4 $X_i \overset{iid}{\sim}$ Gamma(α, β).

$$M_{X_i}(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha} \quad \text{for } t < \beta$$

$$M_{\bar{X}}(t) = \left[M_X(t/n)\right]^n \qquad \text{(from Th}^m\text{ 3 in notes)}$$

$$= \left(1 - \frac{t}{n\beta}\right)^{-n\alpha}.$$

This is mgf of Gamma dist^n with $\alpha' = n\alpha$; $\beta' = n\beta$.

$\Rightarrow \bar{X} \sim$ Gamma$(n\alpha, n\beta)$.

(Note, when we have ΣX_i then only α changes & not β).

5 $X \sim \chi^2_{(p)}$.

$\Rightarrow M_X(t) = (1-2t)^{-p/2}$ for $t < 1/2$.

$Y = cX$.

$M_Y(t) = E[e^{tY}]$

$= E[e^{ctY}]$

$= E_Y[e^{ct}]$

$= M_Y(ct)$

$= (1-2ct)^{-p/2}$ ... i

Recall, the mgf of Gamma distⁿ is

$\left(1 - \frac{t}{\beta}\right)^{-\alpha}$ for $t < \beta$. ... ii

On comparing i & ii we get

$\frac{1}{\beta} = 2c$ & $\alpha = p/2$

$\Rightarrow \alpha = p/2$, $\beta = 1/2c$.

$\Rightarrow Y \sim$ Gamma distⁿ $\left(\alpha = p/2 ; \beta = \frac{1}{2c}\right)$.

6 $X \sim U(0,1)$

$f_X(x) = 1$.

$$M_X(t) = \begin{cases} \dfrac{e^t - 1}{t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

(this can also be derived from using $M_X(t) = E(e^{tX})$ or directly using $M_X(t)$ of $U(a,b)$).

(This info is not required here; extra notes)

$Y = -2 \ln X$

Now, mgf of Y will be

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(-2\ln X)}]$$

$$= E[e^{\ln X^{-2t}}]$$

$$= E[X^{-2t}]$$

(as $E[g(X)] = \int_{x \in \mathcal{X}} g(x) f(x)$ if X is contin.)

$$= \int_0^1 x^{-2t} \cdot 1 \, dx$$

$$= \left. \frac{x^{-2t+1}}{-2t+1} \right|_0^1$$

$$= \left(\frac{1}{1-2t}\right)^{+1}$$

$$= (1-2t)^{-1}$$

$$= (1-2t)^{-2/2}$$

This is mgf of χ^2 with 2 degrees of freedom.

$\Rightarrow Y = -2\ln X \sim \chi^2(2)$.