

Chapter 2 Inverse Trigonometric Functions

EXERCISE 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution:

Let, $\sin^{-1}\left(-\frac{1}{2}\right) = y$

Hence,

$$\begin{aligned}\sin y &= \left(-\frac{1}{2}\right) \\ &= -\sin\left(\frac{\pi}{6}\right) \\ &= \sin\left(-\frac{\pi}{6}\right)\end{aligned}$$

Range of the principal value of $\sin^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, principal value of $\sin^{-1}\left(-\frac{1}{2}\right) = \left(-\frac{\pi}{6}\right)$.

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution:

Let, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$

Hence,

$$\begin{aligned}\cos y &= \left(\frac{\sqrt{3}}{2}\right) \\ &= \cos \frac{\pi}{6}\end{aligned}$$

Range of the principal value of $\cos^{-1}(x)$ is $(0, \pi)$.

Thus, principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{\pi}{6}\right)$

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$.

Solution:

Let, $\operatorname{cosec}^{-1}(2) = y$

Hence,

$$\begin{aligned}\operatorname{cosec} y &= 2 \\ &= \operatorname{cosec}\left(\frac{\pi}{6}\right)\end{aligned}$$

Range of the principal value of $\operatorname{cosec}^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Thus, principal value of $\operatorname{cosec}^{-1}(2) = \left(\frac{\pi}{6}\right)$.

Question 4:

Find the principal value of $\tan^{-1}(-\sqrt{3})$

Solution:

Let, $\tan^{-1}(-\sqrt{3}) = y$

Hence,

$$\begin{aligned}\tan y &= -\sqrt{3} \\ &= -\tan\left(\frac{\pi}{3}\right) \\ &= \tan\left(-\frac{\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\tan^{-1}(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, principal value of $\tan^{-1}(-\sqrt{3}) = \left(-\frac{\pi}{3}\right)$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

Let, $\cos^{-1}\left(-\frac{1}{2}\right) = y$

Hence,

$$\begin{aligned}\cos y &= -\frac{1}{2} \\ &= -\cos\left(\frac{\pi}{3}\right) \\ &= \cos\left(\pi - \frac{\pi}{3}\right) \\ &= \cos\left(\frac{2\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\cos^{-1}(x) = [0, \pi]$

Thus, principal value of $\cos^{-1}\left(-\frac{1}{2}\right) = \left(\frac{2\pi}{3}\right)$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Solution:

Let, $\tan^{-1}(-1) = y$

Hence,

$$\begin{aligned}\tan y &= -1 \\ &= -\tan\left(\frac{\pi}{4}\right) \\ &= \tan\left(-\frac{\pi}{4}\right)\end{aligned}$$

Range of the principal value of $\tan^{-1}(x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, principal value of $\tan^{-1}(-1) = \left(-\frac{\pi}{4}\right)$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

Let, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$

Hence,

$$\begin{aligned}\sec y &= \frac{2}{\sqrt{3}} \\ &= \sec\left(\frac{\pi}{6}\right)\end{aligned}$$

Range of the principal value of $\sec^{-1}(x) = [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Thus, principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{\pi}{6}\right)$.

Question 8:

Find the principal value of $\cot^{-1}(\sqrt{3})$

Solution:

Let, $\cot^{-1}(\sqrt{3}) = y$

Hence,

$$\begin{aligned}\cot y &= \sqrt{3} \\ &= \cot\left(\frac{\pi}{6}\right)\end{aligned}$$

Range of the principal value of $\cot^{-1}(x) = (0, \pi)$

Thus, principal value of $\cot^{-1}(\sqrt{3}) = \left(\frac{\pi}{6}\right)$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution:

Let, $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$

Hence,

$$\begin{aligned}
 \cos y &= -\frac{1}{\sqrt{2}} \\
 &= -\cos\left(\frac{\pi}{4}\right) \\
 &= \cos\left(-\frac{\pi}{4}\right) \\
 &= \cos\left(\pi - \frac{\pi}{4}\right) \\
 &= \cos\left(\frac{3\pi}{4}\right)
 \end{aligned}$$

Range of the principal value of $\cos^{-1}(x) = [0, \pi]$

Thus, principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \left(\frac{3\pi}{4}\right)$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$

Solution:

Let, $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$

Hence,

$$\begin{aligned}
 \operatorname{cosec} y &= -\sqrt{2} \\
 &= -\operatorname{cosec}\left(\frac{\pi}{4}\right) \\
 &= \operatorname{cosec}\left(-\frac{\pi}{4}\right)
 \end{aligned}$$

Range of the principal value of $\operatorname{cosec}^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Thus, principal value of $\operatorname{cosec}^{-1}(-\sqrt{2}) = \left(-\frac{\pi}{4}\right)$.

Question 11:

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Solution:

Let, $\tan^{-1}(1) = x$

Hence,

$$\begin{aligned}\tan x &= 1 \\ &= \tan\left(\frac{\pi}{4}\right)\end{aligned}$$

Therefore,

$$\tan^{-1}(1) = \left(\frac{\pi}{4}\right)$$

Now, let $\cos^{-1}\left(-\frac{1}{2}\right) = y$

Hence,

$$\begin{aligned}\cos y &= -\frac{1}{2} \\ &= -\cos\left(\frac{\pi}{3}\right) \\ &= \cos\left(\pi - \frac{\pi}{3}\right) \\ &= \cos\left(\frac{2\pi}{3}\right)\end{aligned}$$

Therefore,

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Again, let $\sin^{-1}\left(-\frac{1}{2}\right) = z$

Hence,

$$\begin{aligned}\sin z &= -\frac{1}{2} \\ &= -\sin\left(\frac{\pi}{6}\right) \\ &= \sin\left(-\frac{\pi}{6}\right)\end{aligned}$$

Therefore,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Thus,

$$\begin{aligned}
 \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) &= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} \\
 &= \frac{3\pi + 8\pi - 2\pi}{12} \\
 &= \frac{9\pi}{12} \\
 &= \frac{3\pi}{4}
 \end{aligned}$$

Question 12:

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Solution:

Let, $\tan^{-1}(1) = x$

Hence,

$$\begin{aligned}
 \cos x &= \frac{1}{2} \\
 &= \cos\left(\frac{\pi}{3}\right)
 \end{aligned}$$

Therefore,

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Let, $\sin^{-1}\left(\frac{1}{2}\right) = y$

Hence,

$$\begin{aligned}
 \sin y &= \frac{1}{2} \\
 &= \sin\left(\frac{\pi}{6}\right)
 \end{aligned}$$

Therefore,

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Thus

$$\begin{aligned}\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) \\ &= \frac{2\pi}{3}\end{aligned}$$

Question 13:

Find the value of $\sin^{-1} x = y$, then

- (A) $0 \leq y \leq \pi$ (B) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(C) $0 \leq y \leq \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Solution:

It is given that $\sin^{-1} x = y$

Range of the principal value of $\sin^{-1} x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

The answer is B.

Question 14:

Find the value of $\tan^{-1} \sqrt{3} - \sec^{-1}(-2)$ is equal to

- (A) 0 (B) $-\frac{\pi}{3}$
(C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Solution:

Let $\tan^{-1}(\sqrt{3}) = x$

Hence,

$$\begin{aligned}\tan x &= \sqrt{3} \\ &= \tan\left(\frac{\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\tan^{-1} x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Therefore, $\tan^{-1}(\sqrt{3}) = \left(\frac{\pi}{3}\right)$

Let $\sec^{-1}(-2) = y$

Hence,

$$\begin{aligned}\sec y &= (-2) \\ &= -\sec\left(\frac{\pi}{3}\right) \\ &= \sec\left(-\frac{\pi}{3}\right) \\ &= \sec\left(\pi - \frac{\pi}{3}\right) \\ &= \sec\left(\frac{2\pi}{3}\right)\end{aligned}$$

Range of the principal value of $\sec^{-1} x = [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Therefore, $\sec^{-1}(-2) = \frac{2\pi}{3}$

Thus,

$$\begin{aligned}\tan^{-1} \sqrt{3} - \sec^{-1}(-2) &= \frac{\pi}{3} - \frac{2\pi}{3} \\ &= -\frac{\pi}{3}\end{aligned}$$

The answer is B.

EXERCISE 2.2

Question 1:

Prove $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Solution:

Let $x = \sin \theta$

Hence, $\sin^{-1}(x) = \theta$

Now,

$$\begin{aligned} RHS &= \sin^{-1} (3x - 4x^3) \\ &= \sin^{-1} (3\sin \theta - 4\sin^3 \theta) \\ &= \sin^{-1} (\sin 3\theta) \\ &= 3\theta \\ &= 3 \sin^{-1} x \\ &= LHS \end{aligned}$$

Question 2:

Prove $3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$.

Solution:

Let $x = \cos \theta$

Hence, $\cos^{-1}(x) = \theta$

Now,

$$\begin{aligned} RHS &= \cos^{-1} (4x^3 - 3x) \\ &= \cos^{-1} (4\cos^3 \theta - 3\cos \theta) \\ &= \cos^{-1} (\cos 3\theta) \\ &= 3\theta \\ &= 3 \cos^{-1} x \\ &= LHS \end{aligned}$$

Question 3:

Prove $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$.

Solution:

Since we know that $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Now,

$$\begin{aligned}
 LHS &= \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} \\
 &= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \cdot \frac{7}{24}} \\
 &= \tan^{-1} \left(\frac{\frac{48+77}{264}}{\frac{264-14}{264}} \right) \\
 &= \tan^{-1} \left(\frac{125}{250} \right) \\
 &= \tan^{-1} \left(\frac{1}{2} \right) \\
 &= RHS
 \end{aligned}$$

Question 4:

Prove $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$.

Solution:

Since we know that $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$ and $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Now,

$$\begin{aligned}
LHS &= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} \\
&= \tan^{-1} \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7} \\
&= \tan^{-1} \left(\frac{4}{3}\right) + \tan^{-1} \left(\frac{1}{7}\right) \\
&= \tan^{-1} \left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \cdot \frac{1}{7}} \right) \\
&= \tan^{-1} \left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}} \right) \\
&= \tan^{-1} \left(\frac{31}{17} \right) \\
&= RHS
\end{aligned}$$

Question 5:

Write the function in the simplest form: $\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}, x \neq 0$

Solution:

Let $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

Hence,

$$\begin{aligned}
\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} &= \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right) \\
&= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) \\
&= \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\
&= \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\
&= \tan^{-1} \left(\tan \frac{\theta}{2} \right) \\
&= \frac{\theta}{2} \\
&= \frac{1}{2} \tan^{-1} x
\end{aligned}$$

Question 6:

Write the function in the simplest form: $\tan^{-1} \frac{1}{\sqrt{x^2-1}}, |x| > 1$

Solution:

Let $x = \operatorname{cosec} \theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

Hence,

$$\begin{aligned}
\tan^{-1} \frac{1}{\sqrt{x^2-1}} &= \tan^{-1} \frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}} \\
&= \tan^{-1} \left(\frac{1}{\cot \theta} \right) \\
&= \tan^{-1} (\tan \theta) \\
&= \theta \\
&= \operatorname{cosec}^{-1} x \\
&= \frac{\pi}{2} - \sec^{-1} x
\end{aligned}$$

Question 7:

Write the function in the simplest form: $\tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), 0 < x < \pi$

Solution:

Since, $1 - \cos x = 2 \sin^2 \frac{x}{2}$ and $1 + \cos x = 2 \cos^2 \frac{x}{2}$

Hence,

$$\begin{aligned}\tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right) &= \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} \right) \\ &= \tan^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) \\ &= \tan^{-1} \left(\tan \frac{x}{2} \right) \\ &= \frac{x}{2}\end{aligned}$$

Question 8:

Write the function in the simplest form: $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \pi$

Solution:

$$\begin{aligned}\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right) &= \tan^{-1} \left(\frac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}} \right) \\ &= \tan^{-1} \left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} \right) \\ &= \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right) \\ &= \tan^{-1}(1) - \tan^{-1}(\tan x) \\ &= \frac{\pi}{4} - x\end{aligned}$$

Question 9:

Write the function in the simplest form: $\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, |x| < a$

Solution:

Let $x = a \sin \theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{a} \right)$

Hence,

$$\begin{aligned} \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} &= \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right) \\ &= \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right) \\ &= \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right) \\ &= \tan^{-1} (\tan \theta) \\ &= \theta \\ &= \sin^{-1} \frac{x}{a} \end{aligned}$$

Question 10:

Write the function in the simplest form: $\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), a > 0; \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$

Solution:

Let $x = a \tan \theta \Rightarrow \theta = \tan^{-1} \left(\frac{x}{a} \right)$

Hence,

$$\begin{aligned} \tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right) &= \tan^{-1} \left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta} \right) \\ &= \tan^{-1} \left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta} \right) \\ &= \tan^{-1} (\tan 3\theta) \\ &= 3\theta \\ &= 3 \tan^{-1} \frac{x}{a} \end{aligned}$$

Question 11:

Write the function in the simplest form: $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$

Solution:

Let $\sin^{-1} \frac{1}{2} = x$

Hence,

$$\begin{aligned}\sin x &= \frac{1}{2} \\ &= \sin \left(\frac{\pi}{6} \right)\end{aligned}$$

$$\begin{aligned}x &= \frac{\pi}{6} \\ \sin^{-1} \left(\frac{1}{2} \right) &= \frac{\pi}{6}\end{aligned}$$

Therefore,

$$\begin{aligned}\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right] &= \tan^{-1} \left[2 \cos \left(2 \times \frac{\pi}{6} \right) \right] \\ &= \tan^{-1} \left[2 \cos \frac{\pi}{3} \right] \\ &= \tan^{-1} \left[2 \times \frac{1}{2} \right] \\ &= \tan^{-1} [1] \\ &= \frac{\pi}{4}\end{aligned}$$

Question 12:

Find the value of $\cot \left(\tan^{-1} a + \cot^{-1} a \right)$

Solution:

Since $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

Hence,

$$\begin{aligned}\cot \left(\tan^{-1} a + \cot^{-1} a \right) &= \cot \left(\frac{\pi}{2} \right) \\ &= 0\end{aligned}$$

Question 13:

Find the value of $\tan \frac{1}{2} \left(\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right)$, $|x| < 1, y > 0$ and $xy < 1$.

Solution:

Let $x = \tan \theta \Rightarrow \theta = \tan^{-1} x$

Hence,

$$\begin{aligned} \sin^{-1} \frac{2x}{1+x^2} &= \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \\ &= \sin^{-1} (\sin 2\theta) \\ &= 2\theta \\ &= 2 \tan^{-1} x \end{aligned}$$

Now, let $y = \tan \phi \Rightarrow \phi = \tan^{-1} y$

Hence,

$$\begin{aligned} \cos^{-1} \frac{1-y^2}{1+y^2} &= \cos^{-1} \left(\frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \right) \\ &= \cos^{-1} (\cos 2\phi) \\ &= 2\phi \\ &= 2 \tan^{-1} y \end{aligned}$$

Therefore,

$$\begin{aligned} \tan \frac{1}{2} \left(\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right) &= \tan \frac{1}{2} (2 \tan^{-1} x + 2 \tan^{-1} y) \\ &= \tan (\tan^{-1} x + \tan^{-1} y) \\ &= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \\ &= \left(\frac{x+y}{1-xy} \right) \end{aligned}$$

Question 14:

If $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1$, find the value of x .

Solution:

It is given that $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1$

Since we know that $\sin(x+y) = \sin x \cos y + \cos x \sin y$
Therefore,

$$\begin{aligned}\sin\left(\sin^{-1}\frac{1}{5}\right)\cos(\cos^{-1}x) + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) &= 1 \\ \left(\frac{1}{5}\right)\times(x) + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) &= 1 \\ \frac{x}{5} + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin(\cos^{-1}x) &= 1 \quad \dots(1)\end{aligned}$$

Now, let $\sin^{-1}\frac{1}{5} = y \Rightarrow \sin y = \frac{1}{5}$

Then,

$$\begin{aligned}\cos y &= \sqrt{1 - \left(\frac{1}{5}\right)^2} \\ &= \frac{2\sqrt{6}}{5} \\ y &= \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)\end{aligned}$$

Therefore,

$$\sin^{-1}\frac{1}{5} = \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \quad \dots(2)$$

Now, let $\cos^{-1}x = z \Rightarrow \cos z = x$

Then,

$$\begin{aligned}\sin z &= \sqrt{1 - x^2} \\ z &= \sin^{-1}\sqrt{1 - x^2}\end{aligned}$$

Therefore,

$$\cos^{-1}x = \sin^{-1}\sqrt{1 - x^2} \quad \dots(3)$$

From (1), (2) and (3), we have

$$\begin{aligned}
&\Rightarrow \frac{x}{5} + \cos\left(\cos^{-1} \frac{2\sqrt{6}}{5}\right) \sin\left(\sin^{-1} \sqrt{1-x^2}\right) = 1 \\
&\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5} \sqrt{1-x^2} = 1 \\
&\Rightarrow x + 2\sqrt{6} \sqrt{1-x^2} = 5 \\
&\Rightarrow 5 - x = 2\sqrt{6} \sqrt{1-x^2}
\end{aligned}$$

On squaring both the sides

$$\begin{aligned}
25 + x^2 - 10x &= 24 - 24x^2 \\
25x^2 - 10x + 1 &= 0 \\
(5x - 1)^2 &= 0 \\
(5x - 1) &= 0 \\
x &= \frac{1}{5}
\end{aligned}$$

Question 15:

If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x-1} = \frac{\pi}{4}$, find the value of x .

Solution:

It is given that $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x-1} = \frac{\pi}{4}$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$$

Since

Therefore,

$$\begin{aligned}
&\Rightarrow \tan^{-1} \left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2} \right) \left(\frac{x+1}{x+2} \right)} \right) = \frac{\pi}{4} \\
&\Rightarrow \tan^{-1} \left[\frac{(x-1)(x+2) + (x+1)(x-2)}{(x+2)(x-2) - (x-1)(x+1)} \right] = \frac{\pi}{4} \\
&\Rightarrow \tan^{-1} \left[\frac{x^2 + x - 2 + x^2 - x - 2}{x^2 - 4 - x^2 + 1} \right] = \frac{\pi}{4} \\
&\Rightarrow \tan^{-1} \left[\frac{2x^2 - 4}{-3} \right] = \frac{\pi}{4} \\
&\Rightarrow \tan \left[\tan^{-1} \frac{4 - 2x^2}{3} \right] = \tan \frac{\pi}{4} \\
&\Rightarrow \frac{4 - 2x^2}{3} = 1 \\
&\Rightarrow 4 - 2x^2 = 3 \\
&\Rightarrow 2x^2 = 1 \\
&\Rightarrow x^2 = \frac{1}{2} \\
&\Rightarrow x = \pm \frac{1}{\sqrt{2}}
\end{aligned}$$

Question 16:

Find the value of $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$.

Solution:

Since, $\sin \frac{2\pi}{3} = \sin \left(\pi - \frac{\pi}{3} \right)$

Therefore,

$$\begin{aligned}
\sin^{-1} \left(\sin \frac{2\pi}{3} \right) &= \sin^{-1} \left[\sin \left(\pi - \frac{\pi}{3} \right) \right] \\
&= \sin^{-1} \left(\sin \frac{\pi}{3} \right) \\
&= \frac{\pi}{3}
\end{aligned}$$

Question 17:

Find the value of $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$.

Solution:

Since, $\tan \tan^{-1} \left(-\frac{3}{4} \right)$

Therefore,

$$\begin{aligned} \tan^{-1} \left(\tan \frac{3\pi}{4} \right) &= \tan^{-1} \left[-\tan \left(-\frac{3\pi}{4} \right) \right] \\ &= \tan^{-1} \left[-\tan \left(\pi - \frac{\pi}{4} \right) \right] \\ &= \tan^{-1} \left[-\tan \left(\frac{\pi}{4} \right) \right] \\ &= \tan^{-1} \left[\tan \left(-\frac{\pi}{4} \right) \right] \\ &= \left(-\frac{\pi}{4} \right) \end{aligned}$$

Question 18:

Find the value of $\tan \left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2} \right)$.

Solution:

Let $\sin^{-1} \frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Then,

$$\Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5}$$

$$\Rightarrow \sec x = \frac{5}{4}$$

Therefore,

$$\begin{aligned} \tan x &= \sqrt{\sec^2 x - 1} \\ &= \sqrt{\frac{25}{16} - 1} \\ &= \frac{3}{4} \end{aligned}$$

$$x = \tan^{-1} \frac{3}{4}$$

$$\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

Now,

$$\cot^{-1} \frac{3}{2} = \tan^{-1} \frac{2}{3} \quad \dots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned} \tan\left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{2}{3}\right) &= \tan\left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3}\right) \\ &= \tan\left[\tan^{-1} \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right] \\ &= \tan\left(\tan^{-1} \frac{17}{6}\right) \\ &= \frac{17}{6} \end{aligned}$$

Question 19:

$\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ is equal to

- (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Solution:

$$\begin{aligned} \cos^{-1}\left(\cos \frac{7\pi}{6}\right) &= \cos^{-1}\left(\cos \frac{-7\pi}{6}\right) \\ &= \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \\ &= \cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right] \\ &= \frac{5\pi}{6} \end{aligned}$$

Thus, the correct option is B.

Question 20:

$\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

- (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Solution:

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) = x$$

Hence,

$$\begin{aligned}\sin x &= -\frac{1}{2} \\ &= -\sin \frac{\pi}{6} \\ &= \sin\left(-\frac{\pi}{6}\right) \\ x &= -\frac{\pi}{6}\end{aligned}$$

Since, Range of principal value of $\sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Therefore,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Then,

$$\begin{aligned}\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right) &= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) \\ &= \sin\left(\frac{\pi}{2}\right) \\ &= 1\end{aligned}$$

Thus, the correct option is D.

Question 21:

Find the values of $\tan^{-1} \sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

- (A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Solution:

$$\text{Let } \tan^{-1} \sqrt{3} = x$$

Hence,

$$\tan x = \sqrt{3} = \tan \frac{\pi}{3}, \text{ where } \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Therefore, $\tan^{-1} \sqrt{3} = \frac{\pi}{3}$

Now, let $\cot^{-1}(-\sqrt{3}) = y$

Hence,

$$\begin{aligned} \cot y &= (-\sqrt{3}) \\ &= -\cot\left(\frac{\pi}{6}\right) \\ &= \cot\left(\pi - \frac{\pi}{6}\right) \\ &= \cot\left(\frac{5\pi}{6}\right) \end{aligned}$$

Since, Range of principal value of $\cot^{-1} x = (0, \pi)$

Therefore,

$$\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$$

Then,

$$\begin{aligned} \tan^{-1} \sqrt{3} - \cot^{-1}(-\sqrt{3}) &= \frac{\pi}{3} - \frac{5\pi}{6} \\ &= -\frac{\pi}{2} \end{aligned}$$

Thus, the correct option is B.

MISCELLANEOUS EXERCISE

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution:

$$\begin{aligned}\cos^{-1}\left(\cos\frac{13\pi}{6}\right) &= \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] \\ &= \cos^{-1}\left[\cos\frac{\pi}{6}\right] \\ &= \frac{\pi}{6}\end{aligned}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

Solution:

$$\begin{aligned}\tan^{-1}\left(\tan\frac{7\pi}{6}\right) &= \tan^{-1}\left[\tan\left(2\pi - \frac{5\pi}{6}\right)\right] \\ &= \tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right] \\ &= \tan^{-1}\left[\tan\left(\pi - \frac{5\pi}{6}\right)\right] \\ &= \tan^{-1}\left[\tan\frac{\pi}{6}\right] \\ &= \frac{\pi}{6}\end{aligned}$$

Question 3:

Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

Solution:

Let $\sin^{-1}\frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Then,

$$\cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Therefore,

$$\tan x = \frac{3}{4}$$

$$x = \tan^{-1} \frac{3}{4}$$

$$\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

Thus,

$$\begin{aligned} LHS &= 2 \sin^{-1} \frac{3}{5} \\ &= 2 \tan^{-1} \frac{3}{4} \quad [from (1)] \\ &= \tan^{-1} \left(\frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} \right) \\ &= \tan^{-1} \left(\frac{24}{7} \right) \\ &= RHS \end{aligned}$$

Question 4:

Prove that $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$.

Solution:

$$\text{Let } \sin^{-1} \frac{8}{17} = x \Rightarrow \sin x = \frac{8}{17}$$

Then,

$$\cos x = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

Therefore,

$$\tan x = \frac{8}{15}$$

$$x = \tan^{-1} \frac{8}{15}$$

$$\sin^{-1} \frac{8}{17} = \tan^{-1} \frac{8}{15} \quad \dots(1)$$

Now, let $\sin^{-1} \frac{3}{5} = y \Rightarrow \sin y = \frac{3}{5}$

Then,

$$\cos y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Therefore,

$$\tan y = \frac{3}{4}$$

$$y = \tan^{-1} \frac{3}{4}$$

$$\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned} LHS &= \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{8}{15} + \tan^{-1} \frac{3}{4} \\ &= \tan^{-1} \left[\frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \cdot \frac{3}{4}} \right] \\ &= \tan^{-1} \left[\frac{\frac{32 + 45}{60}}{\frac{60 - 24}{60}} \right] \\ &= \tan^{-1} \frac{77}{36} \\ &= RHS \end{aligned}$$

Question 5:

Prove that $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$.

Solution:

Let $\cos^{-1} \frac{4}{5} = x \Rightarrow \cos x = \frac{4}{5}$

Then,

$$\sin x = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

Therefore,

$$\tan x = \frac{3}{4}$$

$$x = \tan^{-1} \frac{3}{4}$$

$$\cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

Now, let $\cos^{-1} \frac{12}{13} = y \Rightarrow \cos y = \frac{12}{13}$

Then,

$$\sin y = \frac{5}{13}$$

Therefore,

$$\tan y = \frac{5}{12}$$

$$y = \tan^{-1} \frac{5}{12}$$

$$\cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned} \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} &= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12} \\ &= \tan^{-1} \left[\frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} \right] \\ &= \tan^{-1} \left[\frac{56}{33} \right] \quad \dots(3) \end{aligned}$$

Now, let $\cos^{-1} \frac{33}{65} = z \Rightarrow \cos z = \frac{33}{65}$

Then,

$$\sin z = \sqrt{1 - \left(\frac{33}{65}\right)^2} = \frac{56}{65}$$

Therefore,

$$\begin{aligned}\tan z &= \frac{33}{56} \\ z &= \tan^{-1} \frac{56}{33} \\ \cos^{-1} \frac{33}{65} &= \tan^{-1} \frac{56}{33} \quad \dots(4)\end{aligned}$$

Thus, by using (3) and (4)

$$\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$$

Hence proved.

Question 6:

Prove that $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$.

Solution:

Let $\cos^{-1} \frac{12}{13} = y \Rightarrow \cos y = \frac{12}{13}$

Then,

$$\sin y = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$$

Therefore,

$$\begin{aligned}\tan y &= \frac{5}{12} \\ y &= \tan^{-1} \frac{5}{12} \\ \cos^{-1} \frac{12}{13} &= \tan^{-1} \frac{5}{12} \quad \dots(1)\end{aligned}$$

Now, let $\sin^{-1} \frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

Then,

$$\cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Therefore,

$$\tan x = \frac{3}{4}$$

$$x = \tan^{-1} \frac{3}{4}$$

$$\sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(2)$$

Now, let $\sin^{-1} \frac{56}{65} = z \Rightarrow \sin z = \frac{56}{65}$
Then,

$$\cos z = \sqrt{1 - \left(\frac{56}{65}\right)^2} = \frac{33}{65}$$

Therefore,

$$\tan z = \frac{56}{33}$$

$$z = \tan^{-1} \frac{56}{33}$$

$$\sin^{-1} \frac{56}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$$

Thus, by using (1) and (2)

$$\begin{aligned} LHS &= \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{3}{4} \\ &= \tan^{-1} \left[\frac{\frac{5}{12} + \frac{3}{4}}{1 - \frac{5}{12} \cdot \frac{3}{4}} \right] \\ &= \tan^{-1} \left[\frac{\frac{20+36}{48}}{\frac{48-15}{48}} \right] \\ &= \tan^{-1} \left(\frac{56}{33} \right) \\ &= \sin^{-1} \frac{56}{65} \quad [Using(3)] \\ &= RHS \end{aligned}$$

Question 7:

Prove that $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$.

Solution:

Let $\sin^{-1} \frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13}$

Then,

$$\cos x = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}$$

Therefore,

$$\tan x = \frac{5}{12}$$

$$x = \tan^{-1} \frac{5}{12}$$

$$\sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12} \quad \dots(1)$$

Now, let $\cos^{-1} \frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$

Then,

$$\sin y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

Therefore,

$$\tan y = \frac{4}{3}$$

$$y = \tan^{-1} \frac{4}{3}$$

$$\cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3} \quad \dots(2)$$

Thus, by using (1) and (2)

$$\begin{aligned}
 RHS &= \sin^{-1} \frac{5}{12} + \cos^{-1} \frac{3}{5} \\
 &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} \\
 &= \tan^{-1} \left(\frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \cdot \frac{4}{3}} \right) \\
 &= \tan^{-1} \left(\frac{63}{16} \right) \\
 &= LHS
 \end{aligned}$$

Question 8:

Prove that $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Solution:

$$\begin{aligned}
 LHS &= \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \\
 &= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \cdot \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \cdot \frac{1}{8}} \right) \\
 &= \tan^{-1} \left(\frac{12}{34} \right) + \tan^{-1} \left(\frac{11}{23} \right) \\
 &= \tan^{-1} \left(\frac{6}{17} \right) + \tan^{-1} \left(\frac{11}{23} \right) \\
 &= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \cdot \frac{11}{23}} \right) \\
 &= \tan^{-1} \left(\frac{325}{325} \right) \\
 &= \tan^{-1} (1) \\
 &= \frac{\pi}{4} \\
 &= RHS
 \end{aligned}$$

Question 9:

Prove that $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in [0, 1]$.

Solution:

Let $x = \tan^2 \theta$

Then,

$$\sqrt{x} = \tan \theta$$

$$\theta = \tan^{-1} \sqrt{x}$$

Therefore,

$$\begin{aligned} \left(\frac{1-x}{1+x} \right) &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \\ &= \cos 2\theta \end{aligned}$$

Thus,

$$\begin{aligned} RHS &= \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) \\ &= \frac{1}{2} \cos^{-1} (\cos 2\theta) \\ &= \frac{1}{2} \times 2\theta \\ &= \theta \\ &= \tan^{-1} \sqrt{x} \\ &= LHS \end{aligned}$$

Question 10:

Prove that $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right)$.

Solution:

$$\begin{aligned} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2} && \text{(by rationalizing)} \\ &= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x - 1+\sin x} \\ &= \frac{2(1+\sqrt{1-\sin^2 x})}{2\sin x} = \frac{1+\cos x}{\sin x} \\ &= \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \cot \frac{x}{2} \end{aligned}$$

Thus,

$$\begin{aligned}
 LHS &= \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) \\
 &= \cot^{-1} \left(\cot \frac{x}{2} \right) \\
 &= \frac{x}{2} \\
 &= RHS
 \end{aligned}$$

Question 11:

Prove that $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{2} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$

Solution:

Let $x = \cos 2\theta \Rightarrow \theta = \frac{1}{2} \cos^{-1} x$

Thus,

$$\begin{aligned}
 LHS &= \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}} \right) \\
 &= \tan^{-1} \left(\frac{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta} \right) \\
 &= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) \\
 &= \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \\
 &= \tan^{-1} 1 - \tan^{-1} (\tan \theta) \\
 &= \frac{\pi}{4} - \theta \\
 &= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \\
 &= RHS
 \end{aligned}$$

Question 12:

Prove that $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3}$

Solution:

$$\begin{aligned} LHS &= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} \\ &= \frac{9}{4} \left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3} \right) \\ &= \frac{9}{4} \left(\cos^{-1} \frac{1}{3} \right) \quad \dots(1) \end{aligned}$$

Now, let $\cos^{-1} \frac{1}{3} = x \Rightarrow \cos x = \frac{1}{3}$
Therefore,

$$\begin{aligned} \sin x &= \sqrt{1 - \left(\frac{1}{3}\right)^2} \\ &= \frac{2\sqrt{2}}{3} \\ x &= \sin^{-1} \frac{2\sqrt{2}}{3} \\ \cos^{-1} \frac{1}{3} &= \sin^{-1} \frac{2\sqrt{2}}{3} \quad \dots(2) \end{aligned}$$

Thus, by using (1) and (2)

$$\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3}$$

Hence proved.

Question 13:

Solve $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$.

Solution:

It is given that $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

Since, $2 \tan^{-1}(x) = \tan^{-1} \frac{2x}{1-x^2}$

Hence,

$$\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = (2 \operatorname{cosec} x)$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow \tan x = \tan \frac{\pi}{4}$$

Therefore,

$$x = n\pi + \frac{\pi}{4}, \text{ where } n \in Z.$$

Question 14:

Solve $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$

Solution:

Since $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$

Hence,

$$\Rightarrow \tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} x = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow x = \tan \frac{\pi}{6}$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin(\tan^{-1} x), |x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$

(B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$

(D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Let $\tan y = x$

Therefore,

$$\sin y = \frac{x}{\sqrt{1+x^2}}$$

Now, let $\tan^{-1} x = y$

Therefore,

$$y = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

Hence,

$$\tan^{-1} x = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

Thus,

$$\begin{aligned} \sin(\tan^{-1} x) &= \sin \left(\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) \right) \\ &= \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

Thus, the correct option is D.

Question 16:

Solve: $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$, then x is equal to

- (A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Solution:

It is given that $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1} x = \cos^{-1}(1-x) \quad \dots(1)$$

Let $\sin^{-1} x = y \Rightarrow \sin y = x$

Hence,

$$\begin{aligned}\cos y &= \sqrt{1-x^2} \\ y &= \cos^{-1}(\sqrt{1-x^2}) \\ \sin^{-1} x &= \cos^{-1} \sqrt{1-x^2}\end{aligned}$$

From equation (1), we have

$$-2 \cos^{-1} \sqrt{1-x^2} = \cos^{-1}(1-x)$$

Put $x = \sin y$

$$\begin{aligned}\Rightarrow -2 \cos^{-1} \sqrt{1-\sin^2 y} &= \cos^{-1}(1-\sin y) \\ \Rightarrow -2 \cos^{-1}(\cos y) &= \cos^{-1}(1-\sin y) \\ \Rightarrow -2y &= \cos^{-1}(1-\sin y) \\ \Rightarrow 1-\sin y &= \cos(-2y) \\ \Rightarrow 1-\sin y &= \cos 2y \\ \Rightarrow 1-\sin y &= 1-2\sin^2 y \\ \Rightarrow 2\sin^2 y - \sin y &= 0 \\ \Rightarrow \sin y(2\sin y - 1) &= 0 \\ \Rightarrow \sin y &= 0, \frac{1}{2}\end{aligned}$$

Therefore,

$$x = 0, \frac{1}{2}$$

When $x = \frac{1}{2}$, it does not satisfy the equation.

Hence, $x = 0$ is the only solution

Thus, the correct option is C.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $\frac{-3\pi}{4}$

Solution:

$$\begin{aligned}\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y} &= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] \\&= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right] \\&= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \\&= \tan^{-1}(1) \\&= \tan^{-1}\left(\tan\frac{\pi}{4}\right) \\&= \frac{\pi}{4}\end{aligned}$$

Thus, the correct option is C.