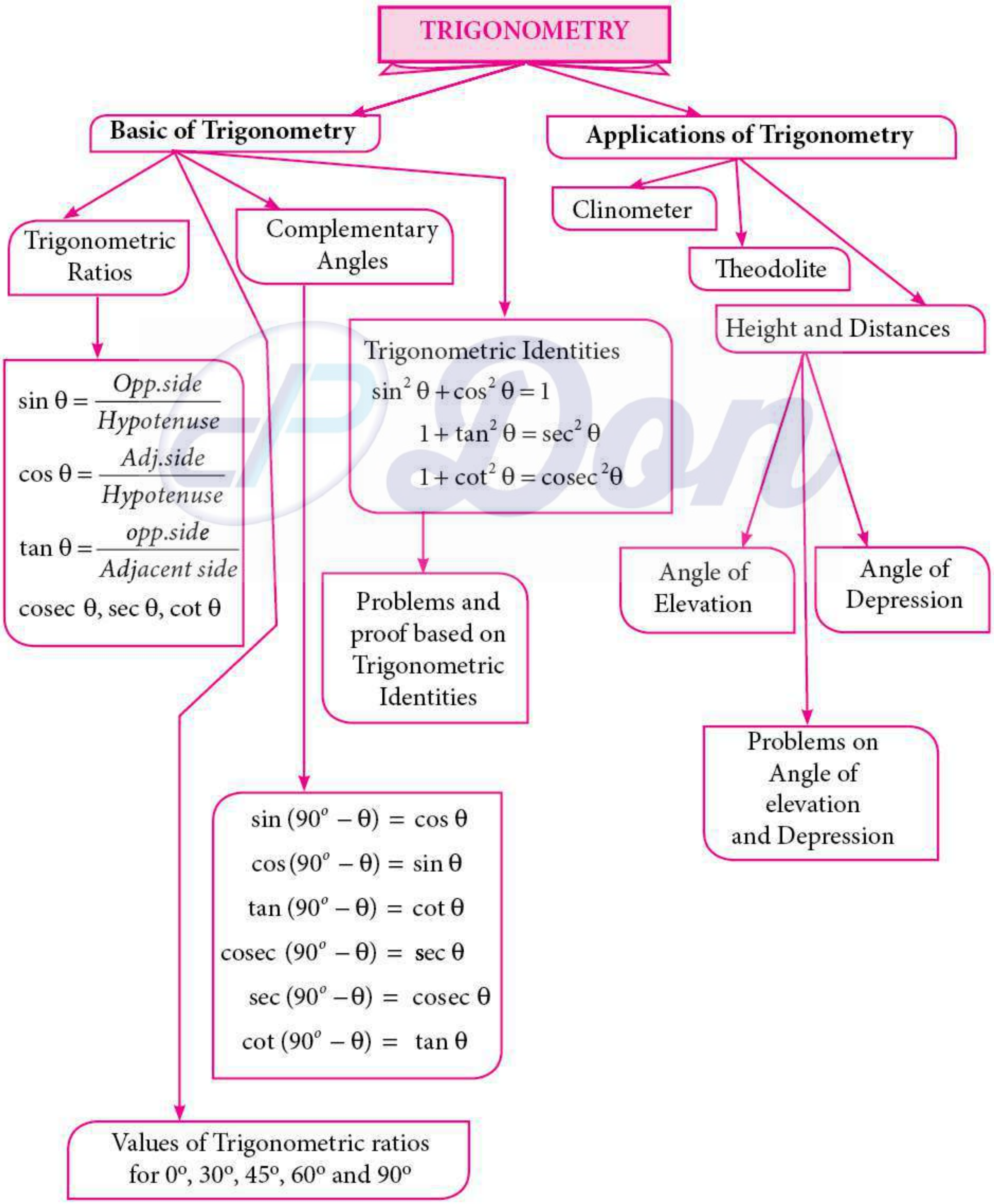
UNIT 6 TRIGONOMETRY

MIND MAP



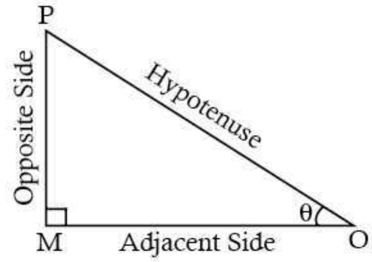
TRIGONOMETRIC RATIOS

Key Points

(i) Trigonometric Ratios

Let $0^o < \theta < 90^o$

1. $\sin\theta = \frac{\textit{Opposite side}}{\textit{Hypotenuse}}$
2. $\cos\theta = \frac{\textit{Adjacent side}}{\textit{Hypotenuse}}$
3. $\tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}}$
4. $\text{cosec}\,\theta = \frac{\textit{Hypotenuse}}{\textit{Opposite side}}$
5. $\sec\theta = \frac{\textit{Hypotenuse}}{\textit{Adjacent side}}$
6. $\cot\theta = \frac{\textit{Adjacent side}}{\textit{Opposite side}}$



Again

7. $\tan\theta = \frac{\sin\theta}{\cos\theta}$
8. $\text{cosec}\,\theta = \frac{1}{\sin\theta}$
9. $\sec\theta = \frac{1}{\cos\theta}$
10. $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(ii) Table For Trigonometric Ratios For 0°, 30°, 45°, 60°, 90°

θ Trigonometric Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\text{cosec}\,\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

(iii) Complementary Angle

1. $\sin(90-\theta) = \cos\theta$
2. $\cos(90-\theta) = \sin\theta$
3. $\tan(90-\theta) = \cot\theta$
4. $\text{cosec}(90-\theta) = \sec\theta$
5. $\sec(90-\theta) = \text{cos}ec\,\theta$
6. $\cot(90-\theta) = \tan\theta$

We can write

1. $(\sin\theta)^2 = \sin^2\theta$
2. $(\cos\theta)^2 = \cos^2\theta$
3. $(\tan\theta)^2 = \tan^2\theta$
4. $(\text{cosec}\,\theta)^2 = \text{cosec}^2\,\theta$
5. $(\sec\theta)^2 = \sec^2\theta$
6. $(\cot\theta)^2 = \cot^2\theta$

(iv) Trigonometric Identities

For all real values of θ

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \text{cosec}^2\theta$

Examples

These identities are termed as fundamental identities of trigonometry.

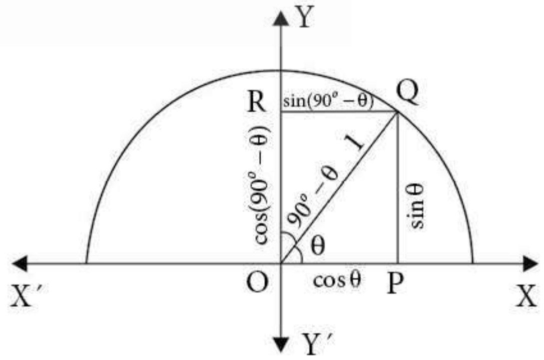
Though the above identities are true for any angle θ, we will consider the following equal forms for $0^o < \theta < 90^o$

1. $\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
2. $\tan^2\theta = \sec^2\theta - 1$ (or) $\sec^2\theta - \tan^2\theta = 1$
3. $\cot^2\theta = \text{cosec}^2\theta - 1$ (or) $\text{cosec}^2\theta - \cot^2\theta = 1$

1. Prove that (i) $\sin(90-\theta) = \cos\theta$ (ii) $\cos(90-\theta) = \sin\theta$.

Visual proof of Trigonometric complementary angle:

Consider a semicircle of radius 1 as shown in the figure.



Let $\angle QOP = \theta$.

Then $\angle QOR = 90^o - \theta$, so that OPQR forms a rectangle.

From triangle OPQ, $\frac{OP}{OQ} = \cos\theta$

But OQ = radius = 1

Therefore OP = OQ $\cos\theta = \cos\theta$

Similarly, $\frac{PQ}{OQ} = \sin\theta$

$\Rightarrow$ PQ $= OQ\sin\theta = \sin\theta$ (since OQ = 1)

OP $= \cos\theta$, PQ $= \sin\theta$ ...(1)

Now, from triangle QOR,

We have $\frac{OR}{OQ} = \cos(90^o - \theta)$

Therefore, OR $= OQ\cos(90^o - \theta)$

$\Rightarrow OR = \cos(90^o - \theta)$

Similarly, $\frac{RQ}{OQ} = \sin(90^o - \theta)$

$$\Rightarrow RQ = \sin(90^o - \theta)$$

$$OR = \cos(90^o - \theta),\ RQ = \sin(90^o - \theta) \qquad \ldots(2)$$

Since OPQR is a rectangle,

$$OP = RQ \text{ and } OR = PQ$$

Therefore From (1) and (2)

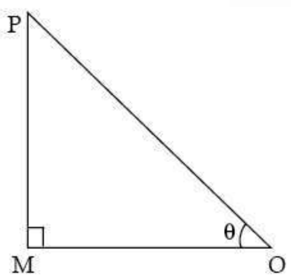
We get, $\sin(90^o - \theta) = \cos\theta$ and $\cos(90^o - \theta) = \sin\theta$

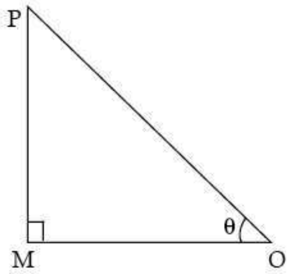
2. Prove the Trigonometric Identities

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

Picture	Identity	Proof
	$\sin^2\theta + \cos^2\theta = 1$	In right triangle OMP, We have $\frac{OM}{OP} = \cos\theta, \frac{PM}{OP} = \sin\theta \qquad \ldots(1)$ By Pythagoras theorem $MP^2 + OM^2 = OP^2 \qquad \ldots(2)$ Dividing each term on both sides of (2) by OP^2, (since $OP \neq 0$) We get, $\frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{OP^2}{OP^2}$ $\Rightarrow \left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = \left(\frac{OP}{OP}\right)^2$ From (1), $(\sin\theta)^2 + (\cos\theta)^2 = 1^2$ Hence $\sin^2\theta + \cos^2\theta = 1$
	$1 + \tan^2\theta = \sec^2\theta$	In right triangle OMP, we have $\frac{MP}{OM} = \tan\theta, \frac{OP}{OM} = \sec\theta \qquad \ldots(3)$ From (2), $MP^2 + OM^2 = OP^2$ Dividing each term on both sides of (2) by OM^2, (since $OM \neq O$) we get, $\frac{MP^2}{OM^2} + \frac{OM^2}{OM^2} = \frac{OP^2}{OM^2}$ $\Rightarrow \left(\frac{MP}{OM}\right)^2 + \left(\frac{OM}{OM}\right)^2 = \left(\frac{OP}{OM}\right)^2$ From (3), $(\tan\theta)^2 + 1^2 = (\sec\theta)^2$ Hence $1 + \tan^2\theta = \sec^2\theta$

	$1+\cot^2\theta = \operatorname{cosec}^2\theta$	In right triangle OMP, we have $\frac{OM}{MP} = \cot\theta, \frac{OP}{MP} = \operatorname{cosec}\theta \quad \dots (4)$ From (2), $MP^2 + OM^2 = OP^2$ Dividing each term on both sides of (2) by MP^2, (since $MP \neq 0$) We get, $\frac{MP^2}{MP^2} + \frac{OM^2}{MP^2} = \frac{OP^2}{MP^2}$ $\Rightarrow \left(\frac{MP}{MP}\right)^2 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2$ From (4), $1^2 + (\cot\theta)^2 = (\operatorname{cosec}\theta)^2$ Hence $1+\cot^2\theta = \operatorname{cosec}^2\theta$

Worked Examples

6.1 **Prove that** $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.

Sol :

$$\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta$$

$$= \tan^2\theta\,(1-\cos^2\theta) = \tan^2\theta\sin^2\theta$$

6.2 **Prove that** $\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$.

Sol :

$$\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

[multiply numerator and denominator by the conjugate of 1 + cos A]

$$= \frac{\sin A\,(1-\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A\,(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A\,(1-\cos A)}{\sin^2 A} = \frac{1-\cos A}{\sin A}$$

6.3 **Prove that** $1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta$.

Sol : $1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta}$

$$= 1 + \frac{\operatorname{cosec}^2\theta - 1}{\operatorname{cosec}\theta + 1} \quad [\text{since } \operatorname{cosec}^2\theta - 1 = \cot^2\theta]$$

$$= 1 + \frac{(\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 1)}{\operatorname{cosec}\theta + 1}$$

$$= 1 + (\operatorname{cosec}\theta - 1) = \operatorname{cosec}\theta$$

6.4 **Prove that** $\sec\theta - \cos\theta = \tan\theta\sin\theta$.

Sol :

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta} \quad [\text{since } 1-\cos^2\theta = \sin^2\theta]$$

$$= \frac{\sin\theta}{\cos\theta} \times \sin\theta = \tan\theta\sin\theta$$

6.5 **Prove that** $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$.

Sol :

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$

[multiply numerator and denominator by the conjugate of $1-\cos\theta$]

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} \quad [\text{since } \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1+\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta$$

6.6 Prove that $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$.

Sol :

$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta} = \cot\theta$$

6.7 Prove that $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B = 1$

Sol :

$\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A\cos^2 B + \sin^2 A\sin^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B$

$= \sin^2 A(\cos^2 B + \sin^2 B) + \cos^2 A(\sin^2 B + \cos^2 B)$

$= \sin^2 A(1) + \cos^2 A(1)$ [since $\sin^2 B + \cos^2 B = 1$]

$= \sin^2 A + \cos^2 A = 1$

6.8 If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, **then prove that** $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Sol :

Now, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

Squaring both sides

$$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$

$$\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$

$$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \quad [\text{since } \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$

$$= \sqrt{2}\sin\theta$$

Therefore $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

6.9 Prove that $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$.

Sol :

$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta\sin^2\theta \times 1}{\sin^2\theta\cos^2\theta} = 1.$$

6.10 Prove that $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\operatorname{cosec} A$.

Sol :

$$\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$

$$= \frac{2\sin A}{1-\cos^2 A} = \frac{2\sin A}{\sin^2 A}$$

$$= 2\operatorname{cosec} A$$

6.11 If $\operatorname{cosec}\theta + \cot\theta = P$ **then prove that** $\cos\theta = \frac{P^2-1}{P^2+1}$.

Sol :

Given $\operatorname{cosec}\theta + \cot\theta = P$ ...(1)

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$ (identity)

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{\operatorname{cosec}\theta + \cot\theta}$$

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{P} \quad ...(2)$$

Adding (1) and (2) we get,

$$2\operatorname{cosec}\theta = P + \frac{1}{P}$$

$$2\operatorname{cosec}\theta = \frac{P^2+1}{P} \quad ...(3)$$

Subtracting (2) from (1), we get,

$$2\cot\theta = P - \frac{1}{P}$$

$$2\cot\theta = \frac{P^2-1}{P} \qquad ...(4)$$

Dividing (4) by (3) we get,

$$\frac{2\cot\theta}{2\operatorname{cosec}\theta} = \frac{P^2-1}{P}\times\frac{P}{P^2+1} \Rightarrow \cos\theta = \frac{P^2-1}{P^2+1}.$$

6.12 Prove that $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$

Sol :

$$\tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A\cos^2 B - \sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A(1-\sin^2 B) - \sin^2 B(1-\sin^2 A)}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$$

6.13 Prove that $\left(\dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\dfrac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2\sin A\cos A$

Sol :

$$\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$$

$$= \left(\frac{(\cos A - \sin A)(\cos^2 A + \sin^2 A + \cos A\sin A)}{\cos A - \sin A}\right) - \left(\frac{(\cos A + \sin A)(\cos^2 A + \sin^2 A - \cos A\sin A)}{\cos A + \sin A}\right)$$

$$\left[\begin{array}{r}\text{since } a^3 - b^3 = (a-b)(a^2+b^2+ab)\\ a^3+b^3 = (a+b)(a^2+b^2-ab)\end{array}\right]$$

$$=(1+\cos A\sin A) - (1-\cos A\sin A) = 2\cos A\sin A$$

6.14 Prove that

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$$

Sol :

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec} A + \cot A - 1}$$

$$= \frac{\sin A(\operatorname{cosec} A + \cot A - 1) + \cos A(\sec A + \tan A - 1)}{(\sec A + \tan A - 1)(\operatorname{cosec} A + \cot A - 1)}$$

$$= \frac{\sin A\operatorname{cosec} A + \sin A\cot A - \sin A + \cos A\sec A + \cos A\tan A - \cos A}{(\sec A + \tan A - 1)(\operatorname{cosec} A + \cot A - 1)}$$

$$= \frac{1 + \cos A - \sin A + 1 + \sin A - \cos A}{\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A} - 1\right)\left(\frac{1}{\sin A} + \frac{\cos A}{\sin A} - 1\right)}$$

$$= \frac{2}{\left(\frac{1+\sin A - \cos A}{\cos A}\right)\left(\frac{1+\cos A - \sin A}{\sin A}\right)}$$

$$= \frac{2\sin A\cos A}{(1+\sin A - \cos A)(1+\cos A - \sin A)}$$

$$= \frac{2\sin A\cos A}{[1+(\sin A - \cos A)][1-(\sin A - \cos A)]}$$

$$= \frac{2\sin A\cos A}{1-(\sin A - \cos A)^2}$$

$$= \frac{2\sin A\cos A}{1-(\sin^2 A + \cos^2 A - 2\sin A\cos A)}$$

$$= \frac{2\sin A\cos A}{1-(1-2\sin A\cos A)}$$

$$= \frac{2\sin A\cos A}{1-1+2\sin A\cos A} = \frac{2\sin A\cos A}{2\sin A\cos A} = 1.$$

6.15 Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2$.

Sol :

$$\text{LHS} = \left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}}$$

$$= \frac{1+\tan^2 A}{\frac{\tan^2 A + 1}{\tan^2 A}} = \tan^2 A \qquad ...(1)$$

$$\text{RHS} = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \left(\frac{1-\tan A}{1-\frac{1}{\tan A}}\right)^2$$

$$= \left(\frac{1-\tan A}{\frac{\tan A - 1}{\tan A}}\right)^2 = (-\tan A)^2 = \tan^2 A \quad ...(2)$$

From (1) and (2), $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2$

6.16 **Prove that** $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}=\sin^2 A\cos^2 A.$

Sol :

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

$$=\frac{\left(1+\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\operatorname{cosec} A)(\sec^2 A+\sec A\operatorname{cosec} A+\operatorname{cosec}^2 A)}$$

$$=\frac{\dfrac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\operatorname{cosec} A)\left(\dfrac{1}{\cos^2 A}+\dfrac{1}{\cos A\sin A}+\dfrac{1}{\sin^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)\left(\dfrac{\sin A}{\sin A\cos A}-\dfrac{\cos A}{\sin A\cos A}\right)}{(\sec A-\operatorname{cosec} A)\left(\dfrac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)(\sec A-\operatorname{cosec} A)}{(\sec A-\operatorname{cosec} A)(1+\sin A\cos A)}\times\sin^2 A\cos^2 A$$

$$=\sin^2 A\cos^2 A$$

6.17 **If** $\dfrac{\cos^2\theta}{\sin\theta}=p$ *and* $\dfrac{\sin^2\theta}{\cos\theta}=q$, **then prove that** $p^2q^2(p^2+q^2+3)=1.$

Sol :

We have $\dfrac{\cos^2\theta}{\sin\theta}=p$ ...(1)

and $\dfrac{\sin^2\theta}{\cos\theta}=q$ ... (2)

$p^2q^2(p^2+q^2+3)=$

$$\left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\times\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2+\left(\frac{\sin^2\theta}{\cos\theta}\right)^2+3\right]$$

[from (1) and (2)]

$$=\left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\times\left[\frac{\cos^4\theta}{\sin^2\theta}+\frac{\sin^4\theta}{\cos^2\theta}+3\right]$$

$$=(\cos^2\theta\times\sin^2\theta)\times\left[\left(\frac{\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

$$=\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta$$

$$=(\cos^2\theta)^3+(\sin^2\theta)^3+3\sin^2\theta\cos^2\theta$$

$$=[(\cos^2\theta+\sin^2\theta)^3-3\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)]+3\sin^2\theta\cos^2\theta$$

$$=1-3\cos^2\theta\sin^2\theta(1)+3\cos^2\theta\sin^2\theta=1.$$

Progress Check

1. The number of trigonometric ratios is _____

Ans : 6

2. $1-\cos^2\theta$ **is _____**

Ans : $\sin^2\theta$

3. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$ **is _____**

Ans : 1

4. $(\cot\theta+\operatorname{cosec}\theta)(\cot\theta-\operatorname{cosec}\theta)$ **is _____**

Ans : -1

5. $\cos 60^\circ\sin 30^\circ+\cos 30^\circ\sin 60^\circ$ **is _____**

Ans : 1

6. $\tan 60^\circ\cos 60^\circ+\cot 60^\circ\sin 60^\circ$ **is _____**

Ans : $\dfrac{\sqrt{3}+1}{2}$

7. $(\tan 45^\circ\cot 45^\circ)+(\sec 45^\circ\operatorname{cosec} 45^\circ)$ **is _____**

Ans : 4

8. (i) $\sec\theta=\operatorname{cosec}\theta$ **if** θ **is _____**

(ii) $\cot\theta=\tan\theta$ **if** θ **is _____**

Ans : 45°, 45°

Thinking Corner

1. When will the values of $\sin\theta$ **and** $\cos\theta$ **be equal?**

Ans : When $\theta=45^\circ$

$$\sin 45^\circ=\cos 45^\circ=\frac{1}{\sqrt{2}}$$

$$\therefore\ \sin\theta=\cos\theta\ \text{for}\ \theta=45^\circ$$

2. For what values of θ, $\sin\theta=2$**?**

Ans : since $\sin\theta=\dfrac{\textit{Opposite side}}{\textit{Hypotenuse}}$,

it takes values From -1 to 1.

$\therefore$ For no real value, $\sin\theta$ equal to 2.

3. Among the six trigonometric quantities, as the value of angle θ increase from 0° to 90°, which of the six trigonometric quantities has undefined values?

Ans : tan 90° is undefined
sec 90° is undefined
cosec 0° is undefined
cot 0° is undefined

4. Is it possible to have eight trigonometric ratios?

Ans : No. Since trigonometric ratios are relation between two of three sides of triangles only 6 combinations are there.

5. Let $0^\circ \le \theta \le 90^\circ$. For what values of θ does

(i) $\sin\theta > \cos\theta$ (ii) $\cos\theta > \sin\theta$

(iii) $\sec\theta = 2\tan\theta$ (iv) $\text{cosec}\,\theta = 2\cot\theta$

Ans :

(i) $\sin 60^\circ = \frac{\sqrt{3}}{2}$; $\cos 60^\circ = \frac{1}{2}$

$\Rightarrow \sin 60^\circ > \cos 60^\circ$

$\sin 90^\circ = 1$; $\cos 90^\circ = 0$

$\Rightarrow \sin 90^\circ > \cos 90^\circ$

$\sin\theta > \cos\theta$ for $\theta = 60^\circ$ and $\theta = 90^\circ$

(ii) $\sin 0^\circ = 0$; $\cos 0^\circ = 1$

$\Rightarrow \cos 0^\circ > \sin 0^\circ$

$\sin 30^\circ = \frac{1}{2}$; $\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos 30^\circ > \sin 30^\circ$

$\therefore \cos\theta > \sin\theta$ for $\theta = 0^\circ$ and $\theta = 30^\circ$

(iii) Given $\sec\theta = 2\tan\theta$

$$\frac{\sec\theta}{\tan\theta} = 2$$

$$\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = 2$$

$$\frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = 2$$

$$\frac{1}{\sin\theta} = 2$$

$$\text{cosec}\,\theta = 2$$

$$\therefore \text{cosec}\,\theta = 2 \text{ for } \theta = 30^\circ$$

$$\therefore \sec\theta = 2\tan\theta \text{ for } \theta = 30^\circ$$

Also $\sec 30^\circ = \frac{2}{\sqrt{3}}$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$2\tan 30^\circ = \frac{2}{\sqrt{3}} = \sec 30^\circ$$

$$\therefore \theta = 30^\circ$$

(iv) $\text{cosec}\,\theta = 2\cot\theta$

$$\frac{1}{\sin\theta} = 2\frac{\cos\theta}{\sin\theta}$$

$$\frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta} = 2$$

$$\frac{1}{\cos\theta} = 2$$

$$\sec\theta = 2$$

$$\sec\theta = 2 \text{ for } \theta = 60^\circ$$

$$\therefore \text{cosec}\,\theta = 2\cot\theta \text{ for } \theta = 60^\circ$$

Exercise 6. 1

1. Prove the following identities.

(i) $\cot\theta + \tan\theta = \sec\theta\,\text{cosec}\,\theta$

(ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

Sol :

(i) $\cot\theta + \tan\theta = \sec\theta\,\text{cosec}\,\theta$

$$\text{LHS} = \cot\theta + \tan\theta$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \frac{1}{\sin\theta} \times \frac{1}{\cos\theta}$$

$$= \text{cosec}\,\theta\sec\theta$$

$$= \sec\theta\,\text{cosec}\,\theta$$

$$= \text{RHS}$$

(ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

$$\text{LHS} = \tan^4\theta + \tan^2\theta$$

$$= \tan^2\theta(\tan^2\theta + 1)$$

$$= \tan^2\theta.\sec^2\theta \quad [\because 1 + \tan^2\theta = \sec^2\theta]$$

$$= \sec^4\theta - \sec^2\theta$$

$[\because \tan^2\theta = \sec^2\theta - 1]$

$$= \text{RHS}$$

2. Prove the following identities.

(i) $\dfrac{1-\tan^2\theta}{\cot^2\theta-1} = \tan^2\theta$ **(ii)** $\dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

Sol :

(i) $\dfrac{1-\tan^2\theta}{\cot^2\theta-1} = \tan^2\theta$

$$\text{LHS} = \frac{1-\tan^2\theta}{\cot^2\theta-1}$$

$$= \frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta}{\sin^2\theta}-1}$$

$$= \frac{\dfrac{(\cos^2\theta-\sin^2\theta)}{\cos^2\theta}}{\dfrac{(\cos^2\theta-\sin^2\theta)}{\sin^2\theta}}$$

$$= \frac{(\cos^2\theta-\sin^2\theta)}{\cos^2\theta} \times \frac{\sin^2\theta}{(\cos^2\theta-\sin^2\theta)}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = \text{RHS}$$

(ii) $\dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

$$\text{LHS} = \frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}$$

[Multiplying the Numerator and Denominator by $1 - \sin\theta$]

$$= \frac{\cos\theta\,(1-\sin\theta)}{1^2-\sin^2\theta}$$

[$\because (a+b)(a-b) = a^2-b^2$]

$$= \frac{\cos\theta\,(1-\sin\theta)}{\cos^2\theta}$$

[$\because 1-\sin^2\theta = \cos^2\theta$]

$$= \frac{(1-\sin\theta)}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta$$

$$= \text{RHS}$$

3. Prove the following identities.

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

Sol :

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$

$$\text{LHS} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$$

[Multiplying the Numerator and denominator by $\sqrt{1-\sin\theta}$]

$$= \sqrt{\frac{1^2-\sin^2\theta}{(1-\sin\theta)^2}} \quad [\because (a+b)(a-b) = a^2-b^2]$$

$$= \sqrt{\frac{\cos^2\theta}{(1-\sin\theta)^2}} \quad [\because 1-\sin^2\theta = \cos^2\theta]$$

$$= \frac{\cos\theta}{1-\sin\theta}$$

$$= \frac{\cos\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}$$

[Multiplying Numerator and denominator by $1+\sin\theta$]

$$= \frac{\cos\theta\,(1+\sin\theta)}{1^2-\sin^2\theta} = \frac{\cos\theta\,(1+\sin\theta)}{\cos^2\theta}$$

[$\because (a+b)(a-b) = a^2-b^2$] [$1-\sin^2\theta = \cos^2\theta$]

$$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta = \text{RHS}$$

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

$$\text{LHS} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}+\sqrt{\frac{1-\sin\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}}$$

$$=\sqrt{\frac{(1+\sin\theta)^2}{1^2-\sin^2\theta}}+\sqrt{\frac{(1-\sin\theta)^2}{1^2-\sin^2\theta}}$$

$$=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}+\sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$=\frac{1+\sin\theta}{\cos\theta}+\frac{1-\sin\theta}{\cos\theta}$$

$$=\frac{1+\sin\theta+1-\sin\theta}{\cos\theta}$$

$$=2\times\frac{1}{\cos\theta}$$

$$=2\sec\theta$$

$$=\text{RHS}$$

4. Prove the following identities.

(i) $\sec^6\theta=\tan^6\theta+3\tan^2\theta\sec^2\theta+1$

(ii) $(\sin\theta+\sec\theta)^2+(\cos\theta+\text{cosec}\,\theta)^2 = 1+(\sec\theta+\text{cosec}\,\theta)^2$.

Sol :

(i) $\sec^6\theta=\tan^6\theta+3\tan^2\theta\sec^2\theta+1$

$$\text{LHS}=\sec^6\theta$$

$$=(\sec^2\theta)^3$$

$$=(1+\tan^2\theta)^3 \qquad [\because 1+\tan^2\theta=\sec^2\theta)$$

$$=1^3+(\tan^2\theta)^3+3(1)(\tan^2\theta)(1+\tan^2\theta)$$

$$[\because (a+b)^3=a^3+b^3+3ab(a+b)]$$

$$=1+\tan^6\theta+3\tan^2\theta(\sec^2\theta)$$

$$[\because 1+\tan^2\theta=\sec^2\theta]$$

$$=\tan^6\theta+3\tan^2\theta\sec^2\theta+1=\text{RHS}$$

(ii) $(\sin\theta+\sec\theta)^2+(\cos\theta+\text{cosec}\,\theta)^2$
$=1+(\sec\theta+\text{cosec}\,\theta)^2$

$$\text{LHS}=(\sin\theta+\sec\theta)^2+(\cos\theta+\text{cosec}\,\theta)^2$$

$$=\sin^2\theta+\sec^2\theta+2\sin\theta\sec\theta+\cos^2\theta+\text{cosec}^2\theta+2\cos\theta\,\text{cosec}\,\theta$$

$$[\because (a+b)^2=a^2+b^2+2ab]$$

$$=(\sin^2\theta+\cos^2\theta)+(\sec^2\theta+\text{cosec}^2\theta)+2(\sin\theta\sec\theta+\cos\theta\,\text{cosec}\,\theta)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\left(\sin\theta\frac{1}{\cos\theta}+\cos\theta\frac{1}{\sin\theta}\right)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$=1+\sec^2\theta+\text{cosec}^2\theta+2\,(\text{cosec}\,\theta\ \sec\theta)$$

$$=1+[\sec^2\theta+\text{cosec}^2\theta+2\sec\theta\,\text{cosec}\,\theta]$$

$$=1+(\sec\theta+\text{cosec}\,\theta)^2=\text{RHS}$$

5. Prove the following identities.

(i) $\sec^4\theta(1-\sin^4\theta)-2\tan^2\theta=1$

(ii) $\dfrac{\cot\theta-\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\text{cosec}\,\theta-1}{\text{cosec}\,\theta+1}$

Sol :

(i) $\sec^4\theta(1-\sin^4\theta)-2\tan^2\theta=1$

$$\text{LHS}=\sec^4\theta(1-\sin^4\theta)-2\tan^2\theta$$

$$=\sec^4\theta(1^2-(\sin^2\theta)^2)-2\tan^2\theta$$

$$=\sec^4\theta(1+\sin^2\theta)(1-\sin^2\theta)-2\tan^2\theta$$

$$[\because a^2-b^2=(a+b)(a-b)]$$

$$=\sec^4\theta(1+\sin^2\theta)\cos^2\theta-2\tan^2\theta$$

$$[\because 1-\sin^2\theta=\cos^2\theta]$$

$$=\frac{1}{\cos^4\theta}(1+\sin^2\theta)\cos^2\theta-2\tan^2\theta$$

$$=\frac{1}{\cos^2\theta}(1+\sin^2\theta)-2\frac{\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1+\sin^2\theta}{\cos^2\theta}-\frac{2\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1^2+\sin^2\theta-2\sin^2\theta}{\cos^2\theta}$$

$$=\frac{1-\sin^2\theta}{\cos^2\theta}$$

$$=\frac{(\cos^2\theta)}{\cos^2\theta}=1=\text{RHS}$$

(ii) $\dfrac{\cot\theta-\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\text{cosec}\,\theta-1}{\text{cosec}\,\theta+1}$

$$\text{LHS}=\frac{\cot\theta-\cos\theta}{\cot\theta+\cos\theta}$$

$$=\frac{\frac{\cos\theta}{\sin\theta}-\cos\theta}{\frac{\cos\theta}{\sin\theta}+\cos\theta}$$

$$= \frac{\frac{\cos\theta - \sin\theta\cos\theta}{\sin\theta}}{\frac{\cos\theta + \sin\theta\cos\theta}{\sin\theta}}$$

$$= \frac{(\cos\theta - \sin\theta\cos\theta)}{\sin\theta} \times \frac{\sin\theta}{(\cos\theta + \sin\theta\cos\theta)}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{\cos\theta\,(1+\sin\theta)}$$

$$\text{LHS} = \frac{1-\sin\theta}{1+\sin\theta} \quad \text{...(1)}$$

$$\text{RHS} = \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$$

$$= \frac{\frac{1}{\sin\theta}-1}{\frac{1}{\sin\theta}+1} = \frac{\frac{1-\sin\theta}{\sin\theta}}{\frac{1+\sin\theta}{\sin\theta}} = \frac{1-\sin\theta}{\sin\theta} \times \frac{\sin\theta}{1+\sin\theta}$$

$$\text{RHS} = \frac{1-\sin\theta}{1+\sin\theta} \quad \text{..(2)}$$

From (1) and (2)

LHS = RHS

6. Prove the following identities.

(i) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

(ii) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2.$

Sol :

(i) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

$$\text{LHS} = \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A - \cos B)(\cos A + \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A - \sin^2 B) + (\cos^2 A - \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{1-1}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{0}{(\cos A + \cos B)(\sin A + \sin B)} = 0 = \text{RHS}$$

(ii) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$

$$\text{LHS} = \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)^3 - 3\sin A\cos A(\sin A + \cos A)}{\sin A + \cos A}$$

$$+ \frac{(\sin A - \cos A)^3 + 3\sin A\cos A(\sin A - \cos A)}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)[(\sin A + \cos A)^2 - 3\sin A\cos A]}{(\sin A + \cos A)}$$

$$+ \frac{(\sin A - \cos A)[(\sin A - \cos A)^2 + 3\sin A\cos A]}{\sin A - \cos A}$$

$$= \sin^2 A + \cos^2 A + 2\sin A\cos A - 3\sin A\cos A$$

$$+ \sin^2 A + \cos^2 A - 2\sin A\cos A + 3\sin A\cos A$$

$$= 2\sin^2 A + 2\cos^2 A$$

$$= 2(\sin^2 A + \cos^2 A) = 2 = \text{RHS}$$

7. (i) If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.

(ii) If $\sqrt{3}\sin\theta - \cos\theta = 0$, then show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Sol :

(i) We have $\sin\theta + \cos\theta = \sqrt{3}$

Squaring on both the sides,

$$(\sin\theta + \cos\theta)^2 = \left(\sqrt{3}\right)^2$$

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$$

$$1 + 2\sin\theta\cos\theta = 3$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$2\sin\theta\cos\theta = 3 - 1$$

$$2\sin\theta\cos\theta = 2$$

$$\sin\theta\cos\theta = 2/2$$

$$\sin\theta\cos\theta = 1 \quad \text{... (1)}$$

Now to prove $\tan\theta + \cot\theta = 1$

$$\text{LHS} = \tan\theta + \cot\theta$$
$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$
$$= \frac{1}{1} \quad [\because \textit{From}\ (1)\ \sin\theta\cos\theta = 1]$$
$$= 1$$
$$\therefore\ \tan\theta + \cot\theta = 1$$

(ii) Given $\sqrt{3}\sin\theta - \cos\theta = 0$

$$\sqrt{3}\sin\theta = \cos\theta$$
$$\frac{\sqrt{3}\sin\theta}{\cos\theta} = 1$$
$$\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$$
$$\tan\theta = \frac{1}{\sqrt{3}}$$
$$\therefore\ \theta = 30^\circ$$
$$\text{LHS} = \tan 3\theta$$
$$= \tan 3(30^\circ)$$
$$= \tan 90^\circ$$
$$= \text{undefined} \quad \dots(1)$$
$$\text{RHS} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$
$$= \frac{3\tan 30^\circ - \tan^3 30^\circ}{1 - 3\tan^2 30^\circ}$$
$$= \frac{3\frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$$
$$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - 3\times\frac{1}{3}}$$
$$= \frac{\frac{3\times 3 - 1}{3\sqrt{3}}}{1-1}$$
$$= \frac{9-1}{3\sqrt{3}(0)} = \frac{8}{0} = \text{undefined} \quad \dots(2)$$

From (1) and (2), LHS = RHS.

8. (i) If $\frac{\cos\alpha}{\cos\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$, prove that $(m^2 + n^2)\cos^2\beta = n^2$.

(ii) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then prove that $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$.

Sol :

Given $\frac{\cos\alpha}{\cos\beta} = m$

$$\frac{\cos\alpha}{\sin\beta} = n$$
$$\text{LHS} = (m^2 + n^2)\cos^2\beta$$
$$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$$
$$= \frac{(\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta)}{\cos^2\beta\sin^2\beta}\cos^2\beta$$
$$= \frac{\cos^2\alpha(\sin^2\beta + \cos^2\beta)}{\sin^2\beta}$$
$$= \frac{\cos^2\alpha}{\sin^2\beta}\ (1) \quad [\because\ \sin^2\theta + \cos^2\theta = 1]$$
$$= \left(\frac{\cos\alpha}{\sin\beta}\right)^2$$
$$= n^2 = \text{RHS}$$

(ii) We have $\cot\theta + \tan\theta = x$

$$\sec\theta - \cos\theta = y$$

Taking $\cot\theta + \tan\theta = x$

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = x$$
$$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = x$$
$$\frac{1}{\sin\theta\cos\theta} = x \quad \dots(1)$$

Also $\sec\theta - \cos\theta = y$

$$\frac{1}{\cos\theta} - \cos\theta = y$$
$$\Rightarrow \frac{1 - \cos^2\theta}{\cos\theta} = y$$
$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = y \quad \dots(2)$$

Now LHS $= (x^2y)^{2/3} - (xy^2)^{2/3}$

$$=\left[\left(\frac{1}{\cos\theta\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)\right]^{\frac{2}{3}}-\left[\left(\frac{1}{\cos\theta\sin\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\right]^{\frac{2}{3}}$$

[$\because$ using (1) *and* (2)]

$$=\left[\frac{1}{\cos^2\theta\sin^2\theta}\times\frac{\sin^2\theta}{\cos\theta}\right]^{\frac{2}{3}}-\left[\frac{1}{\cos\theta\sin\theta}\times\frac{\sin^4\theta}{\cos^2\theta}\right]^{\frac{2}{3}}$$

$$=\left(\frac{1}{\cos^3\theta}\right)^{\frac{2}{3}}-\left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{\frac{2}{3}}$$

$$=\frac{1}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}$$

$$=\sec^2\theta-\tan^2\theta$$

$$=1=\text{RHS} \qquad [\because \sec^2\theta-\tan^2\theta=1]$$

9. (i) If $\sin\theta+\cos\theta=p$ *and* $\sec\theta+\operatorname{cosec}\theta=q$, **then prove that q $(p^2-1)=2p$.**

(ii) If $\sin\theta(1+\sin^2\theta)=\cos^2\theta$, **then prove that** $\cos^6\theta-4\cos^4\theta+8\cos^2\theta=4$

Sol :

(i) $\sin\theta+\cos\theta=p$ and $\sec\theta+\operatorname{cosec}\theta=q$

LHS $= q(p^2-1)$

$$=(\sec\theta+\operatorname{cosec}\theta)[(\sin\theta+\cos\theta)^2-1]$$

$$=\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)(\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1)$$

$$=\left(\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\right)(1+2\sin\theta\cos\theta-1)$$

$$=\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\times 2\sin\theta\cos\theta$$

$$=2(\sin\theta+\cos\theta)=2p=\text{RHS}$$

(ii) Given $\sin\theta(1+\sin^2\theta)=\cos^2\theta$

Squaring on both the sides.

$$\sin^2\theta(1+\sin^2\theta)^2=\cos^4\theta$$

$$\Rightarrow(1-\cos^2\theta)\{1+(1-\cos^2\theta)\}^2=\cos^4\theta$$

$$\left[\begin{array}{l}\because\ \sin^2\theta=1-\cos^2\theta\\ \cos^2\theta=1-\sin^2\theta\end{array}\right]$$

$$\Rightarrow(1-\cos^2\theta)\{1+(1-\cos^2\theta)^2+2(1)(1-\cos^2\theta)\}=\cos^4\theta$$

$$\Rightarrow(1-\cos^2\theta)(1+1+\cos^4\theta-2(1)\cos^2\theta+2-2\cos^2\theta)=\cos^4\theta$$

$$\Rightarrow(1-\cos^2\theta)(4-4\cos^2\theta+\cos^4\theta)=\cos^4\theta$$

$$\Rightarrow 4-4\cos^2\theta+\cos^4\theta-4\cos^2\theta+4\cos^4\theta-\cos^6\theta=\cos^4\theta$$

$$\Rightarrow 4-8\cos^2\theta+5\cos^4\theta-\cos^6\theta=\cos^4\theta$$

$$4-8\cos^2\theta+4\cos^4\theta-\cos^6\theta=0$$

$$4=8\cos^2\theta-4\cos^4\theta+\cos^6\theta$$

$$\Rightarrow\cos^6\theta-4\cos^4\theta+8\cos^2\theta=4$$

10. If $\frac{\cos\theta}{1+\sin\theta}=\frac{1}{a}$, **then prove that** $\frac{a^2-1}{a^2+1}=\sin\theta$.

Sol : Given $\frac{\cos\theta}{1+\sin\theta}=\frac{1}{a}$

$$\therefore\ a=\frac{1+\sin\theta}{\cos\theta}$$

Now LHS $=\frac{a^2-1}{a^2+1}$

$$=\frac{\left(\frac{1+\sin\theta}{\cos\theta}\right)^2-1}{\left(\frac{1+\sin\theta}{\cos\theta}\right)^2+1}$$

$$=\frac{\frac{1^2+\sin^2\theta+2\sin\theta}{\cos^2\theta}-1}{\frac{1^2+\sin^2\theta+2\sin\theta}{\cos^2\theta}+1}$$

$$=\frac{\frac{1+\sin^2\theta+2\sin\theta-\cos^2\theta}{\cos^2\theta}}{\frac{1+\sin^2\theta+2\sin\theta+\cos^2\theta}{\cos^2\theta}}$$

$$=\frac{(1-\cos^2\theta)+\sin^2\theta+2\sin\theta}{\cos^2\theta}\times\frac{\cos^2\theta}{1+(\sin^2\theta+\cos^2\theta)+2\sin\theta}$$

$$=\frac{\sin^2\theta+\sin^2\theta+2\sin\theta}{1+1+2\sin\theta}$$

$$\left[\begin{array}{l}\because\sin^2\theta+\cos^2\theta=1\\ 1-\cos^2\theta=\sin^2\theta\end{array}\right]$$

$$=\frac{2\sin^2\theta+2\sin\theta}{2+2\sin\theta}$$

$$=\frac{2\sin\theta(\sin\theta+1)}{2(1+\sin\theta)}$$

$$=\sin\theta=\text{RHS}$$

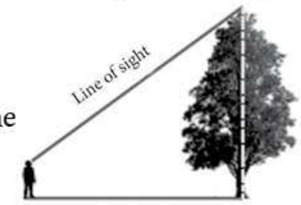
HEIGHT AND DISTANCES

Key Points

Trigonometry is used for finding the heights and distances of various objects without actually measuring them.

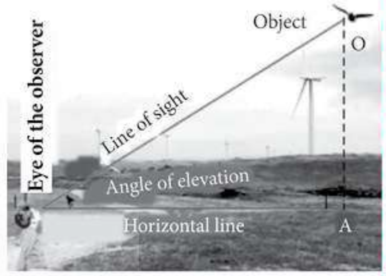
(i) Line of Sight

The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.



(ii) Theodolite

1. Theodolite is an instrument which is used in measuring the angle between an object and the eye of the observer.
2. It has two wheels placed right angles to each other used for measuring horizontal and vertical angles.

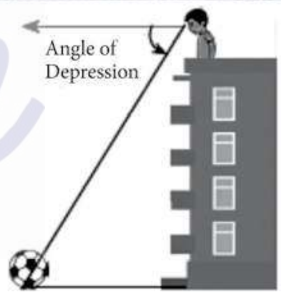


(iii) Angle of Elevation

The angle of elevation is an angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level.

(iv) Angle of Depression

The angle of depression is an angle formed by the line of sight with the horizontal when the point is below the horizontal level.

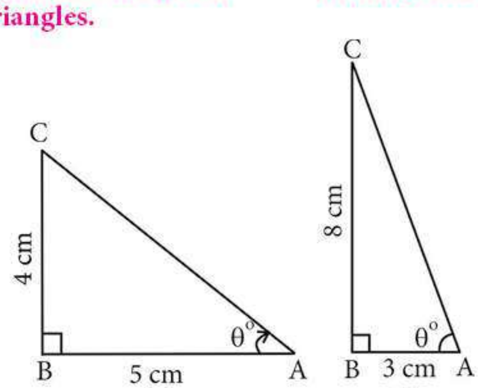


(v) Clinometer

1. The angle of elevation and angle of depression are usually measured by a device called inclinometer or clinometer.
2. From a given point when height of an object increases, the angle of elevation increases.
3. The angle of elevation increases as we move towards the foot of the vertical like building.

Worked Examples

6.18 **Calculate the size of $\angle BAC$ in the given triangles.**



(a) (b)

Sol :

(i) In right triangle ABC [see fig (a)]

$$\tan\theta = \frac{opposite\ side}{adjacent\ side} = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(0.8)$$

$$\theta = 38.7^{\circ} \text{ (since } \tan 38.7^{\circ} = 0.8011)$$

$$\angle BAC = 38.7^{\circ}$$

(ii) In right triangle ABC [see fig (b)]

$$\tan\theta = \frac{8}{3}$$

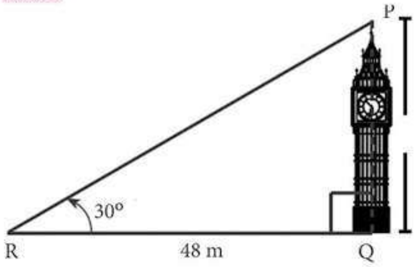
$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = \tan^{-1}(2.66)$$

$$\theta = 69.4^{\circ} \text{ (since } \tan 69.4^{\circ} = 2.6604)$$

$$\angle BAC = 69.4^{\circ}$$

6.19 **A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower.**

Sol :



Let PQ be the height of the tower.

Take PQ = h and QR is the distance between the tower and the point R. In right triangle PQR,

$$\angle PRQ = 30^\circ$$

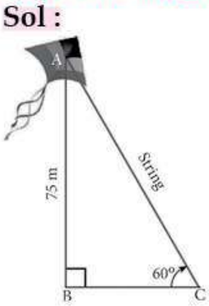
$$\tan\theta = \frac{PQ}{QR}$$

$$\tan 30^\circ = \frac{h}{48} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{48} \Rightarrow h = 16\sqrt{3}$$

Therefore the height of the tower is $16\sqrt{3}$ m.

6.20 **A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.**

Sol :



Let AB be the height of the kite above the ground. Then, AB = 75.

Let AC be the length of the string.

In right triangle ABC, $\angle ACB = 60^\circ$

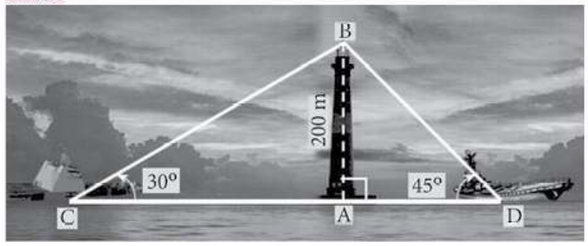
$$\sin\theta = \frac{AB}{AC} \Rightarrow \sin 60^\circ = \frac{75}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{75}{AC} \Rightarrow AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$$

Hence, the length of the string is $50\sqrt{3}$ m.

6.21 **Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45° respectively. If the lighthouse is 200 m high, find the distance between the two ships. $\left(\sqrt{3} = 1.732\right)$**

Sol :



Let AB be the lighthouse and C and D the positions of the two ships.

Then, AB = 200 m,

$\angle ACB = 30^\circ$, $\angle ADB = 45^\circ$

In right triangle BAC, $\tan 30^\circ = \frac{AB}{AC}$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC} \Rightarrow AC = 200\sqrt{3} \quad ...(1)$$

In right triangle BAD, $\tan 45^\circ = \frac{AB}{AD}$

$$1 = \frac{200}{AD} \Rightarrow AD = 200 \quad ...(2)$$

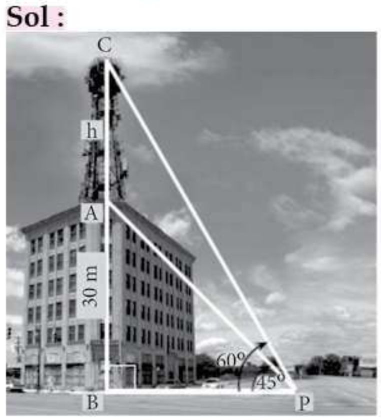
Now, $CD = AC + AD = 200\sqrt{3} + 200$ [by (1) and (2)]

$$CD = 200\left(\sqrt{3}+1\right) = 200 \times 2.732 = 546.4$$

Distance between two ships is 546.4 m.

6.22 **From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are 45° and 60° respectively. Find the height of the tower $\left(\sqrt{3} = 1.732\right)$**

Sol :



Let AC be the height of the tower.

Let AB be the height of the building

Then, AC = h metres, AB = 30 m.

In right triangle CBP, $\angle CPB = 60^\circ$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60^\circ = \frac{AB+AC}{BP} \Rightarrow \sqrt{3} = \frac{30+h}{BP} \quad \text{... (1)}$$

In right triangle ABP, $\angle APB = 45^\circ$

$$\tan\theta = \frac{AB}{BP}$$

$$\tan 45^\circ = \frac{30}{BP} \Rightarrow BP = 30 \quad \text{... (2)}$$

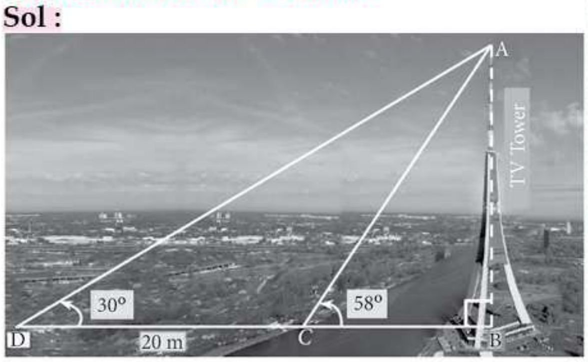
Substituting (2) in (1), we get $\sqrt{3} = \dfrac{30+h}{30}$

$$h = 30\left(\sqrt{3}-1\right)$$
$$= 30\,(1.732 - 1) = 30(0.732) = 21.96$$

Hence, the height of the tower is 21.96 m.

6.23 A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is 58°. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal. (tan 58° = 1.6003)

Sol :



Let AB be the height of the TV tower.

CD = 20 m

Let BC be the width of the canal.

In right triangle ABC, $\tan 58^\circ = \dfrac{AB}{BC}$

$$1.6003 = \frac{AB}{BC} \quad \text{...(1)}$$

In right triangle ABD, $\tan 30^\circ = \dfrac{AB}{BD} = \dfrac{AB}{BC+CD}$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+20} \quad \text{...(2)}$$

Dividing (1) by (2) we get, $\dfrac{1.6003}{\frac{1}{\sqrt{3}}} = \dfrac{BC+20}{BC}$

$$BC = \frac{20}{1.7717} = 11.24 \text{ m} \quad \text{...(3)}$$

$$1.6003 = \frac{AB}{11.29} \text{ [from (1) and (3)]}$$

$$AB = 17.99$$

Hence, the height of the tower is 17.99 m and the width of the canal is 11.24 m.

6.24 An aeroplane sets off from G on a bearing of 24° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 55° and a distance of 180 km away.

(i) How far is H to the North of G?

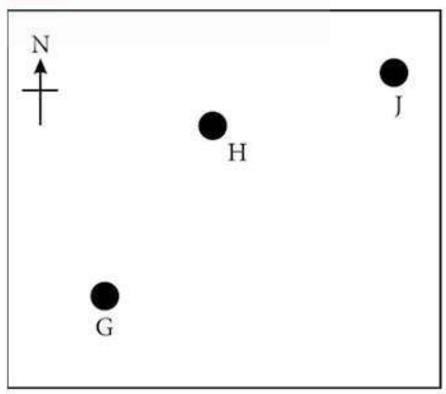
(ii) How far is H to the East of G?

(iii) How far is J to the North of H?

(iv) How far is J to the East of H?

$$\begin{pmatrix} \sin 24^\circ = 0.4067, & \sin 11^\circ = 0.1908 \\ \cos 24^\circ = 0.9135, & \cos 11^\circ = 0.9816 \end{pmatrix}$$

Sol :



(i) In right triangle GOH, $\cos 24^\circ = \dfrac{OG}{GH}$

$0.9135 = \dfrac{OG}{250}$; OG = 228.38 km

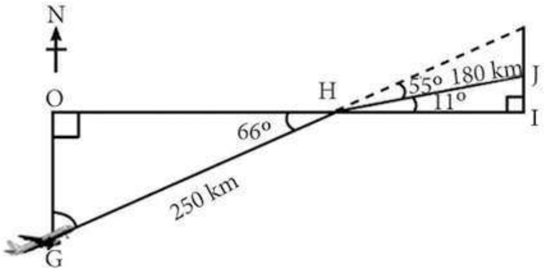
Distance of H to the North of G = 228.38 km

(ii) In right triangle GOH,

$$\sin 24^\circ = \frac{OH}{GH}$$

$$0.4067 = \frac{OH}{250};\ OH = 101.68$$

Distance of H to the East of G = 101.68 km



(iii) In right triangle HIJ,

$$\sin 11^\circ = \frac{IJ}{HJ}$$

$$0.1908 = \frac{IJ}{180};\ IJ = 34.34 \text{ km}$$

Distance of J to the North of H = 34.34 km

(iv) In right triangle HIJ,

$$\cos 11^\circ = \frac{HI}{HJ}$$

$$0.9816 = \frac{HI}{180};\ HI = 176.69 \text{ km}$$

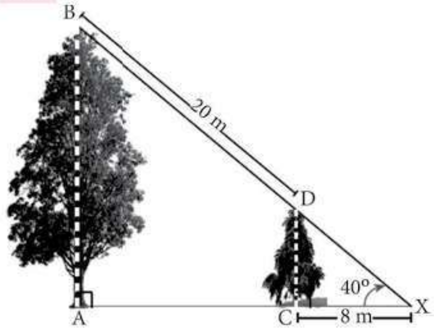
Distance of J to the East of H = 176.69 km

6.25 **Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°. If the horizontal distance between X and the smaller tree is 8 m and the distance the tops of the two trees is 20 m, calculate**

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees. ($\cos 40^\circ = 0.7660$)

Sol :



Let AB be the height of the bigger tree and CD be the height of the smaller tree and X is the point on the ground.

(i) In right triangle XCD, $\cos 40^\circ = \frac{CX}{XD}$

$$XD = \frac{8}{0.7660} = 10.44 \text{ m}$$

Therefore the distance between X and top of the smaller tree = XD = 10.44 m

(ii) In right triangle XAB,

$$\cos 40^\circ = \frac{AX}{BX} = \frac{AC + CX}{BD + DX}$$

$$0.7660 = \frac{AC + 8}{20 + 10.44}$$

$$AC = 23.32 - 8 = 15.32 \text{ m}$$

Therefore the horizontal distance between two trees = AC = 15.32 m

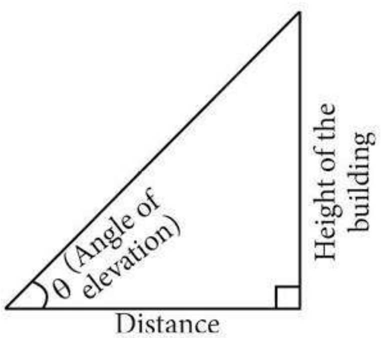
Thinking Corner

1. What type of triangle is used to calculate heights and distances?

Ans : Right angled triangle is used to calculate heights and distances.

2. When the height of the building and distance from the foot of the building are given, which trigonometric ratio is used to find the angle of elevation?

Ans :



If θ is the angle of elevation then the known measures are opposite side and adjacent side.

$\therefore$ tan θ is used to find the angle of elevation.

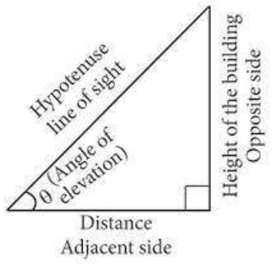
$$\text{i.e.,}\quad \tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} = \frac{\textit{Height of the building}}{\textit{Distance}}$$

3. If the line of sight and angle of elevation is given, then which trigonometric ratio, is used.

(i) to find the height of the building

(ii) to find the distance from the foot of the building.

Ans :



(i) To find the height of the building

$$\sin\theta = \frac{Opposite\ side}{Hypotenuse} \text{ is used.}$$

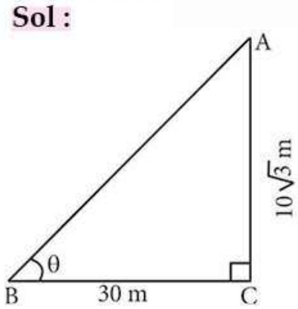
(ii) To find the distance from the foot of the building.

$$\cos\theta = \frac{Adjacent\ side}{Hypotenuse} \text{ is used.}$$

Exercise 6. 2

1. Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height $10\sqrt{3}\,m$.

Sol :



From the right $\Delta\, ABC$

$$\tan\theta = \frac{Opposite\ side}{Adjacent\ side} = \frac{AC}{BC}$$

$$= \frac{10\sqrt{3}\,m}{30\,m} = \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\,\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

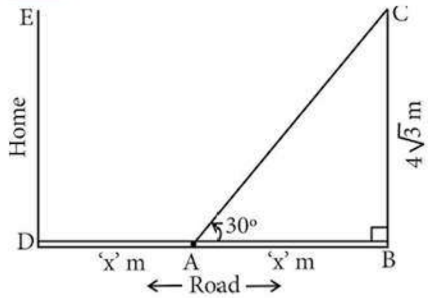
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = 30^\circ$$

$\therefore$ Angle of elevation is 30°

2. A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the road facing a row house. The angle of elevation from the pedestrian to the top of the house is 30°. Find the width of the road.

Sol :



Let AB = x be the distance between foot of the house and the observer at the median of the road.

$\therefore$ DB = 2x is the width of the road.

Height of the house BC = $4\sqrt{3}\,m$

From the right triangle $\Delta\, ABC$

$$\therefore \quad \tan 30^\circ = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{x}$$

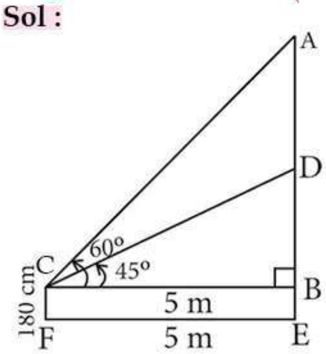
$$x = 4\sqrt{3}\times\sqrt{3} = 4\times 3 = 12 \text{ m}$$

Width of the road $= 2\times x = 2\times 12 = 24$ m

$\therefore$ Width of the road = 24 m.

3. To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? $\left(\sqrt{3} = 1.732\right)$

Sol :



Let CF be the height of the man; AD be the height of the window; BC is the distance between the observer and the house.

From the right triangle ΔCBD

$$\tan 45^\circ = \frac{DB}{BC}$$

$$1 = \frac{DB}{5}$$

$$DB = 5 \text{ m} \qquad \dots (1)$$

From the right triangle CBA

$$\tan 60^\circ = \frac{AB}{CB}$$

$$\sqrt{3} = \frac{AD+DB}{5}$$

$$5\sqrt{3} = AD + 5 \quad [\because \textit{from } (1)\ DB = 5m]$$

$$AD = 5\sqrt{3} - 5 = 5\left(\sqrt{3}-1\right)$$

$$AD = 5\,(1.732 - 1)$$

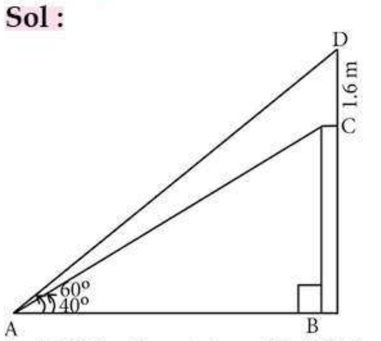
$$\left[Given\ \sqrt{3} = 1.732\right]$$

$$= 5 \times 0.732 = 3.660$$

$\therefore$ Height of the window = 3.66 m

4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 40°. Find the height of the pedestal. $\left(\tan 40^\circ = 0.8391, \sqrt{3} = 1.732\right)$

Sol :



Let CD be the statue of tall 1.6 m.

BC be the pedestal.

From the right triangle $\Delta\ ABC$

$$\tan 40^\circ = \frac{BC}{AB}$$

$$0.8391 = \frac{BC}{AB}$$

$$AB = \frac{BC}{0.8391} \qquad \dots (1)$$

From the right triangle $\Delta\ ABD$

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\sqrt{3} = \frac{BC+CD}{AB}$$

$$1.732 = \frac{BC+1.6}{AB}$$

$$AB = \frac{BC+1.6}{1.732} \qquad \dots (2)$$

From (1) and (2)

$$\frac{BC}{0.8391} = \frac{BC+1.6}{1.732}$$

$$1.732\ BC = 0.8391\ (BC + 1.6)$$

$$1.732\ BC = 0.8391\ BC + (0.8391)\ (1.6)$$

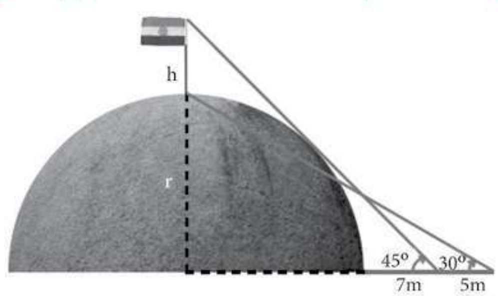
$$1.732\ BC - 0.8391\ BC = 1.34256$$

$$0.8929\ BC = 1.34256$$

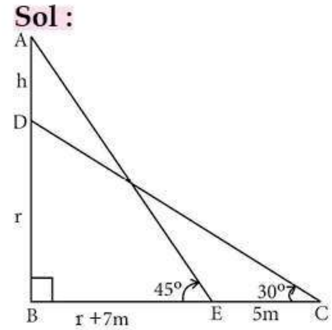
$$BC = \frac{1.34256}{0.8929} = \frac{13425.6}{8929} = 1.5 \text{ m}$$

$\therefore$ Height of the pedestal = 1.5 m

5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle 45° and moving 5 m away from the dome and seeing the bottom of the pole at an angle 30°. Find (i) the height of the pole (ii) radius of the dome. $\left(\sqrt{3} = 1.732\right)$



Sol :



Let BD be the radius of the dome AD is the flag pole of height 'h' m.

(i) From the right triangle $\Delta\, ABE$

$$\tan 45^{o} = \frac{AB}{BE}$$

$$1 = \frac{r+h}{r+7}$$

$$r + 7 = r + h$$

$$r + h - r = 7$$

$$h = 7\text{ m}$$

$\therefore$ Height of the flag pole = 7 m

(ii) From the right triangle $\Delta\, BDC$

$$\tan 30^{o} = \frac{BD}{BC} = \frac{r}{r+7+5}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{r+12}$$

$$r + 12 = \sqrt{3}\, r$$

$$12 = \sqrt{3}\, r - r$$

$$12 = r\,(\sqrt{3}-1)$$

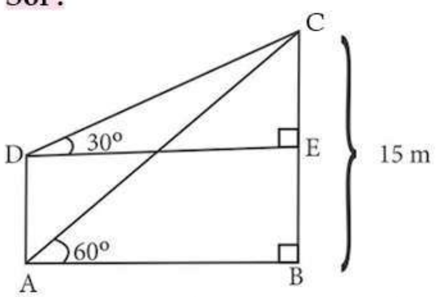
$$r = \frac{12}{1.732-1} = \frac{12}{0.732} = 16.39\text{ m}$$

Height of pole = 7 m

$\therefore$ Radius of dome = 16.39 m.

6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

Sol :



Let AD be the electronic pole and BC be the tower.

From the right triangle $\Delta\, ABC$

$$\tan 60^{o} = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{15}{AB}$$

$$AB = \frac{15}{\sqrt{3}} \qquad \dots (1)$$

From the right triangle $\Delta\, DEC$

$$\tan 30^{o} = \frac{CE}{DE}$$

$$\frac{1}{\sqrt{3}} = \frac{CB - EB}{AB} \qquad [\because\ DE = AB \text{ and } CE = CB - EB]$$

$$\frac{1}{\sqrt{3}} = \frac{15 - DA}{AB} \qquad [\because\ EB = DA]$$

$$AB = (15 - AD)\sqrt{3} \qquad \dots (2)$$

From (1) and (2)

$$\frac{15}{\sqrt{3}} = (15 - AD)\sqrt{3}.$$

$$15 = (15 - AD)\sqrt{3}\,.\sqrt{3}$$

$$= (15 - AD)\,3$$

$$15 = 45 - 3AD$$

$$3AD = 45 - 15$$

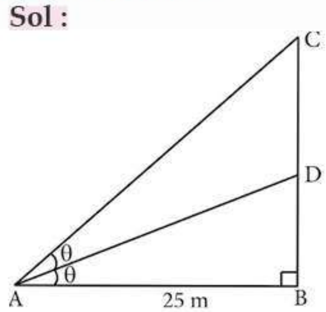
$$3AD = 30$$

$$AD = \frac{30}{3} = 10\text{ m}$$

$\therefore$ Height of the electric pole = 10 m.

7. A vertical pole fixed to the ground is divided in the ratio 1 : 9 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a place on the ground, 25 m away from the base of the pole, what is the height of the pole?

Sol :



Let CB be the pole and point D divides it such that

$$BD : DC = 1 : 9$$

Given that $AB = 25$m

Let the two parts subtend equal angles at point A such that $CAD = \angle BAD = \theta$

By angle Bisector theorem, we have

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$\Rightarrow \frac{1}{9} = \frac{25}{AC}$$

$[\because BD = DC = 1:9 \text{ and } AB = 25m]$

$$AC = 9 \times 25 \text{ m} \quad \ldots(1)$$

From the right triangle ΔABC

$$CB = \sqrt{AC^2 - AB^2} \quad [\because \textit{Pythagorus Theorem}]$$

$$= \sqrt{(25\times 9)^2 - 25^2} \quad [\because \textit{From } (1)]$$

$$= \sqrt{25^2 \times 9^2 - 25^2}$$

$$= \sqrt{25^2(9^2-1)} = 25\sqrt{81-1}$$

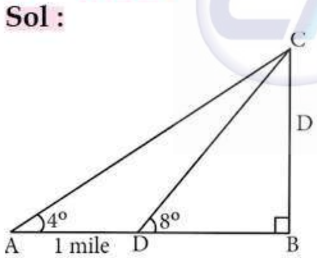
$$= 25 \times \sqrt{80} = 25 \times 4\sqrt{5}$$

$$= 100\sqrt{5}\ m$$

$\therefore$ Height of the pole = $100\sqrt{5}\ m$.

8. A traveler approaches a mountain on highway. He measures the angle of elevation to the peak at each milestone. At two consecutive milestones the angles measured are 4° and 8°. What is the height of the peak if the distance between consecutive milestones is 1 mile (tan 4° = 0.0699, tan 8° = 0.1405)

Sol :



Let BC be the mountain

$$AD = 1 \text{ mile}$$

From the right triangle ΔABC

$$\tan 4^\circ = \frac{BC}{AB}$$

$$0.0699 = \frac{BC}{AD + DB} \quad [\text{Given } \tan 4^\circ = 0.0699]$$

$$0.0699 = \frac{BC}{1 + DB}$$

$$0.0699\,(1 + BD) = BC \quad \ldots (1)$$

From the right triangle ΔDBC

$$\tan 8^\circ = \frac{BC}{BD}$$

$$0.1405 = \frac{BC}{BD}$$

$$BC = 0.1405\ BD \quad \ldots (2)$$

From (1) and (2)

$$0.0699\,(1 + BD) = 0.1405\ BD$$

$$0.0699 + 0.0699\ BD = 0.1405\ BD$$

$$0.0699 = 0.1405\ BD - 0.0699\ BD$$

$$0.0699 = 0.0706\ BD$$

$$\therefore BD = \frac{0.0699}{0.0706} = 0.99 \text{ mile.}$$

From (2)

$$BC = 0.1405 \times 0.99 = 0.1390$$

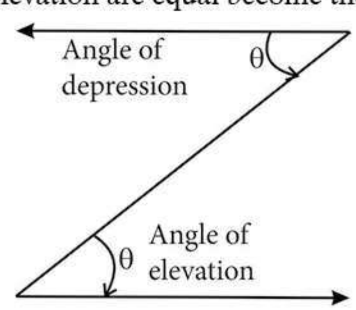
$$= 0.14 \text{ miles}$$

$\therefore$ Height of the peak = 0.14 miles approximately.

PROBLEMS INVOLVING ANGLE OF DEPRESSION

Key Points

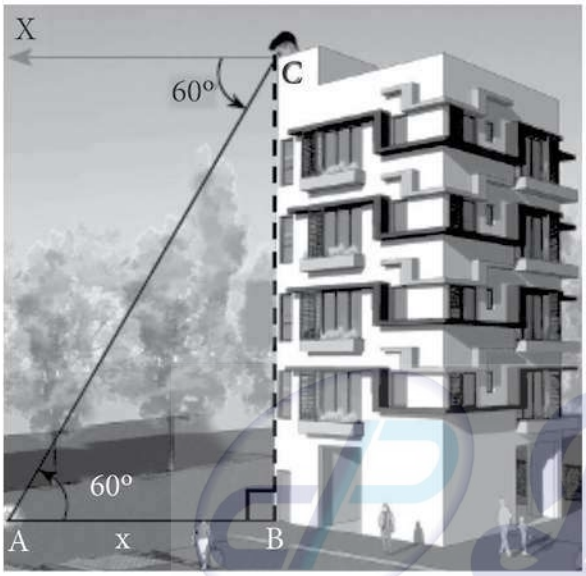
Angle of Depression and Angle of Elevation are equal become they are alternative Angles.



Worked Examples

6.26 **A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60°. Find the distance between the foot of the tower and the ball. ($\sqrt{3}$ = 1.732)**

Sol :



Let BC be the height of the tower and A be the position of the ball lying on the ground. Then,

BC = 20 m and $\angle XCA = 60° = \angle CAB$

Let AB = x metres.

In right triangle ABC,

$$\tan 60° = \frac{BC}{AB}$$

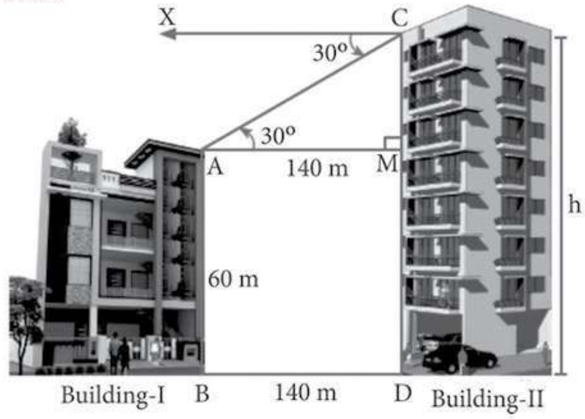
$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{20 \times 1.732}{3} = 11.54 \text{ m}$$

Hence, the distance between the foot of the tower and the ball is 11.54 m.

6.27 **The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30°. If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3}$ = 1.732)**

Sol :



The height of the first building AB = 60 m. Now, AB = MD = 60 m.

Let the height of the second building CD = h.

Distance BD = 140 m

Now, AM = BD = 140 m

From the diagram,

$$\angle XCA = 30° = \angle CAM$$

In right triangle AMC, $\tan 30° = \frac{CM}{AM}$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

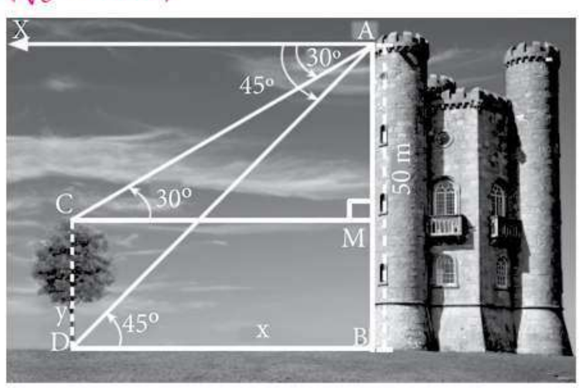
$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3} = \frac{140 \times 1.732}{3}$$

$$CM = 80.78$$

Now, h = CD = CM + MD = 80.78 + 60 = 140.78

Therefore the height of the second building is 140.78 m

6.28 **From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. ($\sqrt{3}$ = 1.732)**



Sol :

The height of the tower AB = 50 m

Let the height of the tree CD = y and BD = x

From the diagram, $\angle XAC = 30^\circ = \angle ACM$ and $\angle XAD = 45^\circ = \angle ADB$

In right triangle ABD,

$$\tan 45^\circ = \frac{AB}{BD} \Rightarrow 1 = \frac{50}{x} \Rightarrow x = 50 \text{ m}$$

In right triangle AMC,

$$\tan 30^\circ = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \qquad [\text{since } DB = CM]$$

$$AM = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3} = \frac{50 \times 1.732}{3} = 28.85 \text{ m}$$

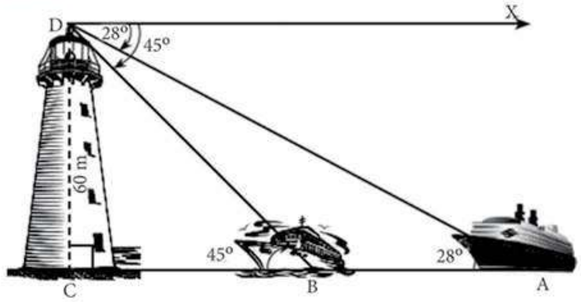
Therefore,

Height of the tree = CD = MB = AB – AM

= 50 – 28.85 = 21.15 m

6.29 **As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are 28° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. ($\tan 28^\circ = 0.5317$)**

Sol :



Let the observer on the lighthouse CD be at D.

Height of the lighthouse CD = 60 m

From the diagram,

$$\angle XDA = 28^\circ = \angle DAC \text{ and}$$
$$\angle XDB = 45^\circ = \angle DBC$$

In right triangle DCB,

$$\tan 45^\circ = \frac{DC}{BC}$$

$$1 = \frac{60}{BC} \Rightarrow BC = 60 \text{ m}$$

In right triangle DCA, $\tan 28^\circ = \dfrac{DC}{AC}$

$$0.5317 = \frac{60}{AC}$$

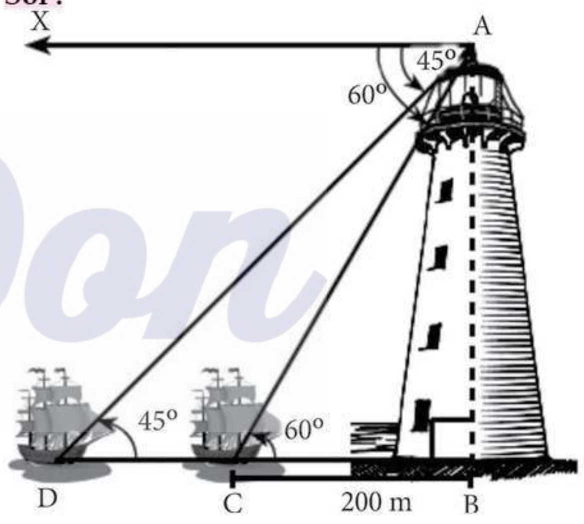
$$AC = \frac{60}{0.5317} = 112.85$$

Distance between the two ships

AB = AC – BC = 52.85 m

6.30 **A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of 60° with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression becomes 45°. What is the approximate speed of the boat (in km/hr), assuming that it is sailing in still water? ($\sqrt{3} = 1.732$)**

Sol :



Let AB be the tower.

Let C and D be the positions of the boat.

From the diagram,

$$\angle XAC = 60^\circ = \angle ACB \text{ and}$$
$$\angle XAD = 45^\circ = \angle ADB, \; BC = 200 \text{ m}$$

In right triangle ABC, $\tan 60^\circ = \dfrac{AB}{BC}$

$$\sqrt{3} = \frac{AB}{200}$$

$$\Rightarrow \quad AB = 200\sqrt{3} \qquad \ldots (1)$$

In right triangle ABD, $\tan 45^\circ = \dfrac{AB}{BD}$

$$\Rightarrow \quad 1 = \frac{200\sqrt{3}}{BD} \qquad [\text{by (1)}]$$

$$\Rightarrow \quad BD = 200\sqrt{3}$$

Now, $\quad CD = BD - BC$

$CD = 200\sqrt{3} - 200 = 200(\sqrt{3}-1) = 146.4$

It is given that the distance CD is covered in 10 seconds.

That is, the distance of 146.4 m is covered in 10 seconds.

Therefore, speed of the boat $= \frac{distance}{time}$

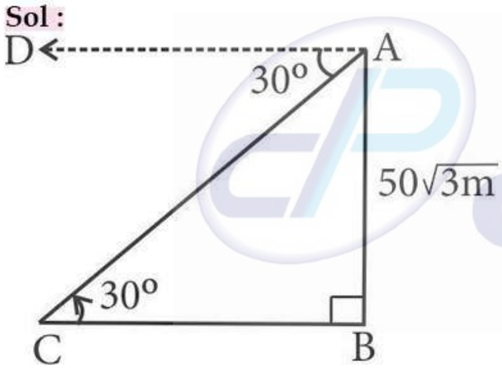
$= \frac{146.4}{10} = 14.64$ m/s $\Rightarrow$ $14.64 \times \frac{3600}{1000}$ km/hr

$= 52.704$ km/hr

Exercise 6. 3

1. From the top of a rock $50\sqrt{3}\,m$ high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.

Sol :



Let AB be the rock.

C be the position of the car.

$$\angle DAC = \angle ACB = 30^\circ$$

In right triangle $\Delta\, ABC$

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{50\sqrt{3}}{BC}$$

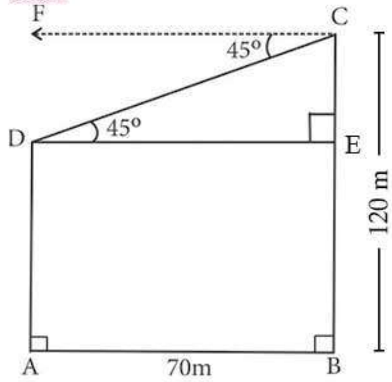
$$BC = 50\sqrt{3} \times \sqrt{3}$$

$$= 50 \times 3 = 150 \text{ m}$$

$\therefore$ Distance of the car from the rock = 150 m.

2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.

Sol :



Let AD is the first building.

BC is the second building.

$$AD = BE = BC - CE$$

$$\angle FCD = \angle CDE = 45^\circ$$

From the right triangle ΔCED

$$\tan 45^\circ = \frac{CE}{DE}$$

$$1 = \frac{CE}{AB} = \frac{CE}{70\,m}$$

$$CE = 70 \text{ m}$$

$\therefore$ $BE = BC - EC$

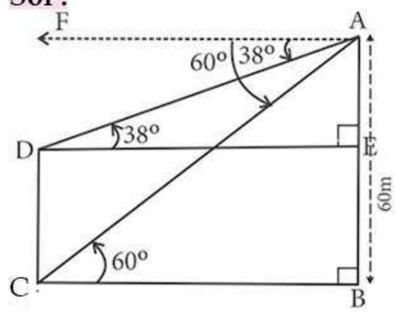
$= 120$ m $- 70$ m $= 50$ m

$\therefore$ $AD = 50$ m

$\therefore$ Height of the first building is 50 m.

3. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60° respectively. Find the height of the lamp post. (tan 38° = 0.7813, $\sqrt{3}$ = 1.732)

Sol :



Let AB be the building of height 60 m.

DC be the lamp post.

$$DC = BE$$

$$\angle FAD = \angle ADE = 38^\circ$$

$$\angle FAC = \angle ACB = 60^\circ$$

In the right triangle ΔADE

$$\tan 38^\circ = \frac{AE}{DE}$$

$$0.7813 = \frac{AE}{CB}$$

$$CB = \frac{AE}{0.7813} \quad \text{... (1)}$$

From the right triangle ΔACB

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{60}{BC}$$

$$BC = \frac{60}{\sqrt{3}} = \frac{60\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$= \frac{60\sqrt{3}}{3}$$

$$CB = 20\sqrt{3} \quad \text{... (2)}$$

From (1) and (2)

$$\frac{AE}{0.7813} = 20\sqrt{3}$$

$$AE = 20\times 1.732\times 0.7813$$

$$= 34.64\times 0.7813 = 27.064232 = 27.06 \text{ m}$$

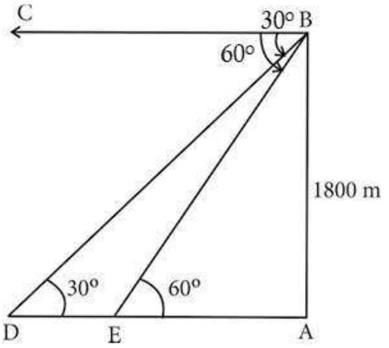
Now height of the lamp post

= DC = EB = AB – AE = 60 – 27.06 = 32.93 m

$\therefore$ Height of the lamp post = 32.93 m

4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3}$ = 1.732)

Sol :



Let AB = 1800 m be the height where the aeroplane is flying D and E are positions of two boats.

$$\angle CBD = \angle BDA = 30^\circ$$

$$\angle CBE = \angle BEA = 60^\circ$$

In right triangle ΔBAE

$$\tan 60^\circ = \frac{BA}{EA}$$

$$\sqrt{3} = \frac{1800}{EA}$$

$$EA = \frac{1800}{\sqrt{3}} \quad \text{... (1)}$$

In right triangle ΔBDA

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{1800}{DE+EA}$$

$$DE + EA = 1800\sqrt{3}$$

$$DE = 1800\sqrt{3} - EA$$

$$DE = 1800\sqrt{3} - \frac{1800}{\sqrt{3}} \quad [\because \text{From (1)}]$$

$$= \frac{1800\sqrt{3}\sqrt{3}-1800}{\sqrt{3}}$$

$$= \frac{1800\times 3-1800}{\sqrt{3}}$$

$$= \frac{5400-1800}{\sqrt{3}} = \frac{3600}{\sqrt{3}}$$

$$= \frac{3600\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{3600\sqrt{3}}{3} = 1200\sqrt{3}$$

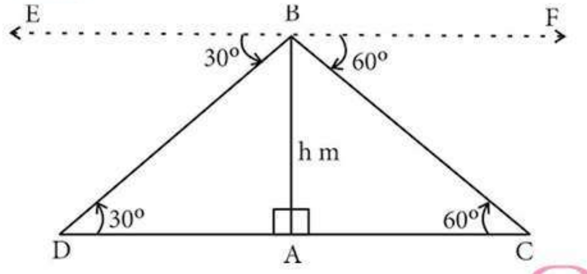
$$= 1200\times 1.732$$

$$DE = 2078.4 \text{ m}$$

$\therefore$ Distance between the boats = 2078.4 m

5. From the top of a lighthouse, the angles of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ships is $\frac{4h}{\sqrt{3}}m$.

Sol :



Let D and C be the positions of two ships AB be the light house of height 'h' m.

$$\angle EBD = \angle BDA = 30^\circ$$
$$\angle FBC = \angle BCA = 60^\circ$$

In right triangle BAC

$$\tan 60^\circ = \frac{AB}{AC}$$
$$\sqrt{3} = \frac{h}{AC}$$
$$AC = \frac{h}{\sqrt{3}} \quad \text{... (1)}$$

In right triangle ΔBAD

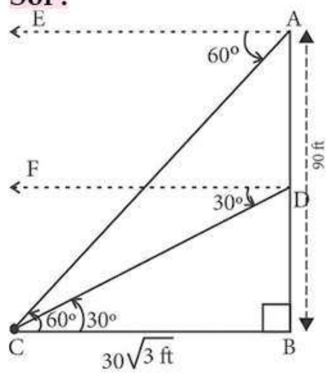
$$\tan 30^\circ = \frac{AB}{AD}$$
$$\frac{1}{\sqrt{3}} = \frac{h}{AD}$$
$$AD = h\sqrt{3} \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow \quad AC + AD = \frac{h}{\sqrt{3}} + h\sqrt{3}$$
$$DC = \frac{h + h\sqrt{3}\sqrt{3}}{\sqrt{3}}$$
$$DC = \frac{h+3h}{\sqrt{3}} = \frac{4h}{\sqrt{3}}\,m$$

$\therefore$ Distance between the ships is $\frac{4h}{\sqrt{3}}\,m$.

6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.

Sol :



Let AB be the building of height 90 ft.
AD is the distance descending by the lift in 2 minutes.

$$\angle EAC = \angle ACB = 60^\circ$$
$$\angle FDC = \angle DCB = 30^\circ$$

In right triangle ΔDCB

$$\tan 30^\circ = \frac{DB}{CB}$$
$$\frac{1}{\sqrt{3}} = \frac{DB}{30\sqrt{3}}$$
$$\frac{30\sqrt{3}}{\sqrt{3}} = DB$$

1 feet	= 30.5 cm
	= 0.305 m

$$DB = 30 \text{ ft}$$
$$\therefore \quad AD = AB - DB = 90 - 30 = 60 \text{ ft}$$

Distance covered $= 60 \times 0.305$ m

Time taken $= 2$ min $= 2 \times 60$ sec

$$\therefore \text{ Speed of the lift} = \frac{Distance}{Time}$$
$$= \frac{60 \times 0.305}{2 \times 60}\,m/s = 0.1525 \text{ m/s} = 0.15 \text{ m/s}$$

Speed of the lift = 0.15 m/s.

Another Method

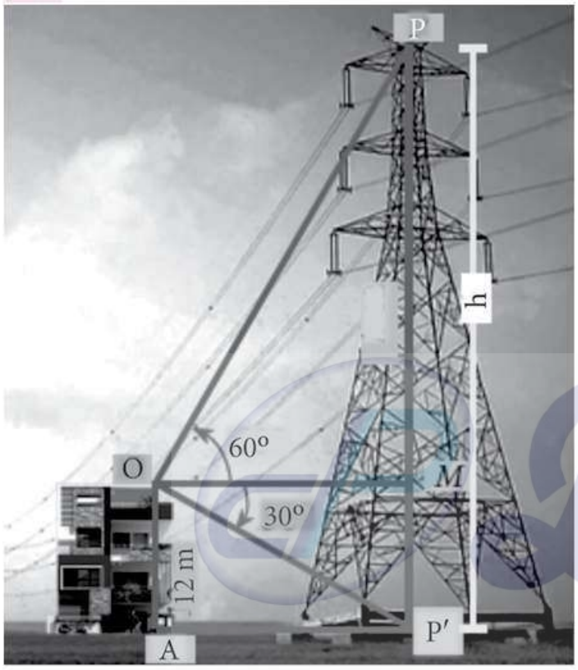
$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{60}{2 \times 60} \text{ ft/sec}$$
$$= \frac{1}{2} \text{ ft/sec}$$
$$= 0.5 \text{ ft/sec}$$

PROBLEMS INVOLVING ANGLE OF ELEVATION AND DEPRESSION

Worked Examples

6.31 **From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.**

Sol :



As shown in Figure, OA is the building, O is the point of observation on the top of the building OA. PP' is the cable tower with P as the top and P' as the bottom.

Then the angle of elevation of P, $\angle MOP = 60°$

And the angle of depression of P', $\angle MOP' = 30°$

Suppose, height of the cable tower

PP' = h metres.

Through O, draw $OM \perp PP'$

$$MP = PP' - MP' = h - OA = h - 12$$

In right triangle OMP, $\frac{MP}{OM} = \tan 60°$

$$\Rightarrow \quad \frac{h-12}{OM} = \sqrt{3}$$

$$\Rightarrow \quad OM = \frac{h-12}{\sqrt{3}} \quad \text{... (1)}$$

Similarly in right triangle OMP', $\frac{MP'}{OM} = \tan 30°$

$$\Rightarrow \quad \frac{12}{OM} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad OM = 12\sqrt{3} \quad \text{... (2)}$$

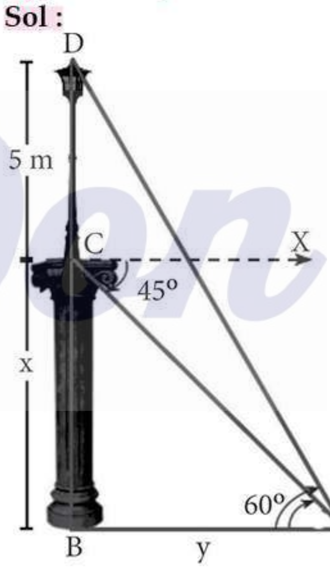
From (1) and (2) we have, $\frac{h-12}{\sqrt{3}} = 12\sqrt{3}$

$$\Rightarrow \quad h - 12 = 12\sqrt{3} \times \sqrt{3} \Rightarrow h = 48$$

Hence, the required height of the cable tower is 48 m.

6.32 **A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point 'A' on the ground is 60° and the angle of depression to the point 'A' from the top of the tower is 45°. Find the height of the tower. ($\sqrt{3}$ = 1.732)**

Sol :



Let BC be the height of the tower and CD be the height of the pole.

Let 'A' be the point of observation.

Let BC = x and AB = y.

From the diagram,

$\angle BAD = 60°$ and $\angle XCA = 45° = \angle BAC$

In right triangle ABC, $\tan 45° = \frac{BC}{AB}$

$$\Rightarrow \quad 1 = \frac{x}{y} \Rightarrow x = y \quad \text{... (1)}$$

In right triangle ABD,

$$\tan 60° = \frac{BD}{AB} = \frac{BC+CD}{AB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x+5}{y} \Rightarrow \sqrt{3}\,y = x + 5$$

$\Rightarrow \quad \sqrt{3}\ x = x + 5 \quad$ [From (1)]

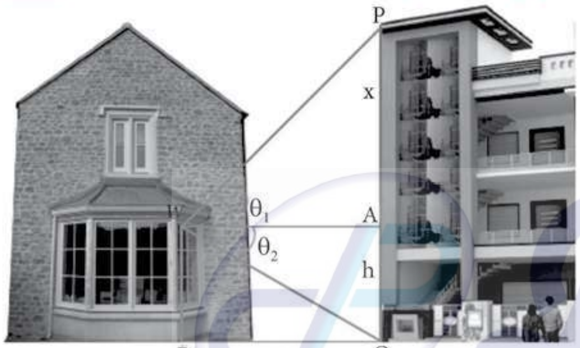
$\Rightarrow x = \dfrac{5}{\sqrt{3}-1} = \dfrac{5}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \dfrac{5\,(1.732+1)}{2}$

$= 6.83$

Hence, height of the tower is 6.83 m.

6.33 From a window (h meters high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is h $\left(1+\dfrac{\cot\theta_2}{\cot\theta_1}\right)$.

Sol :



Let W be the point on the window where the angles of elevation and depression are measured.

Let PQ be the house on the opposite side.

Then WA is the width of the street.

Height of the window = h metres = AQ

(WR = AQ)

Let PA = x metres.

In right triangle PAW,

$$\tan\theta_1 = \frac{AP}{AW}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{x}{AW} \Rightarrow AW = \frac{x}{\tan\theta_1}$$

$$\Rightarrow \quad AW = x\cot\theta \qquad \text{... (1)}$$

In right triangle QAW,

$$\tan\theta_2 = \frac{AQ}{AW}$$

$$\Rightarrow \quad \tan\theta_2 = \frac{h}{AW}$$

$$\Rightarrow \quad AW = h\cot\theta_2 \qquad \text{... (2)}$$

From (1) and (2) we get, x cot θ_1 = h cot θ_2

$$\Rightarrow \quad x = h\frac{\cot\theta_2}{\cot\theta_1}$$

Therefore, height of the opposite house

$$= PA + AQ = x + h = h\frac{\cot\theta_2}{\cot\theta_1} + h$$

$$= h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right)$$

Hence Proved.

Progress Check

1. **The line drawn from the eye of an observer to the point of object is _____**

   Ans : Line of sight

2. **Which instrument is used in measuring the angle between an object and the eye of the observer?**

   Ans : Theodolite

3. **When the line of sight is above the horizontal level, the angle formed is ______**

   Ans : Angle of elevation

4. **The angle of elevation _____ as we move towards the foot of the vertical object (tower).**

   Ans : Increases

5. **When the line of sight is below the horizontal level, the angle formed is _____**

   Ans : Angle of Depression.

Thinking Corner

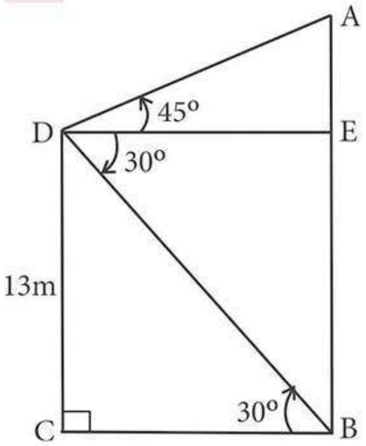
1. **What is the minimum number of measurements required to determine the height or distance or angle of elevation?**

   Ans : Any two measurements are needed in minimum to find the other.

Exercise 6. 4

1. **From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the second tree. ($\sqrt{3}$ = 1.732)**

Sol :



Let CD is the tree of height 13 m.

AB is another tree.

$$\angle ADE = 45^\circ$$

$$\angle EDB = \angle DBC = 30^\circ$$

CB = DE and CD = EB = 13 m

In right triangle ΔAED

$$\tan 45^\circ = \frac{AE}{DE}$$

$$1 = \frac{AE}{DE}$$

$$AE = DE \quad \text{... (1)}$$

In the right triangle ΔDBC

$$\tan 30^\circ = \frac{DC}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{13}{BC}$$

$$BC = 13\sqrt{3}\ m$$

$\therefore$ From (1) AE $= 13\sqrt{3}\ m$

[$\because$ BC = DE = AE]

Height of the tree = AB = AE + EB

$= (13\sqrt{3} + 13)\ m$

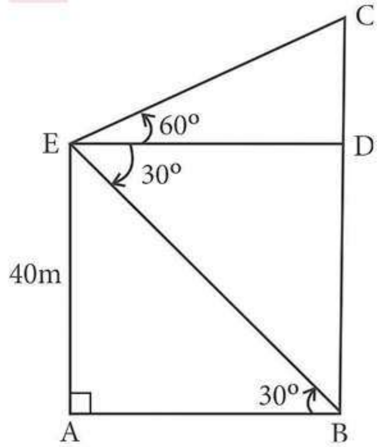
$= (13 \times 1.732 + 13)$ m

$= (22.516 + 13)$ m = 35.516 m = 35.52 m

$\therefore$ Height of the second tree = 35.52 m

2. A man is standing on the deck of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill. $(\sqrt{3} = 1.732)$

Sol :



Let a man is standing on the deck of a ship at a point E

Such that AE = 40 m.

$\therefore$ AE = BD = 40 m

Also AB = ED

Let BC be the height of the hill.

(i) In right triangle ΔABE

$$\tan 30^\circ = \frac{AE}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{40}{AB}$$

$$AB = 40\sqrt{3}\ m \quad \text{... (1)}$$

$$AB = 40 \times 1.732 = 69.28 \text{ m}$$

$\therefore$ Distance of the hill from the ship = 69.28 m

(ii) In the right triangle ΔCDE

$$\tan 60^\circ = \frac{CD}{ED}$$

$$\sqrt{3} = \frac{CD}{40\sqrt{3}} \quad [\because AB = ED = 40\sqrt{3}\ m]$$

CD $= 40\sqrt{3} \times \sqrt{3} = 40 \times 3 = 120$ m

Now height of the hill = BC = BD + DC

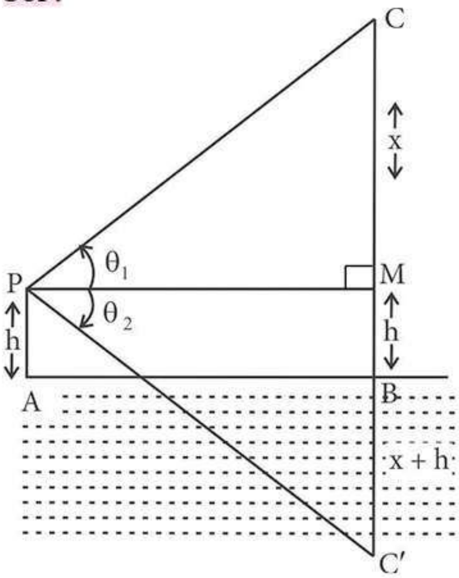
= 40 + 120 = 160 m

$\therefore$ Height of the hill = 160 m

Distance of the hill from ship = 69.28 m.

3. If the angle of elevation of a cloud from a point 'h' metres above a lake is θ_1 and the angle of depression of its reflection in the lake is θ_2. Prove that the height that the cloud is located from the ground is $\frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$.

Sol :



Let θ_1 be the angle of elevation of the cloud from P and θ_2 be the angle of depression.

PA = MB = 'h' m

BC be the height of the cloud from earth.

C' be the reflection of the cloud.

$\therefore \quad BC' = $ 'x + h' m

In right triangle ΔCPM

$$\tan\theta_1 = \frac{CM}{PM}$$

$$\tan\theta_1 = \frac{x}{AB}$$

$$AB = x\cot\theta_1 \qquad \dots(1)$$

In right triangle $\Delta PMC'$

$$\tan\theta_2 = \frac{C'M}{PM}$$

$$= \frac{x+h+h}{AB}$$

$$\tan\theta_2 = \frac{x+2h}{AB}$$

$$AB = (x+2h)\cot\theta_2 \qquad \dots(2)$$

From (1) and (2), we have

$$x\cot\theta_1 = (x+2h)\cot\theta_2$$

$$x\cot\theta_1 = x\cot\theta_2 + 2h\cot\theta_2$$

$$x\cot\theta_1 - x\cot\theta_2 = 2h\cot\theta_2$$

$$x(\cot\theta_1 - \cot\theta_2) = 2h\cot\theta_2$$

$$x\left(\frac{1}{\tan\theta_1} - \frac{1}{\tan\theta_2}\right) = \frac{2h}{\tan\theta_2} \quad \left[\because \frac{1}{\tan\theta} = \cot\theta\right]$$

$$x\left(\frac{\tan\theta_2 - \tan\theta_1}{\tan\theta_1\tan\theta_2}\right) = \frac{2h}{\tan\theta_2}$$

$$x = \frac{2h\tan\theta_1\tan\theta_2}{\tan\theta_2(\tan\theta_2 - \tan\theta_1)}$$

$$x = \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

Hence the height CB of the cloud is given by

$$CB = x + h$$

$$= \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1} + h$$

$$= \frac{2h\tan\theta_1 + h(\tan\theta_2 - \tan\theta_1)}{\tan\theta_2 - \tan\theta_1}$$

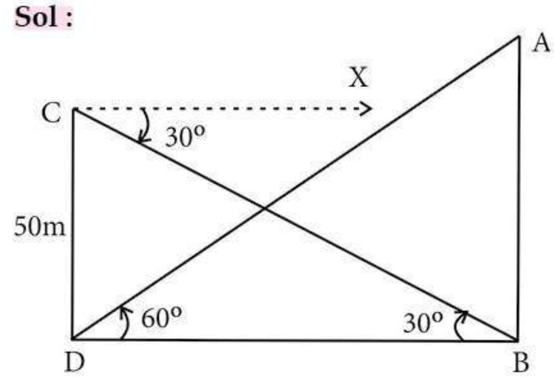
$$= \frac{2h\tan\theta_1 + h\tan\theta_2 - h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

$$= \frac{h\tan\theta_1 + h\tan\theta_2}{\tan\theta_2 - \tan\theta_1} = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

$$\therefore \text{ The required height } = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is 60° and the angle of depression of the foot of the tower from the top of the apartment is 30°. If the height of the apartment is 50 m, find the height of the cell phone tower. According to Radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

Sol :



Let AB be the cell phone tower.

CD be the apartment.

$$\angle XCB = \angle CBD = 30^\circ$$

$$\angle ADB = 60^\circ$$

In right triangle ABD

$$\tan 60^\circ = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{AB}{BD}$$

$$BD = \frac{AB}{\sqrt{3}} \qquad \text{... (1)}$$

In the right triangle ΔCDB

$$\tan 30^\circ = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{BD}$$

$$BD = 50\sqrt{3} \qquad \text{... (2)}$$

From (1) and (2)

$$\frac{AB}{\sqrt{3}} = 50\sqrt{3}$$

$AB = 50 \times \sqrt{3} \times \sqrt{3} = 50 \times 3 = 150$ m

$\therefore$ Height of the cell phone tower = 150 m.

Since height of the tower > 120 m, yes, the tower meets the radiation norms.

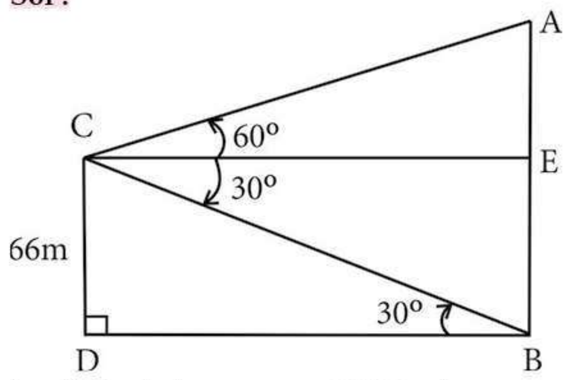
5. The angles of elevation and depression of the top and bottom of a Lamp post from the top of a 66 m high apartment are 60° and 30° respectively. Find

(i) The height of the Lamp post.

(ii) The difference between height of the Lamp post and the apartment.

(iii) The distance between the Lamp post and the apartment. ($\sqrt{3} = 1.732$)

Sol :



Let AB be the lamp post and CD be the apartment given CD = 66 m = EB.

$$\angle ACE = 60^\circ$$

$$\angle ECB = \angle CBD = 30^\circ$$

(i) In the right triangle ΔBDC

$$\tan 30^\circ = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{66}{BD}$$

$$BD = 66\sqrt{3} \qquad \text{... (1)}$$

$$= 66 \times 1.732 = 114.312 \text{ m}$$

$\therefore$ The distance between the lamp post and the apartment = 114.31 m

Now $BD = EC = 114.31$ m

In the right triangle ΔACE

$$\tan 60^\circ = \frac{AE}{CE}$$

$$\sqrt{3} = \frac{AE}{66\sqrt{3}}$$

$$AE = 66\sqrt{3} \times \sqrt{3} \quad \text{[From (1)]}$$

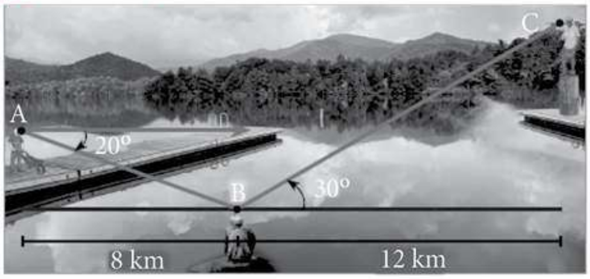
$$= 66 \times 3 = 198 \text{ m}$$

(i) Height of the lamp post = AB
= AE + EB = 198 + 66 = 264 m

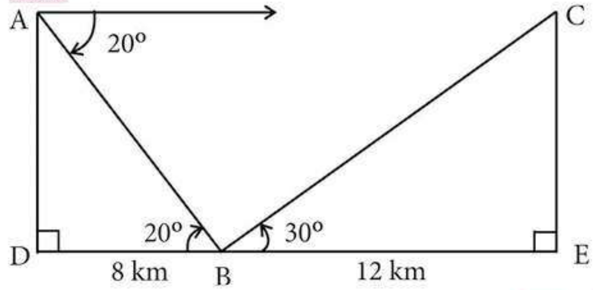
(ii) The difference between lamp post and apartment = 198 m.

(iii) The distance between the lamp post and apartment = 114.31 m.

6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°. Calculate: (i) the vertical height between A and B. (ii) the vertical height between B and C. (tan 20° = 0.3640, $\sqrt{3}$ = 1.732)



Sol :



In the right ΔADB

$$\tan 20° = \frac{AD}{DB}$$

$$0.3640 = \frac{AD}{8}$$

$$AD = 8 \times 0.3640 = 2.91 \text{ km}$$

Vertical height between A and B = 2.91 km.

(ii) In the right ΔCEB

$$\tan 30° = \frac{CE}{BE}$$

$$\frac{1}{\sqrt{3}} = \frac{CE}{12}$$

$$CE = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{12 \times 1.732}{3} = 4 \times 1.732 = 6.928 = 6.93 \text{ km}$$

Vertical height between B and C = 6.93 km.

Exercise 6. 5

Multiple Choice Questions:

1. The value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$ is equal to

(1) $\tan^2\theta$ (2) 1
(3) $cot^2\theta$ (4) 0 **[Ans : (2)]**

Sol :

$$\sin^2\theta + \frac{1}{1+\tan^2\theta} = \sin^2\theta + \frac{1}{\sec^2\theta}$$

$$= \sin^2\theta + \cos^2\theta = 1$$

2. $\tan\theta\, cosec^2\theta - \tan\theta$ is equal to

(1) $\sec\theta$ (2) $\cot^2\theta$
(3) $\sin\theta$ (4) $\cot\theta$ **[Ans : (4)]**

Sol :

$$\tan\theta\, cosec^2\theta - \tan\theta = \tan\theta\,(cosec^2\theta - 1)$$

$$= \tan\theta\cot^2\theta$$

$$= \tan\theta \times \frac{1}{\tan^2\theta}$$

$$= \frac{1}{\tan\theta}$$

$$= \cot\theta$$

3. If $(\sin\alpha + cosec\,\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = k + \tan^2\alpha + \cot^2\alpha$, then the value of k is equal to

(1) 9 (2) 7
(3) 5 (4) 3 **[Ans : (2)]**

Sol : $(\sin\alpha + cosec\,\alpha)^2 + (\cos\alpha + \sec\alpha)^2$

$$= \sin^2\alpha + cosec^2\alpha + 2\sin\alpha\, cosec\,\alpha + \cos^2\alpha + \sec^2\alpha + 2\cos\alpha\sec\alpha$$

$$= (\sin^2\alpha + \cos^2\alpha) + cosec^2\alpha + \sec^2\alpha + 2\sin\alpha\frac{1}{\sin\alpha} + 2\cos\alpha\frac{1}{\cos\alpha}$$

$$= 1 + cosec^2\alpha + \sec^2\alpha + 2 + 2$$

$$= 5 + 1 + \cot^2\alpha + 1 + \tan^2\alpha$$

$$= 7 + \tan^2\alpha + \cot^2\alpha$$

comparing with $k + \tan^2\alpha + \cot^2\alpha$

k = 7

4. If $\sin\theta + \cos\theta = a$ and $\sec\theta + \text{cosec}\,\theta = b$, then the value of b $(a^2 - 1)$ is equal to

(1) 2a (2) 3a
(3) 0 (4) 2ab **[Ans : (1)]**

Sol :

$$b(a^2 - 1) = (\sec\theta + cosec\,\theta)[(\sin\theta + \cos\theta)^2 - 1]$$

$$= (\sec\theta + cosec\,\theta)[\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1]$$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$$

$$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} \times 2\sin\theta\cos\theta = 2a$$

5. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, then $x^2 - \frac{1}{x^2}$ is equal to

(1) 25 (2) $\frac{1}{25}$
(3) 5 (4) 1 **[Ans : (2)]**

Sol : $5x = \sec\theta \Rightarrow 25x^2 = \sec^2\theta$ ... (1)

$\frac{5}{x} = \tan\theta \Rightarrow \frac{25}{x^2} = \tan^2\theta$ ... (2)

Subtract (1) and (2)

$$\Rightarrow 25x^2 - \frac{25}{x^2} = \sec^2\theta - \tan^2\theta$$

$$25\left(x^2 - \frac{1}{x^2}\right) = 1$$

$$x^2 - \frac{1}{x^2} = \frac{1}{25}$$

6. If $\sin\theta = \cos\theta$, then $2\tan^2\theta + \sin^2\theta - 1$ is equal to

(1) $\frac{-3}{2}$ (2) $\frac{3}{2}$

(3) $\frac{2}{3}$ (4) $\frac{-2}{3}$ **[Ans : (2)]**

Sol :

$$\sin\theta = \cos\theta \Rightarrow \theta = 45^o$$
$$= 2\tan^2 45^o + \sin^2 45^o - 1$$
$$= 2(1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 1$$
$$= 2 + \frac{1}{2} - 1$$
$$= 1 + \frac{1}{2}$$
$$= \frac{3}{2}$$

7. If x = $a\tan\theta$ and y = $b\sec\theta$ then

(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(3) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

[Ans : (1)]

Sol : $x = a\tan\theta \qquad y = b\sec\theta$

$$x^2 = a^2\tan^2\theta \qquad y^2 = b^2\sec^2\theta$$
$$\frac{x^2}{a^2} = \tan^2\theta \qquad \frac{y^2}{b^2} = \sec^2\theta$$
$$\sec^2\theta - \tan^2\theta = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

8. $(1+\tan\theta+\sec\theta)(1+\cot\theta-\operatorname{cosec}\theta)$ is equal to

(1) 0 (2) 1

(3) 2 (4) – 1 **[Ans : (3)]**

Sol :

$(1+\tan\theta+\sec\theta)(1+\cot\theta-\operatorname{cosec}\theta)$

$$= \left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)\left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)$$
$$= \left(\frac{(\cos\theta+\sin\theta)+1}{\cos\theta}\right)\left(\frac{(\sin\theta+\cos\theta)-1}{\sin\theta}\right)$$
$$= \frac{(\sin\theta+\cos\theta)^2 - 1^2}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$
$$= \frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$
$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$= 2$$

9. $a\cot\theta + b\operatorname{cosec}\theta$ = p and $b\cot\theta + a\operatorname{cosec}\theta$ = q then $p^2 - q^2$ is equal to

(1) $a^2 - b^2$ (2) $b^2 - a^2$

(3) $a^2 + b^2$ (4) b – a **[Ans : (2)]**

Sol :

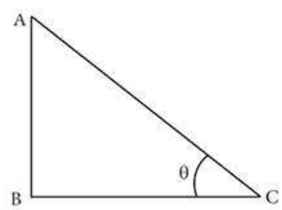
$$p^2 - q^2 = (a\cot\theta + b\operatorname{cosec}\theta)^2 - (b\cot\theta + a\operatorname{cosec}\theta)^2$$
$$= a^2\cot^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\cot\theta\operatorname{cosec}\theta$$
$$-(b^2\cot^2\theta + a^2\operatorname{cosec}^2\theta + 2ab\cot\theta\operatorname{cosec}\theta)$$
$$= a^2\cot^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\cot\theta\operatorname{cosec}\theta$$
$$- b^2\cot^2\theta - a^2\operatorname{cosec}^2\theta - 2ab\cot\theta\operatorname{cosec}\theta$$
$$= a^2(\cot^2\theta - \operatorname{cosec}^2\theta) + b^2(\operatorname{cosec}^2\theta - \cot^2\theta)$$
$$= a^2(-1) + b^2(1)$$
$$= -a^2 + b^2$$
$$= b^2 - a^2$$

10. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3}:1$, then the angle of elevation of the sun has measure

(1) 45° (2) 30°

(3) 90° (4) 60° **[Ans : (4)]**

Sol :



BC =Length of shadow

AB =Height of the tower

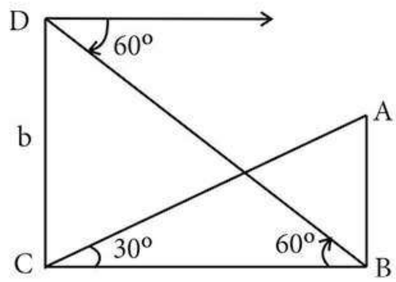
$$\tan\theta = \frac{AB}{BC} = \frac{\sqrt{3}}{1} \text{ (Given)}$$
$$\tan 60^o = \sqrt{3}\ ; \quad \theta = 60^o$$

11. The electric pole subtends an angle of 30° at a point on the same level as its foot. At a second point 'b' metres above the first, the depression of the foot of the tower is 60°. The height of the tower (in metres) is equal to

(1) $\sqrt{3}\,b$ (2) $\dfrac{b}{3}$

(3) $\dfrac{b}{2}$ (4) $\dfrac{b}{\sqrt{3}}$ **[Ans : (2)]**

Sol :



Let AB - tower, DC - electric pole

In $\Delta\, ABC$

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC}$$

$$BC = AB\sqrt{3} \quad \dots (1)$$

In $\Delta\, DCB$

$$\tan 60^\circ = \frac{DC}{BC}$$

$$\sqrt{3} = \frac{b}{BC}$$

$$BC = \frac{b}{\sqrt{3}} \quad \dots (2)$$

From (1) and (2)

$$AB\sqrt{3} = \frac{b}{\sqrt{3}}$$

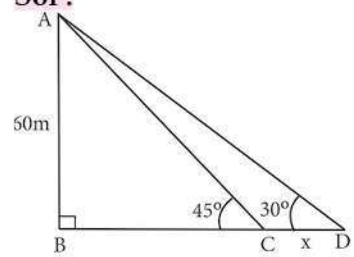
$$AB = \frac{b}{\sqrt{3}\sqrt{3}}$$

$$= \frac{b}{3}$$

12. A tower is 60 m height. Its shadow is x metres shorter when the sun's altitude is 45° than when it has been 30°, then x is equal to

(1) 41.92 m
(2) 43.92 m
(3) 43 m
(4) 45.6 m **[Ans : (2)]**

Sol :



From the figure

AB - tower

BC - shadow when altitude is 45°

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{60}{BC}$$

$$BC = 60 \text{ m} \quad \dots (1)$$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$= \frac{60}{BC + CD}$$

$$\frac{1}{\sqrt{3}} = \frac{60}{60 + x} \quad [\therefore \text{ from (1)}]$$

$$60 + x = 60\sqrt{3}$$

$$x = 60\sqrt{3} - 60$$

$$= 60(\sqrt{3} - 1)$$

$$= 60(1.732 - 1)$$

$$= 60 \times 0.732$$

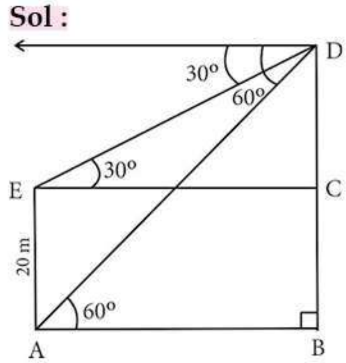
$$= 43.92 \text{ m}$$

13. The angle of depression of the top and bottom of 20 m tall building from the top of a multistoried building are 30° and 60° respectively. The height of the multistoried building and the distance between two buildings (in metres) is

(1) $20, 10\sqrt{3}$ (2) $30, 5\sqrt{3}$

(3) 20, 10 (4) $30, 10\sqrt{3}$ **[Ans : (4)]**

Sol :



From $\Delta\, ABD$

$$\tan 60^{\circ} = \frac{BD}{AB}$$

$$\sqrt{3} = \frac{BD}{AB}$$

$$AB = \frac{BD}{\sqrt{3}} \qquad \text{... (1)}$$

From $\Delta\, DEC$

$$\tan 30^{\circ} = \frac{DC}{EC}$$

$$\frac{1}{\sqrt{3}} = \frac{DC}{AB}$$

$$AB = DC\sqrt{3} \qquad \text{... (2)}$$

(1) and (2) $\Rightarrow$

$$\frac{BD}{\sqrt{3}} = DC\sqrt{3}$$

$$\frac{BC+CD}{\sqrt{3}} = DC\sqrt{3}$$

$$BC + CD = DC \times 3$$

$$BC = 3\,CD - CD$$

$$BC = 2\,CD$$

$$AE = 2\,CD$$

$$20 = 2\,CD$$

$$CD = \frac{20}{2}$$

$$= 10 \text{ m}$$

Height of the building

$$BD = BC + CD = 20 + 10 = 30 \text{ m}$$

Distance between buildings

$$AB = DC\sqrt{3} \qquad [\because \text{ from (2)}]$$

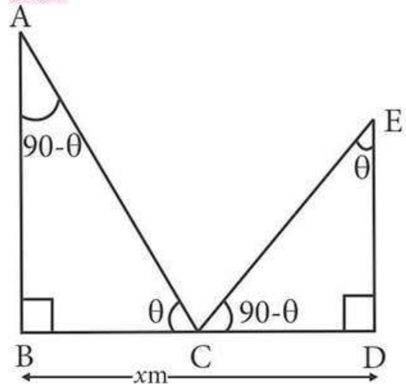
$$= 10\sqrt{3}\, m$$

14. Two persons are standing 'x' metres apart from each other and the height of the first person is double that of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the shorter person (in metres) is

(1) $\sqrt{2}\,x$ (2) $\frac{x}{2\sqrt{2}}$

(3) $\frac{x}{\sqrt{2}}$ (4) $2x$ **[Ans : (2)]**

Sol :



AB and DE - persons

$$AB = 2\,DE$$

$$BD = x \Rightarrow BC = CD = \frac{x}{2}$$

From $\Delta\, ABC$

$$\tan\theta = \frac{AB}{BC}$$

$$\tan\theta = \frac{2\,DE}{BC} = \frac{2\,DE}{\frac{x}{2}} = \frac{4\,DE}{x} \qquad \text{... (1)}$$

From $\Delta\, EDC$

$$\tan\theta = \frac{CD}{ED} = \frac{\frac{x}{2}}{ED}$$

$$\tan\theta = \frac{x}{2\,ED} \qquad \text{... (2)}$$

From (1) and (2)

$$\frac{4\,DE}{x} = \frac{x}{2\,ED}$$

$$8\,DE^2 = x^2$$

$$DE^2 = \frac{x^2}{8}$$

$$DE = \frac{x}{\sqrt{8}}$$

$$DE = \frac{x}{2\sqrt{2}}$$

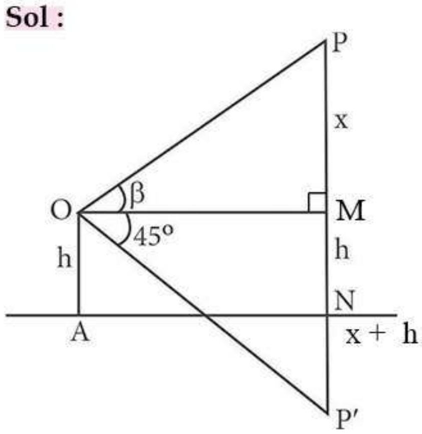
15. The angle of elevation of a cloud from a point h metres above a lake is β. The angle of depression of its reflection in the lake is 45°. The height of location of the cloud from the lake is

(1) $\frac{h(1+\tan\beta)}{1-\tan\beta}$ (2) $\frac{h(1-\tan\beta)}{1+\tan\beta}$

(3) $h\tan(45^{o}-\beta)$ (4) none of these

[Ans : (1)]

Sol :



From $\Delta OMP'$

$$\tan 45° = \frac{h+(x+h)}{OM}$$

$$1 = \frac{x+2h}{OM}$$

$$OM = x + 2h \quad \text{... (1)}$$

From ΔOMP

$$\tan\beta = \frac{x}{OM}$$

$$OM = \frac{x}{\tan\beta} \quad \text{... (2)}$$

From (1) and (2)

$$x + 2h = \frac{x}{\tan\beta}$$

$$\frac{x}{\tan\beta} - x = 2h$$

$$\frac{x - x\tan\beta}{\tan\beta} = 2h$$

$$x(1-\tan\beta) = 2h\tan\beta$$

$$x = \frac{2h\tan\beta}{1-\tan\beta}$$

$$PN = x + h = \frac{2h\tan\beta}{1-\tan\beta} + h$$

$$= \frac{2h\tan\beta + h - h\tan\beta}{1-\tan\beta}$$

$$= \frac{h\tan\beta + h}{1-\tan\beta}$$

$$= \frac{h(1+\tan\beta)}{1-\tan\beta}$$

$\therefore$ Height of the cloud from lake $= \dfrac{h(1+\tan\beta)}{1-\tan\beta}$

UNIT EXERCISE - 6

1. Prove that (i) $\cot^2 A\left(\dfrac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\dfrac{\sin A-1}{1+\sec A}\right)=0.$

(ii) $\dfrac{\tan^2\theta-1}{\tan^2\theta+1} = 1-2\cos^2\theta.$

Sol :

(i) LHS $= \cot^2 A\left(\dfrac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\dfrac{\sin A-1}{1+\sec A}\right)$

$$= \frac{\cot^2 A(\sec A-1)(\sec A+1)+\sec^2 A(\sin A-1)(1+\sin A)}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\cot^2 A(\sec^2 A-1)+\sec^2 A(\sin^2 A-1)}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\cot^2 A\tan^2 A-\sec^2 A(1-\sin^2 A)}{(1+\sin A)(1+\sec A)}$$

$[\because \sec^2 A - \tan^2 A = 1]$

$$= \frac{\cot^2 A\tan^2 A-\sec^2 A\cos^2 A}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\frac{1}{\tan^2 A}\tan^2 A-\frac{1}{\cos^2 A}\cos^2 A}{(1+\sin A)(1+\sec A)}$$

$$= \frac{1-1}{(1+\sin A)(1-\sec A)} = 0 = \text{RHS.}$$

(ii) $\dfrac{\tan^2\theta-1}{\tan^2\theta+1} = 1-2\cos^2\theta$

$$\text{LHS} = \frac{\tan^2\theta-1}{\tan^2\theta+1}$$

$$= \frac{\frac{\sin^2\theta}{\cos^2\theta}-1}{\frac{\sin^2\theta}{\cos^2\theta}+1}$$

$$= \frac{\dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{1} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= (1 - \cos^2\theta) - \cos^2\theta$$

$$[\therefore \sin^2\theta = 1 - \cos^2\theta]$$

$$= 1 - \cos^2\theta - \cos^2\theta$$

$$= 1 - 2\cos^2\theta = \text{RHS}$$

2. Prove that $\left(\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta + \cos\theta}\right)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

Sol :

$$\text{LHS} = \left(\frac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta}\right)^2$$

$$= \left(\frac{(1+\sin\theta) - \cos\theta}{(1+\sin\theta)+\cos\theta} \times \frac{(1+\sin\theta)-\cos\theta}{(1+\sin\theta)-\cos\theta}\right)^2$$

$$= \left[\frac{((1+\sin\theta)-\cos\theta)^2}{(1+\sin\theta)^2 - (\cos\theta)^2}\right]^2$$

$$= \left[\frac{1+\sin^2\theta + 2\sin\theta + \cos^2\theta - 2\cos\theta(1+\sin\theta)}{1+\sin^2\theta + 2\sin\theta - \cos^2\theta}\right]^2$$

$$[\because (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= \left[\frac{\sin^2\theta + 2\sin\theta + 1 + \cos^2\theta - 2\cos\theta - 2\sin\theta\cos\theta}{1 + (1-\cos^2\theta) + 2\sin\theta - \cos^2\theta}\right]^2$$

$$= \left[\frac{(\cos^2\theta + \sin^2\theta) + 2\sin\theta + 1 - 2\cos\theta(1+\sin\theta)}{1+1-2\cos^2\theta + 2\sin\theta}\right]^2$$

$$= \left[\frac{1+1+2\sin\theta - 2\cos\theta(1+\sin\theta)}{2-2\cos^2\theta + 2\sin\theta}\right]^2$$

$$= \left[\frac{2+2\sin\theta - 2\cos\theta(1+\sin\theta)}{2-2\cos^2\theta + 2\sin\theta}\right]^2$$

$$= \left(\frac{2(1+\sin\theta) - 2\cos\theta(1+\sin\theta)}{2(1-\cos^2\theta) + 2\sin\theta}\right)^2$$

$$= \left(\frac{(1+\sin\theta)\,2(1-\cos\theta)}{2\sin^2\theta + 2\sin\theta}\right)^2$$

$$= \left(\frac{2(1+\sin\theta)(1-\cos\theta)}{2\sin\theta(1+\sin\theta)}\right)^2$$

$$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{1-\cos\theta}{1+\cos\theta}$$

$$= \text{RHS}$$

3. If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ **and** $x\sin\theta = y\cos\theta$ **then prove that $x^2 + y^2 = 1$.**

Sol :

We have $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$

$\Rightarrow \quad (x\sin\theta)(\sin^2\theta) + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow \quad x\sin\theta(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta \quad [\because \ x\sin\theta = y\cos\theta]$S

$\Rightarrow \quad x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$\Rightarrow \quad x\sin\theta = \sin\theta\cos\theta$

$x = \cos\theta$ ... (1)

Now, $x\sin\theta = y\cos\theta$

$\Rightarrow \quad \cos\theta\sin\theta = y\cos\theta$

$[\because \ x = \cos\theta$ from (1)$]$

$\Rightarrow \quad y = \sin\theta$ ... (2)

From (1) and (2)

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta$$
$$= 1$$
$$\therefore \quad x^2 + y^2 = 1$$

4. If $a\cos\theta - b\sin\theta$ **= c, then prove that** $(a\sin\theta + b\cos\theta) = \pm\sqrt{a^2 + b^2 - c^2}$ **.**

Sol :

We have $a\cos\theta - b\sin\theta = c$

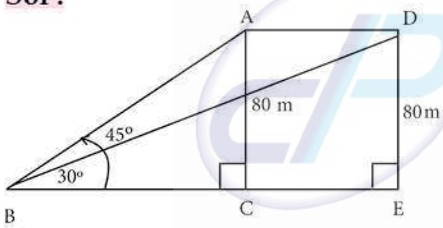
Squaring on both the sides.

$(a\cos\theta - b\sin\theta)^2 = c^2$

$$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta = c^2$$
$$a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$$
$$a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$$
$$a^2 + b^2 - c^2 = a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta$$
$$a^2 + b^2 - c^2 = (a\sin\theta + b\cos\theta)^2$$
$$\pm\sqrt{a^2+b^2-c^2} = a\sin\theta + b\cos\theta$$
$$\therefore\ a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$$

5. A bird is sitting on the top of a 80 m high tree. From a point on the ground, the angle of elevation of the bird is 45°. The bird flies away horizontally in such away that it remained at a constant height from the ground. After 2 seconds, the angle of elevation of the bird from the same point is 30°. Determine the speed at which the bird flies.($\sqrt{3}$ = 1.732)

Sol :



Let the initial position of the bird be A and after two seconds its position is at D.

$$AC = DE = 80 \text{ m}$$
$$\angle ABC = 45^\circ$$
$$\angle DBC = 30^\circ$$

In right ΔABC

$$\tan 45^\circ = \frac{AC}{BC}$$
$$1 = \frac{80}{BC}$$
$$BC = 80 \text{ m}$$

In right triangle ΔDBE

$$\tan 30^\circ = \frac{DE}{BE}$$
$$\frac{1}{\sqrt{3}} = \frac{80}{BC+CE}$$
$$\frac{1}{\sqrt{3}} = \frac{80}{80+CE}$$
$$80 + CE = 80\sqrt{3}$$
$$CE = (80\sqrt{3} - 80)$$
$$= 80(\sqrt{3}-1)$$
$$= 80\ (1.732 - 1)$$
$$= 80 \times 0.732$$
$$CE = 58.56 \text{ m}$$

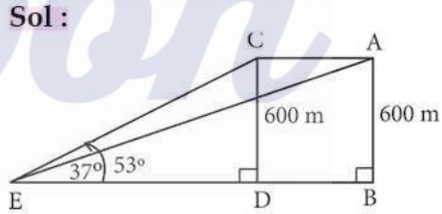
$\therefore$ The bird travelled 58.56 m in 2 seconds.

$$\text{Speed of the bird} = \frac{\textit{Distance travelled}}{\textit{Time taken}}$$
$$= \frac{58.56}{2}$$
$$= 29.28 \text{ m/s}$$

$\therefore$ Speed of flying bird = 29.28 m/s.

6. An aeroplane is flying parallel to the Earth's surface at a speed of 175 m/sec and at a height of 600 m. The angle of elevation of the aeroplane from a point on the Earth's surface is 37° at a given point. After what period of time does the angle of elevation increase to 53°?
(tan 53° = 1.3270, tan 37° = 0.7536)

Sol :



Let A be the initial position of the aeroplane and C be the position of the aeroplane at an angle of elevation 53°

$$\angle AEB = 37^\circ$$
$$\angle CED = 53^\circ$$

In the right triangle ΔAEB

$$\tan 37^\circ = \frac{AB}{BE}$$
$$0.7536 = \frac{600}{BD+DE}$$
$$BD + DE = \frac{600}{0.7536}$$
$$= \frac{60{,}00{,}000}{7536}$$
$$BD + DE = 796.18 \text{ m} \qquad \dots (1)$$

In right ΔCED

$$\tan 53^\circ = \frac{CD}{DE}$$

$$1.3270 = \frac{600}{DE}$$

$$DE = \frac{600}{1.3270}$$

$$= \frac{6,00,000}{1327}$$

$$DE = 452.15 \text{ m} \quad \dots (2)$$

$(1) - (2) \Rightarrow$ BD + DE – DE = 796.18 – 452.15

BD = 344.03 m

$\therefore$ CA = 344.03 m [$\because$ BD = CA]

$\therefore$ Distance travelled by the aeroplane = 344.03 m

Speed of the aeroplane = 175 m/s

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{344.03}{175}$$

= 1.965 seconds.

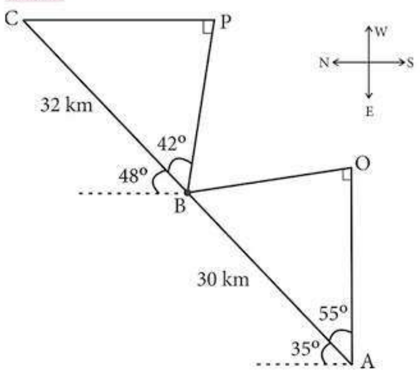
= 1.97 seconds.

$\therefore$ After 1.97 seconds the angle of elevation is 53°.

7. A bird is flying from A towards B at an angle of 35°, a point 30 km away from A. At B it changes its course of flight and heads towards C on a bearing of 48° and distance 32 km away.
(i) How far is B to the North of A?
(ii) How far is B to the West of A?
(iii) How far is C to the North of B?
(iv) How far is C to the East of B?

$$\left(\begin{array}{l}\sin 55^\circ = 0.8192, \cos 55^\circ = 0.5736\\ \sin 42^\circ = 0.6691, \cos 42^\circ = 0.7431\end{array}\right)$$

Sol :



Let A be the initial position of the bird.

B be the position after travelling 30 km at an angle of 35° from A.

Let C be the position after travelling 32 km at an angle of 48° from B.

$\therefore$ $\angle OAP = 55°$

$\angle PBC = 42°$

[$\because$ complementary angle]

(i) In $\Delta\, AOB$

$$\sin 55^\circ = \frac{OB}{AB}$$

$$0.8192 = \frac{OB}{30}$$

$$OB = 30 \times 0.8192 = 24.58 \text{ km}$$

(ii) From the right triangle $\Delta\, AOB$

$$\cos 55^\circ = \frac{OA}{AB}$$

$$0.5736 = \frac{OA}{30}$$

$$OA = 30 \times 0.5736 = 17.21 \text{ km}$$

B is 17.21 km to the West of A.

(iii)

$$\sin 42^\circ = \frac{PC}{BC}$$

$$0.6691 = \frac{PC}{32}$$

$$PC = 32 \times 0.6691 = 21.41 \text{ km}$$

C is 21.41 km to the North of B.

(iv) In the right $\Delta\, BPC$

$$\cos 42^\circ = \frac{BP}{BC}$$

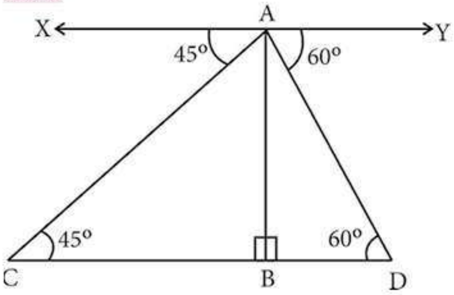
$$0.7431 = \frac{BP}{32}$$

$$BP = 23.78 \text{ km}$$

C is 23.78 km to the East of B.

8. Two ships are sailing in the sea on either side of the lighthouse. The angles of depression of two ships as observed from the top of the lighthouse are 60° and 45° respectively. If the distance between the ships is $200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$ metres, find the height of the lighthouse.

Sol :



Let C and D are two ships.
Let AB be the height of the light house.

$$\angle XAC = \angle ACB = 45^\circ$$
$$\angle YAD = \angle ADB = 60^\circ$$
$$CD = 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)m$$

In the right $\Delta\ ABD$

$$\tan 60^\circ = \frac{AB}{BD}$$
$$\sqrt{3} = \frac{AB}{BD}$$
$$BD = \frac{AB}{\sqrt{3}} \quad \text{... (1)}$$

In the right triangle ABC

$$\tan 45^\circ = \frac{AB}{BC}$$
$$1 = \frac{AB}{BC}$$
$$BC = AB \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow \ BD + BC = \frac{AB}{\sqrt{3}} + AB$$

$$CD = AB\left(\frac{1}{\sqrt{3}}+1\right) \quad [\because\ CB + BD = CD]$$

$$\frac{CD}{\left(\frac{1}{\sqrt{3}}+1\right)} = AB$$

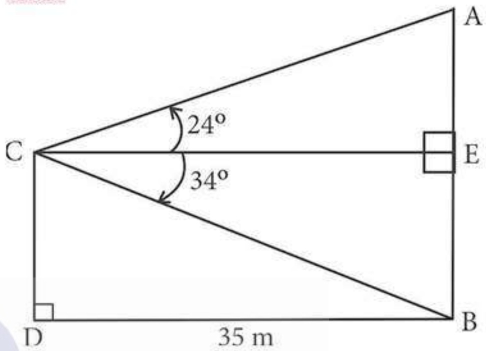
$$AB = \frac{200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)}$$

$$AB = 200 \text{ m}$$

$\therefore$ Height of the light house is 200 m.

9. A building and a statue are in opposite side of a street from each other 35 m apart. From a point on the roof of building the angle of elevation of the top of statue is 24° and the angle of depression of base of the statue is 34°. Find the height of the statue. ($\tan 24^\circ = 0.4452$, $\tan 34^\circ = 0.6745$)

Sol :



Let AB be the statue CD be the building Given BD = 35 m.

$$\angle ACE = 24^\circ$$
$$\angle ECB = 34^\circ$$
$$DB = CE = 35 \text{ m}$$

In right triangle $\Delta\ ECB$

$$\tan 34^\circ = \frac{EB}{CE}$$
$$EB = CE \times \tan 34^\circ$$
$$= 35 \times 0.6745 \text{ m}$$
$$EB = 23.61 \text{ m} \quad \text{... (1)}$$

In right $\Delta\ AEC$

$$\tan 24^\circ = \frac{AE}{EC}$$
$$AE = \tan 24^\circ \times EC$$
$$= 0.4452 \times 35$$
$$AE = 15.58 \text{ m} \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow \quad AE + EB = 15.58 + 23.61$$
$$AB = 39.19 \text{ m}$$

$\therefore$ Height of the statue = 39.19 m.

CREATIVE QUESTIONS

I. Multiple Choice Questions

Trigonometric Ratios

1. If $\tan\theta = \frac{a}{b}$, then $\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta}$ is

(1) $\frac{a^2+b^2}{a^2-b^2}$ (2) $\frac{a^2-b^2}{a^2+b^2}$

(3) $\frac{a+b}{a-b}$ (4) $\frac{a-b}{a+b}$ **[Ans : (1)]**

Sol :

$$\tan\theta = \frac{a}{b}$$

$$\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta} = \frac{a\frac{\sin\theta}{\cos\theta} + b\frac{\cos\theta}{\cos\theta}}{a\frac{\sin\theta}{\cos\theta} - b\frac{\cos\theta}{\cos\theta}}$$

[Dividing by $\cos\theta$]

$$= \frac{a\tan\theta + b}{a\tan\theta - b}$$

$$= \frac{a\times\frac{a}{b} + b}{a\times\frac{a}{b} - b}$$

$$= \frac{\frac{a^2}{b} + b}{\frac{a^2}{b} - b}$$

$$= \frac{\frac{a^2+b^2}{b}}{\frac{a^2-b^2}{b}}$$

$$= \frac{a^2+b^2}{b}\times\frac{b}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2}$$

2. If A and B are complementary angles then

(1) $\sin A = \sin B$ (2) $\cos A = \cos B$

(3) $\tan A = \tan B$ (4) $\sec A = \text{cosec } B$

[Ans : (4)]

Sol :

A and B are complementary angles

$$A + B = 90^\circ$$
$$A = 90^\circ - B$$
$$\text{Sec } A = \text{Sec}(90 - B) = \text{Cosec } B$$

3. If $x\sin(90^\circ-\theta)\cot(90^\circ-\theta) = \cos(90^\circ-\theta)$ then x =

(1) 0 (2) 1

(3) -1 (4) 2 **[Ans : (2)]**

Sol :

$$x\sin(90-\theta)\cot(90^\circ-\theta) = \cos(90^\circ-\theta)$$

$$\Rightarrow \quad x\cos\theta\tan\theta = \sin\theta$$

$$\Rightarrow \quad x = \frac{\sin\theta}{\cos\theta\tan\theta}$$

$$= \frac{\sin\theta}{\cos\theta}\times\cot\theta$$

$$= \tan\theta\cot\theta$$

$$= \tan\theta\times\frac{1}{\tan\theta} = 1$$

4. If x tan 45° cos 60° = sin 60° cot 60°, then x is

(1) 1 (2) $\sqrt{3}$

(3) $\frac{1}{2}$ (4) $\frac{1}{\sqrt{2}}$ **[Ans : (1)]**

Sol :

$$x\tan 45^\circ\cos 60^\circ = \sin 60^\circ\cot 60^\circ$$

$$x\times 1\times\frac{1}{2} = \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{3}}$$

$$x\times\frac{1}{2} = \frac{1}{2}$$

$$x = \frac{2}{2} = 1$$

5. $\frac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ}$ =

(1) $\tan 90^\circ$ (2) 1

(3) $\sin 45^\circ$ (4) $\sin 0^\circ$ **[Ans : (4)]**

Sol :

$$\frac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ} = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

Trigonometric Identities:

6. If $\sec\theta + \tan\theta = x$, then $\sec\theta =$

(1) $\frac{x^2+1}{x}$ (2) $\frac{x^2+1}{2x}$

(3) $\frac{x^2-1}{2x}$ (4) $\frac{x^2-1}{x}$ **[Ans : (2)]**

Sol :

$$\sec\theta + \tan\theta = x$$
$$(\sec\theta + \tan\theta)^2 = x^2$$
$$\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta = x^2$$
$$\sec^2\theta + \sec^2\theta - 1 + 2\sec\theta\tan\theta = x^2$$
$$2\sec^2\theta + 2\sec\theta\tan\theta = x^2 + 1$$
$$2\sec\theta(\sec\theta + \tan\theta) = x^2 + 1$$
$$\sec\theta\ (x) = \frac{x^2+1}{2}$$
$$\sec\theta = \frac{x^2+1}{2x}$$

7. $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} =$

(1) $\sec\theta + \tan\theta$ (2) $\sec\theta - \tan\theta$

(3) $\sec^2\theta + \tan^2\theta$ (4) $\sec^2\theta - \tan^2\theta$

[Ans : (1)]

Sol :

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}$$
$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$
$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$
$$= \frac{1+\sin\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$= \sec\theta + \tan\theta$$

8. $\cos^4 A - \sin^4 A =$

(1) $2\cos^2 A + 1$ (2) $2\cos^2 A - 1$

(3) $2\sin^2 A - 1$ (4) $2\sin^2 A + 1$ **[Ans : (2)]**

Sol :

$$\cos^4 A - \sin^4 A = (\cos^2 A)^2 - (\sin^2 A)^2$$
$$= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= (1)(\cos^2 A - (1 - \cos^2 A))$$
$$= \cos^2 A - 1 + \cos^2 A$$
$$= 2\cos^2 A - 1$$

9. $\frac{\sin\theta}{1+\cos\theta} =$

(1) $\frac{1+\cos\theta}{\sin\theta}$ (2) $\frac{1-\cos\theta}{\cos\theta}$

(3) $\frac{1-\cos\theta}{\sin\theta}$ (4) $\frac{1-\sin\theta}{\cos\theta}$ **[Ans : (3)]**

Sol :

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

10. If $\sin\theta + \sin^2\theta = 1$ then $\cos^2\theta + \cos^4\theta =$

(1) -1 (2) 1

(3) 0 (4) None of these

[Ans : (2)]

Sol :

$$\sin^2\theta + \cos^2\theta = 1$$
$$\text{Taking } \sin\theta + \sin^2\theta = 1$$
$$\sin\theta + (1 - \cos^2\theta) = 1$$
$$\sin\theta + 1 - \cos^2\theta = 1$$
$$\sin\theta = 1 - 1 + \cos^2\theta$$
$$\sin\theta = \cos^2\theta$$
$$\sin^2\theta = \cos^4\theta \quad \ldots (1)$$
$$\text{Now } \sin^2\theta + \cos^2\theta = 1$$

$$1-\cos^2\theta = \sin^2\theta$$

$$1-\cos^2\theta = \cos^4\theta$$

using (1)

$$\cos^4\theta+\cos^2\theta = 1$$

$$\cos^2\theta+\cos^4\theta = 1$$

Heights and Distances - Angle of Elevation:

11. From a given point when height of an object increases the angle of elevation _____

(1) increases
(2) decreases
(3) neither increases nor decreases
(4) equal. **[Ans : (1)]**

12. The ratio of the length of a rod and its shadow is $1:\sqrt{3}$. The angle of elevation of the sum is

(1) 30° (2) 45°
(3) 60° (4) 90° **[Ans : (1)]**

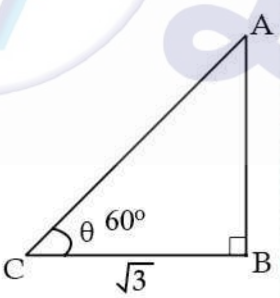
Sol :

$$\tan\theta = \frac{AB}{BC}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \tan 30^\circ$$

$$\theta = 30^\circ$$

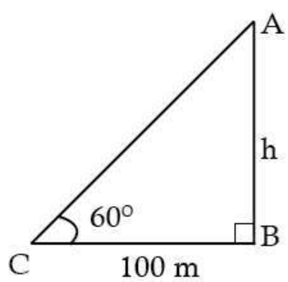


Angle of elevation is 30°

13. If the angle of elevation of a tower from a distance of 100 m from its foot is 60°, then the height of the tower is

(1) $100\sqrt{3}\ m$ (2) $\frac{100}{\sqrt{3}}m$
(3) $50\sqrt{3}\ m$ (4) $\frac{200}{\sqrt{3}}m$ **[Ans : (2)]**

Sol :



AB – tower

From C distance $= 100$ m

Angle of elevation $\theta = 60^\circ$

From right ΔABC

$$\tan\theta = \frac{AB}{BC}$$

$$\tan 60^\circ = \frac{h}{100}$$

$$\sqrt{3} = \frac{h}{100}$$

$$h = 100\sqrt{3}$$

Height of the tower $= 100\sqrt{3}$ m.

14. If the altitude of the sun is at 60°, then the height of the vertical tower that will cast a shadow of length 30 m is

(1) $30\sqrt{3}\ m$ (2) 15 m
(3) $\frac{30}{\sqrt{3}}m$ (4) $15\sqrt{2}\ m$ **[Ans : (1)]**

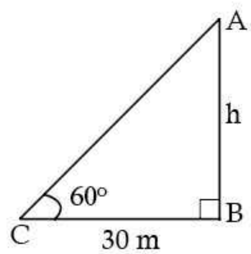
Sol :

From the right ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{h}{30}$$

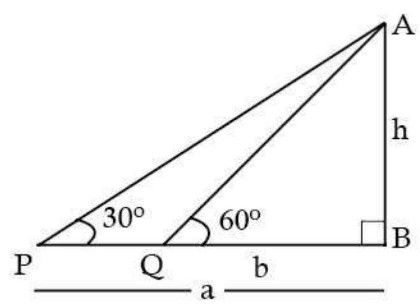
$$h = 30\sqrt{3}\ \text{m}$$



15. The angles of elevation of a tower from two points distant a and b (a>b) from its foot and in the same straight line from if are 30° and 60°, then the height of the tower is

(1) $\sqrt{a+b}$ (2) $\sqrt{ab}$
(3) $\sqrt{a-b}$ (4) $\sqrt{\frac{a}{b}}$ **[Ans : (2)]**

Sol :



$$\tan 30^\circ = \frac{h}{a} = \frac{1}{\sqrt{3}}$$

$$\frac{h}{a} = \frac{1}{\sqrt{3}} \quad \text{... (1)}$$

$$\tan 60^\circ = \frac{AB}{QB}$$

$$\sqrt{3} = \frac{h}{b} \quad \text{... (2)}$$

$$(1) \times (2) \Rightarrow \frac{h}{a} \times \frac{h}{b} = \frac{1}{\sqrt{3}} \times \sqrt{3}$$

$$\frac{h^2}{ab} = 1$$

$$h^2 = ab$$

Height of the tower, h = $\sqrt{ab}$

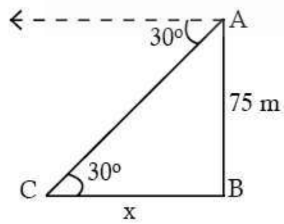
Angle of Depression:

16. The angle of depression of a car, standing on the ground from the top of a 75 m tower is 30°. The distance of the car from the base of the tower in metres is

(1) $25\sqrt{3}$ (2) $50\sqrt{3}$

(3) $75\sqrt{3}$ (4) 150 **[Ans : (3)]**

Sol :



From the right ΔABC

$$\tan 30^\circ = \frac{75}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{75}{x}$$

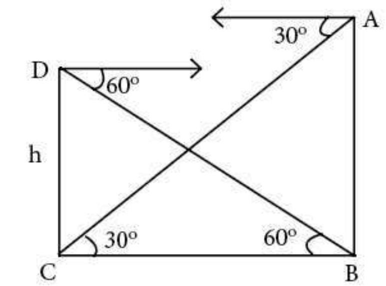
$$x = 75\sqrt{3} \text{ m}$$

17. A tower subtends an angle 30° at a point on the same level as its foot. At a second point h metres above the first the depression of the foot of the tower is 60°. The height of the tower is

(1) $\frac{h}{2} m$ (2) $\sqrt{3}\, h\, m$

(3) $\frac{h}{3} m$ (4) $\frac{h}{\sqrt{3}} m$ **[Ans : (3)]**

Sol :



Let AB be the tower

From the right Δ DCB

$$\tan 60^\circ = \frac{h}{BC}$$

$$\sqrt{3} = \frac{h}{BC}$$

$$BC = \frac{h}{\sqrt{3}} \quad \text{... (1)}$$

From the right ΔCBA

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC}$$

$$BC = AB\sqrt{3} \quad \text{... (2)}$$

From (1) and (2)

$$AB\sqrt{3} = \frac{h}{\sqrt{3}}$$

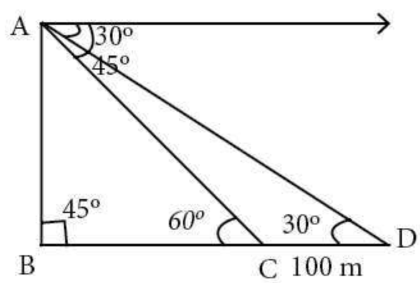
$$AB = \frac{h}{\sqrt{3} \times \sqrt{3}} = \frac{h}{3} \text{ m}$$

18. The angles of depression of two ships from the top of a light house are 45° and 30° towards east. If the ships are 100 m apart, the height of the light house is

(1) $\frac{50}{\sqrt{3}+1} m$ (2) $\frac{50}{\sqrt{3}-1} m$

(3) $50(\sqrt{3}-1)\, m$ (4) $50(\sqrt{3}+1)\, m$ **[Ans : (4)]**

Sol :



Let AB be the light house C and D are two ships.
From the right Δ ABC

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{AB}{BC}$$

$$BC = AB \qquad \dots (1)$$

From the right $\Delta\ ABD$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+CD}$$

$$BC + 100 = \sqrt{3}\,AB \qquad \dots (2)$$

From (1) and (2)

$$\sqrt{3}\,AB = AB + 100$$

$$\sqrt{3}\,AB - AB = 100$$

$$(\sqrt{3}-1)\,AB = 100$$

$$AB = \frac{100}{(\sqrt{3}-1)}$$

$$= \frac{100\times(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{100\ (\sqrt{3}+1)}{(\sqrt{3})^2-1^2}$$

$$= \frac{100\ (\sqrt{3}+1)}{3-1}$$

$$= \frac{100}{2}(\sqrt{3}+1)$$

$$AB = 50(\sqrt{3}+1)\text{ m}$$

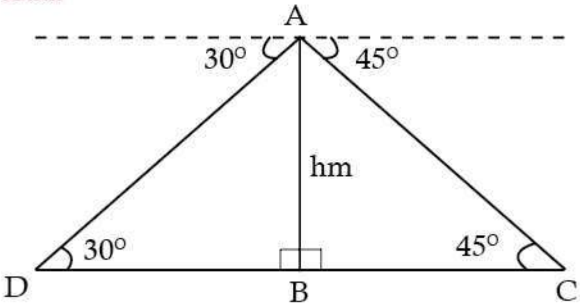
19. If the altitude of the light house is h metres and from it the angle of depression of two ships on opposite sides of the light house are observed to be 30° and 45°, then the distance between the ships are

(1) $(\sqrt{3}+1)h$ metres (2) $(\sqrt{3}-1)h$ metres

(3) $\sqrt{3}\,h$ metres (4) $1+\left(1+\frac{1}{\sqrt{3}}\right)h$ metres

[Ans : (1)]

Sol :



Let AB be the light house C and D be the two ships.
CD is the distance between two ships
From the right $\Delta\ ABD$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{DB}$$

$$DB = h\sqrt{3} \qquad \dots (1)$$

From the right $\Delta\ ABC$

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{h}{BC}$$

$$BC = h \qquad \dots (2)$$

From (1) and (2) $BC + DB = h + h\sqrt{3}$

$$DC = (1+\sqrt{3})\,h$$

$$\text{Distance between ships} = (\sqrt{3}+1)\ h\text{ m}$$

II. Very Short Answer Questions

1. If $\sin\theta+\sin^2\theta=1$ prove that $\cos^2\theta+\cos^4\theta=1$.

Sol :

We have $\sin\theta+\sin^2\theta = 1$

$$\Rightarrow \sin\theta = 1-\sin^2\theta$$

$$\Rightarrow \sin\theta = \cos^2\theta \qquad \dots (1)$$

Now $\cos^2\theta+\cos^4\theta = \cos^2\theta+(\cos^2\theta)^2$

$$\Rightarrow \cos^2\theta+\cos^4\theta = \cos^2\theta+\sin^2\theta$$

$$[\because \textit{From}\ (1)]$$

$$\Rightarrow \cos^2\theta+\cos^4\theta = 1$$

$$[\because \sin^2\theta+\cos^2\theta=1]$$

2. Prove the trigonometrical identity $(1-\sin^2\theta)\sec^2\theta$ = 1.

Sol :

$$\text{LHS} = (1-\sin^2\theta)\sec^2\theta = \cos^2\theta\ \sec^2\theta$$

$$= \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1 = \text{RHS}$$

3. Prove that trigonometrical identity
$\cos^2\theta(1+\tan^2\theta) = 1.$

Sol :

$$\text{LHS} = \cos^2\theta(1+\tan^2\theta)$$

$$= \cos^2\theta\sec^2\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \cos^2\theta\times\frac{1}{\cos^2\theta} = 1 = \text{RHS}$$

4. Prove the trigonometrical identity
$\cos^2\theta + \dfrac{1}{1+\cot^2\theta} = 1.$

Sol :

$$\text{LHS} = \cos^2\theta + \frac{1}{1+\cot^2\theta}$$

$$= \cos^2\theta + \frac{1}{\text{cosec}^2\theta} \qquad [\because 1+\cot^2\theta = \text{cosec}^2\theta]$$

$$= \cos^2\theta + \sin^2\theta \qquad \left[\because \frac{1}{\text{cosec}\,\theta} = \sin\theta\right]$$

$$= 1 = \text{RHS}$$

5. Prove the trigonometrical identity
$\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta.$

Sol :

$$\text{LHS} = \frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$$

$$= \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta}$$

$$= 2\sec^2\theta \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta\right]$$

$$= \text{RHS}$$

6. Prove the trigonometrical identity
$\text{cosec}^2\theta + \sec^2\theta = \text{cosec}^2\theta\sec^2\theta.$

Sol :

$$\text{LHS} = \text{cosec}^2\theta + \sec^2\theta$$

$$= \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta} \qquad \left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta,\ \frac{1}{\cos\theta} = \sec\theta\right]$$

$$= \frac{1}{\sin^2\theta}\ \frac{1}{\cos^2\theta}$$

$$= \text{cosec}^2\theta\sec^2\theta = \text{RHS}$$

7. Prove the trigonometric identity
$\cot^2\theta - \dfrac{1}{\sin^2\theta} = -1.$

Sol :

$$\text{LHS} = \cot^2\theta - \frac{1}{\sin^2\theta}$$

$$= \cot^2\theta - \text{cosec}^2\theta$$

$$= -(\text{cosec}^2\theta - \cot^2\theta) \qquad [\because \text{cosec}^2\theta - \cot^2\theta = 1]$$

$$= -1 = \text{RHS}$$

8. Prove the trigonometric identity
$(1+\tan^2\theta)(1+\sin\theta)(1-\sin\theta) = 1.$

Sol :

$$\text{LHS} = (1+\tan^2\theta)(1+\sin\theta)(1-\sin\theta)$$

$$= (1+\tan^2\theta)\{(1+\sin\theta)(1-\sin\theta)\}$$

$$= (1+\tan^2\theta)(1-\sin^2\theta)$$

$$[\because (a+b)(a-b) = a^2-b^2]$$

$$= (1+\tan^2\theta)(\cos^2\theta)$$

$$= \sec^2\theta\cos^2\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \frac{1}{\cos^2\theta}\times\cos^2\theta = 1 = \text{RHS}$$

9. Prove the trigonometric identity
$(1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta) = 1$

Sol :

$$\text{LHS} = (1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta)$$

$$= (1+\cot^2\theta)(1-\cos^2\theta)$$

$$= \text{cosec}^2\theta\sin^2\theta \qquad \left[\begin{array}{l}\because 1-\cos^2\theta = \sin^2\theta \\ \textit{and } 1+\cot^2\theta = \text{cosec}^2\theta\end{array}\right]$$

$$= \frac{1}{\sin^2\theta}\sin^2\theta = 1 = \text{RHS}$$

10. Prove the trigonometric identity

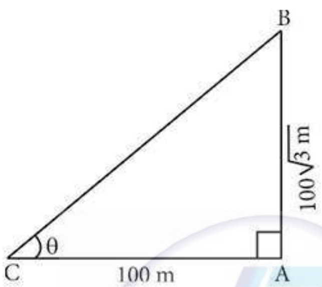
$$\tan^2\theta - \frac{1}{\cos^2\theta} = -1$$

Sol :

$$\begin{aligned} \text{LHS} &= \tan^2\theta - \frac{1}{\cos^2\theta} \\ &= \tan^2\theta - \sec^2\theta = -(\sec^2\theta - \tan^2\theta) \\ &= -1 = \text{RHS} \qquad [\because \sec^2\theta - \tan^2\theta = 1] \end{aligned}$$

11. A tower is $100\sqrt{3}$ metres high. Find the angle of elevation of its top from a point 100 metres away from its foot.

Sol :



Let AB be the tower AC be the distance from the point to the foot of the tower.

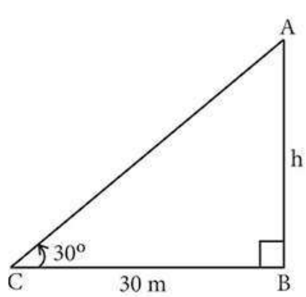
From the right triangle ΔCAB

$$\begin{aligned} \tan\theta &= \frac{BA}{AC} \\ &= \frac{100\sqrt{3}}{100} \\ &= \sqrt{3} \\ \theta &= \tan^{-1}\left(\sqrt{3}\right) \\ \theta &= 60^\circ \end{aligned}$$

Angle of elevation is 60°

12. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower is 30°. Find the height of the tower.

Sol :



Let AB be the tower.

BC is the distance between the point and the foot of the tower.

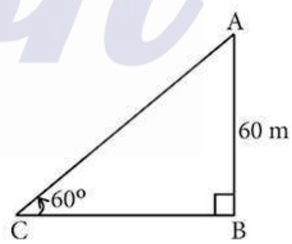
From the right triangle ΔABC

$$\begin{aligned} \tan 30^\circ &= \frac{AB}{BC} \\ \frac{1}{\sqrt{3}} &= \frac{AB}{30} \\ AB &= \frac{30}{\sqrt{3}} \\ &= \frac{3\times 10}{\sqrt{3}} \\ &= \frac{\sqrt{3}\times\sqrt{3}\times 10}{\sqrt{3}} = 10\sqrt{3}\, m \end{aligned}$$

$\therefore$ Height of the tower is $10\sqrt{3}\, m$

13. A kite is flying at a height of 60 m above the ground. The inclination of the string with the ground where its string is tied is 60°. Find the length of the string.

Sol :



Let AB be the height of the kite from the ground.

AC is the length of the string.

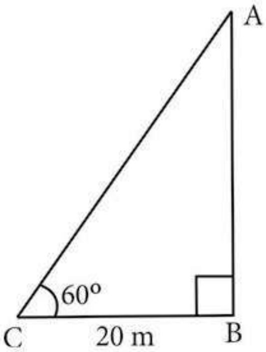
From the right triangle ΔABC

$$\begin{aligned} \sin 60^\circ &= \frac{AB}{AC} \\ \frac{\sqrt{3}}{2} &= \frac{60}{AC} \\ AC &= \frac{2\times 60}{\sqrt{3}} \\ &= \frac{2\times 20\times\sqrt{3}\times\sqrt{3}}{\sqrt{3}} \\ &= 40\sqrt{3}\, m \end{aligned}$$

Length of the string $= 40\sqrt{3}\, m$

14. A tower stands vertically on the ground. From a point on the ground which is 20 m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

Sol :



Let AB is the tower.

In the right triangle ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

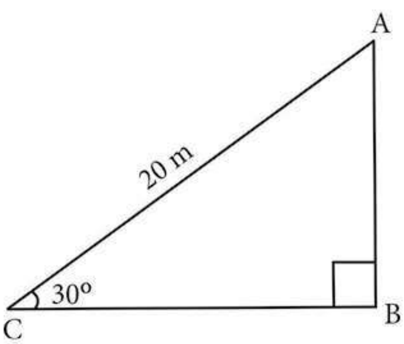
$$\sqrt{3} = \frac{AB}{20}$$

$$AB = 20\sqrt{3}\, m$$

$\therefore$ Height of the tower is $20\sqrt{3}\, m$.

15. A circus artist is climbing a 20 m long rope which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.

Sol :



Let AB be the vertical height of the pole.

AC be the length of the rope.

From the right triangle ΔABC

$$\sin 30^\circ = \frac{AB}{AC}$$

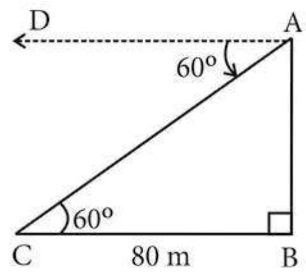
$$\frac{1}{2} = \frac{AB}{20}$$

$$AB = \frac{20}{2} = 10 \text{ m}$$

$\therefore$ Height of the pole = 10 m.

16. The angle between the top of a building and a point 80 m away from the base on level ground is 60°. How tall is the building?

Sol :



Let AB is the building

$$\angle DAC = \angle ACB = 60^\circ$$

In right triangle ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

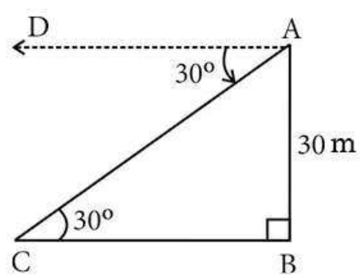
$$\sqrt{3} = \frac{AB}{80}$$

$$AB = 80\sqrt{3}\, m$$

Height of the building is $80\sqrt{3}\, m$.

17. From the top of the tower 30 m height a man is observing the base of a tree at an angle of depression measuring 30°. Find the distance between the tree and the tower.

Sol :



Let AB = 30 m is the height of the tower.

$$\angle DAC = \angle ACB = 30^\circ$$

CB is the distance between the tree and the tower.

From right triangle ΔABC

$$\tan 30^\circ = \frac{AB}{BC}$$

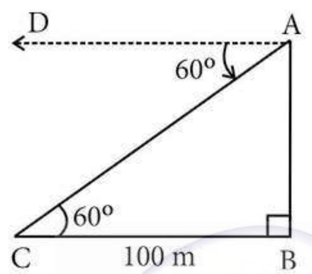
$$\frac{1}{\sqrt{3}} = \frac{30}{BC}$$

$$BC = 30\sqrt{3}\, m$$

$\therefore$ Distance between the tree and tower $= 30\sqrt{3}\, m$.

18. The angle of depression of a vehicle on the ground from the top of a tower is 60°. If the vehicle is at a distance of 100 m away from the building, find the height of the tower.

Sol :



Let AB is the tower.

$$\angle DAC = \angle ACB = 60^\circ$$

In right triangle ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

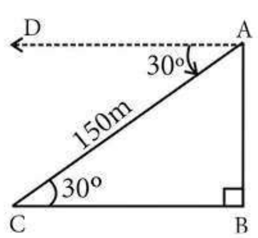
$$\sqrt{3} = \frac{AB}{100}$$

$$AB = 100\sqrt{3}\, m$$

$\therefore$ Height of the tower $= 100\sqrt{3}\, m$.

19. Anu was flying a kite on a hill, but he dumped his kite into the pond below. If the length of the string of his kite is 150 m and the angle of depression from his position to the kite is 30° then how high is the hill where he is standing?

Sol :



Let AB be the hill.

C be the pond.

$$\angle DAC = \angle ACB = 30^\circ$$

Distance between the pond and top of the hill is AC = 150 m.

In right ΔABC

$$\sin 30^\circ = \frac{AB}{AC}$$

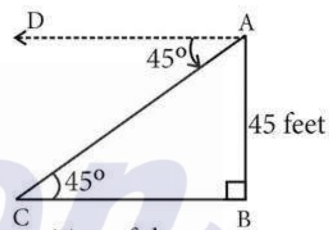
$$\frac{1}{2} = \frac{AB}{150}$$

$$AB = \frac{150}{2} = 75 \text{ m}$$

Height of the hill = 75 m.

20. From the top of a fire tower, a forest ranger sees his partner on the ground at an angle of depression of 45°. If the tower is 45 feet in height, how far is the partner from the base of the tower?

Sol :



Let C be the position of the partner.

AB be the tower.

$$\angle DAC = \angle ACB = 45^\circ$$

In right triangle ΔACB

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{45}{BC}$$

$$BC = 45 \text{ ft}$$

$\therefore$ The partner is 45 ft far from the base of the tower.

III. Short Answer Questions:

1. Prove the trigonometric identity

$$\frac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$$

Sol :

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \frac{\sin\theta\,(1+\cos\theta)}{1-\cos^2\theta} \qquad [\because 1-\cos^2\theta = \sin^2\theta]$$

$$= \frac{1}{\sin\theta}(1+\cos\theta) = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \operatorname{cosec}\theta + \cot\theta = \text{RHS}$$

2. Prove the trigonometric identity

$$\frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta} = \frac{\sec\theta + 1}{\sec\theta - 1}$$

Sol :

$$\text{LHS} = \frac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta} + \sin\theta}{\frac{\sin\theta}{\cos\theta} - \sin\theta}$$

$$= \frac{\sin\theta\left(\frac{1}{\cos\theta}+1\right)}{\sin\theta\left(\frac{1}{\cos\theta}-1\right)} = \frac{\sec\theta+1}{\sec\theta-1} = \text{RHS}$$

3. Prove the trigonometric identity

$$\cot\theta - \tan\theta = \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$$

Sol :

$$\text{LHS} = \cot\theta - \tan\theta$$

$$= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta - (1-\cos^2\theta)}{\sin\theta\cos\theta} \quad [\because \sin^2\theta = 1-\cos^2\theta]$$

$$= \frac{\cos^2\theta - 1 + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

4. Prove the trigonometric identity

$$\tan\theta - \cot\theta = \frac{2\sin^2\theta - 1}{\sin\theta\cos\theta}$$

Sol :

$$\text{LHS} = \tan\theta - \cot\theta$$

$$= \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\ \cos\theta}$$

$$= \frac{\sin^2\theta - (1-\sin^2\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta - 1 + \sin^2\theta}{\sin\theta\cos\theta} = \frac{2\sin^2\theta - 1}{\sin\theta\cos\theta} = \text{RHS}$$

5. Prove the trigonometric identity

$$\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$$

Sol :

$$\text{LHS} = \sec^4\theta - \sec^2\theta$$

$$= \sec^2\theta(\sec^2\theta - 1)$$

$$= (1+\tan^2\theta)(1+\tan^2\theta - 1)$$

$$[\because \sec^2\theta = 1+\tan^2\theta]$$

$$= (1+\tan^2\theta)\tan^2\theta$$

$$= \tan^2\theta + \tan^4\theta = \text{RHS}$$

6. Prove the trigonometric identity

$$(\operatorname{cosec}\theta - \cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$$

Solution:

$$\text{LHS} = (\operatorname{cosec}\theta - \cot\theta)^2$$

$$= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1-\cos\theta}{1+\cos\theta} = \text{RHS}$$

7. Prove the trigonometric identity

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$$

Sol :

$$\text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)}{1+\sin\theta}\times\frac{(1-\sin\theta)}{(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2} = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$$

8. Prove the trigonometric identity

$$\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$$

Sol :

$$\text{LHS} = \frac{1-\sin\theta}{1+\sin\theta} = \frac{1-\sin\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$

$$= \frac{(1-\sin\theta)^2}{1-\sin^2\theta} = \frac{(1-\sin\theta)^2}{\cos^2\theta}$$

$$= \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= (\sec\theta - \tan\theta)^2 = \text{RHS}$$

9. Prove the trigonometric identity

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = (\operatorname{cosec}\theta - \cot\theta)$$

Sol :

$$\text{LHS} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}}$$

$$= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}$$

$$= \sqrt{\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2}$$

$$= \sqrt{(\operatorname{cosec}\theta - \cot\theta)^2}$$

$$= \operatorname{cosec}\theta - \cot\theta = \text{RHS}$$

10. Prove the trigonometric identity

$$\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = 2\sec\theta$$

Sol :

$$\text{LHS} = \frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta}$$

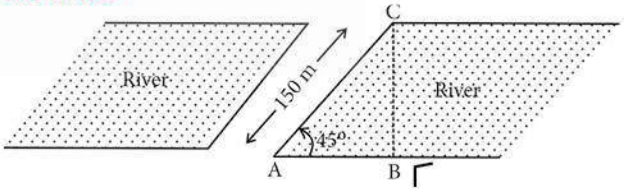
$$= \frac{\cos\theta\,(1+\sin\theta) + \cos\theta\,(1-\sin\theta)}{(1-\sin\theta)\,(1+\sin\theta)}$$

$$= \frac{\cos\theta + \cos\theta\sin\theta + \cos\theta - \cos\theta\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2\cos\theta}{\cos^2\theta}$$

$$= \frac{2}{\cos\theta} = 2\sec\theta = \text{RHS}$$

11. A bridge across a river makes an angle of 45° with the river bank. If the length of the bridge across the river is 150 m, what is the width of the river.



Sol :

Let BC be the width of the river.

AC is the length of the bridge.

From the right triangle CBA

$$\sin 45^\circ = \frac{BC}{AC}$$

$$\frac{1}{\sqrt{2}} = \frac{BC}{150}$$

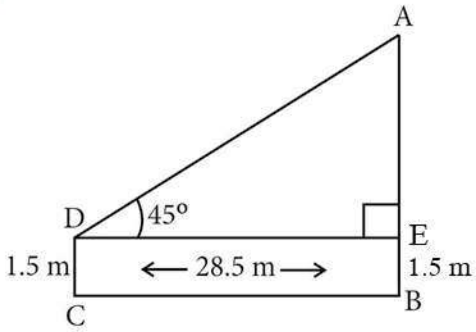
$$BC = \frac{150}{\sqrt{2}}$$

$$= \frac{75\sqrt{2}\,\sqrt{2}}{\sqrt{2}} = 75\sqrt{2}\,m$$

$\therefore$ Width of the river is $75\sqrt{2}\,m$.

12. An observer 1.5 m tall is 28.5 away from a tower. The angle of elevation of the top of the tower from her eyes is 45° what is the height of the tower?

Sol :



Let AB is the tower.

CD is the observer of height 1.5m.

CB is the Distance between the observer and tower.

From the right triangle ΔAED

$$\tan 45^\circ = \frac{AE}{DE}$$

$$1 = \frac{AE}{28.5}$$

$$AE = 28.5 \text{ m}$$

$$AB = AE + BE$$

$$= AE + DC$$

$$= (28.5 + 1.5) \text{ m} = 30 \text{ m}$$

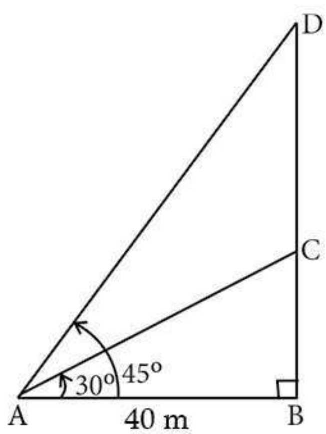
$\therefore$ Height of the tower is 30 m.

13. From a point on the ground 40 m away from the foot of a tower, the angle of elevation of the top of the tower is 30°. The angle of elevation of the top of the water tank on the tower is 45°. Find

(i) The height of the tower and

(ii) The depth of the tank.

Sol :



Let BC is the tower.

CD is the water tank.

In right triangle $\Delta\ ABD$

$$\tan 45^\circ = \frac{BD}{AB}$$

$$1 = \frac{BC+CD}{40}$$

$$BC + CD = 40 \text{ m} \quad \text{...(1)}$$

In the right triangle ΔABC

$$\tan 30^\circ = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{BC}{40}$$

$$BC = \frac{40}{\sqrt{3}} m$$

$$BC = 23.1 \text{ m.}$$

Substituting BC = 23.1 m in (1)

$$23.1 + CD = 40$$

$$CD = 40 - 23.1$$

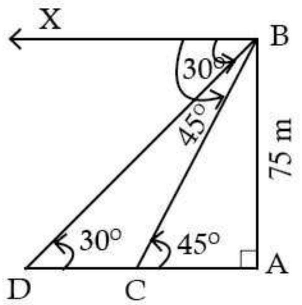
$$= 16.9 \text{ m.}$$

$\therefore$ Height of the tower = 23.1 m.

Depth of the tank = 16.9 m.

14. As observed from the top of a 75 m high lighthouse from the sea level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the light house find the distance between the two ships.

Sol :



Let C and D be the ships.

In right triangle ΔABC

$$\frac{AB}{AC} = \tan 45^\circ$$

$$\frac{75}{AC} = 1$$

$$AC = 75 \text{ m}$$

In right triangle ΔABD

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{75}{AD}$$

$$AD = 75\sqrt{3}$$

$$\therefore CD = AD - AC$$

$$= 75\sqrt{3} - 75 = 75(\sqrt{3}-1)$$

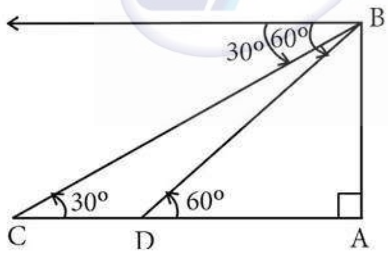
$$= 75\,[1.732 - 1]$$

$$= 75 \times 0.732 = 54.9 \text{ m}$$

$\therefore$ Distance between ships is 54.9 m.

15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30º, which is approaching the foot of the tower with a uniform speed six seconds later, the angle of depression of the car is found to be 60º. Find the time taken by the car to reach the foot of the tower from this point.

Sol :



Let AB be the height of the tower C and D be the two position of the car.

In right ΔABD, we have,

$$\frac{AB}{AD} = \tan 60^\circ$$

$$\frac{AB}{AD} = \sqrt{3}$$

$$AB = \sqrt{3}\ AD \quad \text{... (1)}$$

In right triangle ΔABC

$$\frac{AB}{AC} = \tan 30^\circ$$

$$\frac{AB}{AC} = \frac{1}{\sqrt{3}}$$

$$AB = \frac{AC}{\sqrt{3}} \quad \text{... (2)}$$

From (1) and (2)

$$\sqrt{3}\ AD = \frac{AC}{\sqrt{3}}$$

$$AC = \sqrt{3}\times\sqrt{3}\times AD = 3\ AD$$

$$CD = AC - AD$$

$$= 3\ AD - AD = 2\ AD$$

Since the distance 2AD is covered in 6 second, the distance AD will be covered in 6/2 = 3 seconds.

IV. Long Answer Questions

1. If $\tan^2\theta = 1 - a^2$ prove that

$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)\,\frac{3}{2}$$

Sol :

$$\text{LHS} = \sec\theta + \tan^3\theta \operatorname{cosec}\theta$$

$$= \sec\theta \left\{\frac{\sec\theta + \tan^3\theta \operatorname{cosec}\theta}{\sec\theta}\right\}$$

[$\because$ Multiplying and dividing by $\sec\theta$]

$$= \sec\theta \left\{\frac{\frac{1}{\cos\theta} + \tan^3\theta \operatorname{cosec}\theta}{\frac{1}{\cos\theta}}\right\}$$

$$= \sec\theta \left\{\frac{\frac{1+\tan^3\theta\cos\theta.\operatorname{cosec}\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right\}$$

$$= \sec\theta\ \frac{(1+\tan^3\theta\operatorname{cosec}\theta\sec\theta}{\cos\theta}\times\frac{\cos\theta}{1}$$

$$= \sec\theta\left[1+\tan^3\theta\,\frac{\cos\theta}{\sin\theta}\right]$$

$$= \sec\theta\,(1+\tan^3\theta\cot\theta)$$

$$= \sqrt{1+\tan^2\theta}\left\{1+\tan^3\theta\times\frac{1}{\tan\theta}\right\}$$

$$= \sqrt{1+\tan^2\theta}\,(1+\tan^2\theta)$$

$$= (1+\tan^2\theta)^{\frac{1}{2}}(1+\tan^2\theta)$$

$$= (1+\tan^2\theta)^{\frac{3}{2}} = \left(1+(1-a^2)\right)^{\frac{3}{2}}$$

$$= \left(1+1-a^2\right)^{\frac{3}{2}} = (2-a^2)^{\frac{3}{2}} = \text{RHS}$$

2. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$. prove that $a^2 + b^2 = m^2 + n^2$.

Sol :

Given $a\cos\theta + b\sin\theta = m$

$a\sin\theta - b\cos\theta = n$

$$\text{RHS} = m^2 + n^2$$
$$= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$
$$= a^2\cos^2\theta + b^2\sin^2\theta + ab\sin\theta\cos\theta + a^2\sin^2\theta + b^2\cos^2\theta - ab\sin\theta\cos\theta$$
$$= a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$
$$= a^2 + b^2 = \text{LHS}$$

3. If $\text{cosec}\,\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, prove that $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$.

Sol :

Given $\text{cosec}\,\theta - \sin\theta = m$

$$\Rightarrow \frac{1}{\sin\theta} - \sin\theta = m$$
$$\frac{1-\sin^2\theta}{\sin\theta} = m$$
$$\frac{\cos^2\theta}{\sin\theta} = m$$

Also $\sec\theta - \cos\theta = n$

$$\frac{1}{\cos\theta} - \cos\theta = n$$
$$\Rightarrow \frac{1-\cos^2\theta}{\cos\theta} = n$$
$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = n$$

$$\text{LHS} = (m^2n)^{2/3} + (mn^2)^{2/3}$$
$$= \left(\frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right)^{\frac{2}{3}} + \left(\frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right)^{\frac{2}{3}}$$
$$= (\cos^3\theta)^{\frac{2}{3}} + (\sin^3\theta)^{\frac{2}{3}}$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1 = \text{RHS}$$

4. If tan A = n tan B and sin A = m sin B, Prove that $\cos^2 A = \frac{m^2-1}{n^2-1}$.

Sol : Given $\tan A = n\tan B$

$$\Rightarrow \tan B = \frac{1}{n}\tan A$$
$$\Rightarrow \frac{1}{\tan B} = \frac{n}{\tan A}$$
$$\Rightarrow \cot B = \frac{n}{\tan A} \quad \text{... (1)}$$

Also $\sin A = m\sin B$

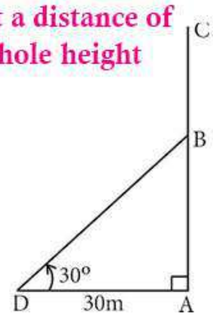
$$\Rightarrow \sin B = \frac{1}{m}\sin A$$
$$\Rightarrow \frac{1}{\sin B} = \frac{m}{\sin A}$$
$$\Rightarrow \text{cosec}\, B = \frac{m}{\sin A} \quad \text{... (2)}$$

We know that $\text{cosec}^2\theta - \cot^2\theta = 1$

Now $\text{cosec}^2 B - \cot^2 B = 1$

$$\Rightarrow \frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$
$$\Rightarrow \frac{m^2}{\sin^2 A} - n^2\frac{\cos^2 A}{\sin^2 A} = 1$$
$$\frac{m^2 - n^2\cos^2 A}{\sin^2 A} = 1$$
$$m^2 - n^2\cos^2 A = \sin^2 A$$
$$m^2 = \sin^2 A + n^2\cos^2 A$$
$$m^2 = 1 - \cos^2 A + n^2\cos^2 A$$
$$\Rightarrow m^2 - 1 = n^2\cos^2 A - \cos^2 A$$
$$\Rightarrow m^2 - 1 = (n^2 - 1)\cos^2 A$$
$$\Rightarrow \frac{m^2-1}{n^2-1} = \cos^2 A$$
$$\cos^2 A = \frac{m^2-1}{n^2-1}$$

5. A tree is broken by the wind. the top struck the ground at an angle of 30° and at a distance of 30 m from the root. Find the whole height of the tree.



Sol :

Let AC be the tree.

BD be the broken part of the tree.

BD = BC

In the right triangle ΔABD

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{30}$$

$$AB = \frac{30}{\sqrt{3}}\,m \qquad ...(1)$$

Also In $\Delta\, ABD$

$$\cos 30^\circ = \frac{AD}{BD}$$

$$\frac{\sqrt{3}}{2} = \frac{30}{BD}$$

$$BD = \frac{2\times 30}{\sqrt{3}}\,m = \frac{60}{\sqrt{3}}\,\text{m} \qquad ...(2)$$

$\therefore$ Height of the tree $= AB + BC$

$= AB + BD$

$= \dfrac{30}{\sqrt{3}} + \dfrac{60}{\sqrt{3}}$

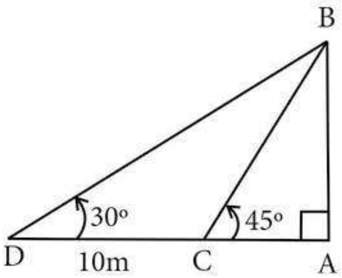
$= \dfrac{30+60}{\sqrt{3}} = \dfrac{90}{\sqrt{3}}$

$= \dfrac{30\times\sqrt{3}\times\sqrt{3}}{\sqrt{3}} = 30\sqrt{3}$ m

$\therefore$ Height of the tree $= 30\sqrt{3}\,m$.

6. The shadow of a vertical tower on level ground increases by 10 m, when the altitude of the sun changes from angle of elevation 45° to 30°. Find the height of the tower, correct to one place of decimal $\left(\sqrt{3} = 1.732\right)$

Sol :



Let AB is the tower.

AC and AD are shadows when the angle of elevation of the sun are 45° and 30° respectively.

$$CD = 10 \text{ m}$$

In $\Delta\, CAB$

$$\tan 45^\circ = \frac{AB}{AC}$$

$$1 = \frac{AB}{AC}$$

$$AC = AB \qquad ...(1)$$

In the right triangle ΔDAB

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{AC+CD} = \frac{AB}{AC+10}$$

$$AC + 10 = AB\sqrt{3} \qquad ...(2)$$

Using (1) $AC + 10 = \sqrt{3}\,AC$

$$\sqrt{3}\,AC - AC = 10$$

$$\left(\sqrt{3}-1\right)AC = 10$$

$$AC = \frac{10}{\sqrt{3}-1}$$

$$AC = \frac{10}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{10\left(\sqrt{3}+1\right)}{3-1}$$

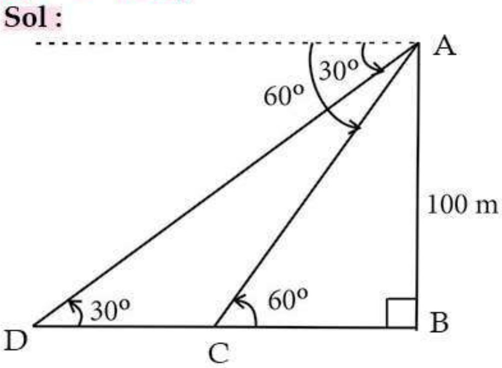
$$= \frac{10\left(\sqrt{3}+1\right)}{2} = 5\left(\sqrt{3}+1\right)$$

$$= 5\,(1.732 + 1) = 13.65 \text{ m}$$

$\therefore$ Height of the tower $= 13.65$ m.

7. As observed from the top of a light house 100 m high above sea level, the angle of depression of a ship sailing directly towards it, changes from 30° to 60°. Determine the distances travelled by the ship during the period of observation [$\sqrt{3}$ = 1.732]

Sol :



Let A represents the position of the observer

$$AB = 100$$

In right triangle ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{100}{BC}$$

$$BC = \frac{100}{\sqrt{3}}$$

$$= \frac{100\sqrt{3}}{3}$$

$$= \frac{100 \times 1.732}{3} = 57.73 \text{ m}$$

In right triangle ΔABD

$$\frac{AB}{BD} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{100}{BD}$$

$$BD = \sqrt{3} \times 100$$

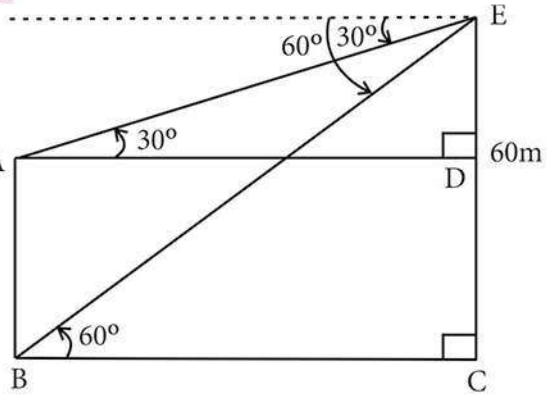
$$= 1.732 \times 100 = 173.2$$

$\therefore$ Distance travelled CD = BD – BC

$$= 173.2 - 57.73 = 115.47 \text{ m.}$$

8. From the top of a building 60 m high, the angles of depression of the top and bottom of a vertical lamp post are observed to be 30° and 60° respectively. Find (i) The horizontal distance between the building and the lamp post. (ii) The height of the lamp post ($\sqrt{3}$ = 1.732)

Sol :



Let CE be the building and AB be the lamp post

$$CE = 60 \text{ m}$$

In right ΔBCE

$$\frac{CE}{BC} = \tan 60^\circ$$

$$\sqrt{3} = \frac{60}{BC}$$

$$BC = \frac{60}{\sqrt{3}}$$

$$= \frac{60\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{60\sqrt{3}}{3}$$

$$BC = 20\sqrt{3}\ m \qquad \text{... (1)}$$

In right triangle ΔADE

$$\tan 30^\circ = \frac{DE}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{DE}{20\sqrt{3}}$$

[From (1) and BC = DE]

$$DE = \frac{20\sqrt{3}}{\sqrt{3}} = 20 \text{ m}$$

Height of the lamp post = AB = CD

= CE – DE

= 60 m – 20 m = 40 m

Distance between the lamp post and building

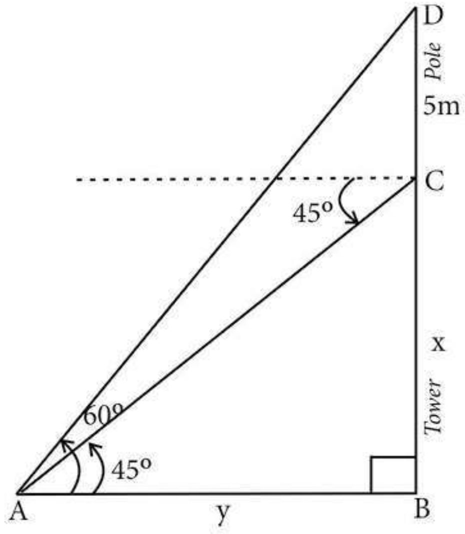
$= 20\sqrt{3}\ m$

$= 20 \times 1.732$ m

= 34.64 m.

9. A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point A on the ground is 60° and the angle of depression of the point A from the top of the tower is 45°. Find the height of the tower.

Sol :



In the figure, let BC be the tower and CD be the pole.

Let BC = 'x' m and AB = 'y' m

In right ΔABC

$$\frac{BC}{AB} = \tan 45^\circ = 1$$

$$BC = AB$$

$$y = x \qquad \ldots (1)$$

In right ΔABD

$$\frac{BD}{AB} = \tan 60^\circ = \sqrt{3}$$

$$\frac{x+5}{y} = \sqrt{3}$$

$$y\sqrt{3} = x + 5$$

$$x\sqrt{3} = x + 5 \quad [\because x = y \text{ from } (1)]$$

$$\sqrt{3}x - x = 5$$

$$(\sqrt{3}-1)x = 5$$

$$x = \frac{5}{\sqrt{3}-1}$$

$$= \frac{5}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{5(\sqrt{3}+1)}{3-1}$$

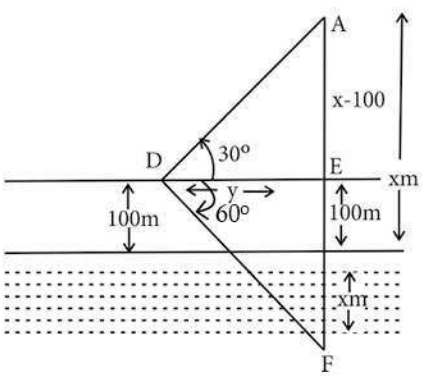
$$= \frac{5(1.732+1)}{2}$$

$$= \frac{5}{2} \times 2.732 = 6.83 \text{ m}$$

$\therefore$ Height of the tower is 6.83 m.

10. From a point 100 m above a lake the angle of elevation of a stationary helicopter is 30° and the angle of depression of reflection of the helicopter in the lake is 60°. Find the height of the helicopter.

Sol :



In the figure A is the stationary helicopter F is its reflection in the lake.

In right ΔAED

$$\tan 30^\circ = \frac{AE}{DE}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{AE}{DE}$$

$$\frac{1}{\sqrt{3}} = \frac{x-100}{y}$$

$$y = (x-100)\sqrt{3} \qquad \ldots (1)$$

In right ΔDEF

$$\tan 60^\circ = \frac{EF}{DE}$$

$$\frac{x+100}{y} = \sqrt{3}$$

$$\sqrt{3}\,y = x + 100$$

$$y = \frac{(x+100)}{\sqrt{3}} \qquad \ldots (2)$$

From (1) and (2) we have

$$\frac{x+100}{\sqrt{3}} = \sqrt{3}(x-100)$$

$$\sqrt{3}\times\sqrt{3}(x-100) = x + 100$$

$$3(x-100) = x + 100$$

$$3x - 300 - x - 100 = 0$$

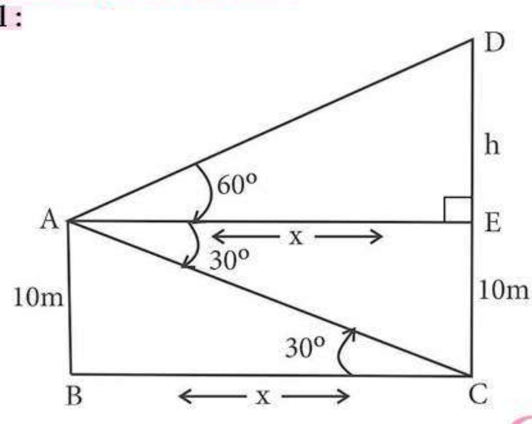
$$2x = 400$$

$$x = 200$$

$\therefore$ Height of the helicopter = 200 m.

11. A man standing on the deck of a ship, which is 10 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill.

Sol :



Let CD be the hill and the man is in A.

$$\angle EAD = 60^\circ ; \angle BCA = 30^\circ$$

In ΔAED

$$\tan 60^\circ = \frac{DE}{EA}$$

$$\sqrt{3} = \frac{h}{x}$$

$$h = \sqrt{3}\, x \quad \ldots (1)$$

In ΔABC

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{10}{x}$$

$$x = 10\sqrt{3} \quad \ldots (2)$$

From (1) and (2) $\quad h = \sqrt{3}\,(10\sqrt{3})$

$[\because x = 10\sqrt{3}$ in (1)]

$$= 10 \times 3 = 30$$

$$CD = CE + ED$$

$$= 10 + 30$$

$$= 40 \text{ m}$$

Distance of the hill from the ship is $10\sqrt{3}\, m$

Height of the hill = 40 m