)^3}$ $M_X'''(t) = \frac{-6\lambda}{(\lambda-t)^4}$

$M_X''''(t) = \frac{24\lambda}{(\lambda-t)^5}$ 0.25

$E[X^4] = \frac{24\lambda}{\lambda^5}$

$= 24\theta^4$ 0.25

Approach-2: (Using Integration)

$E[X^4] = \int_0^\infty x^4 \frac{1}{\theta} e^{-x/\theta} dx$ 0.25

$E[X^4] = 24\theta^4$ 0.25

→ Solving all 4 integration by parts → [0.125 × 4] now, $\mathrm{Var}_\theta[X^2] = 24\theta^4 - (2\theta^2)^2 = 20\theta^4$ 0.25 Marks

so,

$$MSE_\theta[W(X) = X^2] = \mathrm{Var}_\theta[X^2] + \mathrm{Bias}_\theta^2[X^2]$$

$$= 20\theta^4 + (2\theta^2 - \theta)^2$$

$$= 20\theta^4 + 4\theta^4 + \theta^2 - 4\theta^3 = 24\theta^4 - 4\theta^3 + \theta^2$$

0.25 marks

Note: If Someone has done

$$MSE(T(X)) = E[(X^2 - \theta)^2]$$

$$= E[X^4] + \theta^2 - 2\theta E[X^2]$$

will receive 0.25 marks for formula too!!

(c) (2 points) Can we find Cramer-Rao lower bound. If yes, find it. If not, why not. (0.5 → Checking, 1.5 → Calculating CRLB)

Checking if it exists or not: → 0.5

Approach-1:

Regularity Conditions: Show $\mathrm{Var}\left[\frac{\partial}{\partial\theta}(\log f_\theta(x))\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f_\theta(x)\right]$

$-\log\theta - \frac{x}{\theta}$

0.25 formula

: LHS:

$$\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f_\theta(x)\right]$$

$$= E\left[\left(\frac{\partial}{\partial\theta}\log f_\theta x\right)^2\right] - \left(E\left[\frac{\partial}{\partial\theta}\log f_\theta x\right]\right)^2$$

$$= \frac{1}{\theta^2} - \cancel{\frac{2E[X]}{\theta^3}} + \frac{E[X^2]}{\theta^4} - \left(-\frac{1}{\theta} + \frac{E[X]}{\theta^2}\right)^2$$

$$= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} + \frac{2\theta^2}{\theta^4} - \left(\frac{1}{\theta} + \frac{1}{\theta}\right)^2$$

$$= \frac{1}{\theta^2}$$

: RHS:

$$= -E\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right]$$

$$= -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = \frac{1}{\theta^2}$$

LHS = RHS!!

∴ CRLB exists

0.25 Correct with steps

Approach 2: Show it is exponential family & use the fact that CRLB can always be found for exponential family.

0.25

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} = e^{-\log\theta - \frac{x}{\theta}}$$

Compare with $h(x)\, e^{(\eta(\theta)\, T(x) - A(\theta))}$

$h(x) = 1$ $\quad A(\theta) = \log\theta$

$T(x) = x$ $\quad \eta(\theta) = -1/\theta$ $\quad$ 0.25

∴ Yes from exp family of distb

Note: This is formula in general.

Cramer Rao Lower Bound $= \dfrac{\left\{ \frac{d}{d\theta}\left(E_\theta[T(X)]\right)\right\}^2}{E_\theta\left[\left(\frac{\partial}{\partial\theta} \ln f_X(X|\theta)\right)^2\right]}$ $\to$ 0.25 $\to$ 0.25

$$\text{Numerator} = \frac{d}{d\theta}\left(E_\theta[X^2]\right) = \frac{d}{d\theta}(2\theta^2) = 4\theta \quad 0.25$$

$$\text{denominator} = E_\theta\left[\left(\frac{\partial}{\partial\theta}\left(\ln\left(\frac{1}{\theta} e^{-x/\theta}\right)\right)\right)^2\right]$$

Use Fischer's Inequality (holds for exponential)

$$= -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\left(-\ln\theta - \frac{x}{\theta}\right)\right] \quad 0.25$$

$$= E_\theta\left[\frac{\partial^2}{\partial\theta^2}\left(\ln\theta + \frac{x}{\theta}\right)\right] = E_\theta\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right]$$

$$= 1/\theta^2 - 2/\theta^2 = 1/\theta^2 \quad 0.25$$

$$\text{So, CRLB} = \frac{(4\theta)^2}{1/\theta^2} = 16\theta^4 \quad 0.25$$