MTH310/520: Submission 8

Time: 15 Minutes, Marks: 5

April 19, 2024

Name and Roll No:

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1. (5 points) Prove that for any graph G,

   i) $\chi(G) + \chi(\overline{G}) \leq n + 1$

   ii) $\chi(G)\chi(\overline{G}) \geq n$

   where $\overline{G}$ is the complement of G.

Solution.

i) Proceed by induction on n. For $n = 2$, if $\chi(G) = 1$ then $\chi(\overline{G}) = 2$, or vice versa. Therefore the bound holds. Suppose the statement is true for a graph with n vertices. Now consider G to be a graph with $n + 1$ vertices. Let $G' = G \setminus \{v\}$. By inductive hypothesis the bound holds. Adding back v if none or one of $\chi(G')$ or $\chi(\overline{G'})$ increases by one, then the statement holds. Otherwise suppose both increases by one. Let $\chi(G') = k$. That implies v has at most k neighbours in G and $\chi(G') \leq k - 1$. Then v has at most $n - k - 1$ neighbours in $\overline{G}$ and $\chi(\overline{G'}) \leq n - k - 1$. Therefore, $\chi(G) + \chi(\overline{G}) \leq k - 1 + n - k - 1 + 2 = n + 1$.

ii) Let $\chi(G) = k$. Then there exists a color class having at least n/k vertices. On other hand those vertices form a clique in $\overline{G}$. Therefore, $\chi(\overline{G}) \geq \frac{n}{k}$. This implies $\chi(\overline{G}) \leq \frac{n}{\chi(G)}$. This gives the desired bound.

Rubric: +3 for the first part. +2 for the second part.