Q1: $T = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ $\theta \rightarrow$ unknown

where $X_1 \cdots X_n \sim N(\underset{\text{mean}}{\alpha\theta}, \underset{\text{var}}{\theta^2})$

To prove: 1) T is sufficient for θ

2) T is not a complete statistic.

Soln: To prove sufficiency you can either use factorization thm: $\boxed{f_X(x|\theta) = g(t|\theta)\,h(x)}$

or

if $f_{X|T}(x|t)$ is free of θ

$\Rightarrow \dfrac{f_X(x|\theta)}{f_T(t|\theta)}$ is free of θ

to prove for sufficiency, we use fact. thm.

$$f(X_1 \cdots X_n|\theta) = \left(\frac{1}{(2\pi\theta^2)}\right)^{n/2} e^{-\sum_{i=1}^{n} \frac{(X_i-\alpha\theta)^2}{2\theta^2}}$$

$$= \left(\frac{1}{2\pi\theta^2}\right)^{n/2} e^{-\frac{\sum X_i^2}{2\theta^2} + \frac{\alpha}{\theta^2}\sum X_i - \frac{n\alpha^2\theta^2}{2\theta^2}}$$

$$= \underbrace{\left(\frac{1}{2\pi\theta^2}\right)^{n/2} e^{-\frac{\sum x_i^2}{2\theta^2} - \frac{\alpha \sum x_i}{\theta} + \frac{n\alpha^2}{2}}}_{g(t|\theta)} \qquad \underline{h(x) = 1}$$

Sufficiency proved.

$\Rightarrow$ Not complete statistic.

If $\{f_T(t|\theta);\ \theta \in \Theta\}$ is a family of pdf/pmf for a statistic $T = T(X)$ then it is said complete if

$$E_\theta[g(T)] = 0 \ \ \forall \theta \in \Theta \Rightarrow P_\theta(g(T) = 0) = 0 \ \ \forall \theta \in \Theta$$

i.e $g(T) = 0$ almost $\forall \theta \in \Theta$

Q16 Proving noncompleteness for $T = (\sum X_i , \sum X_i^2)$

$X_i \sim N(\alpha\theta, \theta^2)$ $\quad f_\theta(x_i) = \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{1}{2}\frac{(x-\alpha\theta)^2}{\theta^2}}$

$T = (T_1, T_2)$, now

$E(T_1] = E(\sum X_i) = n\alpha\theta$

$E[T_2] = E[\sum X_i^2] = \sum E(X_i^2] = \sum_{i=1}^{n} V(X_i) - ((E(X_i))^2$

$= \sum_{i=1}^{n} \theta^2 + (\alpha\theta)^2 = n(\theta^2 + \alpha^2\theta^2)$

$= n\theta^2(1+\alpha^2)$

$E(T_1^2) = E[(\sum X_i)^2] = E[(n\bar{X})^2]$

$E(\bar{X}) = \frac{n\alpha\theta}{n} = \alpha\theta \quad V(\bar{X}) = \frac{1}{n^2}\theta^2 n = \frac{\theta^2}{n}$

$E(\bar{X}^2] = V(\bar{X}) - (E(\bar{X}))^2 = \theta^2/n + \alpha^2\theta^2$

so, $E(T_1^2] = n^2\left(\frac{\theta^2}{n} + \alpha^2\theta^2\right) = \theta^2(n + n^2\alpha^2)$

we calculated above $E[T_1)$, $E[T_2]$, $E[T_1^2]$ because our goal is to create $g(T) = g(T_1, T_2)$ s.t $E[g(T)] = 0 \ \forall\theta$

$E\left[\frac{T_1^2}{n+n^2\alpha^2} - \frac{T_2}{n(1+\alpha^2)}\right] = \theta^2 - \theta^2 = 0$ ☆

$\therefore$ we take $g(T) = \frac{T_1^2}{n+n^2\alpha^2} - \frac{T_2}{n(1+\alpha^2)}$ (i.e $g(T) \neq 0 \ \forall t$ but $E(g(T)) = 0$)

$$g(T) = \frac{\left(\sum X_i\right)^2}{n(1+n\alpha^2)} - \frac{\sum X_i^2}{n(1+\alpha^2)}$$

g(T) function discussed in class was wrong. It would've worked in case $X_i \sim N(\theta, \theta^2)$

$$g(T) = 2\left(\sum_{i=1}^{n} X_i\right)^2 - (n+1)\sum_{i=1}^{n} X_i^2$$

holds for this

$$E(g(T)) = E\left(2T_1^2 - (n+1)T_2\right)$$

$$= 2\theta^2(n+n^2\alpha^2) - (n+1)n\theta^2(1+\alpha^2)$$

$$= 2\theta^2 n + 2\theta^2 n^2\alpha^2 - n\theta^2(n+n\alpha^2+1+\alpha^2)$$

$$= 2\theta^2 n + 2\theta^2 n^2\alpha^2 - n^2\theta^2 - n^2\theta^2\alpha^2 - n\theta^2 - n\theta^2\alpha^2$$

put $\alpha = 1$ ($\because$ In tut 1 Q1 $X_i \sim N(\alpha\theta, \theta^2)$)

$$= \cancel{2\theta^2 n} + \cancel{2\theta^2 n^2} - \cancel{n^2\theta^2} - \cancel{n^2\theta^2} - \cancel{n\theta^2} - \cancel{n\theta^2}$$

$$= 0$$

Ques2: $X_1 - - X_n$ iid poisson (λ)

$T(X) = \sum_{i=1}^{n} X_i$ is complete ?

$\Rightarrow T(X) = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$

Using defn of completeness,

$$E_\lambda(g(t)) = 0$$

$$\sum_{t=0}^{\infty} g(t) \times \frac{e^{-n\lambda}(n\lambda)^t}{t!} = 0$$

$$g(0) + \frac{g(1)}{1!}(n\lambda) + \cdots = 0$$

[as $n\lambda > 0$] $\Rightarrow g(t) = 0 \quad \forall t$

for the above summation to be zero

$g(t) = 0$ $\therefore$ Proved

Ques 36 $X_1 \cdots X_n$ iid Bernoulli(p)
($0<p<1$)

$T(X) = \sum_{i=1}^{n} X_i$ is a complete statistic
(p is unknown)

so, now,

$E_p[g(T)] = 0 \quad \forall p$

$\sum_{t=0}^{n} g(t) \times {}^nC_t \, p^t (1-p)^{n-t} = 0 \; \forall p$

nonzero

It is only possible when $g(t) = 0 \; \forall t$

$\therefore$ Completeness proved.

Ques 4.6

	$P(X=0)$	$P(X=1)$	$P(X=2)$	
D_1	p	$3p$	$(1-4p)$	$0<p<1/4$
D_2	p	p^2	$1-p-p^2$	$0<p<1/2$

to determine completeness,

① for D_1, $E_p[g(T)]=0$ Consider

(as per D_1)

$$g(0)p + g(1)3p + g(2)(1-4p) = 0$$

$$\underbrace{p[g(0)+3g(1)-4g(2)]} + \underbrace{g(2)} = 0$$

we need both of them to be 0.

i.e $g(2)=0$ and $\underbrace{g(0)+3g(1)=0}$

Here for $g(0), g(1) \neq 0$ this can still hold true

i.e possibility of nontriv. soln exist here

So, $\therefore$ $E_p[g(T)]=0$ holds even when $g(t)\neq 0$, not complete.

② for D_2

$$g(0)p + g(1)p^2 + g(2)(1-p-p^2) = 0$$

$$g(2) + [g(0) - g(2)]p + [g(1) - g(2)]p^2 = 0$$

$\boxed{g(2) = 0}$ & $g(0) - g(2) = 0$

$g(1) - g(2) = 0$

$\Rightarrow g(0) = g(1) = 0$

∴ Complete.

Ques 8 $X \sim f_\theta(x) = \theta x^{\theta-1} e^{-x^\theta}$

$(x>0)$

a) Pdf of $Y = \theta \log_e(X)$ $\boxed{X = e^{Y/\theta}}$

if $X>0$

Proof given in Clarifications PDF.

<u>Transformation formula:</u>

→ Cont RV 'X' with $f_X(x)$ pdf.

we've $Y = g(X)$ (some new RV)

then,
$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

so,
$$\frac{dx}{dy} = \frac{d}{dy}\left(e^{y/\theta}\right) = \frac{1}{\theta}e^{y/\theta}$$

$$f_Y(y) = \theta x^{\theta-1} e^{-x^\theta} \times \frac{1}{\theta} e^{y/\theta}$$

(substitute $x = e^{y/\theta}$)

$$= \theta \left(e^{y/\theta}\right)^{\theta-1} e^{-\left(e^{y/\theta}\right)^\theta} \frac{1}{\theta} e^{y/\theta}$$

$$= e^{y/\theta} \cdot e^{\frac{(\theta-1)y}{\theta}} e^{-e^y} = e^y e^{-e^y}$$

$$= \boxed{e^{y-e^y}} \quad \forall\, y \in \mathbb{R}$$

$\therefore$ $f_Y(y)$ does not depend on θ

$y = \theta \log X$ is not a statistic.!!

② $U = \dfrac{\log X_1}{\log X_2}$, X_1, X_2 random sample from $f_\theta(x)$

show U is ancillary statistic.

$$U = \frac{\theta \log X_1}{\theta \log X_2} = \frac{Y_1}{Y_2}$$, $\therefore$ U will be independent of θ

Also the distbn is independent of θ

$\therefore$ U is ancillary.

Ques 6: $X_1 \ldots \ldots X_n$ random sample drawn from $f_\theta(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} x^{\alpha-1}$

Gamma (α, β)

$\begin{pmatrix} 0 < x < \infty \\ \alpha > 0 \quad \beta > 0 \end{pmatrix}$

Let $y = cX$ for some $c > 0$

now

$\left(x = \frac{y}{c}\right)$

$$f_y(y) = f_x(x) \frac{\partial x}{\partial y}$$

$\left(\frac{\partial x}{\partial y} = \frac{1}{c}\right)$

$$\boxed{= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot e^{-y/c\beta} \cdot \left(\frac{y}{c}\right)^{\alpha-1} \cdot \frac{1}{c}}$$

$\frac{1}{c} f_x\left(\frac{y}{c}\right)$

$\therefore$ Scale family.

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-y/\beta c} \frac{y^{\alpha-1}}{c^\alpha}$$

$$= \frac{1}{\Gamma(\alpha)(\beta c)^\alpha} e^{-y/\beta c} y^{\alpha-1}$$

$\Rightarrow y \sim$ Gamma $(\alpha, \beta c)$

Ques7.

$X \sim f_\theta(x) = \dfrac{1}{\pi(1+x^2)}$ $\quad x \in \mathbb{R}$

$y = cx + d \Rightarrow x = (y-d)/c$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi} \times \frac{1}{\left(1-\frac{(y-d)^2}{c^2}\right)} \times \frac{1}{c}$$

$$= \frac{1}{c\pi} \times \frac{c^2}{(c^2+(y-d)^2)} = \frac{1}{c} f\left(\frac{y-d}{c}\right)$$

$\therefore$ Location scale family.

Ques8.

$X \sim f_{\mu,\beta}(x) = \dfrac{1}{\beta} \dfrac{e^{-\frac{(x-\mu)}{\beta}}}{\left(1+e^{-\frac{(x-\mu)}{\beta}}\right)^2}$ $\quad (x \in \mathbb{R})$

$y = cx + d \Rightarrow \boxed{x = \dfrac{y-d}{c}}$

$$f_Y(y) = f_X(x) \frac{\partial x}{\partial y}$$

$$= \frac{1}{\beta} e^{-\left(\frac{y-d-c\mu}{c\beta}\right)} \times \frac{1}{\left(1- e^{-\frac{(y-d-c\mu)}{c\beta}}\right)^2} \frac{1}{c}$$

$= \frac{1}{c\beta} f\left(\frac{y-d}{c}\right)$ Location-scale!!

Q9) $X_1 --- X_n$ is Random sample from $N(\theta, \theta^2)$ $(\theta > 0)$

i.e, $X_i \overset{iid}{\sim} N(\theta, \theta^2)$ $(\theta > 0)$

$Y = c\cancel{X}$ $\qquad X = y/c$

$$f(y) = \frac{1}{\sqrt{2\pi\theta^2}} \cdot \left(e^{-\frac{1}{2}\left(\frac{(y/c-\theta)^2}{\theta^2}\right)}\right) \cdot \frac{1}{c}$$

$$= \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{1}{2\theta^2c^2}(y-\theta c)^2} \frac{1}{c}$$

$$Y \sim N(\theta c, (\theta c)^2)$$

$\therefore$ belongs to scale family.

$$f_Y(y) = f_X(x) \frac{1}{c}$$