UNIT – 05 MOTION OF SYSTEM OF PARTICLES AND RIGID BODIES

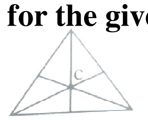
TWO MARKS AND THREE MARKS:

01. Define center of mass.

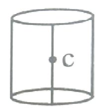
A point where the entire mass of the body appears to be concentrated.

02. Find out the center of mass for the given geometrical structures.

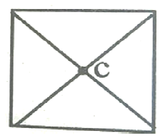
a) Equilateral triangle
Lies in center



b) Cylinder
Lies on its central axis



c) Square
Lies at their diagonals meet



03. Define torque and mention its unit.

Torque is defined as the moment of the external applied force about a point or axis of rotation. The expression for torque is, $\vec{\tau} = \vec{r} x \vec{F}$. Its unit is Nm.

04. What are the conditions in which force cannot produce torque?

The torque is zero when $\vec{r}$ and $\vec{F}$ are parallel or anti-parallel. If parallel, then θ=0 and sin 0 = 0. If anti-parallel, then θ = 180 and sin 180 = 0. Hence, $\tau = 0$. The torque is zero if the force acts at the reference point. i.e. as $\vec{r} = 0$, $\tau = 0$.

05. What is the relation between torque and angular momentum?

An external torque on a rigid body fixed to an axis produces rate of change of angular momentum in the body about that axis. $T = \frac{dL}{dt}$

06. What is equilibrium?

i) A rigid body is said to be in mechanical equilibrium when both its linear momentum and angular momentum remain constant.

ii) When all the forces act upon the object are balanced, then the object is said to be an equilibrium.

07. Give any two examples of torque in day-to-day life.

i) Opening and closing of a door about the hinges ii) Turning of a nut using a wrench
iii) Opening a bottle cap (or) water top

08. How do you distinguish between stable and unstable equilibrium?

Stable equilibrium	Unstable equilibrium
Linear momentum and angular momentum are zero.	Linear momentum and angular momentum are zero.
The body tries to come back to equilibrium if slightly disturbed and released.	The body cannot come back to equilibrium if slightly disturbed and released.
The center of mass of the body shifts slightly higher if disturbed from equilibrium.	The center of mass of the body shifts slightly lower if disturbed from equilibrium.
Potential energy of the body is minimum and it increases if disturbed.	Potential energy of the body is not minimum and it decreases if disturbed

09. Define couple.

Pair of forces which are equal in magnitude but **opposite in direction** and separated by a **perpendicular distance** so that **their lines of action do not coincide** that causes a turning effect is called a couple

10. State principle of moments.

When an object is in equilibrium the sum of the anticlockwise moments about a turning point must be equal to the sum of the clockwise moments.

11. Define center of gravity.

The point at which the entire weight of the body acts irrespective of the position and orientation of the body.

12. Mention any two physical significance of moment of inertia.

i) For rotational motion, moment of inertia is a measure of rotational inertia.

ii) The moment of inertia of a body is not an invariable quantity. It depends not only on the mass of the body, but also on the way the mass is distributed around the axis of rotation.

13. What is radius of gyration?

The radius of gyration of an object is the perpendicular distance from the axis of rotation to an equivalent point mass, which would have the same mass as well as the same moment of inertia of the object.

14. State conservation of angular momentum.

When no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

15. What are the rotational equivalents for the physical quantities, (i) mass and (ii) force?

i) For mass : Moment of inertia , $I = mr^2$ ii) For Force : Torque $\tau = I\,\alpha$

16. What is the condition for pure rolling?

(i) The combination of translational motion and rotational motion about the center of mass. (or) (ii) The momentary rotational motion about the point of contact.

17. What is the difference between sliding and slipping?

Sliding is the case when $v_{CM} > R\omega$ (or $v_{TRANS} > v_{ROT}$). The translation is more than the rotation.

Slipping is the case when $v_{CM} < R\omega$ (or $v_{TRANS} < v_{ROT}$). The rotation is more than the translation.

18. What is rigid body?

A rigid body is the one which maintains its definite and fixed shape even when an external force acts on it.

19. Define Point Mass

A point mass is a hypothetical point particle which has nonzero mass and no size or shape.

20. State the rule which is used to find the direction of torque.

The direction of torque is found using right hand rule. This rule says that if fingers of right hand are kept along the position vector with palm facing the direction of the force and when the fingers are curled the thumb points to the direction of the torque.

21. When will a body have a precession?

The torque about the axis will rotate the object about it and the torque perpendicular to the axis will turn the axis of rotation. When both exist simultaneously on a rigid body, the body will have a precession.

22. State Parallel axis theorem

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the perpendicular distance between the two axes. $I = I_C + MR^2$

23. State Perpendicular axis theorem.

The moment of inertia of a plane laminar body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two perpendicular axes lying in the plane of the body such that all the three axes are mutually perpendicular and have a common point. $I_Z = I_X + I_Y$

24. Give the scalar relation between torque and angular acceleration.

The scalar relation between the torque and angular acceleration is $\tau = I\alpha$

I = Moment of inertia of the rigid body. The torque in rotational motion is equivalent to the force in linear motion.

25. Give the relation between rotational kinetic energy and angular momentum.

The angular momentum of a rigid body is, $L = I\omega$

The rotational kinetic energy of the rigid body is, $KE = \frac{1}{2} I\omega^2$

By multiplying the numerator and denominator of the above equation with I, we get a relation between L and KE as, $KE = \frac{1}{2}\frac{I^2\omega^2}{I}$; $= \frac{1}{2}\frac{(I\omega)^2}{I}$; $KE = \frac{L^2}{2I}$

26. Obtain an expression for the power delivered by torque.

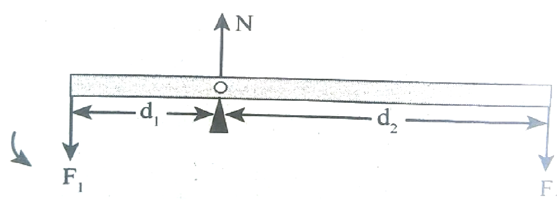
Power delivered is the work done per unit time. If we differentiate the expression for work done with respect to time, we get the instantaneous power (P).

$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} \because$ (dw = τ dθ; P = $\tau\omega$

27. What are the conditions for neutral equilibrium?

1) Linear momentum and angular momentum are zero.
2) The body remains at the same equilibrium if slightly disturbed and released.
3) The center of mass of the body does not shift higher or lower if disturbed from equilibrium.
4) Potential energy remains same even if disturbed.

28. Explain the principle of moments.



1) Consider a light rod of negligible mass which is pivoted at a point along its length. Let two parallel forces F_1 and F_2 act at the two ends at distances d_1 and d_2 from the Point of pivot and the normal reaction force N at the point of pivot as shown in Figure.
2) If the rod has to remain stationary in horizontal position, it should be in translational and rotational equilibrium. Then, both the net force and net torque must be zero.

For net force to be zero, $-F_1 + N - F_2 = 0$

$N = F_1 + F_2$

For net torque to be zero, $d_1 F_1 - d_2F_2 = 0$

$d_1 F_1 = d_2F_2$

The above equation represents the principle of moments.

29. Write the principles used in beam balance and define Mechanical Advantage.

i) This forms the principle for beam balance used for weighing goods with the condition $d_1 = d_2$; $F_1 = F_2$.

$\frac{F_1}{F_2} = \frac{d_2}{d_1}$

ii) If F_1 is the load and F_2 is our effort, we get advantage when, $d_1 < d_2$. This implies that $F_1 > F_2$. Hence, we could lift a large load with small effort. The ratio $\left(\frac{d_2}{d_1}\right)$ is called mechanical advantage of the simple lever. The pivoted point is called fulcrum.

Mechanical Advantage (MA) = $\frac{d_2}{d_1}$

30. Find the expression for radius of gyration.

1) A rotating rigid body with respect to any axis, is considered to be made up of point masses $m_1, m_2, m_3, \dots m_n$ at perpendicular distances (or positions) $r_1, r_2, r_3 \dots r_n$ respectively
2) The moment of inertia of that object can be written as,

$I = \Sigma m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2$

If we take all the n number of individual masses to be equal,

$m = m_1 = m_2 = m_3 = \dots\dots m_n$ then

$I = mr_1^2 + mr_2^2 + mr_3^2 + \dots mr_n^2$

$I = m(r_1^2 + r_2^2 + r_3^2 + \dots r_n^2)$

$= nm\left(\frac{r_1^2+r_2^2+r_3^2+\dots+r_n^2}{n}\right)$

$I = MK^2$, where, nm is the total mass M of the body and K is the radius of gyration.

$$K = \sqrt{\frac{r_1^2+r_2^2+r_3^2+\dots+r_n^2}{n}}$$

3) The expression for radius of gyration indicates that it is the root mean square (rms) distance of the particles of the body from the axis of rotation.

31. Derive an expression for work done by torque

i) Consider a rigid body rotating about a fixed axis. A point P on the body rotating about an axis perpendicular to the plane of the page. A tangential force F is applied on the body.

ii) It produces a small displacement ds on the body. The work done (dw) by the force is, $dw = F\,ds$

iii) As the distance ds, the angle of rotation $d\theta$ and radius r are related by the expression, $ds = r\,d\theta$

The expression for work done now becomes, $dw = F\,ds$; $dw = F\,r\,d\theta$

iv) The term (Fr) is the torque τ produced by the force on the body. $dw = \tau d\theta$

This expression gives the work done by the external torque τ, which acts on the body rotating about a fixed axis through an angle $d\theta$.

32. Write the comparison of translational and rotational quantities?

S. No.	Translational Motion	Rotational motion about a fixed axis
1	Displacement, x	Angular displacement, θ
2	Time, t	Time, t
3	Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
4	Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
5	Mass, m	Moment of inertia, I
6	Force, F = ma	Torque, $\tau = I\,\alpha$
7	Linear momentum, p = mv	Angular momentum, $L = I\omega$
8	Impulse, $F\,\Delta t = \Delta p$	Impulse, $\tau\,\Delta t = \Delta L$
9	Work done, w = F s	Work done, $w = \tau\,\theta$
10	Kinetic energy, KE = ½ Mv^2	Kinetic energy, KE = ½ $I\,\omega^2$
11	Power, P = F v	Power, $P = \tau\omega$

CONCEPTUAL QUESTIONS:

01. When a tree is cut, the cut is made on the side facing the direction in which the tree is required to fall. Why?

The weight of tree exerts a torque about the point where the cut is made. This cause rotation of the tree about the cut.

02. Why does a porter bend forward while carrying a sack of rice on his back?

Due to the added weight of rice sack, centre of gravity of the combined body weight and the carrying weight shifted to new position. Once he bends, the centre of gravity realigns as with the body's axis making his body balanced.

03. Why is it much easier to balance a meter scale on your finger tip than balancing on a Match stick?

Meter scale is longer and larger than a match stick. Meter scale's centre of gravity is higher but match stick has centre of gravity much lower as compared to scale. Higher the centre of gravity easier it is to balance.

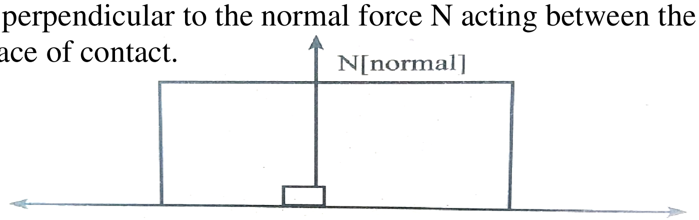
04. Two identical water bottles one empty and the other filled with water are allowed to roll down an inclined plane. Which one of them reaches the bottom first? Explain your answer.

1) Bottle filled with water rolls, faster than the empty bottle. Due to M.I. $I = mr^2$
2) When it rolls, down it possesses translational KE and rotational KE
3) For the empty bottle 100% of the mass of the bottle spins as the bottle rolls.
4) But for full bottle, much of the water in the bottle is efficiency sliding down without spinning.
5) Thus 100% of the mass of the sliding water goes into translational KE and full bottle have a greater speed.

05. A rectangle block rests on a horizontal table. A horizontal force is applied on the block at a height h above the table to move the block. Does the line of action of the normal force N exerted by the table on the block depend on h?

The line of action of normal force N exerted by the table on the block does not depend on h because the reactionary force N exerted by the table which is directed vertically upward and passes through its centre of gravity. Since the block is in equilibrium, $N = mg$.

Friction is always perpendicular to the normal force N acting between the surface. It acts tangential to the surface of contact.



06. Three identical solid spheres move down through three inclined planes A, B and C all same dimensions. A is without friction, B is undergoing pure rolling and C is rolling with slipping. Compare the kinetic energies E_A, E_B and E_C at the bottom.

The KE of A without friction $E_A = \frac{1}{2} m\,(2gh)$

The KE of B undergoes pure rolling $E_B = \frac{1}{2} m\left(\frac{2gh}{1+\frac{K^2}{R^2}}\right)$

The KE of C rolling with slipping $EC = \frac{1}{2} m^2 gh$

07. Give an example to show that the following statement is false. 'any two forces acting on a body can be combined into single force that would have same effect'.

Lifting a table from the floor by two persons pushing the car by two persons.

FIVE MARKS QUESTIONS

01. Explain the types of equilibrium with suitable examples

Translational equilibrium

1) Linear momentum is constant 2) Net force is zero

Rotational equilibrium

1) Angular momentum is constant 2) Net torque is zero

Static equilibrium

1) Linear momentum and angular momentum are zero
2) Net force and net torque are zero

Dynamic equilibrium

1) Linear momentum and angular momentum are constant
2) Net force and net torque are zero

Stable equilibrium

1) Linear momentum and angular momentum are zero
2) The body tries to come back to equilibrium if slightly disturbed and released
3) The center of mass of the body shifts slightly higher if disturbed from equilibrium
4) Potential energy of the body is minimum and it increases if disturbed

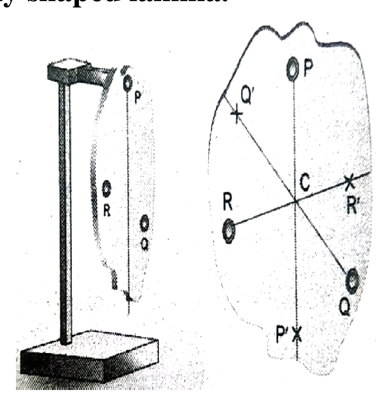
Unstable equilibrium

1) Linear momentum and angular momentum are zero
2) The body cannot come back to equilibrium if slightly disturbed and released
3) The center of mass of the body shifts slightly lower if disturbed from equilibrium
4) Potential energy of the body is not minimum and it decreases if disturbed

Neutral equilibrium

1) Linear momentum and angular momentum are zero
2) The body remains at the same equilibrium if slightly disturbed and released
3) The center of mass of the body does not shift higher or lower if disturbed from equilibrium
4) Potential energy remains same even if disturbed

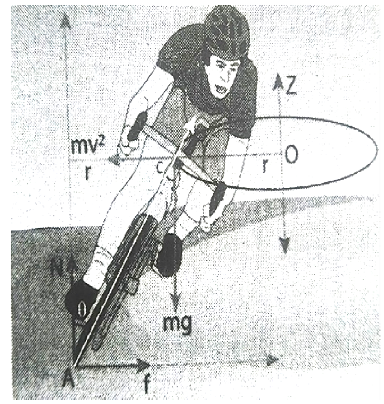
02. Explain the method to find the center of gravity of a irregularly shaped lamina.



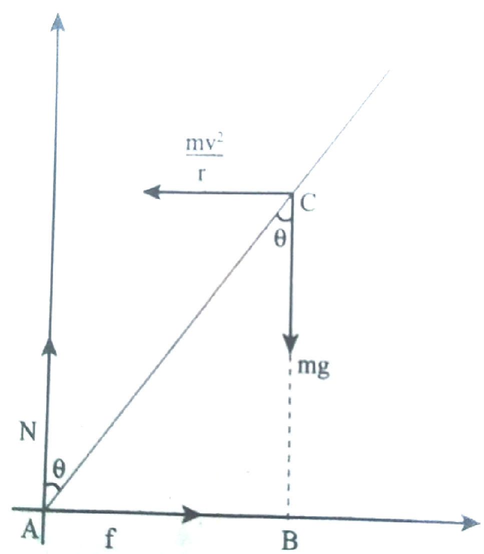
1) The center of gravity of a uniform lamina of even an irregular shape by pivoting it at various points by trial and error.
2) The lamina remains horizontal when pivoted at the point where the net gravitational force acts, which is the center of gravity
3) When a body is supported at the center of gravity, the sum of the torques acting on all the point masses of the rigid body becomes zero. Moreover the weight is compensated by the normal reaction force exerted by the pivot.
4) The body is in static equilibrium and hence it remains horizontal.

5) There is also another way to determine the center of gravity of an irregular lamina. If we suspend the lamina from different points like P, Q, R , the vertical lines PP', QQ', RR' all pass through the center of gravity.
6) Here, reaction force acting at the point of suspension and the gravitational force acting at the center of gravity cancel each other and the torques caused by them also cancel each other.

03. Explain why a cyclist bends while negotiating a curve road? Arrive at the expression for angle of bending for a given velocity.



i) Let us consider a cyclist negotiating a circular level road (not banked) of radius r with a speed v.
ii) The cycle and the cyclist are considered as one system with mass m. The center gravity of the system is C and it goes in a circle of radius r with center at O.
iii) Let us choose the line OC as X-axis and the vertical line through O as Z-axis as shown in Figure
iv) The system as a frame is rotating about Z-axis. The system is at rest in this rotating frame. To solve problems in rotating frame of reference, we have to apply a centrifugal force (pseudo force) on the system which will be $\frac{mv^2}{r}$.This force will act through the center of gravity.
v) The forces acting on the system are, (i) gravitational force (mg), (ii) normal force (N), (iii) frictional force (f) and (iv) centrifugal force $\left(\frac{mv^2}{r}\right)$
vi) As the system is in equilibrium in the rotational frame of reference, the net external force and net external torque must be zero. Let us consider all torques about the point A in Figure
vii) For rotational equilibrium, $\vec{\tau}$net = 0. The torque due to the gravitational force about point A is (mgAB) which causes a clockwise turn that is taken as negative. The torque due to the centripetal force is $\left(\frac{mv^2}{r}BC\right)$which causes an anticlockwise turn that is taken as positive.



$-\ mgAB + \frac{mv^2}{r}BC = 0$; $mg\ AB\ = \frac{mv^2}{r}BC$

From ΔABC,

$AB =\ AC\sin\theta$ and $BC = AC\cos\theta$

$mg\ AC\sin\theta = \frac{mv^2}{r}AC\cos\theta$; $\tan\theta = \frac{v^2}{rg}$

$$\boldsymbol{\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)}$$

viii) While negotiating a circular level road of radius r at velocity v, a cyclist has to bend by an angle θ from vertical given by the above expression to stay in equilibrium (i.e. to avoid a fall).

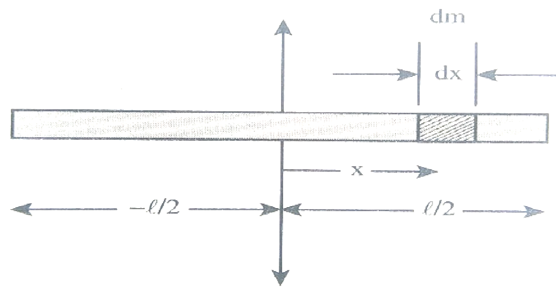
04. Derive the expression for moment of inertia of a rod about its center and perpendicular to the rod.

1) Let us consider a uniform rod of mass (M) and length (l) as shown in Figure . Let us find an expression for moment of inertia of this rod about an axis that passes through the center of mass and perpendicular to the rod.
2) First an origin is to be fixed for the coordinate system so that it coincides with the center of mass, which is also the geometric center of the rod. The rod is now along the x axis.
3) We take an infinitesimally small mass (dm) at a distance (x) from the origin. The moment of inertia (dI) of this mass (dm) about the axis is,

   $dI = (dm)x^2$

   As the mass is uniformly distributed, the mass per unit length (λ) of the rod is,

   $\lambda = \frac{M}{l}$



The (dm) mass of the infinitesimally small length as, dm = λ, dx $= \frac{M}{l}$dx.

The moment of inertia (I) of the entire rod can be found by integrating dI,

$I = \int dI = \int (dm)x^2$;

$\int \left(\frac{M}{l}dx\right) x^2$;

$I = \frac{M}{l}\int x^2 dx$

4) As the mass is distributed on either side of the origin, the limits for integration are taken from $-\frac{l}{2}$ to $\frac{l}{2}$

$I = \frac{M}{l}\int_{\frac{-l}{2}}^{\frac{l}{2}} x^2 dx \qquad = \frac{M}{l}\left[\frac{x^3}{3}\right]_{\frac{-l}{2}}^{\frac{l}{2}}$

$I = \frac{M}{l}\left[\frac{l^3}{24} - \left(-\frac{l^3}{24}\right)\right] \qquad = \frac{M}{l}\left[\frac{l^3}{24} + \frac{l^3}{24}\right]$

$I = \frac{M}{l}\left[2\left(\frac{l^3}{24}\right)\right]$;

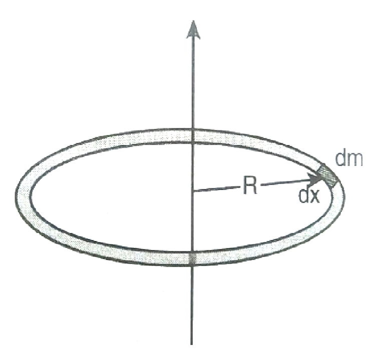
$\mathbf{I = \frac{1}{12} m\mathit{l}^2}$

05. Derive the expression for moment of inertia of a uniform ring about an axis passing through the center and perpendicular to the plane.

1) Consider a uniform ring of mass M and radius R. To find the moment of inertia of the ring about an axis passing through its center and perpendicular to the plane, let us take an infinitesimally small mass (dm) of length (dx) of the ring.
2) This (dm) is located at a distance R, which is the radius of the ring from the axis as shown in Figure

   The moment of inertia (dI) of this small mass (dm) is,

   $dI = (dm)R^2$



The length of the ring is its circumference ($2\pi R$). As the mass is uniformly distributed, the mass per unit length (λ is,

$$\lambda = \frac{M}{2\pi R}$$

The (dm) mass of the infinitesimally small length as, dm = λ, dx $= \frac{M}{2\pi R}$dx.

Now, the moment of inertia (I) of the entire ring is,

$$I = \int dI = \int (dm)R^2 ;$$

$$\int \left(\frac{M}{2\pi R} dx\right) R^2$$

$$I = \frac{MR}{2\pi} \int dx$$

To cover the entire length of the ring, the limits of integration are taken from 0 to $2\pi R$

$$I = \frac{MR}{2\pi} \int_0^{2\pi R} dx ; = \frac{MR}{2\pi} [x]_0^{2\pi R} ;$$

$$= \frac{MR}{2\pi} [2\pi R - 0]$$

$$\mathbf{I = MR^2}$$

06. Derive the expression for moment of inertia of a uniform disc about an axis passing through the center and perpendicular to the plane.

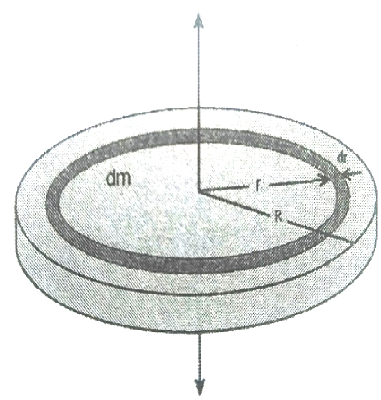
i) Consider a disc of mass M and radius R. This disc is made up of many infinitesimally small rings as shown in Figure. Consider one such ring of mass (dm) and thickness (dr) and radius (r). The moment of inertia (dI) of this small ring is, dI = (dm)r^2

ii) As the mass is uniformly distributed, the mass per unit area (σ) is, $\sigma = \frac{M}{\pi R^2}$

The mass of the infinitesimally small ring is,

$dm = \sigma 2\pi r dr = \frac{M}{\pi R^2} 2\pi r dr$

where, the term ($2\pi r$ dr) is the area of this elemental ring ($2\pi r$ is the length and dr is the thickness) $dm = \frac{2M}{R^2} r dr$. ; $dI = \frac{2M}{R^2} r^3 dr$



The moment of inertia (I) of the entire disc is, $I = \int dI$

$$I = \int_0^R \frac{2M}{R^2} r^3 dr;$$

$$= \frac{2M}{R^2} \int_0^R r^3 dr$$

$$I = \frac{2M}{R^2} \left[\frac{r^4}{4}\right]_0^R ;$$

$$= \frac{2M}{R^2} \left[\frac{R^4}{4} - 0\right]$$

$$\mathbf{I = \tfrac{1}{2} MR^2}$$

07. Discuss conservation of angular momentum with example.

1) When no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum. $\tau = \frac{dL}{dt}$ if $\tau = 0$ then , L = Constant

2) As the angular momentum is L = Iω, the conservation of angular momentum could further be written for initial and final situations as, $I_i\omega_i = I_f\omega_f$ (or) Iω = constant

3) The above equations say that if I increases ω will decrease and vice-versa to keep the angular momentum constant.
4) There are several situations where the principle of conservation of angular momentum is applicable.
5) One striking Example : The ice dancer spins slowly when the hands are stretched out and spins faster when the hands are brought close to the body. Stretching of hands away from body increases moment of inertia, thus the angular velocity decreases resulting in slower spin.
6) When the hands are brought close to the body, the moment of inertia decreases, and thus the angular velocity increases resulting in faster spin.

08. State and prove parallel axis theorem.

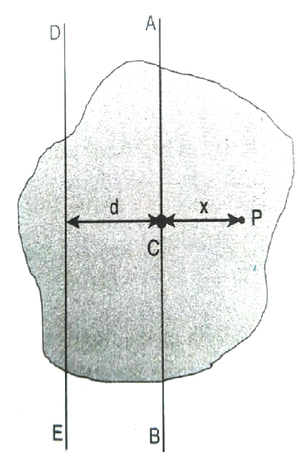
i) Parallel axis theorem states that the moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its center of mass and the product of the mass of the body and the square of the perpendicular distance between the two axes.

ii) If IC is the moment of inertia of the body of mass M about an axis passing through the center of mass, then the moment of inertia I about a parallel axis at a distance d from it is given by the relation, $I = I_C + Md^2$

iii) let us consider a rigid body as shown in Figure. Its moment of inertia about an axis AB passing through the center of mass is I_C. DE is another axis parallel to AB at a perpendicular distance d from AB. The moment of inertia of the body about DE is I. We attempt to get an expression for I in terms of I_C. For this, let us consider a point mass m on the body at position x from its center of mass.

iv) The moment of inertia of the point mass about the axis DE is, $m(x + d)^2$.

The moment of inertia I of the whole body about DE is the summation of the above expression.



$$I = \Sigma m(x + d)^2$$

This equation could further be written as,

$$I = \Sigma m(x^2 + d^2 + 2xd)$$
$$I = \Sigma(mx^2 + md^2 + 2dmx)$$
$$I = \Sigma mx^2 + \Sigma md^2 + 2d\Sigma mx$$

v) Here, Σmx^2 is the moment of inertia of the body about the center of mass. Hence,

$$I_C = \Sigma mx^2$$

The term, $\Sigma mx = 0$ because, x can take positive and negative values with respect to The axis AB. The summation (Σmx) will be zero

Thus, $I = I_C + \Sigma md^2$; $I_C + (\Sigma m)d^2$

vi) Here, Σm is the entire mass M of the object ($\Sigma m = M$)

$$I = I_C + Md^2$$

Hence, the parallel axis theorem is proved.

09. State and prove perpendicular axis theorem.

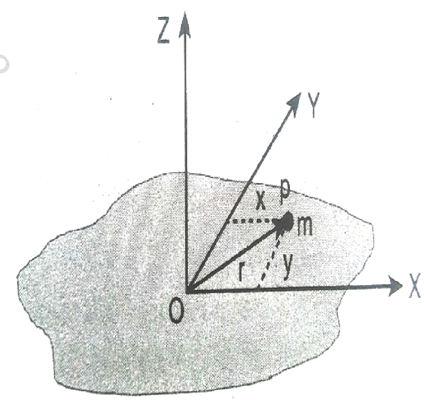
i) The theorem states that the moment of inertia of a plane laminar body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two perpendicular axes lying in the plane of the body such that all the three axes are mutually perpendicular and have a common point.

ii) Let the X and Y-axes lie in the plane and Z-axis perpendicular to the plane of the laminar object. If the moments of inertia of the body about X and Y-axes are I_X and I_Y respectively and I_Z is the moment of inertia about Z-axis, then the perpendicular axis theorem could be expressed as,

$$I_Z = I_X + I_Y$$

iii) To prove this theorem, let us consider a plane laminar object of negligible thickness on which lies the origin (O). The X and Y-axes lie on the plane and Z-axis is perpendicular to it as shown in Figure. The lamina is considered to be made up of a large number of particles of mass m. Let us choose one such particle at a point P which has coordinates (x, y) at a distance r from O.

iv) The moment of inertia of the particle about Z-axis is, mr^2 The summation of the above expression gives the moment of inertia of the entire lamina about Z-axis as,



$$I_Z = \Sigma mr^2$$

Here, $r^2 = x^2 + y^2$

Then, $I_Z = \Sigma m(x^2 + y^2)$

$$I_Z = \Sigma mx^2 + \Sigma my^2$$

In the above expression, the term Σmx^2 is the moment of inertia of the body about the Y-axis and similarly the term Σmy^2 is the moment of inertia about X-axis. Thus,

$$I_X = \Sigma my^2 \text{ and } I_Y = \Sigma mx^2$$

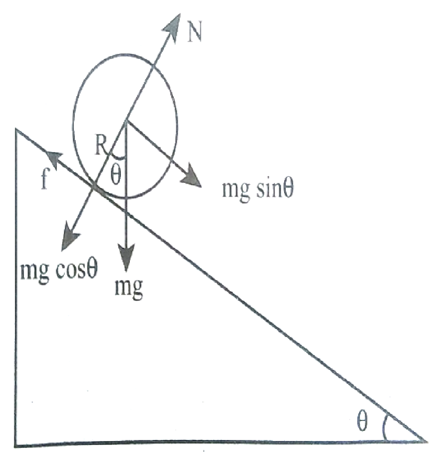
Substituting in the equation for I_z gives, $\mathbf{I_Z = I_X + I_Y}$

Thus, the perpendicular axis theorem is proved.

10. Discuss rolling on inclined plane and arrive at the expression for the acceleration.

1) Let us assume a round object of mass m and radius R is rolling down an inclined Plane without slipping as shown in Figure. There are two forces acting on the object along the inclined plane.

2) One is the component of gravitational force (mg sinθ) and the other is the static frictional force (f). The other component of gravitation force (mg cosθ) is cancelled by the normal force (N) exerted by the plane. As the motion is happening along the incline, we shall write the equation for motion from the free body diagram (*FBD*) of the object.



3) For translational motion, mg sinθ is the supporting force and f is the opposing force,

$$mg\sin\theta - f = ma \quad \text{-----(1)}$$

For rotational motion, let us take the torque with respect to the center of the object.

Then mg sinθ cannot cause torque as it passes through it but the frictional force f can set torque of Rf.

$$Rf = I\alpha$$

4) By using the relation, a = r α, and moment of inertia $I = mK^2$, we get,

$$Rf = mk^2 \frac{a}{R} \; ; \; f = ma\left(\frac{K^2}{R^2}\right)$$

Now equation (1) becomes

$$mg\sin\theta - ma\left(\frac{K^2}{R^2}\right) = ma$$

$$mg\sin\theta = ma + ma\left(\frac{K^2}{R^2}\right)$$

$$a\left(1 + \frac{K^2}{R^2}\right) = g\sin\theta$$

After rewriting it for acceleration, we get, $a = \dfrac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$

5) We can also find the expression for final velocity of the rolling object by using third equation of motion for the inclined plane. $v^2 = u^2 + 2as$. If the body starts rolling from rest, u = 0. When h is the vertical height of the incline, the length of the incline s is, $S = \dfrac{h}{\sin\theta}$; $v^2 = 2\dfrac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}\left(\dfrac{h}{\sin\theta}\right) = \dfrac{2gh}{\left(1+\frac{K^2}{R^2}\right)}$

By taking square root, $v = \sqrt{\dfrac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}$

6) The time taken for rolling down the incline could also be written from first equation of motion as, v = u + at. For the object which starts rolling from rest, u = 0. Then,

$$t = \frac{v}{a} \; ; \; t = \left(\sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}\right)\left(\frac{\left(1+\frac{K^2}{R^2}\right)}{g\sin\theta}\right)$$

$$t = \sqrt{\frac{2h\left(1+\frac{K^2}{R^2}\right)}{g\sin^2\theta}}$$

7) The equation suggests that for a given incline, the object with the least value of radius of gyration K will reach the bottom of the incline first.

11. Derive an expression for the position vector of the center of mass of particle system.

i) To find the center of mass for a collection of n point masses, say, $m_1, m_2, m_3 \ldots m_n$ we have to first choose an origin and an appropriate coordinate system as shown in Figure.

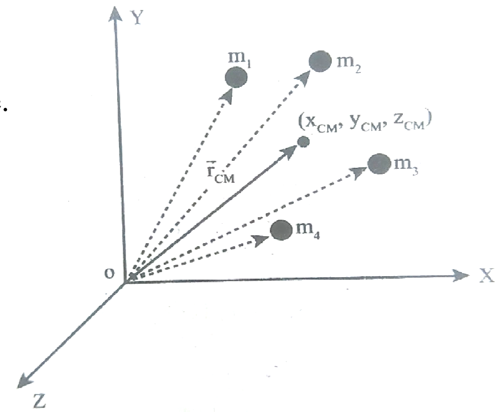
ii) Let, $x_1, x_2, x_3 \ldots x_n$ be the X-coordinates of the positions of these point masses in the X -direction from the origin. The equation for the x coordinate of the center of mass is, $x_{CM} = \dfrac{\Sigma m_i x_i}{\Sigma m_i}$

where, Σm_i is the total mass M of all the particles, ($\Sigma m_i = M$) .

$$\text{Hence, } x_{CM} = \frac{\Sigma m_i x_i}{M}$$

iii) Similarly, we can also find y and z coordinates of the center of mass for these distributed point masses as indicated in Figure.

$y_{CM} = \frac{\Sigma m_i y_i}{M}$; $z_{CM} = \frac{\Sigma m_i z_i}{M}$



iv) Hence, the position of center of mass of these point masses in a Cartesian coordinate system is (x_{CM}, y_{CM}, z_{CM}). In general, the position of center of mass can be written in a vector form as,

$\vec{r}_{CM} = \frac{\Sigma m_i \vec{r}_i}{M}$

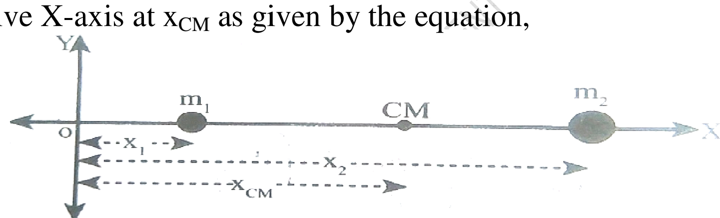
v) where, $\vec{r}_{CM} = x_{CM}\,\hat{\imath} + y_{CM}\,\hat{\jmath} + z_{CM}\,\hat{k}$ is the position vector of the center of mass and $\vec{r}_i = x_i\,\hat{\imath} + y_i\,\hat{\jmath} + z_i\,\hat{k}$ is the position vector of the distributed point mass; where, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along X, Y and Z-axes respectively.

12. Derive an expression for the center of mass of two point masses.

Let the center of mass of two point masses m_1 and m_2, which are at positions x_1 and x_2 respectively on the X-axis. For this case, we can express the position of center of mass in the following three ways based on the choice of the coordinate system.

(i) When the masses are on positive X-axis:

The origin is taken arbitrarily so that the masses m_1 and m_2 are at positions x_1 and x_2 on the positive X-axis as shown in Figure. The center of mass will also be on the positive X-axis at x_{CM} as given by the equation,



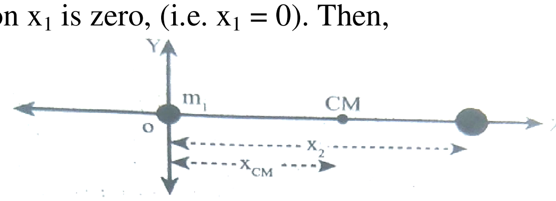
$\mathbf{x_{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

(ii) When the origin coincides with any one of the masses:

The calculation could be minimized if the origin of the coordinate system is made to coincide with any one of the masses as shown in Figure . When the origin coincides with the point mass m_1, its position x_1 is zero, (i.e. $x_1 = 0$). Then,

$x_{CM} = \frac{m_i(0) + m_2 x_2}{m_1 + m_2}$

The equation further simplifies as, $\mathbf{x_{CM}} = \frac{m_2 x_2}{m_1 + m_2}$

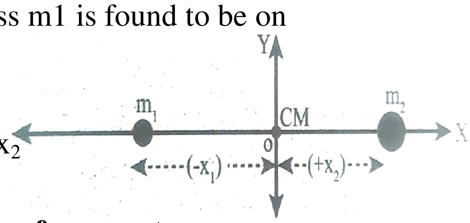


iii) When the origin coincides with the center of mass itself:

If the origin of the coordinate system is made to coincide with the center of mass, then, $x_{CM} = 0$ and the mass m1 is found to be on the negative X-axis as shown in Figure.

Hence, its position x_1 is negative, (i.e. $-x_1$).

$0 = \frac{m_1(-x_1) + m_2 x_2}{m_1 + m_2}$; $0 = m_1(-x_1) + m_2 x_2$



$\mathbf{m_1 x_1 = m_2 x_2}$

The equation given above is known as **principle of moments.**

13. State in the absence of any external force the velocity of the centre of mass remains constant.

1) When a rigid body moves, its center of mass will also move along with the body. For kinematic quantities like velocity (v_{CM}) and acceleration (a_{CM}) of the center of mass, we can differentiate the expression for position of center of mass with respect to time once and twice respectively. For simplicity, let us take the motion along X direction only.

$$\vec{v}_{CM} = \frac{d\vec{x}_{CM}}{dt} \;;= \frac{\Sigma m_i\left(\frac{d\vec{x}_i}{dt}\right)}{\Sigma m_i}$$

$$\boldsymbol{\vec{v}_{CM} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i}}$$

$$\vec{a}_{CM} = \frac{d}{dt}\left(\frac{d\vec{x}_{CM}}{dt}\right) \;; = \left(\frac{d\vec{v}_{CM}}{dt}\right) \;; = \frac{\Sigma m_i\left(\frac{d\vec{v}_i}{dt}\right)}{\Sigma m_i}$$

$$\boldsymbol{\vec{a}_{CM} = \frac{\Sigma m_i a_i}{\Sigma m_i}}$$

2) In the absence of external force, i.e. $\vec{F}_{ext} = 0$, the individual rigid bodies of a system can move or shift only due to the internal forces.

3) This will not affect the position of the center of mass. This means that the center of mass will be in a state of rest or uniform motion. Hence, $\vec{v}_{CM}$ will be zero when center of mass is at rest and constant when center of mass has uniform motion ($\vec{v}_{CM} = 0$ or $\vec{v}_{CM}$ = constant). There will be no acceleration of center of mass, ($\vec{a}_{CM} = 0$) .

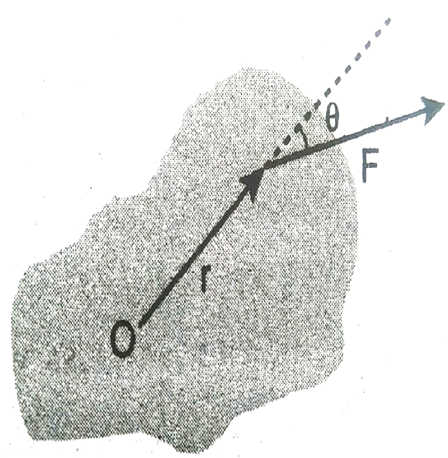
From equation $\mathbf{0} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i}$ (or) constant ; $\boldsymbol{\vec{v}_{CM}} = \frac{\Sigma m_i \vec{v}_i}{\Sigma m_i}$ **;** $\vec{a}_{CM} = 0$

4) Here, the individual particles may still move with their respective velocities and accelerations due to internal forces.

14. Define Torque and derive its expression.

1) Torque is defined as the moment of the external applied force about a point or axis of rotation. The expression for torque is, $\vec{\tau} = \vec{r} \times \vec{F}$

2) where, $\vec{r}$ is the position vector of the point where the force $\vec{F}$ is acting on the body as shown in Figure.



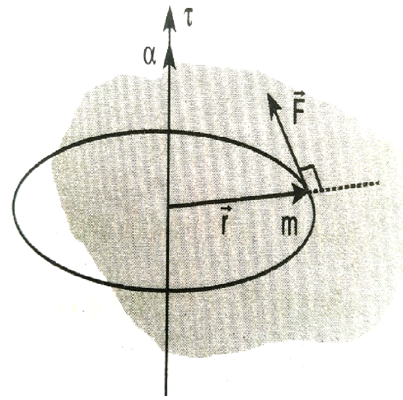
3) Here, the product of $\vec{r}$ and $\vec{F}$ is called the vector product or cross product. The vector product of two vectors results in another vector that is perpendicular to both the vectors. Hence, torque ($\vec{\tau}$) is a vector quantity.

4) Torque has a magnitude ($rF\sin\theta$) and direction perpendicular to , $\vec{r}$ and $\vec{F}$.
Its unit is N m. $\vec{\tau} = (rF\sin\theta)\hat{n}$

5) Here, θ is the angle between $\vec{r}$ and $\vec{F}$ and $\hat{n}$ is the unit vector in the direction of $\vec{\tau}$.

15. Obtain the relation between torque and angular acceleration.

i) Let us consider a rigid body rotating about a fixed axis. A point mass m in the body will execute a circular motion about a fixed axis



ii) A tangential force $\vec{F}$ acting on the point mass produces the necessary torque for this rotation. This force $\vec{F}$ is perpendicular to the position vector $\vec{r}$ of the point mass.

iii) The torque produced by the force on the point mass m about the axis can be written

as, $\tau = r\,F\sin 90 = r\,F$; $(\sin 90 = 1)$

$\tau = rma$ $(F = ma)$

$\tau = rmr\alpha$; $= mr^2\alpha$ $(a = r\,\alpha)$

$\boldsymbol{\tau = mr^2\alpha}$ **----------------(1)**

iv) Hence, the torque of the force acting on the point mass produces an angular acceleration (α) in the point mass about the axis of rotation. In vector notation,

$\vec{\tau} = (mr^2)\,\vec{\alpha}$ --------------(2)

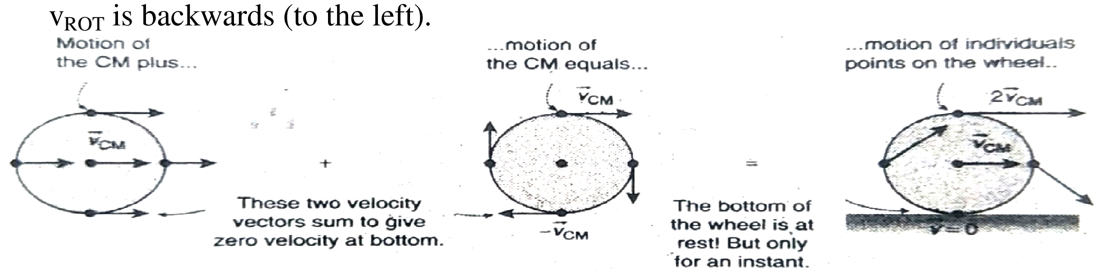
v) The directions of τ and α are along the axis of rotation. If the direction of τ is in the direction of α, it produces angular acceleration. On the other hand if, τ is opposite to α, angular deceleration or retardation is produced on the point mass.

vi) The term mr^2 in equations 1 and 2 is called moment of inertia (I) of the point mass. A rigid body is made up of many such point masses. Hence, the moment of inertia of a rigid body is the sum of moments of inertia of all such individual point masses that constitute the body $(I = \Sigma m_i r_i^2)$ .Hence, torque for the rigid body can be written as,

$\vec{\tau} = (\Sigma m_i r_i^2)\,\vec{\alpha}$; $\boldsymbol{\vec{\tau} = I\,\vec{\alpha}}$

16. Discuss the pure rolling and find the condition for rolling without slipping and sliding.

1) In pure rolling, the point of the rolling object which comes in contact with the surface is at momentary rest.
2) This is the case with every point that is on the edge of the rolling object. As the rolling proceeds, all the points on the edge, one by one come in contact with the surface; remain at momentary rest at the time of contact and then take the path of the cycloid as already mentioned. Hence, we can consider the pure rolling in two different ways.
   (i) The combination of translational motion and rotational motion about the center of mass. (or)
   (ii) The momentary rotational motion about the point of contact.
3) As the point of contact is at momentary rest in pure rolling, its resultant velocity v is zero (v = 0). For example, at the point of contact, v_{TRANS} is forward (to right) and v_{ROT} is backwards (to the left).



4) That implies that, v_{TRANS} and v_{ROT} are equal in magnitude and opposite in direction ($v = v_{TRANS} - v_{ROT} = 0$). Hence, we conclude that in pure rolling, for all the points on the edge, the magnitudes of v_{TRANS} and v_{ROT} are equal ($v_{TRANS} = v_{ROT}$).
As $v_{TRANS} = v_{CM}$ and $v_{ROT} = R\omega$, in pure rolling we have, $v_{CM} = R\omega$
5) For the topmost point, the two velocities v_{TRANS} and v_{ROT} are equal in magnitude and in the same direction (to the right). Thus, the resultant velocity v is the sum of these two velocities, $v = v_{TRANS} + v_{ROT}$. In other form, $v = 2\,v_{CM}$

Sliding

i) Sliding is the case when $v_{CM} > R\omega$ (or $v_{TRANS} > v_{ROT}$). The translation is more than the rotation. This kind of motion happens when sudden break is applied in a moving vehicles, or when the vehicle enters into a slippery road. In this case, the point of contact has more of v_{TRANS} than v_{ROT}.
ii) Hence, it has a resultant velocity v in the forward direction. The kinetic frictional force (f_k) opposes the relative motion. Hence, it acts in the opposite direction of the relative velocity.
iii) This frictional force reduces the translational velocity and increases the rotational velocity till they become equal and the object sets on pure rolling. Sliding is also referred as forward slipping.

Slipping

1) Slipping is the case when $v_{CM} < R\omega$ (or $v_{TRANS} < v_{ROT}$). The rotation is more than the translation. This kind of motion happens when we suddenly start the vehicle from rest or the vehicle is stuck in mud.
2) In this case, the point of contact has more of v_{ROT} than v_{TRANS}. It has a resultant velocity v in the backward direction.
3) The kinetic frictional force (fk) opposes the relative motion. Hence it acts in the opposite direction of the relative velocity.
4) This frictional force reduces the rotational velocity and increases the translational velocity till they become equal and the object sets pure rolling. Slipping is sometimes empahasised as backward slipping.

17. Write an expression for the kinetic energy of a body in pure rolling.

1) The total kinetic energy (KE) as the sum of kinetic energy due to translational motion (KE_{TRANS}) and kinetic energy due to rotational motion (KE_{ROT}).

$$KE = KE_{TRANS} + KE_{ROT}$$

2) If the mass of the rolling object is M, the velocity of center of mass is v_{CM}, its moment of inertia about center of mass is I_{CM} and angular velocity is ω, then

$$KE = ½\, Mv^2_{CM} + ½\, I_{CM}\,\omega^2$$

With center of mass as reference:

3) The moment of inertia (I_{CM}) of a rolling object about the center of mass is, $I_{CM} = MK^2$ and $v_{CM} = R\omega$. Here, K is radius of gyration.

$$KE = ½\, Mv^2_{CM} + ½\,(MK^2)\,\frac{v^2_{CM}}{R^2}$$

$$KE = ½\, Mv^2_{CM} + ½\, Mv^2_{CM}\left(\frac{K^2}{R^2}\right)$$

$$KE = ½\, Mv^2_{CM}\left(1 + \frac{K^2}{R^2}\right)$$

With point of contact as reference:

4) We can also arrive at the same expression by taking the momentary rotation happening with respect to the point of contact (another approach to rolling). If we take the point of contact as O, then,

$$KE = ½\, I_o\, \omega^2$$

5) Here, I_o is the moment of inertia of the object about the point of contact. By parallel axis theorem, $I_o = I_{CM} + MR^2$. Further we can write, $I_o = MK^2 + MR^2$.

With $v_{CM} = R\omega$ or $\omega = \frac{v_{CM}}{R}$

$$KE = ½\,(MK^2 + MR^2)\,\frac{v^2{}_{CM}}{R^2}$$

$$KE = ½\, Mv^2{}_{CM}\left(1 + \frac{K^2}{R^2}\right)$$

6) KE in pure rolling can be determined by any one of the following two cases. (i) The combination of translational motion and rotational motion about the center of mass. (or) (ii) The momentary rotational motion about the point of contact.

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FEAR has two meanings:

Forget **E**verything **A**nd **R**un **or**

Face **E**verything **A**nd **R**ise.

The Choice is yours! Do not be afraid!

"A paper flying in air is due to its luck but
a bird is flying due to its effort. So if luck is not with you,
efforts are always there to support you".

"Hard work is like stairs and luck is like lift.
Sometimes lift may fail but stairs will always take you to the top.
Have a successful life"!

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