पाठ्यक्रम
Course Title ________ Sheet No. ________

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$$f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}$$

$$L = \frac{1}{\theta^n} e^{-\frac{\Sigma x_i}{\theta}}$$

$$l(\theta) = -\frac{\Sigma x_i}{\theta} - n\log\theta$$

$$\frac{\partial l}{\partial \theta} = \frac{\Sigma x_i}{\theta^2} - \frac{n}{\theta} = 0 \quad \Rightarrow \quad \hat{\theta} = \bar{X}$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{2\Sigma x_i}{\theta^3} + \frac{n}{\theta^2}\Big|_{\hat{\theta}}$$

$$\Rightarrow \frac{-2\Sigma x_i}{\bar{x}^3} + \frac{n}{\bar{x}^2}$$

$$\Rightarrow \frac{-2\Sigma x_i + n\bar{x}}{\bar{x}^3}$$

$$= -\frac{n}{\bar{x}^2} < 0$$

$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$ (unrestricted) (or for (H_1))

at $\theta = \theta_0$ $\quad \hat{\theta}_{MLE} = \theta_0$

$$\Rightarrow \lambda(x) = \cancel{\frac{\prod \frac{1}{\theta_0} e^{-x/\theta_0}}{\frac{1}{\bar{X}} e^{-}}} \quad \frac{\frac{1}{(\theta_0)^n} e^{-\frac{\Sigma x_i}{\theta_0}}}{\frac{1}{(\bar{X})^n} e^{-\frac{\Sigma x_i}{\bar{x}}}}$$

(+) Thus reject H_0 if $\bar{X} \le C_1$ or $\bar{X} \ge C_2$.

$\Rightarrow \lambda(x) = \left(\frac{\bar{X}}{\theta_0}\right)^n e^{-\frac{\sum x_i}{\theta_0} + n}$

$\Rightarrow \lambda(x) \le C$ reject H_0

$\Rightarrow \left(\frac{\bar{X}}{\theta_0}\right)^n e^{-\frac{\sum x_i}{\theta_0} + n} \le C$.

$\Rightarrow \cancel{(\bar{x})^n \; n \log \bar{x} - \frac{\sum x_i}{\theta_0} \le c} \quad (\bar{X})^n e^{-\frac{\sum x_i}{\theta_0}} \le C$.

$\Rightarrow \bar{X} \le C_1$ or $e^{-\frac{\sum x_i}{\theta_0}} \le C$

$\Rightarrow \frac{\sum x_i}{\theta_0} \ge \ln C$

$\Rightarrow \bar{X} \ge C_2$ (*)

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$X_i \sim Exp(\theta)$ $\quad Y_j \sim Exp(\mu)$

$f(x_i) = \frac{1}{\theta} e^{-x_i/\theta}$ $\qquad f(y_j) = \frac{1}{\mu} e^{-y_j/\mu}$.

$1 \le i \le n$. $\qquad 1 \le j \le m$.

$L(\theta, \mu \mid x, y) = \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}} \frac{1}{(\mu)^m} e^{-\frac{\sum y_j}{\mu}}$

$\log L = -\frac{\sum x_i}{\theta} - \frac{\sum y_j}{\mu} - n \log \theta - m \log \mu$.

$\frac{dl}{d\mu} = + \frac{\sum y_i}{\mu^2} - \frac{m}{\mu}$.

$\Rightarrow \hat{\mu} = \bar{y}$.

Similarly $\frac{dl}{d\theta} = + \frac{\sum x_i}{\theta^2} - \frac{n}{\theta} = 0$

$\Rightarrow \hat{\theta} = \bar{X}$.

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$\cancel{\frac{\partial^2 l}{\partial\theta^2}\Big|_{\hat\theta=\theta}}$ $\quad \cancel{l \; \frac{\partial^2 l}{\partial\mu^2}\Big|_{\hat\theta}}$

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{-2\Sigma x_i}{\theta^3} + \frac{n}{\theta^2}\Big|_{\hat\theta} = \frac{-2\Sigma x_i}{\bar{x}^3} + \frac{n}{\bar{x}^2} = \frac{-n}{\bar{x}^2}$$

$$\frac{\partial^2 l}{\partial\mu^2} = \frac{-2\Sigma y_i}{\mu^3} + \frac{m}{\mu^2}\Big|_{\hat\mu} = \frac{-2\Sigma y_i}{\bar{y}^3} + \frac{m}{\bar{y}^2} = \frac{-m}{\bar{y}^2}.$$

$$\frac{\partial^2 l}{\partial\theta\partial\mu} = 0 = \frac{\partial^2 l}{\partial\mu\partial\theta}$$

$$\Rightarrow \begin{bmatrix} \frac{-n}{x^2} & 0 \\ 0 & -m/y^2 \end{bmatrix} = \frac{nm}{x^2y^2}$$ is a neg. definite matrix.

(As clearly eigen values are neg.

& $|A_1| < 0 \quad |A_2| > 0$)

Thus unrestricted MLE will be $\hat\theta = \bar{X}$ & $\hat\mu = \bar{Y}$

for $\theta = \mu$ $\cancel{\Rightarrow \hat{\hat\theta} = \hat\theta = \hat{\hat\mu}}$

Now, for restricted MLE. $\cancel{As}$, $\cancel{\theta = \mu}$.

$$\Rightarrow L(\theta) = \cancel{\frac{1}{\theta^{m+n}}} \frac{1}{(\theta)^{m+n}} e^{-\frac{\Sigma x_i + \Sigma y_j}{\theta}}$$

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[redacted]

$$l(\theta) = -(m+n)\ln\theta - \left(\frac{\sum x_i + \sum y_j}{\theta}\right)$$

$$\frac{dl}{d\theta} = \frac{-(m+n)}{\theta} + \frac{\sum x_i + \sum y_j}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \frac{\sum X_i + \sum Y_j}{m+n}$$

$$\left.\frac{d^2 l}{d\theta^2}\right|_{\hat{\theta}} < 0 \quad \text{(verify)}$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{\sum X_i + \sum Y_j}{m+n}$$

$$\Rightarrow \lambda(x) = \frac{\left(\frac{m+n}{\sum X_i + \sum Y_j}\right)^{m+n} \cancel{e^{-\left(\frac{\sum X_i + \sum Y_j}{\sum X_i + \sum Y_j}\right) m+n}}}{\cancel{e^{-(n+m)}} \left(\frac{1}{\bar{X}}\right)^n \left(\frac{1}{\bar{Y}}\right)^m}$$

$$= \frac{(m+n)^{m+n}}{\left(\sum X_i + \sum Y_j\right)^{m+n}} (\bar{X})^n (\bar{Y})^m$$

$$\Rightarrow \lambda(x) \le C \quad \text{reject } H_0$$

$$\Rightarrow \frac{(m+n)^{m+n}}{\left(\sum X_i + \sum Y_j\right)^{m+n}} \frac{\left(\sum X_i\right)^n}{n^n} \frac{\left(\sum Y_j\right)^m}{m^m} \le C$$

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$$\frac{(\sum X_i)^{n/m}\ (\sum y_j)}{(\sum X_i + \sum y_j)^{1+n/m}} \leq C.$$

$$\frac{(\sum X_i)^{n/m}}{(\sum X_i + \sum y_j)^{n+m}} \leq C.$$

$$\frac{\sum X_i}{\sum X_i + \sum Y_j} \leq C$$

Under H_0. $X_i \sim Exp(\theta)$; $Y_i \sim Exp(\mu)$

$\Rightarrow \sum X_i \sim Gamma(n, \theta)$

& $\sum Y_j \sim Gamma(m, \theta)$ as $H_0: \mu = \theta$

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_j} \sim Beta$$

(by transformation)

Hind DO $U = \sum X_i$; $V = \sum X_i + \sum Y_j$

$\sim Exp(n, \theta)$; $\sim Exp(m+n, 2\theta)$

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$$L = \frac{1}{\sigma^n} e^{n\theta/\sigma} \prod_{i=1}^{n} e^{-x_i/\sigma} I_{(\theta,\infty)}(x_i)$$

$$= \frac{1}{\sigma^n} e^{n\theta/\sigma} e^{-\sum x_i/\sigma} \prod_{i=1}^{n} I_{(\theta,\infty)}(x_i)$$

$\hat{\theta} = X_{(1)}$

$$L = \frac{1}{\alpha^n} e^{\frac{n x_{(1)}}{\alpha}} e^{-\sum x_i/\alpha}$$

$$\log L = -n \log \alpha + \frac{n\hat{\theta}}{\alpha} - \frac{\sum x_i}{\alpha}$$

$$= -n \log \alpha + \frac{n X_{(1)}}{\alpha} - \frac{\sum X_i}{\alpha}$$

$$\frac{\partial l}{\partial \alpha} = \frac{-n}{\alpha} - \frac{n X_{(1)}}{\alpha^2} + \frac{\sum X_i}{\alpha^2} = \frac{n(\bar{x} - x_{(1)})}{\alpha^2}$$

$$\Rightarrow \frac{-n}{\alpha} + \frac{\sum X_i - n X_{(1)}}{\alpha^2} = 0$$

$$\Rightarrow \hat{\alpha} = (\bar{X} - X_{(1)})$$

$$\left.\frac{\partial^2 l}{\partial \alpha^2}\right|_{\hat{\alpha}} < 0 \quad \text{at } \hat{\alpha}.$$

$\Rightarrow$ Unrestricted MLE, $\hat{\alpha} = \bar{X} - X_{(1)}$

$\hat{\theta} = X_{(1)}$

Under H_0.

$H_0: \theta \leq 0$.

case i $\quad \theta \leq X_{(1)} < 0 \quad \Rightarrow \hat{\hat{\theta}} = X_{(1)}$

case ii $\quad \theta < 0 < X_{(1)} \quad \Rightarrow \hat{\hat{\theta}} = 0$.

In case i $\quad \alpha(x) = \alpha$.

case ii $\quad \hat{\hat{\theta}} = 0 \quad e^{-\sum x_i/\alpha}$.

$$\Rightarrow L = \frac{1}{\alpha^n} e$$

$$l(\alpha) = -n \log \alpha - \frac{\sum x_i}{\alpha} \Rightarrow -\frac{n}{\alpha} + \frac{\sum x_i}{\alpha^2} = 0$$

$$\hat{\hat{\alpha}} = \bar{X}$$

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$$\Rightarrow \lambda(x) = \frac{\frac{1}{\bar{X}^n} e^{-\Sigma x_i / \bar{x}}}{\left(\frac{1}{\bar{x} - x_{(1)}}\right)^n e^{-n(\bar{x}-x_{(1)})/(\bar{x} - x_{(1)})}}$$

[redacted]

$$= \left(\frac{\bar{x} - x_{(1)}}{\bar{x}}\right)^n$$

$$\Rightarrow \lambda(x) \le C$$

$$\left(1 - \frac{x_{(1)}}{\bar{x}}\right)^n \le C$$

$$\Rightarrow \frac{x_{(1)}}{\bar{x}} \ge C$$ Reject H_0.

[redacted]