

Chapter 7 Integrals

EXERCISE 7.1

Question 1:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\cos 2x) = -2 \sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2} \frac{d}{dx}(\cos 2x)$$

$$\Rightarrow \sin 2x = \frac{d}{dx}\left(-\frac{1}{2} \cos 2x\right)$$

Thus, the anti-derivative of $\sin 2x$ is $-\frac{1}{2} \cos 2x$

Question 2:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\cos 3x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\sin 3x) = 3 \cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3} \frac{d}{dx}(\sin 3x)$$

$$\Rightarrow \cos 3x = \frac{d}{dx}\left(\frac{1}{3} \sin 3x\right)$$

Thus, the anti-derivative of $\cos 3x$ is $\frac{1}{3} \sin 3x$

Question 3:

Find an anti-derivative (or integral) of the following functions by the method of inspection, e^{2x} .

Solution:

$$\Rightarrow \frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2} \frac{d}{dx}(e^{2x})$$

$$\Rightarrow e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Thus, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $(ax+b)^2$.

Solution:

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a} \frac{d}{dx}(ax+b)^3$$

$$\Rightarrow (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Thus, the anti-derivative $(ax+b)^2$ of is $\frac{1}{3a}(ax+b)^3$

Question 5:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x - 4e^{3x}$

Solution:

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Thus, the anti-derivative of $\sin 2x - 4e^{3x}$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$

Find the following integrals in Exercises 6 to 20:

Question 6:

$$\int(4e^{3x} + 1)dx$$

Solution:

$$\begin{aligned}\int (4e^{3x} + 1) dx &= 4 \int e^{3x} dx + \int 1 dx \\&= 4 \left(\frac{e^{3x}}{3} \right) + x + C \\&= \frac{4}{3} e^{3x} + x + C\end{aligned}$$

Question 7:

$$\int x^2 \left(1 - \frac{1}{x^2} \right) dx$$

Solution:

$$\begin{aligned}\int x^2 \left(1 - \frac{1}{x^2} \right) dx &= \int (x^2 - 1) dx \\&= \int x^2 dx - \int 1 dx \\&= \frac{x^3}{3} - x + C\end{aligned}$$

Question 8:

$$\int (ax^2 + bx + c) dx$$

Solution:

$$\begin{aligned}\int (ax^2 + bx + c) dx &= a \int x^2 dx + b \int x dx + c \int 1 dx \\&= a \left(\frac{x^3}{3} \right) + b \left(\frac{x^2}{2} \right) + cx + C \\&= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C\end{aligned}$$

Question 9:

$$\int (2x^2 + e^x) dx$$

Solution:

$$\begin{aligned}\int (2x^2 + e^x) dx &= 2 \int x^2 dx + \int e^x dx \\&= 2 \left(\frac{x^3}{3} \right) + e^x + C \\&= \frac{2}{3} x^3 + e^x + C\end{aligned}$$

Question 10:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

Solution:

$$\begin{aligned}\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx &= \int \left(x + \frac{1}{x} - 2 \right) dx \\&= \int x dx + \int \frac{1}{x} dx - 2 \int 1 dx \\&= \frac{x^2}{2} + \log|x| - 2x + C\end{aligned}$$

Question 11:

$$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$$

Solution:

$$\begin{aligned}\int \frac{x^3 + 5x^2 - 4}{x^2} dx &= \int (x + 5 - 4x^{-2}) dx \\&= \int x dx + 5 \int 1 dx - 4 \int x^{-2} dx \\&= \frac{x^2}{2} + 5x - 4 \left(\frac{x^{-1}}{-1} \right) + C \\&= \frac{x^2}{2} + 5x + \frac{4}{x} + C\end{aligned}$$

Question 12:

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx$$

Solution:

$$\begin{aligned}\int \frac{x^3 + 3x + 4}{\sqrt{x}} dx &= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx \\&= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3(x^{\frac{3}{2}})}{2} + \frac{4(x^{\frac{1}{2}})}{1} + C \\&= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C \\&= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C\end{aligned}$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x-1} dx$$

Solution:

$$\begin{aligned}\int \frac{x^3 - x^2 + x - 1}{x-1} dx &= \int \left[\frac{(x^2 + 1)(x - 1)}{x - 1} \right] dx \\&= \int (x^2 + 1) dx \\&= \int x^2 dx + \int 1 dx \\&= \frac{x^3}{3} + x + C\end{aligned}$$

Question 14:

$$\int (1-x)\sqrt{x} dx$$

Solution:

$$\begin{aligned}\int (1-x)\sqrt{x} dx &= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx \\&= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx \\&= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C \\&= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C\end{aligned}$$

Question 15:

$$\int \sqrt{x}(3x^2 + 2x + 3)dx$$

Solution:

$$\begin{aligned}\int \sqrt{x}(3x^2 + 2x + 3)dx &= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx \\&= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx \\&= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \\&= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C\end{aligned}$$

Question 16:

$$\int (2x - 3\cos x + e^x)dx$$

Solution:

$$\begin{aligned}\int (2x - 3\cos x + e^x)dx &= 2\int xdx - 3\int \cos xdx + \int e^xdx \\&= \frac{2x^2}{2} - 3(\sin x) + e^x + C \\&= x^2 - 3\sin x + e^x + C\end{aligned}$$

Question 17:

$$\int (2x^2 - 3\sin x + 5\sqrt{x})dx$$

Solution:

$$\begin{aligned}\int (2x^2 - 3\sin x + 5\sqrt{x})dx &= 2\int x^2dx - 3\int \sin xdx + 5\int x^{\frac{1}{2}}dx \\&= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \\&= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C\end{aligned}$$

Question 18:

$$\int \sec x (\sec x + \tan x) dx$$

Solution:

$$\begin{aligned}\int \sec x (\sec x + \tan x) dx &= \int (\sec^2 x + \sec x \tan x) dx \\ &= \int \sec^2 x dx + \int \sec x \tan x dx \\ &= \tan x + \sec x + C\end{aligned}$$

Question 19:

$$\int \frac{\sec^2 x}{\csc^2 x} dx$$

Solution:

$$\begin{aligned}\int \frac{\sec^2 x}{\csc^2 x} dx &= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx \\ &= \int \frac{\sin^2 x}{\cos^2 x} dx \\ &= \int \tan^2 x dx \\ &= \int (\sec^2 x - 1) dx \\ &= \int \sec^2 x dx - \int 1 dx \\ &= \tan x - x + C\end{aligned}$$

Question 20:

$$\int \frac{2 - 3 \sin x}{\cos^2 x} dx$$

Solution:

$$\begin{aligned}\int \frac{2 - 3 \sin x}{\cos^2 x} dx &= \int \left(\frac{2}{\cos^2 x} - \frac{3 \sin x}{\cos^2 x} \right) dx \\ &= \int 2 \sec^2 x dx - 3 \int \tan x \sec x dx \\ &= 2 \tan x - 3 \sec x + C\end{aligned}$$

Choose the correct answer in Exercises 21 and 22

Question 21:

The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

- (A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$
- (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$
- (C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$
- (D) $\frac{3}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

Solution:

$$\begin{aligned}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) &= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C\end{aligned}$$

Thus, the correct option is C.

Question 22:

If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$, then $f(x)$ is

$$(A) x^4 + \frac{1}{x^3} - \frac{129}{8}$$

$$(B) x^3 + \frac{1}{x^4} + \frac{129}{8}$$

$$(C) x^4 + \frac{1}{x^3} + \frac{129}{8}$$

$$(D) x^3 + \frac{1}{x^4} - \frac{129}{8}$$

Solution:

$$\text{Given, } \frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$$

$$\text{Anti-derivative of } 4x^3 - \frac{3}{x^4} = f(x)$$

Therefore,

$$f(x) = \int 4x^3 - \frac{3}{x^4} dx$$

$$f(x) = 4 \int x^3 dx - 3 \int (x^{-4}) dx$$

$$f(x) = 4 \left(\frac{x^4}{4} \right) - 3 \left(\frac{x^{-3}}{-3} \right) + C$$

$$f(x) = x^4 + \frac{1}{x^3} + C$$

Also,

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$$

$$\Rightarrow 16 + \frac{1}{8} + C = 0$$

$$\Rightarrow C = -\left(16 + \frac{1}{8} \right)$$

$$\Rightarrow C = -\frac{129}{8}$$

$$\Rightarrow f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$$

Thus, the correct option is A.

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

Question 1:

$$\frac{2x}{1+x^2}$$

Solution:

Put $1+x^2 = t$

Therefore, $2x dx = dt$

$$\begin{aligned}\int \frac{2x}{1+x^2} dx &= \int \frac{1}{t} dt = \log|t| + C \\ &= \log|1+x^2| + C \\ &= \log(1+x^2) + C\end{aligned}$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Solution:

Put $\log|x|=t$

Therefore, $\frac{1}{x} dx = dt$

$$\begin{aligned}\int \frac{(\log|x|)^2}{x} dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{(\log|x|)^3}{3} + C\end{aligned}$$

Question 3:

$$\frac{1}{x+x \log x}$$

Solution:

$$\frac{1}{x+x \log x} = \frac{1}{x(1+\log x)}$$

Put $1+\log x=t$

Therefore, $\frac{1}{x} dx = dt$

$$\int \frac{1}{x(1+\log x)} dx = \int \frac{1}{t} dt = \log|t| + C$$
$$= \log|1+\log x| + C$$

Question 4:

$$\sin x \sin(\cos x)$$

Solution:

Put $\cos x = t$

Therefore, $-\sin x dx = dt$

$$\int \sin x \sin(\cos x) dx = -\int \sin t dt = -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$

Question 5:

$$\sin(ax+b)\cos(ax+b)$$

Solution:

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2}$$
$$= \frac{\sin 2(ax+b)}{2}$$

Put $2(ax+b) = t$

Therefore, $2adx = dt$

$$\int \frac{\sin 2(ax+b)}{2} dx = \frac{1}{2} \int \frac{\sin t dt}{2a}$$
$$= \frac{1}{4a} [-\cos t] + C$$
$$= \frac{-1}{4a} \cos 2(ax+b) + C$$

Question 6:

$$\sqrt{ax+b}$$

Solution:

Put $ax+b = t$

Therefore,

$$\begin{aligned}
&\Rightarrow adx = dt \\
&\Rightarrow dx = \frac{1}{a}dt \\
\int (ax+b)^{\frac{1}{2}} dx &= \frac{1}{a} \int t^{\frac{1}{2}} dt = \frac{1}{a} \left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}} \right) + C \\
&= \frac{2}{3a} (ax+b)^{\frac{3}{2}} + C
\end{aligned}$$

Question 7:

$$x\sqrt{x+2}$$

Solution:

Put, $x+2 = t$

$$\therefore dx = dt$$

$$\Rightarrow \int x\sqrt{x+2} = \int (t-2)\sqrt{t} dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}} \right) dt$$

$$= \int t^{\frac{3}{2}} dt - 2 \int t^{\frac{1}{2}} dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$$x\sqrt{1+2x^2}$$

Solution:

$$\text{Put, } 1+2x^2 = t$$

$$\therefore 4x dx = dt$$

$$\begin{aligned}\Rightarrow \int x \sqrt{1+2x^2} dx &= \int \frac{\sqrt{t}}{4} dt \\ &= \frac{1}{4} \int t^{\frac{1}{2}} dt \\ &= \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{1}{6} (1+2x^2)^{\frac{3}{2}} + C\end{aligned}$$

Question 9:

$$(4x+2) \sqrt{x^2+x+1}$$

Solution:

$$\text{Put, } x^2 + x + 1 = t$$

$$\therefore (2x+1) dx = dt$$

$$\begin{aligned}\int (4x+2) \sqrt{x^2+x+1} dx &= \int 2\sqrt{t} dt \\ &= 2 \int \sqrt{t} dt \\ &= 2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = \frac{4}{3} (x^2 + x + 1)^{\frac{3}{2}} + C\end{aligned}$$

Question 10:

$$\frac{1}{x - \sqrt{x}}$$

Solution:

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}(\sqrt{x}-1)}$$

Put, $(\sqrt{x}-1) = t$

$$\therefore \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{x}(\sqrt{x}-1)} dx = \int \frac{2}{t} dt$$

$$= 2 \log|t| + C$$

$$= 2 \log|\sqrt{x}-1| + C$$

Question 11:

$$\frac{x}{\sqrt{x+4}}, x > 0$$

Solution:

Put, $x+4 = t$

$$\therefore dx = dt$$

$$\int \frac{x}{\sqrt{x+4}} dx = \int \frac{(t-4)}{\sqrt{t}} dt = \int \left(\sqrt{t} - \frac{4}{\sqrt{t}} \right) dt$$

$$= \left(\frac{\frac{3}{2}t^{\frac{3}{2}}}{\frac{3}{2}} \right) - 4 \left(\frac{\frac{1}{2}t^{\frac{1}{2}}}{\frac{1}{2}} \right) + C = \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$$

$$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

Question 12:

$$(x^3 - 1)^{\frac{1}{3}} x^5$$

Solution:

$$\text{Put, } x^3 - 1 = t$$

$$\therefore 3x^2 dx = dt$$

$$\Rightarrow \int (x^3 - 1)^{\frac{1}{3}} x^5 dx = \int (x^3 - 1)^{\frac{1}{3}} x^3 x^2 dx$$

$$\Rightarrow \int t^{\frac{1}{3}} (t+1) \frac{dt}{3} = \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt$$

$$= \frac{1}{3} \left[\frac{\frac{7}{3}}{7} t^{\frac{7}{3}} + \frac{\frac{4}{3}}{4} t^{\frac{4}{3}} \right] + C$$

$$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$$

$$= \frac{1}{7} (x^3 - 1)^{\frac{7}{3}} + \frac{1}{4} (x^3 - 1)^{\frac{4}{3}} + C$$

Question 13:

$$\frac{x^2}{(2+3x^3)^3}$$

Solution:

$$\text{Put, } 2+3x^3 = t$$

$$\therefore 9x^2 dx = dt$$

$$\Rightarrow \int \frac{x^2}{(2+3x^3)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$$

$$= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C$$

$$= -\frac{1}{18} \left(\frac{1}{t^2} \right) + C$$

$$= \frac{-1}{18(2+3x^3)^2} + C$$

Question 14:

$$\frac{1}{x(\log x)^m}, x > 0$$

Solution:

Put, $\log x = t$

$$\begin{aligned} \therefore \frac{1}{x} dx &= dt \\ \Rightarrow \int \frac{1}{x(\log x)^m} dx &= \int \frac{dt}{(t)^m} = \left(\frac{t^{-m+1}}{1-m} \right) + C \\ &= \frac{(\log x)^{1-m}}{(1-m)} + C \end{aligned}$$

Question 15:

$$\frac{x}{9-4x^2}$$

Solution:

Put, $9 - 4x^2 = t$

$$\therefore -8x dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{x}{9-4x^2} dx &= \frac{-1}{8} \int \frac{1}{t} dt \\ &= \frac{-1}{8} \log|t| + C \\ &= \frac{-1}{8} \log|9-4x^2| + C \end{aligned}$$

Question 16:

$$e^{2x+3}$$

Solution:

Put, $2x + 3 = t$

$$\therefore 2dx = dt$$

$$\begin{aligned} \Rightarrow \int e^{2x+3} dx &= \frac{1}{2} \int e^t dt \\ &= \frac{1}{2} (e^t) + C \\ &= \frac{1}{2} e^{(2x+3)} + C \end{aligned}$$

Question 17:

$$\frac{x}{e^{x^2}}$$

Solution:

Put, $x^2 = t$

$$\therefore 2x dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{x}{e^{x^2}} dx &= \frac{1}{2} \int \frac{1}{e^t} dt = \frac{1}{2} \int e^{-t} dt \\ &= \frac{1}{2} \left(\frac{e^{-t}}{-1} \right) + C \\ &= -\frac{1}{2} e^{-x^2} + C \\ &= \frac{-1}{2e^{x^2}} + C \end{aligned}$$

Question 18:

$$\frac{e^{\tan^{-1} x}}{1+x^2}$$

Solution:

Put, $\tan^{-1} x = t$

$$\therefore \frac{1}{1+x^2} dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{e^{\tan^{-1} x}}{1+x^2} dx &= \int e^t dt \\ &= e^t + C \\ &= e^{\tan^{-1} x} + C \end{aligned}$$

Question 19:

$$\frac{e^{2x} - 1}{e^{2x} + 1}$$

Solution:

$$\frac{e^{2x} - 1}{e^{2x} + 1}$$

Dividing Nr and Dr by e^x , we get

$$\frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Let $e^x + e^{-x} = t$

$$\begin{aligned} & (e^x - e^{-x}) dx = dt \\ & \Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx \\ & = \int \frac{dt}{t} \\ & = \log|t| + C \\ & = \log|e^x + e^{-x}| + C \end{aligned}$$

Question 20:

$$\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$$

Solution:

$$\begin{aligned} & \text{Put, } e^{2x} + e^{-2x} = t \\ & (2e^{2x} - 2e^{-2x}) dx = dt \\ & \Rightarrow 2(e^{2x} - e^{-2x}) dx = dt \\ & \Rightarrow \int \left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right) dx = \int \frac{dt}{2t} \\ & = \frac{1}{2} \int \frac{1}{t} dt \\ & = \frac{1}{2} \log|t| + C \\ & = \frac{1}{2} \log|e^{2x} + e^{-2x}| + C \end{aligned}$$

Question 21:

$$\tan^2(2x-3)$$

Solution:

$$\tan^2(2x-3) = \sec^2(2x-3) - 1$$

Put, $2x-3 = t$

$$\begin{aligned}
& \therefore 2dx = dt \\
& \Rightarrow \int \tan^2(2x-3) dx = \int [\sec^2(2x-3) - 1] dx \\
& = \frac{1}{2} \int (\sec^2 t) dt - \int 1 dx = \frac{1}{2} \int \sec^2 t dt - \int 1 dx \\
& = \frac{1}{2} \tan t - x + C \\
& = \frac{1}{2} \tan(2x-3) - x + C
\end{aligned}$$

Question 22:

$$\sec^2(7-4x)$$

Solution:

$$\text{Put, } 7-4x = t$$

$$\therefore -4dx = dt$$

$$\begin{aligned}
& \therefore \int \sec^2(7-4x) dx = \frac{-1}{4} \int \sec^2 t dt \\
& = \frac{-1}{4} (\tan t) + C \\
& = \frac{-1}{4} \tan(7-4x) + C
\end{aligned}$$

Question 23:

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Solution:

$$\text{Put, } \sin^{-1} x = t$$

$$\begin{aligned}
& \frac{1}{\sqrt{1-x^2}} dx = dt \\
& \Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t dt \\
& = \frac{t^2}{2} + C = \frac{(\sin^{-1} x)^2}{2} + C
\end{aligned}$$

Question 24:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$$

Solution:

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$

$$(-3\sin x + 2\cos x)dx = dt$$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2} \int \frac{1}{t} dt$$

$$= \frac{1}{2} \log|t| + C$$

$$= \frac{1}{2} \log|2\sin x + 3\cos x| + C$$

Question 25:

$$\frac{1}{\cos^2 x (1 - \tan x)^2}$$

Solution:

$$\frac{1}{\cos^2 x (1 - \tan x)^2} = \frac{\sec^2 x}{(1 - \tan x)^2}$$

Let $(1 - \tan x) = t$

$$-\sec^2 x dx = dt$$

$$\Rightarrow \int \frac{\sec^2 x}{(1 - \tan x)^2} dx = \int \frac{-dt}{t^2}$$

$$= - \int t^{-2} dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{(1 - \tan x)} + C$$

Question 26:

$$\frac{\cos \sqrt{x}}{\sqrt{x}}$$

Solution:

Let $\sqrt{x} = t$

$$\begin{aligned}\frac{1}{2\sqrt{x}}dx &= dt \\ \Rightarrow \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx &= 2 \int \cos t dt \\ &= 2 \sin t + C \\ &= 2 \sin \sqrt{x} + C\end{aligned}$$

Question 27:

$$\sqrt{\sin 2x} \cos 2x$$

Solution:

Put, $\sin 2x = t$

So, $2 \cos 2x dx = dt$

$$\begin{aligned}\Rightarrow \int \sqrt{\sin 2x} \cos 2x dx &= \frac{1}{2} \int \sqrt{t} dt \\ &= \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{1}{3} t^{\frac{3}{2}} + C \\ &= \frac{1}{3} (\sin 2x)^{\frac{3}{2}} + C\end{aligned}$$

Question 28:

$$\frac{\cos x}{\sqrt{1 + \sin x}}$$

Solution:

Put, $1 + \sin x = t$

$$\begin{aligned}\therefore \cos x dx &= dt \\ \Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx &= \int \frac{dt}{\sqrt{t}} \\ &= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{1+\sin x} + C\end{aligned}$$

Question 29:

$$\cot x \log \sin x$$

Solution:

$$\text{Let } \log \sin x = t$$

$$\begin{aligned}\Rightarrow \frac{1}{\sin x} \cos x dx &= dt \\ \therefore \cot x dx &= dt \\ \Rightarrow \int \cot x \log \sin x dx &= \int t dt \\ &= \frac{t^2}{2} + C \\ &= \frac{1}{2} (\log \sin x)^2 + C\end{aligned}$$

Question 30:

$$\frac{\sin x}{1+\cos x}$$

Solution:

$$\text{Put, } 1+\cos x = t$$

$$\therefore -\sin x dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{\sin x}{1+\cos x} dx &= \int -\frac{dt}{t} \\ &= -\log|t| + C \\ &= -\log|1+\cos x| + C\end{aligned}$$

Question 31:

$$\frac{\sin x}{(1+\cos x)^2}$$

Solution:

Put, $1+\cos x = t$

$$\therefore -\sin x dx = dt$$

$$\begin{aligned} \Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx &= \int -\frac{dt}{t^2} \\ &= -\int t^{-2} dt \\ &= \frac{1}{t} + C \\ &= \frac{1}{(1+\cos x)} + C \end{aligned}$$

Question 32:

$$\frac{1}{1+\cos x}$$

Solution:

$$\begin{aligned} \text{Let } I &= \int \frac{1}{1+\cos x} dx \\ &= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx \\ &= \int \frac{\sin x}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x} dx \\ &= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx \\ &= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx \\ &= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx \end{aligned}$$

$$\text{Let } \sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt$$

$$\begin{aligned} \therefore I &= \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t} \\ &= \frac{x}{2} - \frac{1}{2} \log|t| + C = \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + C \end{aligned}$$

Question 33:

$$\frac{1}{1-\tan x}$$

Solution:

$$\begin{aligned} \text{Put, } I &= \int \frac{1}{1-\tan x} dx \\ &= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{\cos x - \sin x} dx \\ &= \frac{1}{2} \int \frac{2\cos x}{\cos x - \sin x} dx = \frac{1}{2} \int \frac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)} dx \\ &= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx = \frac{x}{2} + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x} dx \end{aligned}$$

$$\text{Put, } \cos x - \sin x = t \Rightarrow (-\sin x - \cos x) dx = dt$$

$$\begin{aligned} \therefore I &= \frac{x}{2} + \frac{1}{2} \int \frac{-dt}{t} = \frac{x}{2} - \frac{1}{2} \log|t| + C \\ &= \frac{x}{2} - \frac{1}{2} \log|\cos x - \sin x| + C \end{aligned}$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x \cos x}$$

Solution:

$$\begin{aligned} \text{Let } I &= \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx = \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx \\ &= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx = \int \frac{\sec^2 x dx}{\sqrt{\tan x}} \end{aligned}$$

$$\text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{\sqrt{t}} \\ &= 2\sqrt{t} + C \\ &= 2\sqrt{\tan x} + C \end{aligned}$$

Question 35:

$$\frac{(1+\log x)^2}{x}$$

Solution:

Put, $1 + \log x = t$

$$\begin{aligned}\therefore \frac{1}{x} dx &= dt \\ \Rightarrow \int \frac{(1 + \log x)^2}{x} dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{(1 + \log x)^3}{3} + C\end{aligned}$$

Question 36:

$$\frac{(x+1)(x+\log x)^2}{x}$$

Solution:

$$\frac{(x+1)(x+\log x)^2}{x} = \left(\frac{x+1}{x}\right)(x+\log x)^2 = \left(1 + \frac{1}{x}\right)(x+\log x)^2$$

Put, $(x + \log x) = t$

$$\begin{aligned}\therefore \left(1 + \frac{1}{x}\right) dx &= dt \\ \Rightarrow \int \left(1 + \frac{1}{x}\right) (x + \log x)^2 dx &= \int t^2 dt \\ &= \frac{t^3}{3} + C \\ &= \frac{1}{3}(x + \log x)^3 + C\end{aligned}$$

Question 37:

$$\frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}$$

Solution:

Put, $x^4 = t$

$$\begin{aligned}\therefore 4x^3 dx &= dt \\ \Rightarrow \int \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8} dx &= \frac{1}{4} \int \frac{\sin(\tan^{-1} t)}{1+t^2} dt \quad \dots(1)\end{aligned}$$

Let $\tan^{-1} t = u$

$$\therefore \frac{1}{1+t^2} dt = du$$

From (1), we get

$$\int \frac{x^3 \sin(\tan^{-1} x^4) dx}{1+x^8} = \frac{1}{4} \int \sin u du$$

$$= \frac{1}{4} (-\cos u) + C$$

$$= -\frac{1}{4} \cos(\tan^{-1} t) + C$$

$$= -\frac{1}{4} \cos(\tan^{-1} x^4) + C$$

Choose the correct answer in Exercises 38 and 39.

Question 38:

$$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx \quad \text{equals}$$

- | | |
|--------------------------------|-------------------------------|
| (A) $10^x - x^{10} + C$ | (B) $10^x + x^{10} + C$ |
| (C) $(10^x - x^{10})^{-1} + C$ | (D) $\log(10^x + x^{10}) + C$ |

Solution:

Put, $x^{10} + 10^x = t$

$$\therefore (10x^9 + 10^x \log_e 10) dx = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log(10^x + x^{10}) + C$$

Thus, the correct option is D.

Question 39:

$$\int \frac{dx}{\sin^2 x \cos^2 x} \quad \text{equals}$$

- | | |
|---------------------------|----------------------------|
| (A) $\tan x + \cot x + C$ | (B) $\tan x - \cot x + C$ |
| (C) $\tan x \cot x + C$ | (D) $\tan x - \cot 2x + C$ |

Solution:

$$\text{Put, } I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{1}{\sin^2 x \cos^2 x} dx$$

$$\begin{aligned}&= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\&= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\&= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\&= \tan x - \cot x + C\end{aligned}$$

Thus, the correct option is B.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

Question 1:

$$\sin^2(2x+5)$$

Solution:

$$\sin^2(2x+5) = \frac{1 - \cos 2(2x+5)}{2} = \frac{1 - \cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5) dx = \int \frac{1 - \cos(4x+10)}{2} dx$$

$$= \frac{1}{2} \int 1 dx - \frac{1}{2} \int \cos(4x+10) dx$$

$$= \frac{1}{2}x - \frac{1}{2} \left(\frac{\sin(4x+10)}{4} \right) + C$$

$$= \frac{1}{2}x - \frac{1}{8} \sin(4x+10) + C$$

Question 2:

$$\sin 3x \cos 4x$$

Solution:

$$\text{Using, } \sin A \cos B = \frac{1}{2} \{ \sin(A+B) + \sin(A-B) \}$$

$$\therefore \int \sin 3x \cos 4x dx = \frac{1}{2} \int \{ \sin(3x+4x) + \sin(3x-4x) \} dx$$

$$= \frac{1}{2} \int \{ \sin 7x + \sin(-x) \} dx$$

$$= \frac{1}{2} \int \{ \sin 7x - \sin x \} dx$$

$$= \frac{1}{2} \int \sin 7x dx - \frac{1}{2} \int \sin x dx$$

$$= \frac{1}{2} \left(\frac{-\cos 7x}{7} \right) - \frac{1}{2} (-\cos x) + C$$

$$= \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$$

Question 3:

$$\cos 2x \cos 4x \cos 6x$$

Solution:

Using, $\cos A \cos B = \frac{1}{2} \{ \cos(A+B) + \cos(A-B) \}$

$$\begin{aligned}\therefore \int \cos 2x (\cos 4x \cos 6x) dx &= \int \cos 2x \left[\frac{1}{2} \{ \cos(4x+6x) + \cos(4x-6x) \} \right] dx \\&= \frac{1}{2} \int \{ \cos 2x \cos 10x + \cos 2x \cos(-2x) \} dx \\&= \frac{1}{2} \int \{ \cos 2x \cos 10x + \cos^2 2x \} dx \\&= \frac{1}{2} \int \left[\left\{ \frac{1}{2} \cos(2x+10x) + \frac{1}{2} \cos(2x-10x) \right\} + \left(\frac{1+\cos 4x}{2} \right) \right] dx \\&= \frac{1}{4} \int (\cos 12x + \cos 8x + 1 + \cos 4x) dx \\&= \frac{1}{4} \left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + x + \frac{\sin 4x}{4} + C \right]\end{aligned}$$

Question 4:

$$\sin^3(2x+1)$$

Solution:

Put, $I = \int \sin^3(2x+1) dx$

$$\begin{aligned}&\Rightarrow \int \sin^3(2x+1) dx = \int \sin^2(2x+1) \sin(2x+1) dx \\&= \int (1 - \cos^2(2x+1)) \sin(2x+1) dx\end{aligned}$$

Let $\cos(2x+1) = t$

$$\Rightarrow -2 \sin(2x+1) dx = dt$$

$$\Rightarrow \sin(2x+1) dx = \frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2} \int (1-t^2) dt$$

$$= \frac{-1}{2} \left\{ t - \frac{t^3}{3} \right\}$$

$$= \frac{-1}{2} \left\{ \cos(2x+1) - \frac{\cos^3(2x+1)}{3} \right\}$$

$$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$

Question 5:

$$\sin^3 x \cos^3 x$$

Solution:

$$\begin{aligned} \text{Let } I &= \int \sin^3 x \cos^3 x dx \\ &= \int \cos^3 x \sin^2 x \sin x dx \\ &= \int \cos^3 x (1 - \cos^2 x) \sin x dx \end{aligned}$$

$$\text{Let } \cos x = t$$

$$\Rightarrow -\sin x dx = dt$$

$$\begin{aligned} \Rightarrow I &= - \int t^3 (1 - t^2) dt \\ &= - \int (t^3 - t^5) dt = - \left\{ \frac{t^4}{4} - \frac{t^6}{6} \right\} + C \\ &= - \left\{ \frac{\cos^4 x}{4} - \frac{\cos^6 x}{6} \right\} + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C \end{aligned}$$

Question 6:

$$\sin x \sin 2x \sin 3x$$

Solution:

$$\text{Using, } \sin A \sin B = \frac{1}{2} \{ \cos(A - B) - \cos(A + B) \}$$

$$\begin{aligned} \therefore \int \sin x \sin 2x \sin 3x dx &= \int \left[\sin x \frac{1}{2} \{ \cos(2x - 3x) - \cos(2x + 3x) \} \right] dx \\ &= \frac{1}{2} \int (\sin x \cos(-x) - \sin x \cos 5x) dx \\ &= \frac{1}{2} \int (\sin x \cos x - \sin x \cos 5x) dx \\ &= \frac{1}{2} \int \frac{\sin 2x}{2} dx - \frac{1}{2} \int \sin x \cos 5x dx \\ &= \frac{1}{4} \left[\frac{-\cos 2x}{2} \right] - \frac{1}{2} \int \left\{ \frac{1}{2} \sin(x + 5x) + \frac{1}{2} \sin(x - 5x) \right\} dx \\ &= \frac{-\cos 2x}{8} - \frac{1}{4} \int (\sin 6x + \sin(-4x)) dx \\ &= \frac{-\cos 2x}{8} - \frac{1}{4} \left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4} \right] + C \\ &= \frac{-\cos 2x}{8} - \frac{1}{8} \left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2} \right] + C \\ &= \frac{1}{8} \left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x \right] + C \end{aligned}$$

Question 7:

$$\sin 4x \sin 8x$$

Solution:

Using, $\sin A \sin B = \frac{1}{2} \{ \cos(A - B) - \cos(A + B) \}$

$$\begin{aligned}\therefore \int \sin 4x \sin 8x dx &= \int \left\{ \frac{1}{2} \cos(4x - 8x) - \frac{1}{2} \cos(4x + 8x) \right\} dx \\ &= \frac{1}{2} \int (\cos(-4x) - \cos 12x) dx \\ &= \frac{1}{2} \int (\cos 4x - \cos 12x) dx \\ &= \frac{1}{2} \left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12} \right]\end{aligned}$$

Question 8:

$$\frac{1 - \cos x}{1 + \cos x}$$

Solution:

$$\begin{aligned}\frac{1 - \cos x}{1 + \cos x} &= \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \quad \left[2 \sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2 \cos^2 \frac{x}{2} = 1 + \cos x \right] \\ &= \tan^2 \frac{x}{2} \\ &= \left(\sec^2 \frac{x}{2} - 1 \right) \\ \therefore \frac{1 - \cos x}{1 + \cos x} dx &= \int \left(\sec^2 \frac{x}{2} - 1 \right) dx \\ &= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x \right] + C \\ &= 2 \tan \frac{x}{2} - x + C\end{aligned}$$

Question 9:

$$\frac{\cos x}{1 + \cos x}$$

Solution:

$$\begin{aligned} \frac{\cos x}{1 + \cos x} &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \quad \left[\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \text{ and } \cos x = 2 \cos^2 \frac{x}{2} - 1 \right] \\ &= \frac{1}{2} \left[1 - \tan^2 \frac{x}{2} \right] \\ \therefore \int \frac{\cos x}{1 + \cos x} dx &= \frac{1}{2} \int \left(1 - \tan^2 \frac{x}{2} \right) dx \\ &= \frac{1}{2} \int \left(1 - \sec^2 \frac{x}{2} + 1 \right) dx \\ &= \frac{1}{2} \int \left(2 - \sec^2 \frac{x}{2} \right) dx \\ &= \frac{1}{2} \left[2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C \\ &= x - \tan \frac{x}{2} + C \end{aligned}$$

Question 10:

$$\sin^4 x$$

Solution:

$$\begin{aligned}\sin^4 x &= \sin^2 x \sin^2 x \\&= \left(\frac{1-\cos 2x}{2}\right) \left(\frac{1-\cos 2x}{2}\right) \\&= \frac{1}{4}(1-\cos 2x)^2 \\&= \frac{1}{4}[1+\cos^2 2x - 2\cos 2x] \\&= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right] \\&= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] \\&= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right] \\&\therefore \int \sin^4 x dx = \frac{1}{4} \int \left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right] dx \\&= \frac{1}{4} \left[\frac{3}{2}x + \frac{1}{2} \left(\frac{\sin 4x}{4} \right) - 2 \times \frac{\sin 2x}{2} \right] + C \\&= \frac{1}{8} \left[3x + \frac{\sin 4x}{4} - 2\sin 2x \right] + C \\&= \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C\end{aligned}$$

Question 11:

$$\cos^4 2x$$

Solution:

$$\begin{aligned}\cos^4 2x &= (\cos^2 2x)^2 \\&= \left(\frac{1+\cos 4x}{2}\right)^2 \\&= \frac{1}{4} [1 + \cos^2 4x + 2\cos 4x] \\&= \frac{1}{4} \left[1 + \left(\frac{1+\cos 8x}{2}\right) + 2\cos 4x\right] \\&= \frac{1}{4} \left[1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right] \\&= \frac{1}{4} \left[\frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right] \\∴ \int \cos^4 2x dx &= \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx \\&= \frac{3}{8}x + \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + C\end{aligned}$$

Question 12:

$$\frac{\sin^2 x}{1 + \cos x}$$

Solution:

$$\begin{aligned}\frac{\sin^2 x}{1 + \cos x} &= \frac{\left(2\sin \frac{x}{2} \cos \frac{x}{2}\right)^2}{2\cos^2 \frac{x}{2}} \quad \left[\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}; \cos x = 2\cos^2 \frac{x}{2} - 1\right] \\&= \frac{4\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \\&= 2\sin^2 \frac{x}{2} \\&= 1 - \cos x \\∴ \int \frac{\sin^2 x}{1 + \cos x} dx &= \int (1 - \cos x) dx \\&= x - \sin x + C\end{aligned}$$

Question 13:

$$\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$

Solution:

$$\begin{aligned}
 \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} &= \frac{-2 \sin \frac{2x+2\alpha}{2} \sin \frac{2x-2\alpha}{2}}{-2 \sin \frac{x+\alpha}{2} \sin \frac{x-\alpha}{2}} \quad \left[\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \right] \\
 &= \frac{\sin(x+\alpha) \sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right)} \\
 &= \frac{\left[2 \sin\left(\frac{x+\alpha}{2}\right) \cos\left(\frac{x-\alpha}{2}\right) \right] \left[2 \sin\left(\frac{x-\alpha}{2}\right) \cos\left(\frac{x+\alpha}{2}\right) \right]}{\sin\left(\frac{x+\alpha}{2}\right) \sin\left(\frac{x-\alpha}{2}\right)} \\
 &= 4 \cos\left(\frac{x+\alpha}{2}\right) \cos\left(\frac{x-\alpha}{2}\right) \\
 &= 2 \left[\cos\left(\frac{x+\alpha}{2} + \frac{x-\alpha}{2}\right) + \cos\left(\frac{x+\alpha}{2} - \frac{x-\alpha}{2}\right) \right] \\
 &= 2 [\cos(x) + \cos\alpha] \\
 &= 2 \cos x + 2 \cos \alpha \\
 \therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx &= \int 2 \cos x + 2 \cos \alpha dx \\
 &= 2 [\sin x + x \cos \alpha] + C
 \end{aligned}$$

Question 14:

$$\frac{\cos x - \sin x}{1 + \sin 2x}$$

Solution:

$$\begin{aligned}
 \frac{\cos x - \sin x}{1 + \sin 2x} &= \frac{\cos x - \sin x}{(\sin^2 x + \cos^2 x) + 2 \sin x \cos x} \quad [\sin^2 x + \cos^2 x = 1; \sin 2x = 2 \sin x \cos x] \\
 &= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}
 \end{aligned}$$

Let $\sin x + \cos x = t$

$$\begin{aligned}
& \therefore (\cos x - \sin x) dx = dt \\
\Rightarrow & \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx \\
& = \int \frac{dt}{t^2} \\
& = \int t^{-2} dt \\
& = -t^{-1} + C \\
& = -\frac{1}{t} + C \\
& = \frac{-1}{\sin x + \cos x} + C
\end{aligned}$$

Question 15:

$$\tan^3 2x \sec 2x$$

Solution:

$$\begin{aligned}
\tan^3 2x \sec 2x &= \tan^2 2x \tan 2x \sec 2x \\
&= (\sec^2 2x - 1) \tan 2x \sec 2x \\
&= \sec^2 2x \tan 2x \sec 2x - \tan 2x \sec 2x \\
\therefore \int \tan^3 2x \sec 2x dx &= \int \sec^2 2x \tan 2x \sec 2x dx - \int \tan 2x \sec 2x dx \\
&= \int \sec^2 2x \tan 2x \sec 2x - \frac{\sec 2x}{2} + C
\end{aligned}$$

$$\text{Let } \sec 2x = t$$

$$\therefore 2 \sec 2x \tan 2x dx = dt$$

$$\begin{aligned}
\therefore \int \tan^3 2x \sec 2x dx &= \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C \\
&= \frac{t^3}{6} - \frac{\sec 2x}{2} + C \\
&= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C
\end{aligned}$$

Question 16:

$$\tan^4 x$$

Solution:

$$\begin{aligned}
 & \tan^4 x \\
 &= \tan^2 x \tan^2 x \\
 &= (\sec^2 x - 1) \tan^2 x \\
 &= \sec^2 x \tan^2 x - \tan^2 x \\
 &= \sec^2 x \tan^2 x - (\sec^2 x - 1) \\
 &= \sec^2 x \tan^2 x - \sec^2 x + 1 \\
 \therefore \int \tan^4 x dx &= \int \sec^2 x \tan^2 x dx - \int \sec^2 x dx + \int 1 dx \\
 &= \int \sec^2 x \tan^2 x dx - \tan x + x + C \quad \dots(1)
 \end{aligned}$$

Consider $\sec^2 x \tan^2 x dx$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we get

$$\int \tan^4 x dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Question 17:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$$

Solution:

$$\begin{aligned}
 \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x} \\
 &= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\
 &= \tan x \sec x + \cot x \operatorname{cosec} x \\
 \therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx &= \int (\tan x \sec x + \cot x \operatorname{cosec} x) dx \\
 &= \sec x - \operatorname{cosec} x + C
 \end{aligned}$$

Question 18:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Solution:

$$\begin{aligned} & \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} \\ &= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \quad [\cos 2x = 1 - 2\sin^2 x] \\ &= \frac{1}{\cos^2 x} = \sec^2 x \\ \therefore & \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C \end{aligned}$$

Question 19:

$$\frac{1}{\sin x \cos^3 x}$$

Solution:

$$\begin{aligned} \frac{1}{\sin x \cos^3 x} &= \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} \\ &= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x} \\ &= \tan x \sec^2 x + \frac{1}{\frac{\sin x \cos x}{\cos^2 x}} \\ &= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x} \end{aligned}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx$$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\begin{aligned} & \Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t dt + \int \frac{1}{t} dt \\ &= \frac{t^2}{2} + \log|t| + C \\ &= \frac{1}{2} \tan^2 x + \log|\tan x| + C \end{aligned}$$

Question 20:

$$\frac{\cos 2x}{(\cos x + \sin x)^2}$$

Solution:

$$\frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2 \cos x \sin x} = \frac{\cos 2x}{1 + \sin 2x}$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos 2x}{(1 + \sin 2x)} dx$$

Let $1 + \sin 2x = t$

$$\Rightarrow 2 \cos 2x dx = dt$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \frac{1}{2} \int \frac{1}{t} dt$$

$$= \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} \log |1 + \sin 2x| + C$$

$$= \frac{1}{2} \log |(\sin x + \cos x)^2| + C$$

$$= \log |\sin x + \cos x| + C$$

Question 21:

$$\sin^{-1}(\cos x)$$

Solution:

$$\sin^{-1}(\cos x)$$

Let $\cos x = t$

$$\text{Then, } \sin x = \sqrt{1 - t^2}$$

$$\Rightarrow (-\sin x) dx = dt$$

$$dx = \frac{-dt}{\sin x}$$

$$dx = \frac{-dt}{\sqrt{1-t^2}}$$

$$\therefore \int \sin^{-1}(\cos x) dx = \int \sin^{-1} t \left(\frac{-dt}{\sqrt{1-t^2}} \right)$$

$$= - \int \frac{\sin^{-1} t}{\sqrt{1-t^2}}$$

Let $\sin^{-1} t = u$

$$\begin{aligned}\Rightarrow \frac{1}{\sqrt{1-t^2}} dt &= du \\ \therefore \int \sin^{-1}(\cos x) dx &= - \int u du \\ &= -\frac{u^2}{2} + C \\ &= -\frac{(\sin^{-1} t)^2}{2} + C \\ &= -\frac{[\sin^{-1}(\cos x)]^2}{2} + C \quad \dots(1)\end{aligned}$$

We know that,

$$\begin{aligned}\sin^{-1} x + \cos^{-1} x &= \frac{\pi}{2} \\ \therefore \sin^{-1}(\cos x) &= \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)\end{aligned}$$

Substituting in equation (1), we get

$$\begin{aligned}\int \sin^{-1}(\cos x) dx &= -\frac{\left[\frac{\pi}{2} - x\right]^2}{2} + C \\ &= -\frac{1}{2} \left(\frac{\pi^2}{4} + x^2 - \pi x \right) + C \\ &= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{\pi x}{2} + C \\ &= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right) \\ &= \frac{\pi x}{2} - \frac{x^2}{2} + C_1\end{aligned}$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Solution:

$$\begin{aligned}
 \frac{1}{\cos(x-a)\cos(x-b)} &= \frac{1}{\sin(a-b)} \left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} \right] \\
 &= \frac{1}{\sin(a-b)} \left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} \right] \\
 &= \frac{1}{\sin(a-b)} \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} \right] \\
 &= \frac{1}{\sin(a-b)} [\tan(x-b) - \tan(x-a)] \\
 \Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)} dx &= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] dx \\
 &= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|] \\
 &= \frac{1}{\sin(a-b)} \left[\log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| \right] + C
 \end{aligned}$$

Choose the correct answer in Exercises 23 and 24.

Question 23:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$$

- is equal to
- (A) $\tan x + \cot x + C$ (B) $\tan x + \sec ec x + C$
 (C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

Solution:

$$\begin{aligned}
 \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx &= \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx \\
 &= \int (\sec^2 x - \sec ec^2 x) dx \\
 &= \tan x + \cot x + C
 \end{aligned}$$

Thus, the correct option is A.

Question 24:

$$\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx \text{ equals}$$

- (A) $-\cot(ex^x) + C$ (B) $\tan(xe^x) + C$
(C) $\tan(e^x) + C$ (D) $\cot(e^x) + C$

Solution:

$$\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$$

Put, $e^x x = t$

$$\Rightarrow (e^x x + e^x \cdot 1) dx = dt$$

$$e^x (x+1) dx = dt$$

$$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)} dx = \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t dt$$

$$= \tan t + C$$

$$= \tan(e^x x) + C$$

Thus, the correct answer is B.

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23

Question 1:

$$\frac{3x^2}{x^6 + 1}$$

Solution:

Put, $x^3 = t$

$$\therefore 3x^2 dx = dt$$

$$\begin{aligned}\Rightarrow \frac{3x^2}{x^6 + 1} dx &= \int \frac{dt}{t^2 + 1} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(x^3) + C\end{aligned}$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Solution:

Put, $2x = t$

$$\therefore 2dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{1+4x^2}} dx &= \frac{1}{2} \int \frac{dt}{\sqrt{1+t^2}} \\ &= \frac{1}{2} \left[\log \left| t + \sqrt{t^2 + 1} \right| \right] + C \quad \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log \left| x + \sqrt{x^2 + a^2} \right| \right] \\ &= \frac{1}{2} \log \left| 2x + \sqrt{4x^2 + 1} \right| + C\end{aligned}$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2 + 1}}$$

Solution:

Put, $2-x = t$

$$\begin{aligned}
&\Rightarrow -dx = dt \\
&\Rightarrow \int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = - \int \frac{1}{\sqrt{t^2 + 1}} dt \\
&= -\log|t + \sqrt{t^2 + 1}| + C \quad \left[\int \frac{1}{\sqrt{x^2 + a^2}} dt = \log|x + \sqrt{x^2 + a^2}| \right] \\
&= -\log|2-x + \sqrt{(2-x)^2 + 1}| + C \\
&= \log \left| \frac{1}{(2-x) + \sqrt{x^2 - 4x + 5}} \right| + C
\end{aligned}$$

Question 4:

$$\frac{1}{\sqrt{9-25x^2}}$$

Solution:

$$\text{Put, } 5x = t$$

$$\therefore 5dx = dt$$

$$\begin{aligned}
&\Rightarrow \int \frac{1}{\sqrt{9-25x^2}} dx = \frac{1}{5} \int \frac{1}{\sqrt{9-t^2}} dt \\
&= \frac{1}{5} \int \frac{1}{\sqrt{3^2-t^2}} dt \\
&= \frac{1}{5} \sin^{-1}\left(\frac{t}{3}\right) + C \\
&= \frac{1}{5} \sin^{-1}\left(\frac{5x}{3}\right) + C
\end{aligned}$$

Question 5:

$$\frac{3x}{1+2x^4}$$

Solution:

$$\text{Let } \sqrt{2}x^2 = t$$

$$\begin{aligned}\therefore 2\sqrt{2}xdx &= dt \\ \Rightarrow \int \frac{3x}{1+2x^4} dx &= \frac{3}{2\sqrt{2}} \int \frac{dt}{1+t^2} \\ &= \frac{3}{2\sqrt{2}} \left[\tan^{-1} t \right] + C \\ &= \frac{3}{2\sqrt{2}} \tan^{-1}(\sqrt{2}x^2) + C\end{aligned}$$

Question 6:

$$\frac{x^2}{1-x^6}$$

Solution:

Put, $x^3 = t$

$$\begin{aligned}\therefore 3x^2 dx &= dt \\ \Rightarrow \int \frac{x^2}{1-x^6} dx &= \frac{1}{3} \int \frac{dt}{1-t^2} \\ &= \frac{1}{3} \left[\frac{1}{2} \log \left| \frac{1+t}{1-t} \right| \right] + C \\ &= \frac{1}{6} \log \left| \frac{1+x^3}{1-x^3} \right| + C\end{aligned}$$

Question 7:

$$\frac{x-1}{\sqrt{x^2-1}}$$

Solution:

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx \quad \dots(1)$$

$$\text{For } \int \frac{x}{\sqrt{x^2-1}} dx, \text{ let } x^2-1=t \Rightarrow 2x dx = dt$$

$$\therefore \int \frac{x}{\sqrt{x^2 - 1}} dx$$

$$= \frac{1}{2} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{2} \left[2t^{\frac{1}{2}} \right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2 - 1}$$

From (1), we get

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-1}} dx &= \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx & \left[\int \frac{x}{\sqrt{x^2-a^2}} dt = \log|x + \sqrt{x^2-a^2}| \right] \\ &= \sqrt{x^2-1} - \log|x + \sqrt{x^2-1}| + C \end{aligned}$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Solution:

$$\text{Put, } x^3 = t \Rightarrow 3x^2 dx = dt$$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}} dx = \frac{1}{3} \int \frac{dt}{\sqrt{t^2+(a^3)^2}}$$

$$= \frac{1}{3} \log|t + \sqrt{t^2 + a^6}| + C$$

$$= \frac{1}{3} \log|x^3 + \sqrt{x^6 + a^6}| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$$

Solution:

$$\text{Put, } \tan x = t$$

$$\begin{aligned}\therefore \sec^2 x dx &= dt \\ \Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx &= \int \frac{dt}{\sqrt{t^2 + 2^2}} \\ &= \log|t + \sqrt{t^2 + 4}| + C \\ &= \log|\tan x + \sqrt{\tan^2 x + 4}| + C\end{aligned}$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Solution:

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x+1 = t$

$$\therefore dx = dt$$

$$\begin{aligned}\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx &= \int \frac{1}{\sqrt{t^2 + 1}} dt \\ &= \log|t + \sqrt{t^2 + 1}| + C \\ &= \log|(x+1) + \sqrt{(x+1)^2 + 1}| + C \\ &= \log|(x+1) + \sqrt{x^2 + 2x + 2}| + C\end{aligned}$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Solution:

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \int \frac{1}{(3x+1)^2 + (2)^2} dx$$

$$\text{Let } (3x+1) = t$$

$$\Rightarrow 3dx = dt$$

$$\Rightarrow \int \frac{1}{(3x+1)^2 + (2)^2} dx = \frac{1}{3} \int \frac{1}{t^2 + 2^2} dt$$

$$= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) \right] + C$$

$$= \frac{1}{6} \left[\tan^{-1} \left(\frac{3x+1}{2} \right) \right] + C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Solution:

$$7-6x-x^2 \text{ can be written as } 7-(x^2+6x+9-9)$$

Thus,

$$7-(x^2+6x+9-9)$$

$$= 16 - (x^2 + 6x + 9)$$

$$= 16 - (x+3)^2$$

$$= (4)^2 - (x+3)^2$$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

$$\text{Let } x+3 = t$$

$$\Rightarrow dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{4} \right) + C$$

$$= \sin^{-1} \left(\frac{x+3}{4} \right) + C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Solution:

$(x-1)(x-2)$ can be written as $x^2 - 3x + 2$

Thus,

$$\begin{aligned} & x^2 - 3x + 2 \\ &= x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2 \\ &= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \\ &= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$\text{Let } \left(x - \frac{3}{2}\right) = t$$

$$\therefore dx = dt$$

$$\begin{aligned} & \Rightarrow \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{1}{2}\right)^2}} dt \\ &= \log \left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2} \right| + C \\ &= \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C \end{aligned}$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Solution:

$$8+3x-x^2 = 8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$$

Thus,

$$\begin{aligned} & 8 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} \right) \\ &= \frac{41}{4} - \left(x - \frac{3}{2} \right)^2 \\ &= \int \frac{1}{\sqrt{8+3x-x^2}} dx = \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} dx \end{aligned}$$

$$\text{Let } \left(x - \frac{3}{2} \right) = t$$

$$\therefore dx = dt$$

$$\begin{aligned} & \Rightarrow \int \frac{1}{\sqrt{\frac{41}{4} - \left(x - \frac{3}{2} \right)^2}} dx = \int \frac{1}{\sqrt{\left(\frac{41}{4} \right) - t^2}} dt \\ &= \sin^{-1} \left(\frac{t}{\sqrt{41}} \right) + C \\ &= \sin^{-1} \left(\frac{x - \frac{3}{2}}{\sqrt{41}} \right) + C \\ &= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}} \right) + C \end{aligned}$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Solution:

$$(x-a)(x-b) = x^2 - (a+b)x + ab$$

Thus,

$$\begin{aligned}
& x^2 - (a+b)x + ab \\
&= x^2 - (a+b)x + \frac{(a+b)^2}{4} - \frac{(a+b)^2}{4} + ab \\
&= \left[x - \left(\frac{a+b}{2} \right) \right]^2 - \frac{(a-b)^2}{4} \\
&\Rightarrow \int \frac{1}{\sqrt{(x-a)(x-b)}} dx = \int \frac{1}{\sqrt{\left\{ x - \left(\frac{a+b}{2} \right) \right\}^2 - \left(\frac{a-b}{2} \right)^2}} dx \\
&\text{Let } x - \left(\frac{a+b}{2} \right) = t \\
&\therefore dx = dt \\
&\Rightarrow \int \frac{1}{\sqrt{t^2 - \left(\frac{a-b}{2} \right)^2}} dt = \int \frac{1}{\sqrt{t^2 - \left(\frac{a-b}{2} \right)^2}} dt \\
&= \log \left| t + \sqrt{t^2 - \left(\frac{a-b}{2} \right)^2} \right| + C \\
&= \log \left| x - \left(\frac{a+b}{2} \right) + \sqrt{(x-a)(x-b)} \right| + C
\end{aligned}$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Solution:

$$\text{Let, } 4x+1 = A \frac{d}{dx}(2x^2+x-3) + B$$

$$\Rightarrow 4x+1 = A(4x+1) + B$$

$$\Rightarrow 4x+1 = 4Ax + A+B$$

Equating the coefficients of x and constant term on both sides, we get

$$4A = 4 \Rightarrow A = 1$$

$$A+B = 1 \Rightarrow B = 0$$

$$\text{Let } 2x^2+x-3 = t$$

$$\therefore (4x+1)dx = dt$$

$$\Rightarrow \int \frac{4x+1}{\sqrt{2x^2+x-3}} dx = \int \frac{1}{\sqrt{t}} dt$$

$$= 2\sqrt{t} + C$$

$$= 2\sqrt{2x^2+x-3} + C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Solution:

$$\text{Put, } x+2 = A \frac{d}{dx}(x^2-1) + B \dots (1)$$

$$\Rightarrow x+2 = A(2x) + B$$

Equating the coefficients of x and constant term on both sides, we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$B = 2$$

From (1) we get

$$\begin{aligned} \Rightarrow \int \frac{x+2}{\sqrt{x^2-1}} dx &= \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}} dx \\ &= \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx + \int \frac{2}{\sqrt{x^2-1}} dx \quad \dots (2) \end{aligned}$$

$$\text{In } \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx, \text{ Let } x^2-1=t \Rightarrow 2xdx=dt$$

$$\frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2} [2\sqrt{t}]$$

$$= \frac{1}{2} [2\sqrt{x^2-1}]$$

$$= \sqrt{x^2-1}$$

$$\text{Then, } \int \frac{2}{\sqrt{x^2-1}} dx = 2 \int \frac{1}{\sqrt{x^2-1}} dx = 2 \log|x+\sqrt{x^2-1}|$$

From equation (2) we get

$$\int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + 2 \log|x+\sqrt{x^2-1}| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Solution:

$$\text{Let } 5x - 2 = A \frac{d}{dx}(1 + 2x + 3x^2) + B$$

$$\Rightarrow 5x - 2 = A(2 + 6x) + B$$

Equating the coefficients of x and constant term on both sides, we get

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \int \frac{\frac{5}{6}(2 + 6x) - \frac{11}{3}}{1 + 2x + 3x^2} dx$$

$$\Rightarrow \frac{5}{6} \int \frac{2 + 6x}{1 + 2x + 3x^2} dx - \frac{11}{3} \int \frac{1}{1 + 2x + 3x^2} dx$$

Let

$$I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx \text{ and } I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$$

$$\therefore \int \frac{5x - 2}{1 + 2x + 3x^2} dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \quad \dots(1)$$

$$I_1 = \int \frac{2 + 6x}{1 + 2x + 3x^2} dx$$

$$\text{Put } 1 + 2x + 3x^2 = t$$

$$\Rightarrow (2 + 6x) dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log|1 + 2x + 3x^2| \quad \dots(2)$$

$$I_2 = \int \frac{1}{1 + 2x + 3x^2} dx$$

$$1 + 2x + 3x^2 \text{ can be written as } 1 + 3\left(x^2 + \frac{2}{3}x\right)$$

Thus,

$$\begin{aligned}
& 1 + 3 \left(x^2 + \frac{2}{3}x \right) \\
& = 1 + 3 \left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} \right) \\
& = 1 + 3 \left(x + \frac{1}{3} \right)^2 - \frac{1}{3} \\
& = \frac{2}{3} + 3 \left(x + \frac{1}{3} \right)^2 \\
& = 3 \left[\left(x + \frac{1}{3} \right)^2 + \frac{2}{9} \right] \\
& = 3 \left[\left(x + \frac{1}{3} \right)^2 + \left(\frac{\sqrt{2}}{3} \right)^2 \right] \\
I_2 & = \frac{1}{3} \int \frac{1}{\left[\left(x + \frac{1}{3} \right)^2 + \left(\frac{\sqrt{2}}{3} \right)^2 \right]} dx \\
& = \frac{1}{3} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}} \right) \right] \\
& = \frac{1}{3} \left[\frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x + 1}{\sqrt{2}} \right) \right] \\
& = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x + 1}{\sqrt{2}} \right) \quad \dots (3)
\end{aligned}$$

Substituting equations (2) and (3) in equation (1), we get

$$\begin{aligned}
\int \frac{5x - 2}{1 + 2x + 3x^2} dx & = \frac{5}{6} \left[\log |1 + 2x + 3x^2| \right] - \frac{11}{3} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x + 1}{\sqrt{2}} \right) \right] + C \\
& = \frac{5}{6} \log |1 + 2x + 3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x + 1}{\sqrt{2}} \right) + C
\end{aligned}$$

Question 19:

$$\frac{6x + 7}{\sqrt{(x-5)(x-4)}}$$

Solution:

$$\frac{6x + 7}{\sqrt{(x-5)(x-4)}} = \frac{6x + 7}{\sqrt{x^2 - 9x + 20}}$$

$$\text{Put, } 6x+7 = A \frac{d}{dx}(x^2 - 9x + 20) + B$$

$$\Rightarrow 6x+7 = A(2x-9) + B$$

Equating the coefficients of x and constant term, we get

$$2A = 6 \Rightarrow A = 3$$

$$-9A + B = 7 \Rightarrow B = 34$$

$$\therefore 6x+7 = 3(2x-9) + 34$$

$$\int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = \int \frac{3(2x-9) + 34}{\sqrt{x^2 - 9x + 20}} dx$$

$$= 3 \int \frac{(2x-9)}{\sqrt{x^2 - 9x + 20}} dx + 34 \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

$$\text{Let } I_1 = \int \frac{2x-9}{\sqrt{x^2 - 9x + 20}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

$$\therefore \int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx = 3I_1 + 34I_2 \quad \dots (1)$$

Then,

$$I_1 = \int \frac{2x-9}{\sqrt{x^2 - 9x + 20}} dx$$

$$\text{Let } x^2 - 9x + 20 = t$$

$$\Rightarrow (2x-9)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}}$$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2 - 9x + 20} \quad \dots (2)$$

and

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}} dx$$

$$x^2 - 9x + 20 = x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4}$$

Thus,

$$\begin{aligned}
 & x^2 - 9x + 20 + \frac{81}{4} - \frac{81}{4} \\
 &= \left(x - \frac{9}{2} \right)^2 - \frac{1}{4} \\
 &= \left(x - \frac{9}{2} \right)^2 - \left(\frac{1}{2} \right)^2 \\
 \Rightarrow I_2 &= \int \frac{1}{\left(x - \frac{9}{2} \right)^2 - \left(\frac{1}{2} \right)^2} dx
 \end{aligned}$$

$$I_2 = \log \left| \left(x - \frac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right| \quad \dots (3)$$

Substituting equations (2) and (3) in (1), we get

$$\begin{aligned}
 \int \frac{6x+7}{\sqrt{x^2-9x+20}} dx &= 3 \left[2\sqrt{x^2-9x+20} \right] + 34 \log \left[\left(x - \frac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right] + C \\
 &= 6\sqrt{x^2 - 9x + 20} + 34 \log \left[\left(x - \frac{9}{2} \right) + \sqrt{x^2 - 9x + 20} \right] + C
 \end{aligned}$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Solution:

$$\text{Consider, } x+2 = A \frac{d}{dx}(4x-x^2) + B$$

$$\Rightarrow x+2 = A(4-2x) + B$$

Equating the coefficients of x and constant term on both sides, we get

$$-2A = 1 \Rightarrow A = -\frac{1}{2}$$

$$4A + B = 2 \Rightarrow B = 4$$

$$\Rightarrow (x+2) = -\frac{1}{2}(4-2x) + 4$$

$$\begin{aligned}
 \therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx &= \int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{(4x-x^2)}} dx \\
 &= -\frac{1}{2} \int \frac{(4-2x)}{\sqrt{(4x-x^2)}} dx + 4 \int \frac{1}{\sqrt{(4x-x^2)}} dx
 \end{aligned}$$

$$\text{Let } I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2} I_1 + 4I_2 \quad \dots(1)$$

Then,

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}} dx$$

$$\text{Let } 4x-x^2 = t$$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\Rightarrow 4x-x^2 = -(-4x+x^2)$$

$$= (-4x+x^2+4-4)$$

$$= 4-(x-2)^2$$

$$= (2)^2 - (x-2)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}} dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad \dots(3)$$

Using equations (2) and (3) in (1), we get

$$\begin{aligned} \int \frac{x+2}{\sqrt{4x-x^2}} dx &= -\frac{1}{2}(2\sqrt{4x-x^2}) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C \\ &= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C \end{aligned}$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Solution:

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2+2x+3}} dx &= \frac{1}{2} \int \frac{2(x+2)}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \frac{1}{2} \int \frac{2}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx \end{aligned}$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \quad \dots(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Put, $x^2 + 2x + 3 = t$

$$\Rightarrow (2x+2)dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\Rightarrow x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = (x+1)^2 + (\sqrt{2})^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} dx = \log \left| (x+1) + \sqrt{x^2+2x+3} \right| \quad \dots(3)$$

Using equations (2) and (3) in (1), we get

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2+2x+3}} dx &= \frac{1}{2} \left[2\sqrt{x^2+2x+3} \right] + \log \left| (x+1) + \sqrt{x^2+2x+3} \right| + C \\ &= \sqrt{x^2+2x+3} + \log \left| (x+1) + \sqrt{x^2+2x+3} \right| + C \end{aligned}$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

Solution:

$$\text{Let } (x+3) = A \frac{d}{dx}(x^2 - 2x - 5) + B$$

$$(x+3) = A(2x-2) + B$$

Equating the coefficients of x and constant term on both sides, we get

$$2A=1 \Rightarrow A=\frac{1}{2}$$

$$-2A+B=3 \Rightarrow B=4$$

$$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5} dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5} dx$$

$$= \frac{1}{2} \int \frac{2x-2}{x^2-2x-5} dx + 4 \int \frac{1}{x^2-2x-5} dx$$

$$\text{Let } I_1 = \int \frac{2x-2}{x^2-2x-5} dx \text{ and } I_2 = \int \frac{1}{x^2-2x-5} dx$$

$$\therefore \int \frac{x+3}{x^2-2x-5} dx = \frac{1}{2} I_1 + 4I_2 \quad \dots (1)$$

$$\text{Then, } I_1 = \int \frac{2x-2}{x^2-2x-5} dx$$

$$\text{Put, } x^2 - 2x - 5 = t$$

$$\Rightarrow (2x-2)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log|x^2 - 2x - 5| \quad \dots (2)$$

$$I_2 = \int \frac{1}{x^2-2x-5} dx$$

$$= \int \frac{1}{(x^2-2x+1)-6} dx$$

$$= \int \frac{1}{(x-1)^2 - (\sqrt{6})^2} dx$$

$$= \frac{1}{2\sqrt{6}} \log \left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right) \quad \dots (3)$$

Substituting (2) and (3) in (1), we get

$$\int \frac{x+3}{x^2-2x-5} dx = \frac{1}{2} \log|x^2-2x-5| + \frac{4}{2\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$$

$$= \frac{1}{2} \log|x^2-2x-5| + \frac{2}{\sqrt{6}} \log \left| \frac{x-1-\sqrt{6}}{x-1+\sqrt{6}} \right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Solution:

$$\text{Let } 5x+3 = A \frac{d}{dx}(x^2 + 4x + 10) + B$$

$$\Rightarrow 5x+3 = A(2x+4) + B$$

Equating the coefficients of x and constant term, we get

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

$$4A + B = 3 \Rightarrow B = -7$$

$$\therefore 5x+3 = \frac{5}{2}(2x+4) - 7$$

$$\begin{aligned} & \Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}} dx \\ &= \frac{5}{2} \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx \\ & \text{Let } I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx \text{ and } I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} dx \end{aligned}$$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2} I_1 - 7I_2 \quad \dots(1)$$

Then,

$$I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}} dx$$

$$\text{Put, } x^2 + 4x + 10 = t$$

$$\therefore (2x+4) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \quad \dots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+6}} dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}} dx$$

$$= \log|(x+2)\sqrt{x^2+4x+10}| \quad \dots(3)$$

Using equations (2) and (3) in (1), we get

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{2} \left[2\sqrt{x^2+4x+10} \right] - 7 \log|(x+2)\sqrt{x^2+4x+10}| + C$$

$$= 5\sqrt{x^2+4x+10} - 7 \log|(x+2)\sqrt{x^2+4x+10}| + C$$

Choose the correct answer in Exercises 24 and 25.

Question 24:

- $\int \frac{dx}{x^2 + 2x + 2}$ equals
- (A) $x \tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$
(C) $(x+1) \tan^{-1} x + C$ (D) $\tan^{-1} x + C$

Solution:

$$\begin{aligned}\int \frac{dx}{x^2 + 2x + 2} &= \int \frac{dx}{(x^2 + 2x + 1) + 1} \\&= \int \frac{1}{(x+1)^2 + (1)^2} dx \\&= [\tan^{-1}(x+1)] + C\end{aligned}$$

Hence, the correct option is B.

Question 25:

- $\int \frac{dx}{\sqrt{9x - 4x^2}}$ equals
- (A) $\frac{1}{9} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$ (B) $\frac{1}{2} \sin^{-1} \left(\frac{8x-9}{8} \right) + C$
(C) $\frac{1}{3} \sin^{-1} \left(\frac{9x-8}{8} \right) + C$ (D) $\frac{1}{2} \sin^{-1} \left(\frac{9x-8}{9} \right) + C$

Solution:

$$\begin{aligned}& \int \frac{dx}{\sqrt{9x-4x^2}} \\&= \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x\right)}} dx \\&= \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64}\right)}} dx \\&= \int \frac{1}{\sqrt{-4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]}} dx \\&= \frac{1}{2} \int \frac{1}{\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2} dx \\&= \frac{1}{2} \left[\sin^{-1} \left(\frac{x - \frac{9}{8}}{\frac{9}{8}} \right) \right] + C \quad \left(\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1} \frac{y}{a} + C \right) \\&= \frac{1}{2} \sin^{-1} \left(\frac{8x - 9}{9} \right) + C\end{aligned}$$

Hence, the correct option is B.

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Solution:

$$\text{Let } \frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\Rightarrow x = A(x+2) + B(x+1)$$

Equating the coefficients of x and constant term, we get

$$A+B=1$$

$$2A+B=0$$

On solving, we get

$$A=-1 \text{ and } B=2$$

$$\begin{aligned} \therefore \frac{x}{(x+1)(x+2)} &= \frac{-1}{(x+1)} + \frac{2}{(x+2)} \\ \Rightarrow \int \frac{x}{(x+1)(x+2)} dx &= \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx \\ &= -\log|x+1| + 2\log|x+2| + C \\ &= \log(x+2)^2 - \log(x+1) + C \\ &= \log \frac{(x+2)^2}{(x+1)} + C \end{aligned}$$

Question 2:

$$\frac{1}{x^2-9}$$

Solution:

$$\text{Let } \frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)}$$

$$1 = A(x-3) + B(x+3)$$

Equating the coefficients of x and constants term, we get

$$A+B=0$$

$$-3A+3B=1$$

On solving, we get

$$A = -\frac{1}{6} \text{ and } B = \frac{1}{6}$$

$$\begin{aligned}\therefore \frac{1}{(x+3)(x-3)} &= \frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \\ \Rightarrow \int \frac{1}{(x^2-9)} dx &= \int \left(\frac{-1}{6(x+3)} + \frac{1}{6(x-3)} \right) dx \\ &= -\frac{1}{6} \log|x+3| + \frac{1}{6} \log|x-3| + C \\ &= \frac{1}{6} \log \frac{|(x-3)|}{|(x+3)|} + C\end{aligned}$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Solution:

$$\text{Let } \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \dots (1)$$

Equating the coefficients of x^2 , x and constant terms, we get

$$A+B+C=0$$

$$-5A-4B-3C=3$$

$$6A+3B+2C=-1$$

Solving these equations, we get

$$A=1, B=-5 \text{ and } C=4$$

$$\begin{aligned}\therefore \frac{3x-1}{(x-1)(x-2)(x-3)} &= \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \\ \Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)} dx &= \int \left\{ \frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{4}{(x-3)} \right\} dx \\ &= \log|x-1| - 5 \log|x-2| + 4 \log|x-3| + C\end{aligned}$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Solution:

$$\text{Let } \frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \dots (1)$$

Equating the coefficients of x^2 , x and constant terms, we get

$$A + B + C = 0$$

$$-5A - 4B - 3C = 1$$

$$6A + 4B + 2C = 0$$

Solving these equations, we get

$$A = \frac{1}{2}, B = -2 \text{ and } C = \frac{3}{2}$$

$$\begin{aligned} & \therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \\ & \Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx = \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx \\ & = \frac{1}{2} \log|x-1| - 2 \log|x-2| + \frac{3}{2} \log|x-3| + C \end{aligned}$$

Question 5:

$$\frac{2x}{x^2 + 3x + 2}$$

Solution:

$$\text{Let } \frac{2x}{x^2 + 3x + 2} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$2x = A(x+2) + B(x+1) \quad \dots(1)$$

Equating the coefficients of x and constant terms, we get

$$A + B = 2$$

$$2A + B = 0$$

Solving these equations, we get

$$A = -2 \text{ and } B = 4$$

$$\begin{aligned} & \therefore \frac{2x}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{4}{(x+2)} \\ & \Rightarrow \int \frac{2x}{(x+1)(x+2)} dx = \int \left\{ \frac{4}{(x+2)} - \frac{2}{(x+1)} \right\} dx \\ & = 4 \log|x+2| - 2 \log|x+1| + C \end{aligned}$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Solution:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we get

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2} \left(\frac{2-x}{x(1-2x)} \right) \dots (1)$$

$$\text{Let } \frac{2-x}{x(1-2x)} = \frac{A}{x} + \frac{B}{(1-2x)}$$

$$\Rightarrow (2-x) = A(1-2x) + Bx$$

Equating the coefficients of x and constant term, we get

$$-2A + B = -1$$

$$\text{And, } A = 2$$

Solving these equations, we get

$$A = 2 \text{ and } B = 3$$

$$\therefore \frac{2-x}{x(1-2x)} = \frac{2}{x} + \frac{3}{(1-2x)}$$

Substituting in equation (1), we get

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{2} \left\{ \frac{2}{x} + \frac{3}{(1-2x)} \right\}$$

$$\Rightarrow \int \frac{1-x^2}{x(1-2x)} dx = \int \left\{ \frac{1}{2} + \frac{1}{2} \left(\frac{2}{x} + \frac{3}{(1-2x)} \right) \right\} dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)} \log|1-2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4} \log|1-2x| + C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Solution:

$$\text{Let } \frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} \dots (1)$$

$$x = (Ax+B)(x-1) + C(x^2+1)$$

$$x = Ax^2 - Ax + Bx - B + Cx^2 + C$$

Equating the coefficients of x^2 , x and constant term, we get

$$A + C = 0$$

$$-A + B = 1$$

$$-B + C = 0$$

On solving these equations, we get

$$A = -\frac{1}{2}, B = \frac{1}{2} \text{ and } C = \frac{1}{2}$$

From equation (1), we get

$$\begin{aligned}
& \because \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x + \frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)} \\
& \Rightarrow \int \frac{x}{(x^2+1)(x-1)} dx = -\frac{1}{2} \int \frac{x}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x-1} dx \\
& = -\frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C \\
& \text{Consider } \int \frac{2x}{x^2+1} dx, \text{ let } (x^2+1) = t \Rightarrow 2x dx = dt \\
& \Rightarrow \int \frac{2x}{x^2+1} dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1| \\
& \therefore \int \frac{x}{(x^2+1)(x-1)} dx = -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C \\
& = \frac{1}{2} \log|x-1| - \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + C
\end{aligned}$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Solution:

$$\text{Let } \frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Equating the coefficients of x^2, x and constant term, we get

$$A + C = 0$$

$$A + B - 2C = 1$$

$$-2A + 2B + C = 0$$

On solving these equations, we get

$$A = \frac{2}{9}, B = \frac{1}{3} \text{ and } C = -\frac{2}{9}$$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x+1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)} dx = \frac{2}{9} \int \frac{1}{(x-1)} dx + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{(x+2)} dx$$

$$= \frac{2}{9} \log|x-1| + \frac{1}{3} \left(\frac{-1}{x-1} \right) - \frac{2}{9} \log|x+2| + C$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Solution:

$$\frac{3x+5}{x^3-x^2-x+1} = \frac{3x+5}{(x-1)^2(x+1)}$$

$$\text{Let } \frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x^2 - 2x + 1) \quad \dots(1)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+C=0$$

$$B-2C=3$$

$$-A+B+C=5$$

On solving these equations, we get

$$A = -\frac{1}{2}, B = 4 \text{ and } C = \frac{1}{2}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)} = \frac{-1}{2(x-1)} + \frac{4}{(x-1)^2} + \frac{1}{2(x+1)}$$

$$\Rightarrow \int \frac{3x+5}{(x-1)^2(x+1)} dx = -\frac{1}{2} \int \frac{1}{(x-1)} dx + 4 \int \frac{1}{(x-1)^2} dx + \frac{1}{2} \int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{2} \log|x-1| + 4 \left(\frac{-1}{x-1} \right) + \frac{1}{2} \log|x+1| + C$$

$$= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| - \frac{4}{(x-1)} + C$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Solution:

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)(x+1)(2x+3)}$$

$$\text{Let } \frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(2x+3)}$$

$$\Rightarrow (2x-3) = A(x-1)(2x-3) + B(x+1)(2x+3) + C(x+1)(x-1)$$

$$\Rightarrow (2x-3) = A(2x^2 + x - 3) + B(2x^2 + 5x + 3) + C(x^2 - 1)$$

$$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$$

Equating the coefficients of x^2, x and constant term, we get

$$2A+2B+C=0$$

$$A+5B=2$$

$$-3A+3B-C=-3$$

On solving, we get

$$A=\frac{5}{2}, B=-\frac{1}{10} \text{ and } C=-\frac{24}{5}$$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x+1)(x-1)(x+1)} dx = \frac{5}{2} \int \frac{1}{(x+1)} dx - \frac{1}{10} \int \frac{1}{(x-1)} dx - \frac{24}{5} \int \frac{1}{(2x+3)} dx$$

$$= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{24}{5 \times 2} \log|2x+3| + C$$

$$= \frac{5}{2} \log|x+1| - \frac{1}{10} \log|x-1| - \frac{12}{5} \log|2x+3| + C$$

Question 11:

$$\frac{5x}{(x+1)(x^2-4)}$$

Solution:

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$$

$$\text{Let } \frac{5x}{(x+1)(x^2-4)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$$

$$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \dots(1)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+B+C=0$$

$$-B+3C=5$$

$$-4A-2B+2C=0$$

On solving, we get

$$A = \frac{5}{3}, B = -\frac{5}{2} \text{ and } C = \frac{5}{6}$$

$$\begin{aligned}\therefore \frac{5x}{(x+1)(x+2)(x-2)} &= \frac{5}{3(x+1)} - \frac{5}{2(x+2)} + \frac{5}{6(x-2)} \\ \Rightarrow \int \frac{5x}{(x+1)(x+2)(x-2)} dx &= \frac{5}{3} \int \frac{1}{(x+1)} dx - \frac{5}{2} \int \frac{1}{(x+2)} dx + \frac{5}{6} \int \frac{1}{(x-2)} dx \\ &= \frac{5}{3} \log|x+1| - \frac{5}{2} \log|x+2| + \frac{5}{6} \log|x-2| + C\end{aligned}$$

Question 12:

$$\frac{x^3 + x + 1}{x^2 - 1}$$

Solution:

On dividing $(x^3 + x + 1)$ by $x^2 - 1$, we get

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

$$\frac{2x + 1}{x^2 - 1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$$2x + 1 = A(x-1) + B(x+1) \quad \dots (1)$$

Equating the coefficients of x and constant term, we get

$$A + B = 2$$

$$-A + B = 1$$

On solving, we get

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\begin{aligned}\Rightarrow \int \frac{x^3 + x + 1}{x^2 - 1} dx &= \int x dx + \frac{1}{2} \int \frac{1}{(x+1)} dx + \frac{3}{2} \int \frac{1}{(x-1)} dx \\ &= \frac{x^2}{2} + \frac{1}{2} \log|x+1| + \frac{3}{2} \log|x-1| + C\end{aligned}$$

Question 13:

$$\frac{2}{(1-x)(1+x^2)}$$

Solution:

$$\text{Let } \frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficients of x^2 , x and constant term, we get

$$A - B = 0$$

$$B - C = 0$$

$$A + C = 2$$

On solving these equations, we get

$$A = 1, B = 1 \text{ and } C = 1$$

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\int \frac{1}{1-x} dx + \frac{1}{2} \int \frac{2x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|x-1| + \frac{1}{2} \log|1+x^2| + \tan^{-1} x + C$$

Question 14:

$$\frac{3x-1}{(x+2)^2}$$

Solution:

$$\text{Let } \frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

$$\Rightarrow 3x-1 = A(x+2) + B$$

Equating the coefficient of x and constant term, we get

$$A = 3$$

$$2A + B = -1 \Rightarrow B = -7$$

$$\begin{aligned} \therefore \frac{3x-1}{(x+2)^2} &= \frac{3}{(x+2)} - \frac{7}{(x+2)^2} \\ \Rightarrow \int \frac{3x-1}{(x+2)^2} dx &= 3 \int \frac{1}{(x+2)} dx - 7 \int \frac{x}{(x+2)^2} dx \\ &= 3 \log|x+2| - 7 \left(\frac{-1}{(x+2)} \right) + C \\ &= 3 \log|x+2| + \frac{7}{(x+2)} + C \end{aligned}$$

Question 15:

$$\frac{1}{x^4-1}$$

Solution:

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(x^2+1)}$$

$$\text{Let } \frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$$

$$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$$

$$1 = A(x^3 + x - x^2 - 1) + B(x^3 + x + x^2 + 1) + Cx^3 + Dx^2 - Cx - D$$

$$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$$A+B+C=0$$

$$-A+B+D=0$$

$$A+B-C=0$$

$$-A+B-D=1$$

On solving, we get

$$A = \frac{-1}{4}, B = \frac{1}{4}, C = 0 \text{ and } D = -\frac{1}{2}$$

$$\begin{aligned} & \therefore \frac{1}{(x^4-1)} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(1+x^2)} \\ & \Rightarrow \int \frac{1}{(x^4-1)} dx = \int \frac{-1}{4(x+1)} dx + \int \frac{1}{4(x-1)} dx - \int \frac{1}{2(1+x^2)} dx \\ & \Rightarrow \int \frac{1}{(x^4-1)} dx = -\frac{1}{4} \log|x+1| + \frac{1}{4} \log|x-1| - \frac{1}{2} \tan^{-1} x + C \\ & = \frac{1}{4} \log \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

Question 16:

$$\frac{1}{x(x^n+1)}$$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution:

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1} , we get

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n = t \Rightarrow nx^{n-1}dx = dt$

$$\therefore \int \frac{1}{x(x^n+1)} dx = \int \frac{x^{n-1}}{x^n(x^n+1)} dx = \frac{1}{n} \int \frac{1}{t(t+1)} dt$$

$$\text{Let } \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$$

$$1 = A(1+t) + Bt \dots (1)$$

Equating the coefficients of t and constant term, we get

$A = 1$ and $B = -1$

$$\begin{aligned}\therefore \frac{1}{t(t+1)} &= \frac{1}{t} - \frac{1}{(1+t)} \\ \Rightarrow \int \frac{1}{x(x^n+1)} dx &= \frac{1}{n} \int \left\{ \frac{1}{t} - \frac{1}{(1+t)} \right\} dx \\ &= \frac{1}{n} [\log|t| - \log|t+1|] + C \\ &= \frac{1}{n} [\log|x^n| - \log|x^n+1|] + C \\ &= \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C\end{aligned}$$

Question 17:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)} \quad [\text{Hint: Put } \sin x = t]$$

Solution:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)} \quad \text{Put, } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \frac{dt}{(1-t)(2-t)}$$

$$\text{Let } \frac{1}{(1-t)(2-t)} = \frac{A}{(1-t)} + \frac{B}{(2-t)}$$

$$1 = A(2-t) + B(1-t) \quad \dots(1)$$

Equating the coefficients of t and constant, we get

$$-2A - B = 0, \text{ and } 2A + B = 1$$

On solving, we get

$$A = 1 \text{ and } B = -1$$

$$\therefore \frac{1}{(1-t)(2-t)} = \frac{1}{(1-t)} - \frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx = \int \left\{ \frac{1}{1-t} - \frac{1}{(2-t)} \right\} dt$$

$$= -\log|1-t| + \log|2-t| + C$$

$$= \log \left| \frac{2-t}{1-t} \right| + C$$

$$= \log \left| \frac{2-\sin x}{1-\sin x} \right| + C$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Solution:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

$$\text{Let } \frac{(4x^2+10)}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10 = (A+C)x^3 + (B+D)x^2 + (4A+3C)x + (4B+3D)$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$$A+C=0$$

$$B+D=4$$

$$4A+3C=0$$

$$4B+3D=10$$

On solving these equations, we get

$$A=0, B=-2, C=0 \text{ and } D=6$$

$$\therefore \frac{(4x^2+10)}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)} \right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx = \int 1 - \left\{ \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)} \right\} dx$$

$$= \int \left\{ 1 + \frac{2}{x^2 + (\sqrt{3})^2} - \frac{6}{x^2 + 2^2} \right\} dx$$

$$= x + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) - 6 \left(\frac{1}{2} \tan^{-1} \frac{x}{2} \right) + C$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Solution:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Put, $x^2 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dt}{(t+1)(t+3)} \quad \dots(1)$$

$$\text{Let } \frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$$

$$1 = A(t+3) + B(t+1) \quad \dots(2)$$

Equating the coefficients of t and constant, we get

$$A+B=0 \text{ and } 3A+B=1$$

On solving, we get

$$A=\frac{1}{2} \text{ and } B=-\frac{1}{2}$$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} + \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$$

$$= \frac{1}{2} \log |t+1| - \frac{1}{2} \log |t+3| + C$$

$$= \frac{1}{2} \log \left| \frac{t+1}{t+3} \right| + C = \frac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C$$

Question 20:

$$\frac{1}{x(x^4-1)}$$

Solution:

$$\frac{1}{x(x^4-1)}$$

Multiplying Nr and Dr by x^3 , we get

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \int \frac{x^3}{x^4(x^4-1)} dx$$

Put, $x^4 = t \Rightarrow 4x^3 = dt$

$$\therefore \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \frac{dt}{t(t-1)}$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{(t-1)}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Equating the coefficients of t and constant, we get

$A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \int \left\{ \frac{-1}{t} + \frac{1}{t-1} \right\} dt$$

$$= \frac{1}{4} \left[-\log|t| + \log|t-1| \right] + C$$

$$= \frac{1}{4} \log \left| \frac{t-1}{t} \right| + C = \frac{1}{4} \log \left| \frac{x^4-1}{x^4} \right| + C$$

Question 21:

$$\frac{1}{(e^x-1)} \quad [\text{Hint: Put } e^x = t]$$

Solution:

Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{1}{(e^x-1)} dx = \int \frac{1}{t-1} \times \frac{dt}{t} = \int \frac{1}{t(t-1)} dt$$

$$\text{Let } \frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$

$$1 = A(t-1) + Bt \quad \dots(1)$$

Equating the coefficients of t and constant, we get

$A = -1$ and $B = 1$

$$\begin{aligned}\therefore \frac{1}{t(t-1)} &= \frac{-1}{t} + \frac{1}{t-1} \\ \Rightarrow \int \frac{1}{t(t-1)} dt &= \log \left| \frac{t-1}{t} \right| + C \\ &= \log \left| \frac{e^x - 1}{e^x} \right| + C\end{aligned}$$

Question 22:

$\int \frac{x dx}{(x-1)(x-2)}$ equals

A. $\log \left| \frac{(x-1)^2}{(x-2)} \right| + C$

B. $\log \left| \frac{(x-2)^2}{(x-1)} \right| + C$

C. $\log \left| \left(\frac{x-1}{x-2} \right)^2 \right| + C$

D. $\log |(x-1)(x-2)| + C$

Solution:

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1) \quad \dots (1)$

Equating the coefficients of x and constant, we get

$A = -1$ and $B = 2$

$$\begin{aligned}\therefore \frac{x}{(x-1)(x-2)} &= \frac{-1}{(x-1)} + \frac{2}{(x-2)} \\ \Rightarrow \int \frac{x}{(x-1)(x-2)} dx &= \int \left\{ \frac{-1}{(x-1)} + \frac{2}{(x-2)} \right\} dx \\ &= -\log|x-1| + 2\log|x-2| + C \\ &= \log \left| \frac{(x-2)^2}{x-1} \right| + C\end{aligned}$$

Thus, the correct option is B.

Question 23:

$$\int \frac{dx}{x(x^2+1)}$$
 equals

- A. $\log|x| - \frac{1}{2} \log(x^2 + 1) + C$
- B. $\log|x| + \frac{1}{2} \log(x^2 + 1) + C$
- C. $-\log|x| + \frac{1}{2} \log(x^2 + 1) + C$
- D. $\frac{1}{2} \log|x| + \log(x^2 + 1) + C$

Solution:

$$\text{Let } \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

Equating the coefficients of x^2, x and constant terms, we get

$$A+B=0$$

$$C=0$$

$$A=1$$

On solving these equations, we get

$$A=1, B=-1 \text{ and } C=0$$

$$\begin{aligned}\therefore \frac{1}{x(x^2+1)} &= \frac{1}{x} + \frac{-x}{x^2+1} \\ \Rightarrow \int \frac{1}{x(x^2+1)} dx &= \int \left\{ \frac{1}{x} - \frac{x}{x^2-1} \right\} dx \\ &= \log|x| - \frac{1}{2} \log|x^2+1| + C\end{aligned}$$

Thus, the correct option is A.

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

Question 1:

$$x \sin x$$

Solution:

$$\text{Let } I = \int x \sin x dx$$

Taking $u = x$ and $v = \sin x$ and integrating by parts,

$$\begin{aligned} I &= x \int \sin x dx - \int \left\{ \left(\frac{d}{dx}(x) \right) \int \sin x dx \right\} dx \\ &= x(-\cos x) - \int 1 \cdot (-\cos x) dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

Question 2:

$$x \sin 3x$$

Solution:

$$\text{Let } I = \int x \sin 3x dx$$

Taking $u = x$ and $v = \sin 3x$ and integrating by parts,

$$\begin{aligned} I &= x \int \sin 3x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sin 3x dx \right\} dx \\ &= x \left(\frac{-\cos 3x}{3} \right) - \int 1 \cdot \left(\frac{-\cos 3x}{3} \right) dx \\ &= \frac{-x \cos 3x}{3} + \frac{1}{3} \int \cos 3x dx \\ &= \frac{-x \cos 3x}{3} + \frac{1}{9} \sin 3x + C \end{aligned}$$

Question 3:

$$x^2 e^x$$

Solution:

$$\text{Let } I = \int x^2 e^x dx$$

Taking $u = x^2$ and $v = e^x$ and integrating by parts, we get

$$\begin{aligned}
I &= x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx \\
&= x^2 e^x - \int 2x e^x dx \\
&= x^2 e^x - 2 \int x e^x dx
\end{aligned}$$

Again using integration by parts, we get

$$\begin{aligned}
&= x^2 e^x - 2 \left[x \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int e^x dx \right\} dx \right] \\
&= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] \\
&= x^2 e^x - 2 \left[x e^x - e^x \right] \\
&= x^2 e^x - 2 x e^x + 2 e^x + C \\
&= e^x (x^2 - 2x + 2) + C
\end{aligned}$$

Question 4:

$$x \log x$$

Solution:

$$\text{Let } I = \int x \log x dx$$

Taking $u = \log x$ and $v = x$ and integrating by parts, we get

$$\begin{aligned}
I &= \log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx \\
&= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx \\
&= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C
\end{aligned}$$

Question 5:

$$x \log 2x$$

Solution:

$$\text{Let } I = \int x \log 2x dx$$

Taking $u = \log 2x$ and $v = x$ and integrating by parts, we get

$$\begin{aligned}
I &= \log 2x \int x dx - \int \left\{ \left(\frac{d}{dx} \log 2x \right) \int x dx \right\} dx \\
&= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx \\
&= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C
\end{aligned}$$

Question 6:

$$x^2 \log x$$

Solution:

$$\text{Let } I = \int x^2 \log x dx$$

Taking $u = \log x$ and $v = x^2$ and integrating by parts, we get

$$\begin{aligned}
I &= \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx \\
&= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \\
&= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx \\
&= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C
\end{aligned}$$

Question 7:

$$x \sin^{-1} x$$

Solution:

$$\text{Let } I = \int x \sin^{-1} x dx$$

Taking $u = \sin^{-1} x$ and $v = x$ and integrating by parts, we get

$$\begin{aligned}
I &= \sin^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x dx \right\} dx \\
&= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} dx - \int \frac{1}{\sqrt{1-x^2}} dx \right\} \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C \\
&= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C \\
&= \frac{1}{4} (2x^2 - 1) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + C
\end{aligned}$$

Question 8:

$$x \tan^{-1} x$$

Solution:

Let $I = \int x \tan^{-1} x dx$

Taking $u = \tan^{-1} x$ and $v = x$ and integrating by parts, we get

$$\begin{aligned}
I &= \tan^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x dx \right\} dx \\
&= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
&= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left\{ \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right\} dx \\
&= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\
&= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left(x - \tan^{-1} x \right) + C \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C
\end{aligned}$$

Question 9:

$$x \cos^{-1} x$$

Solution:

$$\text{Let } I = \int x \cos^{-1} x dx$$

Taking $u = \cos^{-1} x$ and $v = x$ and integrating by parts, we get

$$\begin{aligned}
I &= \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x dx \right\} dx \\
&= \cos^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx \\
&= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx \\
&= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx \\
&= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx \\
&= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x (1)
\end{aligned}$$

Where, $I_1 = \int \sqrt{1-x^2} dx$

$$\Rightarrow I_1 = \sqrt{1-x^2} \int 1 dx - \int \frac{d}{dx} \sqrt{1-x^2} \int 1 dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}} x dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{ \int \sqrt{1-x^2} dx + \int \frac{-dx}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \{I_1 + \cos^{-1} x\}$$

$$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1} x$$

$$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1} x$$

Substituting in (1),

$$\begin{aligned} I &= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \left(\frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \cos^{-1} x \right) - \frac{1}{2} \cos^{-1} x \\ &= \frac{(2x^2-1)}{4} \cos^{-1} x - \frac{x}{4} \sqrt{1-x^2} + C \end{aligned}$$

Question 10:

$$(\sin^{-1} x)^2$$

Solution:

$$\text{Let } I = \int (\sin^{-1} x)^2 \cdot 1 dx$$

Taking $u = (\sin^{-1} x)^2$ and $v = 1$ and integrating by parts, we get

$$\begin{aligned}
I &= \int (\sin^{-1} x)^2 \cdot 1 dx - \int \left\{ \frac{d}{dx} (\sin^{-1} x)^2 \cdot 1 \right\} dx \\
&= (\sin^{-1} x)^2 \cdot x - \int \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \cdot x dx \\
&= x(\sin^{-1} x)^2 + \int \sin^{-1} x \left(\frac{-2x}{\sqrt{1-x^2}} \right) dx \\
&= x(\sin^{-1} x)^2 + \left[\sin^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right] \\
&= x(\sin^{-1} x)^2 + \left[\sin^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} dx \right] \\
&= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - \int 2 dx \\
&= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + C
\end{aligned}$$

Question 11:

$$\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$$

Solution:

$$\text{Let } I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$$

$$I = \frac{-1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1} x dx$$

Taking $u = \cos^{-1} x$ and $v = \left(\frac{-2x}{\sqrt{1-x^2}} \right)$ and integrating by parts, we get

$$\begin{aligned}
I &= \frac{-1}{2} \left[\cos^{-1} x \int \frac{-2x}{\sqrt{1-x^2}} dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int \frac{-2x}{\sqrt{1-x^2}} dx \right\} dx \right] \\
&= \frac{-1}{2} \left[\cos^{-1} x \cdot 2\sqrt{1-x^2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2} dx \right] \\
&= \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + \int 2 dx \right] \\
&= \frac{-1}{2} \left[2\sqrt{1-x^2} \cos^{-1} x + 2x \right] + C \\
&= -\left[\sqrt{1-x^2} \cos^{-1} x + x \right] + C
\end{aligned}$$

Question 12:

$$x \sec^2 x$$

Solution:

$$\text{Let } I = \int x \sec^2 x dx$$

Taking $u = x$ and $v = \sec^2 x$ and integrating by parts, we get

$$\begin{aligned} I &= x \int \sec^2 x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int \sec^2 x dx \right\} dx \\ &= x \tan x - \int 1 \cdot \tan x dx \\ &= x \tan x + \log |\cos x| + C \end{aligned}$$

Question 13:

$$\tan^{-1} x$$

Solution:

$$\text{Let } I = \int 1 \cdot \tan^{-1} x dx$$

Taking $u = \tan^{-1} x$ and $v = 1$ and integrating by parts, we get

$$\begin{aligned} I &= \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1 dx \right\} dx \\ &= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} x dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2}{1+x^2} dx \\ &= x \tan^{-1} x - \frac{1}{2} \log |1+x^2| + C \\ &= x \tan^{-1} x - \frac{1}{2} \log (1+x^2) + C \end{aligned}$$

Question 14:

$$x(\log x)^2$$

Solution:

$$\text{Let } I = \int x(\log x)^2 dx$$

Taking $u = (\log x)^2$ and $v = x$ and integrating by parts, we get

$$\begin{aligned} I &= (\log x)^2 \int x dx - \int \left[\left\{ \frac{d}{dx} (\log x)^2 \right\} \int x dx \right] dx \\ &= \frac{x^2}{2} (\log x)^2 - \left[\int 2 \log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx \right] \\ &= \frac{x^2}{2} (\log x)^2 - \int x \log x dx \end{aligned}$$

Again, using integration by parts, we get

$$\begin{aligned}
I &= \frac{x^2}{2}(\log x)^2 - \left[\log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx \right] \\
&= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right] \\
&= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x dx \\
&= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + C
\end{aligned}$$

Question 15:

$$(x^2 + 1) \log x$$

Solution:

$$\text{Let } I = \int (x^2 + 1) \log x dx = \int x^2 \log x dx + \int \log x dx$$

$$\text{Let } I = I_1 + I_2 \dots \dots \dots (1)$$

$$\text{Where, } I_1 = \int x^2 \log x dx \text{ and } I_2 = \int \log x dx$$

$$I_1 = \int x^2 \log x dx$$

Taking $u = \log x$ and $v = x^2$ and integrating by parts, we get

$$\begin{aligned}
I_1 &= \log x \int x^2 dx - \int \left[\left(\frac{d}{dx} \log x \right) \int x^2 dx \right] dx \\
&= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \\
&= \frac{x^3}{3} \log x - \frac{1}{3} \left(\int x^2 dx \right) \\
&= \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 \dots \dots \dots (2)
\end{aligned}$$

$$I_2 = \int \log x dx$$

Taking $u = \log x$ and $v = 1$ and integrating by parts,

$$\begin{aligned}
I_2 &= \log x \int 1 dx - \int \left[\left(\frac{d}{dx} \log x \right) \int 1 dx \right] dx \\
&= \log x \cdot x - \int \frac{1}{x} \cdot x dx \\
&= x \log x - \int 1 dx \\
&= x \log x - x + C_2 \dots \dots \dots (3)
\end{aligned}$$

Using equations (2) and (3) in (1),

$$\begin{aligned}
I &= \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x \log x - x + C_2 \\
&= \frac{x^3}{3} \log x - \frac{x^3}{9} + x \log x - x + (C_1 + C_2) \\
&= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C
\end{aligned}$$

Question 16:

$$e^x (\sin x + \cos x)$$

Solution:

$$\text{Let } I = \int e^x (\sin x + \cos x) dx$$

$$\text{Let } f(x) = \sin x$$

$$f'(x) = \cos x$$

$$I = \int e^x \{f(x) + f'(x)\} dx$$

$$\text{Since, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore I = e^x \sin x + C$$

Question 17:

$$\frac{xe^x}{(1+x)^2}$$

Solution:

$$\begin{aligned}
\text{Let, } I &= \int \frac{xe^x}{(1+x)^2} dx = \int e^x \left\{ \frac{x}{(1+x)^2} \right\} dx \\
&= \int e^x \left\{ \frac{1+x-1}{(1+x)^2} \right\} dx = \int e^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\} dx
\end{aligned}$$

$$\text{Here, } f(x) = \frac{1}{1+x} \quad f'(x) = \frac{-1}{(1+x)^2}$$

$$\Rightarrow \int \frac{xe^x}{(1+x)^2} dx = \int e^x \{f(x) + f'(x)\} dx$$

$$\text{Since, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore \int \frac{xe^x}{(1+x)^2} dx = e^x \frac{1}{1+x} + C$$

Question 18:

$$e^x \left(\frac{1 + \sin x}{1 + \cos x} \right)$$

Solution:

$$\begin{aligned}
& e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) = e^x \left(\frac{\frac{\sin^2 x}{2} + \frac{\cos^2 x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) \\
& = \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \cos^2 \frac{x}{2}} = \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2 \\
& = \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2 \\
& = \frac{1}{2} e^x \left[1 + \tan \frac{x}{2} \right]^2 \\
& = \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
& = \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
& \frac{e^x (1 + \sin x) dx}{(1 + \cos x)} = e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \dots \dots \dots (1)
\end{aligned}$$

$$\text{Let } \tan \frac{x}{2} = f(x) \quad \text{so } f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

From equation (1), we get

$$\int \frac{e^x(1+\sin x)}{(1+\cos x)}dx = e^x \tan \frac{x}{2} + C$$

Question 19:

$$e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$$

Solution:

$$\text{Let } I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$$

$$\text{Here, } \frac{1}{x} = f(x) \quad f'(x) = \frac{-1}{x^2}$$

It is known that

$$\begin{aligned} & \int e^x \{f(x) + f'(x)\} dx \\ &= e^x f(x) + C \\ \therefore I &= \frac{e^x}{x} + C \end{aligned}$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

Solution:

$$\begin{aligned} \int e^x \left\{ \frac{x-3}{(x-1)^3} \right\} dx &= \int e^x \left\{ \frac{x-1-2}{(x-1)^3} \right\} dx \\ &= \int e^x \left\{ \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right\} dx \end{aligned}$$

$$\text{Let } f(x) = \frac{1}{(x-1)^2}, \quad f'(x) = \frac{-2}{(x-1)^3}$$

It is known that,

$$\begin{aligned} \int e^x \{f(x) + f'(x)\} dx &= e^x f(x) + C \\ \therefore \int e^x \left\{ \frac{(x-3)}{(x-1)^2} \right\} dx &= \frac{e^x}{(x-1)^2} + C \end{aligned}$$

Question 21:

$$e^{2x} \sin x$$

Solution:

$$\text{Let } I = e^{2x} \sin x dx \dots \dots \dots (1)$$

Taking $u = \sin x$ and $v = e^{2x}$ and integrating by parts, we get

$$\begin{aligned} I &= \sin x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \sin x \right) \int e^{2x} dx \right\} dx \\ &\Rightarrow I = \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx \\ &\Rightarrow I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x dx \end{aligned}$$

Again, using integration by parts, we get

$$\begin{aligned}
I &= \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \int e^{2x} dx - \int \left\{ \left(\frac{d}{dx} \cos x \right) \int e^{2x} dx \right\} dx \right] \\
\Rightarrow I &= \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} dx \right] \\
\Rightarrow I &= \frac{e^{2x} \sin x}{2} - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx \right] \\
\Rightarrow I &= \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} I \quad [\text{From (1)}] \\
\Rightarrow I + \frac{1}{4} I &= \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \\
\Rightarrow \frac{5}{4} I &= \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \\
\Rightarrow I &= \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + C \\
\Rightarrow I &= \frac{e^{2x}}{5} [2 \sin x - \cos x] + C
\end{aligned}$$

Question 22:

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

Solution:

$$\text{Let } x = \tan \theta \quad dx = \sec^2 \theta d\theta$$

$$\begin{aligned}
\therefore \sin^{-1} \left(\frac{2x}{1+x^2} \right) &= \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta \\
\int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx &= \int 2\theta \cdot \sec^2 \theta d\theta = 2 \int \theta \cdot \sec^2 \theta d\theta
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
 & 2 \left[\theta \int \sec^2 \theta d\theta - \int \left\{ \left(\frac{d}{d\theta} \theta \right) \int \sec^2 \theta d\theta \right\} d\theta \right] \\
 &= 2 \left[\theta \cdot \tan \theta - \int \tan \theta d\theta \right] \\
 &= 2 \left[\theta \cdot \tan \theta + \log |\cos \theta| \right] + C \\
 &= 2 \left[x \tan^{-1} x + \log \left| \frac{1}{\sqrt{1+x^2}} \right| \right] + C \\
 &= 2x \tan^{-1} x + 2 \log (1+x^2)^{-\frac{1}{2}} + C \\
 &= 2x \tan^{-1} x + 2 \left[\frac{-1}{2} \log (1+x^2) \right] + C \\
 &= 2x \tan^{-1} x - \log (1+x^2) + C
 \end{aligned}$$

Question 23:

$\int x^2 e^{x^3} dx$ equals

A. $\frac{1}{3} e^{x^3} + C$

B. $\frac{1}{3} e^{x^2} + C$

C. $\frac{1}{2} e^{x^3} + C$

D. $\frac{1}{2} e^{x^2} + C$

Solution:

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t$ so, $3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3} \int e^t dt$$

$$= \frac{1}{3} (e^t) + C$$

$$= \frac{1}{3} e^{x^3} + C$$

Thus, the correct option is A.

Question 24:

$$\int e^x \sec x (1 + \tan x) dx$$
 equals

- A. $e^x \cos x + C$
- B. $e^x \sec x + C$
- C. $e^x \sin x + C$
- D. $e^x \tan x + C$

Solution:

$$\int e^x \sec x (1 + \tan x) dx$$

$$\text{Consider, } I = \int e^x \sec x (1 + \tan x) dx = \int e^x (\sec x + \sec x \tan x) dx$$

$$\text{Let } \sec x = f(x) \quad \sec x \tan x = f'(x)$$

$$\text{It is known that, } \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore I = e^x \sec x + C$$

Thus, the correct option is B.

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

Question 1:

$$\sqrt{4-x^2}$$

Solution:

Let $I = \int \sqrt{4-x^2} dx = \int \sqrt{(2)^2 - (x)^2} dx$
 Since, $\sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

$$\begin{aligned}\therefore I &= \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} + C \\ &= \frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \frac{x}{2} + C\end{aligned}$$

Question 2:

$$\sqrt{1-4x^2}$$

Solution:

$$\text{Let, } I = \int \sqrt{1-4x^2} dx = \int \sqrt{(1)^2 - (2x)^2} dx$$

$$\text{Put, } 2x = t \Rightarrow 2dx = dt$$

$$\therefore I = \frac{1}{2} \int \sqrt{(1)^2 - (t)^2} dt$$

Since, $\sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

$$\Rightarrow I = \frac{1}{2} \left[\frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right] + C$$

$$= \frac{t}{4} \sqrt{1-t^2} + \frac{1}{4} \sin^{-1} t + C$$

$$= \frac{2x}{4} \sqrt{1-4x^2} + \frac{1}{4} \sin^{-1} 2x + C$$

$$= \frac{x}{2} \sqrt{1-4x^2} + \frac{1}{4} \sin^{-1} 2x + C$$

Question 3:

$$\sqrt{x^2 + 4x + 6}$$

Solution:

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2 + 4x + 6} dx \\ &= \int \sqrt{x^2 + 4x + 4 + 2} dx \\ &= \int \sqrt{(x^2 + 4x + 4) + 2} dx \\ &= \int \sqrt{(x+2)^2 + (\sqrt{2})^2} dx \end{aligned}$$

$$\text{Since, } \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + C$$

$$\begin{aligned} I &= \frac{(x+2)}{2} \sqrt{x^2 + 4x + 6} + \frac{2}{2} \log|(x+2) + \sqrt{x^2 + 4x + 6}| + C \\ &= \frac{(x+2)}{2} \sqrt{x^2 + 4x + 6} + \log|(x+2) + \sqrt{x^2 + 4x + 6}| + C \end{aligned}$$

Question 4:

$$\sqrt{x^2 + 4x + 1}$$

Solution:

Consider,

$$\begin{aligned} I &= \int \sqrt{x^2 + 4x + 1} dx \\ &= \int \sqrt{(x^2 + 4x + 4) - 3} dx \\ &= \int \sqrt{(x+2)^2 - (\sqrt{3})^2} dx \end{aligned}$$

$$\text{Since, } \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$$

$$\therefore I = \frac{(x+2)}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log|(x+2) + \sqrt{x^2 + 4x + 1}| + C$$

Question 5:

$$\sqrt{1-4x-x^2}$$

Solution:

$$\begin{aligned} \text{Consider, } I &= \int \sqrt{1-4x-x^2} dx \\ &= \int \sqrt{1-(x^2+4x+4-4)} dx \\ &= \int \sqrt{1+4-(x+2)^2} dx \\ &= \int \sqrt{(\sqrt{5})^2 - (x+2)^2} dx \end{aligned}$$

$$\text{Since, } \sqrt{a^2-x^2} dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\therefore I = \frac{(x+2)}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C$$

Question 6:

$$\sqrt{x^2+4x-5}$$

Solution:

$$\begin{aligned} \text{Let } I &= \int \sqrt{x^2+4x-5} dx \\ &= \int \sqrt{(x^2+4x+4)-9} dx = \int \sqrt{(x+2)^2 - (3)^2} dx \end{aligned}$$

$$\text{Since, } \int \sqrt{x^2-a^2} dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$$

$$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log|(x+2)+\sqrt{x^2+4x-5}| + C$$

Question 7:

$$\sqrt{1+3x-x^2}$$

Solution:

$$\begin{aligned} \text{Put, } I &= \int \sqrt{1+3x-x^2} dx \\ &= \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)} dx \\ &= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2} dx = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2} dx \end{aligned}$$

$$\text{Since, } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

$$\therefore I = \frac{x - \frac{3}{2}}{2} \sqrt{1+3x-x^2} + \frac{13}{4 \times 2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}} \right) + C$$

$$= \frac{2x-3}{4} \sqrt{1+3x-x^2} + \frac{13}{8} \sin^{-1} \left(\frac{2x-3}{\sqrt{13}} \right) + C$$

Question 8:

$$\sqrt{x^2 + 3x}$$

Solution:

$$\text{Let } I = \int \sqrt{x^2 + 3x} dx$$

$$= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}} dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$$

$$\text{Since, } \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\therefore I = \frac{\left(x + \frac{3}{2}\right)}{2} \sqrt{x^2 + 3x} - \frac{9}{2} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C$$

$$= \frac{(2x+3)}{4} \sqrt{x^2 + 3x} - \frac{9}{8} \log \left| \left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x} \right| + C$$

Question 9:

$$\sqrt{1 + \frac{x^2}{9}}$$

Solution:

$$\text{Let } I = \int \sqrt{1 + \frac{x^2}{9}} dx = \frac{1}{3} \int \sqrt{9+x^2} dx = \frac{1}{3} \int \sqrt{(3)^2 + x^2} dx$$

$$\text{Since, } \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\therefore I = \frac{1}{3} \left[\frac{x}{2} \sqrt{x^2 + 9} + \frac{9}{2} \log \left| x + \sqrt{x^2 + 9} \right| \right] + C$$

$$= \frac{x}{6} \sqrt{x^2 + 9} + \frac{3}{2} \log \left| x + \sqrt{x^2 + 9} \right| + C$$

Question 10:

$\int \sqrt{1+x^2} dx$ is equal to

- A. $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log|x+\sqrt{1+x^2}| + C$
- B. $\frac{2}{3}(1+x^2)^{\frac{2}{3}} + C$
- C. $\frac{2}{3}x(1+x^2)^{\frac{2}{3}} + C$
- D. $\frac{x^3}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log|x+\sqrt{1+x^2}| + C$

Solution:

$$\text{Since, } \sqrt{a^2+x^2}dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

$$\therefore \int \sqrt{1+x^2}dx = \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log|x+\sqrt{1+x^2}| + C$$

Thus, the correct option is A.

Question 11:

$\int \sqrt{x^2-8x+7}dx$ is equal to

- A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log|x-4+\sqrt{x^2-8x+7}| + C$
- B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log|x+4+\sqrt{x^2-8x+7}| + C$
- C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log|x-4+\sqrt{x^2-8x+7}| + C$
- D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log|x-4+\sqrt{x^2-8x+7}| + C$

Solution:

$$\text{Let } I = \int \sqrt{x^2-8x+7}dx$$

$$= \int \sqrt{(x^2-8x+16)-9}dx$$

$$= \int \sqrt{(x-4)^2 - (3)^2}dx$$

$$\text{Since, } \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$$

$$\therefore I = \frac{(x-4)}{2}\sqrt{x^2-8x+7} - \frac{9}{2}\log|(x-4)+\sqrt{x^2-8x+7}| + C$$

Thus, the correct option is D.

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

Question 1:

$$\int_a^b x dx$$

Solution:

Since, $\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$ where $h = \frac{b-a}{n}$

Here, $a = a, b = b$ and $f(x) = x$

$$\begin{aligned}
 & \therefore \int_a^b x dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [a + (a+h) + (a+2h) + \dots + (a+(n-1)h)] \\
 &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\underbrace{a+a+\dots+a}_{n \text{ times}} + (h+2h+3h+\dots+(n-1)h) \right] \\
 &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [na + h(1+2+3+\dots+(n-1))] \\
 &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + h \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\
 &= (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[na + \frac{n(n-1)h}{2} \right] = (b-a) \lim_{n \rightarrow \infty} \frac{n}{n} \left[a + \frac{(n-1)h}{2} \right] \\
 &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)h}{2} \right] = (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{(n-1)(b-a)}{2n} \right] \\
 &= (b-a) \lim_{n \rightarrow \infty} \left[a + \frac{\left(1 - \frac{1}{n}\right)(b-a)}{2} \right] = (b-a) \left[a + \frac{(b-a)}{2} \right] \\
 &= (b-a) \left[\frac{2a+b-a}{2} \right] \\
 &= \frac{(b-a)(b+a)}{2} \\
 &= \frac{1}{2}(b^2 - a^2)
 \end{aligned}$$

Question 2:

$$\int_0^b (x+1) dx$$

Solution:

Let $I = \int_0^b (x+1) dx$

Since, $\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here, $a = 0, b = 5$ and $f(x) = (x+1)$

$$\Rightarrow h = \frac{5-0}{n} = \frac{5}{n}$$

$$\therefore \int_0^5 (x+1) dx = (5-0) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f\left(\frac{5}{n}\right) + \dots + f\left((n-1)\frac{5}{n}\right) \right]$$

$$= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(\frac{5}{n} + 1 \right) + \dots + \left\{ 1 + \left(\frac{5(n-1)}{n} \right) \right\} \right]$$

$$= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1 + \underbrace{\frac{1}{n}}_{n \text{ times}} + 1 \dots 1 \right) + \left[\frac{5}{n} + 2 \cdot \frac{5}{n} + 3 \cdot \frac{5}{n} + \dots + (n-1) \frac{5}{n} \right] \right]$$

$$= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \{1+2+3\dots(n-1)\} \right]$$

$$= 5 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{5}{n} \cdot \frac{(n-1)n}{2} \right] = 5 \lim_{n \rightarrow \infty} \left[1 + \frac{5(n-1)}{2n} \right]$$

$$= 5 \lim_{n \rightarrow \infty} \left[1 + \frac{5}{2} \left(1 - \frac{1}{n} \right) \right] = 5 \left[1 + \frac{5}{2} \right]$$

$$= 5 \left[\frac{7}{2} \right]$$

$$= \frac{35}{2}$$

Question 3:

$$\int_2^3 x^2 dx$$

Solution:

Since,

$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here, $a = 2, b = 3$ and $f(x) = x^2$

$$\begin{aligned}
\Rightarrow h &= \frac{3-2}{n} = \frac{1}{n} \\
\therefore \int_2^3 x^2 dx &= (3-2) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(2) + f\left(2 + \frac{1}{n}\right) + f\left(2 + \frac{2}{n}\right) + \dots + f\left(2 + (n-1)\frac{1}{n}\right) \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[(2)^2 + \left(2 + \frac{1}{n}\right)^2 + \left(2 + \frac{2}{n}\right)^2 + \dots + \left(2 + \frac{(n-1)^2}{n}\right) \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[2^2 + \left\{ 2^2 + \left(\frac{1}{n}\right)^2 + 2 \cdot 2 \cdot \frac{1}{n} \right\} + \dots + \left\{ (2)^2 + \frac{(n-1)^2}{n^2} + 2 \cdot 2 \cdot \frac{(n-1)}{n} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(2^2 + \underset{n \text{ times}}{\dots} + 2^2 \right) + \left\{ \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n-1}{n}\right)^2 \right\} + 2 \cdot 2 \cdot \left\{ \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{(n-1)}{n} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \left\{ 1^2 + 2^2 + 3^2 + \dots + (n-1)^2 \right\} + \frac{4}{n} \left\{ 1 + 2 + \dots + (n-1) \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{1}{n^2} \left\{ \frac{n(n-1)(2n-1)}{6} \right\} + \frac{4}{n} \left\{ \frac{n(n-1)}{2} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[4n + \frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6} + \frac{4n-4}{2} \right] = \lim_{n \rightarrow \infty} \left[4 + \frac{1}{6} \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) + 2 - \frac{2}{n} \right] \\
&= 4 + \frac{2}{6} + 2 \\
&= \frac{19}{3}
\end{aligned}$$

Question 4:

$$\int_1^4 (x^2 - x) dx$$

Solution:

$$\text{Let } I = \int_1^4 (x^2 - x) dx$$

$$= \int_1^4 x^2 dx - \int_1^4 x dx$$

$$\text{Let } I = I_1 - I_2, \text{ where } I_1 = \int_1^4 x^2 dx \text{ and } I_2 = \int_1^4 x dx \dots (1)$$

$$\text{Since, } \int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(a) + f(a+h) + f(a+(n-1)h) \right], \text{ where } h = \frac{b-a}{n}$$

$$\text{For, } I_1 = \int_1^4 x^2 dx,$$

$a=1, b=4$ and $f(x)=x^2$

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$\begin{aligned} I_1 &= \int_1^4 x^2 dx = (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(1) + f(1+h) + \dots + f(1+(n-1)h) \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left(1 + \frac{3}{n}\right)^2 + \left(1 + 2 \cdot \frac{3}{n}\right)^2 + \dots + \left(1 + \frac{(n-1)3}{n}\right)^2 \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1^2 + \left\{ 1^2 + \left(\frac{3}{n}\right)^2 + 2 \cdot \frac{3}{n} \right\} + \dots + \left\{ 1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2 \cdot (n-1) \cdot 3}{2} \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1^2 + \underset{n \text{ times}}{\dots} + 1^2 \right) + \left(\frac{3}{n} \right)^2 \left\{ 1^2 + 2^2 + \dots + (n-1)^2 \right\} + 2 \cdot \frac{3}{n} \left\{ 1 + 2 + \dots + (n-1) \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9}{n^2} \left\{ \frac{(n-1)(n)(2n-1)}{6} \right\} + \frac{6}{n} \left\{ \frac{(n-1)(n)}{2} \right\} \right] \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{9n}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + \frac{6n-6}{2} \right] \\ &= 3 \lim_{n \rightarrow \infty} \left[1 + \frac{9}{6} \left(1 - \frac{1}{n} \right) \left(2 - \frac{1}{n} \right) + 3 - \frac{3}{n} \right] \\ &= 3 [1+3+3] \\ &= 3[7] \\ I_1 &= 21 \dots (2) \end{aligned}$$

For $I_2 = \int_1^4 x dx$

$a=1, b=4$ and $f(x)=x$

$$\Rightarrow h = \frac{4-1}{n} = \frac{3}{n}$$

$$\therefore I_2 = (4-1) \lim_{n \rightarrow \infty} \frac{1}{n} [f(1) + f(1+h) + \dots + f(a+(n-1)h)]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} [1 + (1+h) + \dots + (1+(n-1)h)]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + \left(1 + \frac{3}{n} \right) + \dots + \left\{ 1 + (n-1) \frac{3}{n} \right\} \right]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\left(1 + 1 + \dots + 1 \right) + \frac{3}{n} (1+2+\dots+(n-1)) \right]$$

$$= 3 \lim_{n \rightarrow \infty} \frac{1}{n} \left[n + \frac{3}{n} \left\{ \frac{(n-1)n}{2} \right\} \right]$$

$$= 3 \lim_{n \rightarrow \infty} \left[1 + \frac{3}{2} \left(1 - \frac{1}{n} \right) \right]$$

$$= 3 \left[1 + \frac{3}{2} \right] = 3 \left[\frac{5}{2} \right]$$

$$I_2 = \frac{15}{2} \quad \dots (3)$$

From equations (2) and (3), we get

$$I = I_1 - I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 5:

$$\int_{-1}^1 e^x dx$$

Solution:

$$\text{Let } I = \int_{-1}^1 e^x dx \quad \dots (1)$$

Since, $\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + f(a+(n-1)h)]$, where $h = \frac{b-a}{n}$

Here, $a = -1, b = 1$ and $f(x) = e^x$

$$\therefore h = \frac{1+1}{n} = \frac{2}{n}$$

$$\begin{aligned}
\therefore I &= (1+1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(-1) + f\left(-1 + \frac{2}{n}\right) + f\left(-1 + 2 \cdot \frac{2}{n}\right) + \dots + f\left(-1 + \frac{(n-1)2}{n}\right) \right] \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} + e^{\left(-1+\frac{2}{n}\right)} + e^{\left(-1+2\frac{2}{n}\right)} + e^{\left(-1+(n-1)\frac{2}{n}\right)} \right] \\
&= 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^{-1} \left\{ 1 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + e^{\frac{6}{n}} + e^{\frac{(n-1)2}{n}} \right\} \right] \\
&= 2 \lim_{n \rightarrow \infty} \frac{e^{-1}}{n} \left[\frac{e^{\frac{2n}{n}} - 1}{e^{\frac{2}{n}} - 1} \right] = e^{-1} \times 2 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{e^2 - 1}{e^{\frac{2}{n}} - 1} \right] \\
&= \frac{e^{-1} \times 2(e^2 - 1)}{\lim_{\frac{2}{n} \rightarrow 0} \left(\frac{e^{\frac{2}{n}} - 1}{\frac{2}{n}} \right) \times 2} \quad \left[\lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right) = 1 \right] \\
&= \frac{e^2 - 1}{e} \\
&= \left(e - \frac{1}{e} \right)
\end{aligned}$$

Question 6:

$$\int_0^4 (x + e^{2x}) dx$$

Solution:

Since,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0, b = 4$ and $f(x) = x + e^{2x}$

$$\begin{aligned}
\therefore h &= \frac{4-0}{n} = \frac{4}{n} \\
\Rightarrow \int_0^4 (x + e^{2x}) dx &= (4-0) \lim_{n \rightarrow \infty} \frac{1}{n} \left[f(0) + f(h) + f(2h) + \dots + f((n-1)h) \right] \\
&= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[(0 + e^0) + (h + e^{2h}) + (2h + e^{4h}) + \dots + ((n-1)h + e^{2(n-1)h}) \right] \\
&= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[1 + (h + e^{2h}) + (2h + e^{4h}) + \dots + ((n-1)h + e^{2(n-1)h}) \right] \\
&= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\{h + 2h + 3h + \dots + (n-1)h\} + (1 + e^{2h} + e^{4h} + \dots + e^{2(n-1)h}) \right]
\end{aligned}$$

$$\begin{aligned}
&= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[h \{1 + 2 + \dots + (n-1)\} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] = 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{h(n-1)n}{2} + \left(\frac{e^{2hn} - 1}{e^{2h} - 1} \right) \right] \\
&= 4 \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{4(n-1)n}{2} + \left(\frac{e^8 - 1}{e^8 - 1} \right) \right] = 4(2) + 4 \lim_{n \rightarrow \infty} \left(\frac{\frac{8}{e^n} - 1}{\frac{8}{n}} \right)^8 \\
&= 8 + \frac{4(e^8 - 1)}{8} \quad \left(\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right) \\
&= 8 + \frac{e^8 - 1}{2} = \frac{15 + e^8}{2}
\end{aligned}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

Question 1:

$$\int_{-1}^1 (x+1) dx$$

Solution:

Let $I = \int_{-1}^1 (x+1) dx$

$$\int (x+1) dx = \frac{x^2}{2} + x = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F(1) - F(-1) \\ &= \left(\frac{1}{2} + 1\right) - \left(\frac{1}{2} - 1\right) \\ &= \frac{1}{2} + 1 - \frac{1}{2} + 1 \\ &= 2 \end{aligned}$$

Question 2:

$$\int_2^3 \frac{1}{x} dx$$

Solution:

Let $I = \int_2^3 \frac{1}{x} dx$

$$\int \frac{1}{x} dx = \log|x| = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F(3) - F(2) \\ &= \log|3| - \log|2| = \log \frac{3}{2} \end{aligned}$$

Question 3:

$$\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$$

Solution:

Let $I = \int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$

$$\begin{aligned} \int (4x^3 - 5x^2 + 6x + 9) dx &= 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{3}\right) + 6\left(\frac{x^2}{2}\right) + 9(x) \\ &= x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x) \end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F(2) - F(1) \\ I &= \left\{ 2^4 - \frac{5(2)^3}{3} + 3(2)^2 + 9(2) \right\} - \left\{ (1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1) \right\} \\ &= \left(16 - \frac{40}{3} + 12 + 18 \right) - \left(1 - \frac{5}{3} + 3 + 9 \right) \\ &= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9 \\ &= 33 - \frac{35}{3} \\ &= \frac{99 - 35}{3} \\ &= \frac{64}{3} \end{aligned}$$

Question 4:

$$\int_0^{\frac{\pi}{4}} \sin 2x dx$$

Solution:

$$\begin{aligned} \text{Let } I &= \int_0^{\frac{\pi}{4}} \sin 2x dx \\ \int \sin 2x dx &= \left(\frac{-\cos 2x}{2} \right) = F(x) \end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F\left(\frac{\pi}{4}\right) - F(0) \\ &= -\frac{1}{2} \left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0 \right] = -\frac{1}{2} \left[\cos\left(\frac{\pi}{2}\right) - \cos 0 \right] \\ &= -\frac{1}{2} [0 - 1] \\ &= \frac{1}{2} \end{aligned}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \cos 2x dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos 2x dx$$

$$\int \cos 2x dx = \left(\frac{\sin 2x}{2} \right) = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2} \left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0 \right] = \frac{1}{2} [\sin \pi - \sin 0]$$

$$= \frac{1}{2} [0 - 0] = 0$$

Question 6:

$$\int_4^5 e^x dx$$

Solution:

$$\text{Let } I = \int_4^5 e^x dx$$

$$\int e^x dx = e^x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(5) - F(4)$$

$$= e^5 - e^4$$

$$= e^4 (e - 1)$$

Question 7:

$$\int_0^{\frac{\pi}{4}} \tan x dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \tan x dx$$

$$\int \tan x dx = -\log |\cos x| = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F\left(\frac{\pi}{4}\right) - F(0) \\
&= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0| = -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1| \\
&= -\log(2)^{-\frac{1}{2}} \\
&= \frac{1}{2}\log 2
\end{aligned}$$

Question 8:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx dx$

$$\int \cos ecx dx = \log|\cos ecx - \cot x| = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) \\
&= \log\left|\cos ec\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\cos ec\frac{\pi}{6} - \cot\frac{\pi}{6}\right| \\
&\quad \log|\sqrt{2}-1| - \log|2-\sqrt{3}| = \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)
\end{aligned}$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F(1) - F(0) \\
&= \sin^{-1}(1) - \sin^{-1}(0) \\
&= \frac{\pi}{2} - 0 \\
&= \frac{\pi}{2}
\end{aligned}$$

Question 10:

$$\int_0^1 \frac{dx}{1+x^2}$$

Solution:

$$\text{Let } I = \int_0^1 \frac{dx}{1+x^2}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F(1) - F(0) \\
&= \tan^{-1}(1) - \tan^{-1}(0) \\
&= \frac{\pi}{4}
\end{aligned}$$

Question 11:

$$\int_2^3 \frac{dx}{x^2 - 1}$$

Solution:

$$\text{Let } I = \int_2^3 \frac{dx}{x^2 - 1}$$

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F(3) - F(2) \\
&= \frac{1}{2} \left[\log \left| \frac{3-1}{3+1} \right| - \log \left| \frac{2-1}{2+1} \right| \right] = \frac{1}{2} \left[\log \left| \frac{2}{4} \right| - \log \left| \frac{1}{3} \right| \right] \\
&= \frac{1}{2} \left[\log \frac{1}{2} - \log \frac{1}{3} \right] \\
&= \frac{1}{2} \left[\log \frac{3}{2} \right]
\end{aligned}$$

Question 12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

$$\int \cos^2 x dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= \left[F\left(\frac{\pi}{2}\right) - F(0) \right] = \frac{1}{2} \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - \left(0 + \frac{\sin 0}{2} \right) \right] \\ &= \frac{1}{2} \left[\frac{\pi}{2} + 0 - 0 - 0 \right] \\ &= \frac{\pi}{4} \end{aligned}$$

Question 13:

$$\int_2^3 \frac{x}{x^2 + 1} dx$$

Solution:

$$\text{Let } I = \int_2^3 \frac{x}{x^2 + 1} dx$$

$$\int \frac{x}{x^2 + 1} dx = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} \log(1 + x^2) = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F(3) - F(2) \\ &= \frac{1}{2} \left[\log(1 + (3)^2) - \log(1 + (2)^2) \right] \\ &= \frac{1}{2} \left[\log(10) - \log(5) \right] \\ &= \frac{1}{2} \log\left(\frac{10}{5}\right) = \frac{1}{2} \log 2 \end{aligned}$$

Question 14:

$$\int_0^1 \frac{2x+3}{5x^2+1} dx$$

Solution:

$$\text{Let } I = \int_0^1 \frac{2x+3}{5x^2+1} dx$$

$$\begin{aligned}
\int \frac{2x+3}{5x^2+1} dx &= \frac{1}{5} \int \frac{5(2x+3)}{5x^2+1} dx = \frac{1}{5} \int \frac{10x+15}{5x^2+1} dx \\
&= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5x^2+1} dx \\
&= \frac{1}{5} \int \frac{10x}{5x^2+1} dx + 3 \int \frac{1}{5\left(x^2+\frac{1}{5}\right)} dx = \frac{1}{5} \log(5x^2+1) + \frac{3}{5} \cdot \frac{1}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} \\
&= \frac{1}{5} \log(5x^2+1) + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5})x \\
&= F(x)
\end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F(1) - F(0) \\
&= \left\{ \frac{1}{5} \log(5+1) + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}) \right\} - \left\{ \frac{1}{5} \log(5 \times 0 + 1) + \frac{3}{\sqrt{5}} \tan^{-1}(0) \right\} \\
&= \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}
\end{aligned}$$

Question 15:

$$\int_0^1 xe^{x^2} dx$$

Solution:

$$\text{Let } I = \int_0^1 xe^{x^2} dx$$

$$\text{Put, } x^2 = t \Rightarrow 2x dx = dt$$

As $x \rightarrow 0, t \rightarrow 0$ and as $x \rightarrow 1, t \rightarrow 1$

$$\therefore I = \frac{1}{2} \int_0^1 e^t dt$$

$$\frac{1}{2} \int e^t dt = \frac{1}{2} e^t = F(t)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \frac{1}{2} e - \frac{1}{2} e^0$$

$$= \frac{1}{2}(e-1)$$

Question 16:

$$\int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$$

Solution:

$$\text{Let } I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3} dx$$

Dividing $5x^2$ by $x^2 + 4x + 3$, we get

$$\begin{aligned} I &= \int_1^2 \left\{ 5 - \frac{20x+15}{x^2 + 4x + 3} \right\} dx \\ &= \int_1^2 5 dx - \int_1^2 \frac{20x+15}{x^2 + 4x + 3} dx \\ &= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2 + 4x + 3} dx \\ I &= 5 - I_1, \text{ where } I_1 = \int_1^2 \frac{20x+15}{x^2 + 4x + 3} dx \quad \dots(1) \end{aligned}$$

$$\text{Let } 20x+15 = A \frac{d}{dx}(x^2 + 4x + 3) + B$$

$$= 2Ax + (4A+B)$$

Equating the coefficients of x and constant term, we get

$$A = 10 \text{ and } B = -25$$

$$\text{Let } x^2 + 4x + 3 = t$$

$$\Rightarrow (2x+4)dx = dt$$

$$\begin{aligned} \Rightarrow I_1 &= 10 \int \frac{dt}{t} - 25 \int \frac{dx}{(x+2)^2 - 1^2} \\ &= 10 \log t - 25 \left[\frac{1}{2} \log \left(\frac{x+2-1}{x+2+1} \right) \right] = \left[10 \log(x^2 + 4x + 3) \right]_1^2 - 25 \left[\frac{1}{2} \log \left(\frac{x+1}{x+3} \right) \right]_1^2 \\ &= [10 \log 15 - 10 \log 8] - 25 \left[\frac{1}{2} \log \frac{3}{5} - \frac{1}{2} \log \frac{2}{4} \right] \\ &= [10 \log(5 \times 3) - 10 \log(4 \times 2)] - \frac{25}{2} [\log 3 - \log 5 - \log 2 + \log 4] \\ &= [10 \log 5 + 10 \log 3 - 10 \log 4 - 10 \log 2] - \frac{25}{2} [\log 3 - \log 5 - \log 2 + \log 4] \\ &= \left[10 + \frac{25}{2} \right] \log 5 + \left[-10 - \frac{25}{2} \right] \log 4 + \left[10 - \frac{25}{2} \right] \log 3 + \left[-10 + \frac{25}{2} \right] \log 2 \\ &= \frac{45}{2} \log 5 - \frac{45}{2} \log 4 - \frac{5}{2} \log 3 + \frac{5}{2} \log 2 \\ &= \frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \end{aligned}$$

Substituting the value I_1 in (1), we get

$$I = 5 - \left[\frac{45}{2} \log \frac{5}{4} - \frac{5}{2} \log \frac{3}{2} \right]$$

$$= 5 - \frac{5}{2} \left[9 \log \frac{5}{4} - \log \frac{3}{2} \right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} (2 \sec^2 x + x^3 + 2) dx$$

$$\int (2 \sec^2 x + x^3 + 2) dx = 2 \tan x + \frac{x^4}{4} + 2x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \left\{ \left(2 \tan \frac{\pi}{4} + \frac{1}{4} \left(\frac{\pi}{4}\right)^4 + 2 \left(\frac{\pi}{4}\right) \right) - (2 \tan 0 + 0 + 0) \right\}$$

$$= 2 \tan \frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

Solution:

$$\text{Let } I = \int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx = - \int_0^{\pi} \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$$

$$= - \int_0^{\pi} \cos x dx$$

$$\int \cos x dx = \sin x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(\pi) - F(0)$$

$$= \sin \pi - \sin 0$$

$$= 0$$

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4} dx$$

Solution:

$$\begin{aligned} \text{Let } I &= \int_0^2 \frac{6x+3}{x^2+4} dx \\ \int \frac{6x+3}{x^2+4} dx &= 3 \int \frac{2x+1}{x^2+4} dx \\ &= 3 \int \frac{2x}{x^2+4} dx + 3 \int \frac{1}{x^2+4} dx \\ &= 3 \log(x^2+4) + \frac{3}{2} \tan^{-1} \frac{x}{2} = F(x) \end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} I &= F(2) - F(0) \\ &= \left\{ 3 \log(2^2+4) + \frac{3}{2} \tan^{-1} \left(\frac{2}{2} \right) \right\} - \left\{ 3 \log(0+4) + \frac{3}{2} \tan^{-1} \left(\frac{0}{2} \right) \right\} \\ &= 3 \log 8 + \frac{3}{2} \tan^{-1} 1 - 3 \log 4 - \frac{3}{2} \tan^{-1} 0 \\ &= 3 \log 8 + \frac{3}{2} \left(\frac{\pi}{4} \right) - 3 \log 4 - 0 \\ &= 3 \log \left(\frac{8}{4} \right) + \frac{3\pi}{8} \\ &= 3 \log 2 + \frac{3\pi}{8} \end{aligned}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$$

Solution:

$$\begin{aligned} \text{Let } I &= \int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx \end{aligned}$$

$$\begin{aligned}
\int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx &= x \int e^x dx - \int \left\{ \left(\frac{d}{dx} x \right) \int e^x dx \right\} dx + \left\{ \frac{-\cos \frac{\pi x}{4}}{\frac{\pi}{4}} \right\} \\
&= xe^x - \int e^x dx - \frac{4}{\pi} \cos \frac{\pi x}{4} \\
&= xe^x - e^x - \frac{4}{\pi} \cos \frac{\pi x}{4} \\
&= F(x)
\end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
I &= F(1) - F(0) \\
&= \left(1 \cdot e^1 - e^1 - \frac{4}{\pi} \cos \frac{\pi}{4} \right) - \left(0 \cdot e^0 - e^0 - \frac{4}{\pi} \cos 0 \right) \\
&= e - e - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} \right) + 1 + \frac{4}{\pi} = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}
\end{aligned}$$

Question 21:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

- A. $\frac{\pi}{3}$
- B. $\frac{2\pi}{3}$
- C. $\frac{\pi}{6}$
- D. $\frac{\pi}{12}$ equals

Solution:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{dx}{1+x^2} &= F(\sqrt{3}) - F(1) \\
&= \tan^{-1} \sqrt{3} - \tan^{-1} 1
\end{aligned}$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Thus, the correct option is D.

Question 22:

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$$

- A. $\frac{\pi}{6}$
 B. $\frac{\pi}{12}$
 C. $\frac{\pi}{24}$
 D. $\frac{\pi}{4}$ equals

Solution:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2 + (3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\begin{aligned} \therefore \int \frac{dx}{(2)^2 + (3x)^2} &= \frac{1}{3} \int \frac{dt}{(2)^2 + t^2} \\ &= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right] \\ &= \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right) \\ &= F(x) \end{aligned}$$

Using second fundamental theorem of calculus, we get

$$\begin{aligned} \int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} &= F\left(\frac{2}{3}\right) - F(0) \\ &= \frac{1}{6} \tan^{-1} \left(\frac{3}{2} \cdot \frac{2}{3} \right) - \frac{1}{6} \tan^{-1} 0 \\ &= \frac{1}{6} \tan^{-1} 1 - 0 \\ &= \frac{1}{6} \times \frac{\pi}{4} \\ &= \frac{\pi}{24} \end{aligned}$$

Thus, the correct option is C.

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

Question 1:

$$\int_0^1 \frac{x}{x^2 + 1} dx$$

Solution:

$$\int_0^1 \frac{x}{x^2 + 1} dx$$

Put, $x^2 + 1 = t \Rightarrow 2x dx = dt$

When, $x = 0, t = 1$ and when $x = 1, t = 2$

$$\begin{aligned}\therefore \int_0^1 \frac{x}{x^2 + 1} dx &= \frac{1}{2} \int_1^2 \frac{dt}{t} \\ &= \frac{1}{2} [\log|t|]_1^2 \\ &= \frac{1}{2} [\log 2 - \log 1] \\ &= \frac{1}{2} \log 2\end{aligned}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi$$

Solution:

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^4 \phi \cos \phi d\phi$$

Let $\sin \phi = t \Rightarrow \cos \phi d\phi = dt$

$$\text{When } \phi = 0, t = 0 \text{ and when } \phi = \frac{\pi}{2}, t = 1$$

$$\begin{aligned}
\therefore I &= \int_0^1 \sqrt{t} (1-t^2)^2 dt \\
&= \int_0^1 t^{\frac{1}{2}} (1+t^4 - 2t^2) dt \\
&= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} \right] dt \\
&= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1 \\
&= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} \\
&= \frac{154 + 42 - 132}{231} = \frac{64}{231}
\end{aligned}$$

Question 3:

$$\int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

Solution:

Consider, $I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$

Let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1} (\sin 2\theta) \sec^2 \theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2 \theta d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \theta \sec^2 \theta d\theta$$

Taking $u = \theta$ and $v = \sec^2 \theta$ and integrating by parts, we get

$$\begin{aligned}
I &= 2 \left[\theta \int \sec^2 \theta d\theta - \int \left\{ \left(\frac{d}{d\theta} \theta \right) \int \sec^2 \theta d\theta \right\} d\theta \right]_0^{\frac{\pi}{4}} \\
&= 2 \left[\theta \tan \theta - \int \tan \theta d\theta \right]_0^{\frac{\pi}{4}} \\
&= 2 \left[\theta \tan \theta + \log |\cos \theta| \right]_0^{\frac{\pi}{4}} \\
&= 2 \left[\frac{\pi}{4} \tan \frac{\pi}{4} + \log \left| \cos \frac{\pi}{4} \right| - \log |\cos 0| \right] = 2 \left[\frac{\pi}{4} + \log \left(\frac{1}{\sqrt{2}} \right) - \log 1 \right] \\
&= 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] \\
&= \frac{\pi}{2} - \log 2
\end{aligned}$$

Question 4:

$$\int_0^2 x \sqrt{x+2} \, dx \quad (\text{Put } x+2=t^2)$$

Solution:

$$\begin{aligned}
&\int_0^2 x \sqrt{x+2} \, dx \\
&\text{Put, } x+2=t^2 \Rightarrow dx = 2tdt
\end{aligned}$$

When $x=0, t=\sqrt{2}$ and when $x=2, t=2$

$$\begin{aligned}
&\therefore \int_0^2 x \sqrt{x+2} \, dx = \int_{\sqrt{2}}^2 (t^2 - 2) \sqrt{t^2} 2tdt \\
&= 2 \int_{\sqrt{2}}^2 (t^2 - 2) t^2 dt \\
&= 2 \int_{\sqrt{2}}^2 (t^4 - 2t^2) dt \\
&= 2 \left[\frac{t^5}{5} - \frac{2t^3}{3} \right]_{\sqrt{2}}^2 \\
&= 2 \left[\frac{32}{5} - \frac{16}{3} - \frac{4\sqrt{2}}{5} + \frac{4\sqrt{2}}{3} \right] = 2 \left[\frac{96 - 80 - 12\sqrt{2} + 20\sqrt{2}}{15} \right] = 2 \left[\frac{16 + 8\sqrt{2}}{15} \right] \\
&= \frac{16(2 + \sqrt{2})}{15} \\
&= \frac{16\sqrt{2}(\sqrt{2} + 1)}{15}
\end{aligned}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$$

Put, $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0, t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$$\begin{aligned} & \Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx = - \int_1^0 \frac{dt}{1 + t^2} \\ &= - \left[\tan^{-1} t \right]_1^0 \\ &= - \left[\tan^{-1} 0 - \tan^{-1} 1 \right] \\ &= - \left[-\frac{\pi}{4} \right] \\ &= \frac{\pi}{4} \end{aligned}$$

Question 6:

$$\int_0^2 \frac{dx}{x + 4 - x^2}$$

Solution:

$$\begin{aligned} & \int_0^2 \frac{dx}{x + 4 - x^2} = \int_0^2 \frac{dx}{-(x^2 - x - 4)} \\ &= \int \frac{dx}{-(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4)} = \int_0^2 \frac{dx}{-\left[\left(x - \frac{1}{2}\right)^2 - \frac{17}{4}\right]} \end{aligned}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

Let $x - \frac{1}{2} = t \Rightarrow dx = dt$

when $x = 0, t = -\frac{1}{2}$ and when $x = 2, t = \frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$\begin{aligned}
&= \left[\frac{1}{2} \left(\frac{\sqrt{17}}{2} \right) \log \frac{\frac{\sqrt{17}}{2} + t}{\frac{\sqrt{17}}{2} - t} \right]_{-\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{\sqrt{17}} \left[\log \frac{\frac{\sqrt{17}}{2} + \frac{3}{2}}{\frac{\sqrt{17}}{2} - \frac{3}{2}} - \log \frac{\frac{\sqrt{17}}{2} - \frac{1}{2}}{\frac{\sqrt{17}}{2} + \frac{1}{2}} \right] \\
&= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right] = \frac{1}{\sqrt{17}} \log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} + 1}{\sqrt{17} - 1} \\
&= \frac{1}{\sqrt{17}} \log \left[\frac{17 + 3 + 4\sqrt{17}}{17 + 3 - 4\sqrt{17}} \right] = \frac{1}{\sqrt{17}} \log \left[\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right] \\
&= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right) = \frac{1}{\sqrt{17}} \log \left[\frac{(5 + \sqrt{17})(5 + \sqrt{17})}{25 - 17} \right] \\
&= \frac{1}{\sqrt{17}} \log \left[\frac{25 + 17 + 10\sqrt{17}}{8} \right] = \frac{1}{\sqrt{17}} \log \left(\frac{42 + 10\sqrt{17}}{8} \right) \\
&= \frac{1}{\sqrt{17}} \log \left(\frac{21 + 5\sqrt{17}}{4} \right)
\end{aligned}$$

Question 7:

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$$

Solution:

$$\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \int_{-1}^1 \frac{dx}{(x^2 + 2x + 1) + 4} = \int_{-1}^1 \frac{dx}{(x+1)^2 + (2)^2}$$

Put, $x+1=t \Rightarrow dx=dt$

When $x=-1, t=0$ and when $x=1, t=2$

$$\begin{aligned}
&\int_{-1}^1 \frac{dx}{(x-1)^2 + (2)^2} = \int_0^2 \frac{dt}{t^2 + 2^2} \\
&= \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]_0^2 = \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0 \\
&= \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}
\end{aligned}$$

Question 8:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Solution:

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$$

Put, $2x = t \Rightarrow 2dx = dt$

When $x = 1, t = 2$ and when $x = 2, t = 4$

$$\therefore \int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2} \right) e^t dt$$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

$$\text{Let } \frac{1}{t} = f(t)$$

$$\text{Then, } f'(t) = -\frac{1}{t^2}$$

$$\Rightarrow \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt = \int_2^4 e^t [f(t) + f'(t)] dt$$

$$= [e^t f(t)]_2^4$$

$$= \left[e^t \cdot \frac{1}{t} \right]_2^4$$

$$= \left[\frac{e^t}{t} \right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2(e^2 - 2)}{4}$$

Question 9:

$$\text{The value of the integral } \int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx \text{ is}$$

- A. 6
B. 0

- C. 3
D. 4

Solution:

$$\text{Consider, } I = \int_{\frac{1}{3}}^1 \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$$

Let $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$

$$\text{When } x = \frac{1}{3}, \theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ and when } x = 1, \theta = \frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow I &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta - \sin^3 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (1 - \sin^2 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^4 \theta} \cos \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}} (\cos \theta)^{\frac{2}{3}}}{\sin^2 \theta \sin^2 \theta} \cos \theta d\theta = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\cos \theta)^{\frac{5}{3}}}{(\sin \theta)^{\frac{5}{3}}} \csc^2 \theta d\theta \\ &= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot \theta)^{\frac{5}{3}} \csc^2 \theta d\theta \end{aligned}$$

Put $\cot \theta = t \Rightarrow -\csc^2 \theta d\theta = dt$

$$\text{When } \theta = \sin^{-1}\left(\frac{1}{3}\right), t = 2\sqrt{2} \quad \text{and when } \theta = \frac{\pi}{2}, t = 0$$

$$\begin{aligned} \therefore I &= - \int_{2\sqrt{2}}^0 (t)^{\frac{5}{3}} dt \\ &= - \left[\frac{3}{8} (t)^{\frac{8}{3}} \right]_{2\sqrt{2}}^0 \\ &= - \frac{3}{8} \left[- (2\sqrt{2})^{\frac{8}{3}} \right]_{2\sqrt{2}}^0 = \frac{3}{8} \left[(\sqrt{8})^{\frac{8}{3}} \right] \\ &= \frac{3}{8} \left[(8)^{\frac{4}{3}} \right] \\ &= \frac{3}{8} [16] \\ &= 6 \end{aligned}$$

Thus, the correct option is A.

Question 10:

If $f(x) = \int_0^x t \sin t dt$, then $f'(x)$ is

A. $\cos x + x \sin x$

B. $x \sin x$

C. $x \cos x$

D. $\sin x + x \cos x$

Solution:

$$f(x) = \int_0^x t \sin t dt$$

Using integration by parts, we get

$$f(x) = t \int_0^x \sin t dt - \int_0^x \left(\frac{d}{dt} t \right) \int \sin t dt dt$$

$$= [t(-\cos t)]_0^x - \int_0^x (-\cos t) dt$$

$$= [-t \cos t + \sin t]_0^x$$

$$= -x \cos x + \sin x$$

$$\Rightarrow f'(x) = -[x(-\sin x)] + \cos x + \cos x$$

$$= x \sin x - \cos x + \cos x$$

$$= x \sin x$$

Thus, the correct option is B.

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2 \left(\frac{\pi}{2} - x\right) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx \quad \dots(1)$$

Let

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) dx}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right) + \cos^{\frac{3}{2}}\left(\frac{\pi}{2} - x\right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x dx}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx \quad \dots(3)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} dx$$

Solution:

Consider,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} dx \dots (1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 \left(\frac{\pi}{2} - x \right) dx}{\sin^5 \left(\frac{\pi}{2} - x \right) + \cos^5 \left(\frac{\pi}{2} - x \right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \cdot dx \Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Question 5:

$$\int_{-5}^5 |x+2| dx$$

Solution:

$$\text{Let } I = \int_{-5}^5 |x+2| dx$$

As, $(x+2) \leq 0$ on $[-5, -2]$ and $(x+2) \geq 0$ on $[-2, 5]$

$$\begin{aligned} \therefore \int_{-5}^5 |x+2| dx &= \int_{-5}^{-2} -(x+2) dx + \int_{-2}^5 (x+2) dx \quad \left(\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \right) \\ I &= -\left[\frac{x^2}{2} + 2x \right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x \right]_{-2}^5 \\ &= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5) \right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2) \right] \\ &= -\left[2 - 4 - \frac{25}{2} + 10 \right] + \left[\frac{25}{2} + 10 - 2 + 4 \right] \\ &= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4 \\ &= 29 \end{aligned}$$

Question 6:

$$\int_2^8 |x-5| dx$$

Solution:

$$\begin{aligned} \text{Consider, } I &= \int_2^8 |x-5| dx \\ \text{As } (x-5) &\leq 0 \text{ on } [2, 5] \text{ and } (x-5) \geq 0 \text{ on } [5, 8] \\ I &= \int_2^5 -(x-5) dx + \int_2^8 (x-5) dx \quad \left(\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \right) \\ &= -\left[\frac{x^2}{2} - 5x \right]_2^5 + \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= -\left[\frac{25}{2} - 25 - 2 + 10 \right] + \left[32 - 40 - \frac{25}{2} + 25 \right] = 9 \end{aligned}$$

Question 7:

$$\int_0^1 x(1-x)^n dx$$

Solution:

$$\text{Consider, } I = \int_0^1 x(1-x)^n dx$$

$$\begin{aligned}
\therefore I &= \int_0^1 (1-x)(1-(1-x))^n dx \\
&= \int_0^1 (1-x)x^n dx = \int_0^1 (x^n - x^{n+1}) dx \\
&= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1 \quad \left(\int_1^a f(x) dx = \int_0^a f(a-x) dx \right) \\
&= \left[\frac{1}{n+1} - \frac{1}{n+2} \right] \\
&= \frac{(n+2)-(n+1)}{(n+1)(n+2)} \\
&= \frac{1}{(n+1)(n+2)}
\end{aligned}$$

Question 8:

$$\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx \dots (1)$$

$$\begin{aligned}
\therefore I &= \int_0^{\frac{\pi}{4}} \log \left(1 + \tan \left(\frac{\pi}{4} - x \right) \right) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right) \\
\Rightarrow I &= \int_0^{\frac{\pi}{4}} \log \left\{ 1 + \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right\} dx \quad \left(\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \right) \\
\Rightarrow I &= \int_0^{\frac{\pi}{4}} \log \left\{ 1 + \frac{1 - \tan x}{1 + \tan x} \right\} dx \Rightarrow I = \int_0^{\frac{\pi}{4}} \log \frac{2}{(1 + \tan x)} dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{4}} \log 2 dx - \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{4}} \log 2 dx - I \quad [\text{from (1)}] \\
\Rightarrow 2I &= \log 2 \left[x \right]_0^{\frac{\pi}{4}} \\
\Rightarrow 2I &= \log 2 \left[\frac{\pi}{4} - 0 \right] \\
I &= \frac{\pi}{8} \log 2
\end{aligned}$$

Question 9:

$$\int_0^2 x\sqrt{2-x}dx$$

Solution:

$$\begin{aligned}
 \text{Consider, } I &= \int_0^2 x\sqrt{2-x}dx \\
 I &= \int_0^2 (2-x)\sqrt{2-(2-x)}dx \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx \right) \\
 &= \int_0^2 (2-x)\sqrt{x}dx \\
 &= \int_0^2 \left\{ 2x^{\frac{1}{2}} - x^{\frac{3}{2}} \right\} dx = \left[2 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^2 \\
 &= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} \right]_0^2 = \frac{4}{3}(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}} \\
 &= \frac{4 \times 2\sqrt{2}}{3} - \frac{2}{5} \times 4\sqrt{2} = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} \\
 &= \frac{40\sqrt{2} - 24\sqrt{2}}{15} = \frac{16\sqrt{2}}{15}
 \end{aligned}$$

Question 10:

$$\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx$$

Solution:

$$\begin{aligned}
 \text{Consider, } I &= \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x) dx \\
 I &= \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log(2 \sin x \cos x)) dx \\
 I &= \int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin x - \log \cos x - \log 2) dx \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \{\log \sin x - \log \cos x - \log 2\} \dots (1) \\
 \text{Since, } & \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx \right) \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \{\log \cos x - \log \sin x - \log 2\} dx \dots (2)
 \end{aligned}$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2) dx$$

$$\Rightarrow 2I = -2 \log 2 \int_0^{\frac{\pi}{2}} 1 \cdot dx$$

$$\Rightarrow I = -\log 2 \left[\frac{\pi}{2} \right]$$

$$\Rightarrow I = \frac{\pi}{2} (-\log 2)$$

$$\Rightarrow I = \frac{\pi}{2} \left[\log \frac{1}{2} \right]$$

$$\Rightarrow I = \frac{\pi}{2} \log \frac{1}{2}$$

Question 11:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$$

Solution:

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$$

As $\sin^2(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x$, therefore $\sin^2 x$ is an even function.

If $f(x)$ is an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

$$I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x dx = 2 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos 2x) dx = \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} - \frac{\sin 2\left(\frac{\pi}{2}\right)}{2} \right] - \left[0 - \frac{\sin 2(0)}{2} \right]$$

$$= \frac{\pi}{2} - \frac{\sin \pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Question 12:

$$\int_0^\pi \frac{x dx}{1 + \sin x}$$

Solution:

$$\text{Let } I = \int_0^\pi \frac{x dx}{1 + \sin x} \quad \dots(1)$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)}{1 + \sin(\pi - x)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)}{1 + \sin x} dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^\pi \frac{x}{1 + \sin x} dx + \int_0^\pi \frac{\pi - x}{1 + \sin x} dx$$

$$\Rightarrow 2I = \int_0^\pi \frac{\pi}{1 + \sin x} dx$$

Multiplying and Dividig by $(1 - \sin x)$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^\pi \{ \sec^2 x - \tan x \sec x \} dx$$

$$\Rightarrow 2I = \pi \left[[\tan x]_0^\pi - [\sec x]_0^\pi \right]$$

$$\Rightarrow 2I = \pi \left[(\tan(\pi) - \tan(0)) - (\sec(\pi) - \sec(0)) \right]$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Question 13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

Solution:

$$\text{Let } I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx \quad \dots(1)$$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, thus $\sin^7 x$ is an odd function.

$f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$$\therefore I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx = 0$$

Question 14:

$$\int_0^{2\pi} \cos^5 x dx$$

Solution:

$$\text{Let } I = \int_0^{2\pi} \cos^5 x dx \dots (1)$$

$$\cos^5(2\pi - x) = \cos^5 x$$

We know that,

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x)$$

$$= 0 \text{ if } f(2a-x) = -f(x)$$

$$\therefore I = 2 \int_0^{2\pi} \cos^5 x dx$$

$$\Rightarrow I = 2(0) = 0 \quad [\cos^5(\pi - x) = -\cos^5 x]$$

Question 15:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x}$$

Solution:

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx \dots (1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx \dots (2)$$

Adding (1) and (2), we get

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{0}{1 + \sin x \cos x} dx \Rightarrow I = 0$$

Question 16:

$$\int_0^\pi \log(1 + \cos x) dx$$

Solution:

Consider, $I = \int_0^\pi \log(1 + \cos x) dx \dots (1)$

$$\Rightarrow I = \int_0^\pi \log(1 + \cos(\pi - x)) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^\pi \log(1 - \cos x) dx \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^\pi \{\log(1 + \cos x) + \log(1 - \cos x)\} dx$$

$$\Rightarrow 2I = \int_0^\pi \log(1 - \cos^2 x) dx$$

$$\Rightarrow 2I = \int_0^\pi \log(\sin^2 x) dx$$

$$\Rightarrow 2I = 2 \int_0^\pi \log(\sin x) dx$$

$$\Rightarrow I = \int_0^\pi \log(\sin x) dx \dots (3)$$

$$\therefore \sin(\pi - x) = \sin x$$

We know that,

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a - x) = f(x)$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log \sin x dx \dots (4)$$

$$\Rightarrow I = 2 \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = 2 \int_0^{\frac{\pi}{2}} \log \cos x dx \dots (5)$$

Adding (4) and (5), we get

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2) dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{2}} (\log 2 \sin x \cos x - \log 2) dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{2}} \log \sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx \\
\text{put, } 2x &= t \Rightarrow 2dx = dt \\
\text{When } x &= 0, t = 0 \\
\therefore I &= \frac{1}{2} \int_0^a \log \sin t dt - \frac{\pi}{2} \log 2 \\
\Rightarrow I &= \frac{1}{2} - \frac{\pi}{2} \log 2 \\
\Rightarrow I &= -\pi \log 2
\end{aligned}$$

Question 17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$$

Solution:

$$\begin{aligned}
\text{Let } I &= \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx \quad \dots(1) \\
\text{We know that, } &\left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)
\end{aligned}$$

$$I = \int \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$$

$$\Rightarrow 2I = \int_0^a 1 dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1| dx$$

Solution:

$$\int_0^4 |x-1| dx$$

Since,

$$(x-1) \leq 0 \text{ when } 0 \leq x \leq 1 \text{ and } (x-1) \geq 0 \text{ when } 1 \leq x \leq 4$$

$$I = \int_0^1 |x-1| dx + \int_1^4 |x-1| dx \quad \left(\int_b^b f(x) dx = \int_b^c f(x) dx + \int_c^b f(x) dx \right)$$

$$\begin{aligned} I &= \int_0^1 -(x-1) dx + \int_1^4 (x-1) dx \\ &= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4 = 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1 \\ &= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1 \\ &= 5 \end{aligned}$$

Question 19:

Show that $\int_0^a f(x)g(x) dx = 2 \int_0^a f(x) dx$ if f and g are defined as $f(x) = f(a-x)$ and $g(x) = (a-x)$

Solution:

Let

$$I = \int_0^a f(x)g(x) dx \dots (1)$$

$$\Rightarrow \int_0^a f(a-x)g(a-x) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow \int_0^a f(x)g(a-x) dx \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\} dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\} dx$$

$$\Rightarrow 2I = \int_0^a f(x) \times 4 dx \quad [g(x) + g(a-x) = 4]$$

$$\Rightarrow I = 2 \int_0^a f(x) dx$$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$ is

- A. 0
- B. 2
- C. π
- D. 1

Solution:

Consider, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x \cos x + \tan^5 x + 1) dx$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dx$$

For $f(x)$ an even function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

If $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx$

And

$$I = 0 + 0 + 0 + 2 \int_0^{\frac{\pi}{2}} 1 dx$$

$$= 2 \left[x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

Thus, the correct is option C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$ is
 A. 2

- B. $\frac{3}{4}$
 C. 0
 D. -2

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \right) dx \quad \dots(1)$$

$$\Rightarrow I = I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)} \right) dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \left(\frac{4+3\cos x}{4+3\sin x} \right) \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left\{ \log \left(\frac{4+3\sin x}{4+3\cos x} \right) + \log \left(\frac{4+3\cos x}{4+3\sin x} \right) \right\} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x} \right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 dx$$

$$\Rightarrow I = 0$$

Thus, the correct option is C.

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 24.

Question 1:

$$\frac{1}{x-x^3}$$

Solution:

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

Let $\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{(1-x)} + \frac{C}{(1+x)}$... (1)

$$\Rightarrow 1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

Equating the coefficients of x^2 , x and constant terms, we get

$$-A + B - C = 0$$

$$B + C = 0$$

$$A = 1$$

On solving these equations, we get

$$A = 1$$

$$B = \frac{1}{2}$$

$$C = -\frac{1}{2}$$

From equation (1), we get

$$\begin{aligned}\frac{1}{x(1-x)(1+x)} &= \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \\ \Rightarrow \int \frac{1}{x(1-x)(1+x)} dx &= \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{(1-x)} dx - \frac{1}{2} \int \frac{1}{(1+x)} dx \\ &= \log|x| - \frac{1}{2} \log|(1-x)| - \frac{1}{2} \log|(1+x)| \\ &= \log|x| - \log\left|(1-x)^{\frac{1}{2}}\right| - \log\left|(1+x)^{\frac{1}{2}}\right| = \log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right| + C \\ &= \log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right| + C = \frac{1}{2} \log\left|\frac{x^2}{1-x^2}\right| + C\end{aligned}$$

Question 2:

$$\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$$

Solution:

$$\begin{aligned}\frac{1}{\sqrt{x+a} + \sqrt{x+b}} &= \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}} \\ &= \frac{\sqrt{x+a} - \sqrt{x+b}}{(x+a) - (x+b)} = \frac{(\sqrt{x+a} - \sqrt{x+b})}{a-b} \\ \Rightarrow \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx &= \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b}) dx \\ &= \frac{1}{(a-b)} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right] = \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C\end{aligned}$$

Question 3:

$$\frac{1}{x\sqrt{ax-x^2}} \quad \left[\text{Hint: } x = \frac{a}{t} \right]$$

Solution:

$$\begin{aligned}\frac{1}{x\sqrt{ax-x^2}} \\ \text{Let } x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2} dt \\ \Rightarrow \int \frac{1}{x\sqrt{ax-x^2}} dx = \int \frac{1}{\frac{a}{t} \sqrt{a \cdot \frac{a}{t} - \left(\frac{a}{t}\right)^2}} \left(-\frac{a}{t^2} dt \right) \\ = -\int \frac{1}{at} \frac{1}{\sqrt{\frac{1}{t} - \frac{1}{t^2}}} dt = -\frac{1}{a} \int \frac{1}{\sqrt{\frac{t^2}{t} - \frac{1}{t}}} dt \\ = -\frac{1}{a} \int \frac{1}{\sqrt{t-1}} dt \\ = -\frac{1}{a} \left[2\sqrt{t-1} \right] + C \\ = -\frac{1}{a} \left[2\sqrt{\frac{a}{x}-1} \right] + C \\ = -\frac{2}{a} \left(\sqrt{\frac{a-x}{x}} \right) + C\end{aligned}$$

Question 4:

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Solution:

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3} , we get

$$\begin{aligned} \frac{x^{-3}}{x^2 x^{-3}(x^4+1)^{\frac{3}{4}}} &= \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2 x^{-3}} \\ &= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5(x^4)^{-\frac{3}{4}}} = \frac{1}{x^5} \left(\frac{x^4+1}{x^4} \right)^{-\frac{3}{4}} \\ &= \frac{1}{x^5} \left(1 + \frac{1}{x^4} \right)^{-\frac{3}{4}} \end{aligned}$$

Let

$$\begin{aligned} \frac{1}{x^4} &= t \Rightarrow -\frac{4}{x^5} dx = dt \Rightarrow \frac{1}{x^5} dx = -\frac{dt}{4} \\ \therefore \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx &= \int \frac{1}{x^5} \left(1 + \frac{1}{x^4} \right)^{-\frac{3}{4}} dx = -\frac{1}{4} \int (1+t)^{-\frac{3}{4}} dt \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{4} \left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}} \right] + C = -\frac{1}{4} \frac{\left(1 + \frac{1}{x^4} \right)^{\frac{1}{4}}}{\frac{1}{4}} + C \\ &= -\left(1 + \frac{1}{x^4} \right)^{\frac{1}{4}} + C \end{aligned}$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} \quad \left[\text{Hint: } \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)} \text{ put } x = t^6 \right]$$

Solution:

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)}$$

$$\text{Let } x = t^6 \Rightarrow dx = 6t^5 dt$$

$$\begin{aligned} \therefore \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= \int \frac{1}{x^{\frac{1}{3}} \left(1 + x^{\frac{1}{6}} \right)} dx = \int \frac{6t^5}{t^2 (1+t)} dt \\ &= 6 \int \frac{t^3}{(1+t)} dt \end{aligned}$$

Adding and Subtracting 1 in Numerator

$$\begin{aligned} &= 6 \int \frac{t^3 + 1 - 1}{1+t} dt \\ &= 6 \int \left(\frac{t^3 + 1}{1+t} - \frac{1}{1+t} \right) dt \end{aligned}$$

$$\text{Using } a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$$

$$\begin{aligned} &= 6 \int \left\{ \frac{(1+t)(t^2 + 1^2 - 1 \times t)}{1+t} - \frac{1}{1+t} \right\} dt \\ &= 6 \int \left\{ (t^2 - t + 1) - \frac{1}{1+t} \right\} dt \\ &= 6 \left[\left(\frac{t^3}{3} \right) - \left(\frac{t^2}{2} \right) + t - \log|1+t| \right] \\ &= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left(1 + x^{\frac{1}{6}} \right) + C \\ &= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left(1 + x^{\frac{1}{6}} \right) + C \end{aligned}$$

Question 6:

$$\frac{5x}{(x+1)(x^2+9)}$$

Solution:

$$\text{Consider, } \frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)} \quad \dots(1)$$

$$\Rightarrow 5x = A(x^2 + 9) + (Bx + C)(x + 1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

Equating the coefficients of x^2 , x and constant term, we get

$$A + B = 0$$

$$B + C = 5$$

$$9A + C = 0$$

On solving these equations, we get

$$A = -\frac{1}{2}$$

$$B = \frac{1}{2}$$

$$C = \frac{9}{2}$$

From equation (1), we get

$$\begin{aligned} \frac{5x}{(x+1)(x^2+9)} &= \frac{-1}{2(x+1)} + \frac{\frac{x}{2} + \frac{9}{2}}{(x^2+9)} \\ \int \frac{5x}{(x+1)(x^2+9)} dx &= \int \left\{ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right\} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{2} \int \frac{x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx = -\frac{1}{2} \log|x+1| + \frac{1}{4} \int \frac{2x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+9| + \frac{9}{2} \cdot \frac{1}{3} \tan^{-1} \frac{x}{3} + C \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log(x^2+9) + \frac{3}{2} \tan^{-1} \frac{x}{3} + C \end{aligned}$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Solution:

$$\frac{\sin x}{\sin(x-a)}$$

Put, $x-a=t \Rightarrow dx=dt$

$$\begin{aligned} \int \frac{\sin x}{\sin(x-a)} dx &= \int \frac{\sin(t+a)}{\sin t} dt \\ &= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt = \int (\cos a + \cot t \sin a) dt \\ &= t \cos a + \sin a \log |\sin t| + C_1 \\ &= (x-a) \cos a + \sin a \log |\sin(x-a)| + C_1 \\ &= x \cos a + \sin a \log |\sin(x-a)| - a \cos a + C_1 \\ &= \sin a \log |\sin(x-a)| + x \cos a + C \end{aligned}$$

Question 8:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Solution:

$$\begin{aligned} \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} &= \frac{e^{4\log x}(e^{\log x} - 1)}{e^{2\log x}(e^{\log x} - 1)} \\ &= e^{2\log x} \\ &= e^{\log x^2} \\ &= x^2 \\ \therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx &= \int x^2 dx = \frac{x^3}{3} + C \end{aligned}$$

Question 9:

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Solution:

$$\begin{aligned} &\frac{\cos x}{\sqrt{4 - \sin^2 x}} \\ \text{Put, } \sin x &= t \Rightarrow \cos x dx = dt \\ \Rightarrow \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx &= \int \frac{dt}{\sqrt{(2)^2 - (t)^2}} \\ &= \sin^{-1}\left(\frac{t}{2}\right) + C \\ &= \sin^{-1}\left(\frac{\sin x}{2}\right) + C \\ &= \frac{x}{2} + C \end{aligned}$$

Question 10:

$$\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$$

Solution:

$$\begin{aligned}
 \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} &= \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x - \sin^2 x \cos^2 x} \\
 &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)}{(\sin^2 x - \sin^2 x \cos^2 x) + (\cos^2 x - \sin^2 x \cos^2 x)} \\
 &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)} \\
 &= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{(\sin^4 x + \cos^4 x)} \\
 &= -\cos 2x \\
 \therefore \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx &= \int -\cos 2x dx = -\frac{\sin 2x}{2} + C
 \end{aligned}$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we get

$$\begin{aligned}
& \frac{1}{\sin(a-b)} \left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} \right] \\
&= \frac{1}{\sin(a-b)} \left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)} \right] \\
&= \frac{1}{\sin(a-b)} \left[\frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} \right] \\
&= \frac{1}{\sin(a-b)} \left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)} \right] \\
&= \frac{1}{\sin(a-b)} [\tan(x+a) - \tan(x+b)] \\
\int \frac{1}{\cos(x+a)\cos(x+b)} dx &= \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)] dx \\
&= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C \\
&= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C
\end{aligned}$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Solution:

$$\begin{aligned}
& \frac{x^3}{\sqrt{1-x^8}} \\
\text{Put, } x^4 &= t \Rightarrow 4x^3 dx = dt \\
\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx &= \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{1}{4} \sin^{-1} t + C \\
&= \frac{1}{4} \sin^{-1}(x^4) + C
\end{aligned}$$

Question 13:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Solution:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\begin{aligned} & \Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{dt}{(t+1)(t+2)} \\ & = \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)} \right] dt \\ & = \log|t+1| - \log|t+2| + C \\ & = \log \left| \frac{t+1}{t+2} \right| + C \\ & = \log \left| \frac{1+e^x}{2+e^x} \right| + C \end{aligned}$$

Question 14:

$$\frac{1}{(x^2+1)(x^2+4)}$$

Solution:

$$\begin{aligned} & \therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)} \\ & \Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1) \\ & \Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D \end{aligned}$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$$A + C = 0$$

$$B + D = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

On solving these equations, we get

$$A = 0$$

$$B = \frac{1}{3}$$

$$C = 0$$

$$D = -\frac{1}{3}$$

From equation (1), we get

$$\begin{aligned}\frac{1}{(x^2+1)(x^2+4)} &= \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)} \\ \int \frac{1}{(x^2+1)(x^2+4)} dx &= \frac{1}{3} \int \frac{1}{x^2+1} dx - \frac{1}{3} \int \frac{1}{x^2+4} dx \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C\end{aligned}$$

Question 15:

$$\cos^3 x e^{\log \sin x}$$

Solution:

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

$$\text{Let } \cos x = t \Rightarrow -\sin x dx = dt$$

$$\begin{aligned}\Rightarrow \int \cos^3 x e^{\log \sin x} dx &= \int \cos^3 x \sin x dx \\ &= - \int t^3 dt \\ &= -\frac{t^4}{4} + C \\ &= -\frac{\cos^4 x}{4} + C\end{aligned}$$

Question 16:

$$e^{3 \log x} (x^4 + 1)^{-1}$$

Solution:

$$e^{3 \log x} (x^4 + 1)^{-1} = e^{\log x^3} (x^4 + 1)^{-1} = \frac{x^3}{(x^4 + 1)}$$

$$\text{Let } x^4 + 1 = t \Rightarrow 4x^3 dx = dt$$

$$\begin{aligned}
&\Rightarrow \int e^{3 \log x} (x^4 + 1)^{-1} dx = \int \frac{x^3}{(x^4 + 1)} dx \\
&= \frac{1}{4} \int \frac{dt}{t} \\
&= \frac{1}{4} \log|t| + C \\
&= \frac{1}{4} \log|x^4 + 1| + C \\
&= \frac{1}{4} \log(x^4 + 1) + C
\end{aligned}$$

Question 17:

$$f'(ax+b)[f(ax+b)]^n$$

Solution:

$$\begin{aligned}
&f'(ax+b)[f(ax+b)]^n \\
\text{Put, } &f(ax+b) = t \Rightarrow af'(ax+b)dx = dt \\
\Rightarrow &\int f'(ax+b)[f(ax+b)]^n dx = \frac{1}{a} \int t^n dt \\
&= \frac{1}{a} \left[\frac{t^{n+1}}{n+1} \right] = \frac{1}{a(n+1)} (f(ax+b))^{n+1} + C
\end{aligned}$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Solution:

$$\begin{aligned}
&\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)}} \\
&= \frac{1}{\sqrt{\sin^4 x \cos \alpha + \sin^3 x \cos x \sin \alpha}} \\
&= \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} = \frac{\csc^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}}
\end{aligned}$$

$$\text{Put, } \cos \alpha + \cot x \sin \alpha = t \Rightarrow -\csc^2 x \sin \alpha dx = dt$$

$$\begin{aligned}
& \therefore \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\cosec^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx \\
&= \frac{-1}{\sin \alpha} \int \frac{dt}{\sqrt{t}} \\
&= \frac{-1}{\sin \alpha} [2\sqrt{t}] + C \\
&= \frac{-1}{\sin \alpha} [2\sqrt{\cos \alpha + \cot x \sin \alpha}] + C \\
&= \frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \frac{\cos x \sin \alpha}{\sin x}} + C \\
&= \frac{-2}{\sin \alpha} \sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C = \frac{-2}{\sin \alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C
\end{aligned}$$

Question 19:

$$\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}, x \in [0,1]$$

Solution:

$$\text{Let } I = \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$$

$$\text{As we know that, } \sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$$

$$\Rightarrow I = \int \frac{\left(\frac{\pi}{2} - \cos^{-1} \sqrt{x}\right) - \cos^{-1} \sqrt{x}}{\frac{\pi}{2}} dx$$

$$\begin{aligned}
&= \frac{2}{\pi} \int \left(\frac{\pi}{2} - 2\cos^{-1} \sqrt{x} \right) dx \\
&= \frac{2}{\pi} \cdot \frac{\pi}{2} \int 1 dx - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \\
&= x - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \quad \dots(1)
\end{aligned}$$

$$\text{Let } I_1 = \int \cos^{-1} \sqrt{x} dx$$

$$\text{Also, let } \sqrt{x} = t \Rightarrow dx = 2tdt$$

$$\begin{aligned}
\Rightarrow I_1 &= 2 \int \cos^{-1} t \cdot t dt \\
&= 2 \left[\cos^{-1} t \cdot \frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right] \\
&= t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} dt \\
&= t^2 \cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt \\
&= t^2 \cos^{-1} t - \int \sqrt{1-t^2} dt + \int \frac{1}{\sqrt{1-t^2}} dt \\
&= t^2 \cos^{-1} t - \frac{1}{2} \sqrt{1-t^2} - \frac{1}{2} \sin^{-1} t + \sin^{-1} t \\
&= t^2 \cos^{-1} t - \frac{1}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t
\end{aligned}$$

From equation (1), we get

$$\begin{aligned}
I &= x - \frac{4}{\pi} \left[t^2 \cos^{-1} t - \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right] \\
&= x - \frac{4}{\pi} \left[x \cos^{-1} \sqrt{x} - \frac{\sqrt{x}}{2} \sqrt{1-x} + \frac{1}{2} \sin^{-1} \sqrt{x} \right] \\
&= x - \frac{4}{\pi} \left[x \left(\frac{\pi}{2} - \sin^{-1} \sqrt{x} \right) - \frac{\sqrt{x-x^2}}{2} + \frac{\pi}{2} \sin^{-1} \sqrt{x} \right] \\
&= x - 2x + \frac{4x}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - \frac{2}{\pi} \sin^{-1} \sqrt{x} \\
&\quad - x + \frac{2}{\pi} \left[(2x-1) \sin^{-1} \sqrt{x} \right] + \frac{2}{\pi} \sqrt{x-x^2} + C \\
&= \frac{2(2x-1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - x + C
\end{aligned}$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Put, $x = \cos^2 \theta \Rightarrow dx = -2 \sin \theta \cos \theta d\theta$

$$I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} (-2\sin\theta\cos\theta) d\theta = -\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}} \sin 2\theta d\theta$$

$$= -2 \int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \right) \cos\theta d\theta$$

$$= -4 \int \sin^2\frac{\theta}{2} \cos\theta d\theta$$

$$= -4 \int \sin^2\frac{\theta}{2} \left(2\cos^2\frac{\theta}{2} - 1 \right) d\theta$$

$$= -4 \int \left(2\sin^2\frac{\theta}{2} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \right) d\theta$$

$$= -8 \int \sin^2\frac{\theta}{2} \cdot \cos^2\frac{\theta}{2} d\theta + 4 \int \sin^2\frac{\theta}{2} d\theta$$

$$= -2 \int \sin^2\frac{\theta}{2} d\theta + 4 \int \sin^2\frac{\theta}{2} d\theta$$

$$= -2 \int \left(\frac{1-\cos 2\theta}{2} \right) d\theta + 4 \int \frac{1-\cos\theta}{2} d\theta$$

$$= -2 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] + 4 \left[\frac{\theta}{2} - \frac{\sin\theta}{2} \right] + C$$

$$= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2\sin\theta + C$$

$$= \theta + \frac{\sin 2\theta}{2} + 2\sin\theta + C$$

$$= \theta + \frac{2\sin\theta\cos\theta}{2} - 2\sin\theta + C$$

$$= \theta + \sqrt{1-\cos^2\theta} \cdot \cos\theta - 2\sqrt{1-\cos^2\theta} + C$$

$$= \cos^{-1}\sqrt{x} + \sqrt{1-x}\sqrt{x} - 2\sqrt{1-x} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$

$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x} e^x$$

Solution:

$$\begin{aligned} I &= \int \left(\frac{2+\sin 2x}{1+\cos 2x} \right) e^x \\ &= \int \left(\frac{2+2\sin x \cos x}{2\cos^2 x} \right) e^x \\ &= \int \left(\frac{1+\sin x \cos x}{\cos^2 x} \right) e^x \\ &= \int (\sec^2 x + \tan x) e^x \end{aligned}$$

$$\text{Let } f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$

$$\begin{aligned} \therefore I &= \int (f(x) + f'(x)) e^x dx \\ &= e^x f(x) + C \\ &= e^x \tan x + C \end{aligned}$$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Solution:

$$\text{Let } \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)} \quad \dots(1)$$

$$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x^2+2x+1)$$

$$\Rightarrow x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$$

$$\Rightarrow x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

Equating the coefficients of x^2 , x and constant term, we get

$$A+C=1$$

$$3A+B+2C=1$$

$$2A+2B+C=1$$

On solving these equations, we get

$$A=-2$$

$$B=1$$

$$C=3$$

From equation (1), we get

$$\begin{aligned} \frac{x^2+x+1}{(x+1)^2(x+2)} &= \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2} \\ \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx &= -2 \int \frac{1}{x+1} dx + 3 \int \frac{1}{(x+2)} dx + \int \frac{1}{(x+1)^2} dx \\ &= -2 \log|x+1| + 3 \log|x+2| - \frac{1}{(x+1)} + C \end{aligned}$$

Question 23:

$$\tan^{-1} \sqrt{\frac{1-x}{1+x}}$$

Solution:

$$I = \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

$$\text{Let } x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$$

$$\begin{aligned} I &= \int \tan^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} (-\sin \theta) d\theta \\ &= - \int \tan^{-1} \sqrt{\frac{2\sin^2 \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}}} \sin \theta d\theta = - \int \tan^{-1} \tan \frac{\theta}{2} \sin \theta d\theta \\ &= -\frac{1}{2} \int \theta \cdot \sin \theta d\theta = -\frac{1}{2} \left[\theta \cdot (-\cos \theta) - \int 1 \cdot (-\cos \theta) d\theta \right] \\ &= -\frac{1}{2} \left[-\theta \cos \theta + \sin \theta \right] \\ &= \frac{1}{2} \theta \cos \theta - \frac{1}{2} \sin \theta \\ &= \frac{1}{2} \cos^{-1} x \cdot x - \frac{1}{2} \sqrt{1-x^2} + C = \frac{x}{2} \cos^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C \\ &= \frac{1}{2} \left(x \cos^{-1} x - \sqrt{1-x^2} \right) + C \end{aligned}$$

Question 24:

$$\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}$$

Solution:

$$\begin{aligned}\frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4} &= \frac{\sqrt{x^2+1}}{x^4} [\log(x^2+1) - \log x^2] \\ &= \frac{\sqrt{x^2+1}}{x^4} \left[\log\left(\frac{x^2+1}{x^2}\right) \right] \\ &= \frac{\sqrt{x^2+1}}{x^4} \log\left(1 + \frac{1}{x^2}\right) \\ &= \frac{1}{x^3} \sqrt{\frac{x^2+1}{x^2}} \log\left(1 + \frac{1}{x^2}\right) \\ &= \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log\left(1 + \frac{1}{x^2}\right)\end{aligned}$$

$$\text{Let } 1 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt$$

$$\therefore I = \int \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log\left(1 + \frac{1}{x^2}\right) dx$$

$$= -\frac{1}{2} \int \sqrt{t} \log t dt = -\frac{1}{2} \int t^{\frac{1}{2}} \log t dt$$

Using integration by parts, we get

$$\begin{aligned}I &= -\frac{1}{2} \left[\log t \cdot \int t^{\frac{1}{2}} dt - \left\{ \left(\frac{d}{dt} \log t \right) \int t^{\frac{1}{2}} dt \right\} dt \right] \\ &= -\frac{1}{2} \left[\log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} dt \right] \\ &= -\frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \log t - \frac{2}{3} \int t^{\frac{1}{2}} dt \right] \\ &= -\frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \log t - \frac{4}{9} t^{\frac{3}{2}} \right] \\ &= -\frac{1}{3} t^{\frac{3}{2}} \log t + \frac{2}{9} t^{\frac{3}{2}} \\ &= -\frac{1}{3} t^{\frac{3}{2}} \left[\log t - \frac{2}{3} \right] \\ &= -\frac{1}{3} \left(1 + \frac{1}{x^2} \right)^{\frac{3}{2}} \left[\log\left(1 + \frac{1}{x^2}\right) - \frac{2}{3} \right] + C\end{aligned}$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-2\sin \frac{x}{2} \cos \frac{x}{2}}{2\sin^2 \frac{x}{2}} \right) dx = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\csc^2 \frac{x}{2}}{2} - \cot \frac{x}{2} \right) dx$$

Let $f(x) = -\cot \frac{x}{2}$

$$\Rightarrow f'(x) = -\left(\frac{1}{2} \csc^2 \frac{x}{2} \right) = \frac{1}{2} \csc^2 \frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x (f(x) + f'(x)) dx$$

$$= \left[e^x f(x) \right]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^x \cot \frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi}$$

$$= - \left[e^{\pi} \times \cot \frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot \frac{\pi}{4} \right]$$

$$= - \left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1 \right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{(\sin x \cos x)}{\cos^4 x}}{\frac{(\cos^4 x + \sin^4 x)}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

Put, $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{4}, t = 1$

$$\begin{aligned}\therefore I &= \frac{1}{2} \int_0^1 \frac{dt}{1+t^2} = \frac{1}{2} \left[\tan^{-1} t \right]_0^1 \\ &= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 \right] \\ &= \frac{1}{2} \left[\frac{\pi}{4} \right] \\ &= \frac{\pi}{8}\end{aligned}$$

Question 27:

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$$

Solution:

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx$$

$$\begin{aligned}
\Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)} dx \\
\Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4\cos^2 x} dx \\
\Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{-3\cos^2 x}{4 - 3\cos^2 x} dx \\
\Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x - 4}{4 - 3\cos^2 x} dx \\
\Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x} dx \\
\Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1 dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3} dx \\
\Rightarrow I &= -\frac{1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1 + \tan^2 x) - 3} dx \\
\Rightarrow I &= -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx \quad \dots (1)
\end{aligned}$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx$

Put, $2\tan x = t \Rightarrow 2\sec^2 x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$

$$\begin{aligned}
\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1 + 4\tan^2 x} dx &= \int_0^{\infty} \frac{dt}{1 + t^2} \\
&= [\tan^{-1} t]_0^{\infty} \\
&= [\tan^{-1}(\infty) - \tan^{-1}(0)] \\
&= \frac{\pi}{2}
\end{aligned}$$

Therefore, from (1), we get

$$I = -\frac{\pi}{6} + \frac{2}{3} \left[\frac{\pi}{2} \right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution:

$$\text{Consider, } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-\sin 2x)}} dx \Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$$

$$\text{Let } (\sin x - \cos x) = t \Rightarrow (\sin x + \cos x)dx = dt$$

$$\text{When } x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2} \right) \text{ and when } x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2} \right)$$

$$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I = \int_{\left(\frac{1+\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\text{As } \frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}, \text{ therefore, } \frac{1}{\sqrt{1-t^2}} \text{ is an even function}$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

We know that if $f(x)$ is an even function, then

$$\Rightarrow I = 2 \int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$= \left[2 \sin^{-1} t \right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$= 2 \sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right)$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

Solution:

$$\begin{aligned} \text{Consider, } I &= \int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}} \\ I &= \int_0^1 \frac{1}{(\sqrt{1+x} - \sqrt{x})} \times \frac{(\sqrt{1+x} + \sqrt{x})}{(\sqrt{1+x} + \sqrt{x})} dx \\ &= \int_0^1 \frac{(\sqrt{1+x} + \sqrt{x})}{1+x-x} dx \\ &= \int_0^1 \sqrt{1+x} dx + \int_0^1 \sqrt{x} dx \\ &= \left[\frac{2}{3}(1+x)^{\frac{3}{2}} \right]_0^1 + \left[\frac{2}{3}(x)^{\frac{3}{2}} \right]_0^1 \\ &= \frac{2}{3} \left[(2)^{\frac{3}{2}} - 1 \right] + \frac{2}{3}[1] \\ &= \frac{2}{3}(2)^{\frac{3}{2}} = \frac{2 \cdot 2\sqrt{2}}{3} \\ &= \frac{4\sqrt{2}}{3} \end{aligned}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Solution:

$$\begin{aligned} \text{Consider, } I &= \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx \\ \text{Put, } \sin x - \cos x &= t \Rightarrow (\cos x + \sin x) dx = dt \end{aligned}$$

$$\text{When } x = 0, t = -1 \text{ and when } x = \frac{\pi}{4}, t = 0$$

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\therefore I = \int_{-1}^0 \frac{dt}{9+16(1-t^2)}$$

$$= \int_{-1}^0 \frac{dt}{9+16-16t^2}$$

$$= \int_{-1}^0 \frac{dt}{25-16t^2} = \int_{-1}^0 \frac{dt}{(5)^2 - (4t)^2}$$

$$= \frac{1}{4} \left[\frac{1}{2(5)} \log \left| \frac{5+4t}{5-4t} \right| \right]_{-1}^0$$

$$= \frac{1}{40} \left[\log(1) - \log \left| \frac{1}{9} \right| \right]$$

$$= \frac{1}{40} \log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Solution:

$$\text{Consider, } I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1}(\sin x) dx$$

$$\text{Put, } \sin x = t \Rightarrow \cos x dx = dt$$

$$\text{When } x = 0, t = 0 \text{ and when } x = \frac{\pi}{2}, t = 1$$

$$\Rightarrow I = 2 \int_0^1 t \tan^{-1}(t) dt \quad \dots (1)$$

$$\text{Consider } \int t \cdot \tan^{-1} t dt = \tan^{-1} t \int t dt - \int \left\{ \frac{d}{dt} (\tan^{-1} t) \int t dt \right\} dt$$

$$= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2} \cdot \frac{t^2}{2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int \frac{t^2 + 1 - 1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int 1 dt + \frac{1}{2} \int \frac{1}{1+t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} t + \frac{1}{2} \tan^{-1} t$$

From equation (1), we get

$$\begin{aligned} \Rightarrow 2 \int_0^1 t \cdot \tan^{-1} t dt &= 2 \left[\frac{t^2 \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2} \tan^{-1} t \right]_0^1 \\ &= \left[\frac{\pi}{4} - 1 + \frac{\pi}{4} \right] = \frac{\pi}{2} - 1 \end{aligned}$$

Question 32:

$$\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$$

Solution:

$$\text{Let } \int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx \quad \dots(1)$$

$$I = \int_0^\pi \left\{ \frac{(\pi-x) \tan(\pi-x)}{\sec(\pi-x) + \tan(\pi-x)} \right\} dx \quad \left(\int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^\pi \left\{ \frac{-(\pi-x) \tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x) \tan x}{(\sec x + \tan x)} dx \quad \dots(2)$$

Adding (1) and (2), we get

$$\begin{aligned}
2I &= \int_0^\pi \frac{\pi \tan x}{\sec x + \tan x} dx \Rightarrow 2I = \pi \int_0^\pi \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx \\
&\Rightarrow 2I = \pi \int_0^\pi \frac{\sin x + 1 - 1}{1 + \sin x} dx \\
&\Rightarrow 2I = \pi \int_0^\pi 1 dx - \pi \int_0^\pi \frac{1}{1 + \sin x} dx \\
&\Rightarrow 2I = \pi \int_0^\pi 1 dx - \pi \int_0^\pi \frac{(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx \\
&\Rightarrow 2I = \pi [x]_0^\pi - \pi \int_0^\pi \frac{1 - \sin x}{\cos^2 x} dx \\
&\Rightarrow 2I = \pi^2 - \pi \int_0^\pi (\sec^2 x - \tan x \sec x) dx \\
&\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^\pi \\
&\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0] \\
&\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1] \\
&\Rightarrow 2I = \pi^2 - 2\pi \\
&\Rightarrow 2I = \pi(\pi - 2)
\end{aligned}$$

$$I = \frac{\pi}{2}(\pi - 2)$$

Question 33:

$$\int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

Solution:

$$\begin{aligned}
\text{Consider, } I &= \int_1^4 [|x-1| + |x-2| + |x-3|] dx \\
&\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x+2| dx + \int_1^4 |x+3| dx \\
I &= I_1 + I_2 + I_3 \quad \dots(1)
\end{aligned}$$

$$\text{Where, } I_1 = \int_1^4 |x-1| dx, I_2 = \int_1^4 |x+2| dx \text{ and } I_3 = \int_1^4 |x+3| dx$$

$$I_1 = \int_1^4 |x-1| dx$$

$(x-1) \geq 0$ for $1 \leq x \leq 4$

$$\therefore I_1 = \int_1^4 (x-1) dx$$

$$\Rightarrow I_1 = \left[\frac{x^2}{2} - x \right]_1^4$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1 \right] = \frac{9}{2} \quad \dots(2)$$

$$I_2 = \int_1^4 |x-2| dx$$

$x-2 \geq 0$ for $2 \leq x \leq 4$ and $x-2 \leq 0$ for $1 \leq x \leq 2$

$$\therefore I_2 = \int_1^2 (2-x) dx + \int_2^4 (x-2) dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} - 2x \right]_2^4 \Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2} \right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$\Rightarrow I_3 = \int_1^4 |x-3| dx$$

$x-3 \geq 0$ for $3 \leq x \leq 4$ and $x-3 \leq 0$ for $1 \leq x \leq 2$

$$\therefore I_3 = \int_1^3 (3-x) dx + \int_3^4 (x-3) dx$$

$$\Rightarrow I_3 = \left[3 - \frac{x^2}{2} \right]_1^3 + \left[\frac{x^2}{2} - 3x \right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2} \right] + \left[8 - 12 - \frac{9}{2} + 9 \right]$$

$$\Rightarrow I_3 = [6 - 4] + \left[\frac{1}{2} \right] = \frac{5}{2} \quad \dots(4)$$

From equations (1), (2), (3) and (4), we get

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

Question 34:

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$$

Solution:

$$\text{Consider, } \int_1^3 \frac{dx}{x^2(x+1)}$$

$$\text{Let, } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2 , x and constant terms, we get

$$A + C = 0$$

$$A + B = 0$$

$$B = 1$$

On solving these equations, we get

$$A = -1$$

$$C = 1$$

$$B = 1$$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Solution:

$$\text{Let } I = \int_0^1 xe^x dx$$

Using integration by parts, we get

$$I = x \int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence proved.

Question 36:

$$\int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Solution:

Consider, $I = \int_{-1}^1 x^{17} \cos^4 x dx$

Let $f(x) = x^{17} \cos^4 x$

$$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$$

$f(x)$ is an odd function.

We know that if $f(x)$ is an odd function, then $\int_{-a}^a f(x) dx = 0$

$$\therefore I = \int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Hence proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x dx$$

$$= \left[-\cos x \right]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence proved.

Question 38:

$$\int_0^{\frac{\pi}{4}} 2 \tan^3 x dx = 1 - \log 2$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} 2 \tan^3 x dx$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} 2 \tan^2 x \cdot \tan x dx = 2 \int_0^{\frac{\pi}{4}} (\sec^2 - 1) \tan x dx \\ &= 2 \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - 2 \int_0^{\frac{\pi}{4}} \tan x dx \\ &= 2 \left[\frac{\tan^2 x}{2} \right]_0^{\frac{\pi}{4}} + 2 \left[\log \cos x \right]_0^{\frac{\pi}{4}} = 1 + 2 \left[\log \cos \frac{\pi}{4} - \log \cos 0 \right] \\ &= 1 + 2 \left[\log \frac{1}{\sqrt{2}} - \log 1 \right] = 1 - \log 2 - \log 1 = 1 - \log 2 \end{aligned}$$

Hence proved.

Question 39:

$$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$$

Solution:

Let $\int_0^1 \sin^{-1} x dx$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 dx$$

Using integration by parts, we get

$$\begin{aligned} I &= \left[\sin^{-1} x \cdot x \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x dx \\ &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx \end{aligned}$$

$$\text{Put, } 1-x^2 = t \Rightarrow -2x dx = dt$$

When $x=0, t=1$ and when $x=1, t=0$

$$\begin{aligned} I &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t}} \\ &= \left[x \sin^{-1} x \right]_0^1 + \frac{1}{2} \left[2\sqrt{t} \right]_0^1 \\ &= \sin^{-1}(1) + [-\sqrt{1}] \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

Hence proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x} dx$ as a limit of a sum.

Solution:

$$\text{Let } I = \int_0^1 e^{2-3x} dx$$

We know that,

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

$$\text{Where, } h = \frac{b-a}{n}$$

Here, $a = 0, b = 1$ and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x} dx = (1-0) \lim_{n \rightarrow \infty} \frac{1}{n} [f(0) + f(0+h) + \dots + f(0+(n-1)h)]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} [e^2 + e^{2-3x} + \dots + e^{2-3(n-1)h}] = \lim_{n \rightarrow \infty} \frac{1}{n} [e^2 \{1 + e^{-3h} + e^{-6h} + e^{-9h} + \dots + e^{-3(n-1)h}\}]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^2 \left\{ \frac{1 - (e^{-3h})^n}{1 - (e^{-3h})} \right\} \right] = \lim_{n \rightarrow \infty} \frac{1}{n} \left[e^2 \left\{ \frac{1 - e^{-\frac{3n}{n}}}{1 - e^{-\frac{3}{n}}} \right\} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{e^2 (1 - e^{-3})}{1 - e^{-\frac{3}{n}}} \right] = e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{e^{-\frac{3}{n}} - 1} \right]$$

$$= e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \left(-\frac{1}{3} \right) \left[\frac{-\frac{3}{n}}{\frac{-3}{e^{-\frac{3}{n}} - 1}} \right] = \frac{e^2 (e^{-3} - 1)}{3} \lim_{n \rightarrow \infty} \left[\frac{\frac{-3}{n}}{e^{-\frac{3}{n}} - 1} \right]$$

$$= \frac{-e^2 (e^{-3} - 1)}{3} (1)$$

$$= \frac{-e^{-1} + e^2}{3}$$

$$= \frac{1}{3} \left(e^2 - \frac{1}{e} \right)$$

Question 41:

$\int \frac{dx}{e^x + e^{-x}}$ is equal to

- A. $\tan^{-1}(e^x) + C$
 B. $\tan^{-1}(e^{-x}) + C$
 C. $\log(e^x - e^{-x}) + C$
 D. $\log(e^x + e^{-x}) + C$

Solution:

Consider, $I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x} + 1} dx$

Put, $e^x = t \Rightarrow e^x dx = dt$

$$\begin{aligned}\therefore I &= \int \frac{dt}{1+t^2} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(e^x) + C\end{aligned}$$

Thus, the correct option is A.

Question 42:

$\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$ is

- A. $\frac{-1}{\sin x + \cos x} + C$
 B. $\log|\sin x + \cos x| + C$
 C. $\log|\sin x - \cos x| + C$
 D. $\frac{1}{(\sin x + \cos x)^2} + C$

Equals to

Solution:

Consider, $I = \int \frac{\cos 2x}{(\sin x + \cos x)^2} dx \Rightarrow I = \int \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)} dx$
 $= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\sin x + \cos x)^2} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Let $\cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt$

$$\begin{aligned}\therefore I &= \int \frac{dt}{t} \\ &= \log|t| + C \\ &= \log|\cos x + \sin x| + C\end{aligned}$$

Thus, the correct option is B.

Question 43:

If $f(a+b-x) = f(x)$, then $\int_a^b xf(x)dx$ is equal to

- A. $\frac{a+b}{2} \int_a^b f(b-x)dx$
- B. $\frac{a+b}{2} \int_a^b f(b+x)dx$
- C. $\frac{b-a}{2} \int_a^b f(x)dx$
- D. $\frac{a+b}{2} \int_a^b f(x)dx$

Solution:

$$\text{Consider, } I = \int_a^b xf(x)dx \quad \dots(1)$$

$$I = \int_a^b (a+b-x)f(a+b-x)dx \quad \left(\int_a^b f(x)dx = \int_a^b f(a+b-x)dx \right)$$

$$\Rightarrow I = \int_a^b (a+b-x)f(x)dx$$

$$\Rightarrow I = (a+b) \int_a^b f(x)dx - I \dots \text{(Using equation (1))}$$

$$\Rightarrow I + I = (a+b) \int_a^b f(x)dx$$

$$\Rightarrow 2I = (a+b) \int_a^b f(x)dx$$

$$\Rightarrow I = \left(\frac{a+b}{2} \right) \int_a^b f(x)dx$$

Thus, the correct option is D.

Question 44:

The value of $\int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$ is

- A. 1

B. 0

C. -1

D. $\frac{\pi}{4}$

Solution:

$$\text{Consider, } I = \int_0^1 \tan^{-1} \left(\frac{2x-1}{1+x-x^2} \right) dx$$

$$\Rightarrow I = \int_0^1 \tan^{-1} \left(\frac{x-(1-x)}{1+x(1-x)} \right) dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x) \right] dx \quad \dots(1)$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x) \right] dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1} x \right] dx$$

$$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x) \right] dx \quad \dots(2)$$

Adding (1) and (2), we get

$$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) + \tan^{-1} x \right) dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Thus, the correct option is B.