# CHAPTER 2: Complex Number

1. $\sqrt{-1} = +i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $\boxed{i^{4n} = 1 \ \forall n \in N}$
2. $|z_1 + z_2| \le |z_1| + |z_2|$ (Triangle inequality)
3) $|z_1 - z_2| \geq ||z_1| - |z_2||$
4. 1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ 2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ 3) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$
   4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$ 5) $Re(z) = \frac{z + \bar{z}}{2}$ 6) $Im(z) = \frac{z - \bar{z}}{2i}$
   7) $\overline{z^n} = (\bar{z})^n \ \forall n \in Z$ 8) $z$ is real iff $z = \bar{z}$
   9) $z$ is purely imaginary iff $-z = \bar{z}$ 10) $\bar{\bar{z}} = z$.
5) 1) $|z| = |\bar{z}|$ 2) $|z_1 z_2| = |z_1||z_2|$ 3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
   4) $|z^n| = |z|^n$ 5) $Re(z) \le |z|$ 6) $Im(z) \le |z|$.
6) $z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$.
7) Square root of $a+ib$ is $\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z| + a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z| - a}{2}}\right)$
   where $z = a+ib$, and $b \neq 0$.
8) 1) Polar form $z = r(\cos\theta + i\sin\theta)$
   2) $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$
   3) $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
   4) $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

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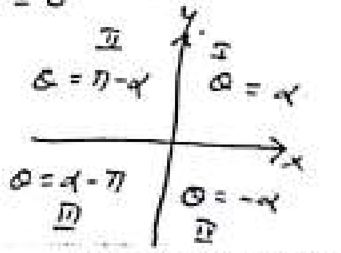
9) De moivre's theorem. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. $\forall n \in Z$.
10) $n^{th}$ roots of a complex number $z = \cos\theta + i\sin\theta$
   $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$. $k = 0, 1, 2 \cdots (n-1)$
11) 1) $1 = \cos 0 + i\sin 0$, 2) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, 3) $-1 = \cos\pi + i\sin\pi$.
12) $\sin\theta + i\cos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$
13) cube root of unity 1) $\omega^3 = 1$ 2) $1 + \omega + \omega^2 = 0$
   3) $\omega = \frac{-1 + i\sqrt{3}}{2}$, 4) $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$.
14) $n^{th}$ root of unity $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$ & $\omega^n = 1$.
15) Principle argument of a complex number



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## Exercise 2.1

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^{4n} = 1$

1. Simplify the following

$i^{1947} + i^{1950} = (i^4)^{486} \cdot i^3 + (i^4)^{487} \cdot i^2$

$= i^3 + i^2 = -i - 1$

$= -1 - i$

2) $i^{1948} - i^{-1869} = (i^4)^{487} - (i)^{-1872} (i)^{+3}$

$= 1 - (1)(-i) = 1 + i$

3) $\sum_{n=1}^{12} i^n = i + i^2 + i^3 + i^4 + \cdots + i^{12}$

$1 + i + i^2 + i^4$
$= (1 + i - 1 - i) = 0$

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8)$
$+ (i^9 + i^{10} + i^{11} + i^{12})$

$= (i + 1 - i + 1) + (i + i^2 + i^3 + i^4) + (i + i^2 + i^3 + i^4)$

$= 0 + 0 + 0 = 0.$

4) $i^{59} + \frac{1}{i^{59}} = i^{59} + i^{-59} = (i^4)^{14}\, i^3 + i^{-60} \times i^1$

$= -i + i = 0.$

5) $i \cdot i^2 \cdot i^3 \cdots i^{2000} = i^{1+2+3+\cdots+2000}$

$\left[\because 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\right]$

$= i^{2000 \times \frac{2001}{2}} = i^{1000 \times 2001}$

$= i^{2001000} = (i^4)^{500250} = 1$

6. $\sum_{n=1}^{10} i^{n+50} = i^{51} + i^{52} + i^{53} + \cdots + i^{60}$

$= i^3 + 1 + i^1 + i^2 + i^3 + i^4 + i + i^2 + i^3 + i^4$

$= -i + 1 + i - 1 - i + 1 + i - 1 - i + 1 = 1 - i$

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## Exercise 2.2

1. Evaluate the following if $z = 5-2i$ and $w = -1+3i$

$\boxed{i^2 = -1}$

(i) $z+w = 5-2i-1+3i = 4+i$

(ii) $z-iw = (5-2i)-i(-1+3i) = 5-2i+i-3i^2$
$= 5-2i+i+3 = 8-i$

(iii) $2z+3w = 2(5-2i)+3(-1+3i) = 10-4i-3+9i$
$= 7+5i$

(iv) $zw = (5-2i)(-1+3i) = -5+15i+2i-6i^2$
$= -5+17i-6(-1) = -5+17i+6 = 1+17i$

(v) $z^2+2zw+w^2 = (5-2i)^2+2(5-2i)(-1+3i)+(-1+3i)^2$
$= 25+4i^2-20i+2(-5+15i+2i-6i^2)+1+3i^2-6i$
$= 25-4-20i-10+30i+4i+12+1-3-6i$
$= 15+8i$

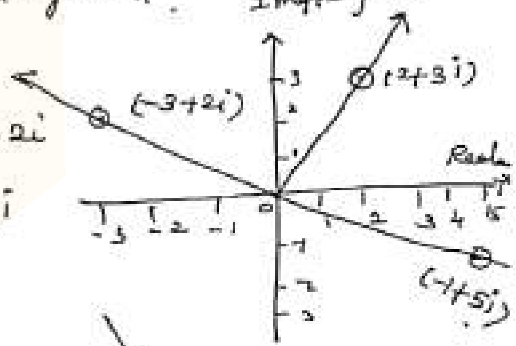
2. Given the complex number $z = 2+3i$ represent the complex numbers in Argand diagram.

(i) $z$, $iz$ and $z+iz$.

$z = 2+3i$

$iz = i(2+3i) = 2i+3i^2 = -3+2i$

$z+iz = (2+3i)-3+2i = -1+5i$

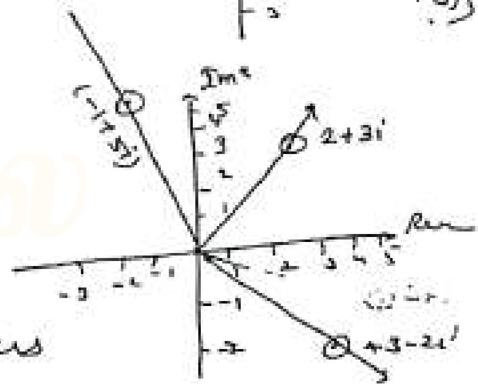



(ii) $z$, $-iz$, $z-iz$

$z = 2+3i$

$-iz = -i(2+3i) = -2i-3i^2 = +3-2i$

$z-iz = 2+3i+3+2i$
$= -1+5i$




3) Find the values of the real numbers $x$ and $y$ if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

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Soln

$(3-i)x-(2-i)y+2i+5 = 2x+(-1+2i)y+3+2i$

$3x-ix-2y+iy+2i+5 = 2x-y+2iy+3+2i$

$(3x-2y+5)+i(-x+y+2) = (2x-y+3)+i(2y+2)$

Equating real & imaginary parts

(i) $3x-2y+5 = 2x-y+3$

$3x-2y+5-2x+y-3=0$

$x-y+2=0$ — (1)

(ii) $-x+y+2 = 2y+2$

$x-y-2+2y+2=0$

$x+y=0$ — (2)

Solve (1) & (2)

$x-y=-2$

$x+y=0$

$2x=-2$

$x=-1$

Put in (2) $x+y=0$

$-1+y=0$

$y=1$

$$\boxed{\therefore x=-1,\ y=1}$$

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## Exercise 2.3

1. If $z_1 = 1-3i$, $z_2 = -4i$ and $z_3 = 5$ show that

(i) $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$

LHS: $(z_1+z_2)+z_3 = (1-3i-4i)+(5) = (1-7i)+5 = 6-7i$

RHS: $z_1+(z_2+z_3) = (1-3i)+(-4i+5) = 6-7i$

$\therefore$ LHS = RHS.

(ii) $(z_1 z_2)z_3 = z_1(z_2 z_3)$

LHS $z_1 z_2 = (1-3i)(-4i) = -4i+12i^2 = -4i-12$

$(z_1 z_2)z_3 = (-12-4i)5 = -60-20i$

RHS $(z_2 z_3) = (-4i)(5) = -20i$

$z_1(z_2 z_3) = (1-3i)(-20i) = -20i+60i^2$

$= -20i-60 = -60-20i$

$\therefore$ LHS = RHS.

2. If $z_1 = 3$ $z_2 = -7i$ and $z_3 = 5+4i$ show that

(i) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$

LHS $z_1(z_2+z_3) = 3(-7i+5+4i) = 3(5-3i)$

$= 15-9i$

RHS $z_1 z_2 = 3(-7i) = -21i$

$z_1 z_3 = 3(5+4i) = 15+12i$

$z_1 z_2 + z_1 z_3 = -21i+15+12i = 15-9i$

LHS = RHS.

(ii) $(z_1+z_2)z_3 = z_1 z_3 + z_2 z_3$

LHS $(z_1+z_2)z_3 = (3-7i)(5+4i)$

$= 15+12i-35i-28i^2$

$= 15-23i+28 = 43-23i$

RHS $z_1 z_3 = 3(5+4i) = 15+12i$

$z_2 z_3 = -7i(5+4i) = -35i-28i^2 = -35i+28$

$z_1 z_3 + z_2 z_3 = 15+12i-35i+28 = 43-23i$

$\therefore$ LHS = RHS.

3) If $z_1 = 2+5i$, $z_2 = -3-4i$ and $z_3 = 1+i$ find the additive and multiplicative inverse of $z_1, z_2$ & $z_3$. (5)

Soln

i) $z_1 = 2+5i$

additive inverse $-z_1 = -2-5i$

multiplicative inverse $\frac{1}{z_1} = \frac{1}{2+5i} \frac{(2-5i)}{(2-5i)} = \frac{2-5i}{2^2+5^2}$

$= \frac{2-5i}{29} = \frac{2}{29} - \frac{5i}{29}$ $[\because (a+ib)(a-ib) = a^2+b^2]$

ii) $z_2 = -3-4i$

additive inverse $-z_2 = 3+4i$

multiplicative inverse $\frac{1}{z_2} = \frac{1}{(-3-4i)} \frac{(-3+4i)}{(-3+4i)}$

$= \frac{-3+4i}{(-3)^2+4^2} = \frac{-3+4i}{9+16} = \frac{-3+4i}{25} = \frac{-3}{25} + i\frac{4}{25}.$

iii) $z_3 = 1+i$

additive inverse $-z_3 = -1-i$

multiplicative inverse $\frac{1}{z_3} = \frac{1}{1+i} \frac{(1-i)}{(1-i)}$

$= \frac{1-i}{1^2+1^2} = \frac{1-i}{2} = \frac{1}{2} - i\left(\frac{1}{2}\right)$

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## Exercise 2.4

1. Write the following in the rectangular form.

(i) $\overline{(5+9i)+(2-4i)} = \overline{7+5i} = 7-5i$

(ii) $\dfrac{10-5i}{6+2i} = \dfrac{10-5i}{6+2i}\dfrac{(6-2i)}{(6-2i)} = \dfrac{60-20i-30i+10i^2}{6^2+2^2}$

$= \dfrac{60-50i-10}{36+4} = \dfrac{+50-50i}{40} = \dfrac{50}{40} - \dfrac{50}{40}i$

$= \dfrac{5}{4} - \dfrac{5}{4}i$

(iii) $\overline{3i} + \dfrac{1}{2-i} = -3i + \dfrac{1}{2-i}\left(\dfrac{2+i}{2+i}\right) = -3i + \dfrac{2+i}{2^2+1^2}$

$= -3i + \dfrac{2+i}{(4+1)} = -3i + \dfrac{2+i}{5}$

$= \dfrac{-15i+2+i}{5} = \dfrac{2-14i}{5} = \dfrac{2}{5} - \dfrac{14}{5}i$

2) If $z = x+iy$ find the following in rectangular form.

(i) $Re\left(\dfrac{1}{z}\right)$

$\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{1}{x+iy}\dfrac{(x-iy)}{(x-iy)} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

$Re\left(\dfrac{1}{z}\right) = \dfrac{x}{x^2+y^2}$.

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(ii) $Re(i\bar{z})$

$z = x+iy$

$\bar{z} = x-iy$

$i\bar{z} = ix - i^2y = ix+y = y+ix$.

$Re(i\bar{z}) = y$.

(iii) $Im(3z+4\bar{z}-4i)$

$3z+4\bar{z}-4i = 3(x+iy)+4(x-iy)-4i$

$= 3x+i3y+4x-i4y-4i$

$= 7x - iy - 4i = 7x + i(-y-4)$

$Im(3z+4\bar{z}-4i) = -y-4.$

3) If $z_1 = 2-i$ & $z_2 = -4+3i$ find the inverse of $z_1 z_2$ & $\frac{z_1}{z_2}$

Soln $z_1 z_2 = (2-i)(-4+3i) = -8+6i+4i-3i^2$

$= -8+10i+3 = -5+10i$

$$\frac{1}{z_1 z_2} = \frac{1}{-5+10i} = \frac{1}{-5+10i} \cdot \frac{(-5-10i)}{(-5-10i)} = \frac{-5-10i}{(-5)^2+10^2}$$

$$= \frac{-5-10i}{25+100} = \frac{-5-10i}{125} = \frac{-5}{125} - \frac{10}{125}i$$

$$= \frac{-1}{25}(1+2i)$$

inverse of $z_1 z_2 = \frac{-1}{25}(1+2i)$

(ii) Inverse of $\frac{z_1}{z_2}$ is $\frac{z_2}{z_1}$

$$\therefore \frac{z_2}{z_1} = \frac{-4+3i}{2-i} = \frac{(-4+3i)}{(2-i)} \times \frac{(2+i)}{(2+i)}$$

$$= \frac{-8-4i+6i+3i^2}{2^2+1^2} = \frac{-8+2i-3}{5} = \frac{-11+2i}{5}$$

4) The complex numbers u, v and w are related by $\frac{1}{u} = \frac{1}{v}+\frac{1}{w}$. If $v = 3-4i$, and $w = 4+3i$ find u in rectangular form.

Soln. $$\frac{1}{v} = \frac{1}{3-4i} \cdot \frac{(3+4i)}{3+4i} = \frac{3+4i}{3^2+4^2} = \frac{3+4i}{9+16} = \frac{3+4i}{25}$$

$$\frac{1}{w} = \frac{1}{4+3i} \cdot \frac{(4-3i)}{(4-3i)} = \frac{4-3i}{4^2+3^2} = \frac{4-3i}{16+9} = \frac{4-3i}{25}$$

$$\frac{1}{u} = \frac{1}{v}+\frac{1}{w} = \frac{3+4i}{25} + \frac{4-3i}{25} = \frac{7-i}{25}$$



5) Prove the following properties.

(i) z is real if and only if $z = \bar{z}$

Soln z is real.

$\therefore y = 0$

$z = x + i0 = x$

$z = x$

$\bar{z} = x$

$\therefore z = \bar{z}$

if $z = \bar{z}$

$z = x+iy \quad \bar{z} = x - iy.$

$\cancel{x}+iy = \cancel{x} - iy$

$2iy = 0$

$\boxed{y=0} \Rightarrow z$ is real.

$\therefore z$ is real iff $z = \bar{z}$.

(i) $Re(z) = \dfrac{z+\bar{z}}{2}.$

$z = x+iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$\dfrac{\bar{z}+z}{2} = x = Re(z)$

(ii) $Im(z) = \dfrac{z-\bar{z}}{2i}$

$z = x+iy, \quad \bar{z} = x - iy$

$z - \bar{z} = x+iy-x+iy$

$z - \bar{z} = 2iy$

$y = \dfrac{z-\bar{z}}{2i}$

$Im(z) = \dfrac{z-\bar{z}}{2i}$

6) Find the least value of the positive integer $n$ for which $(\sqrt{3}+i)^n$

(i) real.

$(\sqrt{3}+i)^2 = (\sqrt{3}+i)(\sqrt{3}+i) = 3 - 1 + 2\sqrt{3}i = 2 + 2\sqrt{3}i$

$= 2(1+\sqrt{3}i)$

$(\sqrt{3}+i)^4 = 2(1+\sqrt{3}i)(2(1+\sqrt{3}i) = 4(1-3+2i\sqrt{3})$

$= 4(-2+2i\sqrt{3}) = 8(-1+i\sqrt{3})$

$(\sqrt{3}+i)^6 = 8(-1+i\sqrt{3})$ ഗ്ഗ $2(1+\sqrt{3}i) = 16(\sqrt{3}^2+1^2)$

$= 16(3+1) = 16(4) = 32. =$ purely real.

$\therefore n=6.$ is the least +ve integer for $(\sqrt{3}+i)^n$ is purely real.

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(ii) $(\sqrt{3}+i)^2 = 2(1+i\sqrt{3})$

$(\sqrt{3}+i)^3 = 2(1+i\sqrt{3})(\sqrt{3}+i) = 2(\sqrt{3}+i+i(3)+i^2\sqrt{3})$

$= 2(\cancel{\sqrt{3}}+i+3i-\cancel{\sqrt{3}}) = 2(4i) = 8i$

= purely imaginary

$\therefore$ $n=3$ is the least +ve integer for purely imaginary.

7) Show that $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary.

(i) Soln

$z$ is purely imaginary $\bar{z} = -z$.

$z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

$\bar{z} = \overline{(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}}$

$= (2-i\sqrt{3})^{10} - (2+i\sqrt{3})^{10}$

$= -[(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}]$

$\bar{z} = -z$. $\therefore$ $z$ is purely imaginary

$\therefore (2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary.

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Soln $\frac{(19-7i)}{9+i} \times \frac{(9-i)}{9-i} = \frac{171-19i-63i-7}{81+1} = \frac{164-82i}{82} = 2-i$

$\frac{(20-5i)}{(7-6i)} \times \frac{(7+6i)}{(7+6i)} = \frac{140+120i-35i+30}{49+36} = \frac{170+85i}{85} = 2+i$

$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$= (2-i)^{12} + (2+i)^{12}$

$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$

$= (2+i)^{12} + (2-i)^{12}$

$\bar{z} = z$

$\therefore$ $z$ is real.

$\therefore \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

## Exercise 2.5

1. Find the modulus of the following complex numbers.

Soln (i) $\frac{2i}{3+4i}$

$\left|\frac{2i}{3+4i}\right| = ?$

$\frac{2i}{3+4i}\frac{(3-4i)}{(3-4i)} = \frac{6i-8i^2}{3^2+4^2} = \frac{6i+8}{16+9} = \frac{8+6i}{25}$

$\left|\frac{2i}{3+4i}\right| = \left|\frac{8+6i}{25}\right| = \frac{1}{25}\sqrt{8^2+6^2} = \frac{1}{25}\sqrt{64+36} = \frac{1}{25}\sqrt{100}$

$= \frac{1}{25}(10) = \frac{2}{5}$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i} = \frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$

$= \frac{2-2i-\cancel{i}+i^2+1+\cancel{i}-2i-2i^2}{1^2+1^2} = \frac{3-4i-1+2}{2}$

$= \frac{4-4i}{2} = \frac{2(2-2i)}{2} = 2-2i$

$|2-2i| = \sqrt{2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$.

(iii) $2i(3-4i)(4-3i) = 2i(\cancel{12}-9i-16i-\cancel{12})$

$= 2i(-25i) = -50i^2 = 50$

$|50| = 50$.

2) For any two complex numbers $z_1$ & $z_2$ such that $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Soln. $|z_1| = |z_2| = 1$

$\left|\frac{z_1+z_2}{1+z_1z_2}\right| = \frac{|z_1+z_2|}{1+|z_1z_2|} \leq \frac{|z_1|+|z_2|}{1+|z_1||z_2|}$

$\leq \frac{1+1}{1+1} \leq 1$

$\left|\frac{z_1+z_2}{1+z_1+z_2}\right| \leq 1$ lies between $[0,1]$

$\therefore \frac{z_1+z_2}{1+z_1z_2}$ is a real number.

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(ii) **Alter** $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

$\bar{z}_1 = \frac{1}{z_1}$ $\;\;|||^{ly}\;\; \bar{z}_2 = \frac{1}{z_2}$

$$\overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)} = \frac{\overline{z_1+z_2}}{\overline{1+z_1z_2}} \qquad (z_1z_2 \neq -1)$$

$$= \frac{\bar{z}_1+\bar{z}_2}{1+\bar{z}_1\bar{z}_2} = \frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1}{z_1}\cdot\frac{1}{z_2}} = \frac{\frac{z_1+z_2}{\cancel{z_1z_2}}}{\frac{1+z_1z_2}{z_1z_2}}$$

$$\overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)} = \frac{z_1+z_2}{1+z_1z_2}$$

$\Rightarrow \frac{z_1+z_2}{1+z_1z_2}$ is real. (if $z$ is real iff $z = \bar{z}$)

3) which one of the points $10-8i$, $11+6i$ is closest to $1+i$

Soln

$$|10-8i| = \sqrt{10^2+8^2} = \sqrt{100+64} = \sqrt{164}$$

$$|11+6i| = \sqrt{121+36} = \sqrt{157}$$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$$

$\therefore 11+6i$ is closest to $1+i$

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4) if $|z| = 3$ show that $7 \le |z+6-8i| \le 13$

Soln $|z+6-8i| \le |z| + |6-8i|$ $\qquad |z_1+z_2| \le |z_1|+|z_2|$

$$\le 3 + \sqrt{6^2+8^2}$$

$$\le 3+\sqrt{36+64} = 3+\sqrt{100} = 3+10 = 13$$

$$\le 13 \quad —(1).$$

$$|z+6-8i| \ge \big|\,|z| - |6-8i|\,\big|$$

$$\ge \left|3 - \sqrt{6^2+8^2}\right| = |3-10| = |-7|$$

$$\ge 7 \quad —(2)$$

by ① & ② $7 \le |z+6-8i| \le 13$.

5) If $|z|=1$ show that $2 \le |z^2-3| \le 4$

soln

$|z^2-3| \ge ||z|^2-3|$
$\ge |1-3|$
$\ge |-2|$
$\ge 2$ — ①

$|z^2-3| \le |z^2|+|-3|$
$\le 1+3$
$|z^2-3| \le 4$. — ②

by ① & ② $2 \le |z^2-3| \le 4$.

6) If $\left|z-\frac{2}{z}\right|=2$ show that the greatest and least value of $|z|$ are $\sqrt{3}+1$ & $\sqrt{3}-1$ respectively.

soln

$$|z| = \left|z-\frac{2}{z}+\frac{2}{z}\right|$$
$$\le \left|z-\frac{2}{z}\right|+\frac{2}{|z|}$$
$$|z| \le 2+\frac{2}{|z|}$$
$$|z|-2-\frac{2}{|z|} \le 0$$
$$\frac{|z|^2-2|z|-2}{|z|} \le 0$$
$$|z|^2-2|z|-2 \le 0$$
$$|z| = \frac{+2 \pm \sqrt{4-4\cdot1\cdot(-2)}}{2\cdot1} = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm\sqrt{12}}{2}$$
$$= \frac{2\pm2\sqrt{3}}{2} = \frac{2(1\pm\sqrt{3})}{2} = 1\pm\sqrt{3}$$
$$= \sqrt{3}+1,\ \sqrt{3}-1.$$

∴ least & greatest values are $\sqrt{3}-1$ & $\sqrt{3}+1$.





7) If $z_1, z_2, z_3$ are three complex numbers such that $|z_1|=1$, $|z_2|=2$, $|z_3|=3$ and $|z_1+z_2+z_3|=1$ show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$

Soln

$|z_1| = 1 \qquad |z_2| = 2 \qquad |z_3| = 3, \; |z_3|^2 = 9$

$|z_1|^2 = 1 \qquad |z_2|^2 = 4 \qquad z_3\bar{z_3} = 3^2$

$z_1\bar{z_1} = 1 \qquad z_2\bar{z_2} = 2^2$

$z_1 = \frac{1}{(\bar{z_1})} \qquad z_2 = \frac{4}{\bar{z_2}} \qquad z_3 = \frac{9}{\bar{z_3}}$

$$z_1 + z_2 + z_3 = \frac{1}{\bar{z_1}} + \frac{4}{\bar{z_2}} + \frac{9}{\bar{z_3}}$$

$$|z_1 + z_2 + z_3| = \left| \frac{\bar{z_2}\bar{z_3} + 4\bar{z_1}\bar{z_3} + 9\bar{z_1}\bar{z_2}}{\bar{z_1}\,\bar{z_2}\,\bar{z_3}} \right|$$

$$= \frac{|9\bar{z_1}\bar{z_2} + 4\bar{z_1}\bar{z_3} + \bar{z_2}\bar{z_3}|}{|\bar{z_1}|\,|\bar{z_2}|\,|\bar{z_3}|}$$

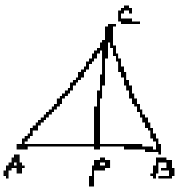
$$1 = \frac{|\overline{9z_1z_2 + 4z_1z_3 + z_2z_3}|}{1\cdot 2\cdot 3}$$

$$6 = |9z_1z_2 + 4z_1z_3 + z_2z_3|$$

$$\therefore |9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$$

Hence proved

8) If the area of the triangle formed by the vertices $z$, $iz$, and $z+iz$ is 50 square units. find the value of $|z|$



Soln Let A, B, C be $z$, $iz$, $z+iz$.

Area of a triangle $= \frac{1}{2} \times (AB) \times PC$.

$AB = |z - iz|$

P is midpoint of $AB = \frac{A+B}{2} = \left[\frac{z+iz}{2}\right]$

$PC = \left|(z+iz) - \left(\frac{z+iz}{2}\right)\right| = \left|\frac{z+iz}{2}\right|$



Area of triangle $= 50$

$\frac{1}{2}\, AB \times PC = 50$

$$\frac{1}{2}|z - iz|\left|\frac{z+iz}{2}\right| = 50$$

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$$\Rightarrow \frac{1}{4}\,|z^2+z^2| = 50$$

$$\frac{2|z|^2}{4} = 50$$

$$\frac{|z|^2}{2} = 50$$

$$|z|^2 = 100$$

$$\boxed{|z| = 10}$$

9) Show that the equation $z^3+2\bar{z}=0$ has five solutio

$$z^3 = -2\bar{z}$$

$$z(z^3) = -2z\bar{z}$$

$$z^4 = -2|z|^2$$

$$|z|^4 + 2|z|^2 = 0$$

$$|z|^2(|z|^2+2) = 0$$

$|z|^2 = 0$ $\qquad$ $|z|^2 = -2$

$|z| = 0$

$z = 0$

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$$z^3 = -2\bar{z} \Rightarrow z^3 = -2\frac{1}{z}.$$

$$z^4 = -2.$$

it has four solution including $z=0$

$z^3+2\bar{z}=0$ It has five solution.

10) Find the square roots of (i) $4+3i$

Soln $\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right]$

$$|z| = |4+3i| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$a=4 \quad b=3.$

$$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{3}{|3|}\sqrt{\frac{5-4}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \pm\left(\frac{3+i}{\sqrt{2}}\right)$$

(ii) $-6+8i$

$$|z| = \sqrt{-6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{8}{8}\sqrt{\frac{10+6}{2}}\right) = \pm\left(\sqrt{2} + i\sqrt{8}\right)$$

$$= \pm\left(\sqrt{2} + i\,2\sqrt{2}\right)$$

(iii) $-5-12i$

$$|z| = \sqrt{-5^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$

$$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} - i\frac{12}{12}\sqrt{\frac{13+5}{2}}\right)$$

$$= \pm\left(\sqrt{4} - i\sqrt{9}\right) = \pm\left(2-i3\right)$$

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## Exercise 2.6

1. If $z = x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$ show that the locus of $z$ is real axis

Soln

$$\left|\frac{z-4i}{z+4i}\right| = 1$$

$z = x+iy$

$$\left|\frac{x+iy-4i}{x+iy+4i}\right| = 1$$

$$\left|\frac{x+i(y-4)}{x+i(y+4)}\right| = 1$$

$$|x+i(y-4)| = |x+i(y+4)|$$

$$\sqrt{\cancel{x^2}+(y-4)^2} = \sqrt{\cancel{x^2}+(y+4)^2}$$

$$\cancel{y^2}+16-8y = \cancel{y^2}+\cancel{16}+8y$$

$$16y = 0$$

$y = 0$ which is real axis.

$\therefore$ The locus of $z$ is real axis

2) If $z = x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$ show that the locus of $z$ is $2x^2+2y^2+x-2y = 0$.

Soln

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{2x+i2y+1}{ix-y+1}$$

put $z = x+iy$

$$= \frac{(1+2x)+i\,2y}{(1-y)+ix} \times \frac{[(1-y)-ix]}{[(1-y)-ix]}$$

$$= \frac{(1+2x)(1-y) - ix(1+2x) + i\,2y(1-y) + 2xy}{(1-y)^2+x^2}$$

$$Im\left(\frac{2z+1}{iz+1}\right) = \frac{-x(1+2x)+2y(1-y)}{(1-y)^2+x^2}$$







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$-2x^2 - x + 2y - 2y^2 = 0$

$2x^2 + 2y^2 - 2y + x = 0$ is the locus of $z$.

3) Obtain the cartesian form of the locus of $z = x+iy$ in each of the following cases.

(i) $[Re(iz)]^2 = 3$

$iz = i(x+iy) = ix - y = -y + ix$.

$Re(iz) = -y$

$[Re(iz)]^2 = 3$

$(-y)^2 = 3$

$\boxed{y^2 - 3 = 0}$

(ii) $Im[(1-i)z+1] = 0$

$(1-i)z+1 = (1-i)(x+iy)+1$

$= x+iy-ix+y+1$

$= (x+y+1)+i(y-x)$

$Im((1-i)z+1) = 0$

$y-x = 0$

$\boxed{x-y = 0}$

(iii) $|z+i| = |z-1|$

$|x+iy+i| = |x+iy-1|$

$|x+i(y+1)| = |(x-1)+iy|$

$\sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$

$\cancel{x^2}+\cancel{y^2}+2y+\cancel{1} = \cancel{x^2}+\cancel{1}-2x+\cancel{y^2}$

$2x+2y = 0$

$\boxed{x+y = 0}$

(iv) ~~$\bar{z} = z^{-1}$~~

~~$z = x+iy$~~

~~$\bar{z} = x-iy$~~

$\bar{z}^1 = \cancel{\frac{1}{z}} = \frac{1}{x+iy}\frac{(x-iy)}{(x-iy)}$

$= \cancel{\frac{x-iy}{x^2+y^2}}$.

(iv) $\bar{z} = z^{-1}$

$\bar{z} = \frac{1}{z}$

$z\bar{z} = 1$

$(x+iy)(x-iy) = 1$

$x^2+y^2 = 1$

$x^2+y^2-1 = 0$ is the locus of $z$.

4. Show that the following equations represent a circle. and find its centre and radius.

$|z-\lambda| = a$
$a \to$ Radius
$\lambda \to$ Centre.

(i) $|z-2-i| = 3$

Soln $|z-(2+i)| = 3$

radius $= 3$

Centre $= 2+i$



(ii) $|2z+2-4i| = 2$

$\cancel{2}|z+1-2i| = \cancel{2}^1$

$2|z+(1-2i)| = 1$

radius $= 1$ x Centre $(1-2i)$

(iii) $|3z-6+12i| = 8.$

$3|z-2+4i| = 8$

$|z-(2-4i)| = 8/3.$

radius $= 8/3$ centre $= 2-4i$

5) Obtain the cartesian equation for the locus of $z=x+iy$ in each of the following cases

(i) $|z-4| = 16.$

$z = x+iy$

$|x+iy-4| = 16$

$|(x-4)+iy| = 16$

$\sqrt{(x-4)^2+y^2} = 16.$

$(x^2+8x+16)+y^2 = 256$

$x^2+y^2+8x-24 = 0$ is the locus of $z$.

(ii) $|z-4|^2 - |z-1|^2 = 16$

$(z-4)(\overline{z-4}) - (z-1)(\overline{z-1}) = 16$ $\qquad |z|^2 = z\bar{z}$

$z = x+iy$

$(x+iy-4)(\overline{x+iy-4}) - (x+iy-1)(\overline{x+iy-1}) = 16$

$(x+iy-4)((x-4)-iy] - ((x-1)+iy)((x-1)-iy)] = 16$

$[(x-4)^2+y^2] - [(x-1)^2+y^2] = 16$

$(x^2-8x+16+y^2) - (x^2-2x+1+y^2) = 16$

$\cancel{x^2} - 8x + \cancel{16} + \cancel{y^2} - \cancel{x^2} + 2x - 1 - \cancel{y^2} - \cancel{16} = 0$

$-6x - 1 = 0$

$6x+1 = 0$ is the locus of $z$.

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# Exercise 2.7

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(i) Write in polar form of the following complex numbers

(i) $2+i2\sqrt{3}$. $= r(\cos\theta + i\sin\theta)$

$$r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+(4\times3)} = \sqrt{4+12} = \sqrt{16} = 4$$

$r\cos\theta = 2$ $\qquad$ $r\sin\theta = 2\sqrt{3}$

$\cos\theta = \frac{2}{4} = \frac{1}{2}$. $\qquad$ $\sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$\alpha = \pi/3$ which is in I quadrant

$\theta = \alpha = \pi/3$. $\qquad$ $\left[\begin{array}{l}\sin\theta = +ve \\ \cos\theta = +ve\end{array}\right]$

$$2+i2\sqrt{3} = 4\left(\cos\pi/3 + i\sin\pi/3\right)$$

$$= 4\left[\cos(2k\pi+\pi/3) + i\sin(2k\pi+\pi/3)\right] \quad \forall k\in z.$$

(ii) $3-i\sqrt{3}$ $= r(\cos\theta + i\sin\theta)$

$$r = \sqrt{3^2+(-\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}.$$

$r\cos\theta = 3$ $\qquad$ $r\sin\theta = -\sqrt{3}$

$\cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin\theta = \frac{-\sqrt{3}}{2\sqrt{3}} = \frac{-1}{2}$

$\alpha = \pi/6$ which is IV quadrant

$\theta = -\alpha = -\pi/6$ $\qquad$ $\left[\begin{array}{l}\sin\theta = -ve \\ \cos\theta = +ve\end{array}\right]$

$$3-i\sqrt{3} = 2\sqrt{3}\left[\cos(-\pi/3) + i\sin(-\pi/3)\right]$$

$$= 2\sqrt{3}\left[\cos(2k\pi-\pi/3) + i\sin(2k\pi-\pi/3)\right] \quad \forall k\in z$$

(iii) $-2-i2$ $= r(\cos\theta + i\sin\theta)$

$$r = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$r\cos\theta = -2$ $\qquad$ $r\sin\theta = -2$

$\cos\theta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$ $\qquad$ $\sin\theta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$.

$\alpha = \pi/4$ which is III quadrant

$\theta = \alpha - \pi$ $\qquad$ $\left[\begin{array}{l}\sin\theta = -ve \\ \cos\theta = -ve\end{array}\right]$

$$= \pi/4 - \pi = \frac{-3\pi}{4}.$$

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$$-2-i2 = 2\sqrt{2}\left[\cos(-3\pi/4)+i\sin(-3\pi/4)\right]$$

$$= 2\sqrt{2}\left[\cos(2k\pi-3\pi/4)+i\sin(2k\pi-3\pi/4)\right] \quad \forall k\in\mathbb{Z}$$

(iv) $\dfrac{i-1}{\cos\pi/3+i\sin\pi/3}$

$-1+i = r(\cos\theta+i\sin\theta)$

$r=\sqrt{-1^2+1^2}=\sqrt{1+1}=\sqrt{2}$

$r\cos\theta=-1$ $\qquad r\sin\theta=1$

$\cos\theta=\frac{-1}{\sqrt{2}}$ $\qquad \sin\theta=1/\sqrt{2}$.

$\theta=\pi/4$ which is II quadrant.

$\sin\theta = +ve$

$\cos\theta = -ve$

$\theta = \pi-\alpha$

$= \pi - \pi/4$

$= 3\pi/4$

$-1+i = \sqrt{2}(\cos 3\pi/4 + i\sin 3\pi/4)$

$$\frac{i-1}{\cos\pi/3+i\sin\pi/3} = \frac{\sqrt{2}(\cos 3\pi/4+i\sin 3\pi/4)}{\cos(\pi/3)+i\sin\pi/3} = \frac{\sqrt{2}\,e^{i3\pi/4}}{e^{i\pi/3}}$$

$$= \sqrt{2}\,e^{i3\pi/4-i\pi/3} = \sqrt{2}\,e^{i\left(\frac{9\pi-4\pi}{12}\right)}$$

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$$= \sqrt{2}\,e^{i5\pi/12}$$

$$= \sqrt{2}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right).$$

$$= \sqrt{2}\left[\cos\left(2k\pi+\frac{5\pi}{12}\right)+i\sin\left(2k\pi+\frac{5\pi}{12}\right)\right] \quad \forall k\in\mathbb{Z}$$

2. Find the rectangular form of the complex numbers

(i) $\left(\cos\pi/6+i\sin\pi/6\right)\left(\cos\pi/12+i\sin\pi/12\right)$

$$= \cos\left(\pi/6+\pi/12\right)+i\sin\left(\pi/6+\pi/12\right)$$

$$= \cos\left(\frac{3\pi}{12}\right)+i\sin\left(\frac{3\pi}{12}\right) = \cos\pi/4+i\sin\pi/4$$

$$= \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$$



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2 (ii) $\dfrac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})} = \dfrac{1}{2}\dfrac{e^{-i\pi/6}}{e^{i\pi/3}} = \dfrac{1}{2} e^{-i\pi/6 - i\pi/3}$

$= \dfrac{1}{2} e^{-i(\pi/6 + \pi/3)} = \dfrac{1}{2} e^{-i\pi/2}$

$= \dfrac{1}{2}(\cos\pi/2 - i\sin\pi/2)$

$= \dfrac{1}{2}(0 - i) = -i/2.$

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3) If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)+\cdots+(x_n+iy_n) = a+ib$ show that (i) $(x_1^2+y_1^2)(x_2^2+y_2^2)+\cdots+(x_n^2+y_n^2) = a^2+b^2$

(ii) $\sum_{r=1}^{n} \tan^{-1}\left(\dfrac{y_r}{x_r}\right) = \tan^{-1}\left(\dfrac{b}{a}\right) + 2k\pi \quad \forall k \in \mathbb{Z}.$

Soln. $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n) = a+ib$ — (1)

Take modulus on both sides

$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)| = |a+ib|$

$|x_1+iy_1|\,|x_2+iy_2|\,|x_3+iy_3|\cdots|x_n+iy_n| = |a+ib|$

$\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}\cdot\sqrt{x_3^2+y_3^2}\cdots\sqrt{x_n^2+y_n^2} = \sqrt{a^2+b^2}$

Take square on both sides.

$(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2) = (a^2+b^2)$

In (1) Take argument on both sides.

$\arg[(x_1+iy_1)(x_2+iy_2)\cdots(x_n+iy_n)] = \arg(a+ib)$

$[\because \arg(z_1\cdot z_2) = \arg z_1 + \arg z_2]$

$\Rightarrow \arg(x_1+iy_1) + \arg(x_2+iy_2) + \cdots + \arg(x_n+iy_n) = \arg(a+ib)$

$[\because \arg(x+iy) = \tan^{-1}\frac{y}{x}]$

$\tan^{-1}\dfrac{y_1}{x_1} + \tan^{-1}\dfrac{y_2}{x_2} + \cdots + \tan^{-1}\dfrac{y_n}{x_n} = \tan^{-1}\dfrac{b}{a}$

In general

$\sum_{r=1}^{n} \tan^{-1}\dfrac{y_r}{x_r} = 2k\pi + \tan^{-1}\left(\dfrac{b}{a}\right) \quad \forall k \in \mathbb{Z}$

Hence proved.

4) If $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$ show that $z = i\tan\theta$.

$$\frac{1+z}{1-z} = (\cos\theta + i\sin\theta)^2$$

$$(1+z)(1+z) = (\cos\theta + i\sin\theta)^2$$

$$(1+z)^2 = (\cos\theta + i\sin\theta)^2$$

$$1+z = \cos\theta + i\sin\theta$$

$$1+z = \frac{\cos\theta}{\cos\theta} + \frac{i\sin\theta}{\cos\theta}$$

$$1+z = 1 + i\tan\theta$$

$$\therefore z = i\tan\theta$$

Hence proved.

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5) If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$ Show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Soln Let us take

$$a = \cos\alpha + i\sin\alpha$$

$$b = \cos\beta + i\sin\beta$$

$$c = \cos\gamma + i\sin\gamma.$$

$$a+b+c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 + i0$$

$$a+b+c = 0$$

if $a+b+c = 0$ We know that

$$a^3 + b^3 + c^3 = 3abc.$$

$$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$

$$= 3\left[(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\right]$$





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$[(\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)]$ 25

$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3\cos(\alpha+\beta+\gamma) + i\,3\sin(\alpha+\beta+\gamma)$

Square real & imaginary parts on both sides

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Hence proved.

6) If $z = x+iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \pi/4$, then show that $x^2+y^2+3x-3y+2=0$.

Soln $z = x+iy$.

$\arg\left(\frac{z-i}{z+2}\right) = \pi/4$ $\quad\left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\right]$

$\arg(z-i) - \arg(z+2) = \pi/4$ $\quad\left[\because \arg(x+iy) = \tan^{-1}\frac{y}{x}\right)$

$\arg(x+iy-i) - \arg(x+iy+2) = \pi/4$

$\arg[x+i(y-1)] - \arg((x+2)+iy) = \pi/4$

$\tan^{-1}\left(\frac{y-1}{x}\right) - \tan^{-1}\frac{y}{x+2} = \pi/4$ $\quad\left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$

$\tan^{-1}\left[\frac{\frac{y-1}{x} - \frac{y}{x+2}}{1+\left(\frac{y-1}{x}\right)\left(\frac{y}{x+2}\right)}\right] = \pi/4.$

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$\frac{\frac{(y-1)(x+2) - xy}{x(x+2)}}{\frac{x(x+2) + y(y-1)}{x(x+2)}} = \tan\pi/4 = 1$

$xy + 2y - x - 2 - xy = x^2 + 2x + y^2 - y$

$x^2+y^2+2x-y-2y+x+2=0$

$x^2+y^2+3x-3y+2=0$ is the locus of $z$.

## Exercise 2.8

1. If $\omega \neq 1$ is a cube root of unity, then show that

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=-1.$$

Soln

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=$$

$$=\frac{a+b\omega+c\omega^2}{\frac{1}{\omega}(b\omega+c\omega^2+a)}=\omega$$

$$\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=\frac{\frac{1}{\omega}(a\omega+b\omega^2+c)}{(c+a\omega+b\omega^2)}=\frac{1}{\omega}\times\frac{\omega^2}{\omega^2}=\frac{\omega^2}{\omega^3}=\omega^2$$

$$LHS=\omega+\omega^2$$
$$=-1$$

$[\because 1+\omega+\omega^2=0,\ \omega+\omega^2=-1]$

$[\because \omega^3=1]$.

2. Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

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Soln

$$\frac{\sqrt{3}}{2}+\frac{i}{2}=r(\cos\theta+i\sin\theta)$$

$$r=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{\frac{4}{4}}=\sqrt{1}=1$$

$$r\cos\theta=\frac{\sqrt{3}}{2} \qquad r\sin\theta=\frac{1}{2}$$

$$\cos\theta=\frac{\sqrt{3}}{2} \qquad \sin\theta=\frac{1}{2}$$

$\alpha=\pi/6$ [which is I quadrant. $\sin\theta=+ve,\ \cos\theta=+ve$]

$\theta=\alpha=\pi/6$.

$$\frac{\sqrt{3}}{2}+\frac{i}{2}=1\left(\cos\pi/6+i\sin\pi/6\right)$$

$$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=\cos 5\pi/6+i\sin 5\pi/6)$$

Similarly $\left(\frac{\sqrt{3}}{2}-i/2\right)^5=\cos 5\pi/6-i\sin 5\pi/6$.

$$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-i/2\right)^5=2\cos 5\pi/6=2\cos(\pi-\pi/6)$$
$$=-2\cos\pi/6=-2\left(\frac{\sqrt{3}}{2}\right)$$
$$=-\sqrt{3}$$ // Proved.

3\. Find the value of $\left(\dfrac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

Soln

$z = \cos\pi/_{10} + i\sin\pi/_{10} = i(\sin\pi/_{10} - i\cos\pi/_{10})$

$\frac{1}{z} = \cos\pi/_{10} - i\sin\pi/_{10}$

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\pi/_{10}-i\cos\frac{\pi}{10}}\right)^{10} = \left(\frac{1+i(\cos\pi/_{10}-i\sin\pi/_{10})}{1-i[\cos\pi/_{10}+i\sin\pi/_{10})}\right)^{10}$$

$$= \left(\frac{1+i\frac{1}{z}}{1-iz}\right)^{10} = \left(\frac{1-\frac{1}{zi}}{1-iz}\right)^{10} = \left(\frac{\frac{zi-1}{zi}}{1-iz}\right)^{10}$$

$i \times \frac{i}{i} = \frac{-1}{i}$

$$= \left(\frac{-1}{zi}\frac{(1-z\cancel{i})}{(1-\cancel{iz})}\right]^{10} = \left(\frac{-1}{iz}\right)^{10} = \frac{1}{i^{10}}\left(\frac{1}{z}\right)^{10}$$

$i^{10} = i^2 = -1$

$$= \frac{1}{-1}\left(\cos\pi/_{10} - i\sin\pi/_{10}\right)^{10}$$

$$= -1(\cos\pi + i\sin(-\pi)]$$

$$= -1\ (-1+i(0)) = (-1)(-1) = 1.$$

4\) If $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$ show that

(i) $\frac{x}{y}+\frac{y}{x} = 2\cos(\alpha-\beta)$

Soln $2\cos\alpha = x+\frac{1}{x}$

$[\because \sin^2\alpha + \cos^2\alpha = 1$

$\cos^2\alpha - 1 = -\sin^2\alpha]$

$2\cos\alpha = \frac{x^2+1}{x}$

$x^2+1 = 2x\cos\alpha$

$x^2 - 2x\cos\alpha + 1 = 0$

$$x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2\cdot 1} = \frac{2\cos\alpha \pm 2\sqrt{\cos^2\alpha - 1}}{2}$$

$$= \frac{2(\cos\alpha \pm \sqrt{-\sin^2\alpha})}{2} = \cos\alpha \pm i\sin\alpha$$

$x = \cos\alpha \pm i\sin\alpha.$

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Similarly $y = \cos\beta \pm i\sin\beta$

$x = \cos\alpha + i\sin\alpha \quad y = \cos\beta + i\sin\beta.$

$$\frac{x}{y} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\alpha} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$\frac{y}{x} = \frac{1}{\frac{x}{y}} = \cos(\alpha-\beta) - i\sin(\alpha-\beta)$$

$$\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha-\beta)$$

(ii) $xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$$\frac{1}{xy} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$xy + \frac{1}{xy} = 2i\sin(\alpha+\beta)$$

(iii) $x^m = (\cos\alpha + i\sin\alpha)^m = \cos m\alpha + i\sin m\alpha$

$y^n = (\cos\beta + i\sin\beta)^n = \cos n\beta + i\sin n\beta$

$x^m y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$

$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

$$\frac{1}{x^m y^n} = \frac{1}{\cos(m\alpha+n\beta) + i\sin(m\alpha+n\beta)}$$

$= \cos(m\alpha+n\beta) - i\sin(m\alpha+n\beta)$

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha+n\beta)$$

(iv) $$\frac{x^m}{y^n} = \frac{\cos(m\alpha) + i\sin m\alpha}{\cos n\beta + i\sin n\beta}$$

$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$$\frac{y^n}{x^m} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta).$$

(29)

5) Solve the equation $z^3+27=0$

Soln

$z^3+27=0$ $\quad [\because -1=\cos\pi+i\sin\pi]$

$z^3=-27$

$z^3=27(-1)$

$z=(27)^{1/3}(-1)^{1/3} = (3^3)^{1/3}[\cos\pi+i\sin\pi]^{1/3}$

$= 3[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/3}$

$= 3\left[\cos\frac{(2k+1)\pi}{3}+i\sin\frac{(2k+1)\pi}{3}\right]$ $\quad k=0,1,2$

$k=0$, $z = 3(\cos\pi/3+i\sin\pi/3) = 3\ \text{cis}\ \pi/3$

$k=1$, $z = 3(\cos 3\pi/3+i\sin 3\pi/3) = 3(-1+i(0)) = -3$

$k=2$, $z = 3(\cos 5\pi/3+i\sin 5\pi/3) = 3\ \text{cis}\ 5\pi/3$.

6) If $\omega\neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3+8=0$ are $-1,\ 1-2\omega,\ 1-2\omega^2$.

Soln $(z-1)^3+8=0$

$(z-1)^3 = -8 = 8(-1)$

$z-1 = 8^{1/3}(-1)^{1/3}$

$z-1 = (2^3)^{1/3}[\cos\pi+i\sin\pi]^{1/3}$

$= 2[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/3}$

$= 2\left[\cos\frac{(2k+1)\pi}{3}+i\sin\frac{(2k+1)\pi}{3}\right]$ $\quad k=0,1,2$.

$k=0$, $\quad z-1 = 2\ \text{cis}\ \pi/3 = 2\left[\frac{1}{2}+\frac{i\sqrt{3}}{2}\right] = -2\left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}\right) = -2\omega^2$

$k=1$, $\quad z-1 = 2\ \text{cis}\ 3\pi/3 = 2[-1+0] = -2$

$k=2$ $\quad z-1 = 2\ \text{cis}\ 5\pi/3 = 2[\cos(\pi-\pi/3)+i\sin(\pi-\pi/3)]$

$= 2[-\cos\pi/3+i\sin\pi/3] = 2\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$

$= -2\omega$

| | | |
|---|---|---|
| $z-1=-2\omega^2$ | $z-1=-2$ | $z-1=-2\omega$ |
| $z=1-2\omega^2$ | $z=-2+1$ | $z=1-2\omega$ |
| | $=-1$ | |

Ans: $z=-1,\ 1-2\omega,\ 1-2\omega^2$.

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7) Find the value of $\sum_{k=1}^{8} \cos\left(\frac{2k\pi}{9}\right) + i\sin\left(\frac{2k\pi}{9}\right)$

Soln

$\sum_{k=1}^{8}\left[\cos\left(\frac{2k\pi}{9}\right) + i\sin\left(\frac{2k\pi}{9}\right)\right]$

$= cis\,\frac{2\pi}{9} + cis\,\frac{4\pi}{9} + cis\,\frac{6\pi}{9} + cis\,\frac{8\pi}{9} \ldots + cis\,\frac{16\pi}{9}$

$= cis\left(\frac{2\pi}{9} + \frac{4\pi}{9} + \frac{6\pi}{9} + \cdots + \frac{16\pi}{9}\right)$

$= cis\left[\frac{2}{9}\pi\,(1+2+3+\cdots+8)\right] = cis\,\frac{2\pi}{9}\left(\frac{8\times 9}{2}\right)$

$= cis\,8\pi = \cos 8\pi + i\sin 8\pi$

$= (-1)^8 + i(0) = 1.$



8) If $\omega \neq 1$ is the cube root of unity show that.

(i) $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6 = 128$

Soln $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6$

$\because 1+\omega+\omega^2 = 0$
$1+\omega = -\omega^2$
$1+\omega^2 = -\omega$
$\omega^3 = 1$

$= (-\omega-\omega)^6 + (-\omega^2-\omega^2)^6$

$= (-2\omega)^6 + (-2\omega^2)^6$

$= 2^6\omega^6 + 2^6\omega^{12} = 2^6\left[\omega^6 + \omega^{12}\right)$

$= 2^6(1+1) = 2^6 \cdot 2^1 = 2^7 = 128$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots(1+\omega^{2^{11}}) = 1$

Soln $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots(1+\omega^{2^{10}})(1+\omega^{2^{11}})$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots(1+\omega)(1+\omega^2)$

$= (-\omega^2)(-\omega)(-\omega^2)(-\omega)\ldots(-\omega^2)(-\omega)$

$\left[\because 1+\omega = -\omega^2\right.$
$1+\omega^2 = -\omega$
$\left.\omega^3 = 1\right]$

$= \omega^3 \cdot \omega^3 \ldots \omega^3 = 1\cdot 1\cdot 1\cdots 1 = 1 = R.H.S.$

9) If $z = 2-2i$ find the rotation of $z$ by $\theta$ radians in the counter clockwise direction about the origin when (i) $\theta = \pi/3$ (ii) $\theta = 2\pi/3$ (iii) $\theta = 3\pi/2$.





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Soln $z = 2-2i$

The rotation of $z$ by $\theta$ in clockwise direction about the origin is $ze^{i\theta}$.

$z = 2-2i = 2(1-i) = r\cos\alpha + i\sin\alpha$.

$r = \sqrt{2}$ $\quad r\cos\alpha = 1 \quad r\sin\alpha = -1$

$\cos\alpha = \frac{1}{\sqrt{2}} \quad \sin\alpha = \frac{-1}{\sqrt{2}}$.



$\alpha = \pi/4$. $\quad \therefore \theta = -\alpha = -\pi/4$

$z = 2\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right) = e^{-i\pi/4}$.

i) $\theta = \pi/3$.

$ze^{i\theta} = 2\sqrt{2}e^{-i\pi/4} \cdot e^{i\pi/3} = 2\sqrt{2}e^{i(-\pi/4+\pi/3)} = 2\sqrt{2}e^{i\left(\frac{-3\pi+4\pi}{12}\right)}$

$= 2\sqrt{2}e^{i\pi/12}$

(ii) $\theta = 2\pi/3$

$ze^{i\theta} = 2\sqrt{2}e^{-i\pi/4} \cdot e^{i2\pi/3} = 2\sqrt{2}e^{i(-\pi/4+\frac{2\pi}{3})} = 2\sqrt{2}e^{i\left(\frac{-3\pi+8\pi}{12}\right)}$

$= 2\sqrt{2}e^{i5\pi/12}$.

(iii) $\theta = 3\pi/2$

$z \cdot e^{i\theta} = 2\sqrt{2}e^{-i\pi/4}e^{i3\pi/2} = 2\sqrt{2}e^{i(-\pi/4+3\pi/2)} = 2\sqrt{2}e^{i\left(\frac{-\pi+6\pi}{12}\right)}$

$= 2\sqrt{2}e^{i5\pi/12}$

10) Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$

Soln $-1 = \cos\pi + i\sin\pi$

$(-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/4}$

$= \cos\left(\frac{2k+1}{4}\right)\pi + i\sin\frac{(2k+1)\pi}{4} \qquad k = 0,1,2,3.$

$k=0$ $\quad \sqrt[4]{-1} = \operatorname{cis}\pi/4 = \cos\pi/4 + i\sin\pi/4 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$k=1$ $\quad \sqrt[4]{-1} = \operatorname{cis}3\pi/4 = \cos 3\pi/4 + i\sin 3\pi/4 = \cos(\pi-\frac{\pi}{4}) + i\sin(\pi-\pi/4)$

$= -\cos\pi/4 + i\sin\pi/4 = \frac{-1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$

$k=2$, $\quad \sqrt[4]{-1} = \cos 5\pi/4 + i\sin 5\pi/4 = \cos(\pi+\pi/4) + i\sin(\pi+\pi/4)$

$= -\cos\pi/4 - i\sin\pi/4 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$k=3$ $\quad \sqrt[4]{-1} = \cos 7\pi/4 + i\sin 7\pi/4 = \cos(2\pi-\pi/4) + i\sin(2\pi-\pi/4)$

$= \cos\pi/4 - i\sin\pi/4 = 1/\sqrt{2} - i\,1/\sqrt{2}$

$\therefore$ Roots are $\pm 1/\sqrt{2} \pm i\frac{1}{\sqrt{2}} = \pm 1/\sqrt{2}\,(1\pm i)$ // Ans.

## Exercise 2.9

1. Choose the correct or the most suitable answer from the given four alternatives.

1) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

$i^n(1+i+i^2+i^3) = i^n(1+i-1-i) = i^n(0) = 0$

Ans ① 0

2) The value of $\sum_{i=1}^{13}(i^n + i^{n-1})$ is

Soln $(i^1+i^0)+(i^2+i^1)+(i^3+i^2)+\cdots+(i^{13}+i^{12})$

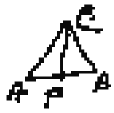
$= i^0 + 2(i^1+i^2+i^3+i^4+i^5+i^6+i^7+\cdots+i^{12}) + i^{13}$

$= 1 + 2(\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1}+\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1}+\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1}) + i$

$= 1 + 2(0) + i = 1+i$

Ans ① $1+i$

3) The area of the triangle formed by the complex numbers $z$, $iz$, and $z+iz$ in the Argand's diagram is



Soln $|A-B| = |z-iz|$

$P$ is mid point of $AB = \frac{A+B}{2} = \frac{z+iz}{2}$

$PC = \left|z+iz-\left(\frac{z+iz}{2}\right)\right| = \left|\frac{z+iz}{2}\right|$

Area of the triangle $= \frac{1}{2} AB \times PC = \frac{1}{2}|z-iz|\left|\frac{z+iz}{2}\right|$

$= \frac{1}{4}|z^2+z^2| = \frac{1}{4}\, 2|z|^2 = \frac{|z|^2}{2}$

(DI)

Ans ① $\frac{|z|^2}{2}$

4) The conjugate of the complex number is $\frac{1}{i-2}$ Then the complex number is

Soln $\overline{\frac{1}{i-2}} = \frac{1}{-i-2} = \frac{-1}{2+i}$

Ans ② $\frac{-1}{i+2}$.

5) If $z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$ then $|z|$ is equal to.

Soln

$|z| = \frac{|\sqrt{3}+i|^3\,|3i+4|^2}{|8+6i|^2} = \frac{\left(\sqrt{\sqrt{3}^2+1^2}\right)^3\left(\sqrt{3^2+4^2}\right)^2}{\left(\sqrt{8^2+6^2}\right)^2} = \frac{2^3(5)^2}{10^2}$

$= \frac{8 \times 25}{100} = 2$

Ans (3) 2

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6) If $z$ is a non zero complex number. Such that $2iz^2=\bar{z}$ then $|z|$ is

Soln $2i\,z^2=\bar{z}$

$2i\,z\cancel{\bar{z}}=\cancel{\bar{z}}$

$2iz=1$

$z=\frac{1}{2i}\times\frac{-i}{-i}=\frac{-i}{2}$

$|z|=\sqrt{\frac{1}{4}}=\frac{1}{2}$ Ans (1) 1/2

7) If $|z-2+i|\le 2$ Then the greatest value of $|z|$ is

Soln $|z-2+i|=|z-(2-i)|\ge|z|-|2-i|$

$\ge |z|-\sqrt{2^2+1^2}$

$2\ge|z|-\sqrt{5}$

$2+\sqrt{5}\ge|z|$

$|z|\le 2+\sqrt{5}$ Ans (4) $\sqrt{5}+2$

8) If $|z-\frac{3}{z}|=2$ then the least value of $|z|$ is

Soln $|z|=|z-\frac{3}{z}+\frac{3}{z}|\le|z-\frac{3}{z}|+|\frac{3}{z}|$

$|z|\le 2+\frac{3}{|z|}$

$|z|-2-\frac{3}{|z|}\le 0$

$|z|^2-2|z|-3\le 0$

$|z|$ is $+3$ or $1$ Ans (5) 1

least value is 1

$-3$ ; $-3$, $1$

9) If $|z|=1$ then the value of $\frac{1+z}{1+\bar{z}}$ is

Soln $|z|=1$, $x^2+y^2=1$

$\frac{1+z}{1+\bar{z}}=\frac{(1+x)+iy}{1+x-iy}\times\frac{(1+x)+iy}{(1+x)+iy}$

$=\frac{(1+x)^2-y^2+2iy(1+x)}{(1+x)^2+y^2}=\frac{1+x^2+2x-y^2+2iy(1+x)}{1+x^2+2x+y^2}$

$\frac{1+z}{1+\bar{z}}=\frac{1+z}{1+\frac{1}{z}}=\frac{1+z}{\frac{z+1}{z}}=\frac{z\cancel{(1+z)}}{\cancel{(1+z)}}=z$

Ans $(z)$

10) The solution of the equation $|z| - z = 1+2i$ is

Soln $|z| - 1 - 2i = z$

$|z|^2 = z\bar{z} = (|z|-1-2i)(|z|-1+2i)$

$= |z|^2 + |z|(-1+2i-1-2i) + (-1-2i)(-1+2i)$

$\cancel{|z|^2} = \cancel{|z|^2} - 2|z| + 5 \Rightarrow 2|z| = 5$

$|z| = 5/2$

put in ①

$z = 5/2 - 1 - 2i = \frac{3}{2} - 2i$

Ans $\frac{3}{2} - 2i$

11) The solution if $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$ then the value of $|z_1+z_2+z_3|$ is

Soln $|z_1| = 1$, $|z_2| = 2$ $|z_3| = 3$

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$|z_1|^2 = 1$ $|z_2|^2 = 4$ $|z_3|^2 = 9$

$z_1\bar{z_1} = 1$, $z_2\bar{z_2} = 4$, $z_3\bar{z_3} = 9$

$z_1 = \frac{1}{\bar{z_1}}$, $z_2 = \frac{4}{\bar{z_2}}$, $z_3 = \frac{9}{\bar{z_3}}$

$z_1+z_2+z_3 = \frac{1}{\bar{z_1}} + \frac{4}{\bar{z_2}} + \frac{9}{\bar{z_3}} = \frac{\bar{z_2}\bar{z_3} + 4\bar{z_1}\bar{z_3} + 9\bar{z_1}\bar{z_2}}{\bar{z_1}\bar{z_2}\bar{z_3}}$

$|z_1+z_2+z_3| = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{z_1z_2z_3} = \frac{6}{1\cdot2\cdot3} = 1$

$|z_1+z_2+z_3| = 1$

Ans. 1

12) If $z$ is a complex number such that $z \in \mathbb{C}/\mathbb{R}$ and $z + 1/z \in \mathbb{R}$ then $|z|$ is

$z = x+iy$

$1/z = x - iy.$

$z + \frac{1}{z} =$

13) $z_1, z_2$ and $z_3$ are complex numbers such that $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|=1$ then $z_1^2+z_2^2+z_3^2$ is

Soln $|z_1|=1 \therefore |z_2|=1 \therefore |z_3|=1$

$\bar{z_1}=\frac{1}{z_1}$ $\quad \bar{z_2}=\frac{1}{z_2}$ $\quad \bar{z_3}=\frac{1}{z_3}$

$(z_1+z_2+z_3) = \frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3} \therefore \frac{|\bar{z_2}\bar{z_3}+\bar{z_1}\bar{z_2}+\bar{z_1}\bar{z_3}|}{|z_1 z_2 z_3|}$

$0 = \frac{|z_1 z_2 + z_3 z_2 + z_1 z_3|}{1} \Rightarrow |z_1z_2+z_2z_3+z_3z_1| = 0$

$\therefore (z_1+z_2+z_3)^2 = z_1^2+z_2^2+z_3^2+2(z_1z_2+z_3z_2+z_3z_1)$

$0 = z_1^2+z_2^2+z_3^2+2(0)$ Ans (0)

$\therefore z_1^2+z_2^2+z_3^2=0$

14) If $\frac{z-1}{z+1}$ is purely imaginary then $|z|$ is

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}\cdot\frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{(x^2-1)-iy(x-1)+iy(x+1)+y^2}{(x+1)^2+y^2}$

it is purely imaginary $Re\left(\frac{z-1}{z+1}\right)=0$

$\frac{x^2-1+y^2}{(x+1)^2+y^2}=0$

$x^2+y^2-1=0$

$x^2+y^2=1$

$|z|^2=1$

$|z|=1$ Ans 1



15) if $z=x+iy$ is a complex number such that $|z+2|=|z-2|$ then the locus of $z$ is

Soln $|z+2|=|z-2|$

$|x+iy+2| = |x+iy-2|$

$\sqrt{(x+2)^2+y^2} = \sqrt{(x-2)^2+y^2}$

$x^2+4x+y^2 = x^2-4x+y^2$

$8x = 0$

$x=0$ which is imaginary axis

Ans (2)

16) The principal argument of $\frac{3}{-1+i}$ is

Soln $\frac{3}{-1+i} \frac{(-1-i)}{(-1-i)} = \frac{3(-1-i)}{-1^2+1^2} = \frac{3}{2}(-1-i)$

$-1-i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{2}$ $\cos\theta = -\frac{1}{\sqrt{2}}$ $\sin\theta = -\frac{1}{\sqrt{2}}$

$\alpha = \pi/4$ which is $\boxed{III}$ quadrant

$\theta = \alpha - \pi = \pi/4 - \pi = -3\pi/4$

$-1-i = \sqrt{2}\left[\cos(-3\pi/4) + i\sin(-3\pi/4)\right]$

$= \frac{3}{2}\sqrt{2}\left(\cos(-3\pi/4) + i\sin(-3\pi/4)\right)$

argument is $-3\pi/4$ Ans (3) $-3\pi/4$.

17) The principal argument of $(\sin 40^\circ + i\cos 40^\circ)^5$ is

$(\sin 40^\circ + i\cos 40^\circ)^5 = (\cos 50^\circ + i\sin 50^\circ)^5$

$= (\cos 250^\circ + i\sin 250^\circ)$

$= \cos(360^\circ - 110^\circ) + i\sin(360^\circ - 110^\circ)$

Principal argument is $(-110^\circ)$

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Ans $-110^\circ$

18) If $(1+i)(1+2i)(1+3i)\cdots(1+ni) = x+iy$ then $2 \cdot 5 \cdot 10 \cdots (1+n^2)$ is

Soln Take modulus on both sides

$\sqrt{1^2+1^2}\,\sqrt{1+2^2}\,\sqrt{1+9}\cdots\sqrt{1+n^2} = \sqrt{x^2+y^2}$

$(2)(5)(10)\cdots(1+n^2) = x^2+y^2$.

Ans ③ $x^2+y^2$.

19) If $\omega \neq 1$ is a cube root of unity and $(1+\omega)^7 = A+B\omega$ Then $(A, B)$ equals

Soln $(1+\omega)^2$

$= 1+\omega^2+2\omega = -\omega+2\omega = \omega$

$[(1+\omega)^2]^3 = (1+\omega)^6 = \omega^6 = 1$

$(1+\omega)^6(1+\omega) = 1+\omega = A+B\omega$

$\therefore A=1 \quad B=1.$

Ans (4) (1,1)

$(A,B) = (1,1)$

20) The principal argument of the complex number $\dfrac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is

$\arg\left(\dfrac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}\right) = \arg(1+i\sqrt{3})^2 - \arg 4i - \arg(1-i\sqrt{3})$

$- \arg(0+4i) - \arg(1-i\sqrt{3})$

$= 2\arg(1+i\sqrt{3}) - \tan^{-1}\left(\frac{4}{0}\right) - \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right)$

$= 2\tan^{-1}\sqrt{3}$

$= 2\pi/3 - \pi/2 + \pi/3 = \pi - \pi/2$

$= \pi/2$

Ans $\pi/2$.

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21) If $\alpha$ & $\beta$ are the roots of $x^2+x+1=0$ then $\alpha^{2020}+\beta^{2020}$ is

Soln $x^2+x+1=0.$

$x = \dfrac{-1\pm\sqrt{1^2-4.1.1}}{2.1} = \dfrac{-1\pm\sqrt{-3}}{2} = \dfrac{-1\pm i\sqrt{3}}{2}$

$\alpha = \dfrac{-1+i\sqrt{3}}{2}, \beta = \dfrac{-1-i\sqrt{3}}{2}$

$\alpha = \omega \qquad \beta = \omega^2.$

$\omega^3 = 1$

$\alpha^{2020}+\beta^{2020} = \omega^{2020}+(\omega^2)^{2020}$

$= \omega^{2020}+\omega^{4040}$

$= \omega^1+\omega^2 = \omega+\omega^2$ Ans (2) (-1)

$= -1$

22) The product of all four values of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}$ is



Soln $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4} = \left(\cos\pi + i\sin\pi\right)^{1/4}$

$= \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4} \quad k = 0, 1, 2, 3$

$k=0$, $cis\,\pi/4$, $k=2 \Rightarrow cis\,5\pi/4$

$k=1$ $cis\,3\pi/4$ $k=3 \Rightarrow cis\,7\pi/4$.

Product of four values $cis\left(\pi/4 + 3\pi/4 + 5\pi/4 + 7\pi/4\right)$

$cis\left(16\pi/4\right) = cis\,4\pi = (-1)^4 = 1$ Ans (3) 1

23) if $\omega \neq$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$ then $k$ is equal to.

Soln $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$

$\left[\because 1+\omega+\omega^2=0,\ \omega = -1-\omega^2\right]$

$\left[\because \omega^3 = 1,\ \omega^6 = 1\right]$

$1(\omega^2-\omega) - 1(\omega-\omega^2) + 1(\omega^2-\omega) = 3k$

$\omega^2 - \omega - \omega + \omega^2 + \omega^2 - \omega = 3k$

$3\omega^2 - 3\omega = 3k$

$\omega^2 - \omega = k$

$k = \left(\frac{-1-i\sqrt{3}}{2}\right) - \left(\frac{-1+i\sqrt{3}}{2}\right) = \frac{\cancel{-1} - i\sqrt{3} \cancel{+1} - i\sqrt{3}}{2}$

$= -\frac{2i\sqrt{3}}{2} = -i\sqrt{3}$ Ans (4) $-\sqrt{3}i$

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24) The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

Soln

$1+\sqrt{3}i = r(\cos\theta + i\sin\theta)$

$r = 2$ $\cos\theta = 1/2$ $\sin\theta = \sqrt{3}/2$ $\therefore \theta = \pi/3$

$1+\sqrt{3}i = 2\left(\cos\pi/3 + i\sin\pi/3\right) = 2e^{i\pi/3}$

$1-\sqrt{3}i = 2e^{-i\pi/3}$

$\frac{1+\sqrt{3}i}{1-\sqrt{3}i} = \frac{\cancel{2}e^{i\pi/3}}{\cancel{2}e^{-i\pi/3}} = e^{i2\pi/3}$

$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(e^{i2\pi/3}\right)^{10}$ Ans $cis\,2\pi/3$

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25) If $\omega = cis \frac{2\pi}{3}$ then the number of distinct roots of

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0.$$

Soln

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$[\because \omega^3 = 1, \; 1+\omega+\omega^2 = 0]$.

$$(z+1)(z^2 + z\omega + \omega^2 z + \cancel{1} - \cancel{1}) - \omega(z\omega + \cancel{\omega^2} - \cancel{\omega^2}) + \omega^2(\omega - z\omega^2 - \omega) = 0$$

$$z^3 + z^2\omega + z^2\omega^2 + z^2 + \cancel{z\omega} + \cancel{\omega^2 z} - \cancel{z\omega^2} - \cancel{z\omega^4} = 0$$

$$z^2(z + \omega + \omega^2 + 1) = 0$$

$$z^2(z + 0) = 0$$

$$z^3 = 0$$

$$z = 0, 0, 0.$$

The number of distinct roots is 1