Chapter-9. XI-Std

Differential Calculus - Limits and Continuity.

Def^n:

Let f be a function of a real variable x. Let c, l be two fixed numbers. If $f(x)$ approaches the value l as x approaches c.

We say l is the limit of the function $f(x)$ as x tends to c.

This is written as $\lim_{x \to c} f(x) = l$.

Note:

(i) There is a difference between '$x \to 0$' and '$x = 0$'.

$x \to 0$ means that x gets nearer and nearer to 0, but never becomes equal to 0. $x = 0$ means that x takes the value 0.

(ii) Left hand limit: $L f(c) = \lim_{x \to c_-} f(x)$

(iii) Right hand Limit $R f(c) = \lim_{x \to c+} f(x)$.

$\therefore \lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x)$.

LHS = RHS = limit.

(iii) Left hand limit x - takes the value $x = c - h$, $h > 0$.

(iv) Right hand limit $x = c + h$, $h > 0$

(v) Left hand limit $\neq$ Right hand limit then $\lim_{x \to x_0} f(x)$ is does not exsit.

Ex: 9.1

In Problems 1-6 complete the table using calculator and use the result to, estimate the limit

1. $\lim_{x \to 2} \frac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Solution:

$\lim_{x \to 2} \frac{x-2}{x^2-x-2}$

$= \lim_{x \to 2} \frac{(x-2)}{(x-2)(x+1)}$

$= \lim_{x \to 2} \frac{1}{x+1}$.

x	1.9	1.99	1.999	2.001	2.01
$f(x)$	$\frac{1}{1.9+1} = \frac{1}{2.9} = 0.34$	$\frac{1}{1.99+1} = \frac{1}{2.99} = 0.33$	$\frac{1}{1.999+1} = \frac{1}{2.999} = 0.33$	$\frac{1}{2.001+1} = \frac{1}{3.001} = 0.33$	$= \frac{1}{2.01+1} = \frac{1}{3.01} = 0.33$

$\lim_{x \to 2} \frac{x-2}{x^2-x-2} = 0.3$.

2. $\lim_{x \to 2} \frac{x-2}{x^2-4}$

Solution:

$\lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2}$

x	1.9	1.99	1.999	2.001	2.01
$f(x)$	$= \frac{1}{1.9+2} = \frac{1}{3.9} = 0.256$	$= \frac{1}{1.99+2} = \frac{1}{3.99} = 0.251$	$\frac{1}{1.999+2} = \frac{1}{3.999} = 0.250$	$\frac{1}{2.001+2} = \frac{1}{4.001} = 0.249$	$\frac{1}{2.01+2} = \frac{1}{4.01} = 0.249$

$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4} = 0.25$

3. $\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{x}}{x}$ H.W. $= 0.288$

4. $\lim_{x \to 0} \frac{\sqrt{1-x} - 2}{x+2} = -0.25$.

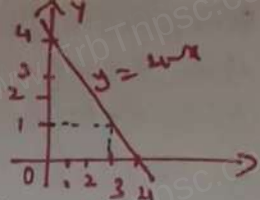
5. $\lim_{x \to 0} \frac{\sin x}{x}$

x	-0.1	-0.01	-0.001
$f(x)$	$= \frac{\sin(0.1)}{-0.1} = -\frac{\sin(0.1)}{-0.1} = 0.998$	$\frac{\sin(-0.01)}{-0.01} = -\frac{\sin(0.01)}{0.01} = 0.999$	$\frac{\sin(-0.001)}{-0.001} = -\frac{\sin 0.001}{-0.001} = 0.9999$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(7) Solution:

$\lim_{x\to 3}(4-x)$

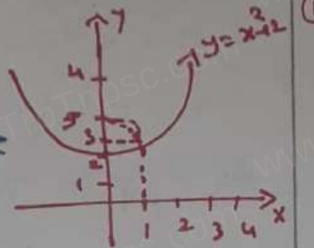


from the graph the value of the function at $x=3$ is $y=f(3)=1$

$\therefore \lim_{x\to 3}(4-x)=1.$

8. Solution:

$\lim_{x\to 1}(x^2+2)$

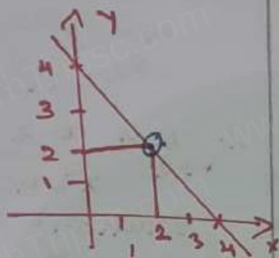


from the graph the value of the function at $x=1$ is $y=f(1)=3$

$\therefore \lim_{x\to 1}(x^2+2)=3.$

9. Solution:

$f(x)=\begin{cases}4-x, & x\neq 2\\ 0 & x=2\end{cases}$



To find $\lim_{x\to 2} f(x)$.

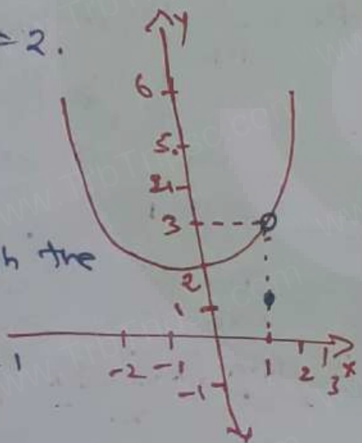
from the figure the value of the function at $x=2$ is

$y=f(2)=2$

$\therefore \lim_{x\to 2} f(x)=2.$

10 Solution.

To find $\lim_{x\to 1} f(x)$



from the graph the value of the function at $x=1$ is $y=f(1)=3$

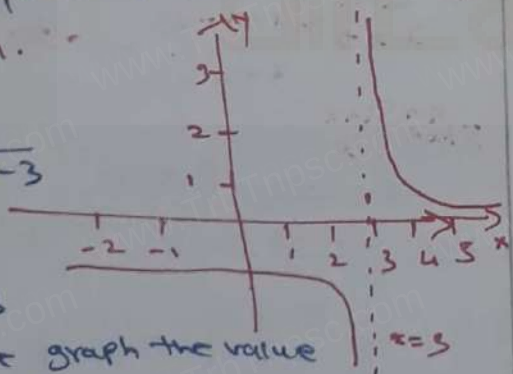
$\therefore \lim_{x\to 1} f(x)=3.$

11 Solution.

To find $\lim_{x\to 3}\frac{1}{x-3}$

$y=f(x)$

$=\frac{1}{x-3}$



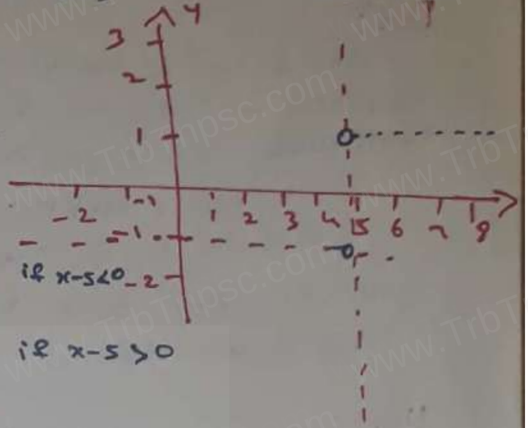
from the graph the value of the function at $x=3$ the curve does not meet the line $x=3$.

$\therefore$ The value of the function is not defined at the point $x=3$.

Hence $\lim_{x\to 3}\frac{1}{x-3}$ does not exist at $x=3$.

(12) Solution:

$\lim_{x\to 5}\frac{|x-5|}{x-5}$



$f(x)=\begin{cases}\frac{-(x-5)}{(x-5)} & \text{if } x-5<0\\ \frac{(x-5)}{(x-5)} & \text{if } x-5>0\end{cases}$

$=\begin{cases}-1 & \text{if } x<5\\ 1 & \text{if } x>5\end{cases}$

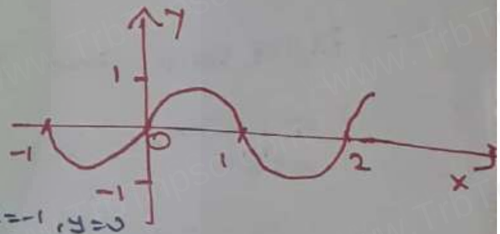
from the graph $x=5$ the curve does not intersect the line $x=5$.

$\therefore$ The value of the function $y=f(x)$ does not exist at $x=5$.

$\therefore \lim_{x\to 5}\frac{|x-5|}{x-5}$ does not exist.

(13)

$\lim_{x\to 1}\sin\pi x$.



$y=\sin\pi x$,

$x=1,\ y=0,$ | $x=-1,\ y=0$

$x=2.\ y=0.$ | $x=-1\ y=0$

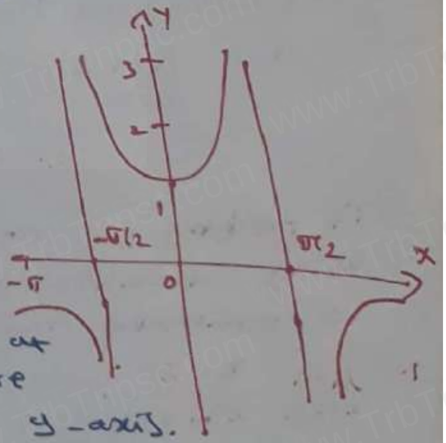
$\therefore$ The curve $y=f(x)$ intersect the $x=1$ at x-axis.

$\therefore y=f(1)=0$

Hence $\lim_{x\to 1}\sin\pi x=0$.

14. Solution.

To find $\lim_{x\to 0}\sec x$.



Let $y=f(x)$

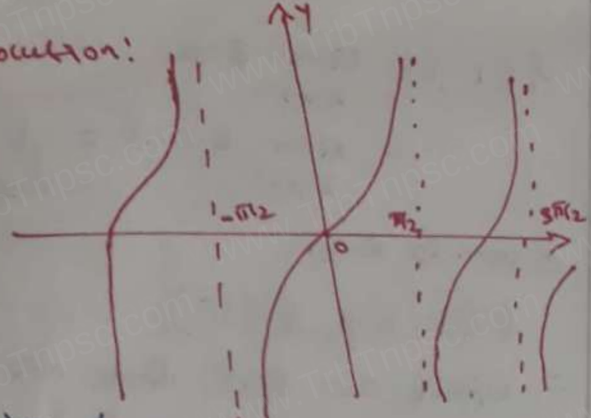
$y=\sec x$.

from the graph at $x=0$ the curve intersect the y-axis.

at $x=0,\ y=1$.

$\therefore \lim_{x\to 0}\sec x=1$.

15. Solution:



$\lim_{x\to\pi/2} \tan x$

$y = f(x) = \overset{\tan x}{\sec x}$

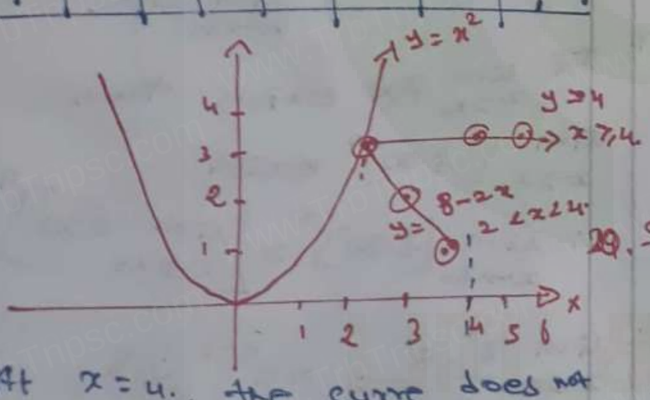
from the graph at $x=\pi/2$, the curve does not intersect the line $x=\pi/2$.

At $x=\pi/2$, the value of the function $y=f(x)$ does not exist.

$\therefore \lim_{x\to\frac{\pi}{2}} \tan x$ does not exist.

16. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8-2x, & 2 < x < 4 \\ 4, & x \ge 4. \end{cases}$

x	-1	0	1	2	3	3.5	4	5	6
$f(x)$	x^2	x^2	x^2	x^2	$8-2x$	$8-2x$	4	4	4
$f(x)$	1	0	1	4	2	1	4	4	4



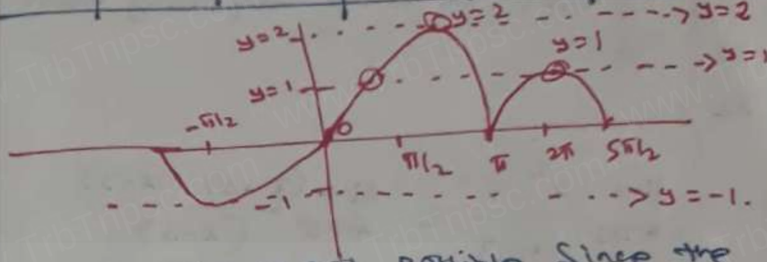
At $x=4$, the curve does not exist.

Hence, except at $x_0=4$ the limit of $f(x)$ exist.

17. Solution:

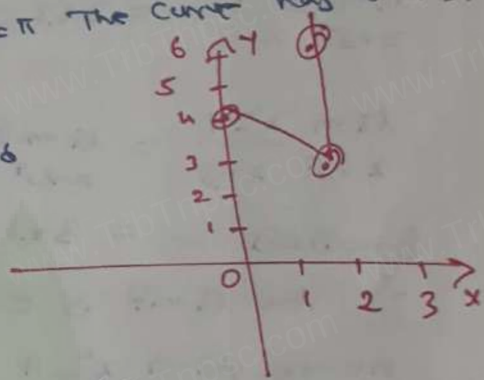
$f(x) = \begin{cases} \sin x, & x < 0 \\ 1-\cos x, & 0 \le x \le \pi \\ \cos x & x > \pi \end{cases}$

x	$-\pi/2$	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$	$\sin x$	$1-\cos x$	$1-\cos x$	$1-\cos x$	$\cos x$	$\cos x$.
$f(x)=$	$\sin(-\pi/2)$ $-\sin\pi/2$ $=-1$	$1-\cos 0$ $=1-1$ $=0$	$1-\cos\pi/2$ $=1-0$ $=1$	$1-\cos\pi$ $1-(-1)$ $=2$	$\cos\frac{3\pi}{2}$ $\cos(\pi+\frac{\pi}{2})$ $=0$	$\cos 2\pi$ $=+1$.



$(\pi/2)$ point is not possible since the range of the curve is $[-1,1]$. Except $x_0=\pi$ The curve has limits.

18. Solution.



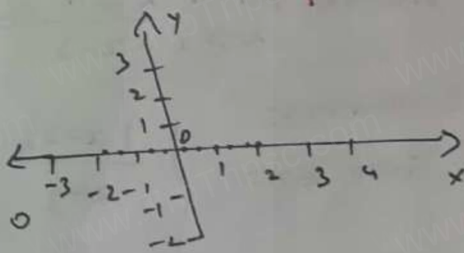
$f(0)$ is undefined.

$\lim_{x\to 0} f(x) = 4.$

$f(2) = 6$

$\lim_{x\to 2} f(x) = 3.$

(ii) 14. Solution:



$f(-2) = 0$

$f(2) = 0$

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ does not exist

20. Solution:

$f(2) = 4.$

Here at $x=2$ the value of the function is given

$\therefore$ we can't conculude anithing about the limit of $f(x)$ as x approaches 2.

19. Solution:

given $\lim_{x\to 8} f(x) = 25$

by def^n. of limit.

$\lim_{x\to 8^-} f(x) = \lim_{x\to 8^+} f(x) = 25$

$\therefore f(8^-) = f(8^+) = 25$

21 Solution:

Given, $\lim_{x \to 2} f(x) = 4$.

Since limit of the function need not be equal to the value of the function,

can't conclude anything about $f(2)$.

22 Solution:

$$\lim_{x \to 3^-} \frac{x^2-9}{x-3} = \lim_{x \to 3^-} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x \to 3^-} (x+3)$$

$$\lim_{x \to 3^-} \frac{x^2-9}{x-3} = 3+3 = 6 \to (1)$$

$$\lim_{x \to 3^+} \frac{x^2-9}{x-3} = \lim_{x \to 3^+} (x+3)$$

$$= 3+3 = 6 \to (2)$$

from (1) & (2)

$\therefore$ The limit is exist

Hence $\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$.

23 Solution:

Given $f(x) = \begin{cases} \frac{|x-1|}{x-1}, & x \neq 1 \\ 0 & x = 1. \end{cases}$

$$f(x) = \begin{cases} \frac{|x-1|}{x-1} & \text{for } x < 1, x > 1 \\ 0 & \text{for } x = 1 \end{cases}$$

$$= \begin{cases} \frac{x-1}{x-1} & \text{for } x > 1 \\ -\frac{(x-1)}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1. \end{cases}$$

$$= \begin{cases} +1 & \text{for } x > 1 \\ -1 & \text{for } x < 1 \\ 0 & \text{for } x = 1. \end{cases}$$

$$\therefore f(1^+) = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} f(1) = 1 \to (1)$$

$$f(1^-) = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} f(-1) = -1. \to (2)$$

from (1) & (2)

$$f(1^-) \neq f(1^+).$$

$\therefore$ limit of $f(x)$ does not exist.

K.S ——

Theorems on limits:

1. If f is a polynomial, then it is always possible to calculate the limit by evaluation.
2. The limit of constant function is that
3. $\lim_{x \to x_0} c f(x) = c \lim_{x \to x_0} f(x)$
4. $\lim_{x \to x_0} [f(x) \pm g(x)] = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x)$.
5. $\lim_{x \to x_0} [f(x) . g(x)] = \lim_{x \to x_0} f(x) . \lim_{x \to x_0} g(x)$.
6. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}, \quad g(x) \neq 0$.
7. If $\lim_{x \to x_0} f(x)$ exists then $\lim_{x \to x_0} [f(x)]^n$ exist.

   $\therefore \lim_{x \to x_0} [f(x)]^n = \left[\lim_{x \to x_0} f(x)\right]^n$.
8. $\lim_{x \to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$ (X)

Proof:

w.k.t $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1})$

$$\lim_{x \to a} x^n - a^n = \lim_{x \to a} (x-a)(x^{n-1} + x^{n-2}a + \dots + a^{n-1})$$

$$\lim_{x \to a} \frac{x^n - a^n}{x-a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + a^{n-1})}{(x-a)}$$

$$= \lim_{x \to a} x^{n-1} + x^{n-2}a + \dots + a^{n-1}.$$

$$= a^{n-1} + a^{n-1} + \dots + a^{n-1} \text{ (n times)}$$

$$= n a^{n-1}.$$

EX: 9.2

Evaluate the following limits.

1. $\lim\limits_{x\to 2} \dfrac{x^4-16}{x-2}$

Solution:

$$\lim_{x\to 2}\frac{x^4-16}{x-2} = \lim_{x\to 2}\frac{(x^2)^2-4^2}{(x-2)}$$

$$= \lim_{x\to 2}\frac{(x^2-4)(x^2+4)}{(x-2)}$$

$$= \lim_{x\to 2}\frac{(x^2-2^2)(x^2+4)}{(x-2)}$$

$$= \lim_{x\to 2}\frac{(x+2)(x-2)(x^2+4)}{(x-2)}$$

$$= \lim_{x\to 2}(x+2)(x^2+4)$$

$$= (2+2)(2^2+4) = 4(4+4)$$

$$= 32.$$

2. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$

$$= \lim_{x\to 1}\frac{x^m-1^m}{x^n-1^n}\times\frac{x-1}{x-1}$$

$$= \lim_{x\to 1}\frac{x^m-1^m}{x-1}\times\frac{x-1}{x^n-1^n}$$

$$= \lim_{x\to 1}\frac{x^m-1^m}{x-1}\times\frac{1}{\frac{x^n-1^n}{x-1}}$$

W.K.T "$\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$"

$$\therefore \quad = m(1)^{m-1}\times\frac{1}{n(1)^{n-1}}$$

$$= \frac{m}{n}$$

3. $\lim\limits_{\sqrt{x}\to 3}\dfrac{x^2-81}{\sqrt{x}-3}$

Let $y=\sqrt{x}$ when $\sqrt{x}\to 3$, $y\to 3$

$$\therefore \lim_{\sqrt{x}\to 3}\frac{x^2-81}{\sqrt{x}-3} = \lim_{\sqrt{x}\to 3}\frac{[(\sqrt{x})^2]^2-3^4}{\sqrt{x}-3}$$

$$= \lim_{\sqrt{x}\to 3}\frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}$$

$$= \lim_{y\to 3}\frac{y^4-3^4}{y-3}$$

$$= 4(3)^{4-1} \qquad \left\{ \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}. \right.$$

$$= 4(3)^3 = 4\times 27$$

$$= 108.$$

4. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$, $x>0$.

Solution:

Solution:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{x+h\to x}\frac{(x+h)^{1/2}-x^{1/2}}{(x+h)-x} \quad \{+x,-x.$$

$\{\because x+h\to x \Rightarrow h\to 0$

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}.$$

$$= \frac{1}{2}\left(x^{\frac{1}{2}-1}\right)$$

$$= \frac{1}{2}x^{-1/2} = \frac{1}{2}\times\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

5. $\lim\limits_{x\to 5}\dfrac{\sqrt{x+4}-3}{x-5}$

Let $y = x+4$. | when $x\to 5$
$x = y-4$. | $\Rightarrow y\to 5+4$
$x-5 = y-4-5$ | $y\to 9$
$x-5 = y-9$

$$\therefore \lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5} = \lim_{y\to 9}\frac{\sqrt{y}-\sqrt{3^2}}{y-9}$$

$$= \lim_{y\to 9}\frac{y^{1/2}-9^{1/2}}{y-9} \qquad \left\{ \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}. \right.$$

$$= \frac{1}{2}(9)^{\frac{1}{2}-1}$$

$$= \frac{1}{2}(9)^{-1/2} = \frac{1}{2}\ \frac{1}{9^{1/2}} = \frac{1}{2}\ \frac{1}{\sqrt{9}}$$

$$= \frac{1}{2}\times\frac{1}{3}$$

$$= \frac{1}{6}.$$

6. $\lim_{x \to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x-2}$

$= \lim_{x \to 2} \left(\dfrac{\frac{2-x}{2x}}{x-2} \right)$

$= \lim_{x \to 2} \dfrac{-(x-2)}{2x(x-2)} = -\lim_{x \to 2} \dfrac{1}{2x}$.

$= \dfrac{-1}{2 \times 2}$

$= \dfrac{-1}{4}$

7. $\lim_{x \to 1} \dfrac{\sqrt{x} - x^2}{1-\sqrt{x}}$

$= \lim_{x \to 1} \dfrac{\sqrt{x} - (\sqrt{x})^4}{1-\sqrt{x}}$

$= \lim_{x \to 1} \dfrac{\sqrt{x}\left(1-(\sqrt{x})^3\right)}{1-\sqrt{x}}$

$= \lim_{x \to 1} \dfrac{\sqrt{x}\left((\sqrt{x})^3 - 1\right)}{\sqrt{x}-1}$

Let $y = \sqrt{x}$, $x \to 1 \Rightarrow y \to \sqrt{1} = 1$.

$\lim_{x \to 1} \dfrac{\sqrt{x} - x^2}{1-\sqrt{x}} = \lim_{y \to 1} \dfrac{y(y^3-1)}{y-1}$

$= \lim_{y \to 1} y \lim_{y \to 1} \dfrac{y^3 - 1^3}{y-1}$

$= (1)(3)(1)^{3-1}$ $\quad \left\{ \lim_{x \to a} \dfrac{x^n - a^n}{x-a} = n a^{n-1} \right.$

$= 3$

8. Solution:

$\lim_{x \to 0} \dfrac{\sqrt{x^2+1} - 1}{\sqrt{x^2+16} - 4}$

$= \lim_{x \to 0} \left[\sqrt{x^2+1} - 1 \times \dfrac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \times \dfrac{1}{\sqrt{x^2+16} - 4} \times \dfrac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4} \right]$

$= \lim_{x \to 0} \left[\dfrac{(\sqrt{x^2+1})^2 - (1)^2}{\sqrt{x^2+1}+1} \times \dfrac{\sqrt{x^2+16}+4}{(\sqrt{x^2+16})^2 - (4)^2} \right]$

$= \lim_{x \to 0} \left[\dfrac{x^2+1-1}{\sqrt{x^2+1}+1} \times \dfrac{\sqrt{x^2+16}+4}{x^2+16-16} \right]$

$= \lim_{x \to 0} \dfrac{x^2}{\sqrt{x^2+1}+1} \times \dfrac{\sqrt{x^2+16}+4}{x^2}$

$= \lim_{x \to 0} \dfrac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1} = \dfrac{\sqrt{0+16}+4}{\sqrt{0+1}+1}$

$= \dfrac{4+4}{1+1} = \dfrac{8}{2}$

$= 4.$

9. Solution:

$\lim_{x \to 0} \dfrac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \dfrac{(\sqrt{1+x} - 1)}{x} \times \dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$= \lim_{x \to 0} \left[\dfrac{1+x-1}{x(\sqrt{1+x}+1)} \right]$

$= \lim_{x \to 0} \left[\dfrac{x}{x(\sqrt{1+x}+1)} \right]$

$= \lim_{x \to 0} \dfrac{1}{\sqrt{1+x}+1} = \dfrac{1}{\sqrt{1+0}+1} = \dfrac{1}{1+1}$

$= \dfrac{1}{2}$

10. $\lim_{x \to 1} \dfrac{\sqrt[3]{7+x^3} - \sqrt{3+x^2}}{x-1}$

$= \lim_{x \to 1} \dfrac{\sqrt[3]{7+x^3} - 2 + 2 - \sqrt{3+x^2}}{x-1}$

$= \lim_{x \to 1} \dfrac{(7+x^3)^{1/3} - 2}{x-1} - \lim_{x \to 1} \dfrac{(3+x^2)^{1/2} - 2}{x-1}$

$= \lim_{x \to 1} \dfrac{(7+x^3)^{1/3} - 8^{1/3}}{x^3-1} \times \dfrac{x^3-1}{x-1}$

$- \lim_{x \to 1} \dfrac{(x+3)^{1/2} - (4)^{1/2}}{x^2-1} \times \dfrac{x^2-1}{x-1}.$

$= \lim_{x \to 1} \dfrac{(7+x^3)^{1/3} - (8)^{1/3}}{(7+x^3) - 8} \times \dfrac{(x-1)(x^2+x+1)}{x-1.}$

$- \lim_{x \to 1} \dfrac{(3+x^2)^{1/2} - (4)^{1/2}}{(3+x^2) - 4} \times \dfrac{(x+1)(x-1)}{(x-1)}$

$= \lim_{x \to 1} \dfrac{(7+x^3)^{1/3} - 8^{1/3}}{(7+x^3) - 8} \times (x^2+x+1)$

$- \lim_{x \to 1} \dfrac{(3+x^2)^{1/2} - (4)^{1/2}}{(3+x^2) - 4} \times (x+1)$

$= \left(\frac{1}{3}\right) 8^{1/3 - 1} (1^2+1+1) - \frac{1}{2} (4)^{1/2 - 1} (1+1)$

$= \frac{1}{3} 8^{-2/3} (3) - \frac{1}{2} 4^{-1/2} \times 2$

$$= (2^3)^{-2/3} - (2^2)^{-1/2}$$

$$= 2^{-2} - 2^{-1} = \frac{1}{2^2} - \frac{1}{2}$$

$$= \frac{1}{4} - \frac{1}{2}$$

$$= -\frac{1}{4}$$

11. $\lim_{x \to 2} \frac{2 - \sqrt{x+2}}{\sqrt[3]{2} - \sqrt[3]{4-x}}$

$$= \lim_{x \to 2} \frac{\sqrt{x+2} - 2}{\sqrt[3]{4-x} - \sqrt[3]{2}} = \lim_{x \to 2} \frac{(x+2)^{1/2} - (2^2)^{1/2}}{(4-x)^{1/3} - (2)^{1/3}}$$

$$= \lim_{x \to 2} \frac{(x+2)^{1/2} - (2^2)^{1/2}}{x-2} \times \frac{x-2}{(4-x)^{1/3} - (2)^{1/3}}$$

$$= \lim_{x \to 2} \frac{(x+2)^{1/2} - (4)^{1/2}}{(x+2) - 4} \times \frac{-[(4-x) - 2]}{(4-x)^{1/3} - (2)^{1/3}}$$

$$= \lim_{x \to 2} \frac{(x+2)^{1/2} - (4)^{1/2}}{(x+2) - 4} \times -\frac{1}{\lim_{x \to 2} \frac{(4-x)^{1/3} - 2^{1/3}}{(4-x) - 2}}$$

$$\left\{ \because \lim_{x \to a} \frac{x^n - a^n}{x-a} = n a^{n-1} \right.$$

$$= \frac{1}{2}(4)^{\frac{1}{2}-1} \times \frac{-1}{\frac{1}{3}(2)^{1/3 - 1}}$$

$$= \frac{1}{2}(4)^{-\frac{1}{2}} \times \frac{-3}{2^{-2/3}} = \frac{1}{2} \times \frac{1}{4^{1/2}} \times (-3) \times 2^{2/3}$$

$$= \frac{-1}{2 \times 2} \times 3 \times 2^{2/3} = \frac{-3}{4}(2^2)^{1/3}$$

$$= \frac{-3}{4} \sqrt[3]{4} \quad //$$

12. $\lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x}$

$$= \lim_{x \to 0} \frac{\sqrt{1+x^2} - 1}{x} \times \frac{\sqrt{1+x^2} + 1}{\sqrt{1+x^2} + 1}$$

$$= \lim_{x \to 0} \left[\frac{1 + x^2 - 1}{x(\sqrt{1+x^2} + 1)} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^2}{x(\sqrt{1+x^2} + 1)} \right]$$

$$= \lim_{x \to 0} \left[\frac{x}{\sqrt{1+x^2} + 1} \right] = \frac{0}{\sqrt{1+0} + 1}$$

$$= \frac{0}{2}$$

$$= 0.$$

13. $\lim_{x \to 0} \frac{\sqrt{1-x} - 1}{x^2}$

$$= \lim_{x \to 0} \left[\frac{\sqrt{1-x} - 1}{x^2} \times \frac{\sqrt{1-x} + 1}{\sqrt{1-x} + 1} \right]$$

$$= \lim_{x \to 0} \left[\frac{1 - x - 1}{x^2(\sqrt{1-x} + 1)} \right]$$

$$= \lim_{x \to 0} \left[\frac{-x}{x^2(\sqrt{1-x} + 1)} \right]$$

$$= -\lim_{x \to 0} \left[\frac{1}{x(\sqrt{1-x} + 1)} \right]$$

$$= -\frac{1}{0(\sqrt{1-0} + 1)} = \frac{-1}{0} = -\infty$$

$\therefore \lim_{x \to 0} \frac{\sqrt{1-x} - 1}{x^2}$ does not exist.

14. $\lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x-5}$

$$= \lim_{x \to 5} \frac{(x-1)^{1/2} - (4)^{1/2}}{(x-1) - 4}$$

Let $x - 1 = y \quad x \to 1$ as $y \to 5 - 1 = 4.$

$$\lim_{y \to 4} \frac{y^{1/2} - (4)^{1/2}}{y - 4} = \frac{1}{2}(4)^{\frac{1}{2} - 1}$$

$$= \frac{1}{2}(4)^{-1/2} = \frac{1}{2} \cdot \frac{1}{4^{1/2}} = \frac{1}{2 \times 2}$$

$$= \frac{1}{4}.$$

15. $\lim_{x \to a} \frac{\sqrt{x-b} - \sqrt{a-b}}{x^2 - a^2}, \ (a > b)$

$$= \lim_{x \to a} \frac{\sqrt{x-b} - \sqrt{a-b}}{x^2 - a^2} \times \frac{\sqrt{x-b} + \sqrt{a-b}}{\sqrt{x-b} + \sqrt{a-b}}$$

$$= \lim_{x \to a} \frac{(x-b) - (a-b)}{(x^2 - a^2)[\sqrt{x-b} + \sqrt{a-b}]}$$

$$= \lim_{x \to a} \frac{x - \cancel{b} - a + \cancel{b}}{(x-a)(x+a)[\sqrt{x-b} + \sqrt{a-b}]}$$

$$= \lim_{x \to a} \frac{(x-a)}{(x-a)(x+a)[\sqrt{x-b} + \sqrt{a-b}]}$$

$$= \lim_{x \to a} \frac{1}{(x+a)[\sqrt{x-b} + \sqrt{a-b}]}$$

$$= \frac{1}{(a+a)(\sqrt{a-b} + \sqrt{a-b})}$$

$$= \frac{1}{2a \cdot 2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-b}}.$$

Ex: 9.3

1. (a) Find the left and right limits of $f(x) = \frac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x=-2$.

(b) $f(x) = \tan x$. at $x = \pi/2$.

Solution:

(a) $f(x) = \frac{x^2-2^2}{(x^2+4x+4)(x+3)} = \frac{(x+2)(x-2)}{(x+2)^2(x+3)}$

$f(x) = \frac{x-2}{(x+2)(x+3)}$

To find Left hand limit:

put $x=-2-h$ where $h>0$

$x \to -2 \quad -2=-2-h \therefore h \to 0$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{h\to 0} \frac{-2-h-2}{(-2-h+2)(-2-h+3)}$

$= \lim\limits_{h\to 0} \frac{-4-h}{-h(1-h)} = \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{4+h}{1-h}\right)$

$= \frac{1}{0}\left(\frac{4+0}{1-0}\right) = \frac{1}{0}$

$\boxed{\lim\limits_{x\to 2^-} f(x) = \infty}$

To find Right hand limit:

Put $x = -2+h$ where $x>0$

when $x\to -2$ then $h \to 0$.

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{h\to 0} \frac{-2+h-2}{(-2+h+2)(-2+h+3)}$.

$= \lim\limits_{h\to 0} \frac{-4+h}{h(1+h)}$.

$= \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{h-4}{1+h}\right) = \frac{1}{0}\left(\frac{0-4}{1+0}\right)$

$= -\infty$

$\lim\limits_{x\to 2^+} f(x) = -\infty$.

(b) $f(x) = \tan x$ at $x = \pi/2$.

put $x = \frac{\pi}{2} - h \quad h>0$

$x \to \frac{\pi}{2}$ as $h \to 0$.

$\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x) = \lim\limits_{h\to 0} \tan\left(\frac{\pi}{2}-h\right)$ $\quad$ $\tan(90-\theta) = \cot\theta$. $\cot(90-\theta) = \tan\theta$

$= \lim\limits_{h\to 0} \cot h$

$= \cot 0 = \infty$

$\lim\limits_{x\to \frac{\pi}{2}^-} \tan x = \infty$

To find right hand limit:

put $x = \frac{\pi}{2} + h \quad h>0$

$x \to \frac{\pi}{2}$ as $h \to 0$

$\lim\limits_{x\to \frac{\pi}{2}^+} \tan x = \lim\limits_{h\to 0} \tan\left(\frac{\pi}{2}+h\right)$

$= \lim\limits_{h\to 0} (-\cot h) = -\cot 0 = -\infty$

2. Evaluate the following limits:

$\lim\limits_{x\to 3} \frac{x^2-9}{x^2(x^2-6x+9)}$.

$= \lim\limits_{x\to 3} \frac{x^2-3^2}{x^2(x-3)^2} = \lim\limits_{x\to 3} \frac{(x+3)(x-3)}{x^2(x-3)^2}$

$= \lim\limits_{x\to 3} \frac{x+3}{x^2(x-3)}$.

put $x = 3-h$, $h>0$

$x\to 3 \quad h \to 0$.

$\therefore \lim\limits_{x\to 3} \frac{x+3}{x^2(x-3)} = \lim\limits_{h\to 0} \cancel{\frac{-h+3}{h^2(h-3)^2}} = \frac{6}{0}$

$= \lim\limits_{h\to 0} \frac{3-h+3}{(3-h)^2(3-h-3)}$

$= \lim\limits_{h\to 0} \frac{6-h}{-h(3-h)^2}$

$= -\lim\limits_{h\to 0} \frac{6-h}{h(3-h)^2} = -\frac{6}{0} = -\infty$

Right hand limit:

Put $x = 3+h \quad h>0$

$x \to 3 \quad h \to 0$

$\lim\limits_{x\to 3^+} \frac{x^2-9}{x^2(x^2-6x+9)} = \lim\limits_{h\to 0} \frac{3+h+3}{(3+h)^2(3+h-3)}$

$= \lim\limits_{h\to 0} \frac{6+h}{h(3+h)^2} = \frac{6+0}{0(3+0)^2} = \frac{6}{0}$

$= \infty$

3. $\lim_{x\to\infty}\left(\frac{3}{x-2}-\frac{2x+11}{x^2+x-6}\right)$ $\quad -6 = -2 \cdot 3$

$=\lim_{x\to\infty}\frac{3}{x-2}-\frac{2x+11}{(x+3)(x-2)}$

$=\lim_{x\to\infty}\frac{3(x+3)-(2x+11)}{(x+3)(x-2)}$

$=\lim_{x\to\infty}\frac{3x+9-2x-11}{(x+3)(x-2)}=\lim_{x\to\infty}\frac{\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$

$=\lim_{x\to\infty}\frac{1}{x+3}=\frac{1}{\infty+3}=\frac{1}{\infty}=0$ //

4. $\lim_{x\to\infty}\frac{x^3+x}{x^4-3x^2+1}$

$=\lim_{x\to\infty}\frac{x^3(1+x/x^3)}{x^4\left(1-\frac{3x^2}{x^4}+\frac{1}{x^4}\right)}$

$=\lim_{x\to\infty}\frac{1+1/x^2}{x\left(1-3/x^2+\frac{1}{x^4}\right)}=\frac{1+\frac{1}{\infty}}{\infty\left(1-\frac{1}{\infty}+\frac{1}{\infty}\right)}$

$=\frac{1+0}{\infty(1-0+0)}=\frac{1}{\infty}=0$ //

5. $\lim_{x\to\infty}\frac{x^4-5x}{x^2-3x+1}$

$=\lim_{x\to\infty}\frac{x^4\left(1-\frac{5x}{x^4}\right)}{x^2\left[1-\frac{3x}{x^2}+\frac{1}{x^2}\right]}=\lim_{x\to\infty}\frac{x^2\left[1-\frac{5}{x^3}\right]}{1-\frac{3}{x}+\frac{1}{x^2}}$

$=\frac{\infty\left[1-5/\infty\right]}{1-\frac{3}{\infty}+\frac{1}{\infty}}=\frac{\infty[1-0]}{1-0+0}=\infty$

6. $\lim_{x\to\infty}\frac{1+x-3x^3}{1+x^2+3x^3}$

$=\lim_{x\to\infty}\frac{x^3\left(\frac{1}{x^3}+\frac{x}{x^3}-3\right)}{x^3\left(\frac{1}{x^3}+\frac{x^2}{x^3}+3\right)}$

$=\lim_{x\to\infty}\frac{\frac{1}{x^3}+\frac{1}{x^2}-3}{\frac{1}{x^3}+\frac{1}{x}+3}=\frac{\frac{1}{\infty}+\frac{1}{\infty}-3}{\frac{1}{\infty}+\frac{1}{\infty}+3}$

$=\frac{0+0-3}{0+0+3}=\frac{-3}{3}=-1.$

$\therefore \lim_{x\to\infty}\frac{1+x-3x^3}{1+x^2+3x^3}=-1.$

7. $\lim_{x\to\infty}\frac{x^3}{2x^2-1}-\frac{x^2}{2x+1}$

$=\lim_{x\to\infty}\left[\frac{x^3(2x+1)-x^2(2x^2-1)}{(2x^2-1)(2x+1)}\right]$

$=\lim_{x\to\infty}\left[\frac{2x^4+x^3-2x^4+x^2}{(2x^2-1)(2x+1)}\right]$

$=\lim_{x\to\infty}\left[\frac{x^3+x^2}{(2x^2-1)(2x+1)}\right]$

$=\lim_{x\to\infty}\left[\frac{x^3(1+x^2/x^3)}{x^2(2-1/x^2)\cdot x(2+1/x)}\right]$

$=\lim_{x\to\infty}\left[\frac{1+1/x}{(2-1/x^2)(2+1/x)}\right]$

$=\frac{1+1/\infty}{(2-1/\infty)(2+1/\infty)}=\frac{1+0}{(2-0)(2+0)}$

$=\frac{1}{(2)(2)}=\frac{1}{4}$

8. (i) $\lim_{n\to\infty}\frac{1+2+3+\dots+n}{3n^2+7n+2}=\frac{1}{6}$

LHS

$\lim_{n\to\infty}\frac{1+2+3+\dots+n}{3n^2+7n+2}=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{3n^2+7n+2}$

$\frac{1}{2}\lim_{n\to\infty}\frac{n.n(1+1/n)}{n^2\left[3+\frac{7n}{n^2}+\frac{2}{n^2}\right]}$

$=\frac{1}{2}\lim_{n\to\infty}\frac{(1+1/n)}{\left(3+7/n+\frac{2}{n^2}\right)}$

$=\frac{1}{2}\left[\frac{1+0}{3+0}\right]=\frac{1}{6}$

(ii) $\lim_{n\to\infty}\frac{1^2+2^2+\dots+(3n)^2}{(1+2+\dots+5n)(2n+3)}=\frac{9}{25}$

LHS

$=\lim_{n\to\infty}\frac{\frac{3n(3n+1)(2\times 3n+1)}{6}}{\frac{5n(5n+1)(2n+3)}{2}}$

$=\lim_{n\to\infty}\frac{3n(3n+1)(6n+1)\times\cancel{2}}{\cancel{6}\times 5n(5n+1)(2n+3)}$

$=\lim_{n\to\infty}\frac{n.n(3+1/n)n(6+1/n)}{5\times n\times n(5+1/n)\times n(2+3/n)}$

$=\lim_{n\to\infty}\frac{(3+1/n)(6+1/n)}{5(5+1/n)(2+3/n)}=\frac{(3+0)(6+0)}{5(5+0)(2+0)}$

$=\frac{18}{50}=\frac{9}{25}$

(iii) $\lim_{n\to\infty}\left[\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{n(n+1)}\right]=1$

$$LHS = \lim_{n\to\infty}\left[\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{n(n+1)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2-1}{1\cdot2}+\frac{3-2}{2\cdot3}+\cdots+\frac{n+1-n}{n(n+1)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2}{1\cdot2}-\frac{1}{1\cdot2}+\frac{3}{2\cdot3}-\frac{2}{2\cdot3}+\cdots+\frac{n+1}{n(n+1)}-\frac{n}{n(n+1)}\right]$$

$$=\lim_{n\to\infty}\left[1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\right]$$

$$=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]=1-\frac{1}{\infty}=1-0$$

$$=1.$$

9. Solution:

given $R(S)=\frac{S}{\alpha S+\beta}$

where S is the number of spawners, R is the number of recruits. α,β are positive constants.

When the number of spawners is sufficiently large $S\to\infty$.

$$\lim_{S\to\infty}R(S)=\lim_{S\to\infty}\frac{S}{\alpha S+\beta}=\lim_{S\to\infty}\frac{S}{S(\alpha+\frac{\beta}{S})}$$

$$=\lim_{S\to\infty}\frac{1}{\alpha+\beta/S}=\frac{1}{\alpha+0}=\frac{1}{\alpha}.$$

When number of spawners is sufficiently large the number of recruits is $\frac{1}{\alpha}$.

10. solution:

Given concentration of salt water after t minutes is $C(t)=\frac{30t}{200+t}$.

To find the concentration of salt water after $t\to\infty$.

$$\lim_{t\to\infty}C(t)=\lim_{t\to\infty}\frac{30t}{200+t}=\lim_{t\to\infty}\frac{30t}{t\left[\frac{200}{t}+1\right]}$$

$$=\lim_{t\to\infty}\frac{30}{\left(\frac{200}{t}+1\right)}=\frac{30}{0+1}$$

$$=30$$

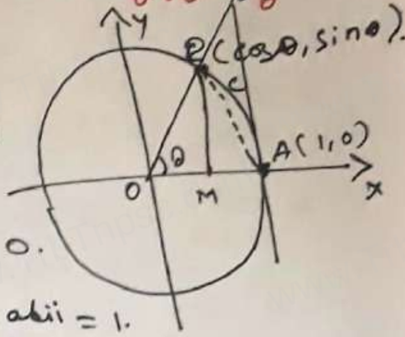
K. Senthil.

Senthil K.

Result 9.1.

(i) $\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$. (ii) $\lim_{\theta\to0}\frac{1-\cos\theta}{\theta}=0$

Proof:



Consider the ~~Δ OAP~~. unit circle with centre O.

$OA=OP=\text{radii}=1.$

In Δ OAP

$$\sin\theta=\frac{opp}{hyp}=\frac{PM}{OP}=\frac{PM}{1}$$

$PM=\sin\theta$

$$\text{Area of } \Delta OAP=\frac{1}{2}\times OA\times PM$$

$$=\frac{1}{2}\times1\times\sin\theta$$

$$=\frac{\sin\theta}{2}\to(1)$$

Consider the circular sector ACP

$$\text{Area of the sector } ACP=\frac{1}{2}r^2\theta$$

$$=\frac{1}{2}\times(OA)^2\times\theta=\frac{1}{2}\times1^2\times\theta$$

$$=\frac{\theta}{2}\to(2)$$

Consider Δ OAB.

$\tan\theta=\frac{AB}{OA}=\frac{AB}{1}$ $AB=\tan\theta$

$$\text{Area of } \Delta OAB=\frac{1}{2}\times OA\times AB$$

$$=\frac{1}{2}\times1\times\tan\theta$$

$$=\frac{\tan\theta}{2}\to(3)$$

from the figure.

Area of Δ OAP ≤ Area of circular sector ≤ Area of Δ OAB.

$$\therefore \frac{\sin\theta}{2}\le\frac{\theta}{2}\le\frac{\tan\theta}{2}$$

$$\frac{\sin\theta}{2}\le\frac{\theta}{2}\le\frac{\sin\theta}{\cos\theta}\times\frac{1}{2}$$

$$\left(\times\frac{2}{\sin\theta}\right)\quad 1\le\frac{\theta}{\sin\theta}\le\frac{1}{\cos\theta}$$

Take reciprocal.

$$\cos\theta\le\frac{\sin\theta}{\theta}\le1.$$

$$\lim_{\theta\to0}\cos\theta\le\lim_{\theta\to0}\frac{\sin\theta}{\theta}\le\lim_{\theta\to0}1.$$

$$1\le\lim_{\theta\to0}\frac{\sin\theta}{\theta}\le1.$$

by Sandwich theorem

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

(ii) $$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0.$$

$$1-\cos\theta = 2\sin^2\theta/2$$

$$\frac{1-\cos\theta}{\theta} = \frac{2\sin^2\theta/2}{\theta}$$

$$= \sin\theta/2 \times 2\sin\theta/2$$

$$\frac{1-\cos\theta}{\theta} = \sin\frac{\theta}{2} \times \frac{\sin\theta/2}{\theta/2}$$

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta \to 0}\left[\sin\frac{\theta}{2} \times \frac{\sin\theta/2}{\theta/2}\right]$$

$$= \lim_{\theta \to 0} \sin\frac{\theta}{2} \times \lim_{\theta \to 0} \frac{\sin\theta/2}{\theta/2}$$

$$= \sin 0 \times (1)$$

$$= 0 \times (1)$$

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$$

Result 9.2

$$\lim_{x \to 0} \frac{e^x-1}{x} = 1.$$

Proof:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 = \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= x\left[\frac{1}{1!} + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right]$$

$$\frac{e^x-1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

$$\lim_{x \to 0} \frac{e^x-1}{x} = \lim_{x \to 0}\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right]$$

$$= 1 + 0 + 0 \cdots 0$$

$$\lim_{x \to 0} \frac{e^x-1}{x} = 1$$

Result 9.3:

$$\lim_{x \to 0} \frac{a^x-1}{x} = \log a, \quad a > 0.$$

Solution:

we know $a^x = e^{\log a^x}$

$$a^x = e^{x\log a}$$

$$a^x - 1 = e^{x\log a} - 1$$

$$\frac{a^x-1}{x} = \frac{e^{x\log a}-1}{x}$$

$$\frac{a^x-1}{x} = \frac{e^{x\log a}-1}{x} \times \frac{\log a}{\log a} \rightarrow (1)$$

Let $y = x\log a$

$x \to 0 \quad y = \log a^x$

$$= \log a^0$$

$$= \log 1$$

$$\boxed{x \to 0 \text{ as } y = 0}$$

$$(1) \Rightarrow \lim_{x \to 0} \frac{a^x-1}{x} = \lim_{y \to 0} \frac{e^y-1}{y} \times \log a$$

$$= (1)\log a$$

$$\lim_{x \to 0} \frac{a^x-1}{x} = \log a$$

Some Important limits:

(i) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1.$

(ii) $\lim_{x \to 0} \frac{\sin^{-1}x}{x} = 1.$

(iii) $\lim_{x \to 0} \frac{\tan^{-1}x}{x} = 1$

(iv) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(v) $\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x = e^k$

(vi) $\lim_{x \to 0} (1+x)^{1/x} = e.$

Continuity :- 9.3

Defⁿ:

If $f(x)$ is continuity the following condition are satisfied.

(i) $f(x)$ must be defined in a neighbourhood of x_0
$f(x_0)$ exists.

(ii) $\lim_{x \to x_0} f(x)$ exists.

i.e) $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$

(iii) $f(x_0) = \lim_{x \to x_0} f(x)$.

Note:-

(i) Constant function is continuous at each point of R.

(ii) Power functions with positive integer exponents are continuous at every point of R.

(iii) Polynomial function.
$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$
are continuous at every point of R.

(iv) $R(x) = \frac{P(x)}{q(x)}$ is cts

(v) The circular function $\sin x$, $\cos x$ are cts at every point of their domain $R = (-\infty, \infty)$

(vi) The n^{th} root function $f(x) = x^{1/n}$ are cts.

(vii) The reciprocal function $f(x) = \frac{1}{x}$ is not cts. on 0.
It is cts $R - \{0\}$.

(viii) e^x is cts on R.

(viii) $f(x) = \log x$ $x > 0$ is cts on $(0, \infty)$.

Algebra of cts functions:-

If f and g are cts at $x = x_0$ then.

(i) $f + g$ is cts at $x = x_0$

(ii) $f - g$ is cts at $x = x_0$

(iv) $f \cdot g$ is cts at $x = x_0$

(v) $\frac{f}{g}$ is cts at $x = x_0$
$g(x) \neq 0$.

EX: 9.4

Evaluate the following limits:

1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{7x}$

$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{7}$ $\left\{\because \lim_{x\to\infty}\left(x+\frac{1}{x}\right)^{x}=e\right.$

$= e^{7}$

2. $\lim_{x\to 0}(1+x)^{\frac{1}{3x}}$ $\left\{\because \lim_{x\to 0}(1+x)^{1/x}=e\right.$

$= \left[\lim_{x\to 0}(1+x)^{1/x}\right]^{1/3} = e^{1/3}$

3. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{m/x}$

Let $A = \lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{m/x}$.

put $\frac{k}{x} = y$

$\frac{1}{x} = \frac{y}{k}$

$(\times m, \cancel{})$ $\frac{m}{x} = \frac{m}{k}\cdot y$

when $x\to\infty$ we have $y\to 0$

$\therefore A = \lim_{y\to 0}(1+y)^{\frac{m}{k}\cdot y}$

$\log A = \log\left[\lim_{y\to 0}(1+y)^{\frac{m}{k}\cdot y}\right]$

$= \lim_{y\to 0}\log(1+y)^{\frac{m}{k}\cdot y}$

$= \lim_{y\to 0}\left[\frac{m}{k}\cdot y\log(1+y)\right]$

$= \frac{m}{k}\times 0\times\log(1+0)$

$\log A = 0$

$A = e^{0}$

$A = 1$

$\therefore \lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{m/x} = 1.$

4. $\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$

Solution:

$\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3} = \lim_{x\to\infty}\left[\frac{2x^2+3+2-2}{2x^2+5}\right]^{8x^2+3+17-17}$

$= \lim_{x\to\infty}\left[\frac{2x^2+5-2}{2x^2+5}\right]^{8x^2+20-17}$

$= \lim_{x\to\infty}\left[\frac{2x^2+5}{2x^2+5} - \frac{2}{2x^2+5}\right]^{4(2x^2+5)-17}$

$= \lim_{x\to\infty}\left[1-\frac{2}{2x^2+5}\right]^{4(2x^2+5)-17}$

Let $y = 2x^2+5$

when $x\to\infty$ $y = 2\times\infty+5 = \infty$

$\therefore x\to\infty$ as $y\to\infty$.

$\therefore \lim_{x\to\infty}\left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3} = \lim_{y\to\infty}\left[1-\frac{2}{y}\right]^{4y-17}$

$= \lim_{y\to\infty}\left[1-\frac{2}{y}\right]^{4y}\times\left[1-\frac{2}{y}\right]^{-17}$

$= \left[\lim_{y\to\infty}\left(1-\frac{2}{y}\right)^{y}\right]^{4}\times\lim_{y\to\infty}\left(1-\frac{2}{y}\right)^{-17}$

$= (e^{-2})^{4}\times\left(1-\frac{2}{\infty}\right)^{17}$ $\left\{\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x} = e^{k}\right.$

$= e^{-8}\times 1-0$

$= e^{-8} = 1/e^{8}$

5. $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x+2}$

$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x+2} = \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}\cdot\left(1+\frac{3}{x}\right)^{2}$

$= e^{3}\times\left(1+\frac{3}{\infty}\right)^{2}$

$= e^{3}(1)$ $\left\{\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x} = e^{k}\right.$

$= e^{3}$

6. $\lim_{x\to 0}\frac{\sin^{3}\frac{x}{2}}{x^{3}}$ $\left\{\because \lim_{x\to 0}\frac{\sin x}{x} = 1.\right.$

$\lim_{x\to 0}\frac{\sin^{3}\frac{x}{2}}{x^{3}} = \lim_{x\to 0}\frac{\sin^{3}\frac{x}{2}}{2^{3}\times\frac{x^{3}}{2^{3}}}$

$= \lim_{x\to 0}\frac{\sin^{3}\frac{x}{2}}{8\left(\frac{x}{2}\right)^{3}} = \frac{1}{8}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^{3}$

$= \frac{1}{8}(1)^{3}$

$= \frac{1}{8}$

7. $\lim_{x\to 0} \frac{\sin \alpha x}{\sin \beta x}$. $\left\{ \lim_{x\to 0} \frac{\sin x}{x} = 1. \right.$

$\lim_{x\to 0} \frac{\sin \alpha x}{\sin \beta x} = \lim_{x\to 0} \frac{\sin \alpha x}{\frac{1}{\alpha}\alpha x} \times \frac{\frac{1}{\beta}(\beta x)}{\sin \beta x}$

$= \frac{\alpha}{\beta} \lim_{x\to 0} \frac{\sin \alpha x}{\alpha x} \times \frac{\beta x}{\sin \beta x}$

$= \frac{\alpha}{\beta} \lim_{x\to 0} \frac{\sin \alpha x}{\alpha x} \times \lim_{x\to 0} \frac{1}{\frac{\sin \beta x}{\beta x}}$

$= \frac{\alpha}{\beta}(1) \times \frac{1}{(1)}$

$= \frac{\alpha}{\beta}$

8. $\lim_{x\to 0} \frac{\tan 2x}{\sin 5x}$ $\left\{ \because \lim_{x\to 0} \frac{\sin x}{x} = 1. \right.$

Solution.

$\lim_{x\to 0} \frac{\tan 2x}{\sin 5x} = \lim_{x\to 0} \frac{\sin 2x}{\cos 2x} \times \frac{1}{\sin 5x}$

$= \lim_{x\to 0} \frac{\sin 2x}{\frac{1}{2}(2x)} \times \frac{1}{\cos 2x} \times \frac{\frac{1}{5}(5x)}{\sin 5x}$

$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{1}{\cos 2x} \times \frac{5x}{\sin 5x}$.

$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x} \lim_{x\to 0} \frac{1}{\cos 2x} \times \lim_{x\to 0} \frac{1}{\frac{\sin 5x}{5x}}$.

$= \frac{2}{5} \times (1) \times \frac{1}{\cos 0} \times \frac{1}{1}$

$= \frac{2}{5}$.

9. $\lim_{\alpha\to 0} \frac{\sin \alpha^n}{(\sin \alpha)^m}$. $\left\{ \lim_{x\to 0} \frac{\sin x}{x} = 1. \right.$

Solution.

$\lim_{\alpha\to 0} \frac{\sin \alpha^n}{(\sin\alpha)^m} = \lim_{\alpha\to 0} \frac{\sin \alpha^n}{\frac{1}{\alpha^n}\alpha^n} \times \frac{\frac{1}{\alpha^m}\times\alpha^m}{(\sin\alpha)^m}$

$= \lim_{\alpha\to 0} \alpha^n \frac{\sin \alpha^n}{\alpha^n} \times \frac{1}{\alpha^m} \times \frac{\alpha^m}{(\sin\alpha)^m}$

$= \lim_{\alpha\to 0} \frac{\alpha^n}{\alpha^m} \frac{\sin \alpha^n}{\alpha^n} \times \frac{1}{\left(\frac{\sin\alpha}{\alpha}\right)^m}$

$= \lim_{\alpha\to 0} \alpha^{n-m} \times \lim_{\alpha\to 0} \frac{\sin \alpha^n}{\alpha^n} \times \frac{1}{\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)^m}$

Case (i) $m = n$.

$\lim_{\alpha\to 0} \frac{\sin \alpha^n}{(\sin\alpha)^n} = \lim_{\alpha\to 0} \alpha^{m-m} \times \lim_{\alpha\to 0} \frac{\sin \alpha^m}{\alpha^m} \times \frac{1}{\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)^m}$

$= \lim_{\alpha\to 0} \alpha^0 \times (1) \times \frac{1}{(1)}$

$= 1 \times 1$

$= 1$.

Case (ii) $m > n$.

$m > n$ then $n - m < 0$.

$\lim_{\alpha\to 0} \frac{\sin \alpha^n}{(\sin\alpha)^m} = \lim_{\alpha\to 0} \alpha^{n-m} \times \lim_{\alpha\to 0} \frac{\sin\alpha^m}{\alpha^m} \times \frac{1}{\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)^m}$

$= \lim_{\alpha\to 0} \frac{1}{\alpha^{m-n}} \times 1 \times 1$

$= \frac{1}{0} \times 1 \times 1$

$= \infty$

Case (iii) $m < n$.

$m < n$ then $n - m > 0$

$\lim_{\alpha\to 0} \frac{\sin \alpha^n}{(\sin\alpha)^m} = \lim_{\alpha\to 0} \alpha^{n-m} \times \lim_{\alpha\to 0} \frac{\sin\alpha^m}{\alpha^m} \times \lim_{\alpha\to 0} \frac{1}{\left(\frac{\sin\alpha}{\alpha}\right)^m}$

$= (0)^{n-m} \times 1 \times 1 = 0$

$\therefore \lim_{\alpha\to 0} \frac{\sin\alpha^n}{(\sin\alpha)^m} = \begin{cases} 1 & \text{if } m = n \\ \infty & \text{if } m > n \\ 0 & \text{if } m < n. \end{cases}$

10. $\lim_{x\to 0} \frac{\sin(a+x) - \sin(a-x)}{x}$.

$\left\{ \sin C - \sin D = 2\cos\frac{C+D}{2} . \sin\frac{C-D}{2} \right.$

$\sin(a+x) - \sin(a-x)$

$= 2\cos\left(\frac{a+x+a-x}{2}\right) \sin\left(\frac{a+x-(a-x)}{2}\right)$

$= 2\cos\left(\frac{2a}{2}\right) \sin\left(\frac{a+x-a+x}{2}\right)$

$= 2\cos a . \sin\frac{2x}{2}$

$= 2\cos a \sin x$.

$\therefore \lim_{x\to 0} \frac{\sin(a+x) - \sin(a-x)}{x}$

$= \lim_{x\to 0} \frac{2\cos a \sin x}{x}$

$= 2\cos a \lim_{x\to 0} \frac{\sin x}{x}$

$= 2\cos a\,(1)$

$= 2\cos a$.

11. $\lim_{x\to 0} \dfrac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b}$ $\quad \left\{\because \lim_{x\to a} \dfrac{x^n-a^n}{x-a} = na^{n-1}\right.$

$\lim_{x\to 0}\left(\dfrac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b}\right) = \lim_{x\to 0}\left[\dfrac{(x^2+a^2)^{1/2}-(a^2)^{1/2}}{(x^2+b^2)^{1/2}-(b^2)^{1/2}}\right]$

$= \lim_{x\to 0} \dfrac{(x^2+a^2)^{1/2}-(a^2)^{1/2}}{(x^2+a^2)-a^2} \times \dfrac{(x^2+b^2)-b^2}{(x^2+b^2)^{1/2}-(b^2)^{1/2}}$

$= \lim_{x\to 0} \dfrac{(x^2+a^2)^{1/2}-(a^2)^{1/2}}{(x^2+a^2)-a^2} \times \lim_{x\to 0} \dfrac{1}{\dfrac{(x^2+b^2)^{1/2}-(b^2)^{1/2}}{(x^2+b^2)-b^2}}$

Let $x^2+a^2=y$ $\quad$ | $x^2+b^2=z$

$x\to 0 \quad y\to a^2$ $\quad$ | $x\to 0$ as $z\to b^2$

$= \lim_{y\to a^2} \dfrac{y^{1/2}-(a^2)^{1/2}}{y-a^2} \times \dfrac{1}{\lim_{z\to b^2}\left(\dfrac{z^{1/2}-(b^2)^{1/2}}{z-b^2}\right)}$

$= \dfrac{1}{2}(a^2)^{\frac{1}{2}-1} \times \dfrac{1}{\frac{1}{2}\times(b^2)^{\frac{1}{2}-1}}$

$= \dfrac{(a^2)^{-1/2}}{(b^2)^{-1/2}} = \dfrac{a^{-1}}{b^{-1}} = \dfrac{b}{a}$

12. $\lim_{x\to 0} \dfrac{2\arcsin x}{3x}$ $\quad \left\{\lim_{x\to 0} \dfrac{\sin^{-1}x}{x} = 1.\right.$

$= \lim_{x\to 0} \dfrac{2}{3}\,\dfrac{\sin^{-1}x}{x}$

$= \dfrac{2}{3}(1) = \dfrac{2}{3}$ //

13. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ $\quad \left\{\because \lim_{x\to 0} \dfrac{\sin x}{x} = 1\right.$

$= \lim_{x\to 0} \dfrac{2\sin^2\frac{x}{2}}{x^2}$ $\quad \left\{1-\cos 2\frac{A}{2} = 2\sin^2\frac{A}{2}\right.$

$= \lim_{x\to 0} \dfrac{2\sin^2\frac{x}{2}}{2^2\times\frac{x^2}{2^2}} = \lim_{x\to 0} \dfrac{2\sin^2\frac{x}{2}}{4\times\frac{x^2}{2^2}}$

$= \dfrac{1}{2}\lim_{x\to 0} \dfrac{\sin^2\frac{x}{2}}{x^2/4}$

$= \dfrac{1}{2}\left[\lim_{x\to 0} \dfrac{\sin x/2}{x/2}\right]^2$

$= \dfrac{1}{2}(1)^2$

$= \dfrac{1}{2}$ //

14. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$ $\quad \left\{\lim_{x\to 0} \dfrac{\sin x}{x} = 1.\right.$

$\lim_{x\to 0} \dfrac{\tan 2x}{x} = \lim_{x\to 0} \dfrac{\sin 2x}{\cos 2x}\times\dfrac{1}{x}$

$= \lim_{x\to 0} \dfrac{\sin 2x}{\frac{1}{2}\times 2x} \times \dfrac{1}{\cos 2x}$

$= 2\lim_{x\to 0} \dfrac{\sin 2x}{2x} \times \lim_{x\to 0}\dfrac{1}{\cos 2x}$

$= 2(1)\times\dfrac{1}{\cos 0}$

$= 2$

15. $\lim_{x\to 0} \dfrac{2^x-3^x}{x}$ $\quad \left\{\because \lim_{x\to 0} \dfrac{a^x-1}{x} = \log a \quad a>0.\right.$

$= \lim_{x\to 0} \dfrac{2^x-1+1-3^x}{x}$

$= \lim_{x\to 0} \dfrac{(2^x-1)-(3^x-1)}{x}$

$= \lim_{x\to 0}\left(\dfrac{2^x-1}{x} - \dfrac{3^x-1}{x}\right)$

$= \lim_{x\to 0} \dfrac{2^x-1}{x} - \lim_{x\to 0}\dfrac{3^x-1}{x}$

$= \log 2 - \log 3$

$= \log\frac{2}{3}$

16. $\lim_{x\to 0} \dfrac{3^x-1}{\sqrt{x+1}-1}$ $\quad \left\{\lim_{x\to 0} \dfrac{a^x-1}{x} = \log a.\right.$

$= \lim_{x\to 0} \dfrac{3^x-1}{\sqrt{x+1}-1} \times \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \lim_{x\to 0} \dfrac{(3^x-1)(\sqrt{x+1}+1)}{x+1-1}$

$= \lim_{x\to 0} \dfrac{(3^x-1)(\sqrt{x+1}+1)}{x}$

$= \lim_{x\to 0}\left(\dfrac{3^x-1}{x}\right) \times \lim_{x\to 0}\left(\sqrt{x+1}+1\right)$

$= \log 3 \times (\sqrt{0+1}+1)$

$= \log 3 + (1+1)$

$= 2\log 3 = \log 3^2 = \log 9.$

17. $\lim_{x\to 0} \frac{1-\cos^2 x}{x \sin 2x}$ $\left\{\because \lim_{x\to 0} \frac{\sin x}{x} = 1.\right.$

$= \lim_{x\to 0} \frac{\sin^2 x}{x\, 2\sin x\cos x}$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x\cos x}.$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \frac{1}{\cos x}$

$= \frac{1}{2} (1)(1)$

$= \frac{1}{2}$

18. $\lim_{x\to\infty} x\left[\cancel{\tfrac{1}{3}}\, 3^{1/x} + 1 - \cos\frac{1}{x} - e^{1/x}\right]$

$\left\{\because \lim_{x\to 0} \frac{e^x-1}{x} = 1,\ \lim_{x\to 0} \frac{a^x-1}{x} = \log a\right.$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$

$\lim_{x\to\infty} x\left[3^{1/x} + 1 - \cos\frac{1}{x} - e^{1/x}\right]$

$= \lim_{x\to\infty}\left[\frac{3^{1/x}+1-\cos 1/x - e^{1/x}}{1/x}\right]$

$= \lim_{x\to\infty}\left[\frac{3^{1/x} - e^{1/x}}{1/x} + \frac{1-\cos 1/x}{1/x}\right]$

$= \lim_{x\to\infty}\left[\frac{3^{1/x}-1+1-e^{1/x}}{1/x} + \frac{1-\cos(\frac{1}{x})}{1/x}\right]$

$= \lim_{x\to\infty}\left[\frac{(3^{1/x}-1)-(e^{1/x}-1)}{1/x} + \frac{1-\cos 1/x}{1/x}\right]$

$= \lim_{x\to\infty}\left[\frac{3^{1/x}-1}{1/x} - \frac{e^{1/x}-1}{1/x} + \frac{1-\cos 1/x}{1/x}\right]$

Let $y = 1/x$ $\quad x\to\infty,\ y = \frac{1}{\infty}$

$y \to 0$

$\therefore \lim_{x\to\infty} x\left[3^{1/x} + 1 - \cos 1/x - e^{1/x}\right]$

$= \lim_{y\to 0}\left[\frac{3^y-1}{y} - \frac{e^y-1}{y} + \frac{1-\cos y}{y}\right]$

$= \lim_{y\to 0} \frac{3^y-1}{y} - \lim_{y\to 0}\frac{e^y-1}{y} + \lim_{y\to 0}\frac{1-\cos y}{y}$

$= \log 3 - 1 + 0$

$= (\log 3) - 1$ //

19. $\lim_{x\to\infty}\left\{x\left[\log(x+a) - \log(x)\right]\right\}$

$\left\{\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.\right.$

$\lim_{x\to\infty}\left\{x\left[\log(x+a) - \log x\right]\right\}$

$= \lim_{x\to\infty} x \log\left(\frac{x+a}{x}\right)$

$= \lim_{x\to\infty} \frac{\log\left(\frac{x}{x} + \frac{a}{x}\right)}{1/x}$

$= \lim_{x\to\infty} \frac{\log\left(1 + a/x\right)}{\frac{1}{a}\times\frac{a}{x}}$

$= a \lim_{x\to\infty} \frac{\log\left(1+\frac{a}{x}\right)}{a/x}$

Let $y = \frac{a}{x}$

$x\to\infty \quad y \to \frac{a}{\infty} = 0$

$\therefore \lim_{x\to\infty}\left\{x\left[\log(x+a) - \log x\right]\right\}$

$= a \lim_{y\to 0} \frac{\log(1+y)}{y}$

$= a(1)$

$= a$

20. $\lim_{x\to\pi} \frac{\sin 3x}{\sin 2x}.$

$= \lim_{x\to\pi} \frac{3\sin x - 4\sin^3 x}{2\sin x\cos x.}$

$= \lim_{x\to\pi}\left[\frac{3\sin x}{2\sin x\cos x} - \frac{\cancel{4}\sin^3 x}{2\sin x\cos x}\right]$

$= \lim_{x\to\pi}\left[\frac{3}{2\cos x} - \frac{2\sin^2 x}{\cos x}\right]$

$= \lim_{x\to\pi} \frac{3}{2\cos x} - \lim_{x\to\pi}\frac{2\sin^2 x}{\cos x}$

$= \frac{3}{2\cos\pi} - \frac{2\sin^2\pi}{\cos\pi}$

$= \frac{3}{2(-1)} - \frac{2(0)}{-1}$

$= \frac{-3}{2}$ //

21. $\lim_{x\to\pi/2}(1+\sin x)^{2\operatorname{cosec}x}$.

$= \lim_{x\to\pi/2}(1+\sin x)^{\frac{2}{\sin x}}$.

Let $y=\sin x$, $x\to\pi/2$, $y=\sin\pi/2$, $y=1$.

$\lim_{x\to\pi/2}(1+\sin x)^{2\operatorname{cosec}x} = \lim_{y\to 1}(1+y)^{2/y} = (1+1)^{2/1}$

$= 2^2$

$= 4$.

22. $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$.

$= \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}\times\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}$

$= \lim_{x\to 0}\frac{(\sqrt{2})^2-(\sqrt{1+\cos x})^2}{(1-\cos^2 x)(\sqrt{2}+\sqrt{1+\cos x})}$

$= \lim_{x\to 0}\frac{2-1-\cos x}{(1^2-\cos^2 x)(\sqrt{2}+\sqrt{1+\cos x})}$

$= \lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)(\sqrt{2}+\sqrt{1+\cos x})}$

$= \lim_{x\to 0}\frac{1}{(1+\cos x)(\sqrt{2}+\sqrt{1+\cos x})}$

$= \frac{1}{1+\cos 0\,(\sqrt{2}+\sqrt{1+\cos 0})}$

$= \frac{1}{1+1\,(\sqrt{2}+\sqrt{2})}$

$= \frac{1}{2\,(2\sqrt{2})}$

$= \frac{1}{4\sqrt{2}}$

23. $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$.

$= \lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$.

$= \lim_{x\to 0}\frac{1+\sin x-1+\sin x}{\frac{\sin x}{\cos x}\times(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$

$= \lim_{x\to 0}\frac{\cos x\times 2\sin x}{\sin x\,[\sqrt{1+\sin x}+\sqrt{1-\sin x}]}$

$= \lim_{x\to 0}\frac{2\cos x}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$

$= \frac{2\cos 0}{\sqrt{1+\sin 0}+\sqrt{1-\sin 0}}$

$= \frac{2(1)}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{1+1} = \frac{2}{2} = 1$.

24. $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$.

$= \lim_{x\to\infty}\left[\frac{x^2-2x+1+2x-2x}{x^2-4x+2}\right]^x$

$= \lim_{x\to\infty}\left[\frac{x^2-4x+2+2x-1}{x^2-4x+2}\right]^x$

$= \lim_{x\to\infty}\left[\frac{x^2-4x+2}{x^2-4x+2}+\frac{2x-1}{x^2-4x+2}\right]^x$

$= \lim_{x\to\infty}\left[1+\frac{2x-1}{x^2-4x+2}\right]^x$

$= \lim_{x\to\infty}\left[1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right]^{\frac{x^2-4x+2}{2x-1}\times\frac{(2x-1)x}{x^2-4x+2}}$

$= \lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right)^{\frac{x^2-4x+2}{2x-1}}\right]^{\frac{(2x-1)x}{x^2-4x+2}}$

$= \left[\lim_{x\to\infty} e\right]^{\frac{(2x-1)x}{x^2-4x+2}}$

$= e^{\lim_{x\to\infty}\frac{2x^2-x}{x^2-4x+2}}$

$= e^{\lim_{x\to\infty}\frac{x^2(2-x/x^2)}{x^2\left(1-\frac{4x}{x^2}+\frac{2}{x^2}\right)}}$

$= e^{\lim_{x\to\infty}\left(\frac{2-1/x}{1-\frac{4}{x}+\frac{2}{x^2}}\right)}$

$= e^{\frac{2-1/\infty}{1-\frac{4}{\infty}+\frac{2}{\infty}}} = e^{\left(\frac{2-0}{1-0}\right)}$

$= e^2$

25. $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$ $\left\{ \because \lim_{x\to 0} \frac{e^x-1}{x} = 1 \right.$ $\lim_{x\to 0}\frac{\sin x}{x} = 1.$

$= \lim_{x\to 0} \frac{e^x - 1/e^x}{\sin x}$

$= \lim_{x\to 0} \frac{\frac{e^x \cdot e^x - 1}{e^x}}{\sin x} = \lim_{x\to 0} \frac{e^{2x}-1}{e^x \sin x}$

$= \lim_{x\to 0} \left[\frac{1}{e^x} \times \frac{e^{2x}-1}{\frac{1}{2}\times 2x} \times \frac{x}{\sin x} \right]$

$= \lim_{x\to 0} \frac{1}{e^x} \times 2 \lim_{x\to 0} \frac{e^{2x}-1}{2x} \times \lim_{x\to 0} \cancel{\frac{1}{\frac{\sin x}{x}}}$

$= \frac{1}{e^0} \times 2 \times (1) \times \frac{1}{(1)}$

$= 2$

26. $\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{x}$ $\left\{ \because \lim_{x\to 0} \frac{e^x-1}{x} = 1. \right.$

$= \lim_{x\to 0} \frac{e^{ax} - 1 + 1 - e^{bx}}{x}$

$= \lim_{x\to 0} \left[\left(\frac{e^{ax}-1}{x} \right) - \left(\frac{e^{bx}-1}{x} \right) \right]$

$= \lim_{x\to 0} \left(\frac{e^{ax}-1}{\frac{1}{a}(ax)} \right) - \lim_{x\to 0} \frac{e^{bx}-1}{\frac{1}{b}(bx)}$

$= a \lim_{x\to 0} \frac{e^{ax}-1}{(ax)} - b \lim_{x\to 0} \frac{e^{bx}-1}{(bx)}$

$= a \times 1 - b \times 1$

$= a - b$

27. $\lim_{x\to 0} \frac{\sin x (1-\cos x)}{x^3}$ $\left\{ \lim_{x\to 0} \frac{\sin x}{x} = 1. \right.$

$\lim_{x\to 0} \frac{\sin x (1-\cos x)}{x^3}$

$= \lim_{x\to 0} \frac{\sin x \times 2 \sin^2 \frac{x}{2}}{x^3}$

$= \lim_{x\to 0} \frac{\sin x}{x} \times 2 \frac{\sin^2 x/2}{x^2}$

$= \lim_{x\to 0} \frac{\sin x}{x} \times 2 \frac{\sin^2 \frac{x}{2}}{2^2 \times \frac{x^2}{2^2}}$

$= \lim_{x\to 0} \frac{\sin x}{x} \times \frac{1}{2} \left(\lim_{x\to 0} \frac{\sin \frac{x}{2}}{x/2} \right)^2$

$= 1 \times 1/2 \times 1$

$= 1/2.$

28. $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$ $\left\{ \lim_{x\to 0} \frac{\sin x}{x} = 1. \right.$

$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$

$= \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{x^3 \cos x.}$

$= \lim_{x\to 0} \frac{\sin x (1-\cos x)}{x \cdot x^2 \cos x.}$

$= \lim_{x\to 0} \frac{\sin x}{x} \frac{2\sin^2 x/2}{x^2} \times \frac{1}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \times \frac{2\sin^2 x/2}{2^2 \times \frac{x^2}{2^2}} \times \frac{1}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \times \frac{1}{2} \left[\lim \frac{\sin x/2}{x/2} \right]^2 \times \lim_{x\to 0} \frac{1}{\cos x}$

$= 1 \times \frac{1}{2} \times 1^2 \times \frac{1}{\cos 0}$

$= \frac{1}{2} \times 1$

$= \frac{1}{2}.$

K.S. M.

Ex: 9.5

1. Prove that $f(x) = 2x^2+3x-5$ is continuous at all points in R.

Solution:

$f(x) = 2x^2+3x-5$.

Clearly $f(x)$ is defined for all points of R. Let x_0 be an arbitrary point in R

Then $f(x_0) = 2x_0^2+3x_0-5 \rightarrow (1)$

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} (2x^2+3x_0-5)$

$= 2x_0^2+3x_0-5 \rightarrow (2)$

from (1) & (2)

$\lim_{x\to x_0} f(x) = f(x_0)$.

$\therefore$ f(x) is defined at all points of R. limit of f(x) exist at all points of R. and is equal to the value of the function f(x).

$\therefore$ f(x) is continuous at all points of R.

2. (i) $x+\sin x$.

Let $f(x) = x+\sin x$

f(x) is defined at all points of R.

Let x_0 be an arbitrary point in R

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} (x+\sin x)$.

$= x_0+\sin x_0 \rightarrow (1)$

$f(x_0) = x_0+\sin x_0 \rightarrow (2)$

from (1) & (2)

$\lim_{x\to x_0} f(x) = f(x_0)$

$\therefore$ At all points of R the limit of f(x) exist and is equal to the value of the function.

f(x) Satisfies all condition for continuity

$\therefore$ f(x) is continuous at all points of f(x).

(ii) $x^2\cos x$.

Let $f(x) = x^2\cos x$

f(x) is defined at all points of R.

Let x_0 be an arbitrary points in R

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x^2\cos x$

$= x_0^2\cos x_0 \rightarrow (1)$

$f(x_0) = x_0^2\cos x_0 \rightarrow (2)$

from (1) & (2)

$\lim_{x\to x_0} x^2\cos x = f(x_0)$.

$\therefore$ The limit is exist.

f(x) Satisfies all condition for continuity.

f(x) is a continuous function in R.

(iii) $e^x\tan x$.

Let $f(x) = e^x\tan x$.

f(x) is defined for all points of R. except at $(2n-1)\frac{\pi}{2}$, $n\in R$.

Let x_0 be an arbitrary point in $R-(2n+1)\frac{\pi}{2}$, $n\in Z$.

$\lim_{x\to x_0} f(x_0) = \lim_{x\to x_0} e^x\tan x$

$= e^{x_0}\tan x_0 \rightarrow (1)$

$f(x_0) = e^{x_0}\tan x_0 \rightarrow (2)$.

from (1) & (2)

$\lim_{x\to x_0} f(x) = f(x_0)$

$\therefore$ limit is exist.

$R-(2n+1)\frac{\pi}{2}$, $n\in Z$ is equal to the value of the function f(x) at the points.

$\therefore$ f(x) satisfies all conditions for continuity.

Hence f(x) is continuous at all points of $R-(2n+1)\frac{\pi}{2}$, $n\in Z$.

(v) $x\log x$.

Let $f(x) = x\log x$.

The function f(x) is defined in the open interval $(0,\infty)$. Since $\log x$ is defined for $x>0$. Let $x_0\in(0,\infty)$

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x\log x = x_0\log x_0 \rightarrow (1)$

$f(x_0) = x_0\log x_0 \rightarrow (2)$

$\therefore \lim_{x\to x_0} f(x) = f(x_0)$.

limit of the function f(x) exist at $x=x_0$.

$\therefore$ f(x) is cts at all points of $(0,\infty)$

(vi) $\frac{\sin x}{x^2}$.

$f(x) = \frac{\sin x}{x^2}$

f(x) is not defined at $x=0$

$\therefore$ f(x) is defined all points of

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{\sin x}{x^2}$$

$$= \frac{\sin x_0}{x_0^2} \longrightarrow (1)$$

$$f(x_0) = \frac{\sin x_0}{x_0^2} \longrightarrow (2)$$

from (1) & (2)

$$\lim_{x \to x_0} f(x) = f(x_0)$$

∴ The limit of the function $f(x)$ exist at $x = x_0$

∴ $f(x)$ is continuous at all points of $R - \{0\}$.

(vii) $\frac{x^2-16}{x+4}$

Let $f(x) = \frac{x^2-16}{x+4}$

$f(x)$ is not defined at $x = -4$.

∴ $f(x)$ is defined for all points of $R - \{-4\}$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{x^2-16}{x+4} = \frac{x_0^2-16}{x_0+4} \rightarrow (1)$$

$$f(x_0) = \frac{x_0^2-16}{x_0+4} \longrightarrow (2)$$

from (1) & (2)

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Limit function is exist $\forall\ x_0 \in R - \{-4\}$.

∴ $f(x)$ is cts all points $R - \{-4\}$.

(viii) $|x+2| + |x-1|$

Let $f(x) = |x+2| + |x-1|$.

$f(x)$ is defined for all points of R.

Let x_0 be an arbitrary point in R. Then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(|x+2| + |x-1| \right)$$

$$= |x_0+2| + |x_0-1| \rightarrow (1)$$

$$f(x_0) = |x_0+2| + |x_0-1| \rightarrow (2)$$

from (1) & (2)

$$f(x_0) = \lim_{x \to x_0} f(x).$$

the limit of the function is exist.

$f(x)$ is cts at all points of R.

ix. $\frac{|x-2|}{|x+1|}$.

Let $f(x) = \frac{|x-2|}{|x+1|}$

$f(x)$ is defined for all points of R except at $x = -1$.

∴ $f(x)$ is defined $R - \{-1\}$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{|x-2|}{|x+1|} = \frac{|x_0-2|}{|x_0+1|} \rightarrow (1)$$

$$f(x_0) = \frac{|x_0-2|}{|x_0+1|} \longrightarrow (2)$$

from (1) & (2) $\lim_{x \to x_0} f(x) = f(x_0)$

∴ limit is exist. at $x = x_0$.

The result is true for all points $R - \{-1\}$.

∴ $f(x)$ is cts $\forall\ x \in R - \{-1\}$

(x) $\cot x + \tan x$.

$f(x) = \cot x + \tan x$.

$f(x)$ is not defined at $x = \frac{n\pi}{2}$ $n \in z$.

∴ $f(x)$ is defined for all points of $R - \{\frac{n\pi}{2}, n \in z\}$.

Let x_0 be an arbitrary point in $R - \{\frac{n\pi}{2}\}$, $n \in z$.

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (\cot x + \tan x)$$

$$= \cot x_0 + \tan x_0.$$

$$f(x_0) = \cot x_0 + \tan x_0$$

$$\therefore f(x_0) = \lim_{x \to x_0} f(x).$$

∴ The limit of the function $f(x)$ exists at $x = x_0$.

It is true $\forall\ x_0 \in R - \{\frac{n\pi}{2}\}\ n \in z$

∴ $f(x)$ is continuous at all points of $R - \{\frac{n\pi}{2}\}, n \in z$.

3. Find the points of discontinuity of the function f. where.

(i) $f(x) = \begin{cases} 4x+5 & \text{if } x \le 3 \\ 4x-5 & \text{if } x > 3. \end{cases}$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (4x+5) = 4\times3+5$

$= 17 \to (1)$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4x-5) = 4(3)-5$

$= 12-5 = 7 \to (2)$

from (1) & (2)

$\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Hence $f(x)$ is not continuous at $x = 3$.

$\therefore x = 3$ is the point of discontinuity.

(ii) $f(x) = \begin{cases} x+2 & \text{if } x \ge 2 \\ x^2 & \text{if } x < 2 \end{cases}$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = (2)^2 = 4.$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x+2 = 2+2 = 4$

and $f(2) = 2+2 = 4.$

$\therefore \lim_{x\to 2} f(x) = f(2).$

$\therefore f(x)$ is continuous at all points in R.

(iii) $f(x) = \begin{cases} x^3 - 3 & x \le 2 \\ x^2+1 & x > 2 \end{cases}$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^3-3 = 2^3-3 = 8-3 = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2+1 = 2^2+1 = 4+1 = 5$

$\lim_{x\to 2} f(x) = 5.$

$f(2) = 2^2+1 = 5.$

$\therefore \lim_{x\to 2} f(x) = f(x)$

$\therefore f(x)$ is cts at all points in R.

(iv) $f(x) = \begin{cases} \sin x & 0 \le x \le \pi/4 \\ \cos x & \pi/4 < x < \pi/2. \end{cases}$

Clearly $f(x)$ is defined at all points of $[0, \pi/2)$

Case (i):

Let $x_0 \in [0, \pi/4]$.

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \sin x = \sin x_0$

$f(x_0) = \sin x_0.$

$\therefore \lim_{x\to x_0} f(x) = f(x_0).$

$f(x)$ is cts at $x = x_0$. $x_0 \in [0, \frac{\pi}{4}]$

Case (ii):

Let $x_0 \in \left(\frac{\pi}{4}, \pi/2\right)$

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \cos x. = \cos x_0.$

$f(x_0) = \cos x_0$

$\lim_{x\to x_0} f(x) = f(x_0)$

$\therefore f(x)$ is cts at $x = x_0$

Hence $f(x)$ is cts at all points $[0, \pi/2)$.

4. $x_0 = 1$. $f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \ne 1. \\ 2 & x = 1. \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} =$

$= \lim_{x\to 1^-} \frac{(x+1)(x-1)}{(x-1)}$

$= \lim_{x\to 1^-} (x+1) = 1+1 = 2$

$\lim_{x\to 1^+} f(x) = 1+1 = 2.$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$

Hence $\lim_{x\to 1} f(x) = 2. \to (1)$

$f(1) = 2 \to (2)$

$\therefore \lim_{x\to 1} f(x) = f(1)$

$\therefore f(x)$ is continuous at $x_0 = 1$.

(ii) Solution:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{\cancel{(x-3)}(3+x)}{\cancel{(x-3)}}$$

$$= \lim_{x\to 3^-} (x+3) = 3+3 = 6 \to (1)$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x+3) = 3+3 = 6$$

$$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 6.$$

$$\therefore \lim_{x\to 3} f(x) = 6. \to (1)$$

$$f(3) = 5 \to (2)$$

$$\therefore \lim_{x\to 3} f(x) \neq f(3)$$

$\therefore f(x)$ is not continuous at $x_0=3$.

(5)

$$f(x) = \begin{cases} \dfrac{x^3-1}{x-1}, & \text{if } x \neq 1 \\ 3 & \text{if } x = 1. \end{cases}$$

Clearly, the given function $f(x)$ is defined at all points of R.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3-1^3}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1} (x^2+x+1) = 1+1+1 = 3.$$

$$\therefore \lim_{x\to 1} \frac{x^3-1}{x-1} = 3,$$

$$f(1) = 3$$

$$\therefore f(1) = \lim_{x\to 1} f(x).$$

$\therefore f(x)$ is cts at all points R

6.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^4-1^4}{x-1} = \frac{(x^2)^2-(1^2)^2}{x-1}$$

$$= \lim_{x\to 1} \frac{(x^2+1)(x^2-1^2)}{(x-1)}$$

$$= \lim_{x\to 1} \frac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x\to 1} (x^2+1)(x+1) = (1+1)(1+1)$$

$$= (2)(2)$$

$$\lim_{x\to 1} \frac{x^4-1}{x-1} = 4.$$

$$f(1) = \alpha.$$

Since $f(x)$ is cts.

$$\therefore \lim_{x\to 1} f(x) = f(1)$$

$$4 = \alpha.$$

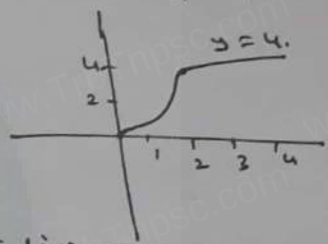
$$\therefore \boxed{\alpha = 4}$$

7. Solution:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 4 & \text{if } x \ge 2. \end{cases}$$

Let $y = f(x)$.

x	-1	0	1	2	3	4	5
$f(x)$	0	0	1	4	4	4	4.



$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$$

$$f(0) = 0^2 = 0.$$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

It is cts at $x=0$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4.$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4 = 4.$$

$$\therefore \lim_{x\to 2} f(x) = f(2).$$

$\therefore f(x)$ is cts at all points.

8. given f, g are cts

$\therefore \lim_{x\to 3} f(x) = f(3) \to (1)$

$\lim_{x\to 3} g(x) = g(3) \to (2)$

given $f(3) = 5$. $\lim_{x\to 3}\left[2f(x) - g(x)\right] = 4$.

$\Rightarrow \lim_{x\to 3} 2f(x) - \lim_{x\to 3} g(x) = 4$.

$2f(3) - g(3) = 4$.

$2(5) - g(3) = 4$.

$10 - 4 = g(3)$

$\boxed{g(3) = 6}$

9. $f(x) = \begin{cases} 2x+1 & \text{if } x \le -1 \\ 3x & \text{if } -1 \le x < 1 \\ 2x-1 & \text{if } x \ge 1. \end{cases}$

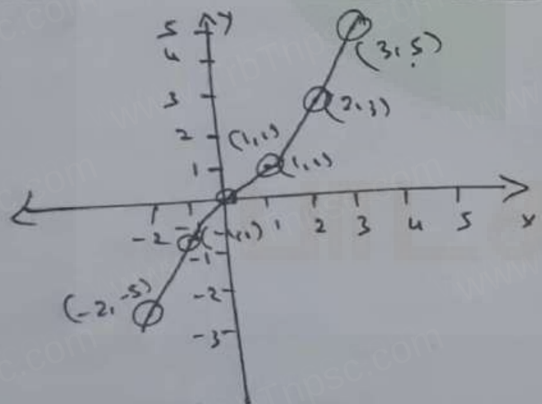
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x = 3(1) = 3$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 2(1)-1 = 2-1 = 1$

$\therefore f(x)$ is not cts at $x=1$.

graph:

x	-2	-1	0	1	2	3
y	$2x+1$	$2x+1$	$3x$	$2x-1$	$2x-1$	$2x-1$
	-3	-1	0	1	3	5



(ii) $f(x) = \begin{cases} (x-1)^3 & \text{if } x < 0 \\ (x+1)^3 & \text{if } x \ge 0 \end{cases}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-1)^3 = (0-1)^3$

$= -1 \to (1)$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1)^3 = 0+1 = 1$

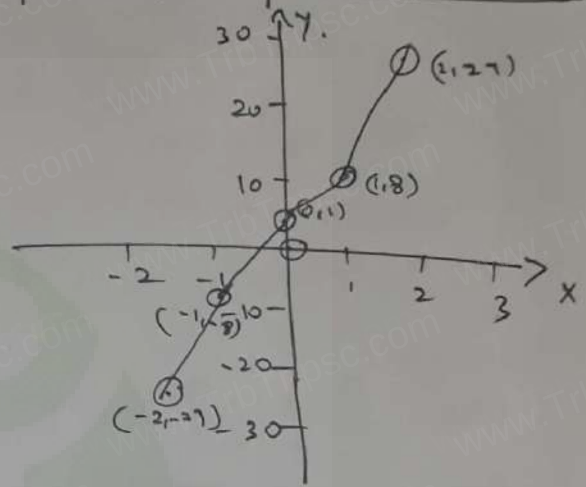
$\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

$f(x)$ is not continuous at $x=0$

graph:

x	-1	-2	0	1	2
y	$(x-1)^3$	$(x-1)^3$	$(x+1)^3$	$(x+1)^3$	$(x+1)^3$
	-8	-27	1	8	27



10. $f(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x < 1 \\ -x^2+4x-2 & 1 \le x < 3 \\ 4-x & x \ge 3 \end{cases}$

(i) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$

$\therefore \lim_{x\to 0} f(x) = 0$

$f(0) = 0$

Hence $\lim_{x\to 0} f(x) = f(0)$.

$\therefore f(x)$ is cts at $x=0$.

(ii). $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = +1$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -x^2+4x-2 = -(1)^2+4\times 1-2$
$= 4-3 = 1$.

$\therefore \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

Hence $\lim_{x\to 1} f(x) = 1$.

$f(1) = (-1)^2 + (4\times 1) - 2$
$= -1+4-2 = 1$.

$\lim_{x\to 1} f(x) = f(1)$

$\therefore f(x)$ is cts at $x=1$

(iii)

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-x^2+4x-2)$

$= (-3)^2+4(-3)-2 = -9 + 12-2$

$= 12-11 = 1.$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4-x) = 4-3 = 1.$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 1.$

Hence $\lim_{x\to 3} f(x) = 1 \rightarrow (1)$

$f(3) = 4-3 = 1.$

$\therefore \lim_{x\to 3} f(x) = f(3)$

$\therefore f(x)$ is cts at $x=3$

Hence $f(x)$ is cts at $x = 0,1,3.$

11. Solution.

$f(x) = \dfrac{x^2-2x-8}{x+2}$, $x_0 = -2.$

$f(x)$ is not defined at $x=-2.$

$\lim_{x\to -2} \dfrac{x^2-2x-8}{x+2} = \lim_{x\to -2} \dfrac{(x+2)(x-4)}{(x+2)}$

$= \lim_{x\to -2} (x-4) = -2-4 = -6.$

$\therefore \lim_{x\to -2} \dfrac{x^2-2x-8}{x+2}$ exists.

Redefine the function $f(x)$ as

$g(x) = \begin{cases} \dfrac{x^2-2x-8}{x+2} & \text{if } x \neq -2 \\ -6 & \text{if } x=-2 \end{cases}$

$\therefore f(x)$ has a removable discontinuity at $x=-2$

Clearly $g(x)$ is continuous on R.

(ii) $f(x) = \dfrac{x^3+64}{x+4}$, $x_0 = -4.$

The function $f(x)$ is not defined at $x=-4.$

$f(x) = \dfrac{x^3+4^3}{(x+4)} = \dfrac{(x+4)(x^2-4x+16)}{(x+4)}$

$\lim_{x\to -4} f(x) = \lim_{x\to -4} (x^2-4x+16)$

$= (-4)^2-4(-4)+16$

$= 16+16+16 = 48.$

Limit the function $f(x)$ exist at $x=-4.$

$\therefore$ The function $f(x)$ has a removable discontinuity at $x=-4.$

Redefine the function $f(x)$ as

$g(x) = \begin{cases} \dfrac{x^3+64}{x+4} & \text{if } x \neq -4 \\ 48 & \text{if } x=-4. \end{cases}$

Clearly $g(x)$ is cts on R.

(iii) $f(x) = \dfrac{3-\sqrt{x}}{9-x}$, $x_0 = 9.$

The function $f(x)$ is not defined at $x=9.$

$\lim_{x\to 9} f(x) = \lim_{x\to 9} \dfrac{3-\sqrt{x}}{9-x}$

$= \lim_{x\to 9} \dfrac{3-\sqrt{x}}{3^2-(\sqrt{x})^2} = \lim_{x\to 9} \dfrac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})}$

$= \lim_{x\to 9} \dfrac{1}{3+\sqrt{x}} = \dfrac{1}{3+\sqrt{9}} = \dfrac{1}{3+3} = \dfrac{1}{6}$

$\therefore$ limit of the function $f(x)$ exist at $x=9.$

Hence the function $f(x)$ has

12. $g(x) = \begin{cases} x^2-b^2 & \text{if } x<4 \\ bx+20 & \text{if } x\geq 4. \end{cases}$

Given g is cts on R.

$\therefore g$ is cts at $x=4.$

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x)$

$\lim_{x\to 4^-} x^2-b^2 = \lim_{x\to 4^+} (bx+20)$

$4^2-b^2 = 4b+20$

$b^2+4b+20-16=0$

$b^2+4b+4=0$

$(b+2)^2=0$

$b+2=0$

$\boxed{b=-2}$

13\. Solution:

$f(x) = x \sin \frac{\pi}{x}$

Define $f(x)$ on a R as

$$f(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$\therefore f(0) = 0$ Then $f(x)$ is cts on R.

14\. $f(x) = \dfrac{x^2-1}{x^3-1}$

$f(x)$ is not defined at $x=1$.

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x^3-1^3}$$

$$= \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)(x^2+x+1)}$$

$$= \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{1+1}{1^2+1+1}$$

$$\lim_{x\to 1} f(x) = \frac{2}{3}$$

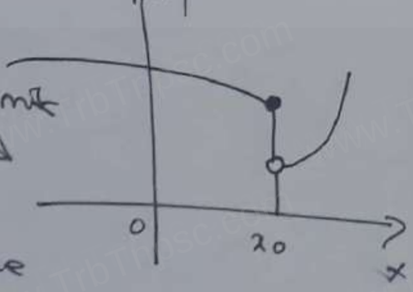
The function $f(x)$ has a removable discontinuity at $x=1$.

Re define $f(x)$ as

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^3-1} & \text{if } x \neq 1 \\ \dfrac{2}{3} & \text{if } x = 1. \end{cases}$$

$\therefore f(1) = \frac{2}{3}$ then $f(x)$ will be cts at $x=1$.

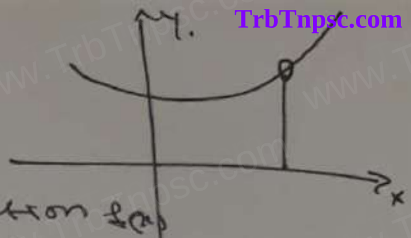
15\. (a)



Left-hand limit and right hand limit does not coincide at $x=x_0$.

It is not continuous.

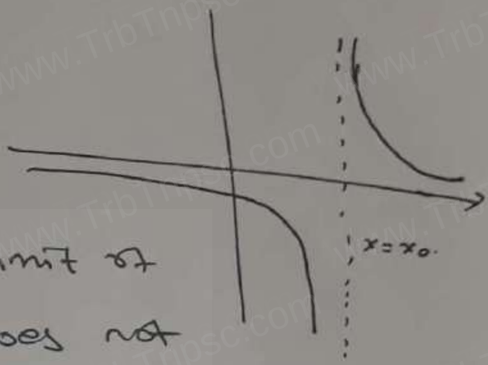
(b)



The function $f(x)$ is not defined at $x=x_0$.

it is not cts.

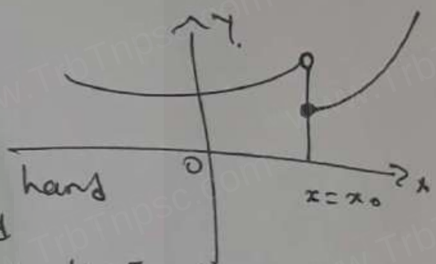
(c)



The limit of $f(x)$ does not exist at $x=x_0$.

It is not cts.

(d)



The left hand limit and right hand limit does not coincide at $x=x_0$.

K. S —

K. SENTHIL. M.Sc, B.Ed.
JOHN BRITTO MATRIC
HR. SEC. SCHOOL
OMALUR-(Tk), SALEM(Dt)
Cell - 9940765925