MTH 377/577 Convex Optimization
Problem set 2: Indicative solutions

1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$ and let $b = (1, 0)$. Does the system $Ax = b$ have a solution where $x \geq 0$? [Use Farkas Lemma]
Ans. Farkas alternative:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} < 0$$

Writing the above as a system of linear inequalities:

$$\begin{gathered} 2y_1 + y_2 \geq 0 \\ 0y_1 + 1y_2 \geq 0 \\ -1y_1 + 2y_2 \geq 0 \\ 1y_1 + 0y_2 < 0 \end{gathered}$$

The above system has a solution: $y = (-1, 10)$. Therefore by Farkas Lemma, $Ax = b$ does not have a non-negative solution.

2. Use Farkas Lemma to decide whether the following system has a non-negative solution:

$$\begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Ans. Follow steps similar to Ans 1. In this case $y = (1, \frac{1}{2})$ is a solution to the Farkas alternative. Therefore, $Ax = b$ does not have a non-negative solution.

3. Let A, B be non-empty, disjoint, convex sets n R^n. Can you claim that there will exist a strict separating hyperplane that separates A and b? If yes, provide an explanation. If no, provide a counterexample.
Ans. No. Atleast one of the sets A, B should be bounded to ensure that a strict separating hyperplane exists. See counterexample in Lecture 7 slides.

4. Consider a non-empty, closed, bounded set C. Suppose I can construct a weak separating hyperplane at every $x \in C$ that is on its boundary and not a point in the interior of C. Is C convex?
Ans. Yes. Here is a simplified indicative proof by contradiction. For the purpose of this question, you can also show the explanation diagrammatically with the correct expressions for the separating hyperplane and the point b.
Proof: Let C be a non-convex, non-empty, closed bounded set that has a weak separating hyperplane at every boundary point. Let $x_1, x_2 \in C$ such that $b = \theta x_1 + (1-\theta)x_2 \notin C$ for some $\theta \in [0, 1]$ and x_0 is a boundary point closest to b. By assumption, $\exists$ weak separating hyperplane (h, β) such that $h^T x \geq \beta \geq h^T b$ for all $x \in C$ and $h^T x_0 = \beta$. Note that for $\theta \in [0, 1]$
$$h^T \theta x_1 \geq \theta\beta \geq h^T \theta b$$
$$h^T (1-\theta)x_2 \geq (1-\theta)\beta \geq h^T (1-\theta)b$$
Adding the above two, we get
$$h^T[\theta x_1 + (1-\theta)x_2] \geq \beta \geq h^T b$$
Since $b = \theta x_1 + (1-\theta)x_2$, the above holds with equality. Note that the above argument holds for all boundary points of C. Therefore, b lies on every weak separating hyperplane for C i.e. is a point of intersection for all weak separating hyperplanes. This is not possible: Case 1: if C is a line passing through b then $b \in C$ and C is convex. Case 2: if the hyperplanes form a cone with origin at b then C is either not bounded above or below.

5. Suppose $C_1, C_2 \in R^n$ are convex sets. Let $C = \{x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$. Is C convex?
Ans. Yes. Let $x = x_1 + x_2$ where $x_1 \in C_1$ and $x_2 \in C_2$. By definition, $x \in C$. Similarly let $y = y_1 + y_2$ where $y_1 \in C_1$, $y_2 \in C_2$ and $y \in C$. Pick any $\theta \in [0, 1]$. Note that $\theta x + (1-\theta)y = \theta(x_1+x_2)+(1-\theta)(y_1+y_2)$. Re-arranging the terms, we get:
$$[\theta x_1 + (1-\theta)y_1] + [\theta x_2 + (1-\theta)y_2]$$

Since C_1, C_2 are convex, $\theta x_1 + (1-\theta)y_1 \in C_1$ and $\theta x_2 + (1-\theta)y_2 \in C_2$. Therefore $[\theta x_1 + (1-\theta)y_1] + [\theta x_2 + (1-\theta)y_2] = \theta x + (1-\theta)y \in C$ for all $\theta \in [0,1]$.

6. "A polyhedron is the intersection of a finite number of half spaces and hyperplanes." Let P be a polyhedron. Show that P **is necessarily convex**.
   Ans. All polyhedra are convex sets. The intersection of a finite number of halfspaces and hyperplanes can be written as a system of weak inequalities: Let $P = \{x \in R^n | Ax \leq b\}$ be a polyhedra. If $x, y \in P$, then $Ax \leq b$ and $Ay \leq b$. Therefore, $A(\theta x + (1-\theta)y) = \theta Ax + (1-\theta)Ay \leq \theta b + (1-\theta)b = b$. Thus $\theta x + (1-\theta)y \in P$.

7. Provide an example of a set for each of the following:
   (a) Closed but not bounded: see example in lecture 7 slide
   (b) Bounded but not convex: $A = \{(x,y) \in R^2 | x^2 + y^2 = 1\}$.
   (c) Convex but not compact: $C = (-5, 5)$ is not compact because it is not closed. It is bounded.

   For which of the above three sets will a separating hyperplane always exist that separates it from a point x that is not in the set?
   Ans. For (c) Convex but not compact: convexity suffices for a weak separating hyperplane to exist between the set and a point not in the set. [Weak separating hyperplane theorem]

   [Note that for two sets and/or for strict hyperplanes, only convexity is not sufficient.]