Integrals

<u>Short Answer Type Questions</u>

1. **Integrate** $\left(\frac{2a}{\sqrt{x}}-\frac{b}{x^2}+3c\sqrt[3]{x^2}\right)$ **w.r.t. x**

Sol. $\int\left(\frac{2a}{\sqrt{x}}-\frac{b}{x^2}+3c\sqrt[3]{x^2}\right)dx$

$=\int 2a(x)^{\frac{-1}{2}}\,dx-\int bx^{-2}dx+\int 3cx^{\frac{2}{3}}dx$

$=4a\sqrt{x}+\frac{b}{x}+\frac{9cx^{\frac{5}{3}}}{5}+C$

2. **Evaluate** $\int\frac{3ax}{b^2+c^2x^2}dx$

Sol. Let $v=b^2+c^2+c^2x^2$, then $dv=2c^2\,xdx$

Therefore, $\int\frac{3ax}{b^2+c^2x^2}dx=\frac{3a}{2c^2}\int\frac{dv}{v}$

$=\frac{3a}{2c^2}\log\left|b^2+c^2x^2\right|+C.$

3. **Verify the following using the concept of integration as an antiderivative.**

$\int\frac{x^3dx}{x+1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C$

Sol. $\frac{d}{dx}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C\right)$

$=1-\frac{2x}{2}+\frac{3x^2}{3}-\frac{1}{x+1}$

$=1-x+x^2-\frac{1}{x+1}=\frac{x^3}{x+1}$

Thus $\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C\right)=\int\frac{x^3}{x+1}dx$

4. **Evaluate** $\int\sqrt{\frac{1+x}{1-x}}dx, x\neq 1.$

Sol. Let $I=\int\sqrt{\frac{1+x}{1-x}}dx=\int\frac{1}{\sqrt{1-x^2}}dx+\int\frac{xdx}{\sqrt{1-x^2}}=\sin^{-1}x+I_1,$

where $I_1=\int\frac{xdx}{\sqrt{1-x^2}}.$

Put $1-x^2=t^2 \Rightarrow -\ 2x\ dx=2t\ dt.$ Therefore

$I_1=-\int dt=-t+C=-\sqrt{1-x^2}+C$

Hence $I=\sin^{-1}x-\sqrt{1-x^2}+C.$

5. **Evaluate** $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \beta > \alpha$**.**

Sol. Put $x-\alpha = t^2. Then\, \beta - x = \beta-(t^2+\alpha) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$ and $dx = 2tdt$. Now

$$I = \int \frac{2t\,dt}{\sqrt{t^2(\beta-\alpha-t^2)}} = \int \frac{2dt}{\sqrt{(\beta-\alpha-t^2)}}$$

$$= 2\int \frac{dt}{\sqrt{k^2-t^2}}, where\ k^2 = \beta-\alpha$$

$$= 2\sin^{-1}\frac{t}{k} + C = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$$

6. **Evaluate** $\int \tan^8 x \sec^4 x dx$

Sol. $I = \int \tan^8 x \sec^4 x dx$

$$= \int \tan^8 x(\sec^2 x)\sec^2 x dx$$

$$= \int \tan^8 x(\tan^2 x + 1)\sec^2 x dx$$

$$= \int \tan^{10} x \sec^2 x dx + \int \tan^8 x \sec^2 x dx$$

$$= \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C.$$

7. **Find** $\int \frac{x^2}{x^4+3x^2+2} dx$

Sol. Put $x^2 = t.\ Then\ 2x\,dx = dt$.

Now $I = \int \frac{x^3 dx}{x^4+3x^2+2} = \frac{1}{2}\int \frac{t\,dt}{t^2+3t+2}$

Consider $\frac{t}{t^2+3t+2} = \frac{A}{t+1} + \frac{B}{t+2}$

Comparing coefficient, we get $A = -1,\ B = 2.$

Then $I = \frac{1}{2}\left[2\int \frac{dt}{t+2} - \int \frac{dt}{t+1}\right]$

$$= \frac{1}{2}\left[2\log|t+2| - \log|t+1|\right]$$

$$= \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$$

8. **Find** $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$

Sol. Dividing numerator and denominator by $cos^2 x,$ we have

$$I = \int \frac{\sec^2 x dx}{2\tan^2 x + 5}$$

Put $tan\, x = t$ so that $sec^2 x\ dx = dt$. Then

$$I=\int\frac{dt}{2t^2+5}=\frac{1}{2}\int\frac{dt}{t^2+\left(\sqrt{\frac{5}{2}}\right)^2}$$

$$=\frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{5}}\right)+C$$

$$=\frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right)+C$$

9. Evaluate $\int_{-1}^{2}(7x-5)dx$ as a limit of sums.

Sol. Here $a=-1,\ b=2$ and $h=\dfrac{2+1}{n}$ i.e, $nh=3$ *and* $f(x)=7x-5$.

Now, we have

$$\int_{-1}^{2}(7x-5)dx=\lim_{h\to 0}h\left[f(-1)+f(-1+h)+f(-1+2h)+\ldots+f(-1+(n-1)h)\right]$$

Now that

$$f(-1)=-7-5=-12$$

$$f(-1+h)=-7+7h-5=-12+7h$$

$$f(-1+(n-1)h)=7(n-1)h-12.$$

Therefore,

$$\int_{-1}^{2}(7x-5)dx=\lim_{h\to 0}h\left[(-12)+(7h-12)+(14h-12)+\ldots+(7(n-1)h-12)\right].$$

$$=\lim_{h\to 0}h\left[7h\left[1+2+\ldots+(n-1)\right]-12n\right]$$

$$=\lim_{h\to 0}h\left[7h\frac{(n-1)n}{2}-.12n\right]=\lim_{h\to 0}\left[\frac{7}{2}(nh)(nh-h)-12nh\right]$$

$$=\frac{7}{2}(3)(3-0)-12\times 3=\frac{7\times 9}{2}-36=\frac{-9}{2}.$$

10. Evaluate $\int_{0}^{\frac{\pi}{2}}\frac{\tan^7 x}{\cot^7 x+\tan^7 x}dx$

Sol. We have

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\tan^7 x}{\cot^7 x+\tan^7 x}dx\ \ldots(1)$$

$$=\int_{0}^{\pi}\frac{\tan^7\left(\frac{\pi}{2}-x\right)}{\cot^7\left(\frac{\pi}{2}-x\right)+\tan^7\left(\frac{\pi}{2}-x\right)}dx\ by\ (P_4)$$

$$= \int_0^{\pi} \frac{\cot^7 (x) dx}{\cot^7 x\, dx + \tan^7 x} \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \text{ which gives } I = \frac{\pi}{4}.$$

11. **Find** $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}} dx$

Sol. We have

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}} = dx \quad ...(1)$$

$$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}} dx \; by \; (P_3)$$

$$\Rightarrow \quad I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x+\sqrt{x}}} dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_2^8 I dx = 8 - 2 = 6$$

Hence, I = 3

12. **Find** $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x\, dx}$

Sol. We have

$$I = \int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x\, dx} = \int_0^{\frac{\pi}{4}} \sqrt{(\sin x + \cos x)^2} dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin x + \cos x) dx$$

$$= (-\cos x + \sin x)_0^{\frac{\pi}{4}}$$

I = 1.

13. **Find** $\int x^2 \tan^{-1} x\, dx.$

Sol. $I = \int x^2 \tan^{-1} x\, dx.$

$$= \tan^{-1} x \int x^2 dx - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log\left|1+x^2\right| + C.$$

14. **Find** $\int \sqrt{10-4x+4x^2}\, dx$

Sol. We have

$$I = \int \sqrt{10-4x+4x^2}\, dx \quad = \int \sqrt{(2x-1)^2 + (3)^2}\; dx$$

Put $t = 2x-1$, then $dt = 2dx$

Therefore, $I = \frac{1}{2} \int \sqrt{t^2 + (3)^2}\, dt$

$$= \frac{1}{2} t \frac{\sqrt{t^2+9}}{2} + \frac{9}{4} \log\left| t + \sqrt{t^2+9} \right| + C$$

$$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2+9} + \frac{9}{4} \log\left| (2x-1) + \sqrt{(2x-1)^2+9} \right| + C$$

Long Answer Type Questions

15. **Evaluate** $\int \frac{x^2 dx}{x^4 + x^2 - 2}$**.**

Sol. Let $x^2 = t$. Then

$$\frac{x^2}{x^4+x^2-2} = \frac{t}{t^2+t-2} = \frac{t}{(t+2)(t-1)} = \frac{A}{t+2} + \frac{B}{t-1}$$

So $t = A(t-1) + B(t+2)$

Comparing coefficients, we get $A = \frac{2}{3}, B = \frac{1}{3}$.

So $\frac{x^2}{x^4+x^2-2} = \frac{2}{3}\frac{1}{x^2+2} + \frac{1}{3}\frac{1}{x^2-1}$

Therefore,

$$\int \frac{x^2}{x^4+x^2-2} dx = \frac{2}{3} \int \frac{1}{x^2+2} dx + \frac{1}{3} \int \frac{dx}{x^2-1}$$

$$= \frac{2}{3} \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + \frac{1}{6} \log\left| \frac{x-1}{x+1} \right| + C$$

16. **Evaluate** $\int \frac{x^3+x}{x^4-9} dx$

Sol. we have

$$I = \int \frac{x^3+x}{x^4-9} dx = \int \frac{x^3}{x^4-9} dx + \frac{x dx}{x^4-9} = I_1 + I_2.$$

Now $I_1 = \int \frac{x^2}{x^4-9}$

Put $t = x^2 - 9$ so that $4x^3 dx = dt$. Therefore

$$I_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C_1 = \frac{1}{4}\log\left|x^4-9\right| + C_1$$

Again, $I_2 = \int \frac{xdx}{x^4-9}$

Put $x^2 = u$ so that $2x\,dx = du$. Then

$$I_2 = \frac{1}{2}\int \frac{du}{u^2-(3)^2} = \frac{1}{2\times 6}\log\left|\frac{u-3}{u+3}\right| + C_2$$

$$= \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C_2.$$

Thus $I = I_1 + I_2$

$$= \frac{1}{4}\log\left|x^4-9\right| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C.$$

17. **Show that** $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Sol. We have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \quad (by\ P4)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x}dx$$

Thus, we get $2I = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\left[\log\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right) - \log\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)-\log\left(\sqrt{2}-1\right)\right] \quad = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{\sqrt{2}} \log\left(\frac{(\sqrt{2}+1)^2}{1}\right) = \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)$$

Hence, $I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

18. **Find** $\int_0^1 x(\tan^{-1} x)^2 \, dx$

Sol. $I = \int_0^1 x(\tan^{-1} x)^2 \, dx$

Integrating by parts, we have

$$I = \frac{x^2}{2}\left[(\tan^{-1} x)^2\right]_0^1 - \frac{1}{2}\int_0^1 x^2 . 2 \frac{\tan^{-1} x}{1+x^2} dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2}{1+x^2} . \tan^{-1} x dx$$

$$= \frac{\pi}{32} - I_1, where \, I_1 = \int_0^1 \frac{x^2}{1+x} \tan^{-1} x dx$$

Now $I_1 = \int_0^1 \frac{x^2+1-1}{1+x^2} \tan^{-1} x \, dx$

$$= \int_0^1 \tan^{-1} x \, dx - \int_0^1 \frac{1}{1+x} \tan^{-1} x dx$$

$$= I_2 - \frac{1}{2}\left((\tan^{-1} x)^2\right)_0^1 \qquad = I_2 - \frac{\pi^2}{32}$$

Here $I_2 = \int_0^1 \tan^{-1} x dx = (x \tan^{-1} x)_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\log\left|1+x^2\right|\right)_0^1 \quad = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Thus $I_1 = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$

Therefore, $I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\log 2 + \frac{\pi^2}{32} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$

$$= \frac{x^2 - 4\pi}{16} + \log\sqrt{2}$$

19. **Evaluate** $\int_{-1}^{2} f(x) dx, where \, f(x) = |x+1| + |x| + |x+1|.$

Sol. We can redefine f as $f(x) = \begin{cases} 2-x, & if \;\; -1 < x \le 0 \\ x+2, & if \;\; 0 < x \le 1 \\ 3x, & if \;\; 1 < x \le 2 \end{cases}$

Therefore, $\int_{-1}^{2} f(x)dx = \int_{-1}^{0}(2-x)dx + \int_{0}^{1}(x+2)dx + \int_{1}^{2}3xdx \ \ (by\ P_2)$

$= \left(2x - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_{0}^{1}\left(\frac{3x^2}{2}\right)_{1}^{2}$

$= 0 - \left(-2 - \frac{1}{2}\right) + \left(\frac{1}{2} + 2\right) + 3\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples from 20 to 30.

20. $\int e^x(\cos x - \sin x)dx$ **is equal to**

(A) $e^x \cos x + C$

(B) $e^x \sin x + C$

(C) $-e^x \cos x + C$

(D) $-e^x \sin x + C$

Sol. (A) is the correct answer since $\int e^x\left[f(x) + f(x)\right]dx = e^x f(x) + C$. Hence $f(x) = cos\,x,\ f'(x) = -sin\,x$.

21. $\int \frac{dx}{\sin^2 x \cos^2 x}$ **is equal to**

(A) $tan\,x + cot\,x + C$

(B) $(tan\,x + cot\,x)^2 + C$

(C) $tan\,x - cot\,x + C$

(D) $(tan\,x - cot\,x)^2 + C$

Sol. (C) is the correct answer, since

$I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x \cos^2 x}$

$= \int \sec^2 x dx + \int \cos ec^2 x dx = \tan x - \cot x + C$

22. **If** $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}}dx = ax + b\ \log\left|4e^x + 5e^{-x}\right| + C$**, then**

(A) $a = \frac{-1}{8}, b = \frac{7}{8}$

(B) $a = \frac{1}{8}, b = \frac{7}{8}$

(C) $a = \frac{-1}{8}, b = \frac{-7}{8}$

(D) $a = \frac{1}{8}, b = \frac{-7}{8}$

Sol. (C) is the correct answer, since differentiating both sides, we have

$$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}},$$

Giving $3e^x - 5e^{-x} = a\left(4e^x + 5e^{-x}\right) + b\left(4e^x - 5e^{-x}\right)$. Comparing coefficients on both sides, we get

$3 = 4a + 4b \; and \; -5 = 5a - 5b$. This verifies $a = \frac{-1}{8}, b = \frac{7}{8}$

23. $\int_{a+c}^{b+c} f(x)dx$ **is equal to**

(A) $\int_a^b f(x-c)dx$

(B) $\int_a^b f(x+c)dx$

(C) $\int_a^b f(x)dx$

(D) $\int_{a-c}^{b-c} f(x)dx$

Sol. (B) is the correct answer, since by putting $x = t + c$, we get

$$I = \int_a^b f(c+t)dt = \int_a^b f(x+c)dx.$$

24. **If** $f \; and \; g$ **are continuous in [0, 1] satisfying** $f(x) = f(a-x)$ **and** $g(x) + g(a-x) = a$, **then** $\int_0^a f(x).g(x)dx$ **then is equal to**

(A) $\frac{a}{2}$

(B) $\frac{a}{2}\int_0^a f(x)dx$

(C) $\int_0^a f(x)dx$

(D) $a\int_0^a f(x)dx$

Sol. (B) is the correct answer. Since $I = \int_0^a f(x).g(x)dx$

$$= \int_0^a f(a-x)g(a-x)dx = \int_0^a f(x)(a - g(x))dx$$

$$= a\int_0^a f(x)dx - \int_0^a f(x).g(x)dx = a\int_0^a f(x)dx - I$$

Or $I = \frac{a}{2}\int_0^a f(x)dx$

25. $x = \int_0^y \frac{dt}{1+9t^2}$ *and* $\frac{d^2 y}{dx^2} = ay,$ **then** a **is equal to**

(A) 3
(B) 6
(C) 9
(D) 1

Sol. (C) is the correct answer, since $x = \int_0^y \frac{dt}{1+9t^2} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$ which gives

$$\frac{d^2 y}{dx^2} = \frac{18y}{2\sqrt{1+9y^2}}.\frac{dy}{dx} = 9y$$

26. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}dx$ **is equal to**

(A) log 2
(B) 2 log 2
(C) $\frac{1}{2}$ **log 2**
(D) 4 log 2

Sol. (B) is the correct answer, since $I = \int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1} + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}dx = 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}dx = 2\int_0^1 \frac{1}{x+1}dx = 2\left|\log|x+1|\right|_0^1 = 2\log 2$$

27. **If** $\int_0^1 \frac{e^t}{1+t}dt = a,$ **then** $\int_0^1 \frac{e^t}{(1+t)^2}$ **is equal to**

(A) $a-1+\frac{e}{2}$

(B) $a+1-\frac{e}{2}$

(C) $a-1-\frac{e}{2}$

(D) $a+1+\frac{e}{2}$

Sol. (B) is the correct answer, since $I = \int_0^1 \frac{e^t}{1+t}dt$

$$= \left| \frac{1}{1+t} e^t \right|_0^1 + \int_0^1 \frac{e^1}{(1+t)^2} dt = a(given)$$

Therefore, $\int_0^1 \frac{e^t}{(1+t)^2} = a - \frac{e}{2} + 1.$

28. $\int_{-2}^{2} |x \cos \pi x| dx$ **is equal to**

(A) $\frac{8}{\pi}$

(B) $\frac{4}{\pi}$

(C) $\frac{2}{\pi}$

(D) $\frac{1}{\pi}$

Sol. (A) is the correct answer, since $I = \int_{-2}^{2} |x \cos \pi x| dx \quad = 2\int_0^2 |x \cos \pi x| dx$

$$= 2\left\{ \int_0^{\frac{1}{2}} |x \cos \pi x| \, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x \cos \pi x| \, dx + \int_{\frac{3}{2}}^{2} |x \cos \pi x| \, dx \right\} = \frac{8}{\pi}.$$

Fill in the blanks in each of the Examples 29 to 32.

29. $\int \frac{\sin^6 x}{\cos^8 x} dx =$ ____________**.**

Sol. $\frac{\tan^7 x}{7} + C$

30. $\int_{-a}^{a} f(x) dx = 0$ **if** f **is an** ____________ **function.**

Sol. Odd.

31. $\int_0^{2a} f(x) dx = 2\int_0^a f(x) dx, \; if \;\; f \;\; (2a - x) =$

Sol. $f\left(x\right).$

32. $\int_0^{\frac{\pi}{2}} \frac{\sin^n x dx}{\sin^n x + \cos^\pi x} =$ ______________**.**

Sol. $\frac{\pi}{4}.$

Integrals
Objective Type Questions

Choose the correct option from given four options in each of the Exercises from 48 to 63.

48. $\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx$ **is equal to**

(A) $2(\sin x+x\cos\theta)+C$

(B) $2(\sin x-x\cos\theta)+C$

(C) $2(\sin x+2x\cos\theta)+C$

(D) $2(\sin x-2x\cos\theta)+C$

Sol. (A) Let $I=\int \frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}dx$

$$=\int \frac{(2\cos^2 x-1-2\cos^2\theta+1)}{\cos x-\cos\theta}dx$$

$$=2\int \frac{(\cos x+\cos\theta)(\cos x-\cos\theta)}{(\cos x-\cos\theta)}dx$$

$$=2\int(\cos x+\cos\theta)\,dx$$

$$=2(\sin x+x\cos\theta)+C$$

49. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ **is equal to**

(A) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$

(B) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

(C) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$

(D) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

Sol. (C) Let $I=\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(x-a-x+b)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(b-a)} \int [\cot(x-b) - \cot(x-a)]\, dx$$

$$= \frac{1}{\sin(b-a)} [\log|\sin(x-b)| - \log|\sin(x-a)|] + C$$

$$= \cos ec(b-a) \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + C$$

50. $\int \tan^{-1}\sqrt{x}\, dx$ **is equal to**

(A) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(B) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(C) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$

(D) $\sqrt{x} - (x+1)\tan^{-1}\sqrt{x} + C$

Sol. (A) Let $I = \int 1.\tan^{-1}\sqrt{x}\, dx$

$$= \tan^{-1}\sqrt{x}.\ x - \frac{1}{2}\int \frac{1}{(1+x)}.\frac{2}{\sqrt{x}} dx$$

$$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{2}{\sqrt{x}(1+x)} dx$$

$$Put\ x = t^2 \Rightarrow dx = 2t\, dt$$

$$\therefore\ I = x\tan^{-1}\sqrt{x} - \int \frac{t}{t(1+t^2)} dt$$

$$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2} dt$$

$$= x\tan^{-1}\sqrt{x} - \int \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1} t + C$$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

51. $\int e^x \left(\frac{1-x}{1+x^2}\right)$ **is equal to**

(A) $\frac{e^x}{1+x^2} + C$

(B) $\frac{-e^x}{1+x^2} + C$

(C) $\frac{e^x}{(1+x^2)^2} + C$

(D) $\frac{-e^x}{(1+x^2)^2} + C$

Sol. (c) **Answer not given**

52. $\int \frac{x^9}{(4x^2+1)^6}dx$ **is equal to**

(A) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(B) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(C) $\frac{1}{10x}(1+4)^{-5}+C$

(D) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

Sol. (D) Let $I=\int \frac{x^9}{(4x^2+1)^6}dx=\int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)}dx$

$$=\int \frac{dx}{x^3\left(4+\frac{1}{x^2}\right)^6}$$

$$Put\ \ 4+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$$

$$\Rightarrow \frac{1}{x^3dx}=-\frac{1}{2}dt$$

$$\therefore\ I=-\frac{1}{2}\int \frac{dt}{t^6}=-\frac{1}{2}\left[\frac{t^{-6+1}}{-6+1}\right]+C$$

$$=\frac{1}{10}\left[\frac{1}{t^5}\right]+C=\frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5}+C$$

53. **If** $\int \frac{dx}{(x+2)(x^2+1)}=a\log\left|1+x^2\right|+b\tan^{-1}x+\frac{1}{5}\log\left|x+2\right|+C$, **then**

(A) $a=\frac{-1}{10}, b=\frac{-2}{5}$

(B) $a=\frac{1}{10}, b=-\frac{2}{5}$

(C) $a=\frac{-1}{10}, b=\frac{2}{5}$

(D) $a=\frac{1}{10}, b=\frac{2}{5}$

Sol. (C) Given that, $\int \frac{dx}{(x+2)(x^2+1)}=a\log|1+x^2|+b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$

Now, $I=\int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x+2)$$

$$\Rightarrow 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$\Rightarrow 1 = (A+B)x^2 + (2B+C)x + A + 2C$$

$$\Rightarrow A + B = 0, A + 2C = 1, 2B + C = 0$$

We have, $A = \frac{1}{5}, B = -\frac{1}{5}\ and\ C = \frac{2}{5}$

$$\therefore \int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5}\int \frac{1}{x+2}dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1}dx$$

$$= \frac{1}{5}\int \frac{1}{x+2}dx - \frac{1}{5}\int \frac{x}{1+x^2}dx + \frac{1}{5}\int \frac{2}{1+x^2}dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1}x + C$$

$$\therefore b = \frac{2}{5}\ and\ a = \frac{-1}{10}$$

54. $\int \frac{x^3}{x+1}$ **is equal to**

(A) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

(B) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(C) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

(D) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

Sol. (D) Let $I = \int \frac{x^3}{x+1}dx$

$$= \int\left((x^2 - x + 1) - \frac{1}{(x+1)}\right)dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

55. $\int \frac{x+\sin x}{1+\cos x}dx$ **is equal to**

(A) $\log|1+\cos x| + C$

(B) $\log|x+\sin x| + C$

(C) $x - \tan\frac{x}{2} + C$

(D) $x.\tan\frac{x}{2}+C$

Sol. (D) Let $I=\int\frac{x+\sin x}{1+\cos x}dx$

$=\int\frac{x}{1+\cos x}dx+\int\frac{\sin x}{1+\cos x}dx$

$=\int\frac{x}{2\cos^2 x/2}dx+\int\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}dx$

$=\frac{1}{2}\int x\sec^2 x/2\,dx+\int\tan x/2\,dx$

$=\frac{1}{2}\left[x.\tan x/2.2-\int\tan\frac{x}{2}.2\,dx\right]+\int\tan\frac{x}{2}dx$

$=x.\tan\frac{x}{2}+C$

56. **If** $\int\frac{x^3dx}{\sqrt{1+x^2}}=a(1+x^2)^{\frac{3}{2}}+b\sqrt{1+x^2}+C$**, then**

(A) $a=\frac{1}{3},\quad b=1$

(B) $a=\frac{-1}{3},\quad b=1$

(C) $a=\frac{-1}{3},\quad b=-1$

(D) $a=\frac{1}{3},\quad b=-1$

Sol. (D) Let $I=\int\frac{x^3}{\sqrt{1+x^2}}dx=a(1+x^2)^{3/2}+b\sqrt{1+x^2}+C$

$\because\ I=\int\frac{x^3}{\sqrt{1+x^2}}dx=\int\frac{x^2.x}{\sqrt{1+x^2}}dx$

$Put\ 1+x^2=t^2$

$\Rightarrow 2x\,dx=2t\,dt$

$\therefore\ I=\int\frac{t(t^2-1)}{t}dt=\frac{t^3}{3}-t+C$

$=\frac{1}{3}(1+x^2)^{3/2}-\sqrt{1+x^2}+C$

$\therefore\ a=\frac{1}{3}\,and\,b=-1$

57. $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{1+\cos 2x}$ **is equal to**

(A) 1

(B) 2

(C) 3
(D) 4

Sol. (A) Let $I=\int_{-\pi/4}^{\pi/4}\frac{dx}{1+\cos 2x}=\int_{-\pi/4}^{\pi/4}\frac{dx}{2\cos^2 x}$

$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx=\int_{0}^{\pi/4}\sec^2 x\,dx=[\tan x]_0^{\pi/4}=1$

58. $\int_0^{\frac{\pi}{2}}\sqrt{1-\sin 2x}dx$ **is equal to**

(A) $2\sqrt{2}$
(B) $2\left(\sqrt{2}+1\right)$
(C) 2
(D) $2\left(\sqrt{2}-1\right)$

Sol. (D) Let $I=\int_0^{\pi/2}\sqrt{1-\sin 2x}\,dx$

$=\int_0^{\pi/4}\sqrt{(\cos x-\sin x)^2}\,dx+\int_{\pi/4}^{\pi/2}\sqrt{(\sin x-\cos x)^2}\,dx$

$=[\sin x+\cos x]_0^{\pi/4}+[-\cos x-\sin x]_{\pi/4}^{\pi/2}$

$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-0-1+\left(-0-1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)$

$=2\sqrt{2}-2=2(\sqrt{2}-1)$

59. $\int_0^{\frac{\pi}{2}}\cos x\, e^{\sin x}dx$ **is equal to_________.**

Sol. Let $I=\int_0^{\pi/2}\cos x\, e^{\sin x}\,dx$

$Put\ \sin x=t\Rightarrow \cos x\,dx=dt$

$As\ \ x\to 0, then\, t\to 0$

$and\ x\to \pi/2, then\, t\to 1$

$\therefore\ I=\int_0^1 e^t\,dt=[e^t]_0^1$

$=e^1-e^0=e-1$

60. $\int\frac{x+3}{(x+4)^2}e^x dx=$ **__________.**

Sol. Let $I=\int\frac{x+3}{(x+4)^2}e^x\,dx$

$=\int\frac{e^x}{(x+4)}-\int\frac{e^x}{(x+4)^2}dx$

$=\int e^x\left(\frac{1}{(x+4)}-\frac{1}{(x+4)^2}\right)dx$

$$= e^x\left(\frac{1}{x+4}\right)+C \quad [\because \int e^x\{f(x)+f'(x)\}dx - e^x f(x)+C]$$

Fill in the blanks in each of the following Exercise 60 to 63.

61. **If** $\int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$**, the a = __________.**

Sol. Let $I = \int_0^a \frac{1}{1+4x^2}dx = \frac{\pi}{8}$

$$Now, \int_0^a \frac{1}{4\left(\frac{1}{4}+x^2\right)}dx = \frac{2}{4}[\tan^{-1} 2x]_0^a$$

$$= \frac{1}{2}\tan^{-1} 2a - 0 = \pi/8$$

$$\frac{1}{2}\tan^{-1} 2a = \frac{\pi}{8}$$

$$\Rightarrow \tan^{-1} 2a = \pi/4$$

$$\Rightarrow 2a = 1$$

$$\therefore a = \frac{1}{2}$$

62. $\int \frac{\sin x}{3+4\cos^2 x}dx =$ __________.

Sol. Let $I = \int \frac{\sin x}{3+4\cos^2 x}dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\therefore I = \int \frac{dt}{3+4t^2} = -\frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= -\frac{1}{4}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{2t}{\sqrt{3}} + C$$

$$= -\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + C$$

63. **The value of** $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx$ **is __________.**

Sol. We have, $f(x) = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx$

$$f(-x) = \int_{-\pi}^{\pi} \sin^3(-2) - \cos^2(-x)\, dx$$

$$= -f(x)$$

Since, $f(x)$ is an odd function.

$$\therefore \int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx = 0$$

Integrals
Short Answer Type Questions

Verify the following:

1. $\int \frac{2x-1}{2x+3}dx = x - \log\left|(2x+3)^2\right| + C$

Sol. Let $I = \int \frac{2x-1}{2x+3}dx = \int \frac{2x+3-3-1}{2x+3}dx$

$= \int 1dx - 4\int \frac{1}{2x+3}dx = x - \int \frac{4}{2\left(x+\frac{3}{2}\right)}dx$

$= x - 2\log + \left|\left(x+\frac{3}{2}\right)\right| C' = x - 2\log\left|\left(\frac{2x+3}{2}\right)\right| + C$

$= x - 2\log\left|(2x+3)\right| + 2\log 2 + C' \quad \left[\because \log\frac{m}{n} = \log m - \log n\right]$

$= x - \log\left|(2x+3)^2\right| + C \quad [\because C = 2\log 2 + C]$

2. $\int \frac{2x+3}{x^2+3x}dx = \log\left|x^2+3x\right| + C$

Sol. Let $I = \int \frac{2x+3}{x^2+3x}dx$

$Put \;\; x^2+3x = t$

$\Rightarrow (2x+3)dx = dt$

$\therefore I = \int \frac{1}{t}dt = \log|t| + C$

$= \log\left|\left(x^2+3x\right)\right| + C$

Evaluate the following:

3. $\int \frac{\left(x^2+2\right)dx}{x+1}$

Sol. Let $I = \int \frac{x^2+2}{x+1}dx$

$= \int \left(x - 1 + \frac{3}{x+1}\right)dx$

$= \int (x-1)dx + 3\int \frac{1}{x+1}dx$

$= \frac{x^2}{2} - x + 3\log\left|(x+1)\right| + C$

4. $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}dx$

Sol. Let $I=\int\left(\frac{e^{6\log x}-e^{5\log x}}{e^{4\log x}-e^{3\log x}}\right)dx$

$=\int\left(\frac{e^{\log x^6}-e^{\log x^5}}{e^{\log x^4}-e^{\log x^3}}\right)dx\quad [\because a\log b=\log b^a]$

$=\int\left(\frac{x^6-x^5}{x^4-x^3}\right)dx\quad [\because e^{\log x}=x]$

$=\int\left(\frac{x^3-x^2}{x-1}\right)dx=\int\frac{x^2(x-1)}{x-1}dx$

$=\int x^2dx=\frac{x^3}{3}+C$

5. $\int\frac{(1+\cos x)}{x+\sin x}dx$

Sol. Consider that, $I=\int\frac{(1+\cos x)}{(x+\sin x)}dx$

Let $x+\sin x=t\Rightarrow(1+\cos x)dx=dt$

$\therefore\ I=\int\frac{1}{t}dt=\log|t|+C$

$=\log|(x+\sin x)|+C$

6. $\int\frac{dx}{1+\cos x}$

Sol. Let $I=\int\frac{dx}{1+\cos x}=\int\frac{dx}{1+2\cos^2\frac{x}{2}-1}$

$=\frac{1}{2}\int\frac{1}{\cos^2\frac{x}{2}}dx=\frac{1}{2}\int\sec^2\frac{x}{2}dx$

$=\frac{1}{2}.\tan\frac{x}{2}.2+C=\tan\frac{x}{2}+C\quad [\because\int\sec^2 x\,dx=\tan x]$

7. $\int\tan^2 x\sec^4 x dx$

Sol. Let $I=\int\tan^2 x\sec^4 x\,dx$

Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$

$\therefore\ I=\int t^2(1+t^2)dt=\int(t^2+t^4)dt$

$=\frac{t^3}{3}+\frac{t^5}{5}+C=\frac{\tan^5 x}{5}+\frac{\tan^3 x}{3}+C$

8. $\int\frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}$

Sol. Let $I=\int\frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}dx=\int\frac{(\sin x+\cos x)}{\sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}}dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int 1\, dx = x + C$$

9. $\int \sqrt{1 + \sin x} dx$

Sol. Let $I = \int \sqrt{1 + \sin x}\, dx$

$$= \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} dx \quad \left[\because \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1\right]$$

$$= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} dx = \int \left(\sin \frac{x}{2} \cos \frac{x}{2}\right) dx$$

$$= -\cos \frac{x}{2}.2 + \sin \frac{x}{2}.2 + C = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C$$

10. $\int \frac{x}{\sqrt{x} + 1} dx$ (Hint: $Put \sqrt{x} = z$)

Sol. Let $I = \int \frac{x}{\sqrt{x} + 1} dx$

$$Put \ \ \sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow dx = 2\sqrt{x} dt$$

$$\therefore I = 2 \int \left(\frac{x\sqrt{x}}{t+1}\right) dt = 2 \int \frac{t^2.t}{t+1} dt = 2 \int \frac{t^3}{t+1} dt$$

$$= 2 \int \frac{t^3 + 1 - 1}{t+1} dt = 2 \int \frac{(t+1)(t^2 - t + 1)}{t+1} dt - 2 \int \frac{1}{t+1} dt$$

$$= 2 \int (t^2 - t + 1) dt - 2 \int \frac{1}{t+1} dt$$

$$= 2\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log\left|(t+1)\right|\right] + C$$

$$= 2\left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log\left|(\sqrt{x} + 1)\right|\right] + C$$

11. $\int \sqrt{\frac{a+x}{a-x}}$

Sol. Let $I = \int \sqrt{\frac{a+x}{a-x}} dx$

$$Put \ \ x = a \cos 2\theta$$

$$\Rightarrow dx = -a. \sin 2\theta.2.d\theta$$

$$\therefore I = -2 \int \sqrt{\frac{a + a \cos 2\theta}{a - a \cos 2\theta}}.a \sin 2\theta d\theta$$

$$\left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1} \frac{x}{a} \Rightarrow \theta = \frac{1}{2} \cos^{-1} \frac{x}{a}\right]$$

$$= -2a\int\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\sin 2\theta d\theta = -2a\int\sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}\sin 2\theta d\theta$$

$$= -2a\int \cot\theta.\sin 2\theta d\theta = -2a\int\frac{\cos\theta}{\sin\theta}.2\sin\theta.\cos\theta d\theta$$

$$= -4a\int\cos^2\theta d\theta = -2a\int(1+\cos 2\theta)d\theta$$

$$= -2a\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= -2a\left[\frac{1}{2} + \cos^{-1}\frac{x}{a} + \frac{1}{2}\sqrt{1-\frac{x^2}{a^2}}\right] + c$$

$$= -a\left[\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{1-\frac{x^2}{a^2}}\right] + C$$

12. $\int\frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}dx$ (Hint: $Put\ x = z^4$)

Sol. Let $I = \int\frac{x^{1/2}}{1+x^{3/4}}dx$

$Put\ \ x = t^4 \Rightarrow dx = 4t^3 dt$

$$\therefore\ I = 4\int\frac{t^2(t^3)}{1+t^3}dt = 4\int\left(t^2 - \frac{t^2}{1+t^3}\right)dt$$

$$I = 4\int t^2 dt - 4\int\frac{t^2}{1+t^3}dt$$

$$I = I_1 - I_2$$

$$I_1 = 4\int t^2 dt = 4.\frac{t^3}{3} + C_1 = \frac{4}{3}x^{3/4} + C_1$$

$$Now,\ I_2 = 4\int\frac{t^2}{1+t^3}dt$$

$$Again,\ put\ \ 1+t^3 = z \Rightarrow 3t^2 dt = dz$$

$$\Rightarrow t^2 dt = \frac{1}{3}dz = \frac{4}{3}\int\frac{1}{z}dz$$

$$= \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log\left|(1+t^3)\right| + C_2$$

$$= \frac{4}{3}\log\left|(1+x^{3/4}\right| + C_2$$

$$\therefore\ I = \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log\left|(1+x^{3/4})\right| - C_2$$

$$= \frac{4}{3}x^{3/4} - \log|(1+x^{3/4}) + C \qquad [\because C = C_1 - C_2]$$

13. $\int \frac{\sqrt{1+x^2}}{x^4}dx$

Sol. Let $I = \int \frac{\sqrt{1+x^2}}{x^4}dx = \int \frac{\sqrt{1+x^2}}{x}.\frac{1}{x^3}\,dx$

$= \int \sqrt{\frac{1+x^2}{x^2}}.\frac{1}{x^3}dx = \int \sqrt{\frac{1}{x^2}+1}.\frac{1}{x^3}dx$

$Put\ \ 1+\frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3}\,dx = 2t\,dt$

$\Rightarrow -\frac{1}{x^3} = t\,dt$

$\therefore\ I = -\int t^2 dt = -\frac{t^3}{3} + C = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2} + C$

14. $\int \frac{dx}{\sqrt{16-9x^2}}$

Sol. Let $I = \int \frac{dx}{\sqrt{16-9x^2}} = \int \frac{dx}{\sqrt{(4)^2-(3x)^2}}dx = \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$

15. $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let $I = \int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left(t^2-\frac{3}{2}t\right)}}$

$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t^2-2.\frac{1}{2}.\frac{3}{2}t\right)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$

$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$

$= \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}$

$= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{t-\frac{3}{4}}{\frac{3}{4}}\right) + C = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4t-3}{3}\right) + C$

16. $\int \frac{3x-1}{x^2+9}dx$

Sol. Let $I = \int \frac{3x-1}{\sqrt{x^2+9}}dx$

$$I = \int \frac{3x-1}{\sqrt{x^2+9}}dx - \int \frac{1}{\sqrt{x^2+9}}dx$$

$$I = I_1 - I_2$$

Now, $I_1 = \int \frac{3x}{\sqrt{x^2+9}}$

Put $x^2+9 = t^2 \Rightarrow 2x\,dx = 2t\,dt \Rightarrow xdx = tdt$

$$\therefore\ I_1 = 3\int \frac{t}{t}dt$$

$$= 3\int dt = 3t + C_1 = 3\sqrt{x^2+9} + C_1$$

and $I_2 = \int \frac{1}{\sqrt{x^2+9}}dx = \int \frac{1}{\sqrt{x^2+(3)^2}}dx$

$$= \log\left|x+\sqrt{x^2+9}\right| + C_2$$

$$\therefore\ I = 3\sqrt{x^2+9} + C_1 - \log\left|x+\sqrt{x^2+9}\right| - C_2$$

$$= 3\sqrt{x^2+9} - \log\left|x+\sqrt{x^2+9}\right| + C\ \ [\because C = C_1 - C_2]$$

17. $\int \sqrt{5-2x+x^2}\,dx$

Sol. Let $I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{x^2-2x+1+4d\,x}$

$$= \int \sqrt{(x-1)^2+(2)^2}\,dx = \int \sqrt{(2)^2+(x-1)^2}\,dx$$

$$= \frac{x-1}{2}\sqrt{2^2+(x-1)^2} + 2\log|\,x-1+\sqrt{2^2+(x-1)^2}\,| + C$$

$$= \frac{x-1}{2}\sqrt{5-2x+x^2} + 2\log|\,x-1+\sqrt{5-2x+x^2}\,| + C$$

18. $\int \frac{x}{x^4-1}dx$

Sol. Let $I = \int \frac{x}{x^4-1}dx$

Put $x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{1}{2}dt$

$$\therefore\ I = \frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{2}.\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C\ \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{4}[\log|x^2-1| - \log|x^2+1|] + C$$

19. $\int \frac{x^2}{1-x^4}dx\ put\ x^2 = t$

Sol. Let $I = \int \frac{x^2}{1-x^4}dx$

$$= \int \frac{\left(\frac{1}{2}+\frac{x^2}{2}-\frac{1}{2}+\frac{x^2}{2}\right)}{(1-x^2)(1+x^2)}dx \quad [\because a^2-b^2=(a+b)(a-b)]$$

$$= \int \frac{\frac{1}{2}(1+x^2)-\frac{1}{2}(1-x^2)}{(1-x^2)(1+x^2)}dx$$

$$= \int \frac{\frac{1}{2}(1+x^2)}{(1-x^2)(1+x^2)}dx - \frac{1}{2}\int \frac{(1-x^2)}{(1-x^2)(1+x^2)}dx$$

$$= \frac{1}{2}\int \frac{1}{1-x^2}dx - \frac{1}{2}\int \frac{1}{1+x^2}dx = \frac{1}{2}.\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C_1 - \frac{1}{2}\tan^{-1}x + C_2$$

$$= \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C \quad [\because C = C_1 + C_2]$$

20. $\int \sqrt{2ax-x^2}\,dx$

Sol. Let $I = \int \sqrt{2ax-x^2}\,dx = \int \sqrt{-(x^2-2ax)dx}$

$$= \int \sqrt{-(x^2-2ax+a^2-a^2)}dx = \int \sqrt{-(x-a)^2-a^2}\,dx$$

$$= \int \sqrt{a^2-(x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

21. $\int \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$

Sol. Let $I = \int \frac{\sin^{-1}x}{(1-x^2)^{3/4}}dx = \int \frac{\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}dx$

$$Put\ \sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$$

$$and\quad x = \sin t \Rightarrow 1-x^2 = \cos^2 t$$

$$\Rightarrow \cos t = \sqrt{1-x^2}$$

$$\therefore\ I = \int \frac{t}{\cos^2 t}dt = \int t.\sec^2 t dt$$

$$= t.\int \sec^2 t dt - \int\left(\frac{d}{dt}t.\int \sec^2 t\,dt\right)dt$$

$$= t.\tan t - \int 1.\tan t\,dt$$

$$= t\tan t + \log|\cos t| + C \qquad [\because \int \tan x dx = -\log|\cos x| + C$$

$$= \sin^{-1} x.\frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C$$

22. $\int \frac{(\cos 5x + \cos 4x)}{1 - 2\cos 3x} dx$

Sol. Let $I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx = \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{1 - 2\left(2\cos^2\frac{3x}{2} - 1\right)} dx$

$$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}.\cos\frac{C-D}{2} \ \ and \ \ \cos 2x = 2\cos^2 x - 1\right]$$

$$\therefore \ I = \int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}} dx = -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}}{4\cos^2\frac{3x}{2} - 3} dx$$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}} dx \quad \left[\text{multiply and divide by} \cos\frac{3x}{2}\right]$$

$$= -\int \frac{2\cos\frac{9x}{2}.\cos\frac{x}{2}.\cos\frac{3x}{2}}{\cos 3.\frac{3x}{2}} dx = -\int 2\cos\frac{3x}{2}.\cos\frac{x}{2} dx$$

$$= -\int \left\{\cos\left(\frac{3x}{2} + \frac{x}{2}\right) + \cos\left(\frac{3x}{2} - \frac{x}{2}\right)\right\} dx$$

$$= -(\cos 2x + \cos x) dx$$

$$= -\left[\frac{\sin 2x}{2} + \sin x\right] + C$$

$$= -\frac{1}{2}\sin 2x - \sin x + C$$

23. $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Sol. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x.\cos^2 x} dx$

$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)}{\sin^2 x.\cos^2 x} dx$$

$$= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x.\cos^2 x} dx - \int \frac{\sin^2 x\cos^2 x}{\sin^2 x.\cos^2 x} dx$$

$$= \int \tan^2 x\, dx + \int \cot^2 x\, dx - \int 1\, dx$$

$$= \int (\sec^2 x - 1) dx + \int (\cos ec^2 x - 1) dx - \int 1\, dx$$

$$= \int \sec^2 x dx + \int \cos ec^2 x dx - 3\int dx$$

$$I = \tan x - \cot x - 3x + C$$

24. $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Sol. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$

Put $x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt$

$$\therefore \ I = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C$$

$$= \frac{2}{3} \sin^{-1} \frac{x^{3/2}}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \sqrt{\frac{x^3}{a^3}} + C$$

25. $\int \frac{\cos x - \cos 2x}{1 - \cos x}$

Sol. Let $I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx = \int \frac{2\sin\frac{3x}{2}.\sin\frac{x}{2}}{1 - 1 + 2\sin^2\frac{x}{2}} dx$

$$= 2\int \frac{\sin\frac{3x}{2}.\sin\frac{x}{2}}{2\sin^2\frac{x}{2}} dx = \int \frac{\sin\frac{3x}{2}}{\sin\frac{x}{2}} dx$$

$$= \int \frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}} dx \ \ [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= 3\int dx - 4\int \sin^2\frac{x}{2} dx = 3\int dx - 4\int \frac{1 - \cos x}{2} dx$$

$$= 3\int dx - 2\int dx + 2\int \cos x \, dx$$

$$= \int dx + 2\int \cos x \, dx = x + 2\sin x + C = 2\sin x + x + C$$

26. $\int \frac{dx}{x\sqrt{x^4 - 1}}$ (Hint: $Put \; x^2 = \sec\theta$)

Sol. Let $I = \int \frac{dx}{x\sqrt{x^4 - 1}}$

Put $x^2 = \sec\theta \Rightarrow \theta = \sec^{-1} x^2$

$\Rightarrow \ 2x\,dx = \sec\theta.\tan\theta\, d\theta$

$$\therefore \ I = \frac{1}{2} \int \frac{\sec\theta.\tan\theta}{\sec\theta\tan\theta} d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$= \frac{1}{2}\sec^{-1}(x^2) + C$

Evaluate the following as limit of sums:

27. $\int_0^2 \left(x^2+3\right) dx$

Sol. Let $I = \int_0^2 (x^2+3)dx$

Here, $a=0, b=2$ *and* $h = \frac{b-a}{n} = \frac{2-0}{n}$

$\Rightarrow\ h = \frac{2}{n} \Rightarrow\ nh = 2 \Rightarrow f(x) = (x^2+3)$

Now, $\int_0^2 (x^2+3)dx = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0+(n-1)h\}]$ ...(i)

$\because\ f(0) = 3$

$\Rightarrow\ f(0+h) = h^2+3, f(0+2h) = 4h^2+3 = 2^2h^2+3$

$f[0+(n-1)h] = (n^2-2n+1)h+3 = (n-1)^2 h + 3$

Form Eq. (i)

$\int_0^2 (x^2+3)dx = \lim_{h\to 0} h[3+h^2+3+2^2h^2+3+3^2h^2+3+\ldots+(n-1)^2h^2+3]$

$= \lim_{h\to 0} h[3n + h^2\{1^2+2^2+\ldots+(n-1)^2\}]$

$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n-1)(2n-2+1)(n-1+)}{6}\right)\right]$ $\left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$

$= \lim_{h\to 0} h\left[3n + h^2\left(\frac{(n^2-n)(2n-1)}{6}\right)\right]$

$= \lim_{h\to 0} h\left[3n + \frac{h^2}{6}(2n^3-n^2-2n^2+n)\right]$

$= \lim_{h\to 0}\left[3nh + \frac{2n^3h^3 - 3n^2h^2.h + nh.h^2}{6}\right]$

$= \lim_{h\to 0}\left[3.2 + \frac{2.8 - 3.2^2.h + 2.h^2}{6}\right] = \lim_{h\to 0}\left[6 + \frac{16-12h+2h^2}{6}\right]$

$= 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}$

28. $\int_0^2 e^x dx$

Sol. Let $I = \int_0^2 e^x dx$

Here, $a = 0$ *and* $b = 2$

$\therefore\ h = \frac{b-a}{n}$

$\Rightarrow nh = 2$ *and* $f(x) = e^x$

Now, $\int_0^2 e^x dx = \lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+...+f\{0+(n-1)h\}]$

$\therefore\ I = \lim_{h\to 0} h[1+e^h+e^{2h}+...+e^{(n-1)h}]$

$= \lim_{h\to 0} h\left[\frac{1.(e^h)^n-1}{e^h-1}\right] = \lim_{h\to 0} h\left(\frac{e^{nh}-1}{e^h-1}\right)$

$= \lim_{h\to 0} h\left(\frac{e^2-1}{e^h-1}\right)$

$= e^2 \lim_{h\to 0}\frac{h}{e^h-1} - \lim_{h\to 0}\frac{h}{e^h-1} \qquad \left[\because \lim_{h\to 0}\frac{h}{e^h-1}=1\right]$

$= e^2-1 = e^2-1$

Evaluate the following;

29. $\int\limits_0^1 \frac{dx}{e^x+e^{-x}}$

Sol. Let $I = \int_0^1 \frac{dx}{e^x+e^{-x}} = \int_0^1 \frac{e^x}{1+e^{2x}}dx$

Put $e^x = t$

$\Rightarrow\ e^x dx = dt$

$\therefore\ I = \int_1^e \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^e$

$= \tan^{-1} e - \tan^{-1} 1$

$= \tan^{-1} e - \frac{\pi}{4}$

30. $\int\limits_0^{\frac{\pi}{2}} \frac{\tan x dx}{1+m^2\tan^2 x}$

Sol. Let $I = \int_0^{\pi/2} \frac{\tan x\, dx}{1+m^2\tan^2 x}dx$

$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2.\frac{\sin^2 x}{\cos^2 x}}dx$

$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x+m^2\sin^2 x}{\cos^2 x}}dx$

$= \int_0^{\pi/2} \frac{\sin x\cos x\, dx}{1-\sin^2 x+m^2\sin^2 x}dx$

$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x(1-m^2)}dx$

Put $\sin^2 x = t$

$$\Rightarrow 2\sin x\cos x\,dx = dt$$

$$\therefore\ I = \frac{1}{2}\int_0^1 \frac{dt}{1-t(1-m^2)}$$

$$= \frac{1}{2}\left[-\log|1-t(1-m^2)|.\frac{1}{1-m^2}\right]_0^1$$

$$= \frac{1}{2}\left[-\log|1-1+m^2|.\frac{1}{1+m^2}+\log|1|.\frac{1}{1-m^2}\right]$$

$$= \frac{1}{2}\left[-\log|m^2|.\frac{1}{1-m^2}\right] = \frac{2}{2}.\frac{\log m}{(m^2-1)}$$

$$= \log\frac{m}{m^2-1}$$

31. $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

Sol. Let $I = \displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} = \int_1^2 \frac{dx}{\sqrt{2x-x^2-2+x}}$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2-3x+2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left[x^2-2.\frac{3}{2}x+\left(\frac{3}{2}\right)^2+2-\frac{9}{4}\right]}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left\{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2\right\}}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= [\sin^{-1}(2x-3)]_1^2 = \sin^{-1}1-\sin^{-1}(-1)$$

$$= \frac{\pi}{2}+\frac{\pi}{2}\quad\left[\because \sin\frac{\pi}{2}=1 \text{ and } \sin(-\theta)=-\sin\theta\right]$$

$$= \pi$$

32. $\displaystyle\int_0^1 \frac{xdx}{\sqrt{1+x^2}}$

Sol. Let $I = \displaystyle\int_0^1 \frac{x}{\sqrt{1+x^2}}dx$

Put $1+x^2 = t^2$

$\Rightarrow 2x\,dx = 2tdt$

$\Rightarrow x\,dx = t\,dt$

$\therefore \; I = \int_1^{\sqrt{2}} \frac{t\,dt}{t}$

$= [t]_1^{\sqrt{2}} = \sqrt{2} - 1$

33. $\int_0^{\pi} x \sin x \cos^2 x\,dx$

Sol. Let $I = \int_0^{\pi} x \sin x \cos^2 x\,dx \;\; \ldots(i)$

and $I = \int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^2(\pi - x)dx$

$\Rightarrow I = \int_0^{\pi} (\pi - x)\sin x \cos^2 x\,dx \;\;\; \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi} \pi \sin x \cos^2 x\,dx$

Put $\cos x = t$

$\Rightarrow -\sin x\,dx = dt$

$As \;\; x \to 0, \;\; then \; t \to 1$

$and \;\; x \to \pi, \; then \; t \to -1$

$\therefore \; I = -\pi \int_1^{-1} t^2 dt \;\; \Rightarrow \;\; I = -\pi \left[\frac{t^3}{3}\right]_1^{-1}$

$\Rightarrow 2I = -\frac{\pi}{3}[-1-1] \;\Rightarrow\; 2I = \frac{2\pi}{3}$

$\therefore \; I = \frac{\pi}{3}$

34. $\int_0^{\frac{1}{2}} \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}}$ (Hint: $let\; x = \sin\theta$)

Sol. Let $I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin\theta$

$\Rightarrow \;\; dx = \cos\theta\, d\theta$

$As \;\; x \to 0, then\, \theta \to 0$

$and\; x \to \frac{1}{2}, then\, \theta \to \frac{\pi}{6}$

$\therefore \; I = \int_0^{\pi/6} \frac{\cos\;\theta}{(1+\sin^2\theta)\cos\theta} d\theta = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta} d\theta$

$= \int_0^{\pi/6} \frac{1}{\cos^2\theta(\sec^2\theta + \tan^2\theta)} d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta} d\theta$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta+\tan^2\theta}\,d\theta$$

$$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+2\tan^2\theta}\,d\theta$$

Again, put $\tan\theta = t$

$$\Rightarrow \sec^2\theta\,d\theta = dt$$

As $\theta \to 0,$ *then* $t \to 0$

and $\theta \to \frac{\pi}{6},$ *then* $t \to \frac{1}{\sqrt{3}}$

$$\therefore\; I = \int_0^{1/\sqrt{3}} \frac{dt}{1+2t^2} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

$$= \frac{1}{2}.\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}[\tan^{-1}(\sqrt{2}t)]_0^{1/\sqrt{3}}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - 0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Integrals
Long Answer Type Questions

35. $\int \frac{x^2 dx}{x^4 - x^2 - 12}$

Sol. Let $I = \int \frac{x^2}{x^4 - x^2 - 12} dx$

$= \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$

$= \int \frac{x^2 dx}{x^2(x^2-4) + 3(x^2-4)}$

$= \int \frac{x^2 dx}{(x^2-4)(x^2+3)}$

Now, $\frac{x^2}{(x^2-4)(x^2+3)}$ $[let\ x^2 = t]$

$\Rightarrow \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$

$\Rightarrow t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get

$A + B = 1$

$\Rightarrow 3A - 4B = 0$

$\Rightarrow 3(1-B) - 4B = 0$

$\Rightarrow 3 - 3B - 4B = 0$

$\Rightarrow 7B = 3$

$\Rightarrow B = \frac{3}{7}$

$If\ B = \frac{3}{7}, then\ A + \frac{3}{7} = 1$

$\Rightarrow A = 1 - \frac{3}{7} = \frac{4}{7}$

$\frac{x^2}{(x^2-4)(x^2+3)} = \frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)}$

$\therefore\ I = \frac{4}{7}\int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7}\int \frac{1}{x^2 + (\sqrt{3})^2} dx$

$= \frac{4}{7} . \frac{1}{2.2} \log\left|\frac{x-2}{x+2}\right| + \frac{3}{7} . \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + C$

$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$

36. $\int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$

Integrals
Long Answer Type Questions

35. $\int \frac{x^2 dx}{x^4 - x^2 - 12}$

Sol. Let $I = \int \frac{x^2}{x^4 - x^2 - 12} dx$

$= \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx$

$= \int \frac{x^2 dx}{x^2(x^2-4) + 3(x^2-4)}$

$= \int \frac{x^2 dx}{(x^2-4)(x^2+3)}$

Now, $\frac{x^2}{(x^2-4)(x^2+3)}$ $[let\ x^2 = t]$

$\Rightarrow \frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$

$\Rightarrow t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get

$A + B = 1$

$\Rightarrow 3A - 4B = 0$

$\Rightarrow 3(1-B) - 4B = 0$

$\Rightarrow 3 - 3B - 4B = 0$

$\Rightarrow 7B = 3$

$\Rightarrow B = \frac{3}{7}$

$If\ B = \frac{3}{7}, then\ A + \frac{3}{7} = 1$

$\Rightarrow A = 1 - \frac{3}{7} = \frac{4}{7}$

$\frac{x^2}{(x^2-4)(x^2+3)} = \frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)}$

$\therefore\ I = \frac{4}{7}\int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7}\int \frac{1}{x^2 + (\sqrt{3})^2} dx$

$= \frac{4}{7} \cdot \frac{1}{2.2} \log\left|\frac{x-2}{x+2}\right| + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1}\frac{x}{\sqrt{3}} + C$

$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C$

36. $\int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$

Sol. Let $I=\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$

Now, $\frac{x^2}{(x^2+a^2)(x^2+b^2)}$ $[let\, x^2=t]$

$=\frac{t}{(t+a^2)(t+b^2)}=\frac{A}{(t+a^2)}+\frac{B}{(t+b^2)}$

$t=A(t+b^2)+B(t+a^2)$

On comparing the coefficient of t, we get

$A+B=1$ ...(i)

$b^2A+a^2B=0$

$\Rightarrow b^2(1-B)+a^2B=0$

$\Rightarrow b^2-b^2B+a^2B=0$

$\Rightarrow b^2+(a^2-b^2)B=0$

$\Rightarrow B=\frac{-b^2}{a^2-b^2}=\frac{b^2}{b^2-a^2}$

$From\, Eq.(i) \quad A+\frac{b^2}{b^2-a^2}=1$

$\Rightarrow A=\frac{b^2-a^2-b^2}{b^2-a^2}=\frac{-a^2}{b^2-a^2}$

$\therefore \quad I=\int \frac{-a^2}{(b^2-a^2)(x^2+a^2)}dx+\int \frac{b^2}{b^2-a^2}.\frac{1}{x^2-b^2}dx$

$=\frac{-a^2}{(b^2-a^2)}\int \frac{1}{x^2+a^2}dx+\frac{b^2}{b^2-a^2}\int \frac{1}{x^2+b^2}dx$

$=\frac{-a^2}{(b^2-a^2)}.\frac{1}{a}\tan^{-1}\frac{x}{a}+\frac{b^2}{b^2-a^2}.\frac{1}{b}\tan^{-1}\frac{x}{b}$

$=\frac{1}{b^2-a^2}\left[-a\tan^{-1}\frac{x}{a}+b\tan^{-1}\frac{x}{a}\right]$

$=\frac{1}{a^2-b^2}\left[a\tan^{-1}\frac{x}{a}-b\tan^{-1}\frac{x}{a}\right]$

37. $\int_0^{\pi}\frac{x}{1+\sin x}$

Sol. Let $I=\int_0^{\pi}\frac{x}{1+\sin x}dx$ ...(i)

and $I=\int_0^{\pi}\frac{\pi-x}{1+\sin(\pi-x)}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin x}dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$
$$= \pi\int_0^{\pi} \frac{(1-\sin x)\,dx}{(1+\sin x)(1-\sin x)}$$
$$= \pi\int_0^{\pi} \frac{(1-\sin x)dx}{\cos^2 x}$$
$$= \pi\int_0^{\pi} (\sec^2 x - \tan x.\sec x)\,dx$$
$$= \pi\int_0^{\pi} \sec^2 x\,dx - \pi\int_0^{\pi} \sec x\,x.\tan x\,dx$$
$$= \pi[\tan x]_0^{\pi} - \pi[\sec x]_0^{\pi}$$
$$= \pi[\tan x - \sec x]_0^{\pi}$$
$$= \pi[\tan\pi - \sec\pi - \tan 0 - \sec 0]$$
$$\Rightarrow 2I = \pi[0+1-0+1]$$
$$2I = 2\pi$$
$$\therefore I = \pi$$

38. $\displaystyle\int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$

Sol. Let $\displaystyle I = \int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Now, $\displaystyle\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}$

$\Rightarrow 2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

Put $x = 3, then$

$6-1 = C(3-1)(3+2)$

$\Rightarrow 5 = 10C \Rightarrow C = \dfrac{1}{2}$

Again, put x = 1, then

$2-1 = A(1+2)(1-3)$

$\Rightarrow 1 = -6A \Rightarrow A = -\dfrac{1}{6}$

Now, put x = - 2, then

$-4-1 = B(-2-1)(-2-3)$

$\Rightarrow -5 = 15B \Rightarrow B = -\dfrac{1}{3}$

$$\therefore\ I = -\frac{1}{6}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x-3}\,dx$$

$$= -\frac{1}{6}\log|(x-1)| - \frac{1}{3}\log|(x+2)| + \frac{1}{2}\log|(x-3)| + C$$

$$= -\log|(x-1)|^{1/6} - \log|(x+2)|^{1/3} + \log|(x-3)|^{1/2} + C$$

$$= \log\left|\frac{\sqrt{x-3}}{(x-1)^{1/6}(x+2)^{1/3}}\right| + C$$

39. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

Sol. Let $I = \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

$$= \int e^{\tan^{-1}x}\left(\frac{1+x^2}{1+x^2} + \frac{x}{1+x^2}\right)dx$$

$$= \int e^{\tan^{-1}x}dx + \int \frac{x e^{\tan^{-1}x}}{1+x^2}dx$$

$I = I_1 + I_2 \quad ...(i)$

Now, $I_2 = \int \frac{x e^{\tan^{-1}x}}{1+x^2}dx$

Put $\tan^{-1}x = t \Rightarrow x = \tan t$

$\Rightarrow \frac{1}{1+x^2}dx = dt$

$\therefore \; I = \int \underset{I}{\tan t} \,.\, \underset{II}{e^t} dt$

$= \tan t . e^t - \int \sec^2 t . e^t dt + C$

$= \tan t . e^t - \int (1+\tan^2 t)e^t dt + C \;\; [\because \sec^2\theta = 1 + \tan^2\theta]$

$\text{I}_2 = \tan t . e^t - \int (1+x^2)\frac{e^{\tan^{-1}x}}{1+x^2}dx + C$

$I_2 = \tan t . e^t - \int e^{\tan^{-1}x}dx + C$

$\therefore \; I = \int e^{\tan^{-1}x}dx + \tan t . e^t - \int e^{\tan^{-1}x}dx + C$

$= \tan t . e^t + C$

$= x e^{\tan^{-1}x} + C$

40. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$ (Hint: $Put\; x = a\tan^2\theta$)

Sol. Let $I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

Put $x = a\tan^2\theta$

$\Rightarrow dx = 2a\tan\theta\sec^2\theta\, d\theta$

$$\therefore\ I=\int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}(2a\tan\theta.\sec^2\theta)\,d\theta$$

$$=2a\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\tan\theta.\sec^2\theta d\theta$$

$$=2a\int \sin^{-1}(\sin\theta)\tan\theta.\sec^2\theta\,d\theta$$

$$=2a\int \underset{I}{\theta}.\underset{II}{\tan\theta\sec^2\theta}\,d\theta$$

$$=2a\left[\theta.\int\tan\theta.\sec^2\theta\,d\theta-\int\left(\frac{d}{d\theta}\theta.\int\tan\theta.\sec^2\theta\,d\theta\right)d\theta\right]$$

$$\left[\begin{array}{l}Put \qquad \tan\theta=t\\ \Rightarrow\ \sec\theta.\tan\theta.d\theta=dt\\ \Rightarrow\int\tan\theta\sec^2\theta\,d\theta=\int t\,dt\end{array}\right]$$

$$=2a\left[\theta.\frac{\tan^2\theta}{2}-\int\frac{\tan^2\theta}{2}d\theta\right]$$

$$=a\theta\tan^2\theta-a\int(\sec^2\theta-1)d\theta$$

$$=a\theta.\tan^2\theta-a\tan\theta+a\theta+C$$

$$=a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}}+\tan^{-1}\sqrt{\frac{x}{a}}\right]+C$$

41. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sqrt{1+\cos x}}{(1-\cos)^{\frac{5}{2}}}$

Sol. Let $\displaystyle I=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx$

$$=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^2\sqrt{1+\cos x}}dx$$

$$=\int_{\pi/3}^{\pi/2}\frac{1}{(1-\cos^2 x)}dx=\int_{\pi/3}^{\pi/2}\frac{1}{\sin^2 x}dx$$

$$=\int_{\pi/3}^{\pi/2}\cos ec^2x\,dx=[-\cot x]_{\pi/3}^{\pi/2}$$

$$=-\left[\cot\frac{\pi}{2}-\cot\frac{\pi}{3}\right]=-\left[0-\frac{1}{\sqrt{3}}\right]=+\frac{1}{\sqrt{3}}$$

Alternate Method

Let $\displaystyle I=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}dx=\int_{\pi/3}^{\pi/2}\frac{\left(2\cos^2\frac{x}{2}\right)^{1/2}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}dx$

$$=\frac{\sqrt{2}}{4\sqrt{2}}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}dx=\frac{1}{4}\int_{\pi/3}^{\pi/2}\frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}dx$$

Put $\sin\frac{x}{2}=t$

$$\Rightarrow \cos\frac{x}{2}.\frac{1}{2}dx=dt$$

$$\Rightarrow \cos\frac{x}{2}dx=2dt$$

$$As\ x\to\frac{\pi}{3},\ then\ t\to\frac{1}{2}$$

$$and\ x\to\frac{\pi}{2},\ then\ t\to\frac{1}{\sqrt{2}}$$

$$\therefore\ I=\frac{2}{4}\int_{1/2}^{1/\sqrt{2}}\frac{dt}{t^5}=\frac{1}{2}\left[\frac{t^{-5+1}}{-5+1}\right]_{1/2}^{1/\sqrt{2}}$$

$$=-\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4}-\frac{1}{\left(\frac{1}{2}\right)^4}\right]$$

$$=-\frac{1}{8}(4-16)=\frac{12}{8}=\frac{3}{2}$$

42. $\int e^{-3x}\cos^3 x\,dx$

Sol. Let $I=\int \underset{II}{e^{-3x}}\ \underset{I}{\cos^3 x}\,dx$

$$=\cos^3 x\int e^{-3x}dx-\int\left(\frac{d}{dx}\cos^3 x\int e^{-3x}dx\right)dx$$

$$=\cos^3 x.\frac{e^{-3x}}{-3}-\int(-3\cos^2 x)\sin x.\frac{e^{-3x}}{-3}dx$$

$$=-\frac{1}{3}\cos^3 xe^{-3x}-\int\cos^2 x\sin xe^{-3x}dx$$

$$=-\frac{1}{3}\cos^3 xe^{-3x}-\int(1-\sin^2 x)\sin x\,e^{-3x}dx$$

$$=-\frac{1}{3}\cos^3 xe^{-3x}-\int\sin x\,e^{-3x}dx+\int\underset{I}{\sin^3 x}\,\underset{II}{e^{-3x}}dx$$

$$=-\frac{1}{3}\cos^3 xe^{-3x}-\int\sin x\,e^{-3x}dx+\sin^3 x.\frac{e^{-3x}}{-3}-\int 3\sin^2 x\cos x.\frac{e^{-3x}}{-3}dx$$

$$=-\frac{1}{3}\cos^3 xe^{-3x}-\int\sin x\,e^{-3x}dx-\frac{1}{3}\sin^3 x\,e^{-3x}+\int(1-\cos^2 x)\cos x\,e^{-3x}dx$$

$$I=-\frac{1}{3}\cos^3 x\, e^{-3x}-\int \underset{I}{\sin x}\, \underset{II}{e^{-3x}}-\frac{1}{3}\sin^3 x\, e^{-3x}+\int \cos x\, e^{-3x}dx-\int \cos^3 x\, e^{-3x}dx$$

$$2I=\frac{e^{-3x}}{3}[\cos^3 x+\sin^3 x]-\left[\sin x.\frac{e^{-3x}}{-3}-\int \cos x.\frac{e^{-3x}}{-3}\,dx\right]+\int \cos x\, e^{-3x}dx$$

$$2I=\frac{e^{-3x}}{-3}[\cos^3 x+\sin^3 x]+\frac{1}{3}\sin x.e^{-3x}-\frac{1}{3}\int \cos x.e^{-3x}dx+\int \cos x\, e^{-3x}dx$$

$$2I=\frac{e^{-3x}}{-3}[\cos^3 x+\sin^3 x]+\frac{1}{3}\sin x\, e^{-3x}+\frac{2}{3}\int \cos x\, e^{-3x}dx$$

Now, let $I_1=\int \underset{I}{\cos x}\, \underset{II}{e^{-3x}}dx$

$$I_1=\cos x.\frac{e^{-3x}}{-3}-\int (-\sin x).\frac{e^{-3x}}{-3}\,dx$$

$$I_1=\frac{-1}{3}\cos x.e^{-3x}-\frac{1}{3}\int \sin x.e^{-3x}dx$$

$$=-\frac{1}{3}\cos x.e^{-3x}-\frac{1}{3}\left[\sin x.\frac{e^{-3x}}{-3}-\int \cos x.\frac{e^{-3x}}{-3}\,dx\right]$$

$$=-\frac{1}{3}\cos x.e^{-3x}+\frac{1}{9}\sin x.e^{-3x}-\frac{1}{9}\int \cos x.e^{-3x}dx$$

$$I_1+\frac{1}{9}I_1=-\frac{1}{3}e^{-3x}.\cos x+\frac{1}{9}\sin x.e^{-3x}$$

$$\left(\frac{10}{9}\right)I_1=-\frac{1}{3}e^{-3x}.\cos x+\frac{1}{9}\sin x.e^{-3x}$$

$$I_1=\frac{-3}{10}e^{-3x}.\cos x+\frac{1}{10}e^{-3x}\sin x$$

$$2I=-\frac{1}{3}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{1}{3}\sin x.e^{-3x}-\frac{3}{10}e^{-3x}.\cos x+\frac{1}{10}e^{-3x}.\sin x+C$$

$$\therefore\ I=-\frac{1}{6}e^{-3x}[\sin^3 x+\cos^3 x]+\frac{13}{30}e^{-3x}.\sin x-\frac{3}{10}e^{-3x}.\cos x+C$$

$$\left[\begin{array}{l}\because \sin 3x=3\sin x-4\sin^3 x\\ and\ \cos 3x=4\cos^3 x-3\cos x\end{array}\right]$$

$$=\frac{e^{-3x}}{24}[\sin 3x-\cos 3x]+\frac{3e^{-3x}}{40}[\sin x-3\cos x]+C$$

43. $\int \sqrt{\tan x}\,dx$ (Hint: $Put\tan x=t^2$)

Sol. Let $I=\int \sqrt{\tan x}\,dx$

Put $\tan x=t^2 \Rightarrow \sec^2 x\,dx=2t\,dt$

$$\therefore\ I=\int t.\frac{2t}{\sec^2 x}dt=2\int \frac{t^2}{1+t^4}dt$$

$$= \int \frac{(t^2+1)+(t^2-1)}{(1+t^4)}\,dt$$

$$= \int \frac{(t^2+1)}{1+t^4}\,dt + \int \frac{(t^2-1)}{1+t^4}\,dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt + \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$= \int \frac{1-\left(-\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2} + \int \frac{1+\left(-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$Put\ \ u = t-\frac{1}{t} \Rightarrow du = \left(1+\frac{1}{t^2}\right)dt$$

$$and\ \ v = t+\frac{1}{t} \Rightarrow dv = \left(1-\frac{1}{t^2}\right)dt$$

$$\therefore\ I = \int \frac{du}{u^2+(\sqrt{2})^2} + \int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right| + C$$

44. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$ **(Hint: Divide Numerator and Denominator by** $cos^4 x$**)**

Sol. Let $I = \displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$

Divide numerator and denominator by $cos^4 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^4 x\,dx}{(a^2+b^2\tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1+\tan^2 x)\sec^2 x\,dx}{(a^2+b^2\tan^2 x)^2}$$

Put $\tan x = t$

$\Rightarrow \sec^2 x\,dx = dt$

$As\ \ x \to 0,\ then\ t \to 0$

$and\ \ x \to \dfrac{\pi}{2},\ then\ t \to \infty\ \ I = \displaystyle\int_0^{\infty} \frac{(1+t^2)}{(a^2+b^2t^2)^2}$

Now, $\dfrac{1+t^2}{(a^2+b^2t^2)^2}$ $[let\ t^2 = u]$

$$\frac{1+u}{(a^2+b^2u)^2}=\frac{A}{(a^2+b^2u)}+\frac{B}{(a^2+b^2u)^2}$$

$$\Rightarrow\ 1+u=A(a^2+b^2u)+B$$

On comparing the coefficient of x and constant term on both sides, we get

$a^2A+B=1\ \ldots(i)$

and $b^2A=1\ \ldots(ii)$

$$\therefore\ A=\frac{1}{b^2}$$

Now, $\dfrac{a^2}{b^2}+B=1$

$$\Rightarrow\ B=1-\frac{a^2}{b^2}=\frac{b^2-a^2}{b^2}$$

$$\therefore\ I=\int_0^\infty\frac{(1+t^2)}{(a^2+b^2t^2)^2}$$

$$=\frac{1}{b^2}\int_0^\infty\frac{dt}{a^2+b^2t^2}+\frac{b^2-a^2}{b^2}\int_0^\infty\frac{dt}{(a^2+b^2t^2)^2}$$

$$=\frac{1}{b^2}\int_0^\infty\frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)}+\frac{b^2-a^2}{b^2}\int_0^\infty\frac{dt}{(a^2+b^2t^2)^2}$$

$$=\frac{1}{ab^3}\left[\tan^{-1}\left(\frac{tb}{a}\right)\right]_0^\infty+\frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}.\frac{1}{a^3b}\right)$$

$$=\frac{1}{ab^3}[\tan^{-1}\infty-\tan^{-1}0]+\frac{\pi}{4}.\frac{b^2-a^2}{(a^3b^3)}$$

$$=\frac{\pi}{2ab^3}+\frac{\pi}{4}.\frac{b^2-a^2}{(a^3b^3)}$$

$$=\pi\left(\frac{2a^2+b^2-a^2}{4a^3b^3}\right)=\frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right)$$

45. $\displaystyle\int_0^1 x\log(1+2x)dx$

Sol. $I=\int_0^1 x\log(1+2x)dx$

$$=\left[\log(1+2x)\frac{x^2}{2}\right]_0^1-\int\frac{1}{1+2x}.2.\frac{x^2}{2}dx$$

$$=\frac{1}{2}[x^2\log(1+2x)]_0^1-\int\frac{x^2}{1+2x}dx$$

$$= \frac{1}{2}[\log 3 - 0] - \left[\int_0^1 \left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x} \right) dx \right]$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_0^1 \frac{x}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2x+1-1)}{(2x+1)}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2} - 0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|(1+2x)|]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3$$

$$= \frac{3}{8}\log 3$$

46. $\int_0^{\pi} x \log \sin x\,dx$

Sol. Let $I = \int_0^{\pi} x \log \sin x\,dx \quad ...(i)$

$$I = \int_0^{\pi} (\pi - x)\log \sin(\pi - x)dx$$

$$= \int_0^{\pi} (\pi - x)\log \sin x\,dx \quad ...(ii)$$

$$2I = \pi\int_0^{\pi} \log \sin x\,dx \quad ...(iii)$$

$$2I = 2\pi\int_0^{\pi/2} \log \sin x\,dx \quad \left[\because \int_0^{2a} f(x) = 2\int_0^{a} f(x)\,dx\right]$$

$$I = \pi\int_0^{\pi/2} \log \sin x\,dx \quad ...(iv)$$

$$Now, \quad I = \pi\int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx \quad ...(v)$$

On adding Eqs. (iv) and (v), we get

$$2I = \pi\int_0^{\pi/2} (\log \sin x + \log \cos x)dx$$

$$2I = \pi\int_0^{\pi/2} \log \sin x \cos x\,dx$$

$$= \pi\int_0^{\pi/22} \log \frac{2\sin x \cos x}{2}\,dx$$

$$2I = \pi\int_0^{\pi/2} (\log \sin 2x - \log 2)dx$$

$$2I = \pi\int_0^{\pi/2} \log \sin 2x\, dx - \pi\int_0^{\pi/2} \log 2\, dx$$

$$Put\ 2x = t \Rightarrow dx = \frac{1}{2} dt$$

$$As\ \ x \to 0, then\, t \to 0$$

$$and\ x \to \frac{\pi}{2}, then\, t \to \pi$$

$$\therefore\ \ 2I \frac{\pi}{2}\int_0^{\pi} \log \sin t\, dt - \frac{\pi^2}{2}\log 2$$

$$\Rightarrow\ 2I = \frac{\pi}{2}\int_0^{\pi} \log \sin x\, dx - \frac{\pi^2}{2}\log 2$$

$$\Rightarrow\ 2I = I - \frac{\pi^2}{2}\log 2\ \ [form\ \ Eq.(iii)]$$

$$\therefore\ I = -\frac{\pi^2}{2}\log 2 = \frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$$

47. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin x + \cos x)\, dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x) dx\ \ ...(i)$

$$I = \int_{-\pi/4}^{\pi/4} \log\left\{\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right\} dx$$

$$= \int_{-\pi/4}^{\pi/4} \log\{\sin(-x) + \cos(-x)\}\, dx$$

$$and\ \ I = \int_{-\pi/4}^{\pi 4} \log(\cos x - \sin x)\, dx\ \ ...(ii)$$

From Eqs. (i) and (ii),

$$2I = \int_{-\pi/4}^{\pi 4} \log \cos 2x\, dx$$

$$2I = \int_0^{\pi/4} \log \cos 2x\, dx \qquad ...(iii)$$

$$\left[\because \int_{-a}^{a} f(x) dx = 2\int_0^a f(x), if\ \ f(-x) = f(x)\right]$$

$$Put\ \ 2x = t\ \Rightarrow\ dx = \frac{dt}{2}$$

$$As\ \ x \to 0, then\ \ t \to 0$$

$$and\ x \to \frac{\pi}{4}, then\ \ t \to \frac{\pi}{2}$$

$$2I = \frac{1}{2}\int_0^{\pi/2} \log \cos t\, dt\ \ ...(iv)$$