प्रतिलोम वृत्तीय फलन

Ex 2.1

प्रश्न 1. निम्नलिखित कोणों के मुख्य मान ज्ञात कीजिए:

(i) $\sin^{-1}(1)$ **(ii)** $\cos^{-1}\left(-\frac{1}{2}\right)$

(iii) $\sec^{-1}\left(-\sqrt{2}\right)$ **(iv)** $\operatorname{cosec}^{-1}(1)$

(v) $\cot^{-1}\left(-\sqrt{\frac{1}{3}}\right)$ **(vi)** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

हल : (i) $\sin^{-1}(1)$

$\sin^{-1}$ की मुख्य मान शाखा $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ है।

माना $\sin^{-1}(1) = x$

$\Rightarrow \sin x = 1$

$$\Rightarrow \quad \sin x = \sin\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{2}$$

$\therefore \sin^{-1}(1)$ का मुख्य मान $= \frac{\pi}{2}$

(ii) $\cos^{-1}\left(-\frac{1}{2}\right)$

$\cos^{-1}$ की मुख्य मान शाखा $[0, \pi]$ है।

माना $\cos^{-1}\left(-\frac{1}{2}\right) = x$

$$\Rightarrow \quad \cos x = -\frac{1}{2} = -\cos\frac{\pi}{3}$$

$$\Rightarrow \quad \cos x = \cos\left[\pi - \frac{\pi}{3}\right] = \cos\frac{2\pi}{3}$$

जहाँ $\frac{2\pi}{3} \in [0, \pi]$

$$\Rightarrow \quad x = \frac{2\pi}{3}$$

$\therefore \cos^{-1}\left(-\frac{1}{2}\right)$ का मुख्य मान $= \frac{2\pi}{3}$

(iii) $\sec^{-1}(-\sqrt{2})$

sec^{-1} का मुख्य मान शाखा $[0, \pi] - \{\frac{\pi}{2}\}$ है।

माना $\sec^{-1}(-\sqrt{2}) = x$

$\Rightarrow$ $\sec x = -\sqrt{2} = -\sec\frac{\pi}{4}$

$\Rightarrow$ $\sec x = \sec\left(\pi - \frac{\pi}{4}\right) = \sec\frac{3\pi}{4}$

जहाँ $\frac{3\pi}{4} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\Rightarrow$ $x = \frac{3\pi}{4}$

$\therefore \sec^{-1}(-\sqrt{2})$ का मुख्य मान $= \frac{3\pi}{4}$

(iv) $\text{cosec}^{-1}(-1)$

cosec^{-1} की मुख्य मान शाखा $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ है।

माना $\text{cosec}^{-1}(-1) = x$

$\Rightarrow$ $\text{cosec}\, x = -1 = -\text{cosec}\frac{\pi}{2}$

$\Rightarrow$ $x = -\frac{\pi}{2}$

$\text{cosec}^{-1}(1)$ का मुख्य मान $= -\frac{\pi}{2}$

(v) $\cot^{-1}(-\frac{1}{\sqrt{3}})$

$\cot^{-1}$ का मुख्य मान शाखा $[0, \pi]$ है।

माना $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = x$

$\Rightarrow$ $\cot x = -\frac{1}{\sqrt{3}} = \cot\left(\pi - \frac{\pi}{3}\right)$

$\Rightarrow$ $x = \frac{2\pi}{3}$ जहाँ $\frac{2\pi}{3} \in [0, \pi]$

$\therefore \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ का मुख्य मान $= \frac{2\pi}{3}$

(vi) $\tan^{-1}(\frac{1}{\sqrt{3}})$

$\tan^{-1}$ की मुख्य मान शाखा $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ है।

माना $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = x$

$\Rightarrow \quad \tan x = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6}$ जहाँ $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ का मुख्य मान $= \frac{\pi}{6}$

प्रश्न 2.

$$2\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$$

हल : LHS

$= 2\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} - \tan^{-1}\frac{1}{7}$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$= \tan^{-1}\frac{1}{1 - \frac{1}{4}} - \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{4}{3} - \frac{1}{7}}{1 + \frac{4}{3} \times \frac{1}{7}}$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$= \tan^{-1}\frac{28-3}{21+4} = \tan^{-1}\frac{25}{25}$

$$= 2\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} - \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{1-\frac{1}{4}} - \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\times\frac{1}{7}}$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\frac{28-3}{21+4} = \tan^{-1}\frac{25}{25}$$

इति सिद्धम्।

प्रश्न 3.

$$\tan^{-1}\frac{17}{19} - \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{1}{7}$$

हल : LHS

$$= \tan^{-1}\frac{17}{19} - \tan^{-1}\frac{2}{3}$$

$$= \tan^{-1}\frac{\frac{17}{19}-\frac{2}{3}}{1+\frac{17}{19}\times\frac{2}{3}}$$

$$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1} \frac{\frac{51-38}{19 \times 3}}{\frac{57+34}{19 \times 3}}$$

$$= \tan^{-1} \frac{13}{91} = \tan^{-1} \frac{1}{7}$$

$= R.H.S.$

इति सिद्धम्।

प्रश्न 4.

$$\cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} = \sin^{-1} \frac{3}{5}$$

हल : LHS

$$= \cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5}$$

$$= \cos^{-1} \frac{63}{65} + \cos^{-1} \left(\frac{1-\left(\frac{1}{5}\right)^2}{1+\left(\frac{1}{5}\right)^2} \right)$$

$$\left[\because 2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right]$$

$$= \cos^{-1} \frac{63}{65} + \cos^{-1} \left(\frac{\frac{24}{25}}{\frac{26}{25}} \right)$$

$$= \cos^{-1} \frac{63}{65} + \cos^{-1} \frac{12}{13}$$

$$= \cos^{-1} \left[\frac{63 \times 12}{65 \times 13} - \sqrt{\left\{1-\left(\frac{63}{65}\right)^2\right\}\left\{1-\left(\frac{12}{13}\right)^2\right\}} \right]$$

$$\left[\because \cos^{-1}x+\cos^{-1}y=\left[xy-\sqrt{\left(1-x^2\right)\left(1-y^2\right)}\right]\right]$$

$$= \cos^{-1}\left[\frac{63\times 12}{65\times 13}-\frac{\sqrt{\left\{(65)^2-(63)^2\right\}\{169-144\}}}{65\times 13}\right]$$

$$= \cos^{-1}\left[\frac{63\times 12}{65\times 13}-\frac{\sqrt{128\times 2\times 25}}{65\times 13}\right]$$

$$= \cos^{-1}\left[\frac{63\times 12-2\times 8\times 5}{65\times 13}\right]$$

$$= \cos^{-1}\left[\frac{756-80}{65\times 13}\right]$$

$$= \cos^{-1}\left[\frac{676}{65\times 13}\right]$$

$$= \cos^{-1}\frac{4}{5}$$

$$= \sin^{-1}\sqrt{1-\left(\frac{4}{5}\right)^2} \qquad \left[\because\ \cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\sqrt{\frac{25-16}{25}}$$

$$= \sin^{-1}\sqrt{\frac{9}{25}}$$

$$= \sin^{-1}\frac{3}{5}$$

= R.H.S.

इति सिद्धम्।

प्रश्न 5. $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3) = 15$

हल : माना $\tan^{-1}2 = \theta \Rightarrow \tan\theta = 2$

$\therefore \sec^2\theta = 1 + \tan^2\theta$

$= 1 + (2)^2 = 1 + 4 = 5$

∴ $\sec^2(\tan^{-1}2) = 5$ …(i)

माना $\cot^{-1}3 = \Phi$

⇒ $\cot \Phi = 3$

∴ $\text{cosec}^2 \Phi = 1 + \cot^2 \Phi$

$= 1 + (3)^2 = 1 + 9 = 10$

∴ $\text{cosec}^2(\cot^{-1}3) = 10$ …(ii)

(i) और (ii) को जोड़ने पर

$\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3) = 5 + 10$

$\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3) = 15.$

इति सिद्धम्

प्रश्न 6.

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$$

हल : (i) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$

माना $\tan^{-1}x = \theta$

∴ $x = \tan\theta$

$$\text{R.H.S.} = \sin^{-1}\frac{2x}{1+x^2}$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

$$= \text{L.H.S.}$$

(ii) $$\text{R.H.S.} = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x$$

$$= \text{L.H.S.}$$

अतः $$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$$

$$= \cos^{-1}\frac{1-x^2}{1+x^2}$$

इति सिद्धम्।

प्रश्न 7.

$$\tan^{-1}\sqrt{\frac{ax}{bc}} + \tan^{-1}\sqrt{\frac{bx}{ca}} + \tan^{-1}\sqrt{\frac{cx}{ab}} = \pi$$

जहाँ $a + b + c = x$

हल : LHS

$$= \tan^{-1}\left(\frac{ax}{bc}\right) + \tan^{-1}\left(\frac{bx}{ca}\right) + \tan^{-1}\left(\frac{cx}{ab}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{\frac{ax}{bc}} + \sqrt{\frac{bx}{ca}} + \sqrt{\frac{ca}{ab}} - \sqrt{\frac{ax}{bc}} \cdot \sqrt{\frac{bx}{cx}} \cdot \sqrt{\frac{cx}{ab}}}{1 - \sqrt{\frac{ax}{bc}}\sqrt{\frac{bx}{ca}} - \sqrt{\frac{bx}{ca}} \cdot \sqrt{\frac{bx}{ab}} - \sqrt{\frac{cx}{ab}}\sqrt{\frac{ax}{bc}}}\right]$$

$$\left(\because \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

$$= \tan^{-1}\left[\frac{\frac{a\sqrt{x} + b\sqrt{x} + c\sqrt{x}}{\sqrt{abc}} - \frac{\sqrt{x}\sqrt{x}\sqrt{x}}{\sqrt{abc}}}{1 - \frac{x}{c} - \frac{x}{a} - \frac{x}{b}}\right]$$

$$= \left[\frac{\frac{\sqrt{x}(a+b+c) - x\sqrt{x}}{\sqrt{abc}}}{\left(1 - \frac{x}{a} - \frac{x}{b} - \frac{x}{c}\right)}\right]$$

$$= \left[\frac{\frac{x\sqrt{x} - x\sqrt{x}}{\sqrt{abc}}}{\left(1 - \frac{x}{a} - \frac{x}{b} - \frac{x}{c}\right)}\right] \qquad (\because a + b + c = x)$$

$= \tan^{-1} (0)$
$= \tan^{-1} (\pi)$
$= \pi$
= RHS
इति सिद्धम्।

प्रश्न 8.

$$\frac{1}{2}\tan^{-1}x = \cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}$$

हल : माना $\tan^{-1} x = \theta$

$x = \tan\theta$

$$\text{R.H.S.} = \cos^{-1}\left\{\frac{1+\sqrt{1+\tan^2\theta}}{2\sqrt{1+\tan^2\theta}}\right\}^{\frac{1}{2}}$$

$$= \cos^{-1}\left\{\frac{1+\sec\theta}{2\sec\theta}\right\}^{\frac{1}{2}} = \cos^{-1}\left(\frac{\cos\theta+1}{2}\right)^{\frac{1}{2}}$$

$$= \cos^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2}\right)^{\frac{1}{2}} = \cos^{-1}\cos\frac{\theta}{2} = \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x = \text{L.H.S.}$$

इति सिद्धम्।

प्रश्न 9. यदि $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, तो सिद्ध कीजिए कि $x^2 + y^2 + x^2 + 2xyz = 1$.

हल : $\because \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \cos^{-1}[xy - \sqrt{1 - x^2}\sqrt{1 - y^2}] = \cos^{-1}(-z)$

$\Rightarrow [\because \cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1 - x^2}\sqrt{1 - x^2}]$ तथा $(\cos^{-1}(-x) = \pi - \cos^{-1}x)$

$\Rightarrow xy - \sqrt{1 - x^2}\sqrt{1 - x^2} = (-z)$

$\Rightarrow xy + z = \sqrt{1 - x^2}\sqrt{1 - x^2}$

$\Rightarrow (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$

$\Rightarrow z^2 + 2xyz = 1 - y^2 - x^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

इति सिद्धम्।

प्रश्न 10. यदि $\sin^{-1} + x + \sin^{-1} y + \sin^{-1} z = \pi$, तो सिद्ध कीजिए कि $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

हल : माना
$\sin^{-1} x = A \therefore x = \sin A$
$\sin^{-1} y = B \therefore y = \sin B$
$\sin^{-1} z = C \therefore z = \sin C$
$\therefore \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$
$A + B + C = \pi$

$\therefore$ **सिद्ध करना है कि**

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

L.H.S.

$$= \sin A\sqrt{1-\sin^2 A} + \sin B\sqrt{1-\sin^2 B} + \sin C\sqrt{1-\sin^2 C}$$
$$= \sin A \cos A + \sin B \cos B + \sin C \cos C$$
$$= \frac{1}{2}[2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C]$$
$$= \frac{1}{2}\{\sin 2A + \sin 2B + \sin 2C]$$
$$= \frac{1}{2}\left[2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C\right]$$
$$\left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$\Rightarrow \frac{1}{2}$ [2 sin (A + B) cos (A − B) + 2 sin C cos C]
$\Rightarrow$ sin (A + B) cos (A − B) + sin C cos C
$\Rightarrow$ sin (π − C) cos (A − B) + sin cos C
$\Rightarrow$ sin C cos (A − B) + sin C cos C
$\Rightarrow$ sin ([cos (A − B) + cos {π − (A + B)}]
$\Rightarrow$ sin ([cos (A − B) − cos (A + B)]
$\Rightarrow$ sin ([2 sin A sin B]
[$\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$]
= 2 sin A sin B sin C
= 2 xyz
= R.H.S.

इति सिद्धम्

प्रश्न 11. यदि $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, तो सिद्ध कीजिए कि xy + yz + zx = 1.

हल : प्रश्नानुसार

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$

$$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-xz}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{x+y+z-xyz}{1-xy-yz-xz} = \tan\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{x+y+z-xyz}{1-xy-yz-xz} = \infty$$

$\Rightarrow$ 1 - xy - yz - zx = 0

$\Rightarrow$ xy + yz + zx = 1

इति सिद्धम्

प्रश्न 12. यदि

$$\frac{1}{2}\sin^{-1}\frac{2x}{1-x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2} + \frac{1}{3}\tan^{-1}\frac{3z-z^3}{1-3z^2} = 5\pi,$$

तो सिद्ध कीजिए कि x + y + z = xyz.

हल : माना x = tanA, y = tanB, z = tanC

$$\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) + \frac{1}{3}\tan^{-1}\left(\frac{3z-z^3}{1-3z^2}\right) = 5\pi$$

$$\Rightarrow \frac{1}{2}\sin^{-1}\left(\frac{2\tan A}{1+\tan^2 A}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2 B}{1+\tan^2 B}\right)$$

$$+\frac{1}{3}\tan^{-1}\left(\frac{3\tan C - \tan^3 C}{1-3\tan^2 C}\right) = 5\pi$$

$$\Rightarrow \frac{1}{2}\sin^{-1}(\sin 2A) + \frac{1}{2}\cos^{-1}(\cos 2B)$$

$$+\frac{1}{3}\tan^{-1}(\tan 3C) = 5\pi$$

$$\Rightarrow \quad \frac{1}{2}(2A) + \frac{1}{2}(2B) + \frac{1}{2}(3C) = 5\pi$$

$$\Rightarrow \quad A + B + C = 5\pi$$

$$\Rightarrow \quad A + B = 5\pi - C$$

$$\tan(A + B) = \tan(5\pi - C)$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \quad \frac{x + y}{1 - xy} = -x$$

($\tan A$, $\tan B$ व $\tan C$ के मान रखने पर)

$\Rightarrow x + y = -2(1 - xy) = -z + xyz$

$x + y + z = xyz.$

इति सिद्धम्

प्रश्न 13. यदि

$$\sec^{-1}\left(\sqrt{1+x^2}\right) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right) + \cot^{-1}\left(\frac{1}{z}\right) = 3\pi,$$

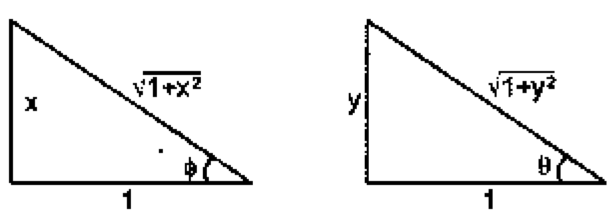
तो सिद्ध कीजिए कि $x + y + z = xyz$.

हल : प्रश्नानुसार

$$\sec^{-1}(\sqrt{1+x^2}) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right) + \cot^{-1}\left(\frac{1}{z}\right) = 3\pi$$

$$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) + \tan^{-1}\frac{y}{\sqrt{1+y^2}}\ (2) = 3\pi$$

पाइथागोरस प्रमेय तथा त्रिकोणमितीय अनुपातों से,



जहाँ $$\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \theta = \tan^{-1}\left(\frac{x}{1}\right)$$

तथा $\sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right) = \phi = \tan^{-1}\left(\frac{y}{1}\right)$

$\Rightarrow \tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = 3\pi$

$\Rightarrow \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = 3\pi$

$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan(3\pi) = 0$

x + y + z - xyz = 0

x + y + z = xyz

इति सिद्धम्।।

प्रश्न 14. सिद्ध कीजिए कि

$\tan^{-1} x + \cot^{-1}(x + 1) = \tan^{-1}(x^2 + x + 1)$

हल : LHS

$= \tan^{-1} x + \cot^{-1}(x + 1)$

$= \tan^{-1} x + \tan^{-1}\left(\frac{1}{x+1}\right)$

$= \tan^{-1}\left(\frac{x + \frac{1}{x+1}}{1 - x \times \frac{1}{x+1}}\right)$

$= \tan^{-1}\left(\frac{x^2+x+1}{x+1-x}\right)$

$= \tan^{-1}(x^2 + x + 1)$

= RHS

इति सिद्धम्।।

प्रश्न 15. यदि $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$, समान्तर श्रेढ़ी में हो, तो सिद्ध कीजिए कि $y^2(x + z) + 2y(1 - xz) - x - z = 0$

हल : $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$, समान्तर श्रेढ़ी में हैं, अतः

∴ $$\tan^{-1} y = \frac{\tan^{-1} x + \tan^{-1} z}{2}$$

∴ $$\tan^{-1} z + \tan^{-1} x = 2\tan^{-1} y$$

⇒ $$\tan^{-1}\left(\frac{z+x}{1-zx}\right) = 2\tan^{-1} y$$

⇒ $$\tan^{-1}\left(\frac{z+x}{1-zx}\right) = \tan^{-1}\left(\frac{2y}{1-y^2}\right)$$

⇒ $$\frac{x+x}{1-zx} = \frac{2y}{1-y^2}$$

⇒ $(z + x) (1 - y^2) = 2y (1 - zx)$

⇒ $z + x - y^2(x + z) = 2y (1 - xz)$

⇒ $y^2(x + 2) + 2y (1 - xz) - x - z = 0$

इति सिद्धम्।।

प्रश्न 16. यदि $x^3 + px^2 + qx + p = 0$ के मूल α, β, γ हो, तो सिद्ध कीजिए कि एक विशेष परिस्थिति के अलावा $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1}\gamma = n\pi$ और वह विशेष स्थिति भी ज्ञात कीजिए जब ऐसा नहीं होता है।

हल : दिया है :

α, β, γ समीकरण : $x^3 + px^2 + qx + p = 0$ के मूल हैं; अत:

$$\alpha + \beta + \gamma = -\left(\frac{x^2 \text{ का गुणांक}}{x^3 \text{ का गुणांक}}\right) = -\frac{p}{1} = -p$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \left(\frac{x^2 \text{ का गुणांक}}{x^3 \text{ का गुणांक}}\right) = \frac{q}{1} = q$$

तथा $$\alpha\beta\gamma = -\left(\frac{\text{अचर पद}}{x^3 \text{ का गुणांक}}\right) = -p$$

$$\text{L.H.S.} = \tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma$$

$$= \tan^{-1}\left[\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-(\alpha\beta+\beta\gamma+\gamma\alpha)}\right]$$

$= \tan^{-1} (0)$ [∵ $\alpha + \beta + \gamma = \alpha\beta\gamma = -p$]

$= n\pi$

= RHS

प्रश्न 17.

$$\sec^{-1}\left(\frac{x}{a}\right)-\sec^{-1}\left(\frac{x}{b}\right)=\sec^{-1}b-\sec^{-1}a.$$

हल :

$$\sec^{-1}\left(\frac{x}{a}\right)-\sec^{-1}\left(\frac{x}{b}\right)=\sec^{-1}(b)-\sec^{-1}(a)$$

$$\Rightarrow \sec^{-1}\left(\frac{x}{a}\right)+\sec^{-1}(a)=\sec^{-1}\left(\frac{x}{b}\right)+\sec^{-1}(b)$$

$$\Rightarrow \cos^{-1}\left(\frac{a}{x}\right)+\cos^{-1}\left(\frac{1}{a}\right)=\cos^{-1}\left(\frac{b}{x}\right)+\cos^{-1}\left(\frac{1}{b}\right)$$

$$\Rightarrow \cos^{-1}\left[\frac{a}{x}\cdot\frac{1}{a}-\sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\left(\frac{1}{a}\right)^2}\right]$$

$$=\cos^{-1}\left[\frac{b}{x}\cdot\frac{1}{b}-\sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\left(\frac{1}{b}\right)^2}\right]$$

$$\Rightarrow \frac{1}{x}-\sqrt{1-\frac{a^2}{x^2}-\frac{1}{a^2}+\frac{1}{x^2}}=\frac{1}{x}-\sqrt{1-\frac{b^2}{x^2}-\frac{1}{b^2}+\frac{1}{x^2}}$$

$$1-\frac{a^2}{x^2}-\frac{1}{a^2}+\frac{1}{x^2}=1-\frac{b^2}{x^2}-\frac{1}{b^2}+\frac{1}{x^2}$$

$$\Rightarrow \qquad \frac{b^2}{x^2}+\frac{1}{b^2}=\frac{a^2}{x^2}+\frac{1}{a^2}$$

$$\Rightarrow \qquad \frac{b^2}{x^2}-\frac{a^2}{x^2}=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\Rightarrow \qquad (b^2-a^2)=x^2\left(\frac{b^2-a^2}{a^2b^2}\right)$$

$$\Rightarrow \qquad x^2=a^2b^2$$

$$\Rightarrow \qquad x=\pm ab.$$

प्रश्न 18.

$$\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}.$$

हल :

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x$$
$$\tan^{-1}(-x)=-\tan^{-1}x]$$

$$\Rightarrow \pi-2\tan^{-1}x-2\tan^{-1}x=\frac{2\pi}{3}$$

$$\Rightarrow \quad 4\tan^{-1}x=\pi-2\frac{\pi}{3}$$

$$\Rightarrow \quad 4\tan^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{12} \quad \Rightarrow x=\tan\frac{\pi}{12}$$

अतः $\quad x=\tan\left(\frac{\pi}{12}\right)$

प्रश्न 19.

$$\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$$

हल :

$$:\tan^{-1}\frac{1}{2x}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$$

$$\because \quad \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\therefore \tan^{-1}\left[\frac{\frac{1}{1+2x}+\frac{1}{4x+1}}{1-\left(\frac{1}{1+2x}\right)\left(\frac{1}{4x+1}\right)}\right]=\tan^{-1}\frac{2}{x^2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{4x+1+2x+1}{(1+2x)(1+4x)}}{\frac{(1+2x)(1+4x)-1}{(1+2x)(1+4x)}}\right]=\tan^{-1}\frac{2}{x^2}$$

$$\Rightarrow \tan^{-1}\left[\frac{6x+2}{1+2x+4x+8x^2-1}\right]=\tan^{-1}\frac{2}{x^2}$$

$$\Rightarrow \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right)=\tan^{-1}\frac{2}{x^2}$$

$$\Rightarrow \frac{3x+1}{4x^2+3x}=\frac{2}{x^2}$$

$$\Rightarrow 3x^3+x^2=8x^2+6x$$

⇒ $x(3x^2 - 7x - 6) = 0$
⇒ $x(3x^2 - 9x + 2x - 6) = 0$
⇒ $x[3x(x - 3) + 2(x - 3) = 0$
⇒ $x(x - 3)(3x + 2) = 0$
⇒ $x = 0, x = 3, x = -\frac{2}{3}$

प्रश्न 20.

$$\tan^{-1}\frac{x+7}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi-\tan^{-1}7$$

हल :

$$\tan^{-1}\frac{x+7}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi-\tan^{-1}7$$

$$\tan^{-1}\left[\frac{\frac{x+7}{x-1}+\frac{x-1}{x}}{1-\left(\frac{x+7}{x-1}\right)\left(\frac{x-1}{x}\right)}\right]=\pi-\tan^{-1}7$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x(x+7)+(x-1)^2}{x(x-1)}}{\frac{x(x-1)-(x+7)(x-1)}{x(x-1)}}\right]=\pi-\tan^{-1}7$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x^2+7x+x^2-2x+1}{x^2-x-x^2-7x+x+7}\right]=\pi-\tan^{-1}7$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2+5x+1}{-7x+7}\right)+\tan^{-1}7=\pi$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{2x^2+5x+1}{-7x+7}+7}{1-\left(\frac{2x^2+5x+1}{-7x+7}\right)\times 7}\right]=\pi$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2x^2+5x+1-49x+49}{-7x+7-14x^2-35x-7}\right]=\pi$$

$$\Rightarrow \quad \frac{2x^2-44x+50}{-14x^2-42x}=\tan\pi$$

$$\Rightarrow \quad 2x^2-44x+50=0$$

$$\Rightarrow \quad x^2-22x+25=0$$

$$\Rightarrow \quad x=\frac{22\pm\sqrt{484-4\times 25}}{2\times 1}$$

$$\Rightarrow \quad x=\frac{22\pm\sqrt{484-100}}{2}$$

$$\Rightarrow \quad x=\frac{22\pm 8\sqrt{6}}{2}$$

$$\Rightarrow \quad x=11\pm 4\sqrt{6}$$

प्रश्न 21.

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cot^{-1}x=\frac{\pi}{4}.$$

हल :

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cot^{-1}x=\frac{\pi}{4}$$

$\because \quad \sin^{-1}\left(\frac{1}{x}\right)=y=\operatorname{cosec}^{-1}x$

$\therefore \quad \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)=\operatorname{cosec}^{-1}(\sqrt{5})$

माना $\quad \operatorname{cosec}^{-1}\sqrt{5}=y$

$\therefore \quad \operatorname{cosec}=\sqrt{5}$

$\therefore \quad \operatorname{cosec}^2 y=5$

$1+\cot^2 y=5$

$\cot^2 y=4$

$\cot y=\pm 2$

$y=\cot^{-1}(2)$

(+ चिह्न लेने पर)

$\therefore \quad y+\cot^{-1}x=\frac{\pi}{4}$

$\Rightarrow \quad \cot^{-1}(2)+\cot^{-1}x=\frac{\pi}{4}$

$\Rightarrow \quad \cot^{-1}\left(\frac{2x-1}{2+x}\right)=\frac{\pi}{4}$

$\left[\because \cot^{-1}x+\cot^{-1}y=\cot^{-1}\left(\frac{xy-1}{x+y}\right)\right]$

$\Rightarrow \quad \frac{2x-1}{2+x}=\cot\frac{\pi}{4} \Rightarrow \frac{2x-1}{2+x}=1$

⇒ 2x - 1 = 2 + x
⇒ 2x - x = 2 + 1
⇒ x = 3

प्रश्न 22.

$$3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$$

हल :

$$3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right)-\tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad 3\tan^{-1}\left[\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})}\right] = \tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad 3\tan^{-1}\left[\frac{2-\sqrt{3}}{4-(\sqrt{3})^2}\right] = \tan^{-1}\left[\frac{\frac{1}{x}+\frac{1}{3}}{1-\frac{1}{x}\times\frac{1}{3}}\right]$$

$$\Rightarrow \quad 3\tan^{-1}(2-\sqrt{3}) = \tan^{-1}\left(\frac{3+x}{3x-1}\right)$$

$$\Rightarrow \tan^{-1}\left\{\frac{3(2-\sqrt{3})-(2-\sqrt{3})^3}{1-3(2-\sqrt{3})^2}\right\} = \frac{3+x}{3-1}$$

$$\left[\because \ 3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right]$$

$$\Rightarrow \quad \frac{3+x}{3x-1} = \left[\frac{3(2-\sqrt{3})-(8-3\sqrt{3}-12\sqrt{3}+18)}{1-3(4+3-4\sqrt{3})}\right]$$

$$\Rightarrow \quad \frac{3+x}{3x-1} = \frac{6-3\sqrt{3}-8+3\sqrt{3}-18+12\sqrt{3}}{1-3(7-4\sqrt{3})}$$

$$\Rightarrow \quad \frac{3+x}{3x-1} = \frac{-20+12\sqrt{3}}{-20+12\sqrt{3}}$$

$$\Rightarrow \quad \frac{3+x}{3x-1} = \frac{20+12\sqrt{3}}{20+12\sqrt{3}} \Rightarrow \frac{3+x}{3x-1} = 1$$

$\Rightarrow 3 + x = 3x - 1$

$\Rightarrow 2x = 4$

$\Rightarrow x = 2$

प्रश्न 23. sin 2 (cos^{-1} {cot (2 tan^{-1} x)} = 0

हल : दी गई समीकरण है

$\sin 2 [\cos^{-1} \{\cot (2 \tan^{-1} x)\}] = 0$

$$\Rightarrow \sin 2\left[\cos^{-1}\left\{\cot\left(\tan^{-1}\frac{2x}{1-x^2}\right)\right\}\right] = 0$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \sin 2\left[\cos^{-1}\left\{\cot\left(\cot^{-1}\frac{1-x^2}{2x}\right)\right\}\right] = 0$$

$$\left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \sin 2\left[\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$$

$$\Rightarrow \sin 2\left[\sin^{-1}\left\{\sqrt{\frac{6x^2-1-x^4}{4x^2}}\right\}\right] = 0$$

$$\left[\because \cos^{-1}y = \sin^{-1}\sqrt{1-y^2}\right]$$

$$\Rightarrow \sin\left[\sin^{-1}\left\{2\left(\frac{\sqrt{6x^2-1-x^4}}{2x}\right)\left(\sqrt{1-\frac{6x^2-1-x^4}{4x^2}}\right)\right\}\right] = 0$$

$$\left[\because 2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})\right]$$

$$2\left(\frac{\sqrt{6x^2-1-x^4}}{2x}\right)\left(\sqrt{\frac{4x^2-6x^2+1+x^4}{4x^2}}\right) = 0$$

$$\left[\because \sin(\sin^{-1}x) = x\right]$$

$\Rightarrow \sqrt{6x^2-1-x^4}\sqrt{-2x^2+x^4+1} = 0$

$\Rightarrow (6x^2 - 1 - x^4)(x^4 - 2x^2 + 1) = 0$

$\Rightarrow 6x^2 - 1 - x^4 = 0$ ....(1)

$\Rightarrow x^4 - 2x^2 + 1 = 0$ ...(2)

समीकरण (1) से,

$\Rightarrow 6x^2 - 1 - x^4 = 0$

$\Rightarrow x^4 - 6x^2 + 1 = 0$

$\Rightarrow x^4 - 2 \times 3x^2 + (3)^2 - 8 = 0$

$\Rightarrow (x^2 - 3)^2 = 8$

$\Rightarrow x^2 - 3 = \pm 2\sqrt{2}$

$\Rightarrow x^2 = 3 \pm 2\sqrt{2}$

$\Rightarrow x^2 = 1 + 2 \pm 2\sqrt{2}$

$\Rightarrow x^2 = (1)^2 + (\sqrt{2})^2 \pm 2\sqrt{2}$
$\Rightarrow x^2 = (1 \pm \sqrt{2})^2$
$\Rightarrow x = \pm (1+\sqrt{2})$
समीकरण (2) से
$\Rightarrow x^4 - 2x^2 + 1 = 0$
$\Rightarrow (x^2)^2 - 2x^2 + (1)^2 = 0$
$\Rightarrow (x^2 - 1)^2 = 0$
$\Rightarrow x^2 = 1$
$\Rightarrow x = \pm 1$

प्रश्न 24.

$$\tan^{-1}\left(\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{6}\right) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{4}$$

हल :

$$\tan\left(\frac{\frac{1}{x}+\frac{1}{4}}{1-\frac{1}{x}\times\frac{1}{4}}\right) + \tan^{-1}\left\{\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right\} + \tan^{-1}\frac{1}{6} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{4+x}{4x-1}\right) + \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\frac{1}{6} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{4+x}{4x-1}\right) + \tan^{-1}\left\{\frac{\frac{5}{12}+\frac{1}{6}}{1-\frac{5}{12}\times\frac{1}{6}}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{4+x}{4x-1}\right) + \tan^{-1}\left(\frac{30+12}{72-5}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{4+x}{4x-1}\right) + \tan^{-1}\left(\frac{42}{67}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{4+x}{4x-1}+\frac{42}{67}}{1-\left(\frac{4+x}{4x-1}\right)\left(\frac{42}{67}\right)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{67(4+x)+42(4x-1)}{(4x-1)67-42(4+x)}=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{268+67x+168x-42}{268x-67-168-42x}=1$$

$$\Rightarrow 268+67x+168x-42$$
$$=268x-67-168-42x$$

$$\Rightarrow \quad 277x-268x=-503+42$$

$$\Rightarrow \quad 9x=-461$$

$$\Rightarrow \quad x=\frac{-461}{9}$$

प्रश्न 25.

$\sin^{-1}x - \sin^{-1}y = \frac{2\pi}{3}$; $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$

हल : प्रश्नानुसार,

$\sin^{-1}x - \sin^{-1}y = \frac{2\pi}{3}$ ...(i)

$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ ....(ii)

$$\left(\frac{\pi}{2}-\sin^{-1}x\right)-\left(\frac{\pi}{2}-\sin^{-1}y\right)=\frac{\pi}{3}$$

समीकरण (ii) में $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ सूत्र के प्रयोग से

$$\Rightarrow \frac{\pi}{2}-\sin^{-1}-\frac{\pi}{2}+\sin^{-1}y=\frac{\pi}{3}$$

$$\Rightarrow \quad -\sin^{-1}x+\sin^{-1}y=\frac{\pi}{3} \qquad ...(iii)$$

समीकरण (i) व (iii) से

$$2\sin^{-1}y=\frac{2\pi}{3}+\frac{\pi}{3}=\pi$$

$$\Rightarrow \quad \sin^{-1} y = \frac{\pi}{2}$$

$$\Rightarrow \quad y = \sin \frac{\pi}{2} = 1 \qquad \text{...(iv)}$$

समी. (i) व (iv) से,

$$\sin^{-1} x + \tan^{-1} (1) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \sin^{-1} x + \frac{\pi}{2} = \frac{2\pi}{3}$$

$$\Rightarrow \quad \sin^{-1} x = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \sin \frac{\pi}{6} = \frac{1}{2}$$

अत: $x = \frac{1}{2}, y = 1.$

Miscellaneous Exercise

प्रश्न 1. $\tan^{-1}(-1)$ का मुख्य मान है
(a) 45°
(b) 135°
(c) – 45°
(d) – 60°

हल :
$\because \tan^{-1}(-x) = -\tan^{-1} x$
$\therefore \tan^{-1}(-1) = -\tan^{-1}(1)$
माना $\tan^{-1} 1 = \theta$
$\therefore \tan\theta = 1$
$\tan\theta = \tan 45°$
$\therefore \theta = 45°$
$\therefore \tan^{-1}(-1) = -45°$
अत: सही विकल्प (c) है।

प्रश्न 2.

2 $\tan^{-1}(1/2)$ बराबर है

(a) $\cos^{-1}\left(\frac{3}{5}\right)$ **(b)** $\cos^{-1}\left(\frac{3}{4}\right)$

(c) $\cos^{-1}\left(\frac{5}{3}\right)$ **(d)** $\cos^{-1}\left(\frac{1}{2}\right)$

हल :

$2\tan^{-1}\left(\frac{1}{2}\right)$

$$\left\{\text{सूत्र : } 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \text{ से}\right\}$$

$$= \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}\right)$$

$$= \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{3}{5}\right)$$

अत: सही विकल्प (a) है।

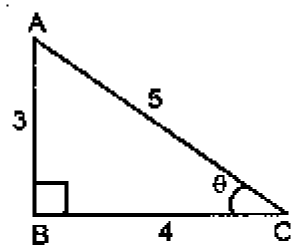
प्रश्न 3. यदि $\tan^{-1}(3/4) = \theta$, तो $\sin\theta$ का मान है

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{4}{3}$ (d) $\frac{1}{4}$

हल :

प्रश्नानूसार,

$$\tan^{-1}\left(\frac{3}{4}\right) = \theta$$



$$\Rightarrow \qquad \tan\theta = \frac{3}{4}$$

समकोण त्रिभुज बनाने पर

$$\sin\theta = \frac{3}{5}$$

$$\therefore \qquad \sin^{-1} = \left(\frac{3}{5}\right)$$

अतः सही उत्तर का विकल्प (b) है।

प्रश्न 4. $\cot[\tan^{-1}\alpha + \cot^{-1}\alpha]$ का मान है

(a) 1

(b) ∞

(c) 0

(d) इनमें से कोई नहीं

हल : $\cos(\tan^{-1}\alpha + \cot^{-1}\alpha)$.

$= \cot\frac{\pi}{2} (\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2})$

= 0

अत: सही उत्तर का विकल्प (c) है।

प्रश्न 5. यदि $sin^{-1}\frac{1}{2} = x$ तो का व्यापक मान है

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $\frac{\pi}{6}$

(c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi(-1)^n \frac{\pi}{6}$

हल : दिया है,

$$\sin^{-1}\left(\frac{1}{2}\right) = x \Rightarrow \sin x = \frac{1}{2} = \sin^{-1}\frac{\pi}{6}$$

$$\Rightarrow \qquad x = \frac{\pi}{6}$$

$\therefore x$ का व्यापक मान $\theta = n\pi + (-1)^n \frac{\pi}{6}$

अत: सही उत्तर का विकल्प (d) है।

प्रश्न 6. 2 tan ($\tan^{-1} x + \tan^{-1} x^3$) का मान है

(a) $\frac{2x}{1-x^2}$
(b) $1 + x^2$
(c) $2x$
(d) इनमें से कोई नहीं

हल :

$$= 2\tan\left\{\tan^{-1}\left(\frac{x+x^3}{1-x\times x^3}\right)\right\}$$

$$= 2\tan\left\{\tan^{-1}\left(\frac{x(1+x^2)}{1-x^4}\right)\right\}$$

$$= 2\tan\left\{\tan^{-1}\frac{x(1+x^2)}{(1-x^2)(1+x^2)}\right\}$$

$$= 2\frac{x}{1-x^2} = \frac{2x}{1-x^2}$$

अतः सही विकल्प (a) है।

प्रश्न 7. यदि $\tan^{-1}(3x) + \tan^{-1}(2x) = \frac{\pi}{2}$. तो x का मान

(a) $\frac{1}{6}$ **(b)** $\frac{1}{3}$ **(c)** $\frac{1}{10}$ **(d)** $\frac{1}{2}$

हल : $\tan^{-1}(3x) + \tan^{-1}(2x) = \frac{\pi}{2}$

$$\Rightarrow \qquad \tan^{-1}\left\{\frac{3x+2x}{1-3x\times 2x}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \qquad \left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$

$\Rightarrow \frac{5x}{1-6x^2} = 1$
$\Rightarrow 1 - 6x^2 = 5x$
$\Rightarrow 6x^2 + 5x - 1 = 0$
$\Rightarrow 6x^2 + 6x - x - 1 = 0$
$\Rightarrow 6x(x + 1) - 1(x + 1) = 0$
$\Rightarrow (x + 1)(6x - 1) = 0$
$\Rightarrow x = -1, x = \frac{1}{6}$
अतः सही विकल्प (a) है।

प्रश्न 8.
$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
का मान है

(a) $\frac{\pi}{2}$ **(b)** $\frac{\pi}{3}$ **(c)** $\frac{2\pi}{3}$ **(d)** π

हल :
$$\sin^{-1}\frac{\sqrt{3}}{2} + 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$$
अतः सही विकल्प (c) है।

प्रश्न 9. यदि $\tan^{-1}(1) + \cos^{-1}(\frac{1}{\sqrt{2}}) = \sin^{-1} x$, तो x का मान है
(a) -1
(b) 0
(c) 1
(d) $-\frac{1}{2}$

हल :

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1} x$$

$$\sin^{-1} x = \frac{\pi}{4} + \frac{\pi}{4}$$

$$\sin^{-1} x = \frac{\pi}{2}$$

$$x = \sin\frac{\pi}{2}$$

$$x = 1$$

अतः सही विकल्प (c) है।

प्रश्न 10. यदि $\cot^{-1} x + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$ तो x का मान है

(a) 1

(b) 3

(c) $\frac{1}{3}$

(d) इनमें से कोई नहीं

हल :

$$\cot^{-1}(x) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \cot - 1\, x = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad \cot^{-1} = \cot^{-1}\frac{1}{3}$$

अतः तुलना से, $x = \frac{1}{3}$

अतः सही विकल्प (c) है।

प्रश्न 11. यदि $4 \sin^{-1} x + \cos^{-1} x = \pi$, तो x का मान कीजिए।

हल :

4 sin^{-1} x + cos^{-1} x = π

$$\Rightarrow \quad 4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \pi$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad 3\sin^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \sin\frac{\pi}{6} \Rightarrow x = \frac{1}{2}$$

प्रश्न 12.

$$\cos\left[\left(\frac{\pi}{2}\right) + \sin^{-1}\left(\frac{1}{3}\right)\right]$$

का मान कीजिए।

हल :

$$\cos\left[\left(\frac{\pi}{2}\right) + \sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= -\sin\left(\sin^{-1}\frac{1}{3}\right) \quad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$

$$= -\frac{1}{3}$$

प्रश्न 13. यदि

$$\sin^{-1}\left(\frac{3}{4}\right) + \sec^{-1}\left(\frac{4}{3}\right) = x$$

तो x का मान कीजिए।

हल :

$$\sin^{-1}\left(\frac{3}{4}\right) + \sec^{-1}\left(\frac{4}{3}\right) = x$$

$$\sin^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{3}{4}\right) = x$$

$$\frac{\pi}{2} = x \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$x = \frac{\pi}{2}$$

प्रश्न 14.

$$\sin^{-1}\left(\frac{4}{5}\right)+2\tan^{-1}\left(\frac{1}{3}\right)$$

का मान कीजिए।

हल :

$$\sin^{-1}\left(\frac{4}{5}\right)+2\tan^{-1}\left(\frac{1}{3}\right)$$

$$=\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left\{\frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2}\right\}$$

$$\because 2\tan^{-1}x=\cos^{-1}\left\{\frac{1-x^2}{1+x^2}\right\}$$

$$=\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{\frac{8}{9}}{\frac{10}{9}}\right)$$

$$=\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)$$

$$=\frac{\pi}{2}$$

प्रश्न 15.

यदि

$$\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{x}\right)=90^\circ,$$

तो x का मान कीजिए।

हल :

$$\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{x}\right) = 90°$$

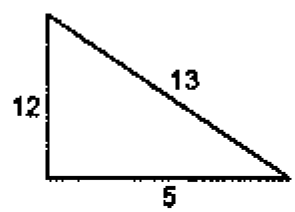
$$\Rightarrow \sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{12}{x}\right)= \sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{5}{13}\right)$$

$$\left[\because \sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{5}{13}\right)=\frac{\pi}{2}\right]$$

$$\Rightarrow \quad \sin^{-1}\frac{12}{x} = \cos^{-1}\frac{5}{13}$$

$$\Rightarrow \quad \sin^{-1}\frac{12}{x} = \sin^{-1}\frac{12}{13}$$

$\cos^{-1}\left(\frac{5}{13}\right)$ के लिए



तुलना से, $\quad x = 13$

प्रश्न 16.

सिद्ध कीजिए कि :

$$\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$$

हल :

$$\text{L.H.S.} = \sin^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{12}{13}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$$\left[\because \cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{3}{13}\right)$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\frac{25}{169}}-\frac{5}{13}\sqrt{1-\frac{9}{25}}\right]$$

$$\left[\because \sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)\right]$$

$$= \left[\frac{3}{5}\times\frac{12}{13} - \frac{5}{12}\cdot\frac{4}{5}\right]$$

$$= \sin^{-1}\left(\frac{16}{65}\right) = \text{R.H.S.}$$

इति सिद्धम्।

प्रश्न 17. यदि $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, तो सिद्ध कीजिए : $x + y + z = xyz$

हल : $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) \tan^{-1} = \pi$$

$$\Rightarrow \quad \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan \pi$$

$x + y + z - xyz = 0 \times (1 - xy - yz + xyz)$

$x + y + z - xyz = 0$

अत: $x + y + z = xyz$

इति सिद्धम्।

प्रश्न 18. सिद्ध कीजिए कि :

$\tan^{-1} [\frac{1}{2}\tan 2A] + \tan^{-1} (\cot A) + \tan^{-1} (\cot A) = 0.$

हल :

$\tan^{-1} [\frac{1}{2} \tan 2A] + \tan^{-1} (\cot A) + \tan^{-1} (\cot A) = 0.$

$$= \tan^{-1}\left(\frac{1}{2}\times\frac{2\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{1}{\tan A}\right) + \tan^{-1}\left(\frac{1}{\tan^2 A}\right)$$

माना $\tan A = x$

तब $\quad$ $$\text{L.H.S.} = \tan^{-1}\left(\frac{x}{1-x^2}\right) + \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{x^2}\right)$$

$$= \tan^{-1}\left\{\frac{\frac{x}{1-x^2}+\frac{1}{x}}{1-\left(\frac{x}{1-x^2}\right)\frac{1}{x}}\right\} + \tan^{-1}\left(\frac{1}{x^2}\right)$$

$$= \tan^{-1}\left\{\frac{\frac{x^2+1-x^2}{x(1-x^2)}}{\frac{1-x^2-1}{x(1-x^2)}}\right\} + \tan^{-1}\left(\frac{1}{x^2}\right)$$

$$= \tan^{-1}\left(\frac{1}{-x^2}\right) + \tan^{-1}\left(\frac{1}{x^2}\right)$$

$$= -\tan^{-1}\left(\frac{1}{x^2}\right) + \tan^{-1}\left(\frac{1}{x^2}\right)$$

$$[\because \tan^{-1}(-\theta) = -\tan^{-1}\theta]$$

$$= 0$$

$$= \text{R.H.S.}$$

इति सिद्धम्।

प्रश्न 19. सिद्ध कीजिए कि :
$\tan^{-1} x = 2\tan^{-1}[\text{cosec}(\tan^{-1} x) - \tan(\cot^{-1} x)]$

हल : माना $\tan^{-1}\theta$

$$\Rightarrow \quad x = \tan\theta = \cot\left(\frac{\pi}{2}-\theta\right)$$

$$\Rightarrow \quad \cot^{-1} x = \frac{\pi}{2}-\theta$$

$$\text{R.H.S.} = 2\tan^{-1}\left[\text{cosec}\,\theta - \tan\left(\frac{\pi}{2}-\theta\right)\right]$$

[समी. (i) से]

$$= 2\tan^{-1}[\text{cosec}\,\theta - \cot\theta]$$

$$= 2\tan^{-1}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= 2\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= 2\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = 2\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= 2 \times \frac{\theta}{2} = \theta = \tan^{-1} x = \text{L.H.S.}$$

= $\tan^{-1} x$
= RHS
इति सिद्धम्।

प्रश्न 20. यदि $\Phi = tan^{-1}\frac{x\sqrt{3}}{2K-x}$ और $\theta = tan^{-1}\frac{2x-K}{K\sqrt{3}}$ तो सिद्ध कीजिए Φ – θ का मान 30° है।
हल :

दिया है, $\phi = \tan^{-1}\frac{x\sqrt{3}}{2K-x}$

$\therefore \quad \tan\phi = \frac{x\sqrt{3}}{2K-x}$

तथा $\quad \theta = \tan^{-1}\frac{2x-K}{K\sqrt{3}}$

$\therefore \quad \tan\theta = \frac{2x-K}{K\sqrt{3}}$

$\because \quad \tan(\phi-\theta) = \frac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta}$

$$\tan(\phi-\theta) = \frac{\frac{x\sqrt{3}}{2K-x} - \frac{2x-K}{K\sqrt{3}}}{1+\left(\frac{x\sqrt{3}}{2K-x}\right)\left(\frac{2x-K}{K\sqrt{3}}\right)}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{\dfrac{3Kx - (2K - x)(2x - K)}{(2K - x)K\sqrt{3}}}{\dfrac{(2K - x)K\sqrt{3} + x\sqrt{3}(2x - K)}{(2K - x)K\sqrt{3}}}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{3Kx - (4xK - 2K^2 - 2x^2 + Kx)}{2\sqrt{3}K^2 - \sqrt{3}Kx + 2\sqrt{3}x^2 - \sqrt{3}Kx}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{3Kx - 4Kx + 2K^2 + 2x^2 - Kx}{2\sqrt{3}K^2 - 2\sqrt{3}Kx + 2\sqrt{3}x^2}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{2K^2 + 2x^2 - 2Kx}{2\sqrt{3}K^2 + 2\sqrt{3}x^2 - 2\sqrt{3}Kx}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{2(K^2 + x^2 - Kx)}{2\sqrt{3}(K^2 + x^2 - Kx)}$$

$$\Rightarrow \quad \tan(\phi - \theta) = \frac{1}{\sqrt{3}}$$

$\tan(\Phi - \theta) = \tan 30°$
$\therefore \Phi - \theta = 30°$
इति सिद्धम्।

प्रश्न 21.
सिद्ध कीजिए कि :

$$2\tan^{-1}\left[\tan(45° - \alpha)\tan\frac{\beta}{2}\right] = \cos^{-1}\left(\frac{\sin 2\alpha + \cos\beta}{1 + \sin 2\alpha\cos\beta}\right)$$

हल : LHS

$= 2\tan^{-1}[\tan(45° - \alpha)\tan\beta/2]$

$$= 2\tan^{-1}\left[\left(\frac{\tan 45° - \tan\alpha}{1 + \tan 45°\tan\alpha}\right)\tan\frac{\beta}{2}\right]$$

$$= 2\tan^{-1}\left[\left(\frac{1 - \tan\alpha}{1 + \tan\alpha}\right)\tan\frac{\beta}{2}\right]$$

$$= 2\tan^{-1}\left[\left(\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}\right)\tan\frac{\beta}{2}\right]$$

$$= \cos^{-1}\left[\frac{1-\left\{\left(\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)\tan\dfrac{\beta}{2}\right\}^2}{1+\left\{\left(\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)\tan\dfrac{\beta}{2}\right\}^2}\right]$$

$$\left[\because\ 2\tan^{-1}x = \cos^{-1}\left\{\frac{1-x^2}{1+x^2}\right\}\right]$$

$$= \cos^{-1}\left[\frac{(\cos\alpha+\sin\alpha)^2\cos^2\dfrac{\beta}{2}-(\cos\alpha-\sin\alpha)^2\sin^2\dfrac{\beta}{2}}{(\cos\alpha+\sin\alpha)^2\cos^2\dfrac{\beta}{2}+(\cos\alpha-\sin\alpha)^2\sin^2\dfrac{\beta}{2}}\right]$$

$$= \cos^{-1}\left[\frac{(1+\sin 2\alpha)\left(\dfrac{1+\cos\beta}{2}\right)-(1-\sin 2\alpha)\left(\dfrac{1-\cos\beta}{2}\right)}{(1+\sin 2\alpha)\left(\dfrac{1+\cos\beta}{2}\right)+(1-\sin 2\alpha)\left(\dfrac{1-\cos\beta}{2}\right)}\right]$$

$$= \cos^{-1}\left[\frac{\dfrac{1}{2}\{(1+\sin 2\alpha+\cos\beta+\sin 2\alpha\cos\beta) - (1-\sin 2\alpha-\cos\beta+\sin 2\alpha\cos\beta)\}}{\dfrac{1}{2}\{(1+\sin 2\alpha+\cos\beta+\sin 2\alpha\sin\beta + 1-\sin 2\alpha-\cos\beta+\sin 2\alpha\cos\beta)\}}\right]$$

$$= \cos^{-1}\left[\frac{1-\sin 2\alpha+\cos\beta+\sin 2\alpha\cos\beta-1+\sin 2\alpha+\cos\beta-\sin 2\alpha\cos\beta}{1+\sin 2\alpha+\cos\beta+\sin 2\alpha\cos\beta+1-\sin 2\alpha-\cos\beta+\sin 2\alpha\cos\beta}\right]$$

$$= \cos^{-1}\left[\frac{2(\sin 2\alpha+\cos\beta)}{2+2\sin 2\alpha\cos\beta}\right]$$

$$= \cos^{-1}\left(\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right) = \text{R.H.S.}$$

इति सिद्धम् ।