MTH 372 (2025) : Extra Questions

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1. Let $X_1, \ldots, X_n$ be i.i.d. from Normal (μ, σ^2), both unknown. Find minimal sufficient statistic(s) (MSS) for (μ, σ^2).

2. Let $X_1, \ldots, X_n$ be i.i.d. from Uniform $(\theta, \theta + 1), -\infty < \theta < \infty$. Find a minimal sufficient statistic for θ.

3. Let $X_1, \ldots, X_n$ be i.i.d. from Uniform $(\alpha, \theta), \theta$ is unknown and α is known. Answer the following questions

   (a) Find a sufficient statistic for θ.

   (b) Find method of moment estimator (MME) for θ.

   (c) Find maximum likelihood estimator (MLE) for θ.

   (d) Will Cramér- Rao inequality be applicable here. If yes, how will you apply it. If not, then explain why can't it be applied.

4. Given below are distribution and an estimator for the unknown parameter of the distribution. Find the corresponding mean squared error (MSE).

   (a) Let $X_1, \ldots, X_n$ be i.i.d. Binomial (n, p), where n is known but p is unknown and the estimator is $T(X) = \overline{X}$.

   (b) Let X be an observation from the following p.m.f.

   $$P_\theta(X = 1) = \frac{1 - \theta}{2}, \;\; P_\theta(X = 2) = \frac{1}{2}, \;\; P_\theta(X = 3) = \frac{\theta}{2}.$$

   The estimator is $T(X) = X - \frac{3}{2}$.

5. Let $X_1, \ldots, X_n$ be i.i.d. with the following pdf

   $$f_\theta(x_i) = \theta e^{-\theta x_i}, \;\; \theta > 0, x_i > 0.$$

   Answer the following questions

(a) Find a minimal sufficient statistic for θ.

(b) What will be an MLE of 2θ.

(c) Is $\overline{X}$ complete for θ.

(d) Does it belong to the exponential family. If yes, show how. If not, show why not.

(e) Does it belong to the location-scale family. If yes, show how. If not, show why not.

(f) Find a uniform minimum variance unbiased estimator (UMVUE) for $1/\theta$.

6. Let $X_1, \ldots, X_n$ be a random sample whose pdf is given by

$$f_\theta(x) = \frac{\theta^x \log(\theta)}{\theta - 1}, \quad \text{if } 0 < x < 1 \text{ and } \theta > 1.$$

(a) Show that the joint distribution belongs to the exponential family.

(b) Find the complete and sufficient statistic $T(X)$ for this family.

7. Suppose that X is a discrete random variable with the following probability mass function:

X	0	1	2	3
P(X)	$2\theta/3$	$\theta/3$	$2(1\text{-}\theta)/3$	$(1\text{-}\theta)/3$

where $0 \leq \theta \leq 1$ is a parameter. The following 10 independent observations were taken from such a distribution: (3,0,2,1,3,2,1,0,2,1). What is the maximum likelihood estimate of θ.

8. Let $X \sim$ Bernoulli(p), where $p \in [1/4, 3/4]$. Find the MLE of p. Will the MLE exist for $p \in (1/4, 3/4)$.

9. Let X be a random variable with the following p.m.f.

$$P_\theta(X = -1) = \theta, \quad P_\theta(X = x) = (1 - \theta)^2 \theta^x$$
$$\text{where } x = 0, 1, 2, \ldots \quad \text{and } 0 < \theta < 1.$$

Solve the following

(a) Find a sufficient statistic for θ.

(b) Is X complete for $0 < \theta < 1$.

(c) Find an unbiased estimator of $(1-\theta)^2$.

(d) Using the unbiased estimator of $(1-\theta)^2$, construct a statistic such that is complete for $0 < \theta < 1$.

(e) Find the UMVUE of $(1-\theta)^2$.

10. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables from Uniform $(-\theta, \theta)$. Answer the following.

    (a) What is sufficient statistic of θ.

    (b) Is it complete for $T = max|X_i|$.

    (c) Does it belong to the exponential family. Explain.

    (d) Is T an unbiased estimator of θ. If not, then find one.

    (e) How will you apply Cramèr - Rao inequality to it.

11. Suppose scores on exams in statistics are normally distributed with an unknown population mean and a population standard deviation of 3 points. A random sample of 36 scores is taken and gives a sample mean (sample mean score) of 68. Find a 90%confidence interval estimate for the population mean exam score (the mean score on all exams).
(Answer: (67.1775, 68.8225).)

12. Suppose you do a study of acupuncture to determine how effective it is in relieving pain. You measure sensory rates for 15 subjects with the results given below. Use the sample data to construct a 95% confidence interval for the mean sensory rate for the population (assumed normal) from which you took the data.
8.6; 9.4; 7.9; 6.8; 8.3; 7.3; 9.2; 9.6; 8.7; 11.4; 10.3; 5.4; 8.1; 5.5; 6.9
(Answer : (7.30, 9.15).)

13. It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2. That is, the random noise added to the signal is an $N(0,\ 4)$ random variable. There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today. Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$. Test using

i) $\alpha = 0.05$; ii) $\alpha = 0.10$.
(Answer: i) Fail to reject null; ii) reject null.)

14. A public health official claims that the mean home water use is 350 gallons a day. To verify this claim, a study of 20 randomly selected homes was instigated with the result that the average daily water uses of these 20 homes were as follows:
340 344 356 386 332 402 362 322 318 360 362 354 340 372 338 375 364 355 324 370
Do the data contradict the official's claim? (Use $\alpha = 0.05$ and assume the assumptions required are satisfied.)
(Answer : fail to reject null)