समाकलन

Ex 9.1

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

(a) $\sqrt[3]{x^2}$ (b) e^{3x}

(c) $\left(\frac{1}{2}\right)^x$ (d) $a^{2\log_a x}$

हल :

(a) $$\int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx$$
$$= \frac{x^{2/3+1}}{\frac{2}{3}+1} + C$$
$$= \frac{x^{5/3}}{5/3} + C$$
$$= \frac{3}{5}x^{5/3} + C$$

(b) $$\int e^{3x}\, dx = \frac{e^{3x}}{3} + C$$

(c) $$\int \left(\frac{1}{2}\right)^x dx = \left(\frac{1}{2}\right)^x \Big/ \log_e \frac{1}{2} + C$$
$$= \frac{\left(\frac{1}{2}\right)^x}{\log_e\left(\frac{1}{2}\right)} + C$$

(d) $$\int a^{2\log_a x}\, dx = \int a^{\log_a x^2}\, dx$$
$$= \int x^2\, dx$$
$$= \frac{x^3}{3} + C$$

प्रश्न 2. निम्न समाकलों के मान ज्ञात कीजिए

$$\int\left(5\cos x-3\sin x+\frac{2}{\cos^2 x}\right)dx$$

हल:

$$\int\left(5\cos x-3\sin x+\frac{2}{\cos^2 x}\right)dx$$

$= 5\int\cos x\,dx - 3\int\sin x\,dx + 2\int\sec^2 x\,dx$
$= 5\sin x - 3(-\cos x) + 2\tan x + c$
$= 5\sin x + 3\cos x + 2\tan x + c$

प्रश्न 3.

$$\int\frac{x^3-1}{x^2}dx$$

हल:

$$\int\frac{x^3-1}{x^2}dx = \int\frac{x^3}{x^2}dx-\int\frac{1}{x^2}dx$$

$$= \int x\,dx-\int x^{-2}\,dx$$

$$= \frac{x^2}{2}-\frac{x^{-2+1}}{-2+1}$$

$$= \frac{x^2}{2}+x^{-1}+c$$

$$= \frac{x^2}{2}+\frac{1}{x}+c$$

प्रश्न 4. $\int(\sec^2 x + \operatorname{cosec}^2 x)\,dx$

हल : $\int(\sec^2 x + \operatorname{cosec}^2 x)\,dx$
$= \int\sec^2 x + \int\operatorname{cosec}^2\,dx$
$= \tan x - \cot x + c$

प्रश्न 5. $\int(1 + x) \sqrt{x}\, dx$

हल: $\int(1 + x) \sqrt{x}\, dx$

$$= \int (x^{1/2} + x^{3/2})\, dx$$

$$= \int x^{1/2}\, dx + \int x^{3/2}\, dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + c$$

$$= \frac{2}{3} x^{3/2} + \frac{2}{5} x^{5/2} + c$$

प्रश्न 6. $\int a^x\, da$

हल: $\int a^x\, da =$

$$= \frac{a^{x+1}}{x+1}$$

प्रश्न 7.

$$\int \frac{x^2}{1+x^2}\, dx$$

हल:

$$\int \frac{x^2}{1+x^2}\, dx = \int \frac{x^2+1-1}{1+x^2}\, dx$$

$$= \int \frac{1+x^2}{1+x^2}\, dx - \int \frac{1}{1+x^2}\, dx$$

$$= \int 1 . dx - \int \frac{1}{1+x^2}\, dx$$

$$= x - \tan^{-1} x + c$$

प्रश्न 8.

$$\int \frac{\cos^2 x}{1+\sin x}\, dx$$

हल:

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{1-\sin^2 x}{1+\sin x}\,dx$$

$$= \int \frac{(1-\sin x)(1+\sin x)}{(1+\sin x)}\,dx$$

= ∫(1 − sin x) dx
= ∫dx − ∫sin x dx
= x − (-cos x) + c
= x + cos x + c

प्रश्न 9. ∫sec x (sec x + tan x) dx

हल: ∫sec x (sec x + tan x) dx
= ∫sec² x dx + ∫sec x tan x dx
= tan x + sec x + c

प्रश्न 10. ∫($\sin^{-1}$ x + $\cos^{-1}$ x) dx

हल: ∫($\sin^{-1}$ x + $\cos^{-1}$ x) dx

$$= \int\left(\frac{\pi}{2}\right)dx \quad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{3}\right]$$

$$= \frac{\pi}{2}\int dx$$

$$= \frac{\pi}{2}x + c$$

प्रश्न 11.

$$\int \frac{x^2-1}{x^2+1}\,dx$$

हल:

$$\int \frac{x^2-1}{x^2+1}\,dx = \int \frac{x^2+1-2}{1+x^2}\,dx$$

$$= \int \frac{x^2+1}{1+x^2}\,dx - 2\int \frac{1}{1+x^2}\,dx$$

$$= \int dx - 2\int \frac{1}{1+x^2}\,dx$$

= x − 2 $\tan^{-1}$ x + c

प्रश्न 12. $\int \tan^2 x\, dx$

हल: $\int \tan^2 x\, dx = \int(\sec^2 x - 1)dx$
$= \int\sec^2 - \int dx$
$= \tan x - x + c$

प्रश्न 13. $\int\cot^2 x\, dx$

हल: $\int(\text{cosec}^2 x - 1)dx$
$= \int\text{cosec}^2\, dx - \int dx$
$= -\cot x - x + c$

प्रश्न 14.

$$\int \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

हल:

$$\int \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \int \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}\, dx$$
$$= \int \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}\, dx$$
$$= \int (1+x)^{1/2}\, dx + \int x^{1/2}\, dx$$
$$= \frac{(1+x)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + c$$
$$= \frac{2}{3}(1+x)^{3/2} + \frac{2}{3}x^{3/2} + c$$

प्रश्न 15. $\int(\tan^2 x - \cot^2 x)\, dx$

हल: $\int(\tan^2 x - \cot^2 x)\, dx$
$= \int(\sec^2 x - 1 - \text{cosec}^2 x + 1)\, dx$
$= \int\sec^2 x\, dx - \int\text{cosec}^2 x\, dx$
$= \tan x + \cot x + c$

प्रश्न 16.

$$\int \frac{\sin x}{1+\sin x}\, dx$$

हल:

$$\int \frac{\sin x}{1+\sin x} dx$$
$$= \int \frac{1+\sin x-1}{1+\sin x} dx$$
$$= \int \frac{1+\sin x}{1+\sin x} dx - \int \frac{1}{1+\sin x} dx$$
$$= \int dx - \int \frac{1}{1+\sin x} + \frac{1-\sin x}{1-\sin x} dx$$
$$= \int dx - \int \frac{1-\sin x}{1-\sin^2 x} dx$$
$$= \int dx - \int \frac{1-\sin x}{\cos^2 x} dx$$
$= \int x - \int \sec^2 x \, dx + \int \tan x \sec x \, dx$
$= x - \tan x + \sec x + c$

प्रश्न 17.
$$\int \frac{1}{1-\cos x} dx$$
हल:
$$\int \frac{1}{1-\cos x} dx$$
$$= \int \frac{1}{1+\cos x} \times \frac{1+\cos x}{1+\cos x} dx$$
$$= \int \left(\frac{1+\cos x}{1-\cos^2 x}\right) dx$$
$$= \int \left(\frac{1+\cos x}{\sin^2 x}\right) dx$$
$$= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx$$
$= \int \operatorname{cosec}^2 x \, dx + \int \operatorname{cosec} x \cot x \, dx$
$= -\cot x - \operatorname{cosec} x + c$

प्रश्न 18.
$$\int \left[1 + \frac{1}{1+x^2} + \frac{3}{x\sqrt{x^2-1}} + 2^x\right] dx$$

हल:

$$\int\left[1+\frac{1}{1+x^2}+\frac{3}{x\sqrt{x^2-1}}+2^x\right]dx$$

$$= \int 1\,dx+\int\frac{1}{1+x^2}\,dx+3\int\frac{1}{x\sqrt{x^2-1}}\,dx+\int 2^x\,dx$$

$$= \left[x+\tan^{-1}x+3\sec^{-1}x+\frac{2^x}{\log_e 2}+c\right]$$

प्रश्न 19. ∫cot x (tan x – cosec x) dx

हल: ∫cot x (tan x – cosec x) dx
= ∫cot x tan x dx – ∫cot x cosec x dx
= ∫1 dx – ∫cosec x cot x dx
= x + cosec x + c

प्रश्न 20.

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

हल:

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int\left(x+\frac{1}{x}+2\times\sqrt{x}\times\frac{1}{\sqrt{x}}\right)dx$$

$$= \int\left(x+\frac{1}{x}+2\right)dx$$

$$= \int x\,dx+\int\frac{1}{x}\,dx+\int 2\,dx$$

$$= \frac{x^2}{2}+\log|x|+2x+c$$

प्रश्न 21. ∫$\log_x$ x dx

हल: ∫$\log_x$ x dx

$$= \int\frac{\log x}{\log_e x}\,dx$$

=∫1 dx
= x + c

प्रश्न 22.

$$\int \sqrt{1+\cos 2x}\, dx$$

हल:

$$\int \sqrt{1+\cos 2x}\, dx$$

$$= \int \sqrt{1+2\cos^2 x-1}\, dx$$

$$= \int \sqrt{2\cos^2 x}\, dx$$

$= \sqrt{2}\int \cos x\, dx$

$= \sqrt{2} \sin x + c$

प्रश्न 23.

$$\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$$

हल:

$$\int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx - \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx$$

$= \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx$

$= -\cot x - \tan x + c$

प्रश्न 24.

$$\int \frac{3\cos x + 4}{\sin^2 x}\, dx$$

हल:

$$\int \frac{3\cos x + 4}{\sin^2 x}\, dx$$

$= 3\int \operatorname{cosec} x \cot x\, dx + 4\int \operatorname{cosec}^2 x\, dx$

$= -3 \operatorname{cosec} x + 4(-\cot x) + c$

$= -3 \operatorname{cosec} x - 4 \cot x + c$

Ex 9.2

निम्न फलनों को x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) $\int x \sin x^2\, dx$
(b) $\int x\sqrt{x^2+1}\, dx$

हल : (a) $\int x \sin x^2\, dx$

माना $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x \sin x^2\, dx = \int \sin x^2 . x\, dx$$

$$= \int \sin t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}(-\cos t) + C$$

$$= -\frac{1}{2}\cos x^2 + C$$

(b) $\int x\sqrt{x^2+1}\, dx$

माना $x^2 + 1 = t$

$\Rightarrow 2x\, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1} \cdot x\, dx$$

$$= \int t^{1/2}\frac{dt}{2} = \frac{1}{2}t^{3/2} \times \frac{2}{3} + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

प्रश्न 2.

(a) $\int \frac{e^x - \sin x}{e^x + \cos x}$ **(b)** $\int \frac{e^x}{\sqrt{1+e^x}}$

हल :

(a)

$$\int \frac{e^x - \sin x}{e^x + \cos x} dx$$

माना $e^x + \cos x = t$

$\Rightarrow (e^x - \sin x) = dt$

$$\int \frac{(e^x - \sin x)}{(e^x + \cos x)} dx = \int \frac{dt}{t}$$

$= \log |t| + C$

$= \log |e^x + \cos x| + C$

(b)

$$\int \frac{e^x}{\sqrt{1+e^x}} dx$$

माना

$1 + e^x = t$

$e^x \, dx = dt$

$$\int \frac{e^x}{\sqrt{1+e^x}} dx = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} \, dt$$

$$= \frac{t^{1/2}}{1/2} + C = 2(1 + e^x)^{1/2} + C$$

$$= 2\sqrt{1+e^x} + C$$

प्रश्न 3.

(a) $\int \sqrt{e^x + 1} \, dx$ **(b)** $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} dx$

हल :

(a)

$$\int \sqrt{e^x+1}\, dx$$

माना $\sqrt{e^x+1} = y$

$\Rightarrow \quad e^x + 1 = y^2$

$\Rightarrow \quad e^x = y^2 - 1$

$\Rightarrow \quad e^x\, dx = 2y\, dy$

$\Rightarrow \quad dx = \dfrac{2y}{e^x} dy$

$\Rightarrow \quad dx = \dfrac{2y}{y^2-1} dy$

$$\int \sqrt{e^x+1}\, dx = \int y \cdot \frac{2y}{y^2-1}\, dy$$

$$= 2\int \frac{y^2}{y^2-1}\, dy$$

$$= 2\int \frac{(y^2-1)+1}{y^2-1}\, dy$$

$$= 2\int 1\, dy + 2\int \frac{1}{y^2-1}\, dy$$

$$= 2y + 2 \times \frac{1}{2}\log\left(\frac{y-1}{y+1}\right) + C$$

$$= 2y + \log\left(\frac{y-1}{y+1}\right) + C$$

$$= 2\sqrt{e^x+1} + \log\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right) + C$$

$$= 2\sqrt{e^x+1} + \log\left[\frac{(\sqrt{e^x+1}-1)(\sqrt{e^x+1}+1)}{(\sqrt{e^x+1}+1)(\sqrt{e^x+1}+1)}\right] + C$$

$$= 2\sqrt{(e^x+1)} + \log\left[\frac{e^x+1-1}{e^x+1+1+2\sqrt{e^x+1}}\right] + C$$

$$= 2\sqrt{e^x+1} + \log\left[\frac{e^x}{e^x+2+2\sqrt{e^x+1}}\right] + C$$

(b)

$$\int \frac{e^{\sqrt{x}} \cos (e^{\sqrt{x}})}{\sqrt{x}} dx$$

माना $\qquad e^{\sqrt{x}} = t$

$\Rightarrow \qquad e^{\sqrt{x}} . \frac{d}{dx}(\sqrt{x}) = dt$

$\Rightarrow \qquad e^{\sqrt{x}} . \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \qquad \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2dt$

$$\int \frac{\cos (e^{\sqrt{x}}) . e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int \cos t \, dt = 2 \sin t + C$$

$$= 2 \sin (e^{\sqrt{x}}) + C$$

प्रश्न 4.

(a) $\frac{1}{x(1+\log x)}$ (b) $\frac{(1+\log x)^3}{x}$

हल :

(a) $\int \frac{1}{x(1+\log x)} dx$

माना $\qquad 1 + \log x = t$

$\Rightarrow \qquad \frac{1}{x} dx = dt$

$$\int \frac{1}{(1+\log x)} \cdot \frac{1}{x} dx = \int \frac{1}{t} dt$$

$$= \log |t| + C$$

$$= \log |1 + \log x| + C$$

(b) $\int \frac{(1+\log x)^3}{x} dx$

माना $\qquad 1 + \log x = t$

$$\Rightarrow \qquad \frac{1}{x}\,dx = dt$$

$$\int (1+\log x)^3 \cdot \frac{1}{x}\,dx = \int t^3\,dt = \frac{1}{4}t^4 + C$$

$$= \frac{1}{4}(1+\log x)^4 + C$$

प्रश्न 5.

(a) $\int \frac{e^{m \tan^{-1} x}}{1+x^2}\,dx$ **(b)** $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx$

हल :

(a) $\int \frac{e^{m \tan^{-1} x}}{1+x^2}\,dx$

माना $\qquad m \tan^{-1} x = t$

$$\Rightarrow \qquad \tan^{-1} x = \frac{t}{m}$$

$$\Rightarrow \qquad \frac{1}{1+x^2}\,dx = \frac{dt}{m}$$

$$\int \frac{e^{m \tan^{-1} x}}{1+x^2}\,dx = \int e^t \frac{dt}{m}$$

$$= \frac{1}{m}e^t + C$$

$$= \frac{1}{m}e^{m \tan^{-1} x} + C$$

(b) $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx = \int \frac{\sin^p x}{\cos^p x . \cos^2 x}\,dx$

$$= \int \tan^p x \sec^2 x\,dx$$

माना $\qquad \tan x = t$

$$\sec^2 x\,dx = dt$$

$$= \int t^p\,dt = \frac{t^{p+1}}{p+1} + C$$

$$= \frac{(\tan x)^{p+1}}{p+1} + C$$

प्रश्न 6.

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$ **(b)** $\int \frac{1+\cos x}{\sin x \cos x} dx$

हल :

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$

$$= \int \frac{1}{\sqrt{1+2\cos^2 x-1}} dx$$

$$= \int \frac{1}{\sqrt{2\cos^2 x}} dx$$

$$= \frac{1}{\sqrt{2}} \int \sec x \, dx$$

$$= \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + C$$

(b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

$$= \int \frac{1}{\sin x \cos x} dx + \int \frac{\cos x}{\sin x \cos x} dx$$

$$= 2\int \frac{1}{\sin 2x} dx + \int \frac{1}{\sin x} dx$$

$$= 2\int \operatorname{cosec} 2x \, dx + \int \operatorname{cosec} x \, dx$$

$$= \frac{2\log|\operatorname{cosec} 2x - \cot 2x|}{2} + \log|\operatorname{cosec} x - \cot x| + C$$

= log|cosec 2x – cot 2x| + log |cosec x – cot x |+ C

प्रश्न 7.

(a) ∫sin 3x sin 2x dx

(b) ∫$\sqrt{1-sinxdx}$

हल : (a) ∫sin 3x sin 2x dx

$$= \int \frac{1}{2}[\cos(3x-2x) - \cos(3x+2x)]\,dx$$

$$= \frac{1}{2}\int [\cos x - \cos 5x]\,dx$$

$$= \frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right) + C$$

$$= \frac{1}{2}\left(\sin x - \frac{1}{5}\sin 5x\right) + C$$

(b) $\int \sqrt{1-\sin x}\,dx$

$$= \int \sqrt{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)}\,dx$$

$$= \int \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}\,dx$$

$$= \int \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)dx$$

$$= \int \sin\frac{x}{2}\,dx - \int \cos\frac{x}{2}\,dx$$

$$= \frac{-\cos\frac{x}{2}}{\left(\frac{1}{2}\right)} - \frac{\sin\frac{x}{2}}{\left(\frac{1}{2}\right)} + C$$

$$= \pm 2\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + C$$

प्रश्न 8. (a) $\int\cos^4 x\,dx$
(b) $\int\sin^3 x\,dx$

हल : (a) $\int\cos x^4\,dx$
$= -\int(\cos^2 x)^2\,dx$

$$= \int\left(\frac{1+\cos 2x}{2}\right)^2 dx \quad \left(\because \cos^2 A = \frac{1+\cos 2A}{2}\right)$$
$$= \frac{1}{4}\int(1+\cos 2x)^2\ dx$$
$$= \frac{1}{4}\int(1+2\cos 2x+\cos^2 2x)\ dx$$
$$= \frac{1}{4}\int 1+2\cos 2x+\frac{1+\cos 4x}{2}\ dx$$
$$= \frac{1}{8}\int(2+4\cos 2x+1+\cos 4x)\ dx$$
$$= \frac{1}{8}\int(\cos 4x+4\cos 2x+3)\ dx$$
$$= \frac{1}{8}\left[\frac{\sin 4x}{4}+\frac{4\sin 2x}{2}+3x\right]+C$$
$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x+2\sin 2x+3x\right]+C$$

(b) $\int \sin^3 x\ dx$

$$= \int\left(\frac{3}{4}\sin x-\frac{1}{4}\sin 3x\right)dx$$
$$(\because\ \sin 3x = 3\sin x - 4\sin^3 x)$$
$$= \frac{3}{4}\int\sin x\ dx-\frac{1}{4}\int\sin 3x\ dx$$

$$= \frac{-3}{4}\cos x+\frac{\cos 3x}{4\times 3}+C$$
$$= \frac{-3}{4}\cos x+\frac{\cos 3x}{12}+C$$

प्रश्न 9.

(a) $\int\frac{1}{\sin x\cos^3 x}\ dx$ (b) $\int\frac{(1+x)e^x}{\cos^2(xe^x)}\ dx$

हल :

(a) $\int \frac{1}{\sin x \cos^3 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x.\cos^3 x} dx$$

$$= \int \left[\frac{\sin^2 x}{\sin x.\cos^3 x} + \frac{\cos^2 x}{\sin x.\cos^3 x}\right] dx$$

$$= \int \left[\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x.\cos^2 x}\right] dx$$

$$= \int \left[\tan x.\sec^2 + \frac{1}{\tan x} \cdot \sec^2 x\right] dx$$

$$= \int \left(\tan x + \frac{1}{\tan x}\right) \sec^2 x \, dx$$

माना tan x = t

$\sec^2 x \, dx = dt$

$$= \int \left(t + \frac{1}{t}\right) dt$$

$$= \frac{1}{2}t^2 + \log|t| + C$$

$$= \frac{1}{2} \tan^2 x + \log|\tan x| + C$$

$$= \log|\tan x| + \frac{1}{2} \tan^2 x + C$$

(b) $\int \frac{(1+x)e^x}{\cos^2 (xe^x)} dx$

माना $xe^x = t$

$\Rightarrow (xe^x + e^x.1)dx = dt$

$\Rightarrow (1 + x)e^x \, dx = dt$

$$\therefore \quad \int \frac{(1+x)e^x}{\cos^2(xe^x)}\,dx = \int \frac{1}{\cos^2 t}\,dt$$

$= \int \sec^2 t\, dt = \tan t + c$

$= \tan(xe^x) + c$

प्रश्न 10.

$$\text{(a)} \int \frac{1}{1-\tan x}\,dx \qquad \text{(b)} \int \frac{1}{1+\cot x}\,dx$$

हल :

$$\text{(a)} \int \frac{1}{1-\tan x}\,dx$$

$$= \int \frac{1}{1-\frac{\sin x}{\cos x}}\,dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{2\cos x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{\sin x + \cos x}{\cos x - \sin x}\,dx$$

माना $\cos x - \sin x = t$

$\Rightarrow$ (sin x + cos x) dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\cos x - \sin x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

(b) $\int \frac{1}{1+\cot x}\,dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}\,dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{(\sin x - \cos x)}{(\sin x + \cos x)}\,dx$$

माना sin x + cos x = t

$\Rightarrow$ -cos x + sin x dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

प्रश्न 11.

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$ (b) $\int \frac{1-\tan x}{1+\tan x} dx$

हल :

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}} dx$

माना tan x = t

⇒ $\sec^2 x$ dx = dt

$$= \int \frac{\sec^2 x . \sec^2 x \, dx}{\sqrt{\tan x}}$$

$$= \int \frac{(1+\tan^2 x)}{\sqrt{\tan x}} \sec^2 x \, dx$$

$$= \int \frac{1+t^2}{t^{1/2}} dt$$

$$= \int (t^{-1/2} + t^{2-1/2}) \, dt$$

$$= \int t^{-1/2} \, dt + \int t^{3/2} \, dt$$

$$= \frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C$$

(b) $$\int \left(\frac{1-\tan x}{1+\tan x}\right) dx = \int \left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right) dx$$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

माना sin x + cos x = t

(cos x − sin x) dx = dt

$= \int \frac{dt}{t}$

= log|t| + C

= log |sin x + cos x| + C

प्रश्न 12.

(a) $\int \frac{\sin(x-a)}{\sin(x+a)} dx$ (b) $\int \frac{\sin x}{\sin(x-a)} dx$

हल :

(a) $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

माना x + a = t

dx = dt

x = t − a

$$\therefore \int \frac{\sin(x-a)}{\sin(x+a)} dx$$

$$= \int \frac{\sin(t-a-a)}{\sin t} dt = \int \frac{\sin(t-2a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2a - \cos t \sin 2a}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2a}{\sin t} dt - \int \sin 2a \cdot \frac{\cos t}{\sin t} dt$$

$= \cos 2a \int dt - \sin 2a \int \cot dt$

$= (\cos 2a)t - \sin 2a \log |\sin t| + C1$

$= (x + a) \cos 2a - \sin 2a \log |\sin (x + a)| + C1$

$= x \cos 2a - \sin 2a \log |\sin(x + a)| + a \cos 2a + C1$

$= x \cos 2a - \sin 2a \log |\sin (x + a)| + C$

(जहाँ $C = a \cos 2a + C1$)

(b) $\int \frac{\sin x}{\sin(x-a)} dx$

माना x − a = t

x = t + a

dx = dt

$$= \int \frac{\sin(t+a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a + \sin a \cos t}{\sin t}$$

$$= \int \left(\frac{\sin t \cos a}{\sin t} + \frac{\sin a \cos t}{\sin t} \right) dt$$

$= \int \cos a \, dt + \int \sin a \cot dt$

$= \cos a.t + \sin a \log |\sin t|$

= (x – a) cos a + sin a log |sin (x – a)| + C1
= x cos a + sin a log |sin (x – a)| + (-a cos a + C1)
= x cos a + sin a log |sin (x -a)| + C
(जहाँ C = -a cos a + C1)

प्रश्न 13.

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$

हल :

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

$$= \int \frac{\sin (5x-3x)}{\sin 5x \sin 3x} dx$$

$$= \int \left[\frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} - \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x}\right] dx$$

$$= \int \cot 3x \, dx - \int \cot 5x \, dx$$

$$= \frac{\log|\sin 3x|}{3} - \frac{\log|\sin 5x|}{5} + C$$

$$= \frac{1}{3}\log|\sin 3x| - \frac{1}{5}\log|\sin 5x| + C$$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$

$$= \int \frac{\sin\left\{\left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)\right\}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} dx$$

$$= \int \left[\frac{\sin\left(x-\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} + \frac{\cos\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)} \right] dx$$

$$= \int \left[\cot\left(x+\frac{\pi}{6}\right) dx + \cot\left(x-\frac{\pi}{6}\right) \right] dx$$

$$= \log\left|\sin\left(x+\frac{\pi}{6}\right)\right| + \log\left|\sin\left(x-\frac{\pi}{6}\right)\right| + C$$

$$= \log\left[\sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)\right] + C$$

प्रश्न 14.

(a) $\int \frac{1}{3\sin x + 4\cos x}\, dx$

(b) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

हल :

(a) $\int \frac{1}{3\sin x + 4\cos x}\, dx$

माना 4 = sin θ तथा 3 = r cos θ

तब्र $r^2 \sin^2 \theta + r^2 \cos^2 \theta = 3^2 + 4^2 = 5^2$

$\Rightarrow r = 5$

तथा $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{4}{3} \Rightarrow \tan\theta = \dfrac{4}{3}$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore \qquad I = \int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{r\cos\theta\sin x + r\sin\theta\cos x}\,dx$$

$$= \frac{1}{r}\int \frac{1}{\sin(\theta + x)}\,dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(\theta + x)\,dx$$

$$= \frac{1}{5}\log|\operatorname{cosec}(\theta + x) - \cot(\theta + x)| + C$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{\theta + x}{2}\right)\right|$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{x + \tan^{-1}\left(\frac{4}{3}\right)}{2}\right)\right| + C$$

(b) $\displaystyle\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b-x+a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right] dx$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)]\, dx$$

$$= \frac{1}{\sin(a-b)} [\log|\sin(x-a)| - \log|\sin(x-b)| + C$$

$$= \frac{1}{\sin(a-b)} \left[\log \frac{|\sin(x-a)|}{|\sin(x-b)|}\right] + C$$

$$= \operatorname{cosec}(a-b) \log \left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$$

प्रश्न 15.

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

(b) $\int \frac{\sec x}{\sin(2x+\alpha) + \sin\alpha} dx$

हल :

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x} dx$

माना $a \cos^2 x + b \sin^2 x = t$

$\Rightarrow (-2a \cos x \sin x + 2b \sin x \cos x)\, dx = dt$

$\Rightarrow (2b \sin \cos x - 2a \sin x \cos x)\, dx = dt$

$\Rightarrow 2(b - a) \sin x \cos x\, dx = dt$

$$\Rightarrow \quad \sin x \cos x\, dx = \frac{1}{2(b-a)}\, dt$$

$$\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$$

$$= \int \frac{1}{t} \times \frac{1}{2(b-a)}\, dt$$

$$= \frac{1}{2(b-a)} \log|t| + C$$

$$= \frac{1}{2(b-a)} \log|a\cos^2 x + b\sin^2 x| + C$$

(b) $\displaystyle\int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\, dx$

माना $\displaystyle I = \int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\, dx$

$$= \int \frac{\sec x}{\sqrt{2\sin\left(\frac{2x+\alpha+\alpha}{2}\right)\cos\left(\frac{2x+\alpha-\alpha}{2}\right)}}\, dx$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= \int \frac{\sec x}{\sqrt{2\sin(x+\alpha)\cos x}}\, dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos x(\sin x\cos\alpha + \cos x\sin\alpha)}}\, dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos^2 x(\tan x\cos\alpha + \sin\alpha)}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec x}{\cos x\sqrt{\tan x\cos\alpha + \sin\alpha}}\, dx$$

$$I = \frac{1}{\sqrt{2}} \int \frac{\sec^2 x}{\sqrt{\tan x\cos\alpha + \sin\alpha}}\, dx \qquad \ldots(1)$$

माना $\tan x \cos\alpha + \sin\alpha = t$ ...(2)

$\Rightarrow \quad \sec^2 x \cos\alpha\, dx = dt$

$\Rightarrow \quad \sec^2 x\, dx = \dfrac{dt}{\cos\alpha}$

समी. (2) का उपयोग (1) में करने पर,

$$I = \frac{1}{\sqrt{2}} \int \frac{dt}{\cos\alpha\sqrt{t}}$$

$$= \frac{1}{\sqrt{2}.\cos\alpha} \int \frac{1}{\sqrt{t}}\, dt$$

$$= \frac{1}{\sqrt{2}.\cos\alpha} \left(\frac{t^{-1/2+1}}{-1/2+1} \right) + C$$

$$= \frac{2}{\sqrt{2}\cos\alpha} t^{1/2} + C$$

$$= \frac{\sqrt{2}}{\cos\alpha} (\tan x \cos\alpha + \sin\alpha)^{1/2} + C$$

$$\Rightarrow \quad I = \sqrt{2}\, \sec\alpha\, (\tan x \cos\alpha + \sin\alpha)^{1/2} + C$$

$$\therefore \int \frac{\sec x}{\sqrt{\sin(2x+\alpha) + \sin\alpha}}\, dx = \sqrt{2}\, \sec\alpha\, (\tan x \cos\alpha + \sin\alpha)^{1/2} + C$$

प्रश्न 16.

(a) $\displaystyle\int \frac{1}{\sqrt{\cos^3 x \sin(x+a)}}\, dx$

(b) $\displaystyle\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\, dx$

हल :

(a) $\displaystyle\int \frac{1}{\sqrt{\cos^3 x.\sin(x+a)}}\, dx$

$$= \int \frac{1}{\cos^2 x \sqrt{\left(\dfrac{\sin(x+a)}{\cos x} \right)}}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\left(\frac{\sin x \cos a}{\cos x} + \frac{\cos x \sin a}{\cos x}\right)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x . \cos a + \sin a}} dx$$

माना $\tan x \cos a + \sin a = t$

$$\cos a \sec^2 x \, dx = dt$$

$$\sec^2 x \, dx = \frac{1}{\cos a} dt$$

$$= \int \frac{1}{\sqrt{t}} \times \frac{1}{\cos a} dx$$

$$= \frac{1}{\cos a}\left(\frac{t^{-1/2+1}}{-1/2+1}\right) + C$$

$$= \frac{1}{\cos a} \times \frac{t^{1/2}}{1/2} + C$$

$$= \frac{2}{\cos a}\sqrt{\tan x \cos a + \sin a} + C$$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2 \alpha + 1)}{\cos x - \cos \alpha} dx$$

$$= 2 \int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha} dx$$

$$= 2 \int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{(\cos x - \cos \alpha)} dx$$

$= 2\int (\cos x + \cos \alpha)\, dx$

$= 2\int \cos x \, dx + 2\int \cos \alpha \, dx$

$= 2 \sin x + 2 \cos \alpha \int dx$

$= 2 \sin x + 2x \cos \alpha + C$

$= 2(\sin x + x \cos \alpha) + C$

Ex 9.1

निम्न फलनों को x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) $\int x \sin x^2 \, dx$
(b) $\int x\sqrt{x^2 + 1} \, dx$

हल : (a) $\int x \sin x^2 \, dx$

माना $x^2 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x \sin x^2 \, dx = \int \sin x^2 . x \, dx$$

$$= \int \sin t \cdot \frac{dt}{2}$$

$$= \frac{1}{2}(-\cos t) + C$$

$$= -\frac{1}{2}\cos x^2 + C$$

(b) $\int x\sqrt{x^2 + 1} \, dx$

माना $x^2 + 1 = t$

$\Rightarrow 2x \, dx = dt$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

$$\int x\sqrt{x^2+1} \, dx = \int \sqrt{x^2+1} \cdot x \, dx$$

$$= \int t^{1/2} \frac{dt}{2} = \frac{1}{2} t^{3/2} \times \frac{2}{3} + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

प्रश्न 2.

(a) $\displaystyle\int \frac{e^x - \sin x}{e^x + \cos x}$ **(b)** $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x}}$

हल :

(a)

$$\int \frac{e^x - \sin x}{e^x + \cos x} dx$$

माना $e^x + \cos x = t$

$\Rightarrow (e^x - \sin x) = dt$

$$\int \frac{(e^x - \sin x)}{(e^x + \cos x)} dx = \int \frac{dt}{t}$$

$= \log |t| + C$

$= \log |e^x + \cos x| + C$

(b)

$$\int \frac{e^x}{\sqrt{1+e^x}} dx$$

माना

$1 + e^x = t$

$e^x \, dx = dt$

$$\int \frac{e^x}{\sqrt{1+e^x}} dx = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} \, dt$$

$$= \frac{t^{1/2}}{1/2} + C = 2(1 + e^x)^{1/2} + C$$

$$= 2\sqrt{1+e^x} + C$$

प्रश्न 3.

(a) $\int \sqrt{e^x + 1} \, dx$ **(b)** $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} dx$

हल :

(a)

$$\int \sqrt{e^x + 1} \, dx$$

माना $\sqrt{e^x + 1} = y$

$\Rightarrow e^x + 1 = y^2$

$\Rightarrow e^x = y^2 - 1$

$\Rightarrow e^x \, dx = 2y \, dy$

$\Rightarrow dx = \frac{2y}{e^x} dy$

$$\Rightarrow \qquad dx = \frac{2y}{y^2-1}\,dy$$

$$\int\sqrt{e^x+1}\,dx = \int y\cdot\frac{2y}{y^2-1}\,dy$$

$$= 2\int\frac{y^2}{y^2-1}\,dy$$

$$= 2\int\frac{(y^2-1)+1}{y^2-1}\,dy$$

$$= 2\int 1\,dy + 2\int\frac{1}{y^2-1}\,dy$$

$$= 2y + 2\times\frac{1}{2}\log\left(\frac{y-1}{y+1}\right)+C$$

$$= 2y + \log\left(\frac{y-1}{y+1}\right)+C$$

$$= 2\sqrt{e^x+1}+\log\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)+C$$

$$= 2\sqrt{e^x+1}+\log\left[\frac{(\sqrt{e^x+1}-1)(\sqrt{e^x+1}+1)}{(\sqrt{e^x+1}+1)(\sqrt{e^x+1}+1)}\right]+C$$

$$= 2\sqrt{(e^x+1)}+\log\left[\frac{e^x+1-1}{e^x+1+1+2\sqrt{e^x+1}}\right]+C$$

$$= 2\sqrt{e^x+1}+\log\left[\frac{e^x}{e^x+2+2\sqrt{e^x+1}}\right]+C$$

(b)

$$\int\frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}\,dx$$

माना $\qquad e^{\sqrt{x}} = t$

$$\Rightarrow \qquad e^{\sqrt{x}}.\frac{d}{dx}(\sqrt{x}) = dt$$

$$\Rightarrow \qquad e^{\sqrt{x}}.\frac{1}{2\sqrt{x}}\,dx = dt$$

$$\Rightarrow \qquad \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2dt$$

$$\int \frac{\cos(e^{\sqrt{x}}).e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \cos t\,dt = 2\sin t + C$$

$$= 2\sin(e^{\sqrt{x}}) + C$$

प्रश्न 4.

(a) $\frac{1}{x(1+\log x)}$ **(b)** $\frac{(1+\log x)^3}{x}$

हल :

(a) $\int \frac{1}{x(1+\log x)}\,dx$

माना $\qquad 1 + \log x = t$

$$\Rightarrow \qquad \frac{1}{x}\,dx = dt$$

$$\int \frac{1}{(1+\log x)} \cdot \frac{1}{x}\,dx = \int \frac{1}{t}\,dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

(b) $\int \frac{(1+\log x)^3}{x}\,dx$

माना $\qquad 1 + \log x = t$

$$\Rightarrow \qquad \frac{1}{x}\,dx = dt$$

$$\int (1+\log x)^3 \cdot \frac{1}{x}\,dx = \int t^3\,dt = \frac{1}{4}t^4 + C$$

$$= \frac{1}{4}(1+\log x)^4 + C$$

प्रश्न 5.

(a) $\int \frac{e^{m\tan^{-1}x}}{1+x^2}\,dx$ **(b)** $\int \frac{\sin^p x}{\cos^{p+2} x}\,dx$

हल :

(a) $\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$

माना $\quad m \tan^{-1} x = t$

$\Rightarrow \quad \tan^{-1} x = \frac{t}{m}$

$\Rightarrow \quad \frac{1}{1+x^2} dx = \frac{dt}{m}$

$$\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx = \int e^t \frac{dt}{m}$$

$$= \frac{1}{m} e^t + C$$

$$= \frac{1}{m} e^{m \tan^{-1} x} + C$$

(b) $\int \frac{\sin^p x}{\cos^{p+2} x} dx = \int \frac{\sin^p x}{\cos^p x . \cos^2 x} dx$

$$= \int \tan^p x \sec^2 x \, dx$$

माना $\quad \tan x = t$

$\sec^2 x \, dx = dt$

$$= \int t^p \, dt = \frac{t^{p+1}}{p+1} + C$$

$$= \frac{(\tan x)^{p+1}}{p+1} + C$$

प्रश्न 6.

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$ (b) $\int \frac{1+\cos x}{\sin x \cos x} dx$

हल :

(a) $\int \frac{1}{\sqrt{1+\cos 2x}} dx$

$$= \int \frac{1}{\sqrt{1+2\cos^2 x - 1}} dx$$

$$= \int \frac{1}{\sqrt{2}\cos^2 x} dx$$

$$= \frac{1}{\sqrt{2}} \int \sec x \, dx$$

$$= \frac{1}{\sqrt{2}} \log|\sec x + \tan x| + C$$

(b) $\int \frac{1 + \cos x}{\sin x \cos x} dx$

$$= \int \frac{1}{\sin x \cos x} dx + \int \frac{\cos x}{\sin x \cos x} dx$$

$$= 2\int \frac{1}{\sin 2x} dx + \int \frac{1}{\sin x} dx$$

$$= 2\int \operatorname{cosec} 2x \, dx + \int \operatorname{cosec} x \, dx$$

$$= \frac{2\log|\operatorname{cosec} 2x - \cot 2x|}{2} + \log|\operatorname{cosec} x - \cot x| + C$$

= log|cosec 2x – cot 2x| + log |cosec x – cot x |+ C

प्रश्न 7.
(a) ∫sin 3x sin 2x dx
(b) ∫$\sqrt{1 - sin x dx}$

हल : (a) ∫sin 3x sin 2x dx

$$= \int \frac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)] dx$$

$$= \frac{1}{2} \int [\cos x - \cos 5x] dx$$

$$= \frac{1}{2}\left(\sin x - \frac{\sin 5x}{5}\right) + C$$

$$= \frac{1}{2}\left(\sin x - \frac{1}{5}\sin 5x\right) + C$$

(b) $\int \sqrt{1 - \sin x} \, dx$

$$= \int \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}\right)} dx$$

$$= \int \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}\, dx$$

$$= \int \left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) dx$$

$$= \int \sin\frac{x}{2}\, dx - \int \cos\frac{x}{2}\, dx$$

$$= \frac{-\cos\frac{x}{2}}{\left(\frac{1}{2}\right)} - \frac{\sin\frac{x}{2}}{\left(\frac{1}{2}\right)} + C$$

$$= \pm 2\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + C$$

प्रश्न 8. (a) $\int \cos^4 x\, dx$
(b) $\int \sin^3 x\, dx$

हल : (a) $\int \cos x^4\, dx$
$= -\int (\cos^2 x)^2\, dx$

$$= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \quad \left(\because \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$= \frac{1}{4}\int (1+\cos 2x)^2\, dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\, dx$$

$$= \frac{1}{8}\int (2 + 4\cos 2x + 1 + \cos 4x)\, dx$$

$$= \frac{1}{8}\int (\cos 4x + 4\cos 2x + 3)\, dx$$

$$= \frac{1}{8}\left[\frac{\sin 4x}{4} + \frac{4\sin 2x}{2} + 3x\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x + 2\sin 2x + 3x\right] + C$$

(b) $\int \sin^3 x\, dx$

$$= \int\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)dx$$
$$(\because \sin 3x = 3\sin x - 4\sin^3 x)$$
$$= \frac{3}{4}\int \sin x\, dx - \frac{1}{4}\int \sin 3x\, dx$$
$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{4 \times 3} + C$$
$$= \frac{-3}{4}\cos x + \frac{\cos 3x}{12} + C$$

प्रश्न 9.

(a) $\int \frac{1}{\sin x \cos^3 x}\, dx$ (b) $\int \frac{(1+x)e^x}{\cos^2(xe^x)}\, dx$

हल :

(a) $\int \frac{1}{\sin x \cos^3 x}\, dx$
$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x.\cos^3 x}\, dx$$
$$= \int\left[\frac{\sin^2 x}{\sin x.\cos^3 x} + \frac{\cos^2 x}{\sin x.\cos^3 x}\right]dx$$
$$= \int\left[\frac{\tan x}{\cos^2 x} + \frac{1}{\tan x.\cos^2 x}\right]dx$$
$$= \int\left[\tan x.\sec^2 + \frac{1}{\tan x}\cdot \sec^2 x\right]dx$$
$$= \int\left(\tan x + \frac{1}{\tan x}\right)\sec^2 x\, dx$$

माना tan x = t

$\sec^2 x\, dx = dt$

$$= \int\left(t + \frac{1}{t}\right) dt$$

$$= \frac{1}{2}t^2 + \log|t| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

$$= \log|\tan x| + \frac{1}{2}\tan^2 x + C$$

(b) $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$

माना $xe^x = t$

$\Rightarrow (xe^x + e^x.1)dx = dt$

$\Rightarrow (1 + x)e^x\, dx = dt$

$$\therefore \int \frac{(1+x)e^x}{\cos^2(xe^x)} dx = \int \frac{1}{\cos^2 t} dt$$

$= \int \sec^2 t\, dt = \tan t + c$

$= \tan(xe^x) + c$

प्रश्न 10.

(a) $\int \frac{1}{1-\tan x} dx$ **(b)** $\int \frac{1}{1+\cot x} dx$

हल :

(a) $\int \frac{1}{1-\tan x} dx$

$$= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int \frac{2 \cos x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int \frac{\cos x - \sin x + \cos x + \sin x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} \int 1\, dx + \frac{1}{2} \int \frac{\cos x + \sin x}{\cos x - \sin x}\, dx$$

$$= \frac{1}{2} x + \frac{1}{2} \int \frac{\sin x + \cos x}{\cos x - \sin x} dx$$

माना cosx – sinx = t

⇒ (sin x + cos x) dx = dt

$$= \frac{1}{2} x + \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} x + \frac{1}{2} \log |t| + C$$

$$= \frac{1}{2} x + \frac{1}{2} \log |\cos x - \sin x| + C$$

$$= \frac{1}{2} [x + \log |\sin x - \cos x| + C$$

(b) $\int \frac{1}{1 + \cot x}\, dx$

$$= \int \frac{1}{1 + \frac{\cos x}{\sin x}}\, dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{2 \sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int \frac{\sin x + \cos x + \sin x - \cos x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)}\, dx$$

माना sin x + cos x = t

⇒ -cos x + sin x dx = dt

$$= \frac{1}{2}x + \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \frac{1}{2}x + \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

$$= \frac{1}{2}[x + \log|\sin x - \cos x| + C$$

प्रश्न 11.

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$ (b) $\int \frac{1 - \tan x}{1 + \tan x}\,dx$

हल :

(a) $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$

माना tan x = t

⇒ $\sec^2 x\,dx = dt$

$$= \int \frac{\sec^2 x.\sec^2 x\,dx}{\sqrt{\tan x}}$$

$$= \int \frac{(1 + \tan^2 x)}{\sqrt{\tan x}}\sec^2 x\,dx$$

$$= \int \frac{1 + t^2}{t^{1/2}}\,dt$$

$$= \int (t^{-1/2} + t^{2-1/2})\,dt$$

$$= \int t^{-1/2}\,dt + \int t^{3/2}\,dt$$

$$= \frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2} + C$$

$$= 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C$$

(b) $\int\left(\frac{1-\tan x}{1+\tan x}\right)dx = \int\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)dx$

$$= \int\frac{\cos x-\sin x}{\cos x+\sin x}\,dx$$

माना sin x + cos x = t

(cos x − sin x) dx = dt

$= \int \frac{dt}{t}$

= log|t| + C

= log |sin x + cos x| + C

प्रश्न 12.

(a) $\int\frac{\sin(x-a)}{\sin(x+a)}dx$ (b) $\int\frac{\sin x}{\sin(x-a)}dx$

हल :

(a) $\int\frac{\sin(x-a)}{\sin(x+a)}dx$

माना x + a = t

dx = dt

x = t − a

$\therefore \int\frac{\sin(x-a)}{\sin(x+a)}dx$

$$= \int\frac{\sin(t-a-a)}{\sin t}dt = \int\frac{\sin(t-2a)}{\sin t}dt$$

$$= \int\frac{\sin t\cos 2a-\cos t\sin 2a}{\sin t}dt$$

$$= \int\frac{\sin t\cos 2a}{\sin t}dt - \int\sin 2a\cdot\frac{\cos t}{\sin t}dt$$

= cos 2a ∫ dt − sin 2a ∫ cot dt

= (cos 2a)t − sin 2a log |sin t| + C1

= (x + a) cos 2a − sin 2a log |sin (x + a) | + C1

= x cos 2a − sin 2a log |sin(x + a)| + a cos 2a + C1

= x cos 2a − sin 2a log |sin (x + a)| + C

(जहाँ C = a cos 2a + C1)

(b) $\int \frac{\sin x}{\sin (x-a)} dx$

माना x − a = t

x = t + a

dx = dt

$= \int \frac{\sin (t+a)}{\sin t} dt$

$= \int \frac{\sin t \cos a + \sin a \cos t}{\sin t}$

$= \int \left(\frac{\sin t \cos a}{\sin t} + \frac{\sin a \cos t}{\sin t} \right) dt$

= ∫cos a dt + ∫sin a cot dt

= cos a.t + sin a log |sin t|

= (x − a) cos a + sin a log |sin (x − a)| + C1

= x cos a + sin a log |sin (x − a)| + (-a cos a + C1)

= x cos a + sin a log |sin (x -a)| + C

(जहाँ C = -a cos a + C1)

प्रश्न 13.

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

(b) $\int \frac{\sin 2x}{\sin \left(x - \frac{\pi}{6} \right) \sin \left(x + \frac{\pi}{6} \right)} dx$

हल :

(a) $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$

$$= \int \frac{\sin (5x - 3x)}{\sin 5x \sin 3x} dx$$

$$= \int \left[\frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} - \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x} \right] dx$$

$$= \int \cot 3x \, dx - \int \cot 5x \, dx$$

$$= \frac{\log |\sin 3x|}{3} - \frac{\log |\sin 5x|}{5} + C$$

$$= \frac{1}{3} \log |\sin 3x| - \frac{1}{5} \log |\sin 5x| + C$$

(b) $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$

$$= \int \frac{\sin\left\{\left(x-\frac{\pi}{6}\right)+\left(x+\frac{\pi}{6}\right)\right\}}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}dx$$

$$= \int\left[\frac{\sin\left(x-\frac{\pi}{6}\right)\cos\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}+\frac{\cos\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}{\sin\left(x-\frac{\pi}{6}\right)\sin\left(x+\frac{\pi}{6}\right)}\right]dx$$

$$= \int\left[\cot\left(x+\frac{\pi}{6}\right)dx+\cot\left(x-\frac{\pi}{6}\right)\right]dx$$

$$= \log\left|\sin\left(x+\frac{\pi}{6}\right)\right|+\log\left|\sin\left(x-\frac{\pi}{6}\right)\right|+C$$

$$= \log\left[\sin\left(x+\frac{\pi}{6}\right)\sin\left(x-\frac{\pi}{6}\right)\right]+C$$

प्रश्न 14.

(a) $\int \frac{1}{3\sin x+4\cos x}\ dx$

(b) $\int \frac{1}{\sin(x-a)\sin(x-b)}\ dx$

हल :

(a) $\int \frac{1}{3\sin x+4\cos x}\ dx$

माना 4 = sin θ तथा 3 = r cos θ

तब्र $r^2 \sin^2\theta + r^2\cos^2\theta = 3^2 + 4^2 = 5^2$

$\Rightarrow r = 5$

तथा $\frac{r\sin\theta}{r\cos\theta}=\frac{4}{3} \Rightarrow \tan\theta=\frac{4}{3}$

$$\Rightarrow \qquad \theta=\tan^{-1}\left(\frac{4}{3}\right)$$

$$\therefore \qquad I=\int\frac{1}{3\sin x+4\cos x}\,dx$$

$$=\int\frac{1}{r\cos\theta\sin x+r\sin\theta\cos x}\,dx$$

$$=\frac{1}{r}\int\frac{1}{\sin(\theta+x)}\,dx$$

$$=\frac{1}{r}\int\operatorname{cosec}(\theta+x)\,dx$$

$$=\frac{1}{5}\log|\operatorname{cosec}(\theta+x)-\cot(\theta+x)|+C$$

$$=\frac{1}{5}\log\left|\tan\left(\frac{\theta+x}{2}\right)\right|$$

$$=\frac{1}{5}\log\left|\tan\left(\frac{x+\tan^{-1}\left(\frac{4}{3}\right)}{2}\right)\right|+C$$

(b) $\int\frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin(x-b-x+a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\,dx$$

$$=\frac{1}{\sin(a-b)}\int\left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)}-\frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\right]dx$$

$$=\frac{1}{\sin(a-b)}\int[\cot(x-a)-\cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-b)}[\log|\sin(x-a)| - \log|\sin(x-b)| + C$$

$$= \frac{1}{\sin(a-b)}\left[\log\frac{|\sin(x-a)|}{|\sin(x-b)|}\right] + C$$

$$= \operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$$

प्रश्न 15.

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

(b) $\int \frac{\sec x}{\sin(2x+\alpha)+\sin\alpha}\, dx$

हल :

(a) $\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$

माना $a\cos^2 x + b\sin^2 x = t$

$\Rightarrow (-2a\cos x\sin x + 2b\sin x\cos x)\, dx = dt$

$\Rightarrow (2b\sin\cos x - 2a\sin x\cos x)\, dx = dt$

$\Rightarrow 2(b-a)\sin x\cos x\, dx = dt$

$$\Rightarrow \sin x\cos x\, dx = \frac{1}{2(b-a)}\, dt$$

$$\int \frac{\sin x \cos x}{a\cos^2 x + b\sin^2 x}\, dx$$

$$= \int \frac{1}{t} \times \frac{1}{2(b-a)}\, dt$$

$$= \frac{1}{2(b-a)}\log|t| + C$$

$$= \frac{1}{2(b-a)}\log|a\cos^2 x + b\sin^2 x| + C$$

(b) $\int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\,dx$

माना $I = \int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\,dx$

$$= \int \frac{\sec x}{\sqrt{2\sin\left(\frac{2x+\alpha+\alpha}{2}\right)\cos\left(\frac{2x+\alpha-\alpha}{2}\right)}}\,dx$$

$$\left[\because \sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= \int \frac{\sec x}{\sqrt{2\sin(x+\alpha)\cos x}}\,dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos x(\sin x\cos\alpha + \cos x\sin\alpha)}}\,dx$$

$$= \int \frac{\sec x}{\sqrt{2\cos^2 x(\tan x\cos\alpha + \sin\alpha)}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec x}{\cos x\sqrt{\tan x\cos\alpha+\sin\alpha}}\,dx$$

$$I = \frac{1}{\sqrt{2}}\int \frac{\sec^2 x}{\sqrt{\tan x\cos\alpha+\sin\alpha}}\,dx \qquad \ldots(1)$$

$$\left.\begin{aligned} &\text{माना } \tan x\cos\alpha + \sin\alpha = t \\ \Rightarrow \quad &\sec^2 x\cos\alpha\, dx = dt \\ \Rightarrow \quad &\sec^2 x\, dx = \frac{dt}{\cos\alpha} \end{aligned}\right\} \qquad \ldots(2)$$

समी. (2) का उपयोग (1) में करने पर,

$$I = \frac{1}{\sqrt{2}} \int \frac{dt}{\cos\alpha\sqrt{t}}$$

$$= \frac{1}{\sqrt{2}.\cos\alpha} \int \frac{1}{\sqrt{t}}\, dt$$

$$= \frac{1}{\sqrt{2}.\cos\alpha}\left(\frac{t^{-1/2+1}}{-1/2+1}\right)+C$$

$$= \frac{2}{\sqrt{2}\cos\alpha} t^{1/2} + C$$

$$= \frac{\sqrt{2}}{\cos\alpha}(\tan x\cos\alpha + \sin\alpha)^{1/2} + C$$

$$\Rightarrow \quad I = \sqrt{2}\sec\alpha(\tan x\cos\alpha + \sin\alpha)^{1/2} + C$$

$$\therefore \int \frac{\sec x}{\sqrt{\sin(2x+\alpha)+\sin\alpha}}\, dx = \sqrt{2}\sec\alpha(\tan x\cos\alpha + \sin\alpha)^{1/2} + C$$

प्रश्न 16.

(a) $\int \frac{1}{\sqrt{\cos^3 x \sin(x+a)}}\, dx$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\, dx$

हल :

(a) $\int \frac{1}{\sqrt{\cos^3 x.\sin(x+a)}}\, dx$

$$= \int \frac{1}{\cos^2 x\sqrt{\left(\frac{\sin(x+a)}{\cos x}\right)}}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\left(\frac{\sin x\cos a}{\cos x} + \frac{\cos x\sin a}{\cos x}\right)}}\, dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x.\cos a + \sin a}} dx$$

माना $\tan x \cos a + \sin a = t$

$$\cos a \sec^2 x \, dx = dt$$

$$\sec^2 x \, dx = \frac{1}{\cos a} dt$$

$$= \int \frac{1}{\sqrt{t}} \times \frac{1}{\cos a} dx$$

$$= \frac{1}{\cos a}\left(\frac{t^{-1/2+1}}{-1/2+1}\right) + C$$

$$= \frac{1}{\cos a} \times \frac{t^{1/2}}{1/2} + C$$

$$= \frac{2}{\cos a}\sqrt{\tan x \cos a + \sin a} + C$$

(b) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$

$$= \int \frac{(2\cos^2 x - 1 - 2\cos^2 \alpha + 1)}{\cos x - \cos \alpha} dx$$

$$= 2\int \frac{\cos^2 x - \cos^2 \alpha}{\cos x - \cos \alpha} dx$$

$$= 2\int \frac{(\cos x + \cos \alpha)(\cos x - \cos \alpha)}{(\cos x - \cos \alpha)} dx$$

$= 2\int (\cos x + \cos \alpha)\, dx$
$= 2\int \cos x \, dx + 2\int \cos \alpha \, dx$
$= 2 \sin x + 2 \cos \alpha \int dx$
$= 2 \sin x + 2x \cos \alpha + C$
$= 2(\sin x + x \cos \alpha) + C$

Ex 9.3

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

(a) $\int \frac{1}{50+2x^2}\,dx$ (b) $\int \frac{1}{\sqrt{32-2x^2}}\,dx$

हल :

(a) $\int \frac{1}{50+2x^2}\,dx$

$$= \frac{1}{2}\int \frac{1}{25+x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{5^2+x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+5^2}\,dx$$

$$= \frac{1}{2}\times\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

$$= \frac{1}{10}\tan^{-1}\left(\frac{x}{5}\right)+C$$

(b) $\int \frac{1}{\sqrt{32-2x^2}}\,dx = \int \frac{1}{\sqrt{2}\sqrt{(4^2-x^2)}}\,dx$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x}{4}\right)+C$$

प्रश्न 2.

(a) $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$ (b) $\frac{1}{\sqrt{1+4x^2}}\,dx$

हल :

(a) $\int \frac{1}{\sqrt{1-e^{2x}}}\,dx$

$= \int \frac{1}{\sqrt{(1)^2-(e^x)^2}}\,dx$

माना $\qquad e^x = t$

$\Rightarrow \qquad e^x\,dx = dt$

$\Rightarrow \qquad dx = \frac{dt}{e^x}$

$\Rightarrow \qquad dx = \frac{dt}{t}$

$$= \int \frac{1}{\sqrt{1-t^2}}\,\frac{dt}{t}$$

$$= \int \frac{1}{t\sqrt{1-t^2}}\,dt$$

$$= \log|1-\sqrt{1-t^2}| + C$$

$$= \log|1-\sqrt{1-e^{2x}}| + C$$

(b) $\int \frac{1}{\sqrt{1+4x^2}}\,dx$

$$= \int \frac{1}{\sqrt{(1)^2+(2x)^2}}\,dx$$

$$= \frac{1}{2}\log|2x+\sqrt{1+(2x)^2}|+C$$

$$= \frac{1}{2}\log|2x+\sqrt{4x^2+1}|+C$$

प्रश्न 3.

(a) $\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx \quad$ (b) $\int \frac{1}{\sqrt{(2-x)^2+1}}\,dx$

हल :

(a) $\int \frac{1}{\sqrt{a^2 - b^2x^2}}\,dx$

$$= \int \frac{1}{\sqrt{a^2 - (bx)^2}}\,dx$$

$$= \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

(b) $\int \frac{1}{\sqrt{(2-x)^2 + 1}}\,dx$

$$= -[\log|(2-x) + \sqrt{(2-x)^2 + 1}|] + C$$

$$= -[\log|(2-x) + \sqrt{4 + x^2 - 4x + 1}|] + C$$

$$= -[\log|(2-x) + \sqrt{x^2 - 4x + 5}|] + C$$

प्रश्न 4.

(a) $\int \frac{x^2}{\sqrt{x^6 + 4}}\,dx$ (b) $\int \frac{x^4}{\sqrt{1 - x^{10}}}\,dx$

हल :

(a) $\int \frac{x^2}{\sqrt{x^6 + 4}}\,dx$

$$= \int \frac{x^2\,dx}{\sqrt{(x^3)^2 + 2^2}}$$

माना $x^3 = t$

$\Rightarrow$ $3x^2\,dx = dt$

$\Rightarrow$ $x^2\,dx = \frac{dt}{3}$

$$= \frac{1}{3}\int \frac{1}{\sqrt{t^2 + 2^2}}$$

$$= \frac{1}{3}\log|t + \sqrt{t^2 + 2^2}| + C$$

$$= \frac{1}{3}\log|x^3 + \sqrt{x^6 + 4}| + C$$

(b) $\int \frac{x^4}{\sqrt{1-x^{10}}}\,dx$

$$= \int \frac{x^4}{\sqrt{1-(x^5)^2}}\,dx$$

माना $\qquad x^5 = t$

तो $\qquad 5x^4\,dx = dt$

या $\qquad x^4\,dx = \frac{dt}{5}$

$$= \frac{1}{5}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{5}\sin^{-1} t + C$$

$$= \frac{1}{5}\sin^{-1}(x^5) + C$$

प्रश्न 5.

(a) $\int \frac{1}{x^2+6x+8}\,dx$ (b) $\frac{1}{\sqrt{2x^2-x+2}}\,dx$

हल :

(a) $\int \frac{1}{x^2+6x+8}\,dx$

$$= \int \frac{1}{x^2+2\times 3x+3^2-1}\,dx$$

$$= \int \frac{1}{(x+3)^2-1}\,dx$$

$$= \frac{1}{2}\log\left|\frac{(x+3)-1}{(x+3)+1}\right| + C$$

$$\left(\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right)$$

$$= \frac{1}{2}\log\left|\frac{x+2}{x+4}\right| + C$$

(b) $\int \frac{1}{\sqrt{2x^2 - x + 2}}\,dx = \int \frac{1}{\sqrt{2\left(x^2 - \frac{1}{2}x + 1\right)}}\,dx$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} + 1}}\,dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}}\,dx$$

सूत्र $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + C$ से

$$= \frac{1}{\sqrt{2}} \log\left|\left(x - \frac{1}{4}\right) + \sqrt{x^2 - \frac{1}{2}x + 1}\right| + C$$

प्रश्न 6.

(a) $\int \frac{e^x}{e^{2x} + 2e^x \cos x + 1}\,dx$

(b) $\int \frac{1 + \tan^2 x}{\sqrt{\tan^2 x + 3}}\,dx$

हल :

(a) $\int \frac{e^x}{e^{2x} + 2e^x \cos x + 1}\,dx$

$$= \int \frac{e^x}{\{(e^x)^2 + 2.e^x \cos x + \cos^2 x\} + \sin^2 x}\,dx$$

$$= \int \frac{e^x}{(e^x + \cos x)^2 + \sin^2 x}\,dx$$

$$= \frac{1}{\sin x} \tan^{-1}\left(\frac{e^x + \cos x}{\sin x}\right) + C$$

(b) $\int \frac{1 + \tan^2 x}{\sqrt{\tan^2 x + 3}}\,dx$

$$= \int \frac{\sec^2 x\,dx}{\sqrt{(\sqrt{3})^2 + \tan^2 x}}$$

माना tanx = t

$\sec^2 x\, dx = dt$

$$= \int \frac{dt}{\sqrt{(\sqrt{3})^2 + t^2}}$$

$$= \log|t + \sqrt{t^2+3}| + C$$

$$= \log|\tan x + \sqrt{\tan^2 x + 3}| + C$$

प्रश्न 7.

(a) $\int \frac{1}{\sqrt{3x-2-x^2}}\, dx$

(b) $\int \frac{1}{\sqrt{4+8x-5x^2}}\, dx$

हल :

(a) $\int \frac{1}{\sqrt{3x-2-x^2}}\, dx$

$$\therefore \quad 3x-2-x^2 = -(x^2-3x+2)$$

$$= -\left(x^2 - 2\times\frac{3}{2}\times x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2\right)$$

$$= -\left[\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} + 2\right]$$

$$= -\left[\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}\right] = \frac{1}{4} - \left(x-\frac{3}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{3x-2-x^2}}\, dx = \int \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2}}\, dx$$

$$= \sin^{-1}\left(\frac{x-3/2}{1/2}\right) + C$$

$$= \sin^{-1}(2x-3) + C$$

(b) $\int \frac{1}{\sqrt{4+8x-5x^2}}\,dx$

$\therefore\ 4+8x-5x^2 = -(5x^2-8x-4)$

$= -5\left(x^2 - \frac{8}{5}x - \frac{4}{5}\right)$

$= -5\left(x^2 - 2\times\frac{4}{5}\times x + \left(\frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2 - \frac{4}{5}\right)$

$= -5\left[\left(x-\frac{4}{5}\right)^2 - \frac{16}{25} - \frac{4}{5}\right]$

$= -5\left[\left(x-\frac{4}{5}\right)^2 - \left(\frac{6}{5}\right)^2\right] = 5\left(\frac{6}{5}\right)^2 - 5\left(x-\frac{4}{5}\right)^2$

$$\int \frac{1}{\sqrt{4+8x-5x^2}}\,dx = \int \frac{1}{\sqrt{5\left(\frac{6}{5}\right)^2 - 5\left(x-\frac{4}{5}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{\left(\frac{6}{5}\right)^2 - \left(x-\frac{4}{5}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{x-4/5}{6/5}\right) + C$$

$$= \frac{1}{\sqrt{5}} \sin^{-1}\left(\frac{5x-4}{6}\right) + C$$

प्रश्न 8.

(a) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

(b) $\int \frac{1}{\sqrt{x^2+2ax+b^2}}\,dx$

हल :

(a) $\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx = \int \frac{\sin x + \cos x}{1-(1-\sin 2x)}\,dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

माना $\sin x - \cos x = t$

$(\cos x + \sin x)dx = dt$

$$= \int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sin^{-1} t + C$$

$$= \sin^{-1} (\sin x - \cos x) + C$$

(b) $\int \frac{1}{\sqrt{x^2 + 2ax + b^2}} dx$

$$= \int \frac{1}{\sqrt{(x^2 + 2ax + a^2) - a^2 + b^2}} dx$$

$$= \int \frac{1}{\sqrt{(x+a)^2 + (b^2 - a^2)}} dx$$

$$= \log \left| (x+a) + \sqrt{x^2 + 2ax + b^2} \right| + C$$

प्रश्न 9.

(a) $\int \sqrt{\frac{a-x}{x}} dx$ **(b)** $\int \sqrt{\frac{a+x}{a-x}} dx$

हल :

(a) $\int \sqrt{\frac{a-x}{x}} dx$

माना $x = a \cos^2\theta$

$dx = a\, 2\cos\theta\, (-\sin\theta)\, d\theta$

$dx = -2a \sin\theta \cos\theta\, d\theta$

$\therefore \quad \cos^2\theta = \frac{x}{a}$ तथा $\sin^2\theta = \frac{a^2 - x^2}{a^2}$

$\therefore \quad \cos\theta = \sqrt{\frac{x}{a}}$

$\Rightarrow \quad \theta = \cos^{-1}\sqrt{\frac{x}{a}}$

$$\int \sqrt{\frac{a-x}{x}}\, dx = \int \sqrt{\frac{a - a\cos^2\theta}{a\cos^2\theta}}\, (-2a \sin\theta \cos\theta)\, d\theta$$

$$= \int \sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}}\,(-2a \sin\theta\cos\theta)\, d\theta$$

$$= \int -2a\sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \times \sin\theta\cos\theta\, d\theta$$

$$= -2a \int \frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta\, d\theta$$

$$= -2a \int \sin^2\theta\, d\theta = -2a\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -2a\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right] + C$$

$$= \frac{a}{2}\sin 2\theta - a\theta + C$$

$$= \frac{1}{2}a\,[\sin 2\theta - 2\theta] + C$$

$$= \frac{1}{2}a\left[\sin 2\left(\cos^{-1}\sqrt{\frac{x}{a}}\right) - 2\cos^{-1}\sqrt{\frac{x}{a}}\right] + C$$

(b) $\int \sqrt{\left(\frac{a+x}{a-x}\right)}\, dx$

माना $x = a\cos\theta$

$dx = -a\sin\theta\, d\theta$

$$\therefore \int \sqrt{\left(\frac{a+x}{a-x}\right)}\, dx = \int \sqrt{\left(\frac{a+a\cos\theta}{a-a\cos\theta}\right)} \cdot (-a\sin\theta\, d\theta)$$

$$= -a\int \sqrt{\left(\frac{1+\cos\theta}{1-\cos\theta}\right)} \cdot \sin\theta\, d\theta$$

$$= -a\int \sqrt{\left[\frac{1+(2\cos^2\theta/2-1)}{1-(1-2\sin^2\theta/2)}\right]} \cdot 2\sin\theta/2\cos\theta/2\, d\theta$$

$$= -2a\int \frac{\cos\theta/2}{\sin\theta/2} \cdot \sin\theta/2\cos\theta/2\, d\theta$$

$$= -a\int 2\cos^2\theta/2\, d\theta = -a\int (1+\cos\theta)\, d\theta$$

$$= -a[\theta + \sin\theta] + c = -a\theta - a\sqrt{1-\cos^2\theta} + c$$

$$= -a\cos^{-1}(x/a) - a\sqrt{[1-(x/a)^2]} + c$$

$$= -a\cos^{-1}(x/a) - \sqrt{a^2 - x^2} + c$$

प्रश्न 10.

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\, dx$$

हल :

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}}\, dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\, dx$$

$$= \frac{2}{3} \int \frac{1}{\sqrt{(a^{3/2})^2 - t^2}}\, dt \qquad \text{माना } x^{3/2} = t$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{t}{a^{3/2}}\right) + c \qquad \frac{3}{2}\, x^{1/2}\, dx = dt$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + c \qquad \sqrt{x}\, dx = \frac{2}{3}\, dt$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x}{a}\right)^{3/2} + c$$

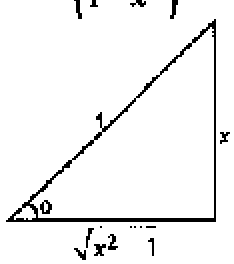
प्रश्न 11.

(a) $\int \frac{1}{(1 - x^2)^{3/2}}\, dx$

(b) $\int \frac{x + 1}{\sqrt{x^2 + 1}}\, dx$

हल :

(a) $\int \frac{1}{(1 - x^2)^{3/2}}\, dx$



माना $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int \frac{1}{(1-\sin^2\theta)^{3/2}} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos^2\theta} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cos^3\theta} d\theta = \int \sec^2\theta d\theta$$

$$= \tan\theta + C$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

(b) $$\int \frac{x+1}{\sqrt{x^2+1}} dx = \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= I_1 + I_2$$

$$I_1 = \int \frac{x}{\sqrt{x^2+1}} dx$$

माना $x^2 + 1 = t$

$2x \, dx = dt$

$$x \, dx = \frac{1}{2} dt$$

$\Rightarrow$ $$I_1 = \frac{1}{2} \int \frac{1}{t^{1/2}} dt$$

$\Rightarrow$ $$I_1 = \frac{1}{2} \left[\frac{t^{-1/2+1}}{-1/2+1}\right] + C_1$$

$\Rightarrow$ $$I_1 = \frac{1}{2} \left[\frac{t^{1/2}}{1/2}\right] + C_1$$

$\Rightarrow$ $$I_1 = \sqrt{t} + C_1$$

$\Rightarrow$ $$I_1 = \sqrt{x^2+1} + C_1 \qquad \text{...(ii)}$$

$$I_2 = \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= \log\left|x + \sqrt{x^2+1}\right| + C_2$$

$$\int \frac{x+1}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + \log\left|x+\sqrt{x^2+1}\right| + (C_1+C_2)$$

$$= \sqrt{x^2+1} + \log\left(x+\sqrt{x^2+1}\right) + C,$$

जहाँ $(C_1 + C_2) = C$

प्रश्न 12.

(a) $\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$

(b) $\int \frac{1}{\sqrt{2x-x^2}}\,dx$

हल :

(a) $\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$

$x = \alpha \cos^2\theta + \beta \sin^2\theta$

$dx = (\beta - \alpha) \sin 2\theta \, d\theta$

$\alpha(1 - \sin^2 \theta) + \beta \sin^2\theta = x$

$\alpha + \beta \sin^2\theta - \alpha \sin^2\theta = x$

$(\beta - \alpha) \sin^2\theta = x - \alpha$

$$\Rightarrow \quad \sin^2\theta = \frac{x-\alpha}{\beta-\alpha}$$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

प्रश्नानुसार,

$$\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$$

$$= \int \frac{(\beta-\alpha)\sin 2\theta}{\sqrt{\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}\,d\theta$$

$$\Rightarrow \qquad \sin^2\theta = \frac{x-\alpha}{\beta-\alpha}$$

$$\Rightarrow \qquad \sin\theta = \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\Rightarrow \qquad \theta = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

प्रश्नानुसार,

$$\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$$

$$= \int \frac{(\beta-\alpha)\sin 2\theta}{\sqrt{(\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}\,d\theta$$

$$= \int \frac{(\beta-\alpha)\sin 2\theta}{\sqrt{(\beta\sin^2\theta-\alpha\sin^2\theta)(\beta\cos^2\theta-\alpha\cos^2\theta)}}\,d\theta$$

$$= \int \frac{(\beta-\alpha)\cdot\sin 2\theta}{\sqrt{(\beta-\alpha)\sin^2\theta\times(\beta-\alpha)\cos^2\theta}}\,d\theta$$

$$= \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\sin\theta\cos\theta}\,d\theta$$

$$= \int 2\,d\theta = 2\theta + C = \sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$$

(b) $$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{-(x^2-2x+1-1)}}\,dx$$

$$= \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \sin^{-1}(x-1) + C$$

प्रश्न 13.

(a) $$\int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx$$

(b) $$\int \frac{\cos x}{\sqrt{4-\sin 2x}}\,dx$$

हल :

(a) $\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{x^2-3x+2}} dx$

$$= \int \frac{1}{\sqrt{x^2-2\times\frac{3}{2}\times x+\frac{9}{4}-\frac{9}{4}+2)}} dx$$

$$= \int \frac{1}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2} dx$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

(b) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\ dx$

माना $\quad \sin x = t$

$\cos x\, dx = dt$

$$= \int \frac{dt}{\sqrt{2^2-t^2}}$$

$$= \sin^{-1} \frac{t}{2}+C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right)+C$$

Ex 9.4

निम्नलिखित फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.

$$\int \frac{1}{16-9x^2}\,dx$$

हल:

$$\int \frac{1}{16-9x^2}\,dx$$

$$= \int \frac{1}{4^2-(3x)^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{4\times 3} \log\left[\frac{4+3x}{4-3x}\right] + C$$

$$\left(\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + C\right)$$

$$= \frac{1}{24}\log\left[\frac{4+3x}{4-3x}\right] + C$$

प्रश्न 2.

$$\int \frac{1}{x^2-36}\,dx$$

हल :

$$\int \frac{1}{x^2-36}\,dx = \int \frac{1}{x^2-6^2}\,dx$$

$$= \frac{1}{2\times 6}\log\left|\frac{x-6}{x+6}\right| + C$$

$$= \frac{1}{12}\log\left|\frac{x-6}{x+6}\right| + C$$

प्रश्न 3.

$$\int \frac{3x}{(x+1)(x-2)}\,dx$$

हल : $\int \frac{3x}{(x+1)(x-2)}dx$

$\therefore \frac{3x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$

$\Rightarrow 3x = Ax - 2A + Bx + B$

$\Rightarrow 3x = (A + B)x - 2A + B$

तुलना करने पर, $A + B = 3$ ....(1)

तथा $-2A + B = 0$

$B = 2A$ ....(2)

समी. (2) से B का मान समी. (1) में रखने पर,

$\Rightarrow A + 2A = 3 \Rightarrow A - 1$

माना $B = 2 \times 1 = 2$

$\therefore \frac{3x}{(x+1)(x-2)} = \frac{1}{x+1} + \frac{2}{x-2}$

$\int \frac{3x}{(x+1)(x-2)}dx = \int \frac{1}{x+1}dx + 2\int \frac{1}{(x-2)}dx$

$= \log|x+1| + 2\log|x-2| + C$

प्रश्न 4.

$\int \frac{3x-2}{(x+1)^2(x+3)}dx$

हल :

$$\int \frac{3x-2}{(x+1)^2(x+3)}dx$$

$\therefore \frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\Rightarrow 3x - 2 = A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2$

$\Rightarrow 3x - 2 = A(x^2 + 4x + 3) + Bx + 3B + C(x^2 + 2x + 1)$

$\Rightarrow 3x - 2 = (A + C)x^2 + (4A + B + 2C)x + 3A + 3B + C$

तुलना से,

$A + C = 0$ तो $A = -C$ ....(i)

$4A + B + 2C = 3$

$4A + B - 2A = 3$

$24 + B = 3$ ...(ii)

$3A + 3B + C = -2$

$3A + 3B - A = -2$

$2A + 3B = -2$ ...(iii)
समी. (iii) में से (ii) को घटाने पर
$(2A + 3B) - (2A + B) = -2 - 3$
$2B = -5 \Rightarrow B = -\frac{5}{2}$
$2A + B = 3$

$$\therefore \qquad 2A = 3 - B \quad \Rightarrow \quad 2A = 3 + \frac{5}{2}$$

$$\Rightarrow \qquad 2A = \frac{11}{2}, \quad \Rightarrow \quad A = \frac{11}{4}$$

अत: $A = \frac{11}{4}, \; B = -\frac{5}{2}, C - \frac{11}{4}$

$$\therefore \int \frac{3x - x}{(x+1)^2 \, (x+3)} \, dx$$

$$= \int \frac{A}{x+1} \, dx + \int \frac{B}{(x+1)^2} \, dx + \int \frac{C}{x+3} \, dx$$

$$= \frac{11}{4} \int \frac{1}{x+1} \, dx - \frac{5}{2} \int \frac{1}{(x+1)^2} \, dx - \frac{11}{4} \int \frac{1}{x+3} \, dx$$

$$= \frac{11}{4} \log |x+1| + \frac{5}{2} \times \frac{1}{(x+1)} - \frac{11}{4} \log |x+3| + C$$

$$\Rightarrow \frac{11}{4} \log \left| \frac{x+1}{x+3} \right|_2 + \frac{5}{2} \frac{1}{(x+1)} + C$$

प्रश्न 5.

$$\int \frac{x^2}{(x+1)(x-2)(x-3)} \, dx$$

हल :

$$\int \frac{x^2}{(x+1)(x-2)(x-3)} \, dx$$

माना

$$\frac{x^2}{(x+1)(x-2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$x^2 = A\,(x - 2)(x - 3) + B\,(x + 1)\,(x - 3) + C\,(x + 1]\,(x - 2)$
$x^2 = A(x^2 - 2x - 3x + 6) + B\,(x^2 + x - 3x - 3) + C\,(x^2 + x - 2x - 2)$

$x^2 = A(x^2 - 5x + 6) + B(x^2 - 2x - 3) + C(x^2 - x - 2)$
$x^2 = (A + B + C)x^2 + (-5A - 2B - C)x + (6A - 3B - 2C)$
दोनों पक्षों में x के गुणांकों की तुलना करने पर,
$A + B + C = 1$ ...(1)
$-5A - 2B - C = 0$ ...(2)
$6A - 3B - 2C = 0$ ...(3)
समी. (1), (2) व (3) को हल करने पर

$$A = \frac{1}{12}, B = -\frac{4}{3} \text{ और } C = \frac{9}{4}$$

$$\therefore \int \frac{x^2}{(x+1)(x-2)(x-3)} dx$$

$$= \frac{1}{12} \int \frac{1}{x+1} dx - \frac{4}{3} \int \frac{1}{x-2} dx + \frac{9}{4} \int \frac{1}{x-3} dx$$

$$= \frac{1}{12} \log |x+1| - \frac{4}{3} \log |x-2| + \frac{9}{4} \log |x-3| + C$$

प्रश्न 6.

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

हल:

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

माना $x^2 = y$

$$\frac{x^2}{x^4 - x^2 + 12} = \frac{y}{y^2 - y - 12} = \frac{y}{y(y-4) + 3(y-4)}$$

$$= \frac{y}{y(y-4) + 3(y-4)}$$

$$= \frac{y}{(y+3)(y-4)}$$

$$\frac{y}{(y+3)(y-4)} = \frac{A}{(y+3)} + \frac{B}{(y-4)}$$

$y = A(y - 4) + B(y + 3)$
$y = Ay - 4A + By + 3B$

y = Ay + By + (-4A + 3B)
y = (A + B)y + (-4A + 3B)

तुलना करने पर A + B = 1, − 4A + 3B = 0

हल करने पर, A = 3/2, B = 4/7

$$\int \frac{x^2}{x^4 - x^2 - 12} dx = \frac{3}{7} \int \frac{1}{x^2 + 3} dx + \frac{4}{7} \int \frac{1}{x^2 - 4} dx$$

$$= \frac{3}{7}\left[\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\right] + \frac{4}{7}\left[\frac{1}{2 \times 2} \log\left|\frac{x-2}{x+2}\right|\right] + c$$

$$= \frac{\sqrt{3}}{7} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{7} \log\left|\frac{x-2}{x+2}\right| + c$$

प्रश्न 7.

$$\int \frac{1}{x^3 - x^2 - x + 1} dx$$

हल:

$$\int \frac{1}{x^3 - x^2 - x + 1} dx$$

$$\frac{1}{x^3 - x^2 - x + 1} = \frac{1}{x^2(x-1) - (x-1)}$$

$$= \frac{1}{(x^2 - 1)(x-1)}$$

$$= \frac{1}{(x-1)^2 (x+1)}$$

$$\text{माना } \frac{1}{(x-1)^2 (x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

1 = A (x − 1) (x + 1) + B (x + 1} + C (x − 1)²
1 = A (x² − 1) + B (x + 1) + C(x² − 2x + 1)
1 = (A + C)x² + (B − 2C)x − A + B + C

तुलना करने पर , A + C = 0,B − 2C = 0
− A + B + C = 1

हल करने पर, $A = -\frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{4}$

$$\int \frac{1}{(x^3 - x^2 - x + 1)} dx$$

$$= -\frac{1}{4} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{(x-1)^2} dx + \frac{1}{4} \int \frac{1}{(x+1)} dx$$

$$= -\frac{1}{4} \log|x-1| + \frac{1}{2} \left[-\frac{1}{(x-1)}\right] \frac{1}{4} \log|x+1| + C$$

$$= \frac{1}{4} \log \frac{x+1}{x-1} - \frac{1}{2(x-1)} + C$$

प्रश्न 8.

$$\int \frac{x^2}{(x+1)(x-2)} dx$$

हल:

$$\int \frac{x^2}{(x+1)(x-2)} dx$$

$$\frac{x^2}{(x+1)(x-2)} = \frac{(x^2 - x - 2) + (x + 2)}{(x+1)(x-2)}$$

$$\therefore (x+1)(x-2) = x^2 - x - 2$$

तथा अंश की बात हर को घात से बड़ी या बराबर नहीं होनी चाहिए:
∴ घात का संयोजन किया गया है।

$$= 1 + \frac{x+2}{(x+1)(x-2)}$$

अब माना

$$\frac{x+2}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$\Rightarrow \quad x + 2 = A(x-2) + B(x+1)$

$\Rightarrow \quad x + 2 = (A + B)x - 2A + B$

तुलना से, $A + B = 1, \quad -2A + B = 2$

हल करने पर, $A = -\frac{1}{3}, B = \frac{4}{3}$

$$\therefore \int\frac{x^2}{(x+1)(x-2)}dx$$

$$= \int dx + \left(-\frac{1}{3}\right)\int\frac{1}{x+1}dx + \frac{4}{3}\int\frac{1}{x-2}dx$$

$$= x - \frac{1}{3}\log|x+1| + \frac{4}{3}\log|x-2| + C$$

$$= x + \frac{4}{3}\log|x-2| - \frac{1}{3}\log(x+1) + C$$

$$= x + \frac{1}{3}\log\frac{(x-2)^4}{|x+1|} + C$$

प्रश्न 9.

$$\int\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

हल:

$$\int\frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

माना $x^2 = y$

$$\frac{x^2}{(x^2+d^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)}$$

$$\Rightarrow \frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$$

$$\Rightarrow \quad y = A(y+b^2) + B(y+a^2)$$

$$\Rightarrow \quad y = (A+B)y + (Ab^2 + Ba^2)$$

तुलना से, $A + B = 1, Ab^2 + Ba^2 = 0$

हल करने पर, $A = \frac{a^2}{a^2-b^2}$ तथा $B = \frac{b^2}{a^2-b^2}$

$$\Rightarrow \quad \int\frac{y}{(y+a^2)(y+b^2)}dy$$

$$= \frac{a^2}{a^2-b^2} \int \frac{dy}{(y+a^2)} - \frac{b^2}{a^2-b^2} \int \frac{dy}{y+b^2}$$

$$\Rightarrow \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

$$= \frac{a^2}{a^2-b^2} \int \frac{1}{x^2+a^2} dx - \frac{b^2}{a^2-b^2} \int \frac{1}{x^2+b^2} dx$$

$$= \frac{a^2}{a^2-b^2} \left(\frac{1}{a} \tan^{-1} \frac{x}{a}\right) - \frac{b^2}{a^2-b^2} \times \frac{1}{b} \tan^{-1} \frac{x}{b} + C$$

$$= \frac{a}{a^2-b^2} \tan^{-1} \frac{x}{a} - \frac{b}{a^2-b^2} \tan^{-1} \frac{x}{b} + C$$

$$= \frac{1}{a^2-b^2} \left[a \tan^{-1} \frac{x}{a} - b \tan^{-1} \frac{x}{b}\right] + C$$

प्रश्न 10.

$$\int \frac{x+1}{x^3+x^2-6x} dx$$

हल:

$$\int \frac{x+1}{x^3+x^2-6x} dx$$

माना $\frac{x+1}{x^3+x^2-6x} = \frac{x+1}{x(x^2+x-6)}$

$$= \frac{x+1}{x(x^2+3x-2x-6)}$$

$$= \frac{x+1}{x(x+3)(x-2)}$$

अब $\frac{x+1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x-2)}$

$x + 1 = A (x + 3) (x - 2) + Bx (x - 2) + Cx (x + 3)$
$x + 1 = A (x^2 + 3x - 2x - 6) + B(x^2 - 2x) + C(x^2 + 3x)$
$x + 1 = A (x^2 + x - 6) + B(x^2 - 2x) + (C x^2 + 3Cx)$
$x + 1 = (A + B + C)x^2 + (A - 2B + 3C)x - 6A$

तुलना करने पर,

A + B + C = 0, A − 2B + 3C = 1, − 6A = 1
हल करने पर,

$$A = -\frac{1}{6}, B = -\frac{2}{15}, C = \frac{3}{10}$$

$$\Rightarrow \int\frac{x+1}{x^3+x^2-6x}dx = -\frac{1}{6}\int\frac{1}{x}dx - \frac{2}{15}\int\frac{1}{x+3}dx]$$

$$+\frac{3}{10}\int\frac{1}{x-2}dx$$

$$= -\frac{1}{6}\log|x| - \frac{2}{15}\log|x+3| + \frac{3}{10}\log|x-2| + C$$

प्रश्न 11.

$$\int\frac{x^2-8x+4}{x^3-4x}dx$$

हल:

$$\int\frac{x^2-8x+4}{x^3-4x}dx$$

$$\text{माना } \frac{x^2-8x+4}{x^3-4x} = \frac{x^2-8x+4}{x(x^2-4)}$$

$$= \frac{x^2-8x+4}{x(x-2)(x+2)}$$

$$\Rightarrow \frac{x^2-8x+4}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

⇒ $x^2 - 8x + 4 = A(x^2 - 4) = A(x^2 - 4) + B(x^2 + 2x) + C(x^2 - 2x)$
⇒ $x^2 - 8x + 4 = (A + B + C)x^2 + (2B - 2C)x - 4A$
तुलना करने पर
A + B + C = 1, 2B − 2 C = − 8, − 4 A = 4
हल करने पर, A = -1, B = − 1, C = 3

$$\Rightarrow \int\frac{x^2+8x+4}{x^3-4x}dx = -1\int\frac{1}{x}dx - \int\frac{1}{(x-2)}dx + 3$$

$$\int\frac{1}{(x+2)}dx$$

= − log | x | − log | x − 2 | + 3 log | x + 2 | + C

प्रश्न 12.

$$\int \frac{1}{(x-1)^2(x+2)} dx$$

हल:

$$\int \frac{1}{(x-1)^2(x+2)} dx$$

माना

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$\Rightarrow 1 = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$
$\Rightarrow 1 = A(x^2 - x + 2x - 2) + B(x + 2) + C(x^2 - 2x + 1)$
$\Rightarrow 1 = A(x^2 + x - 2) + B(x + 2) + C(x^2 - 2x + 1)$
$\Rightarrow 1 = (A + C)x^2 + (A + B - 2C)x - 2A + 2B + C$
तुलना करने पर,
$A + C = 0, A + B - 2C = 0, -2A + 2B + C = 1$
हल करने पर,

$$A = \frac{-1}{9},\ B = \frac{1}{3},\ C = \frac{1}{9}$$

$$\int \frac{1}{(x-1)^2(x+2)} dx$$

$$= \frac{-1}{9}\int \frac{1}{x-1} dx + \frac{1}{3}\int \frac{1}{(x-1)^2} dx + \frac{1}{9}\int \frac{1}{x+2} dx$$

$$= \frac{-1}{9}\log|x-1| - \frac{1}{3}\frac{1}{(x-1)} + \frac{1}{9}\log|x+2| + C$$

$$= \frac{1}{9}\log\frac{|x+2|}{|x-1|} - \frac{1}{3(x-1)} + C$$

प्रश्न 13.

$$\int \frac{1-3x}{1+x+x^2+x^3} dx$$

हल:

$$\int \frac{1-3x}{1+x+x^2+x^3} dx$$

$$\frac{1-3x}{1+x+x^2+x^3} = \frac{1-3x}{(1+x)+x^2(1+x)}$$

$$= \frac{1-3x}{(1+x)(1+x^2)}$$

माना $$\frac{1-3x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$\Rightarrow 1 - 3x = A(1 + x^2) + (Bx + C)(1 + x)$

$\Rightarrow 1 - 3x = A + Ax^2 + Bx + C + Bx^2 + Cx$

$\Rightarrow 1 - 3x = Ax^2 + Bx^2 + Bx + Cx + A + C$

$\Rightarrow 1 - 3x = (A + B)x^2 + (B + C)x + A + C$

तुलना करने पर,

$A + B = 0, B + C = -3, A + C = 1$

हल करने पर

$A = 2, B = -2, C = -1$

$$\therefore \int \frac{1-3x}{1+x+x^2+x^3} dx$$

$$= 2\int \frac{1}{(x+1)} dx + \int \frac{-2x+(-1)}{1+x^2} dx$$

$$= 2\int \frac{1}{1+x} dx - \int \frac{2x+1}{1+x^2} dx$$

$$= 2\int \frac{1}{1+x} dx - \int \frac{2x}{1+x^2} dx - \int \frac{1}{1+x^2} dx$$

$$= 2 \log |1+x| - \log |1+x^2| - \tan^{-1} x + C$$

$$= \log \frac{|1+x|^2}{|1+x^2|} - \tan^{-1} x + C$$

प्रश्न 14.

$$\int \frac{1+x^2}{x^5-x} dx$$

हल:

$$\int \frac{1+x^2}{x^5-x}\,dx$$

$$\frac{1+x^2}{x^5-x} = \frac{1+x^2}{x(x^4-1)} = \frac{1+x^2}{x(x^2-1)(x^2+1)}$$

$$= \frac{1+x^2}{x(x-1)(x+1)(x^2+1)} = \frac{1}{x(x-1)(x+1)}$$

अब $\frac{1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$\Rightarrow \quad \frac{1}{x(x-1)(x+1)} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x-1)(x+1)}$

$\Rightarrow \quad 1 = (A+B+C)x^2+(B-C)x-A$

तुलना करने पर,

$$A+B+C=0,\ B-C=0,\ -A=1$$

हल करने पर,

$$A=-1,\ B=1/2\ C=1/2$$

$$\therefore \int \frac{1+x^2}{x^5-x}\,dx = -\int\frac{1}{x}\,dx+\frac{1}{2}\int\frac{1}{x-1}+\frac{1}{2}\int\frac{1}{x+1}\,dx$$

$$= -\log|x|+\frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|\,C$$

$$= \frac{1}{2}\log|x-1|+\frac{1}{2}\log|x+1|-\log|x|+C$$

$$= \frac{1}{2}\log\left|\frac{(x-1)(x+1)}{x^2}\right|+C$$

$$= \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right|+C$$

प्रश्न 15.

$$\int \frac{x^2+5x+3}{x^2+3x+2}\,dx$$

हल:

$$\int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\, dx$$

$$\frac{x^2 + 5x + 3}{x^2 + 3x + 2} = \frac{(x^2 + 3x + 2) + (2x + 1)}{(x^2 + 3x + 2)}$$

$$= 1 + \frac{2x + 1}{x^2 + 3x + 2}$$

$$= 1 + \frac{2x + 1}{(x + 1)(x + 2)}$$

माना $\dfrac{2x + 1}{(x + 1)(x + 2)} = \dfrac{A}{x + 1} + \dfrac{B}{x + 2}$

$\Rightarrow 2x + 1 = Ax + Bx + 2A + B$

$\Rightarrow 2x + 1 = (A + B)x + (2A + B)$

तुलना करने पर,

$A + B = 2, 2A + B = 1$

हल करने पर,

$A = -1, B = 3$

$$\therefore \int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\, dx$$

$$= \int 1\, dx + (-1)\int \frac{1}{x + 1}\, dx + 3\int \frac{1}{(x + 2)}\, dx$$

$= x - \log|x + 1| + 3 \log|x + 2|$

प्रश्न 16.

$$\int \frac{x - 1}{(x + 1)(x^2 + 1)}\, dx$$

हल:

$$\int \frac{x - 1}{(x + 1)(x^2 + 1)}\, dx$$

माना $\dfrac{x - 1}{(x + 1)(x^2 + 1)} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 + 1}$

$(x - 1) = A(x^2 + 1) + (Bx + C)(x + 1)$

$x - 1 = Ax^2 + A + Bx^2 + Cx + Bx + C$

$x - 1 = (A + B)x^2 + (B + C)x + A + C$

तुलना करने पर,

$A + B = 0, B + C = 1, A + C = -1$

हल करने पर,
A = − 1, B = 1, C = 0

$$\int \frac{x-1}{(x+1)(x^2+1)}\,dx = -\int \frac{1}{x+1}\,dx + \int \frac{x}{x^2+1}\,dx$$

$$= -\log|x+1| + \frac{1}{2}\log|x^2+1| + C$$

$$= -\log|x+1| + \log\sqrt{x^2+1} + C$$

$$= \log\frac{\sqrt{x^2+1}}{|x+1|} + C$$

प्रश्न 17.

$$\int \frac{1}{(1+e^x)(1-e^{-x})}\,dx$$

हल :

$$\int \frac{1}{(1+e^x)(1-e^{-x})}\,dx = \int \frac{e^x}{(e^x+1)(e^x-1)}\,dx$$

माना $\qquad e^x = t \Rightarrow e^x\,dx = dt$

$$= \int \frac{dt}{(t+1)(t-1)}$$

माना $\dfrac{1}{(t+1)(t-1)} = \dfrac{A}{t+1} + \dfrac{B}{t-1}$

$\Rightarrow \qquad 1 = A(t-1) + B(t+1)$

$\Rightarrow \qquad 1 = (A+B)t + (-A+B)$

तुलना से, $\quad A + B = 0, \; -A + B = 1$

$$2B = 1 \Rightarrow B = \frac{1}{2},\; A = -\frac{1}{2}$$

$$\therefore \int \frac{1}{(1+e^x)(1-e^{-x})}\,dx = \int \frac{-1/2}{(t+1)}\,dt + \int \frac{1}{t-1}\,dt$$

$$= -\frac{1}{2}\int \frac{1}{t+1}\,dt + \frac{1}{2}\int \frac{1/2}{t-1}\,dt$$

$$= -\frac{1}{2}\log|t+1| + \frac{1}{2}\log|t-1| + C$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^x-1}{e^x+1}\right| + C$$

प्रश्न 18.

$$\int \frac{1}{(e^x-1)^2}\,dx$$

हल :

$$\int \frac{1}{(e^x-1)^2}\,dx$$

(e^x से अंश व हर में गुणा करने पर)

माना $e^x - 1 = t$, तो $e^x = t + 1$

$e^x\,dx = dt$

$$\Rightarrow \int \frac{e^x}{e^x(e^x-1)^2}\,dx = \int \frac{dt}{(t+1)t^2}$$

अब $$\frac{1}{(t+1)t^2} = \frac{A}{t+1} + \frac{B}{t} + \frac{C}{t^2}$$

$1 = At^2 + Bt(t - 1) + C(t + 1)$

$1 = At^2 + Bt^2 + Bt + Ct + C$

$1 = (A + B)t^2 - (B + C)t + C$

तुलना से, $A + B = 0, B + C = 0, C = 1$

हल करने पर,

$C = 1, B = - 1, A = 1$

अतः $$\int \frac{1}{(e^x-1)^2}\,dx = \int \frac{1}{t+1}\,dt - \int \frac{1}{t}\,dt + \int \frac{1}{t^2}\,dt$$

$$= \log|t+1| - \log|t| - \frac{1}{t} + C$$

$$= \log\left|\frac{t+1}{t}\right| - \frac{1}{t} + C$$

$$= \log\left|\frac{e^x-1+1}{e^x-1}\right| - \frac{1}{e^x-1} + C$$

$$= \log\left|\frac{e^x}{e^x-1}\right| - \frac{1}{e^x-1} + C$$

प्रश्न 19.

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

हल :

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

माना
$e^x = t$
$e^x\,dx = dt$

$$\int \frac{e^x}{(e^x)^2+5e^x+6}\,dx = \int \frac{dt}{t^2+5t+6}$$

$$= \int \frac{1}{(t+3)(t+2)}\,dt$$

पुन: माना $\frac{1}{(t+3)(t+2)} = \frac{A}{t+3}+\frac{B}{t+2}$

$\Rightarrow 1 = At + 2A + Bt + 3B$
$\Rightarrow 1 = (A + B)t + (2A + 3B)$
तुलना करने पर,
$A + B = 0,\ 2A + 3B = 1$
हल करने पर,
$A = -1,\ B = 1$

$$\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$$

$$= \int \frac{dt}{t^2+5t+6} = \int \frac{dt}{(t+3)(t+2)}$$

$$= \int \frac{(-1)}{t+3}\,dt + \int \frac{1}{t+2}\,dt$$

$$= -\log|t+3| + \log|t+2| + C$$

$$= \log\left|\frac{t+2}{t+3}\right| + C$$

$$= \log\left|\frac{e^x+2}{e^x+3}\right| + C$$

प्रश्न 20.

$$\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)} dx$$

हल :

$$\int \frac{\sec^2 x}{(2+\tan x)(3+\tan x)} dx$$

माना

tan x = t

$\sec^2 x \, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x \, dx}{(2+\tan x)(3+\tan x)}$$

$$= \int \frac{dt}{(2+t)(3+t)}$$

माना $\frac{1}{(2+t)(3+t)} = \frac{A}{2+t} + \frac{B}{3+t}$

⇒ 1 = 3A + At + 2B + Bt

⇒ 1 = (A + B)t + 3A + 2B

तुलना करने पर,

A + B = 0

तथा 3A + 2B = 1

हल करने पर, A = 1, B = -1

$$\int \frac{\sec^2 x \, dx}{(2+\tan x)(3+\tan x)} = \int \frac{dt}{(2+t)(3+t)}$$

$$= \int \frac{1}{2+t} dt - \int \frac{1}{3+t} dt$$

$$= \log|2+t| - \log|3+t| + C$$

$$= \log\left|\frac{2+\tan x}{3+\tan x}\right| + C$$

प्रश्न 21.

$$\int \frac{1}{x(x^5+1)} dx$$

हल:

$$\int \frac{1}{x(x^5+1)}\,dx$$

(अंश व हर में x^4 से गुणा करने पर)

$$= \int \frac{x^4}{x^5(x^5+1)}\,dx$$

माना $x^5 = t$

$$5x^4\,dx = dt \text{ या } x^4\,dx = dt$$

$$= \frac{1}{5}\int \frac{dt}{t(t+1)}$$

माना $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$

$\Rightarrow 1 = At + A + Bt$

$\Rightarrow 1 = (A + B)t + A$

तुलना करने पर

$A + B = 0, A = 1$

हल करने पर,

$A = 1, B = -1$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int \frac{1}{t(t+1)}\,dt = \int \frac{1}{t}\,dx - \int \frac{1}{t+1}\,dt$$

$$= \log|t| - \log|t+1| + C$$

अतः $\int \frac{1}{x(x^5+1)}\,dx$

$$= \frac{1}{5}\log|x^5| - \frac{1}{5}\log|x^5+1| + C$$

प्रश्न 22.

$$\int \frac{1}{x(a+bx^n)}\,dx$$

हल:

$$\int \frac{1}{x(a+bx^n)}dx = \int \frac{x^{n-1}}{x.x^{n-1}(a+bx^n)}dx$$

$$= \int \frac{x^{n-1}}{x^n(a+bx^n)}dx$$

$$a + bx^n = t$$

$$bn\, x^{n-1}\, dx = dt$$

$$\Rightarrow \quad x^{n-1}\, dx = \frac{1}{bn}dt$$

$$\Rightarrow \quad \int \frac{1}{x(a+bx^n)}dx = \int \frac{1}{\left(\frac{t-a}{b}\right)} \cdot \frac{1}{bn}dt$$

$$= \frac{1}{n}\int \frac{1}{t(t-a)}dt$$

$$= \frac{1}{n}\left[\int \frac{-1}{at}dt + \int \frac{1}{a(t-a)}dt\right]$$

$$= \frac{1}{n}\left[-\frac{1}{a}\log t + \frac{1}{a}\log|t-a|\right] + C$$

$$= \frac{1}{an}\log\left|\frac{t-a}{t}\right| + C$$

$$= \frac{1}{an}\log\left|\frac{a+bx^n-a}{a+bx^n}\right| + C$$

$$= \frac{1}{an}\log\left|\frac{bx^n}{a+bx^n}\right| + C$$

प्रश्न 23.

$$\int \frac{8}{(x+2)(x^2+4)}dx$$

हल:

$$\int \frac{8}{(x+2)(x^2+4)}dx$$

माना $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$

$\Rightarrow 8 = (x^2 + 4) = (Bx + C)(x + 2)$

$= A(x^2 + 4) + Bx^2 + Cx + 2Bx + 2C$
$= (A + B)x^2 + (2B + C)x + 4A + 2$
तुलना करने पर,
$A + B = 0, 2B + C = 0, 4A + 2C = 8$
हल करने पर,
$A = 1, B = -1, C = 2$

$$\Rightarrow \int \frac{8}{(x+2)(x^2+4)} dx$$

$$= \int \frac{1}{x+2} dx + \int \frac{-x+2}{x^2+4} dx$$

$$= \int \frac{1}{x+2} dx - \int \frac{x}{x^2+4} dx + 2\int \frac{1}{x^2+4} dx$$

$$= \log|x+2| - \frac{1}{2} \log|x^2+4| + 2 \times \frac{1}{2} \tan^{-1} \frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2} \log|x^2+4| + \tan^{-1} \frac{x}{2} + C$$

प्रश्न 24.

$$\int \frac{1-\cos x}{\cos x(1+\cos x)} dx$$

हल : $\int \frac{1-\cos x}{\cos x(1+\cos x)} dx$

$$= \int \frac{1}{\cos x(1+\cos x)} dx - \int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{\cos x} dx - \int \frac{1}{1+\cos x} dx - \int \frac{1}{1+\cos x} dx$$

$$= \int \sec x \, dx - \int \sec^2 \frac{x}{2} dx$$

$$= \log|\sec x + \tan x| - 2 \tan\left(\frac{x}{2}\right) + C$$

Ex 9.5

निम्न फलनों का x के सापेक्ष समाकलन कीजिए-

प्रश्न 1.

$$\frac{1}{x^2+2x+10}$$

हल :

$$\int \frac{1}{x^2+2x+10}\,dx$$

$$= \int \frac{1}{x^2+2\cdot 1\cdot x+1^2+9}\,dx$$

$$= \int \frac{1}{(x+1)^2+3^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right)+C$$

प्रश्न 2.

$$\frac{1}{2x^2+x-1}$$

हल :

$$\int \frac{1}{2x^2+x-1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}x-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x^2+2\cdot\frac{1}{4}\cdot x+\frac{1}{16}\right)-\frac{1}{16}-\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{2 \times \left(\frac{3}{4}\right)} \log \left| \frac{\left(x + \frac{1}{4}\right) - \frac{3}{4}}{x + \frac{1}{4} + \frac{3}{4}} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{4x - 2}{4x + 4} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{2x - 1}{2x + 2} \right| + C$$

प्रश्न 3.

$$\int \frac{1}{9x^2 - 12x + 8} dx$$

हल :

$$\int \frac{1}{9x^2 - 12x + 8} dx$$

$$= \frac{1}{9} \int \frac{1}{x^2 - \frac{12}{9} x + \frac{8}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{x^2 - \frac{4}{3} x + \frac{8}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{\left(x^2 - 2 \times \frac{2}{3} \times x + \frac{4}{9}\right) + \frac{4}{9}} dx$$

$$= \frac{1}{9} \int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} dx$$

$$= \frac{1}{9} \times \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \left(\frac{x - \frac{2}{3}}{\frac{2}{3}} \right) + C$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x - 2}{2} \right) + C$$

प्रश्न 4.

$$\int \frac{1}{3+2x-x^2}$$

हल :

$$\int \frac{1}{3+2x-x^2}\,dx$$

$$= \int \frac{1}{4-1+2x-x^2}\,dx$$

$$= \int \frac{1}{2^2-(x-1)^2}\,dx$$

$$= \frac{1}{2\times 2}\log\left|\frac{2+x-1}{2-x+1}\right|+C$$

$$= \frac{1}{4}\log\left|\frac{x+1}{3-x}\right|+C$$

प्रश्न 5.

$$\int \frac{x}{x^4+x^2+1}\,dx$$

हल :

$$\int \frac{x}{x^4+x^2+1}\,dx$$

माना $x^2 = t$, तो $2x\,dx = dt$

$$x\,dx = \frac{1}{2}\,dt$$

$$= \frac{1}{2}\int \frac{dt}{t^2+t+1}$$

$$= \frac{1}{2}\int \frac{1}{\left(t^2+2\times\frac{1}{2}+t+\frac{1}{4}\right)+\frac{3}{4}}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \frac{1}{2} \times \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \left|\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left|\frac{2t+1}{\sqrt{3}}\right| + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left|\frac{2x^2+1}{\sqrt{3}}\right| + C$$

प्रश्न 6.

$$\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

हल :

$$\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

sin x = t
cos x dx = dt

$$= \int \frac{dt}{t^2 + 4t + 5}$$

$$= \int \frac{dt}{(t^2 + 2 \cdot 2t + 4) + 1}$$

$$= \int \frac{dt}{(t+2)^2 + 1}$$

$$= \frac{1}{1} \tan^{-1} \frac{t+2}{1} + C$$

= $\tan^{-1}(t+2)+c$
= $\tan^{-1}(\sin x+2)+c$

प्रश्न 7.

$$\int \frac{x-3}{x^2 + 2x - 4} dx$$

हल :

$$\int \frac{x-3}{x^2 + 2x - 4} dx$$

$x - 3 = A \frac{d}{dx}(x^2 + 2x - 4) + B$
$\Rightarrow x - 3 = A(2x - 2) + B$
$\Rightarrow x - 3 = 2Ax + (2A + B)$
तुलना करने पर,

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$2A + B = -3$$

$$2 \times \frac{1}{2} + B = -3 \Rightarrow B = -4$$

$$x - 3 = \frac{1}{2}(2x + 2) - 4$$

$$\therefore \int \frac{x-3}{x^2+2x-4}\,dx = \int \frac{\frac{1}{2}(2x+2)-4}{x^2+2x-4}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x-4}\,dx - \int \frac{4}{x^2+2x-4}\,dx$$

$$= \frac{1}{2}\log|x^2+2x-4| - 4\int \frac{1}{(x^2+2\cdot1\cdot x+1)-5}\,dx$$

$$= \frac{1}{2}\log|x^2+2x-4| - 4\int \frac{1}{(x+1)^2-(\sqrt{5})^2}\,dx$$

$$= \frac{1}{2}\log|x^2+2x-4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$$

प्रश्न 8.

$$\int \frac{3x+1}{2x^2-2x+3}$$

हल :

$$\int \frac{3x+1}{2x^2-2x+3}\,dx$$

$3x + 1 = A \frac{d}{dx}(2x^2 - 2x + 3) + B$
$\Rightarrow 3x + 1 = A(4x - 2) + B$
$\Rightarrow 3x + 1 = 4Ax - 2A + B$
तुलना करने पर,
$4A = 3 \Rightarrow A = \frac{3}{4}$

$$-2A + B = 1 \Rightarrow -2 \times \frac{3}{4} + B = 1$$

$$B = 1 + \frac{3}{2} = \frac{5}{2}$$

$$3x + 1 = \frac{3}{4}(4x-2) + \frac{5}{2}$$

$$\int \frac{3x+1}{2x^2-2x+3}\, dx = \int \frac{\frac{3}{4}(4x-2)+\frac{5}{2}}{2x^2-2x+3}\, dx$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\, dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\, dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{4}\int \frac{1}{x^2-x+\frac{3}{2}}\, dx + C_1 \qquad \text{...(i)}$$

$$\int \frac{1}{x^2-x+\frac{3}{2}}\, dx = \int \frac{1}{x^2-2\times\frac{1}{2}\times x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{3}{2}}\, dx$$

$$= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{5}}{2}\right)^2}\, dx$$

$$= \frac{1}{\left(\frac{\sqrt{5}}{2}\right)} \tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right) + C_2$$

$$= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2 \qquad \text{...(ii)}$$

समीकरण (i) व (ii) से

$$\int \frac{3x+1}{2x^2-2x+3}\, dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{4}\times\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_1 + C_2$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C$$

(जहाँ $C = C_1 + C_2$)

प्रश्न 9.

$$\int \frac{x+1}{x^2+4x+5}\, dx$$

हल :

$$\int \frac{x+1}{x^2+4x+5}$$

माना x + 1 = $A\,\frac{d}{dx}(x^2+4x+5)+B$

$\Rightarrow x+1 = A(2x+4)+B$

$\Rightarrow 2A = 1 \Rightarrow A = 1/2$

$\Rightarrow 4A + B = 1 \Rightarrow B = -1$

$\therefore x+1 = \frac{1}{2}(2x+4) - 1$

$$= \int \frac{\frac{1}{2}(2x+4)-1}{x^2+4x+5}$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\, dx - \int \frac{1}{x^2+4x+5}\, dx$$

$$= \frac{1}{2}\log|x^2+4x+5| - \int \frac{1}{(x+2)^2+1}\, dx$$

$$= \frac{1}{2}\log|x^2+4x+5| - \tan^{-1}(x+2) + C$$

प्रश्न 10.

$$\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\, dx$$

हल :

$$\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\, dx$$

माना sinx = t

cos x dx = dt

$$= \int \frac{(3t-2)}{4+t^2-4t}\, dt$$

$$= \int \frac{(3t-2)}{t^2-4t+4}\, dt$$

$$= \int \frac{3t-2}{(t-2)^2}\, dt$$

$$= \int \frac{(3t-6)+4}{(t-2)^2}\, dt$$

$$= \int \frac{3(t-2)}{(t-2)^2}\, dt + \int \frac{4}{(t-2)^2}$$

$$= 3\int \frac{1}{(t-2)}\, dt + 4\int (t-2)^{-2}\, dt$$

$$= 3\log|t-2| - \frac{4}{(t-2)} + C$$

$$= 3\log(2-t| + \frac{4}{(t-2)} + C$$

$$= 3\log|2-\sin x| + \frac{4}{(2-\sin x)} + C$$

$$(\because\ 0 \le \sin x \le 1)$$

प्रश्न 11.

$$\int \frac{1}{2e^{2x}+3e^{x}+1}\, dx$$

हल :

माना $I = \int \frac{1}{2e^{2x}+3e^{x}+1}\, dx$

$$= \int \frac{e^{-x}\cdot e^{-x}\, dx}{2+3e^{-x}+e^{-2x}}$$

(e^{-2x} से हर व अंश में गुणा करने पर)

माना $I = \int \frac{1}{2e^{2x}+3e^{x}+1}\, dx$

$$= \int \frac{e^{-x}\cdot e^{-x}\, dx}{2+3e^{-x}+e^{-2x}}$$

$$= \int \frac{e^{-x} \cdot e^{-x}\, dx}{e^{-2x} + 3e^{-x} + 2}$$

माना $e^{-x} = t$

$-e^{-x}\, dx = dt$

$e^{-x}\, dx = -dt$

t = At + 2A + Bt + B

t = (A +B)t + (2A + B)

तुलना से, A + B = 1

2A + B = 0

B = -2A

A + (-2A) = 1

A − 2A = 1

− 2A = 1

− A = 1 या A = -1

B = -2 x (- 1) = 2 या B = 2

$$\Rightarrow \qquad I = -\int \frac{-1}{t+1}\, dt - \int \frac{2}{t+2}\, dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log \frac{|t+1|}{|t+2|^2} + C$$

$$= \log \left| \frac{t+1}{t^2 + 4t + 4} \right| + C$$

$$= \log \left| \frac{e^{-x} + 1}{e^{-2x} + 4e^{-x} + 4} \right| + C$$

प्रश्न 12.

$$\int \frac{1}{\sqrt{4x^2 - 5x + 1}}\, dx$$

हल :

$$\int \frac{1}{\sqrt{4x^2 - 5x + 1}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2 - \frac{5}{4}x + \frac{1}{4}}}\, dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2 - 2\times\frac{5}{8}\times x + \frac{25}{64} + \frac{1}{4} - \frac{25}{64}}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{5}{8}\right)^2 - \frac{9}{64}}}\,dx$$

$$= \frac{1}{2}\log\left|\left(x-\frac{5}{8}\right) + \sqrt{\left(x-\frac{5}{8}\right)^2 - \frac{9}{64}}\right| + C$$

$$= \frac{1}{2}\log\left|\left(x-\frac{5}{8}\right) + \sqrt{x^2 - \frac{5}{4}x + \frac{1}{4}}\right| + C$$

प्रश्न 13.

$$\int \frac{1}{\sqrt{5x-6-x^2}}$$

हल :

$$\int \frac{1}{\sqrt{5x-6-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{25}{4} - \frac{24}{4} - \left(x^2 + \frac{25}{4} - 5x\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{1}{4} - \left(x-\frac{5}{2}\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x-5/2}{1/2}\right) + C$$

$$= \sin^{-1}\left(\frac{(2x-5)/2}{1/2}\right)$$

$$= \sin^{-1}(2x-5) + C$$

प्रश्न 14.

$$\int \frac{1}{\sqrt{1-x-x^2}}$$

हल :

$$\int \frac{1}{\sqrt{1-x-x^2}}\,dx$$

$$1-x-x^2 = 1+\frac{1}{4}-\left(\frac{1}{4}+x+x^2\right)$$

$$= \frac{5}{4}-\left(\frac{1}{2}+x\right)^2$$

$$\int \frac{1}{\sqrt{1-x-x^2}}\,dx = \int \frac{1}{(\sqrt{5}/2)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{5}/2}\right)$$

$$= \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)+C$$

प्रश्न 15.

$$\int \frac{1}{\sqrt{4+3x-2x^2}}$$

हल :

$$\int \frac{1}{\sqrt{4+3x-2x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{4-(2x^2-3x)}}\,dx$$

$$= \int \frac{1}{\sqrt{4-2\left(x^2-\frac{2\times 3}{4}x+\frac{9}{16}-\frac{9}{16}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{4+\frac{9}{8}-2\left(x-\frac{3}{4}\right)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{\frac{41}{8} - 2\left(x - \frac{3}{4}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2}}\, dx$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x - \frac{3}{4}}{\sqrt{41/4}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4x - 3}{\sqrt{41}}\right) + C$$

प्रश्न 16.

$$\int \frac{x + 2}{\sqrt{x^2 - 2x + 4}}\, dx$$

हल :

$$\int \frac{x + 2}{\sqrt{x^2 - 2x + 4}}\, dx$$

माना

$x^2 - 2x + 4 = t$

$(2x - 2) = t$

$2(x - 1)dx = dt$

$$\Rightarrow \quad (x - 1)\, dx = \frac{dt}{2}$$

$$= \int \frac{x - 1 + 3}{\sqrt{x^2 - 2x + 4}}\, dx$$

$$= \int \frac{x - 1}{\sqrt{x^2 - 2x + 4}}\, dx + \int \frac{3}{\sqrt{x^2 - 2x + 4}}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{t}} + \int \frac{3}{\sqrt{(x - 1)^2 + (\sqrt{3})^2}}\, dx$$

$$= \frac{1}{2} \frac{t^{-1/2 + 1}}{-1/2 + 1} + 3 \log \left| (x - 1) + \sqrt{(x - 1)^2 + (\sqrt{3})^2} \right| + C$$

$$= \frac{1}{2} \frac{t^{1/2}}{1/2} + 3 \log \left| (x - 1) + \sqrt{x^2 - 2x + 4} \right| + C$$

$$= \sqrt{(x^2 - 2x + 4)} + 3 \log \left| (x - 1) + \sqrt{x^2 - 2x + 4} \right| + C$$

प्रश्न 17.

$$\int \frac{x+1}{\sqrt{x^2-x+1}}\,dx$$

हल :

$$\int \frac{x+1}{\sqrt{x^2-x+1}}$$

$x + 1 = A\frac{d}{dx}(x^2 - x + 1) + B$
$x + 1 = A(2x - 1) + B$
$x + 1 = 2Ax - A + B$

तुलना करने पर,

$2A = 1, A = 1/2$
$-A + B = 1$
$\Rightarrow B = \frac{3}{4}$

$$\int \frac{x+1}{\sqrt{x^2-x+1}}\,dx = \frac{1}{2}\int \frac{(2x-1)}{\sqrt{x^2-x+1}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2-x+1}}\,dx$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+1}\right| + C$$

प्रश्न 18.

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$$

हल :

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$$

$x + 3 = A\frac{d}{dx}(x^2 + 2x + 2) + B$
$\Rightarrow x + 3 = A(2x + 2) + B$
$\Rightarrow x + 3 = 24x + 2A + B$

तुलना करने पर,

2A = 1, A = $\frac{1}{2}$
2A + B = 3, B = 2

$$\int \frac{x+3}{\sqrt{x^2+2x+2}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x+2)}{\sqrt{x^2+2x+2}}\,dx + 2\int \frac{1}{\sqrt{x^2+2x+2}}\,dx$$

$$= \sqrt{(x^2+2x+2)} + 2\int \frac{1}{\sqrt{(x+1)^2+1}}$$

$$= \sqrt{(x^2+2x+2)} + 2\log|(x+1)+\sqrt{x^2+2x+2}| + C$$

प्रश्न 19.

$$\int \sqrt{\sec x - 1}\ dx$$

हल :

$$\int \sqrt{\sec x - 1}\,dx$$

$$= \int \sqrt{\frac{1}{\cos x} - 1}\,dx$$

$$= \int \sqrt{\frac{1-\cos x}{\cos x} \times \frac{1+\cos x}{1+\cos x}}\,dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x\,(1+\cos x)}}\,dx$$

$$= \int \frac{\sin x}{\sqrt{\cos x\,(1+\cos x)}}\,dx$$

माना $\cos = t$

$$-\sin x\,dx = dt$$

$$\sin x\,dx = -dt$$

$$= -\int \frac{1}{\sqrt{t(1+t)}}\,dx$$

$$= -\int \frac{1}{\sqrt{t^2+t+1/4-1/4}}\,dt$$

$$= -\int \frac{1}{\sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dt$$

$$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{t^2+t}\right| + C$$

$$= -\log\left|\left(\cos x+\frac{1}{2}\right) + \sqrt{\cos^2 x+\cos x}\right| + C$$

प्रश्न 20.

$$\int \sqrt{\frac{\sin (x-\alpha)}{\sin (x+\alpha)}}$$

हल :

माना $I = \int \sqrt{\frac{\sin (x-\alpha)}{\sin (x+\alpha)}}\, dx$

$$= \int \sqrt{\frac{\sin (x-\alpha).\sin (x-\alpha)}{\sin (x+\alpha).\sin (x-\alpha)}}\, dx$$

$$= \int \sqrt{\frac{\sin^2 (x-\alpha)}{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$= \int \frac{\sin (x-\alpha)}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx$$

$$= \int \frac{\sin x \cos \alpha - \cos x \sin \alpha}{\sqrt{\cos^2 \alpha - \cos^2 x}}\, dx$$

$$= \cos \alpha \underbrace{\int \frac{\sin x}{\sqrt{\cos^2 \alpha - \cos^2 x}}\, dx}_{I_1}$$

$$- \sin \alpha \underbrace{\int \frac{\cos x}{\sqrt{\cos^2 \alpha - \cos^2 x}}\, dx}_{I_2}$$

$$I_1 = \int \frac{\sin x}{\sqrt{\cos^2 \alpha - \cos^2 x}}\, dx$$

माना $\cos x = t \Rightarrow \sin x\, dx = -\, dt$

$$I_1 = \int \frac{-dt}{\sqrt{\cos^2 \alpha - t^2}}$$

$$I_1 = -\sin^{-1}\left(\frac{t}{\cos \alpha}\right)$$

$$I_1 = -\sin^{-1}\left(\frac{\cos x}{\cos \alpha}\right)$$

तथा $$I_2 = \int \frac{\cos x \, dx}{\sqrt{\sin^2 x - \sin^2 \alpha}}$$

माना $\sin x = y$

$\cos x \, dx = dy$

$$I_2 = \int \frac{dy}{\sqrt{y^2 - \sin^2 \alpha}}$$

$$= \log\left[y + \sqrt{(y^2 - \sin^2 \alpha)}\right]$$

$$I_2 = \log\left[\sin x + \sqrt{\sin^2 x - \sin^2 \alpha}\right]$$

$$\Rightarrow \quad I = -\cos \alpha \sin^{-1}\left(\frac{\cos x}{\cos \alpha}\right)$$

$$- \sin \alpha \log\left[\sin x + \sqrt{\sin^2 x - \sin^2 \alpha}\right] + C$$

प्रश्न 21.

$$\int \frac{x^3}{x^2 + x + 1} dx$$

हल :

$$\int \frac{x^3}{x^2 + x + 1} dx$$

$$= \int \frac{x^3 + x^2 - x^2 - x + x}{x^2 + x + 1} dx$$

$$= \int \frac{x^3 + x^2 + x - x^2 - x}{x^2 + x + 1} dx$$

$$= \int \left[\frac{x(x^2 + x + 1)}{(x^2 + x + 1)} - \frac{(x^2 + x + 1) - 1}{x^2 + x + 1}\right] dx$$

$$= \int x\,dx - \int dx + \int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{x^2}{2} - x + \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{2}}}\right) + C$$

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

प्रश्न 22.

$$\int \frac{e^x}{e^{2x}+6e^x+5}\,dx$$

हल :

$$\int \frac{e^x}{e^{2x}+6e^x+5}\,dx$$

माना $e^x = t$

$e^x\,dx = dt$

$$= \int \frac{dt}{t^2+6t+5}$$

$$= \int \frac{dt}{t^2+2\cdot3\cdot t+9-4}$$

$$= \int \frac{1}{(t+3)^2-2^2}\,dt$$

$$= \frac{1}{2\times2}\log\left|\frac{t+3-2}{t+3+2}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right| + C$$

Ex 9.6

निम्न फलनों का x के सापेक्ष समाकलन कीजिए

प्रश्न 1.
(a) $\int x \cos$
(b) $\int x \sec^2 x$

हल : (a) $\int x \cos dx$

$$= x \int \cos x \, dx - \int \left[\frac{d}{dx}(x) . \int \cos x \, dx \right] dx$$

$= x \sin x - \int 1.\sin x \, dx$
$= x \sin x - (-\cos x) + C$
$= x \sin x + \cos x + C$

(b) $\int x \sec^2 x \, dx$

$$= x \int \sec^2 x \, dx - \int \left[\frac{d}{dx}(x) \int \sec^2 x \, dx \right] dx$$

$= x \tan x - \int 1.\tan x \, dx$
$= x \tan x + \log |\cos x| + C$
$= x \tan x - \log |\sec x| + C$

प्रश्न 2.
(a) $x^3 e^{-x}$
(b) $x^3 \sin x$

हल : (a) $\int x^3 e^{-x}$

$$= x^3 \int e^{-x} \, dx - \int \left[\frac{d}{dx} x^3 . \int e^{-x} \, dx \right] dx$$

$$= x^3 \frac{e^{-x}}{-1} - \int 3x^2 \frac{e^{-x}}{-1} dx$$

$= - x^3 e^{-x} + 3\int x^2 e^{-x} \, dx$ ....(i)
$\int x^2 e^{-x} \, dx$

$$= x^2 \int e^{-x} \, dx - \int \left[\frac{d}{dx}(x^2) \int e^{-x} \, dx \right] dx$$

$= - x^2 e^{-x} + \int 2x e^{-x} \, dx$
$= - x^2 e^{-x} + 2\int x e^{-x} \, dx$ ......(ii)
$\int x e^{-x} \, dx$

$$= x\int e^{-x}\,dx - \int \frac{d}{dx}(x)\int e^{-x}\,dx$$

$= -xe^{-x} + \int e^{-x}\,dx$
$= -xe^{-x} - e^{-x} + C$ .....(iii)

समी. (i), (ii) व (iii) को हल करने पर

$\int x^3e^{-x}\,dx = -x^3e^{-x} + 3[-x^2e^{-x} + 2\int xe^{-x}\,dx]$
$= -x^3e^{-x} + 3(-x^2e^{-x} + 2(-xe^{-x} - e^{-x} + C1))$
$= -x^3e^{-x} - 3x^2e^{-x} + 6(-xe^{-x} - e^{-x} + C1)$
$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + 6C1$
$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C$
$= -e\,[x^3 + 3x^2 + 6x + 6] + C$

(b) $\int x^3 \sin x$

$$= x^3\int \sin x\,dx - \int\left(\frac{d}{dx}(x^3)\int \sin x\,dx\right)dx$$

$$= x^3(-\cos x) - 3\int x^2(-\cos x)\,dx$$

$$= -x^3\cos x + 3\int \underset{\mathrm{I}}{x^2}\ \underset{\mathrm{II}}{\cos x}\,dx$$

$$= -x^3\cos x + 3\left[x^2\int \cos x\,dx - \int\left\{\frac{d}{dx}(x^2)\int \cos x\,dx\right\}dx\right]$$

$$= -x^3\cos x + 3\left[x^2\sin x - 2\int \underset{\mathrm{I}}{x}\ \underset{\mathrm{II}}{\sin x}\,dx\right]$$

$$= -x^3\cos x + 3x^2\sin x - 6\left[x\int \sin x\,dx - \left\{\int \frac{d}{dx}(x)\int \sin x\,dx\right\}dx\right]$$

$= -x^3\cos x + 3x^2\sin x - 6(-x\cos x + \int \cos x\,dx)$
$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

प्रश्न 3.
(a) $\int x^3 (\log x)^2\,dx$
(b) $\int x^3 e_x 2\,dx.$
हल :

(a) $\int x^3 (\log x)^2 dx$

$$= (\log x)^2 \int x^3\, dx - \int\left[\frac{d}{dx}(\log x)^2 \int x^3\, dx\right] dx$$

$$= \frac{(\log x)^2\ x^4}{4} - \int\left[2 \log x \cdot \frac{1}{x} \times \frac{x^4}{4}\right] dx$$

$$= \frac{(\log x)^2\ x^4}{4} - \frac{1}{2}\int \underset{\mathrm{I}}{\log x}.\underset{\mathrm{II}}{x^3}\, dx$$

$$= \frac{1}{4}(\log x)^2\ x^4$$

$$- \frac{1}{2}\left[\log x \int x^3\, dx - \int\left\{\frac{d}{dx}\log x \int x^3\, dx\right\} dx\right]$$

$$= \frac{1}{4}(\log x)^2\ x^4 - \frac{1}{2}\left[\log x\, \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx\right]$$

$$= \frac{1}{4}(\log x)^2 x^4 - \frac{1}{2}\left[\frac{1}{4} \log x \cdot x^4 - \frac{1}{4}\int x^3\, dx\right]$$

$$= \frac{1}{4}(\log x)^2 x^4 - \frac{1}{2}\left[\frac{1}{4} \log x \cdot x^4 - \frac{1}{4} \times \frac{x^4}{4}\right] + C$$

$$= \frac{1}{4}(\log x)^2 x^4 - \frac{1}{8}\log x \cdot x^4 + \frac{1}{32}x^4 + C$$

$$= \frac{x^4}{4}\left[(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\right] + C$$

(b) $\int x^3 e_x 2\, dx$.

माना $x^2 = t$

$\Rightarrow 2x\, dx = dt$

$\Rightarrow x\, dx = \frac{dt}{2}$

$$= \int t e^t \frac{dt}{2}$$

$$= \frac{1}{2} \int \underset{\mathrm{I}}{t} . \underset{\mathrm{II}}{e^t} \, dt$$

$$= \frac{1}{2}\left[t \int e^t \, dt - \int \left[\frac{d}{dt}(t) \int e^t \, dt\right] dt\right]$$

$$= \frac{1}{2}\left[te^t - \int e^t \, dt\right]$$

$$= \frac{1}{2}[te^t - e^t]$$

$$= \frac{e^t}{2}[t - 1] + C$$

$$= \frac{1}{2} e^{x^2}(x^2 - 1) + C$$

प्रश्न 4. (a) $\int e^{2x} e_e x \, dx$
(b) $\int (\log x)^2 \, dx$

हल : (a) $\int e^{2x} e_e x \, dx$
माना $e^x = t$
$e^x \, dx = dt$

$$= \int e^x . e^{e^x} . e^x \, dx$$

$$= \int \underset{\mathrm{I}}{t} \underset{\mathrm{II}}{e^t} \, dt$$

$$= t \int e^t \, dt - \int \left\{\frac{d}{dt}(t) \int e^t \, dt\right\} dt$$

$= t e^t - \int 1 e^t \, dt = te^t - e^t + c$
$= e^t (t - 1) + c$
$= e_e x (e^x - 1) + c$

(b) $I = \int (\log x)^2 \, dx = \int (\log x)^2 . 1 \, dx$
[$(\log x)^2$ को प्रथम फलन तथा 1 को द्वितीय फलन लेने पर]

$$= (\log x)^2 \int 1\, dx - \int \left[\frac{d}{dx} (\log x)^2 \int 1\, dx \right] dx$$

$$= (\log x)^2\, x - \int \left[(2 \log x) . \frac{1}{x} \times x\, dx \right] dx$$

$$= x(\log x)^2 - 2 \int \underset{\text{I}}{\log x} . \underset{\text{II}}{1}\, dx$$

(log x को प्रथम फलन तथा 1 को द्वितीय फलन लेने पर)

$$= x (\log x)^2 - 2 \left[\log x \int 1\, dx - \int \frac{d}{dx} \log x \int 1\, dx \right] dx$$

$$= x (\log x)^2 - 2 \left[\log x \int dx - \int \left\{ \frac{d}{dx} (\log x) . \int dx \right\} dx \right]$$

$$= x (\log x)^2 - 2 \left[x \log x - \int \frac{1}{x} x\, dx \right]$$

$= x (\log x)^2 - 2x \log x + 2x + C$

प्रश्न 5.

(a) $\int \cos^{-1} x\, dx$ **(b)** $\int \operatorname{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

हल :

(a) $I = \int \cos^{-1} x\, dx$

माना $\cos^{-1} x = t$

तब $x = \cos t$

$dx = - \sin t\, dt$

$\therefore I = - \int t \sin t\, dt$

(t को प्रथम फलन तथा sin t को द्वितीय फलन लेने पर)

$$= - \left[t \int \sin t\, dt - \int \left\{ \frac{d}{dt} (t) \int \sin t\, dt \right\} dt \right]$$

$$= - t (- \cos t) + \int 1 . (- \cos t)\, dt$$

$$= t \cos t - \int \cos t\, dt = t \cos t - \sin t + C$$

$$= x \cos^{-1} x - \sqrt{1 - x^2} + C$$

(b) $\int \operatorname{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

माना $I = \int \operatorname{cosec}^{-1} \sqrt{\frac{x+a}{x}}\, dx$

$$= \sin^{-1} \sqrt{\frac{x}{x+a}}\, dx$$

माना $x = a \tan^2\theta$

$\therefore dx = 2a \tan\theta \sec^2\theta \, d\theta$

अतः $I = \int \sin^{-1}\sqrt{\dfrac{a\tan^2\theta}{a(1+\tan^2\theta)}}\, 2a\tan\theta\sec^2\theta\, d\theta$

$$= 2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \sin^{-1}(\sin\theta)\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \theta(\tan\theta\sec^2\theta)\, d\theta$$

$$= a\int \underset{\text{I}}{\theta}\underset{\text{II}}{(2\tan\theta\sec^2\theta)}\, d\theta$$

$$= a[\theta\tan^2\theta) - \int 1\cdot\tan^2\theta\, d\theta]$$

(θ को पहला एवं $\tan\theta\sec^2\theta$ को दूसरा फलन मानने पर)

खण्डशः समाकलन से,

$$= a\left[\theta\tan^2\theta - \int(\sec^2\theta - 1)\right]d\theta$$

$$= a\theta\tan^2\theta - \tan\theta + \theta$$

$$= a[\theta(1+\tan^2\theta) - \tan\theta)] + C$$

$$= a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(1+\frac{x}{a}\right) - \sqrt{\frac{x}{a}}\right] + C$$

$$= \left[(a+x)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax}\right] + C$$

प्रश्न 6.

(a) $\int \sin^{-1}(3x - 4x^3)\, dx$ **(b)** $\int \dfrac{x}{1+\cos x}\, dx$

हल : (a) $I = \int \sin^{-1}(3x - 4x^3)\, dx$

माना $x = \sin t$, तब $dx = \cos t\, dt$

$\therefore I = \int \sin^{-1}(3\sin t - 4\sin^3 t).\cos t\, dt$

$= \int \sin^{-1}(\sin 3t)\cos t\, dt$

$= \int 3t\cos t\, dt$

$= 3\int t \cos t\, dt$

(t को प्रथम फलन तथा cos t को द्वितीय फलन लेने पर)

$$= 3\left[t \int \cos t\, dt - \int \left\{\frac{d}{dt}(t) \int \cos t\, dt\right\} dt\right]$$

$$= 3t \sin t - 3 \int 1.\sin t\, dt$$

$$= 3t \sin t - 3\,(-\cos t) + C$$

$$= 3t \sin t + 3 \cos t + C$$

$$= 3\,(\sin^{-1} x)\,(x) + 3\sqrt{1-\sin^2 t} + C$$

$$= 3x \sin^{-1} x + 3\sqrt{1-x^2} + C$$

(b) $\int \frac{x}{1+\cos x} dx$

$$I = \int \frac{x}{1+\cos x} dx = \int \frac{x}{2\cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 \frac{x}{2}} dx$$

(x को प्रथम फलन तथा $\sec^2 \frac{x}{2}$ को द्वितीय फलन लेने पर)।

$$= \frac{1}{2}\left[x \int \sec^2 \frac{x}{2} dx - \int \left\{\frac{d}{dx}(x) \int \sec^2 \frac{x}{2} dx\right\} dx\right]$$

$$= \frac{1}{2}\left[x \times 2\tan \frac{x}{2} - \int 1 \cdot 2\tan \frac{x}{2} dx\right]$$

$$= x \tan \frac{x}{2} - \int \tan \frac{x}{2} dx$$

$$= x \tan \frac{x}{2} - 2\log\left|\sec \frac{x}{2}\right| + C$$

प्रश्न 7.

(a) $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$ **(b)** $\int \cos \sqrt{x}\, dx$

हल : (a) माना $x = \cos\theta$

$dx = -\sin\theta\, d\theta$,

$$\therefore \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

$$= \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta\, d\theta)$$

$$= -\int \tan^{-1}\sqrt{\frac{1-1+2\sin^2\theta/2}{1+2\cos^2\theta/2-1}}\,(\sin\theta\, d\theta)$$

$$= -\int \tan^{-1}\sqrt{\tan^2\theta/2}\,\sin\theta\, d\theta$$

$$= -\int \tan^{-1}(\tan\theta/2)\,\sin\theta\, d\theta$$

$$= -\int \frac{\theta}{2}\sin\theta d\theta = -\frac{1}{2}\int \theta\sin\theta\, d\theta$$

$$= -\frac{1}{2}\left[\theta\int(\sin\theta) - \int\left[\frac{d}{dx}\theta\int\sin\theta\cdot\theta\right]\right]d\theta$$

$$-\frac{1}{2}\left[-\theta\cos\theta + \int 1.\cos\theta\, d\theta\right] + C$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sin\theta] + C$$

$$= -\frac{1}{2}[-\theta\cos\theta + \sqrt{1-\cos^2\theta}] + C$$

$$= -\frac{1}{2}[-x\cos^{-1}x + \sqrt{1-x^2}] + C$$

$$= \frac{1}{2}[x\cos^{-1}x - \sqrt{1-x^2}] + C$$

(b) $\int\cos\sqrt{x}\,dx$

माना $\sqrt{x} = t$, $x = t^2$

$dx = 2t\,dt$

$\int\cos\sqrt{x}\,dx = \int\cos t.2t\,dt$

$$= 2\int \underset{\mathrm{I}}{t}\cdot\underset{\mathrm{II}}{\cos t}\,dt$$

$$= 2\left[t\int\cos t\,dt - \int\left\{\frac{d}{dt}(t).\int\cos t\,dt\right\}dt\right]$$

$$= 2\left[t\sin t - \int\sin t\,dx\right]$$

$$= 2[t\sin t + \cos t] + C$$

$$= 2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$$

प्रश्न 8.

(a) $\int \frac{x}{1+\sin x} dx$ (b) $\int x^2 \tan^{-1} x \, dx$

हल :

(a) $\int \frac{x}{1+\sin x} dx$

$$= \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \int \frac{x(1-\sin x)}{(1-\sin^2 x)} dx$$

$$= \int \frac{x(1-\sin x)}{\cos^2 x} dx$$

$$= \int \frac{x}{\cos^2 x} dx - \int x \tan x \sec x \, dx$$

$= \int x \sec^2 x \, dx - \int x \tan x \sec x \, dx$

$= I_1 - I_2$ …….(i)

$I_1 = \int x \sec^2 x \, dx$

$$I_1 = x \int \sec^2 x \, dx - \int \left\{ \frac{d}{dx} x \int \sec^2 x \, dx \right\} dx$$

$= x \tan x - \int 1.\tan x \, dx$

$= x \tan x - \int \tan x \, dx$

$= x \tan x - (-\log |\cos x|) + C_1$

$= x \tan x + \log |\cos x| + C_1$ ….(ii)

$I_2 = \int x.\tan x \sec x \, dx$

$$= x \int \tan x \sec x \, dx - \int \left(\frac{d}{dx} x \cdot \int \tan x \sec x \, dx \right) dx$$

$= x \sec x - \int 1.\sec dx$

$= x \sec x - \int \sec x \, dx$

= x sec x – log | sec x + tan x | + C2 …(iii)

समीकरण (i), (ii) व (iii) से,

$$\int \frac{x}{1+\sin x}\,dx$$

$$= [x\tan x + \log|\cos x| + C_1] - [x\sec x - \log|\sec x + \tan x| + C_2]$$

$$= x\tan x + \log|\cos x| - x\sec x + \log|\sec x + \tan x| + (C_1 - C_2)$$

$$= x(\tan x - \sec x) + \log\{\cos x(\sec x + \tan x)\} + C$$

$$= x(\tan x - \sec x) + \log(1 + \sin x) + C,$$

(जहाँ $C = C_1 + C_2$)

$$= x\left(\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right) + \log(1+\sin) + C$$

$$= \frac{-x(1-\sin x)}{\cos x} + \log(1+\sin x) + C$$

(b) $\int \underset{\text{II}}{x^2}\ \underset{\text{I}}{\tan^{-1} x}\,dx$

माना प्रथम फलन $\tan^{-1} x$ तथा द्वितीय फलन x^2 है।

$\therefore \int x^2 \tan^{-1} x\,dx$

$$= (\tan^{-1} x)\cdot\frac{x^3}{3} - \int \frac{1}{1+x^2}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{3}\int\left[x - \frac{x}{x^2+1}\right]dx$$

$$= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x}{x^2+1}\,dx$$

$$= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{3}\left(\frac{1}{2}x^2\right) + \frac{1}{3}\left[\frac{1}{2}\log(x^2+1)\right] + C$$

$$= \frac{1}{3}x^3\tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\log(x^2+1) + C$$

प्रश्न 9.

$$\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

हल :

$$I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

$\sin^{-1} x = \theta$ रखने पर

$$\frac{1}{\sqrt{1-x^2}} = d\theta$$

$$\therefore \quad I = \int \theta \cdot \sin\theta \, d\theta \quad (\because \sin^{-1} x = \theta \text{ से } x = \sin\theta)$$

$$= \theta \cdot \int \sin\theta \, d\theta - \int \frac{d}{d\theta} \cdot \theta \left[\int \sin\theta \, d\theta \right] d\theta$$

$$= \theta(-\cos\theta) - \int 1 \cdot (-\cos\theta) \, d\theta$$

$$= -\theta\cos\theta + \sin\theta + C$$

$$= -\sin^{-1} x \sqrt{1-\sin^2\theta} + x + C$$

$$= -\sin^{-1} \sqrt{1-x^2\theta} + x + C$$

$$= x - \sqrt{1-x^2} \sin^{-1} x + C$$

प्रश्न 10.

$$\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$$

हल :

$$\int \frac{x \tan^{-1} x \, dx}{(1+x^2)^{3/2}}$$

माना $x = \tan\theta$

$$= \int \frac{\tan\theta . \tan^{-1}(\tan\theta)}{(1+\tan^2\theta)^{3/2}} \cdot \sec^2\theta \, d\theta$$

$$= \int \frac{\tan\theta \cdot \theta \sec^2\theta \, d\theta}{\sec^3\theta} = \int \frac{\theta \tan\theta}{\sec\theta}$$

$$= \int \theta \sin\theta \, d\theta = \theta \int \sin\theta \, d\theta - \int \left[\frac{d}{d\theta}(\theta) \int \sin\theta \right] d\theta$$

$= \theta(-\cos\theta) + \int \cos\theta \, d\theta$

$= -\theta\cos\theta + \sin\theta + c$

$= \sin\theta - \theta\cos\theta + c$

= cos θ [tanθ − θ] + c

$$= \frac{\tan\theta - \theta}{\sec\theta} + C = \frac{-\theta + \tan\theta}{\sqrt{1+\tan^2\theta}} + C$$

$$= \frac{-\tan^{-x}}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$$

प्रश्न 11. ∫ e^x (cot x + log sin x) dx

हल : ∫ e^x (cot x + log sin x) dx

माना I = ∫e^x [log |sin x| + cot x]dx

= ∫e^x log | sinx | dx + ∫e^x cot x dx

= ∫log | sin x | e^x dx + ∫e^x cot x dx

अब ∫log |sin x| e^x dx

$$= \log|\sin x| \cdot \int e^x\, dx - \int\left[\frac{d}{dx}|\sin x| \cdot \int e^x\, dx\right] dx$$

$$= \log|\sin x| \cdot e^x - \int \frac{1 \cdot \cos x}{\sin x} e^x\, dx$$

$$= \log|\sin x| \cdot e^x - \int \frac{\cot x}{\sin x} e^x\, dx$$

∴ I = e^x log | sin x |e^x − ∫e^x cotx dx + ∫e^x cotx dx

= log | sin x |e^x + C

= e^x log | sin x | + C

प्रश्न 12.

$$\int \frac{2x + \sin 2x}{1 + \cos 2x} dx$$

हल :

$$\int \frac{2x + \sin 2x}{1 + \cos 2x} dx$$

$$= \int \frac{2x + 2\sin x \cos x}{2\cos^2 x} dx$$

= ∫x sec² x dx + ∫tan x dx

= x tan x − ∫tan x dx + ∫tan x dx

= x tan + C

प्रश्न 13.

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$$

हल :

माना $I = \int e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx$

$$= \int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$$

$$= \int e^x\left(\frac{1}{2}\text{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right) dx$$

$$I = \int e^x\left\{\left(-\cot\frac{x}{2}\right)+\frac{1}{2}\text{cosec}^2\frac{x}{2}\right\} dx$$

$$= -\int e^x\cot\frac{x}{2}\,dx+\frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx$$

$$= -\left\{\cot\frac{x}{2}e^x-\int-\text{cosec}^2\frac{x}{2}\cdot\frac{1}{2}e^x\,dx\right\}$$

$$+\frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx$$

(केवल पहले भाग का खण्डशः समाकलन करने पर)

$$= -e^x\cot\frac{x}{2}-\frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx+\frac{1}{2}\int e^x\,\text{cosec}^2\frac{x}{2}\,dx+C$$

$$= -e^x\cot\frac{x}{2}+C$$

प्रश्न 14.

$$\int e^x\left[\log x+\frac{1}{x^2}\right]dx$$

हल :

$$\int e^x \left[\log x + \frac{1}{x^2}\right] dx$$

$$= \int e^x . \log x \, dx + \int e^x . \frac{1}{x^2} \, dx$$

$$= \int \log x \cdot e^x \, dx + \int x^{-2} \cdot e^x \, dx \qquad \text{...(i)}$$

$$= I_1 + I_2 \qquad \text{माना}$$

$$I_1 = \int \underset{\text{I}}{\log x} . \underset{\text{II}}{e^x} \, dx$$

$$= \log x \int e^x \, dx - \int \left[\frac{d}{dx}(\log x) . \int e^x \, dx\right] dx$$

$$= \log x . e^x - \int \underset{\text{I}}{\frac{1}{x}} . \underset{\text{II}}{e^x} \, dx$$

$$= \log x . e^x - \frac{1}{x} . \int e^x \, dx + \int \left[\frac{d}{dx}\left(\frac{1}{x}\right) . \int e^x \, dx\right] dx$$

$$= e^x . \log x - \frac{1}{x} e^x + \int -\frac{1}{x^2} e^x \, dx$$

$$= e^x . \log x - \frac{1}{x} e^x - \int x^{-2} e^x \, dx \qquad \text{...(ii)}$$

समीकरण (i) व (ii) से,

$$\int e^x \left(\log x + \frac{1}{x^2}\right) dx$$

$$= e^x \log x - \frac{1}{x} e^x - \int x^{-2} e^x \, dx + \int x^{-2} e^x \, dx + C$$

$$= e^x \log x - \frac{1}{x} e^x + C$$

$$= e^x \left[\log x - \frac{1}{x}\right] + C$$

प्रश्न 15. ∫eˣ [log (sec x + tan x) + sec x) dx

हल :

= $\int e^x$ [log (sec x + tan x) + sec x) dx

= $\int e^x$ log (sec x + tan x) dx + $\int e^x$ sec x dx

$= \int \log(\sec x + \tan x)\, e^x\, dx + \int \sec x e^x\, dx$

$$= \log(\sec x + \tan x) \int e^x\, dx - \int \left[\frac{d}{dx} \log(\sec x + \tan x) \int e^x\, dx \right] dx + \int \sec x . e^x\, dx$$

$$= \log|\sec x + \tan x|\, e^x - \int \frac{1}{(\sec x + \tan x)} \times (\sec x \tan x) + \sec^2 x |\, e^x\, dx + \int \sec x . e^x\, dx + C$$

$$= e^x \log|\sec x + \tan x| - \int \frac{\sec x (\tan x + \sec x)}{(\sec x + \tan x)} e^x\, dx + \int \sec x\, e^x\, dx + C$$

$= e^x \log|\sec x + \tan x| - \int \sec x e^x\, dx + \int \sec x e^x\, dx + C$

$= e^x \log|\sec x + \tan x| + C$

प्रश्न 16. $\int e^x (\sin x + \cos x) \sec^2 x\, dx$

हल : $\int e^x (\sin x + \cos x) \sec^2 x\, dx$

$$= e^x \left(\frac{\cos x}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$= \int e^x (\sec x + \sec x \tan)\, dx$

$= e^x \sec x + C$ [$\because e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$, $f(x) = \sec x$, $f'(x) = \sec x \tan x$]

प्रश्न 17.

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$$

हल :

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$$

$$= \int e^x \cdot \frac{1}{x^2}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= \int e^x x^{-2}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= \int \underset{\mathrm{I}}{x^{-2}}\, \underset{\mathrm{II}}{e^x}\, dx - 2\int \frac{e^x}{x^3}\, dx$$

$$= x^{-2} \int e^x\, dx - \int\left[\frac{d}{dx}(x^{-2}) \int e^x\, dx\right] dx$$

$$-2 \int \frac{e^x}{x^3}\, dx + C$$

$$= x^{-2} e^x - \int -2x^{-3} e^x\, dx - 2\int \frac{e^x}{x^3}\, dx + C$$

$$= x^{-2}e^x + 2\int \frac{e^x}{x^3}\, dx - 2\int \frac{e^x}{x^3}\, dx + C$$

$$= x^{-2}e^x + C$$

प्रश्न 18.

$$\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$$

हल :

माना $I = \int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx = \int e^x \left\{\frac{1-2x+x^2}{(1+x^2)^2}\right\} dx$

$$= \int e^x \left\{\frac{1+x^2+(-2x)}{(1+x^2)^2}\right\} dx$$

$$= \int e^x \left[\frac{1}{1+x^2} + \frac{(-2x)}{(1+x^2)^2}\right] dx$$

$$= \int e^x \frac{1}{1+x^2}\, dx + \int e^x \frac{(-2x)}{(1+x^2)^2}\, dx$$

$$= \frac{1}{1+x^2} e^x - \int \frac{(-2x)}{(1+x^2)^2} e^x\, dx + \int e^x \frac{(-2x)}{(1+x^2)^2}\, dx$$

$$= \frac{e^x}{1+x^2} + C$$

प्रश्न 19.

$$\int \cos 2\theta . \log \left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$$

हल :

$$I = \int \cos 2\theta . \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$$

$\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$ प्रथम फलन मानकर, खण्डश: समाकलन पर,

$$= \frac{\sin 2\theta}{2} \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$-\frac{1}{2}\int \frac{d}{d\theta}\left[\log\left(\frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}\right)\sin 2\theta\right] \quad ...(i)$$

लेकिन

$$= \frac{d}{d\theta}[\log(\cos\theta + \sin\theta) - \log(\cos\theta - \cos\theta)]$$

$$= \frac{1}{(\cos\theta + \sin\theta)}(-\sin\theta + \cos\theta) - \frac{(-\sin\theta\cos\theta)}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta - \sin\theta) - (\cos\theta + \sin\theta)(-\sin\theta - \cos\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}$$

$$= \frac{(\cos^2\theta - \cos\theta\sin\theta - \sin\theta\cos\theta + \sin^2\theta + \cos\theta\sin\theta + \cos^2\theta + \sin^2\theta + \cos\theta\sin\theta)}{(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{2(\cos^2\theta + \sin^2\theta)}{\cos 2\theta} = \frac{2}{\cos 2\theta}$$

समी. (i) में उपरोक्त मान रखने पर,

$$I = \frac{1}{2}\sin 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) - \frac{1}{2}\int \sin 2\theta \cdot \frac{2}{\cos 2\theta}$$

$$= \frac{1}{2}\sin 2\theta \log\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) + \frac{1}{2}\log(\cos 2\theta) + C$$

प्रश्न 20.

$$\int \frac{x^2}{(x\cos x - \sin x)^2} dx$$

हल :

$$\int \frac{x^2}{(x\cos x - \sin x)^2}\,dx$$

$$= \underset{\text{I}}{x \operatorname{cosec} x} \cdot \underset{\text{II}}{\frac{x\sin x}{(x\cos x - \sin x)^2}}$$

$$= x \operatorname{cosec} x \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$$

$$- \int \frac{d}{dx}(x \operatorname{cosec} x) \int \left[\frac{x\sin x}{(x\cos x - \sin x)^2}\,dx\right]$$

$$= x \operatorname{cosec} x \cdot \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$$

$$- \int (\operatorname{cosec} x - \cot x.\operatorname{cosec} x \cdot \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx \quad \text{...(i)}$$

माना $\qquad I_1 = \int \frac{x\sin x}{(x\cos x - \sin x)^2}\,dx$

तथा $\qquad x\cos x - \sin x = u$

$(x(-\sin x) + \cos x - \cos x)\,dx = du$

$\Rightarrow \qquad -x \sin dx = du$

$\Rightarrow \qquad x \sin dx = -du$

$\Rightarrow \qquad I_1 = \int \frac{-du}{u^2}$

$\qquad = -\int u^2\,du = 1 \times \left(-\frac{1}{u}\right)$

$\Rightarrow \qquad I_1 = \frac{1}{u} + C_1$

$\Rightarrow \qquad I_1 = \frac{1}{x\cos x - \sin x} \qquad \text{...(ii)}$

समीकरण (i) व (ii) से,

$$\int \frac{x^2}{(x\cos x - \cos x)^2}\,dx$$

$$= x \operatorname{cosec} x \frac{1}{(x\cos x - \sin x)}$$

$$- \int(\operatorname{cosec} x - \cot x \operatorname{cosec} x) \cdot \frac{1}{(x\cos x - \sin x)} dx$$

$$= \frac{x \operatorname{cosec} x}{(x\cos x - \sin x)} - \int \frac{\operatorname{cosec} x(1 - x\cot x)}{(x\cos x - \sin x)} dx$$

$$= \frac{x \operatorname{cosec} x}{(x\cos x - \sin x)} - \int \frac{\operatorname{cosec} \cdot (x\cot x - 1)}{(x\cos x - \sin x)} dx$$

$$= \frac{x \operatorname{cosec} x}{(x\cos x - \sin x)} + \int \frac{\operatorname{cosec} x \cdot (x\cos x - \sin x)}{\sin x(x\cos x - \sin x)} dx$$

$$= \frac{x \operatorname{cosec} x}{x\cos x - \sin x} + \int \operatorname{cosec}^2 x \, dx$$

$$= \frac{x \operatorname{cosec} x}{x\cos x - \sin x} - \cot x + C$$

$$= \frac{x}{\sin x(x\cos x - \sin x)} - \frac{\cos x}{\sin x} + C$$

$$= \frac{x - \cos x\,(x\cos x - \sin x)}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{x - x\cos^2 x + \sin x\cos x}{\sin x(x\cos x - \sin x)}$$

$$= \frac{x(1 - \cos^2 x) + \sin x\cos x}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{\sin x(\sin x + \cos x)}{\sin x(x\cos x - \sin x)} + C$$

$$= \frac{x\sin x + \cos x}{x\cos x - \sin x} + C$$

प्रश्न 21.

$$\int \cos^{-1}\left(\frac{1}{x}\right) dx$$

हल :

माना $I = \int \cos^{-1}\frac{1}{x}\,dx$

$$= \int \sec^{-1} x\,dx \quad \left(\because \cos^{-1}\frac{1}{x} = \sec^{-1} x\right)$$

$$= \int \sec^{-1} x \cdot 1\,dx$$

Sec^{-1} x को प्रथम तथा 1 को द्वितीय फलन लेकर खण्डशः माकलन करने पर,

$$I = \sec^{-1} x \int dx - \int\left[\frac{d}{dx}(\sec^{-1} x)\cdot \int dx\right]dx$$

$$= x\sec^{-1} x - \int \frac{1}{x\sqrt{x^2-1}}\,x\,dx$$

$$= x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}}$$

$$= x\sec^{-1} x - \log\left|x + \sqrt{x^2-1}\right| + C$$

प्रश्न 22. ∫ $(\sin^{-1} x)^2$ dx

हल : माना $I = \int(\sin^{-1} x)^2\,dx$

माना $\sin^{-1} x = \theta$

$x = \sin\theta$

$\therefore dx = c0s\,\theta\,d\theta$

$\therefore I = \int \theta^2 \cdot \cos\theta\,a\theta$

(θ^2 को प्रथम फलन एवं $\cos\theta$ को द्वितीय फलन लेने पर)

$= \theta^2 \sin\theta - \int 2\theta \sin\theta\,d\theta + C$

$= \theta^2 \sin\theta - 2\int\theta \sin\theta + C$

$= \theta^2 \sin\theta - 2[\theta - \cos\theta) - \int 1.(-\cos\theta)\,d\theta]+C$

(पुन: θ को प्रथम तथा $\sin\theta$ को द्वितीय फलन लेने पर)।

$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\int\cos\theta\,d\theta + C$

$= \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C$

$$= \theta^2 \sin\theta + 2\theta\sqrt{1-\sin^2\theta} - 2\sin\theta + C$$

$$= x(\sin^{-1} x)^2 + 2\sin^{-1} x \cdot \sqrt{1-x^2} - 2x + C$$

Ex 9.7

निम्नलिखित फलनों के समाकलन कीजिए :

प्रश्न 1. $\int e^{2x} \cos x \, dx$

हल : $\int e^{2x} \cos x \, dx$

माना $I = \int e^{2x} . \cos x \, dx$

$$\Rightarrow \quad I = e^{2x} \int \cos x \, dx - \int \left[\frac{d}{dx} e^{2x} \int \cos x \, dx \right] dx$$

$$\Rightarrow \quad I = e^{2x} \sin x - 2 \int \underset{\mathrm{I}}{e^{2x}} \underset{\mathrm{II}}{\sin x} \, dx$$

$$\Rightarrow \quad I = e^{2x} \sin x - 2 \left[e^{2x} \int \sin x \, dx - \int \left\{ \frac{d}{dx} e^{2x} \int \sin 2x \, dx \right\} dx \right]$$

$$\Rightarrow \quad I = e^{2x} \sin x - 2 \left[e^{2x} (-\cos x) - 2 \int e^{x} (-\cos x) \, dx \right] + C$$

$$\Rightarrow \quad I = e^{2x} \sin x - 2 \left[-e^{2x} \cos 2x + 2 \int e^{x} \cos x \, dx \right] + C$$

$$\Rightarrow \quad I = e^{2x} \sin x + 2 e^{2x} \cos 2x - 4 \int e^{x} \cos x \, dx + C$$

$$\Rightarrow \quad I = e^{2x} \sin x + 2e^{2x} \cos x - 4I + C$$

$$\Rightarrow \quad 5I = e^{2x} \sin x + 2e^{2x} \cos x + C$$

$$\Rightarrow \quad I = \frac{1}{5} e^{2x} (\sin 2x + 2 \cos x) + C$$

$$\therefore \quad \int e^{2x} \cos x \, dx = \frac{1}{5} e^{2x} (2 \cos x + \sin 2x) + C$$

प्रश्न 2.

$\int \sin (\log x) \, dx$

हल :

माना $I = \int \sin (\log x) \, dx$

माना $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$\therefore \quad I = \int \underset{II}{\sin t}\, \underset{I}{e^t}\, dt$$

$$= -e^t \cos t - \int e^t (-\cos t)\, dt$$

(खंडश: समाकलन से)

$$= -e^t \cos t + \int e^t \cos t\, dt$$

$$= -e^t \cos t + \left\{e^t \sin t - \int e^t \sin t\, dt\right\}$$

$$= -e^t \cos t + e^t \sin t - I$$

$$\Rightarrow \quad 2I = e^t (\sin t - \cos t)$$

$$\Rightarrow \quad I = \frac{e^t}{2} (\sin t - \cos t) + C$$

$$= \frac{x}{2} \{\sin (\log x) - \cos (\log x)\} + C$$

प्रश्न 3.

$$\int \frac{e^{a \tan^{-1} x}}{(1 + x^2)^{\frac{3}{2}}}\, dx$$

हल :

$$\int \frac{e^{a \tan^{-1} x}}{(1 + x^2)^{3/2}}\, dx$$

माना $\tan^{-1} x = t$

$x = \tan t$

$$\frac{1}{1 + x^2}\, dx = dt$$

$$\text{अत:} \quad I = \int \frac{e^{at}}{\sqrt{1 + \tan^2 t}}\, dt$$

$$\Rightarrow \quad I = \int e^{at} \cos t\, dt$$

$$\Rightarrow \quad I = e^{at} \int \cos t\, dt - \int \left[\frac{d}{dx}(e^{at}) \int \cos t\, dt\right] dt$$

$$\Rightarrow \quad I = e^{at} \sin t - \int a e^{at} \sin t\, dt$$

$$\Rightarrow \quad I = e^{at}\sin t - a\left[e^{at}\int \sin t\, dt - \int\left[\frac{d}{dt}(e^{at})\int \sin t\, dt\right]dt\right]$$

$$\Rightarrow \quad I = e^{at}\sin t + ae^{at}\cos t - a\int(ae^{at}\cos t)dt$$

$$\Rightarrow \quad I = e^{at}\sin t + ae^{at}\cos t - a^2\int e^{at}\cos t\, dt$$

$$\Rightarrow \quad I = e^{at}\sin t + ae^{at}\cos t - a^2 I$$

$$\Rightarrow \quad I(1 + a^2) = e^{at}\sin + ae^{at}\cos t$$

$$\Rightarrow \quad I = \frac{e^{at}}{(1+a^2)}[\sin t + a\cos t]$$

यहाँ $\tan x = x$, $\sin t = \dfrac{x}{\sqrt{1+x^2}}$, $\cos t = \dfrac{1}{\sqrt{1+x^2}}$

$$\Rightarrow \quad I = \frac{e^{at}}{1+a^2}\left[\frac{x}{\sqrt{1+x^2}} + \frac{a}{\sqrt{1+x^2}}\right] + C$$

$$\Rightarrow \quad I = \frac{e^{at}}{1+a^2}\left[\frac{x+a}{\sqrt{x^2+1}}\right] + C$$

यही अभीष्ट हल है।

प्रश्न 4.

$$\int e^{\frac{x}{\sqrt{2}}}\cos(x+\alpha)$$

हल :

$$\int e^{x/\sqrt{2}}\cos(x+\alpha)\, dx \qquad \text{(माना)}$$

$$I = \int \underset{\text{I}}{\cos}(x+\alpha)\underset{\text{II}}{e}^{x/\sqrt{2}}\, dx$$

$$= \cos(x+\alpha)\int e^{x/\sqrt{2}}\, dx - \int\left[\frac{d}{dx}[\cos(x+\alpha)\int e^{x/\sqrt{2}}\, dx\right]dx$$

$$= \cos(x+\alpha)\frac{e^{x/\sqrt{2}}}{1/\sqrt{2}} - \int\left[\{-\sin(x+\alpha)\}\frac{e^{x/\sqrt{2}}}{1/\sqrt{2}}\right]dx$$

$$= \sqrt{2}\cos(x+\alpha)e^{x/\sqrt{2}} + \sqrt{2}\int\left[\underset{\text{I}}{\sin}(x+\alpha)\underset{\text{II}}{e}^{x/\sqrt{2}}\right]dx$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos (x+\alpha) + \sqrt{2}\Big[\sin (x+\alpha) \int e^{x/\sqrt{2}}\, dx - \int \frac{d}{dx}\{\sin (x+\alpha)\} \int e^{x/\sqrt{2}}\, dx\Big] dx$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos (x+\alpha) + \sqrt{2}\left[\sin (x+\alpha) \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}} - \int \cos (x+\alpha) \frac{e^{x/\sqrt{2}}}{1/\sqrt{2}}\, dx\right]$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos (x+\alpha) + \sqrt{2}\left[\sqrt{2} \sin (x+\alpha) e^{x/\sqrt{2}} - \sqrt{2} \int \cos (x+\alpha) e^{x/\sqrt{2}}\, dx\right]$$

$$= \sqrt{2}e^{x/\sqrt{2}} \cos (x+\alpha) + 2\sin (x+\alpha) e^{x/\sqrt{2}} - 2\int \cos (x+\alpha) e^{x/\sqrt{2}}\, dx + C$$

$$3I = 2e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}} \cos (x+\alpha) + \sin (x+\alpha)\right] + C$$

$$\Rightarrow \int e^{x/\sqrt{2}} \cos (x+\alpha)\, dx = \frac{2}{3} e^{x/\sqrt{2}}\left[\frac{1}{\sqrt{2}} \cos (x+a) + \sin (x+\alpha)\right] + C$$

प्रश्न 5.

$\int e^x \sin^2 x\, dx$

हल : माना $\int e^x \sin^2 x\, dx = I$

$I = \int e^x \sin^2 x\, dx$

$$= \int e^x \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int e^x\, dx - \frac{1}{2}\int \underset{\mathrm{I}}{e^x} \underset{\mathrm{II}}{\cos} 2x\, dx \qquad \ldots\text{(i)}$$

पुनः माना $I_1 = \int e^x \cos 2x\, dx$

$$I = \frac{1}{2}\int e^x - \frac{1}{2} I_1 \qquad \ldots\text{(ii)}$$

$$I_1 = \left[e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin 2x}{2}\, dx\right]$$

$$= \left[\frac{1}{2}e^x \sin 2x - \frac{1}{2}\int \underset{\mathrm{I}}{e^x} \underset{\mathrm{II}}{\sin}\, 2x\, dx\right]$$

$$= \frac{1}{2}\left[e^x \sin 2x - \left\{\frac{-e^x \cos 2x}{2} - \int \frac{e^x(-\cos 2x)}{2}\right\} dx\right]$$

$$= \frac{1}{2}\left[e^x \sin 2x + \frac{1}{2}e^x \cos 2x - \frac{1}{2}\int e^x \cos 2x\right] dx$$

$$= \frac{1}{2}\left[e^x \sin 2x + \frac{1}{2}e^x \cos 2x - \frac{1}{2}I_1\right] + C$$

$$\Rightarrow \qquad I_1 = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}I_1 + C$$

$$\Rightarrow \left(1+\frac{1}{4}\right)I_1 = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x + C$$

$$\Rightarrow \qquad I_1 = \frac{4}{5\times 2}e^x \sin 2x + \frac{4}{5\times 4}e^x \cos 2x + C$$

$$\Rightarrow \qquad I_1 = \frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x + C$$

I_1 का मान समीकरण (ii) में रखने पर,

$$I = \frac{1}{2}\int e^x\ dx - \frac{1}{2}\left[\frac{2}{5}e^x\ \sin 2x + \frac{1}{5}e^x\ \cos 2x + C\right]$$

$$= \frac{1}{2}e^x - \frac{1}{5}e^x\ \sin 2x - \frac{1}{10}e^x\ \cos 2x + C$$

$$\Rightarrow \quad I = \frac{1}{2}e^x - \frac{1}{5}e^x\ \sin 2x - \frac{1}{10}e^x\ \cos 2x + C$$

$$\Rightarrow \int e^x\ \sin^2 x\ dx$$

$$= \frac{1}{2}e^x - \frac{1}{5}e^x \sin 2x - \frac{1}{10}e^x\ \cos 2x\ dx + C$$

$$= \frac{e^x}{2} - \frac{e^x}{10}[\cos 2x + 2\sin 2x] + C$$

प्रश्न 6. ∫e^a sin^-1x dx

हल : ∫e^a sin^-1x dx

माना a sin^-1 x = t

$$x = \sin\left(\frac{t}{a}\right)$$

$$dx = \cos\left(\frac{t}{a}\right)\cdot\frac{1}{a}\,dt$$

$$\Rightarrow \qquad dx = \frac{1}{a}\cos\left(\frac{t}{a}\right)dt$$

$$I = \int e^{a\sin^{-1}x}\,dx$$

$$= \int e^t\cdot\left(\frac{1}{a}\right)\cos\frac{t}{a}\,dt$$

$$= \frac{1}{a}\int e^t\cos\frac{t}{a}\,dt$$

सूत्र $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos ax + b\sin bx) + C$ से

$$= \frac{1}{a}\cdot\frac{e^t}{\left(1+\frac{1}{a^2}\right)}\left[\cos t + \frac{1}{a}\sin b\left(\frac{t}{a}\right)\right] + C$$

$$= \frac{1}{a} \times \frac{e^t}{\left(\frac{1+a^2}{a^2}\right)}\left[\cos t + \frac{1}{a}\sin b\left(\frac{t}{a}\right)\right] + C$$

$$= \frac{e^t a^2}{a^2(1+a^2)}\left[a\cos t + \sin\frac{t}{a}\right] + C$$

$$= \frac{e^t}{(1+a^2)}\left[a\cos t + \sin\frac{t}{a}\right] + C$$

$$= \frac{e^{a\sin^{-1}x}}{(1+a^2)}\{a\sqrt{1-x^2} + x] + C$$

प्रश्न 7.

$$\int \cos\left(b\cos\frac{x}{a}\right)dx$$

हल :

$$\int \cos\left(b\log\frac{x}{a}\right)dx$$

माना $\quad b\log\frac{x}{a} = t$

$\Rightarrow \quad \log\frac{x}{a} = \frac{t}{b}$

$\Rightarrow \quad \frac{x}{a} = e^{t/b}$

$\Rightarrow \quad x = ae^{t/b}$

$$dx = \frac{a}{b}e^{t/b}\,dt$$

$$\int \cos\left(b\cos\frac{x}{a}\right)dx = \int \cos t\,\frac{a}{b}\,e^{t/b}\,dt$$

$$= \frac{a}{b}\int e^{t/b}\cos t\,dt$$

$$= \frac{a}{b} \frac{e^{t/b}}{\left(\frac{1}{b^2}+1\right)} (\cos t + \sin t) + C$$

$$= \frac{a \times b^2}{b(1+b^2)} e^{\frac{b \log}{b}\left(\frac{x}{a}\right)} \left[\cos\left(b \log \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

$$= \frac{a}{1+b^2} e^{\log \frac{x}{a}} \left[\cos\left(b \cos \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

$$= \frac{x}{1+b^2} \left[\cos\left(b \cos \frac{x}{a}\right) + \sin\left(b \log \frac{x}{a}\right)\right] + C$$

प्रश्न 8. $\int e^{4x} \cos 4x \cos 2x\, dx$

हल : $\int e^{4x} \cos 4x \cos 2x\, dx$

$$= \int e^{4x} \frac{1}{2}\{\cos(4x+2x) + \cos(4x-2x)\}\, dx$$

$$= \frac{1}{2}\int e^{4x} \cos 6x\, dx + \frac{1}{2}\int e^{4x} \cos 2x\, dx$$

$$= \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \ldots(i)$$

$$I_1 = \int \underset{II}{e^{4x}} \underset{I}{\cos 6x}\, dx$$

$$= \frac{e^{4x}}{(4^2+6^2)} [4\cos 6x + 6\sin 6x] + C_1$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C\right]$$

$$= \frac{e^{4x}}{52} [4\cos 6x + \sin 6x] + C_1 \qquad \ldots(ii)$$

$$I_2 = \int e^{4x} \cos 2x\, dx$$

$$= \frac{e^{4x}}{4^2+2^2} [4\cos 2x + 2\sin 2x] + C_2$$

$$= \frac{e^{4x}}{20} [4\cos 2x + 2\sin 2x] + C_2 \qquad \ldots(iii)$$

$$I = \frac{1}{2}\times\frac{e^{4x}}{52}[4\cos 6x + 6\sin 6x]$$

$$+\frac{1}{2}\frac{e^{4x}}{20}[4\cos 2x + 2\sin 2x] + C_1 + C_2$$

$$= \frac{e^{4x}}{8}\left[\frac{1}{13}(4\cos 6x + 6\sin 6x)\right.$$

$$\left.+\frac{1}{5}(4\cos 2x + 2\sin 2x)\right] + C$$

(जहाँ $C_1 + C_2 = C$)

प्रश्न 9.

$\sqrt{2x - x^2}$

हल :

$$\int\sqrt{2x - x^2}\,dx = \int\sqrt{-(x^2 - 2x)}\,dx$$

$$= \int\sqrt{-(x^2 - 2x + 1 - 1)}\,dx$$

$$= \int\sqrt{-(x^2 - 2x + 1) + 1}\,dx$$

$$= \int\sqrt{1^2 - (x-1)^2}\,dx$$

$$= \frac{1}{2}(x-1)\sqrt{2x - x^2} + \frac{1}{2}\times 1^2 \sin^{-1}\left(\frac{x-1}{1}\right) + C$$

$$= \frac{1}{2}(x-1)\sqrt{2x - x^2} + \frac{1}{2}\sin^{-1}(x-1) + C$$

प्रश्न 10.

$\int\sqrt{x^2 + 4x + 6}\,dx$

हल :

$$\int \sqrt{x^2+4x+6}\, dx = \int \sqrt{x^2+2.2x+2^2+2}\, dx$$

$$= \int \sqrt{(x+2)^2+(\sqrt{2})^2}\, dx$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+6}+\frac{1}{2}$$

$$+\frac{1}{2}\times(\sqrt{2})^2 \log\left[(x+2)+\sqrt{x^2+4x+6}\right]+C$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+6}$$

$$+\log\left[(x+2)+\sqrt{x^2+4x+6}\right]+C$$

प्रश्न 11.

$$\int \sqrt{x^2+6x+4}\, dx$$

हल :

$$\int \sqrt{x^2+6x+4}\, dx$$

$$= \int \sqrt{x^2+2.3x+3^2-(\sqrt{5})^2}\, dx$$

$$= \int \sqrt{(x+3)^2-(\sqrt{5})^2}\, dx$$

$$= \frac{1}{2}(x+3)\sqrt{x^2+6x+4}$$

$$-\frac{1}{2}(\sqrt{5})^2 \log\left[(x+3)+\sqrt{x^2+6x+4}\right]$$

$$= \frac{1}{2}(x+3)\sqrt{x^2+6x+4}$$

$$-\frac{5}{2}\log\left[(x+3)+\sqrt{x^2+6x+4}\right]+C$$

प्रश्न 12.

$$\int \sqrt{2x^2+3x+4}\, dx$$

हल :

$$\int \sqrt{2x^2+3x+4}\, dx$$

$$= \sqrt{2} \int \sqrt{x^2+\frac{3}{2}x+2}\, dx$$

$$= \sqrt{2} \int \sqrt{x^2+2.\frac{3}{4}x+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+2}\, dx$$

$$= \sqrt{2} \int \sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\, dx$$

$$= \frac{1}{2}\frac{4x+3}{4}\sqrt{2x^2+3x+4}$$

$$+\frac{23}{32}\log\left(\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+C$$

$$= \frac{4x+3}{4}\sqrt{2x^2+3x+4}$$

$$+\frac{23}{16\sqrt{2}}\log\left(\frac{4x+3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+C$$

प्रश्न 13.

$$\int x^2\sqrt{(a^6-x^6)}\, dx$$

हल :

$$\int x^2\sqrt{(a^6-x^6)}\, dx$$

$$= \int \sqrt{(a^3)^2-(x^3)^2}\, x^2\, dx$$

माना $x^3 = t,\ 3x^2\, dx = dt$

$$x^2\, dx = \frac{dt}{3}$$

$$= \int \sqrt{(a^3)^2-t^2}\,\frac{dt}{3}$$

$$= \frac{1}{3}\int \sqrt{(a^3)^2-t^2}\, dt$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{(a^3)^2 - t^2} + \frac{1}{2}(a^3)^2 \sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{(a^6 - t^2)} + \frac{a^6}{2} \sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{6}\left[x^3\sqrt{(a^6 - x^6)} + a^6 \sin^{-1}\left(\frac{x^3}{a^3}\right)\right] + C$$

प्रश्न 14.

$$\int (x+1)\sqrt{x^2+1}\, dx$$

हल :

$$\int (x+1)\sqrt{x^2+1}\, dx$$

$$= \int x\sqrt{x^2+1}\, dx + \int \sqrt{x^2+1}\, dx$$

$$= I_1 + I_2 \qquad \text{(माना)} \quad \ldots(i)$$

$$I_1 = \int x\sqrt{(x^2+1)}\, dx$$

माना $x^2 + 1 = t$

$$2x\, dx = dt$$

$$x\, dx = \frac{dt}{2}$$

$$= \int \sqrt{(x^2+1)}.x\, dx$$

$$= \int \sqrt{t}\, \frac{dt}{2} = \frac{1}{2}\int t^{1/2}\, dt$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} + C_1 = \frac{1}{3}t^{3/2} = \frac{1}{3}(x^2+1)^{3/2} + C_1 \qquad \ldots(ii)$$

$$I_2 = \int \sqrt{x^2+1}\, dx$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log\left(x + \sqrt{x^2+1}\right) + C_2 \qquad \ldots(iii)$$

$$\int (x+1)\sqrt{x^2+1}\, dx$$

$$= \frac{1}{3}(x^2+1)^{3/2} + \frac{1}{2}x\sqrt{(x^2+1)} + \frac{1}{2}\log\left(x + \sqrt{x^2+1}\right) + C$$

(जहाँ $C = C_1 + C_2$)

प्रश्न 15.

$$\int\sqrt{1-4x-x^2}\,dx$$

हल : माना

$$I=\int\sqrt{1-4x-x^2}\,dx$$

$$=\int\sqrt{1-(x^2+4x)}\,dx$$

$$=\int\sqrt{1-(x^2+4x+4-4)}\,dx$$

$$=\int\sqrt{1-(x+2)^2+4}\,dx$$

$$=\int\sqrt{5-(x+2)^2}\,dx$$

$$=\int\sqrt{(\sqrt{5})^2-(x+2)^2}\,dx$$

$$\therefore I=\frac{(x+2)}{2}\sqrt{5-(x+2)^2}+\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+C$$

$$I=\frac{5}{2}\sin^{-1}\frac{x+2}{\sqrt{5}}+\frac{x+2}{2}\sqrt{1-4x-x^2}+C$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C\right]$$

प्रश्न 16.

$$\int\sqrt{4-3x-2x^2}\,dx$$

हल :

$$\int\sqrt{4-(2x^2+3x)}\,dx$$

$$=\int\sqrt{4-(2x^2+3x)}\,dx$$

$$=\int\sqrt{4-2\left(x^2+\frac{3}{2}x\right)}\,dx$$

$$=\sqrt{2}\int\sqrt{2-\left(x^2+\frac{3}{2}x\right)}\,dx$$

$$=\sqrt{2}\int\sqrt{2-\left[x^2+2.\frac{3}{4}x+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}\,dx$$

$$= \sqrt{2}\int\sqrt{\left(2+\frac{9}{16}\right)-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \sqrt{2}\int\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \frac{1}{\sqrt{2}}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}$$

$$+\sqrt{2}\times\frac{1}{2}\times\left(\frac{\sqrt{41}}{4}\right)^2\sin^{-1}\left[\frac{\left(x+\frac{3}{4}\right)}{\left(\frac{\sqrt{41}}{4}\right)}\right]+C$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{4-3x-2x^2}\right.$$

$$\left.+\frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right)\right]+C$$

Miscellaneous Exercise

निम्नलिखित के मान ज्ञात कीजिए

प्रश्न 1.
∫[1 + 2 tan x (tan x + see x) dx

हल :
∫[1 + 2 tan x (tan x + sec x)] dx
= ∫(1 + 2 tan² x + 2 tan x sec x] dx
= ∫[2(1 + tan² x) + 2 sec x tan x − 1] dx
= 2∫(sec² x + sec x tan x) dx − ∫dx
= 2(tan x + sec x) − x + C

प्രश्न 2.
$\int e^x \sin^3 x\, dx$

हल :
$\int e^x \sin^3 x\, dx$

$$= \int e^x\left[\frac{1}{4}(3\sin x - \sin 3x)\right]dx$$

$$= \frac{3}{4}\int e^x \sin x\, dx - \frac{1}{4}\int e^x \sin 3x\, dx \qquad \ldots(i)$$

$$= \frac{3}{4}\cdot\frac{e^x}{2}(\sin x - \cos x) - \frac{1}{4}\cdot\frac{e^x}{1+3^2}[\sin 3x - 3\cos 3x] + C$$

$$= \frac{3}{8}\, e^x(\sin x - \cos x) - \frac{e^x}{40}[\sin 3x - 36\cos 3x] + C$$

$$\left(\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C\right)$$

$$= \frac{e^x}{40}[(15\sin x - 15\cos x) - (\sin 3x - \cos 3x)] + C$$

प्रश्न 3.
∫ x² log (1 − x²)dx

हल :

∫ $x^2 \log(1 - x^2)dx$

$$= \log(1-x^2)\int x^2\,dx - \int\left\{\frac{d}{dx}\log(1-x^2)\int x^2\,dx\right\}dx$$

$$= \log(1-x^2)\cdot\frac{x^3}{3} - \int\frac{1}{1-x^2}\cdot(-2x)\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) + \frac{2}{3}\int\frac{x^4}{1-x^2}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int\frac{x^4}{x^2-1}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int\left(x^2+1+\frac{1}{x^2-1}\right)dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\int x^2\,dx - \frac{2}{3}\int 1\,dx - \frac{2}{3}\int\frac{1}{x^2-1}\,dx$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\cdot\frac{x^3}{3} - \frac{2}{3}x - \frac{2}{3}\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$

$$= \frac{x^3}{3}\log(1-x^2) - \frac{2}{3}\left(x+\frac{x^3}{3}\right) + \frac{1}{3}\log\left|\frac{1+x}{1-x}\right| + c$$

प्रश्न 4.

$$\int\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x+a}}\,dx$$

हल :

माना, $I = \int\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x+a}}\,dx$

$$= \int\frac{\sqrt{x}}{\sqrt{x+a}}\,dx - \sqrt{a}\int\frac{1}{\sqrt{x+a}}\,dx$$

$$= I_1 - \sqrt{a}I_2$$

$$I_1 = \int\frac{\sqrt{x}}{\sqrt{x+a}}\,dx$$

माना $\sqrt{x+a} = t$

$\Rightarrow \quad x+a = t^2$

$x = t^2 - a$

$\Rightarrow \quad I_1 = \int \frac{\sqrt{t^2-a}}{t} 2t\, dt$

$= 2\int \sqrt{t^2-(\sqrt{a})^2}\, dt$

$= 2\left[\frac{1}{2}\sqrt{t^2-a} - \frac{a}{2}\log\left[t+\sqrt{t^2-a}\right]\right] + c_1$

$= \sqrt{(x+a)}\ \sqrt{x} - a \log \sqrt{x+a} + \sqrt{x} + c_1$

$I_2 = \int \frac{1}{\sqrt{x+a}}\, dx = \int (x+a)^{-1/2}\, dx$

$= \frac{(x+a)^{-1/2+1}}{-1/2+1} + c_2$

$= \frac{(x+a)^{1/2}}{1/2} + c_2$

$= 2\sqrt{(x+a)} + c_2$

अत: $\quad I = I_1 - \sqrt{a} I_2$

$= \sqrt{(x+a)}\ \sqrt{x} - a \log (\sqrt{x+a}) + \sqrt{x}$

$- 2\sqrt{a}\ \sqrt{x+a} + C_1 - \sqrt{a}\, C_2$

$= \sqrt{x+a}\ \sqrt{x} - a \log \sqrt{x+a} + \sqrt{x} - 2\sqrt{a}\ \sqrt{x+a} + C$

यही अभीष्ट हल है।

प्रश्न 5.

$$\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\, dx$$

हल :

$$\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x).(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x - 2\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x).(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x}\, dx$$

$$= \int (\sin^2 x - \cos^2 x)\, dx$$

$$= -\int (\cos^2 x - \sin^2 x)\, dx = -\int \cos 2x\, dx$$

$$= -\frac{\sin 2x}{2} + C = -\frac{1}{2}\sin 2x + C$$

प्रश्न 6.

$$\int \frac{x}{1 + \sin x}\, dx$$

हल :

$$\int \frac{x}{1 + \sin x}\, dx = \int \frac{x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)}\, dx$$

$$= \int \frac{x(1 - \sin x)}{1 - \sin^2 x}\, dx$$

$$= \int \frac{x - x\sin x}{\cos^2 x}\, dx$$

$$= \int x\sec^2 x\, dx - \int x \tan x \sec x\, dx$$

$$= \left[x\int \sec^2 x\, dx - \int\left[\frac{d}{dx}(x)\int \tan x \sec x\, dx\right] dx\right]$$

$$-\left[x\int \tan x \sec x\, dx - \int\left[\frac{d}{dx}(x)\int \tan x \sec x\, dx\right] dx\right]$$

= [x tan x – ∫tan x dx] – [xsec x – ∫sec x dx]
= [x tan x – log| sec x |] – [xsec x – log| sec x + tan x |] + C
= x tan x – log| sec x | – x sec x + log x + tan x + C
= x | tan x – sec x | – log | sec x | + log | sec x + tan x | + C

प्रश्न 7.

$$\int \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

हल :

$$\int \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

$$= \int \frac{1}{(x+\sqrt{a^2-x^2})} \times \frac{(x-\sqrt{a^2-x^2})}{(x-\sqrt{a^2-x^2})}$$

$$= \int \frac{x-\sqrt{a^2-x^2}}{x^2+(a^2-x^2)}\,dx$$

$$= \frac{1}{a^2}\int [x-\sqrt{a^2-x^2}]\,dx$$

$$= \frac{1}{a^2}\left[\frac{x^2}{2}-\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}\right]+C$$

$$= \frac{x^2}{2a^2}-\frac{x}{2a^2}\sqrt{(a^2-x^2)}+\frac{1}{2}\sin^{-1}\frac{x}{a}+C$$

प्रश्न 8.

$$\int \frac{2x-1}{(1+x)^2}\,dx$$

हल :

$$\int \frac{2x-1}{(1+x)^2}\,dx$$

$$= \int \frac{(2x+2)-3}{(x+1)^2}\,dx$$

$$= 2\int \frac{1}{(x+1)}\,dx - 3\int \frac{1}{(x+1)^2}\,dx$$

$$= 2\log|x+1| + 3\frac{1}{(x+1)} + C$$

$$= 2\log|x+1| + \frac{3}{(x+1)} + C$$

प्रश्न 9.

$$\int \frac{1}{\cos 2x + \cos 2\alpha}\, dx$$

हल :

$$\int \frac{1}{\cos 2x + \cos 2\alpha}\, dx$$

$$= \frac{1}{2} \int \frac{1}{\cos (x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2\sin 2\alpha} \int \frac{\sin 2\alpha}{\cos(x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2\sin 2\alpha} \int \left[\frac{\sin (x+\alpha)-(x-\alpha)}{\cos(x+\alpha) \cos (x-\alpha)}\right] dx$$

$$= \frac{1}{2\sin 2\alpha} \int \frac{\sin (x+\alpha) \cos (x-\alpha) - \cos(x+\alpha)\sin(x-\alpha)}{\cos(x+\alpha) \cos (x-\alpha)}\, dx$$

$$= \frac{1}{2} \operatorname{cosec} 2a \int [\tan(x+\alpha) - \tan(x-\alpha)]\, dx$$

$$= \frac{1}{2} \operatorname{cosec} 2\alpha \cdot \log \left|\frac{\sec (x+\alpha)}{\sec(x-\alpha)}\right| + C$$

प्रश्न 10.

$$\int \sin^{-1} \left(\frac{2x}{1+x^2}\right) dx$$

हल :

$$\int \sin^{-1} \left(\frac{2x}{1+x^2}\right) dx$$

माना $\quad x = \tan \theta$

$$dx = \sec^2 \theta\, d\theta$$

$$= \int \sin^{-1} \left(\frac{2 \tan \theta}{1+\tan^2 \theta}\right) \sec^2 \theta\, d\theta$$

$$= \int \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= 2\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= 2\left[\theta\int \sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta\int \sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta.\tan\theta - \int 1.\tan\theta\, d\theta\right]$$

$$= 2[\theta\tan\theta - \log\sec\theta]$$

$$= 2x\tan^{-1}x - 2\log\sqrt{1+x^2} + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

$$\therefore \int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2x\tan^{-1}x - \log(1+x^2) + C$$

प्रश्न 11.

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}\, dx$$

हल :

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}\, dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x - 1}}\, dx$$

$$= \int \frac{(\sin x - \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}}\, dx$$

माना $\sin x + \cos x = t$

$(\cos x - \sin x)\, dx = dt$

$(\sin x - \cos x)\, dx = -dt$

$$= -\int \frac{dt}{\sqrt{t^2-1}}$$

$$= -\log(t + \sqrt{t^2-1}) + C$$

$$= -\log(\sin x + \cos x + \sqrt{\sin 2x}) + C$$

प्रश्न 12.

$$\int \frac{\sin 2x}{\sin 4x + \cos 4x} dx$$

हल :

$$\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$= \int \frac{2\tan x \sec^2 x}{\tan^4 x + 1} dx$$

(अंश व हर में $\cos^4 x$ से भाग करने पर)

अब $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ रखने पर

$$= 2 \int \frac{t}{t^4 + 1} dt$$

पुन: $t^2 = u$ रखने पर, $2t\, dt = du$

$$= 2 \int \frac{t}{u^2 + 1} \cdot \frac{du}{2} = \int \frac{1}{u^2 + 1} du$$

$= \tan^{-1} u + c = \tan^{-1} t^2 + C$
$= \tan^{-1} (\tan^2 t) + C$

प्रश्न 13.

$$\int \frac{1 + x}{(2 + x)^2} dx$$

हल :

$$\int \frac{1 + x}{(2 + x)^2} dx$$

$$= \int \frac{(x + 2) - 1}{(x + 2)^2} dx$$

$$= \int \frac{x + 2}{(x + 2)^2} dx - \int \frac{1}{(x + 2)^2} dx$$

$$= \int \frac{1}{x + 2} dx - \int (x + 2)^{-2} dx$$

$$= \log |x + 2| + \frac{1}{(x + 2)} + C$$

प्रश्न 14.

$$\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

हल :

$$\frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}$$

$$(\because \ a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

$$= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \left(\frac{1}{\sin^2 x \cos^2 x} - 3 \right) dx$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3 \right) dx$$

= ∫secx² dx + ∫cosec² xdu − 3 ∫dx

= tan x − cotx − 3x + C

प्रश्न 15.

$$\int \frac{\tan^{-1} x}{x^2} dx$$

हल :

$$\int \frac{\tan^{-1} x}{x^2} dx$$

$$= \int \underset{\mathrm{I}}{\tan^{-1} x} . \underset{\mathrm{II}}{x^{-2}} \ dx$$

$$= \tan^{-1} x \int x^{-2} \ dx - \int \left[\frac{d}{dx}(\tan^{-1} x) \int x^{-2} \ dx \right] dx$$

$$= \tan^{-1} x \cdot \frac{x^{-2+1}}{-2+1} - \int \frac{1}{1+x^2} \cdot \frac{x^{-2+1}}{-2+1} dx$$

$$= -\frac{1}{x} \tan^{-1} x + \int \frac{1}{x(1+x^2)} dx \qquad \ldots(\mathrm{i})$$

माना $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$\Rightarrow \quad 1 = A(1+x^2) + (Bx+C)x$

$\Rightarrow \quad 1 = A + Ax^2 + Bx^2 + Cx$

$\Rightarrow \quad 1 = (A+B)x^2 + Cx + A.$

तुलना करने पर,

$$A + B = 0, C = 0, A = 1$$

हल करने पर

$$A = 1, B = -1, C = 0$$

$\therefore \quad \frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

$$\int \frac{1}{x(1+x^2)}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx$$

माना $1 + x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$$= \log x - \frac{1}{2}\int \frac{1}{t}\,dt$$

$$= \log x - \frac{1}{2}\log|t| + C$$

$$= \log x - \log|\sqrt{t}| + C$$

$$= \log x - \log|\sqrt{1+x^2}| + C$$

$$= \log\left|\frac{x}{\sqrt{1+x^2}}\right| + C$$

$\therefore \quad \int \frac{\tan^{-1}x}{x^2}\,dx = -\frac{\tan^{-1}x}{x} + \log\left|\frac{x}{\sqrt{1+x^2}}\right| + C$

प्रश्न 16.

$$\int \frac{1}{\sin^2 x + \sin 2x}\,dx$$

हल :

$$\int \frac{1}{\sin^2 x + \sin 2x} dx$$

$$= \int \frac{1}{\sin x(\sin x + 2\cos x)} dx$$

$$= \int \frac{\operatorname{cosec} x}{\cos x\,(\tan x + 2)} dx$$

$$= \int \frac{\operatorname{cosec} x . \sec x}{(\tan x + 2)} dx$$

$$= \int \frac{\cot x \sec^2 x}{(\tan x + 2)} dx$$

माना tan x + 2 = y

$\sec^2 x\, dx = dy$

tan x + 2

tan x = y − 2

$$\Rightarrow \qquad \cot x = \frac{1}{y-2}$$

$$I = \int \frac{1}{y(y-2)} dy$$

अब $$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}$$

$$\Rightarrow \qquad 1 = A(y-2) + By$$

$$\Rightarrow \qquad 1 = (A+B)y - 2A$$

तुलना से, $A + B = 0, \; -2A = 1$

$$A = -\frac{1}{2} \text{ तथा } B = \frac{1}{2}$$

$$I = \int \frac{-1/2}{y} dy + \frac{1}{2} \int \frac{1}{y-2} dy$$

$$= \frac{1}{2}\log(y-2) - \frac{1}{2}\log y + C$$

$$= \frac{1}{2} \log \frac{y-2}{y} + C$$

$$= \frac{1}{2}\log\left(\frac{\tan x}{\tan x + 2}\right)$$

प्रश्न 17.

$$\int \frac{1}{4x^2-4x+3}\,dx$$

हल :

$$\int \frac{1}{4x^2-4x+3}\,dx$$

$$= \int \frac{1}{(2x)^2-2\cdot 1\cdot 2x+1+2}\,dx$$

$$= \int \frac{1}{(2x-1)^2+(\sqrt{2})^2}\,dx$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right)+C$$

प्रश्न 18.

$$\int \frac{1}{x[6(\log x)^2+7(\log x)+2}\,dx$$

हल :

माना log x = t

$\frac{1}{x}$dx = dt

$$= \int \frac{dt}{6t^2+7t+2}$$

$$= \int \frac{dt}{6t^2+3t+4t+2}$$

$$= \int \frac{dt}{3t(2t+1)+2(2t+1)}$$

$$= \int \frac{dt}{(2t+1)(3t+2)}$$

माना $\frac{1}{(2t+1)(3t+2)} = \frac{A}{2t+1}+\frac{B}{3t+2}$

1 = A(3t + 2) + B(2t + 1)

1 = (3A + 2B)t + (2A + B)

तुलना करने पर,

3A + 2B = 0 तथा 2A + B = 1

हल करने पर,
A = 2, B = − 3

$$\Rightarrow \int \frac{dt}{(2t+1)(3t+2)}$$

$$= 2\int \frac{1}{(2t+1)}dt - 3\int \frac{1}{(3t+2)}dt$$

$$= \frac{2\log|2t+1|}{2} - \frac{3\log|3t+2|}{3} + C$$

$$= \log|2t+1| - \log|3t+2| + C$$

$$= \log\left|\frac{2t+1}{3t+2}\right| + C$$

$$= \log\left|\frac{2\log x+1}{3\log x+2}\right| + C$$

$$\therefore \int \frac{1}{x[6(\log x)^2 + 7(\log x) + 2} dx$$

$$= \log\left|\frac{2\log x+1}{3\log x+2}\right| + C$$

प्रश्न 19.

$$\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}} dx$$

हल :

$$\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}} dx$$

माना $sec^2 2x = t$
$4 \sin 2x . \cos 2x \, dx = dt$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{2^2 - t^2}}$$

$$= \frac{1}{4} \sin^{-1} \frac{t}{2} + C$$

$$= \frac{1}{4} \sin^{-1}\left(\frac{\sin^2 2x}{2}\right) + C$$

प्रश्न 20.

$$\int \frac{\sin x+\cos x}{9+16 \sin 2x} dx$$

हल :

$$\int \frac{\sin x+\cos x}{9+16 \sin 2x} dx$$

माना sin − cos x = t

(sin x + cos x) dx = dt

⇒ $(\sin x - \cos x)^2 = t^2$

⇒ $\sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

⇒ $1 - \sin 2x = t^2$

⇒ $\sin 2x = 1 - t^2$

$$= \int \frac{dt}{9+16(1-t^2)}$$

$$= \int \frac{dt}{25+16t^2}$$

$$= \frac{1}{16} \int \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{16} \times \frac{1}{2\times\left(\frac{5}{4}\right)} \log \left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right| + C$$

$$= \frac{1}{40} \log \left|\frac{5+4t}{5-4t}\right| + C$$

$$= \frac{1}{40} \log \left|\frac{5+4(\sin x-\cos x)}{5-4(\sin x-\cos x)}\right| + C$$

प्रश्न 21.

$$\int \frac{3x-1}{(x-2)^2} dx$$

हल :

$$\int \frac{3x-1}{(x-2)^2}\,dx$$

$$= \int \frac{(3x-6)+5}{(x-2)^2}\,dx$$

$$= \int \frac{3(x-2)}{(x-2)^2}\,dx + 5\int \frac{1}{(x-2)^2}\,dx$$

$$= 3\int \frac{1}{(x-2)}\,dx + 5\int \frac{1}{(x-2)^2}\,dx$$

$$= 3\log|x-2| - 5\frac{1}{(x-2)} + C$$

$$\therefore \quad \int \frac{3x-1}{(x-2)^2}\,dx = 3\log|x-2| - \frac{5}{(x-2)} + C$$

प्रश्न 22.

$$\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$$

का मान है

(a) tan x + x + C
(b) cot x + x + C
(c) tan x − x + C
(d) cot x − x + C

हल :

$$\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$$

$$= \int \frac{1-1+2\sin^2 x}{1+2\cos^2 x-1}\,dx$$

$$= \int \tan^2 x\,dx$$

$$= \int (\sec^2 x-1)\,dx$$

$= \int \sec^2 x\,dx - \int 1\,dx$

= tan x − x + C
अत: विकल्प (c) सही है।

प्रश्न 23.

$$\int \frac{1}{\sqrt{32-2x^2}}\,dx$$

का मान है

(a) $\sin^{-1}\left(\frac{x}{4}\right)+C$ (b) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x}{4}\right)+C$

(c) $\sin^{-1}\left(\frac{\sqrt{2}x}{4}\right)+C$ (d) $\cos^{-1}\left(\frac{x}{4}\right)+C$

हल :

$$\int \frac{1}{\sqrt{32-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(4)^2-x^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{4}+C$$

अतः विकल्प (b) सही है।

प्रश्न 24.

∫log x dx बराबर है

(a) $x\log(xe)+C$ (b) $x\log x+C$

(c) $x\log\left(\frac{x}{e}\right)+C$ (d) $\log\frac{x}{e}$

हल :

∫log x dx

= ∫log x.1 dx

$$= \log x\int 1\,dx - \int\left[\frac{d}{dx}\log x.\int 1\,dx\right]dx$$

$$= \log x\cdot x - \int\frac{1}{x}\cdot x\,dx$$

= x log x − x loge + C

= x log$\frac{x}{e}$ + c

अत: विकल्प (c) सही है।

प्रश्न 25.

$$\int \frac{dx}{x(x+1)}$$

बराबर है

(a) $\log\left(\frac{x}{x+1}\right)+C$ (b) $\log\left(\frac{x+1}{x}\right)+C$

(c) $\frac{1}{2}\log\left(\frac{x}{x+1}\right)+C$ (d) $\frac{1}{2}\log\left(\frac{x+1}{x}\right)+C$

हल :

माना है।

$$\frac{1}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$$

⇒ 1 = A(x + 1) + B(x)

⇒ 1 = (A + B)x + A

तुलना से, A = 1, A + B = 0

हल करने पर,

A = 1, B = -1

$$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$$

$$\int\frac{1}{x(x+1)}\,dx=\int\frac{1}{x}\,dx-\int\frac{1}{x+1}\,dx$$

$$=\log|x|-\log|x+1|+C$$

$$=\log\left|\frac{x}{x+1}\right|+C$$

अत: विकल्प (a) सही है।