MTH310/520: Submission 3

Time: 15 Minutes, Marks: 5

February 16, 2024

Name and Roll No:

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1. (5 points) Let T be a tree with n vertices, $n \geq 2$. For a positive integer i, let p_i be the number of vertices of T of degree i. Prove that $p_1 - p_3 - 2p_4 - \ldots - (n-3)p_{n-1} = 2$, and use this to give an alternative proof of the end-vertex lemma.

   Solution 1: Proceed by induction on n. When $n = 2$ we have $p_1 = 2$. Suppose the statement is true for $n = k$. Let G be a graph with $n = k+1$. We remove a vertex v of degree 1 from the tree. By induction hypothesis $p_1 - p_3 - 2p_4 - \ldots - (n-3)p_{n-1} = 2$ holds. Adding back v to G we observe that p_1 is increased by 1 but for some $i \in [1 \ldots n]$, p_i decreases by one and p_{i+1} increases by one. Therefore, we now have $LHS = (p_1+1) - \ldots - (i-2)(p_i+1) - (i-1)(p_i+1) - \ldots - (n-3)p_{n-1}$. Observe that total change in this quantity is $1 - (i-2) - (i-1) = 0$. Hence there is no change in RHS and the statement holds.

   Solution 2: We can also prove it algebraically. Let n and m denote the number of vertices and edges respectively.

$$
\begin{aligned}
& p_1 - p_3 - 2p_4 - \ldots - (n-3)p_{n-1} \\
=& (2p_1 - p_1) + (2p_2 - 2p_2) + (2p_3 - 3p_3) + (2p_4 - 4p_4) - \ldots + (2p_{n-3} - (n-1)p_{n-1}) \\
=& 2\sum_{i=1}^{n-1} p_i - \sum_{i=1}^{n-1} ip_i \\
=& 2n - \sum_{v \in V} deg(v) \\
=& 2n - 2m \\
=& 2n - 2(n-1) = 2
\end{aligned}
$$

   Finally we observe that $p_1 \geq 2$, otherwise $LHS < 2$. Hence, end-vertex lemma follows.