Chapter 9 Differential Equations

EXERCISE 9.1

Question 1:

Determine order and degree (if defined) of differential equation $\frac{d^4y}{dx^4}+\sin(y''')=0$.

Solution:

$\Rightarrow \frac{d^4y}{dx^4}+\sin(y''')=0$

$\Rightarrow y''''+\sin(y''')=0$

Highest order derivative in the differential equation is y''''. Its order is four.
Differential equation is not a polynomial equation in its derivatives. Its degree is not defined.

Question 2:

Determine order and degree (if defined) of differential equation $y'+5y=0$.

Solution:

$y'+5y=0$

Highest order derivative in the differential equation is y'. Its order in one.

It is a polynomial equation in y'. Highest power y' is 1. Its degree is one.

Question 3:

Determine order and degree (if defined) of differential equation $\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$.

Solution:

$\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$

Highest order derivative in the given differential equation is $\frac{d^2s}{dt^2}$. Its order is two.

It is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$.

The power $\frac{d^2s}{dt^2}$ is 1. Degree is one.

Question 4:

Determine order and degree (if defined) of differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$.

Solution:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

Highest order derivative in the given differential equation is $\frac{d^2y}{dx^2}$. Order is 2.
Given differential equation is not a polynomial equation in its derivatives. Degree is not defined.

Question 5:

Determine order and degree (if defined) of differential equation $\left(\frac{d^2y}{dx^2}\right)^2 = \cos 3x + \sin 3x$.

Solution:

$$\left(\frac{d^2y}{dx^2}\right)^2 = \cos 3x + \sin 3x$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - \cos 3x - \sin 3x = 0$$

Highest order derivative in the given differential equation is $\frac{d^2y}{dx^2}$. Its order is two.

It is a polynomial equation in $\frac{d^2y}{dx^2}$ and the power is 1. Its degree is 1.

Question 6:

Determine order and degree (if defined) of differential equation $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$.

Solution:

$$(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$$

Highest order derivative present in the differential equation is y'''. Its order is three.
Given differential equation is a polynomial equation in y''', y'' and y'.
Highest power raised to y''' is 2. Its degree is 2.

Question 7:

Determine order and degree (if defined) of differential equation $y''' + 2y'' + y' = 0$.

Solution:

$y''' + 2y'' + y' = 0$

Highest order derivative present in the differential equation is y'''. Its order is 3.

It is a polynomial equation in y''', y'' and y'. The highest power y''' is 1. Its degree is 1.

Question 8:

Determine order and degree (if defined) of differential equation $y' + y = e'$.

Solution:

$y' + y = e'$

$\Rightarrow y' + y - e' = 0$

Highest order derivative present in the differential equation is y'. Its order is one.

Given differential equation is a polynomial equation in y' and the highest power is one. Its degree is one.

Question 9:

Determine order and degree (if defined) of differential equation $y'' + (y')^2 + 2y = 0$.

Solution:

$y'' + (y')^2 + 2y = 0$

Highest order derivative present in the differential equation is y''. Its order is two.

Given differential equation is a polynomial equation in y'' and y', the highest power y'' is one. Its degree is one.

Question 10:

Determine order and degree (if defined) of differential equation $y'' + 2y' + \sin y = 0$.

Solution:

$y'' + 2y' + \sin y = 0$

Highest order derivative present in the differential equation is y''. Its order is two.

This is a polynomial equation in y'' and y' the highest power y'' is one. Its degree is one.

Question 11:

The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$ is

(A) 3 (B) 2 (C) 1 (D) not defined

Solution:

$$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$$

Differential equation is not a polynomial equation in its derivatives. Its degree is not defined.
Thus, the Correct option is D.

Question 12:

The order of the differential equation $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$.

(A) 2 (B) 1 (C) 0 (D) not defined

Solution:

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

Highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Its order is two.
Thus, the correct option is A.

EXERCISE 9.2

Question 1:
Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y = e^x + 1 \quad : \quad y'' - y' = 0$

Solution:

$$y = e^x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x + 1\right)$$

$$\Rightarrow y' = e^x \qquad \ldots(1)$$

$$\frac{d}{dx}(y') = \frac{d}{dx}\left(e^x\right)$$

$$\Rightarrow y'' = e^x \qquad \ldots(2)$$

From (1) and (2)

$$y'' - y' = e^x - e^x$$
$$= 0$$

Thus, the given function is the solution of corresponding differential equation.

Question 2:
Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y = x^2 + 2x + c \quad : \quad y' - 2x - 2 = 0$

Solution:

$$y = x^2 + 2x + c$$

$$y' = \frac{d}{dx}\left(x^2 + 2x + c\right)$$

$$\Rightarrow y' = 2x + 2$$

Therefore,

$$y' - 2x - 2 = 2x + 2 - 2x - 2$$
$$= 0$$

Thus, the given function is the solution of the differential equation.

Question 3:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y=\cos x+C \quad : \quad y'+\sin x=0$

Solution:

$$y=\cos x+C$$

$$y'=\frac{d}{dx}(\cos x+C)$$

$$\Rightarrow y'=-\sin x$$

Therefore,

$$\begin{aligned}y'+\sin x&=-\sin x+\sin x\\&=0\end{aligned}$$

Thus, the given function is the solution of the differential equation.

Question 4:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y=\sqrt{1+x^2} \quad : \quad y'=\frac{xy}{1+x^2}$

Solution:

$$y=\sqrt{1+x^2}$$

$$\begin{aligned}y'&=\frac{d}{dx}\left(\sqrt{1+x^2}\right)\\&=\frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)\\&=\frac{2x}{2\sqrt{1+x^2}}\\&=\frac{x}{2\sqrt{1+x^2}}\\&=\frac{x}{(1+x^2)}\times\sqrt{1+x^2}\\&=\frac{x}{(1+x^2)}.y\\&=\frac{xy}{1+x^2}\end{aligned}$$

Thus, the given function is the solution of the differential equation.

Question 5:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y = Ax \quad : \quad xy' = y\left(x \neq 0\right)$

Solution:

$$y = Ax$$

$$y' = \frac{d}{dx}\left(Ax\right)$$

$$= A$$

Therefore,

$$xy' = xA$$

$$= Ax$$

$$= y$$

Thus, the given function is the solution of the differential equation.

Question 6:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y = x\sin x \quad : \quad xy' = y + x\sqrt{x^2 - y^2}\ \left(x \neq 0\ and\ x > y\right)\ or\ x < -y$

Solution:

$$y = x\sin x$$

$$y' = \frac{d}{dx}\left(x\sin x\right)$$

$$= \sin x.\frac{d}{dx}\left(x\right) + x.\frac{d}{dx}\left(\sin x\right)$$

$$= \sin x + x\cos x$$

Therefore,

$$xy' = x\left(\sin x + x\cos x\right)$$

$$= x\sin x + x^2\cos x$$

$$= y + x^2.\sqrt{1 - \sin^2 x}$$

$$= y + x^2\sqrt{1 - \left(\frac{y}{x}\right)^2}$$

$$= y + x\sqrt{x^2 - y^2}$$

Thus, the given function is the solution of the differential equation.

Question 7:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$$xy = \log y + C \quad : \quad y' = \frac{y^2}{1 - xy} \left(xy \neq 1 \right)$$

Solution:

$$xy = \log y + C$$

$$\Rightarrow \frac{d}{dx}\left(xy \right) = \frac{d}{dx}\left(\log y \right)$$

$$\Rightarrow y\frac{d}{dx}\left(x \right) + x.\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow y + xy' = \frac{1}{y}.y'$$

$$\Rightarrow y^2 + xyy' = y'$$

$$\Rightarrow \left(xy - 1 \right) y' = -y^2$$

$$\Rightarrow y' = \frac{y^2}{1 - xy}$$

Thus, the given function is the solution of the differential equation.

Question 8:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$$y - \cos y = x \quad : \quad \left(y \sin y + \cos y + x \right) y' = y$$

Solution:

$$y - \cos y = x$$

$$\Rightarrow \frac{dy}{dx} - \frac{d}{dx}\left(\cos y \right) = \frac{d}{dx}(x)$$

$$\Rightarrow y' - \left(-\sin y \right).y' = 1$$

$$\Rightarrow y'\left(1 + \sin y \right) = 1$$

$$\Rightarrow y' = \frac{1}{1 + \sin y}$$

Therefore,

$$\left(y\sin y+\cos y+x\right)y'=\left(y\sin y+\cos y+y-\cos y\right)\times\frac{1}{1+\sin y}$$
$$=y\left(1+\sin y\right).\frac{1}{1+\sin y}$$
$$=y$$

Thus, the given function is the solution of the differential equation.

Question 9:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$x+y=\tan^{-1}y \quad : \quad y^2y'+y^2+1=0$

Solution:

$$x+y=\tan^{-1}y$$
$$\Rightarrow\frac{d}{dx}\left(x+y\right)=\frac{d}{dx}\left(\tan^{-1}y\right)$$
$$\Rightarrow 1+y'=\left[\frac{1}{1+y^2}\right]y'$$
$$\Rightarrow y'\left[\frac{1}{1+y^2}-1\right]=1$$
$$\Rightarrow y'\left[\frac{1-\left(1+y^2\right)}{1+y^2}\right]=1$$
$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right]=1$$
$$\Rightarrow y'=\frac{-\left(1+y^2\right)}{y^2}$$

Therefore,

$$y^2y'+y^2+1=y^2\left[\frac{-(1+y^2)}{y^2}\right]+y^2+1$$
$$=-1-y^2+y^2+1$$
$$=0$$

Thus, the given function is the solution of the differential equation.

Question 10:

Verify that the given function (explicit or implicit) is a solution of the corresponding differential equation:

$y=\sqrt{a^2-x^2}\ x\in(-a,a) \quad : \quad x+y\dfrac{dy}{dx}=0\,(y\neq 0)$

Solution:

$$y=\sqrt{a^2-x^2}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}.\frac{d}{dx}\left(a^2-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}.(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}}$$

Therefore,

$$x+y\frac{dy}{dx}=x+\sqrt{a^2-x^2}\times\frac{-x}{\sqrt{a^2-x^2}}$$

$$=x-x$$

$$=0$$

Thus, the given function is the solution of the differential equation.

Question 11:

The numbers of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0 (B) 2 (C) 3 (D) 4

Solution:

We know that the number of constants in the general solution of a differential equation of order *n* is equal to its order.

The number of constants in general equation of fourth order differential equation is 4.

Thus, the correct option is D.

Question 12:

The numbers of arbitrary constants in the particular solution of a differential equation of third order are:

(A) 3 (B) 2 (C) 1 (D) 0

Solution:

In a particular solution of a differential equation, there are no arbitrary constants.

Thus, the correct option is D.

EXERCISE 9.3

Question 1:

Form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

$$\frac{x}{a}+\frac{y}{b}=1$$

Solution:

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$$

$$\Rightarrow 0+\frac{1}{b}y''=0$$

$$\Rightarrow \frac{1}{b}y''=0$$

$$\Rightarrow y''=0$$

Thus, the differential equation of the given curve is $y''=0$.

Question 2:

Form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

$$y^2=a\left(b^2-x^2\right)$$

Solution:

$$y^2=a\left(b^2-x^2\right)$$

$$\Rightarrow 2y\frac{dy}{dx}=a(-2x)$$

$$\Rightarrow 2yy'=-2ax$$

$$\Rightarrow yy'=-ax$$

$$\Rightarrow y'.y'+yy''=-a$$

$$\Rightarrow \frac{(y')^2+yy''}{yy'}=\frac{-a}{-ax}$$

$$\Rightarrow xyy''+x(y')^2-yy'=0$$

Thus, the differential equation of the given curve is $xyy''+x(y')^2-yy'=0$.

Question 3:

Form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

$y = ae^{3x} + be^{-2x}$

Solution:

$$y = ae^{3x} + be^{-2x}$$
$$y' = 3ae^{3x} - 2be^{-2x} \qquad \ldots(1)$$
$$y'' = 9ae^{3x} + 4be^{-2x} \qquad \ldots(2)$$

Subtracting (1) from (2)

$$y'' - y' = 9ae^{3x} + 4be^{-2x} - 3ae^{3x} + 2be^{-2x}$$
$$y'' - y' = 6ae^{3x} + 6be^{-2x}$$
$$y'' - y' = 6\left(ae^{3x} + be^{-2x}\right)$$
$$y'' - y' = 6y \qquad \left(\because y = ae^{3x} + be^{-2x}\right)$$
$$y'' - y' - 6y = 0$$

Thus, the differential equation of the given curve is $y'' - y' - 6y = 0$.

Question 4:

Form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

$y = e^{2x}(a + bx)$

Solution:

$$y = e^{2x}(a + bx)$$
$$y' = 2e^{2x}(a + bx) + e^{2x}b$$
$$\Rightarrow y' = e^{2x}(2a + 2bx + b)$$
$$y' - 2y = e^{2x}(2a + 2bx + b) - e^{2x}(2a + 2bx)$$
$$\Rightarrow y' - 2y = be^{2x} \qquad \ldots(1)$$
$$\Rightarrow y'' - 2y' = 2be^{2x} \qquad \ldots(2)$$

Dividing (2) by (1)

$$\frac{y'' - 2y'}{y' - 2y} = 2$$
$$\Rightarrow y'' - 2y' = 2y' - 4y$$
$$\Rightarrow y'' - 4y' + 4y = 0$$

Thus, the differential equation of the given curve is $y''-4y'+4y=0$.

Question 5:
Form a differential equation representing the given family of curves by eliminating arbitrary constants a and b.

$y=e^{3x}\left(a\cos x+b\sin x\right)$

Solution:
$$y=e^{x}\left(a\cos x+b\sin x\right)$$
$$y'=e^{x}\left(a\cos x+b\sin x\right)+e^{x}\left(-a\sin x+b\cos x\right)$$
$$\Rightarrow y'=e^{x}\left[(a+b)\cos x-(a-b)\sin x\right]$$
$$\Rightarrow y''=e^{x}\left[(a+b)\cos x-(a-b)\sin x\right]+e^{x}\left[-(a+b)\sin x-(a-b)\cos x\right]$$
$$y''=e^{x}\left[2b\cos x-2a\sin x\right]$$
$$y''=2e^{x}\left(b\cos x-a\sin x\right)$$
$$\Rightarrow \frac{y''}{2}=e^{x}\left(b\cos x-a\sin x\right)$$
$$\frac{y''}{2}+y=e^{x}\left(b\cos x-a\sin x\right)+e^{x}\left(a\cos x+b\sin x\right)$$
$$y+\frac{y''}{2}=e^{x}\left[(a+b)\cos x-(a-b)\sin x\right]$$
$$\Rightarrow y+\frac{y''}{2}=y'$$
$$\Rightarrow 2y+y''=2y'$$
$$\Rightarrow y''-2y'+2y=0$$

Thus, the differential equation of the given curve is $y''-2y'+2y=0$.

Question 6:
Form the differential equation of the family of circles touching the y-axis at origin.

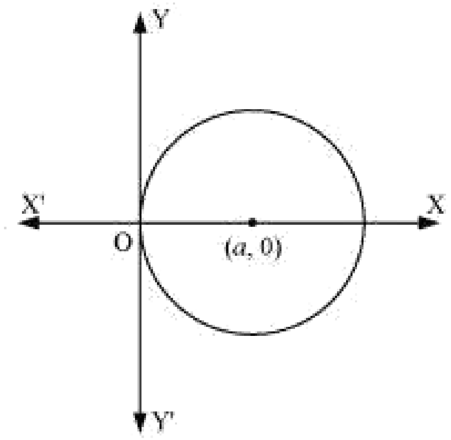
Solution:
Centre of circle touching the y-axis at origin lies on the x-axis.

Let $(a,0)$ be the centre of the circle.

Since it touches the y-axis at origin, its radius is a.

Equation of the circle with centre $(a,0)$ and radius a is

$$(x-a)^2 + y^2 = a^2$$
$$\Rightarrow x^2 + y^2 = 2ax \qquad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$2x + 2yy' = 2a$$
$$\Rightarrow x + yy' = a$$

Putting the value of a in equation (1)

$$\Rightarrow x^2 + y^2 = 2(x + yy')x$$
$$\Rightarrow 2xyy' + x^2 = y^2$$

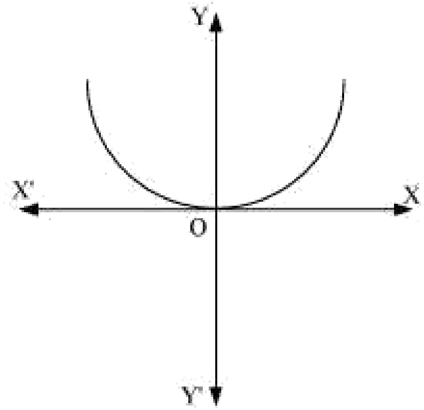
Question 7:

Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

Solution:

The equation of the parabola having the vertex at origin and the axis along the positive y-axis is

$$x^2 = 4ay \qquad \ldots(1).$$



Differentiating equation (1) with respect to x, we get:

$2x = 4ay'$

$\Rightarrow \frac{2x}{x^2} = \frac{4ay'}{4ay}$

$\Rightarrow \frac{2}{x} = \frac{y'}{y}$

$\Rightarrow xy' = 2y$

$\Rightarrow xy' - 2y = 0$

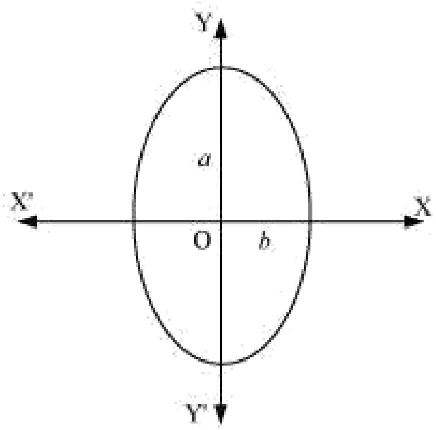
Question 8:

Form the differential equation of the family of ellipses having foci o*n*-axis and centre at origin.

Solution:

The equation of the family of ellipses having foci on the *y*-axis and the centre at origin is as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$\frac{2x}{b^2}+\frac{2yy'}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad ....(2)$$

Differentiating equation (2) with respect to x, we get:

$$\frac{1}{b^2}+\frac{y'.y'+y.y''}{a^2}=0$$

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}\left(y'^2+yy''\right)=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}\left(y'^2+yy''\right)$$

Substituting this value in equation (2), we get:

$$\Rightarrow x\left[-\frac{1}{a^2}\left(y'^2+yy''\right)\right]+\frac{yy'}{a^2}=0$$

$$\Rightarrow -xy'^2-xyy''+yy'=0$$

$$\Rightarrow xyy''+xy'^2-yy'=0$$

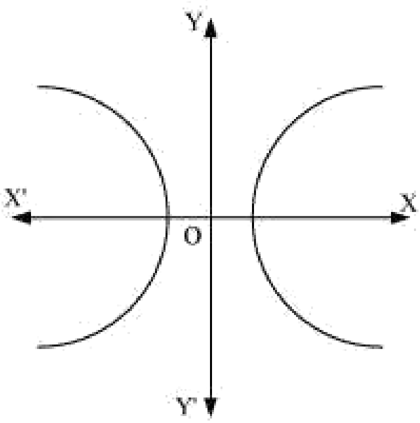
Question 9:

Form the differential equation of the family of hyperbolas having foci on *x*-axis and centre at origin.

Solution:

The equation of the family of hyperbolas with the centre at origin and foci along the *x*-axis is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad ...(1)$$



Differentiating equation (1) with respect to x, we get:

$$\frac{2x}{b^2}+\frac{2yy'}{a^2}=0$$

$$\Rightarrow \frac{x}{b^2}+\frac{yy'}{a^2}=0 \qquad \ldots(2)$$

Differentiating equation (2) with respect to x, we get:

$$\frac{1}{b^2}+\frac{y'.y'+y.y''}{a^2}=0$$

$$\frac{1}{b^2}=-\frac{y'.y'+y.y''}{a^2}$$

Substituting this value in equation (2), we get:

$$x\left[-\frac{y'.y'+y.y''}{a^2}\right]+\frac{yy'}{a^2}=0$$

$$\frac{x}{a^2}[y'.y'+y.y'']-\frac{yy'}{a^2}=0$$

$$\Rightarrow xy'^2+xyy''-yy'=0$$

$$\Rightarrow xyy''+xy'^2-yy'=0$$

Question 10:

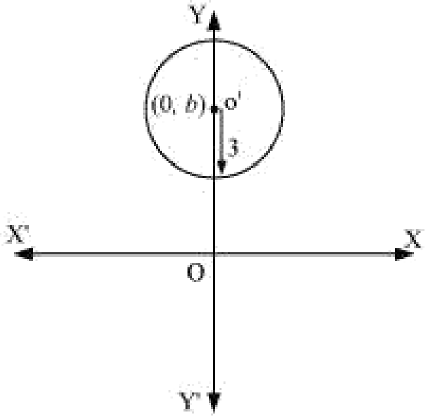
Form the differential equation of the family of circles having centre on *y*-axis and radius 3 units.

Solution:

Let centre of the circle on *y*-axis be $(0,b)$.

$$x^2+(y-b)^2=3^2$$

$$\Rightarrow x^2+(y-b)^2=9 \qquad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$2x+2(y-b)y'=0$$
$$\Rightarrow (y-b).y'=-x$$
$$\Rightarrow (y-b)=\frac{-x}{y'}$$

Substituting this value in equation (1), we get:

$$x^2+\left(\frac{-x}{y'}\right)^2=9$$
$$\Rightarrow x^2\left[1+\frac{1}{(y')^2}\right]=9$$
$$\Rightarrow x^2\left[(y')^2+1\right]=9(y')^2$$
$$\Rightarrow (x^2-9)(y')^2+x^2=0$$

Question 11:

Which of the following differential equations has $y=c_1e^x+c_2e^{-x}$ as the general solution?

(A) $\frac{d^2y}{dx^2}+y=0$ (B) $\frac{d^2y}{dx^2}+y=0$ (C) $\frac{d^2y}{dx^2}+1=0$ (D) $\frac{d^2y}{dx^2}-1=0$

Solution:

$$y = c_1 e^x + c_2 e^{-x}$$
$$\Rightarrow \frac{dy}{dx} = c_1 e^x - c_2 e^{-x}$$
$$\Rightarrow \frac{d^2 y}{dx^2} = c_1 e^x + c_2 e^{-x}$$
$$\Rightarrow \frac{d^2 y}{dx^2} = y$$
$$\Rightarrow \frac{d^2 y}{dx^2} - y = 0$$

Thus, the correct option is B.

Question 12:

Which of the following differential equations has $y = x$ as one of its particular solution?

(A) $\frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$

(B) $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + xy = x$

(C) $\frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

(D) $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + xy = 0$

Solution:

$$\frac{dy}{dx} = 1$$
$$\frac{d^2 y}{dx^2} = 0$$

Therefore,

$$\begin{aligned} \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy &= 0 - x^2.1 + x.x \\ &= -x^2 + x^2 \\ &= 0 \end{aligned}$$

Thus, the correct option is C.

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

Question 1:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

Solution:

$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan^2\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=\left(\sec^2\frac{x}{2}-1\right)$$

$$\Rightarrow dy=\left(\sec^2\frac{x}{2}-1\right)dx$$

Integrating both sides, we get:

$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$\Rightarrow y=\int\sec^2\frac{x}{2}dx-\int dx$$

$$\Rightarrow y=2\tan\frac{x}{2}-x+C$$

Question 2:

$$\frac{dy}{dx}=\sqrt{4-y^2}\ (-2<y<2)$$

Solution:

$$\frac{dy}{dx}=\sqrt{4-y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$

Integrating both sides, we get:

$$\int \frac{dy}{\sqrt{4-y^2}} = \int dx$$
$$\Rightarrow \sin^{-1}\frac{y}{2} = x + C$$
$$\Rightarrow \frac{y}{2} = \sin(x+C)$$
$$\Rightarrow y = 2\sin(x+C)$$

Question 3:

$$\frac{dy}{dx} + y = 1 (y \neq 1)$$

Solution:

$$\frac{dy}{dx} + y = 1 (y \neq 1)$$
$$\Rightarrow dy + y dx = dx$$
$$\Rightarrow dy = (1-y)dx$$
$$\Rightarrow \frac{dy}{1-y} = dx$$

Integrating both sides, we get:

$$\int \frac{dy}{1-y} = \int dx$$
$$\Rightarrow -\log(y-1) = x + \log C$$
$$\Rightarrow -\log C - \log(y-1) = x$$
$$\Rightarrow -[\log C + \log(y-1)] = x$$
$$\Rightarrow \log C(y-1) = -x$$
$$\Rightarrow C(y-1) = e^{-x}$$
$$\Rightarrow y = 1 + \frac{1}{C}e^{-x}$$
$$\Rightarrow y = 1 + Ae^{-x} \qquad \left(\text{where } A = \frac{1}{C}\right)$$

Question 4:

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

Solution:

$$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$$

Dividing both sides by $\tan x \tan y$

$$\Rightarrow \frac{\sec^2 x \tan y dx + \sec^2 y \tan x dy}{\tan x \tan y} = \frac{0}{\tan x \tan y}$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides, we get:

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy \qquad \ldots(1)$$

Let $\tan x = t$

$$\Rightarrow \frac{d}{dx}(\tan x) = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x = \frac{dt}{dx}$$

$$\Rightarrow \sec^2 x dx = dt$$

Now,

$$\int \frac{\sec^2 x}{\tan x} dx = \int \frac{1}{t} dt$$

$$= \log t$$

$$= \log(\tan x) \qquad \ldots(2)$$

Similarly,

$$\int \frac{\sec^2 y}{\tan y} dy = \log(\tan y) \qquad \ldots(3)$$

Using (1), (2) and (3)

$$\Rightarrow \log(\tan x) = -\log(\tan y) + \log C$$

$$\Rightarrow \log(\tan x) = \log\left(\frac{C}{\tan y}\right)$$

$$\Rightarrow \tan x = \frac{C}{\tan y}$$

$$\Rightarrow \tan x \tan y = C$$

Question 5:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

Solution:

$$\left(e^{x}+e^{-x}\right)dy-\left(e^{x}-e^{-x}\right)dx=0$$
$$\Rightarrow\left(e^{x}+e^{-x}\right)dy=\left(e^{x}-e^{-x}\right)dx$$
$$\Rightarrow dy=\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx$$

Integrating both sides, we get:

$$\int dy=\int\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]dx \qquad \ldots(1)$$

Let $\left(e^{x}+e^{-x}\right)=t$

$$\Rightarrow\frac{d}{dx}\left(e^{x}+e^{-x}\right)=\frac{dt}{dx}$$
$$\Rightarrow\left(e^{x}-e^{-x}\right)dx=dt$$

Putting these values in equation (1), we get:

$$\int dy=\int\frac{1}{t}dt+C$$
$$\Rightarrow y=\log(t)+C$$
$$\Rightarrow\log\left(e^{x}+e^{-x}\right)+C$$

Question 6:

$$\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$$

Solution:

$$\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$$
$$\Rightarrow\frac{dy}{1+y^{2}}=\left(1+x^{2}\right)dx$$

Integrating both sides, we get:

$$\int\frac{dy}{1+y^{2}}=\int\left(1+x^{2}\right)dx$$
$$\Rightarrow\tan^{-1}y=\int dx+\int x^{2}dx$$
$$\Rightarrow\tan^{-1}y=x+\frac{x^{3}}{x}+C$$

Question 7:

$y\log y dx - x dy = 0$

Solution:

$y\log y dx - x dy = 0$

$\Rightarrow y\log y dx = x dy$

$\Rightarrow \dfrac{dy}{y\log y} = \dfrac{dx}{x}$

Integrating both sides, we get:

$$\int \frac{dy}{y\log y} = \int \frac{dx}{x} \qquad \ldots(1)$$

Let $\log y = t$

$\Rightarrow \dfrac{d}{dy}(\log y) = \dfrac{dt}{dy}$

$\Rightarrow \dfrac{1}{y} = \dfrac{dt}{dy}$

$\Rightarrow \dfrac{1}{y} dy = dt$

Putting these values in equation (1), we get:

$\int \dfrac{dt}{t} = \int \dfrac{dx}{x}$

$\Rightarrow \log t = \log x + \log C$

$\Rightarrow \log(\log y) = \log Cx$

$\Rightarrow \log y = Cx$

$\Rightarrow y = e^{Cx}$

Question 8:

$x^5 \dfrac{dy}{dx} = -y^5$

Solution:

$$x^5\frac{dy}{dx}=-y^5$$
$$\Rightarrow\frac{dy}{y^5}=-\frac{dx}{x^5}$$
$$\Rightarrow\frac{dx}{x^5}+\frac{dy}{y^5}=0$$

Integrating both sides, we get:

$$\int\frac{dx}{x^5}+\int\frac{dy}{y^5}=k$$
$$\Rightarrow\int x^{-5}dx+\int y^{-5}dy=k$$
$$\Rightarrow\frac{x^{-4}}{-4}+\frac{y^{-4}}{-4}=k$$
$$\Rightarrow x^{-4}+y^{-4}=-4k$$
$$\Rightarrow x^{-4}+y^{-4}=C\qquad(\text{where } C=-4k)$$

Question 9:

$$\frac{dy}{dx}=\sin^{-1}x$$

Solution:

$$\frac{dy}{dx}=\sin^{-1}x$$
$$\Rightarrow dy=\sin^{-1}xdx$$

Integrating both sides, we get:

$$\int dy=\int\sin^{-1}xdx$$
$$\Rightarrow y=\int\left(\sin^{-1}x.1\right)dx$$
$$\Rightarrow y=\sin^{-1}x.\int(1)dx-\int\left[\left(\frac{d}{dx}\left(\sin^{-1}x\right)\int(1)dx\right)\right]dx$$
$$\Rightarrow y=x\sin^{-1}x+\int\frac{-x}{\sqrt{1-x^2}}dx\qquad\ldots(1)$$

Let $1-x^2=t$

$$\Rightarrow \frac{d}{dx}\left(1-x^2\right)=\frac{dt}{dx}$$
$$\Rightarrow -2x=\frac{dt}{dx}$$
$$\Rightarrow xdx=-\frac{1}{2}dt$$

Putting these values in equation (1), we get:

$$\Rightarrow y=x\sin^{-1}x+\int\frac{1}{2\sqrt{t}}dt$$
$$\Rightarrow y=x\sin^{-1}x+\frac{1}{2}\int(t)^{\frac{-1}{2}}dt$$
$$\Rightarrow y=x\sin^{-1}x+\frac{1}{2}.\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right)+C$$
$$\Rightarrow y=x\sin^{-1}x+\sqrt{t}+C$$
$$y=x\sin^{-1}x+\sqrt{1-x^2}+C$$

Question 10:

$$e^x\tan ydx+\left(1-e^x\right)\sec^2 ydy=0$$

Solution:

$$e^x\tan ydx+\left(1-e^x\right)\sec^2 ydy=0$$
$$\left(1-e^x\right)\sec^2 ydy=-e^x\tan ydx$$
$$\frac{\sec^2 y}{\tan y}dy=\frac{-e^x}{1-e^x}dx$$

Integrating both sides, we get:

$$\int\frac{\sec^2 y}{\tan y}dy=\int\frac{-e^x}{1-e^x}dx \qquad \ldots(1)$$

Let $\tan y=u$

$$\Rightarrow \frac{d}{dy}(\tan y)=\frac{du}{dy}$$
$$\Rightarrow \sec^2 y=\frac{du}{dy}$$
$$\Rightarrow \sec^2 ydy=du$$

Integrating both sides, we get:

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{du}{u}$$
$$= \log u$$
$$= \log(\tan y) \qquad \ldots(2)$$

Now, let $\left(1-e^x\right)=t$

$$\frac{d}{dx}\left(1-e^x\right)=\frac{dt}{dx}$$
$$\Rightarrow -e^x = \frac{dt}{dx}$$
$$\Rightarrow -e^x dx = dt$$

Integrating both sides, we get:

$$\int \frac{-e^x}{1-e^x} dx = \int \frac{dt}{t}$$
$$= \log t$$
$$= \log\left(1-e^x\right) \qquad \ldots(3)$$

Using (1), (2) and (3)

$$\Rightarrow \log(\tan y) = \log\left(1-e^x\right)+\log C$$
$$\Rightarrow \log(\tan y) = \log\left[C\left(1-e^x\right)\right]$$
$$\Rightarrow \tan y = C\left(1-e^x\right)$$

For each of the differential equations in Exercises 11 to 14, find a particular solution satisfying the given condition:

Question 11:

$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x\,;\,y=1$ when $x=0$

Solution:

$$\left(x^3+x^2+x+1\right)\frac{dy}{dx}=2x^2+x;\,y=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}$$
$$\Rightarrow dy=\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{2x^2+x}{\left(x^3+x^2+x+1\right)}dx \qquad \ldots(1)$$

Let $\dfrac{2x^2+x}{(x+1)\left(x^2+1\right)}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1} \qquad \ldots(2)$

$$\Rightarrow\frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{Ax^2+A+(Bx+C)(x+1)}{(x+1)\left(x^2+1\right)}$$

$$\Rightarrow 2x^2+x=Ax^2+A+Bx^2+Bx+Cx+C$$

$$\Rightarrow 2x^2+x=(A+B)x^2+(B+C)x+(A+C)$$

Comparing the coefficients of x^2 and x, we get:

$A+B=2$

$B+C=2$

$A+C=0$

Therefore,

$A=\dfrac{1}{2}, B=\dfrac{3}{2}$ and $C=\dfrac{-1}{2}$

Substituting these values in (2), we get:

$$\frac{2x^2+x}{(x+1)\left(x^2+1\right)}=\frac{1}{2}\frac{1}{(x+1)}+\frac{1}{2}\frac{(3x-1)}{\left(x^2+1\right)}$$

Hence, equation (1) becomes:

$$\int dy=\frac{1}{2}\int\frac{1}{(x+1)}dx+\frac{1}{2}\int\frac{(3x-1)}{\left(x^2+1\right)}dx$$

$$\begin{aligned}
y&=\frac{1}{2}\log(x+1)+\frac{3}{2}\int\frac{x}{x^2+1}dx-\frac{1}{2}\int\frac{x}{x^2+1}dx\\
&=\frac{1}{2}\log(x+1)+\frac{3}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\tan^{-1}x+C\\
&=\frac{1}{4}\left[2\log(x+1)+3\log\left(x^2+1\right)\right]-\frac{1}{2}\tan^{-1}x+C\\
&=\frac{1}{4}\left[\log(x+1)^2+\log\left(x^2+1\right)^3\right]-\frac{1}{2}\tan^{-1}x+C\\
&=\frac{1}{4}\left[\log(x+1)^2\left(x^2+1\right)^3\right]-\frac{1}{2}\tan^{-1}x+C
\end{aligned}$$

Now, $y=1$ when $x=0$

$$\Rightarrow 1=\frac{1}{4}\log(1)-\frac{1}{2}\tan^{-1}0+C$$
$$\Rightarrow 1=\frac{1}{4}\times 0-\frac{1}{2}\times 0+C$$
$$\Rightarrow C=1$$

Thus, $y=\frac{1}{4}\left[\log(x+1)^2(x^2+1)^3\right]-\frac{1}{2}\tan^{-1}x+1$

Question 12:

$x(x^2-1)\frac{dy}{dx}=1;\ y=0$ when $x=2$

Solution:

$$x(x^2-1)\frac{dy}{dx}=1$$
$$\Rightarrow dy=\frac{dx}{x(x^2-1)}$$
$$\Rightarrow dy=\frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy=\int\frac{1}{x(x-1)(x+1)}dx \qquad \ldots(1)$$

Let $\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} \qquad \ldots(2)$

$$\Rightarrow \frac{1}{x(x-1)(x+1)}=\frac{A(x-1)(x+1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)}$$
$$\Rightarrow \frac{1}{x(x-1)(x+1)}=\frac{(A+B+C)x^2+(B-C)x-A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2 and x, we get:

$$A=-1$$
$$B-C=0$$
$$A+B+C=0$$

Therefore,

$A=-1,\ B=\frac{1}{2}$ and $C=\frac{1}{2}$

Substituting these values in (2), we get:

$$\Rightarrow \frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

Hence, equation (1) becomes:

$$\int dy=-\int \frac{1}{x}dx+\frac{1}{2}\int \frac{1}{x-1}dx+\frac{1}{2}\int \frac{1}{x+1}dx$$

$$\Rightarrow y=-\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+\log k$$

$$\Rightarrow y=\frac{1}{2}\log\left[\frac{k^2(x-1)(x+1)}{x^2}\right]$$

Now, $y=0$ when $x=2$

$$\Rightarrow 0=\frac{1}{2}\log\left[\frac{k^2(2-1)(2+1)}{4}\right]$$

$$\Rightarrow \log\left(\frac{3k^2}{4}\right)=0$$

$$\Rightarrow \frac{3k^2}{4}=1$$

$$\Rightarrow 3k^2=4$$

$$\Rightarrow k^2=\frac{4}{3}$$

Thus,

$$y=\frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$=\frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

$$=\frac{1}{2}\log\left[\frac{4}{3}.\frac{(x^2-1)}{x^2}\right]$$

$$=\frac{1}{2}\log\left(\frac{x^2-1}{x^2}\right)-\frac{1}{2}\log\frac{3}{4}$$

Question 13:

$\cos\left(\frac{dy}{dx}\right)=a\left(a\in R\right);\ y=1$ when $x=0$

Solution:

$$\cos\left(\frac{dy}{dx}\right)=a$$
$$\Rightarrow\frac{dy}{dx}=\cos^{-1}a$$
$$\Rightarrow dy=\cos^{-1}a\,dx$$

Integrating both sides, we get:

$$\int dy=\cos^{-1}a\int dx$$
$$\Rightarrow y=\cos^{-1}a.x+C$$
$$\Rightarrow y=x\cos^{-1}a+C$$

Now, $y=1$ when $x=0$

$$\Rightarrow 1=0.\cos^{-1}a+C$$
$$\Rightarrow C=1$$

Thus,

$$y=x\cos^{-1}a+1$$
$$\Rightarrow\frac{y-1}{x}=\cos^{-1}a$$
$$\Rightarrow\cos\left(\frac{y-1}{x}\right)=a$$

Question 14:

$\frac{dy}{dx}=y\tan x;\ y=1$ when $x=0$

Solution:

$$\frac{dy}{dx}=y\tan x$$
$$\Rightarrow\frac{dy}{y}=\tan x dx$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = \int \tan x dx$$
$$\Rightarrow \log y = \log(\sec x) + \log C$$
$$\Rightarrow \log y = \log(\sec x.C)$$
$$\Rightarrow y = C \sec x$$

Now, $y = 1$ when $x = 0$
$$\Rightarrow 1 = C \times \sec 0$$
$$\Rightarrow 1 = C \times 1$$
$$\Rightarrow C = 1$$

Thus, $y = \sec x$

Question 15:

Find the equation of a curve passing through the point $(0,0)$ and whose differential equation is $y' = e^x \sin x$.

Solution:

$$y' = e^x \sin x$$
$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$
$$\Rightarrow dy = e^x \sin x dx$$

Integrating both sides, we get:
$$\int dy = \int e^x \sin x dx$$
$$y = \int e^x \sin x dx \qquad \ldots(1)$$

Let $I = \int e^x \sin x dx$

$$\Rightarrow I = \sin x \int e^x dx - \int \left[\frac{d}{dx}(\sin x) . \int e^x dx \right] dx$$

$$\Rightarrow I = \sin x . e^x - \int \cos x . e^x dx$$

$$\Rightarrow I = \sin x . e^x - \left[\cos x \int e^x dx - \int \left(\frac{d}{dx}(\cos x) . \int e^x dx \right) dx \right]$$

$$\Rightarrow I = \sin x . e^x - \left[\cos x e^x - \int (-\sin x) e^x dx \right]$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - I$$

$$\Rightarrow 2I = e^x (\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get:

$$y = \frac{e^x (\sin x - \cos x)}{2} + C$$

Since, the curve passes through $(0,0)$, we have:

$$\Rightarrow 0 = \frac{e^0 (\sin 0 - \cos 0)}{2} + C$$

$$\Rightarrow 0 = \frac{1(0-1)}{2} + C$$

$$\Rightarrow C = \frac{1}{2}$$

Thus,

$$\Rightarrow y = \frac{e^x (\sin x - \cos x)}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = e^x (\sin x - \cos x) + 1$$

$$\Rightarrow 2y - 1 = e^x (\sin x - \cos x)$$

$$\Rightarrow 2y - 1 = e^x (\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x (\sin x - \cos x)$

Question 16:

For the differential equation $xy \frac{dy}{dx} = (x+2)(y+2)$. Find the solution curve passing through the point $(1,-1)$.

Solution:

$$xy\frac{dy}{dx}=(x+2)(y+2)$$
$$\Rightarrow\left(\frac{y}{y+2}\right)dy=\left(\frac{x+2}{2}\right)dx$$

Integrating both sides, we get:

$$\int\left(1-\frac{2}{y+2}\right)dy=\int\left(1+\frac{2}{x}\right)dx$$
$$\Rightarrow\int dy-2\int\frac{1}{y+2}dy=\int dx+2\int\frac{1}{x}dx$$
$$\Rightarrow y-2\log(y+2)=x+2\log x+C$$
$$\Rightarrow y-x-C=\log x^2+\log(y+2)^2$$
$$\Rightarrow y-x-C=\log\left[x^2(y+2)^2\right]$$

Since, the curve passes through $(1,-1)$, we have:

$$\Rightarrow 1-1-C=\log\left[(1)^2(-1+2)^2\right]$$
$$\Rightarrow -2-C=\log 1$$
$$\Rightarrow -2-C=0$$
$$\Rightarrow C=-2$$

Thus, $y-x+2=\log\left[x^2(y+2)^2\right]$ is the required solution of the curve.

Question 17:

Find the equation of a curve passing through the point $(0,-2)$ given that at any point (x,y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Solution:

Let x and y be x-coordinate and y-coordinate of the curve, respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by the relation, $\frac{dy}{dx}$

Therefore

$$\Rightarrow y.\frac{dy}{dx} = x$$
$$\Rightarrow ydy = xdx$$

Integrating both sides, we get:

$$\int ydy = \int xdx$$
$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$
$$\Rightarrow y^2 - x^2 = 2C$$

Since, the curve passes through $(0,-2)$, we have:

$$\Rightarrow (-2)^2 - 0^2 = 2C$$
$$\Rightarrow 2C = 4$$

Thus, $y^2 - x^2 = 4$ is the required equation of the curve.

Question 18:

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining ate point of contact to the point $(-4,-3)$. Find the equation of the curve given that it passes through $(-2,1)$.

Solution:

(x, y) is point of contact of curve and tangent.

Slope (m_1) of segment joining (x, y) and $(-4,-3)$ is $\frac{y+3}{x+4}$

We know that the slope of a tangent to the curve in the coordinate axis is given by the relation, $\frac{dy}{dx}$.

Therefore, slope (m_2) of tangent is $\frac{dy}{dx}$.

Since, $m_2 = 2m_1$

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$
$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$
$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$
$$\Rightarrow \log(y+3) = \log C(x+4)^2$$
$$\Rightarrow y+3 = C(x+4)^2$$

Since, the curve passes through $(-2,1)$, we have:

$$\Rightarrow 1+3 = C(-2+4)^2$$
$$\Rightarrow 4 = 4C$$
$$\Rightarrow C = 1$$

Thus, $y+3=(x+4)^2$ is the required equation of the curve.

Question 19:

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds, it is 6 units. Find the radius of balloon after t seconds.

Solution:

Let the rate of change of volume of the balloon be k.

$$\Rightarrow \frac{dV}{dt} = k$$
$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k$$
$$\Rightarrow \frac{4}{3}\pi .3r^2.\frac{dr}{dt} = k$$
$$\Rightarrow 4\pi r^2 dr = kdt$$

Integrating both sides, we get:

$$\int 4\pi r^2 dr = \int kdt$$
$$\Rightarrow 4\pi \frac{r^3}{3} = kt + C$$
$$\Rightarrow 4\pi r^3 = 3(kt+C)$$

At $t=0, r=3$

$$\Rightarrow 4\pi \times 27 = 3(k \times 0 + C)$$
$$\Rightarrow 108\pi = 3C$$
$$\Rightarrow C = 36\pi$$

Now, at $t = 3, r = 6$
$$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + C)$$
$$\Rightarrow 864\pi = 3(3k + 36\pi)$$
$$\Rightarrow 3k = 288\pi - 36\pi = 252\pi$$
$$\Rightarrow k = 84\pi$$

Hence,
$$4\pi r^3 = 3[84\pi t + 36\pi]$$
$$4\pi r^3 = 4\pi(63t + 27)$$
$$r^3 = 63t + 27$$
$$r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$.

Question 20:

In a bank, principle increases continuously at the rate of $r\%$ per year. Find the value of r if ₹ 100 doubles itself in 10 years. $(\log_e 2 = 0.6931)$

Solution:

Let p, t and r represent the principle, time and rate of interest respectively.
The principle increases continuously at the rate of $r\%$ per year.
$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$
$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:
$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$
$$\Rightarrow \log p = \frac{rt}{100} + k$$
$$\Rightarrow p = e^{\frac{rt}{100} + k}$$

It is given that $p=100$ when $t=0$

Therefore,

$\Rightarrow 100=e^{k}$

Now, if $t=10$ then $p=2\times 100=200$

Hence,

$$200=e^{\frac{r}{10}+k}$$

$$\Rightarrow 200=e^{\frac{r}{10}}.e^{k}$$

$$\Rightarrow 200=e^{\frac{r}{10}}.100$$

$$\Rightarrow e^{\frac{r}{10}}=2$$

$$\Rightarrow \frac{r}{10}=\log_e 2$$

$$\Rightarrow \frac{r}{10}=0.6931$$

$$\Rightarrow r=6.931$$

Thus, the rate of interest, $r=6.931\%$.

Question 21:

In a bank, principle increases continuously at the rate of 5% per year. An amount of ₹1000 is deposited with this bank, how much will it worth after 10 years. $\left(e^{0.5}=1.648\right)$

Solution:

Let p and t be the principle and time, respectively.

The principle increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{dp}{dt}=5\%\times p$$

$$\Rightarrow \frac{dp}{dt}=\left(\frac{5}{100}\right)p$$

$$\Rightarrow \frac{dp}{dt}=\frac{p}{20}$$

$$\Rightarrow \frac{dp}{p}=\frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = \frac{t}{20} + C$$

$$\Rightarrow p = e^{\frac{t}{20}+C}$$

Now, $p = 1000$ when $t = 0$

Therefore,

$$\Rightarrow 1000 = e^{C}$$

Now, at $t = 10$ and $e^{C} = 1000$

$$\Rightarrow p = e^{\frac{10}{20}+C}$$

$$\Rightarrow p = e^{0.5} \times e^{C}$$

$$\Rightarrow p = 1.648 \times 1000$$

$$\Rightarrow p = 1648$$

Thus, after 10 years the amount will worth ₹1648.

Question 22:

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let y be the number of bacteria at any instant t.

Rate of growth of the bacteria is proportional to the number present.

$$\Rightarrow \frac{dy}{dt} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k\int dt$$

$$\Rightarrow \log y = kt + C$$

Let y_0 be the number of bacteria at $t=0$.

$$\Rightarrow \log y_0 = C$$
$$\Rightarrow \log y = kt + \log y_0$$
$$\Rightarrow \log y - \log y_0 = kt$$
$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$
$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right)$$

Since, the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$
$$\Rightarrow \frac{y}{y_0} = \frac{11}{10}$$

Taking log on both the sides

$$\Rightarrow \log\left(\frac{y}{y_0}\right) = \log\left(\frac{11}{10}\right)$$
$$\Rightarrow kt = \log\left(\frac{11}{10}\right) \qquad \left[\because \log\left(\frac{y}{y_0}\right) = kt\right]$$
$$\Rightarrow k.2 = \log\left(\frac{11}{10}\right)$$
$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore,

$$\Rightarrow \frac{1}{2}\log\left(\frac{11}{10}\right)t = \log\left(\frac{y}{y_0}\right)$$
$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)}$$

Now, let the time when the number of bacteria increases from 1,00,000 to 2,00,000 be t_1

Therefore, $y = y_0$ at $t = t_1$

Hence,

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)}$$

$$= \frac{2\log\left(\frac{200000}{100000}\right)}{\log\left(\frac{11}{10}\right)}$$

$$= \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Thus, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours, the number of bacteria increases from 1,00,000 to 2,00,000.

Question 23:

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Solution:

$$\frac{dy}{dx} = e^{x+y} = e^x . e^y$$

$$\Rightarrow \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow e^{-y} dy = e^x dx$$

Integrating both sides, we get:

$$\int e^{-y} dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + k$$

$$\Rightarrow e^x + e^{-y} = -k$$

$$\Rightarrow e^x + e^{-y} = C \qquad (where,\ C = -k)$$

Thus, the correct option is (A).

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

Question 1:

$$\left(x^2+xy\right)dy=\left(x^2+y^2\right)dx$$

Solution:

$\left(x^2+xy\right)dy=\left(x^2+y^2\right)dx$ can be written as:

$$\frac{dy}{dx}=\frac{x^2+y^2}{x^2+xy}$$

$$F(x,y)=\frac{x^2+y^2}{x^2+xy}$$

$$F(\lambda x,\lambda y)=\frac{(\lambda x)^2+(\lambda y)^2}{(\lambda x)^2+(\lambda x)(\lambda y)}=\frac{x^2+y^2}{x^2+xy}=\lambda^0 F(x,y)$$

Equation is a homogeneous equation.

Let $y=vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{(1+v^2) - v(1+v)}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$\Rightarrow \left(\frac{1+v}{1-v}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2-1+v}{1-v}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1-v} - 1\right)dv = \frac{dx}{x}$$

$$\Rightarrow -2\log(1-v) - v = \log x - \log C$$

$$\Rightarrow v = -2\log(1-v) - \log x + \log C$$

$$\Rightarrow v = \log\left[\frac{C}{x(1-v)^2}\right]$$

$$\Rightarrow \frac{y}{x} = \log\left[\frac{C}{x\left(1-\frac{y}{x}\right)^2}\right]$$

$$\Rightarrow \frac{y}{x} = \log\left[\frac{Cx}{(x-y)^2}\right]$$

$$\Rightarrow \frac{Cx}{(x-y)^2} = e^{\frac{y}{x}}$$

$$\Rightarrow (x-y)^2 = Cxe^{\frac{-y}{x}}$$

Question 2:

$$y' = \frac{x+y}{x}$$

Solution:

$$y' = \frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x}$$

$$F(x, y) = \frac{x+y}{x}$$

$$F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x} = \frac{x+y}{x} = \lambda^0 F(x, y)$$

Equation is a homogeneous equation.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

$$\Rightarrow \int dv = \int \frac{dx}{x}$$

$$v = \log|x| + C$$

$$\Rightarrow \frac{y}{x} = \log|x| + C$$

$$\Rightarrow y = x\log|x| + Cx$$

Question 3:

$$(x-y)dy - (x+y)dx = 0$$

Solution:

$$(x-y)dy - (x+y)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \quad ....(1)$$

Let $F(x, y) = \dfrac{x+y}{x-y}$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x+y}{x-y} = \lambda^0 F(x, y)$$

Equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{(1+v^2)}dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \frac{dx}{x}$$

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left[\frac{x^2+y^2}{x^2}\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\log(x^2+y^2) - \log x^2\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2) + C$$

Question 4:

$(x^2 - y^2)dx + 2xydy = 0$

Solution:

$$(x^2 - y^2)dx + 2xydy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x^2-y^2)}{2xy}$$

Let $F(x,y) = -\frac{(x^2-y^2)}{2xy}$

$$\therefore F(\lambda x, \lambda y) = \left[\frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}\right] = \frac{-(x^2-y^2)}{2xy} = \lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = -\left[\frac{x^2 - (vx)^2}{2x.(vx)}\right]$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(1+v^2)}{2v}$$

$$\Rightarrow \frac{2v}{1+v^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \log(1+v^2) = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow 1 + v^2 = \frac{C}{x}$$

$$\Rightarrow \left[1 + \frac{y^2}{x^2}\right] = \frac{C}{x}$$

$$\Rightarrow x^2 + y^2 = Cx$$

Question 5:

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$$

Solution:

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$$

$$\frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$$

Let $F(x,y) = \dfrac{x^2 - 2y^2 + xy}{x^2}$

$$\therefore F(\lambda x, \lambda y) = \frac{(\lambda x)^2 - 2(\lambda y)^2 + (\lambda x)(\lambda y)}{(\lambda x)^2} = \frac{x^2 - 2y^2 + xy}{x^2} = \lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - 2(vx)^2 + x.(vx)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 - 2v^2 + v$$

$$\Rightarrow x\frac{dv}{dx} = 1 - 2v^2$$

$$\Rightarrow \frac{dv}{1 - 2v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{dv}{2\left(\frac{1}{2} - v^2\right)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2}\right] = \frac{dx}{x}$$

$$\frac{1}{2}\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}} + v}{\frac{1}{\sqrt{2}} - v}\right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}} + \frac{y}{x}}{\frac{1}{\sqrt{2}} - \frac{y}{x}}\right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x + \sqrt{2}y}{x - \sqrt{2}y}\right| = \log|x| + C$$

Question 6:

$$xdy - ydx = \sqrt{x^2 + y^2}\,dx$$

Solution:

$$xdy - ydx = \sqrt{x^2 + y^2}\,dx$$

$$\Rightarrow xdy = \left[y + \sqrt{x^2 + y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Let, $F(x,y)=\dfrac{y+\sqrt{x^2+y^2}}{x}$

$\therefore F(\lambda x,\lambda y)=\dfrac{(\lambda x)+\sqrt{(\lambda x)^2(\lambda y)^2}}{\lambda x}=\dfrac{y+\sqrt{x^2+y^2}}{x}=\lambda^0 F(x,y)$

Given differential equation is a homogeneous equation.

Let $y=vx$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{d}{dx}(vx)$

$\Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\Rightarrow v+x\dfrac{dv}{dx}=\dfrac{vx+\sqrt{x^2+(vx)^2}}{x}$

$\Rightarrow v+x\dfrac{dv}{dx}=v+\sqrt{1+v^2}$

$\Rightarrow \dfrac{dv}{\sqrt{1+v^2}}=\dfrac{dx}{x}$

$\log\left|v+\sqrt{1+v^2}\right|=\log|x|+\log C$

$\Rightarrow \log\left|\dfrac{y}{x}+\sqrt{1+\dfrac{y^2}{x^2}}\right|=\log|Cx|$

$\Rightarrow \log\left|\dfrac{y+\sqrt{x^2+y^2}}{x}\right|=\log|Cx|$

$\Rightarrow y+\sqrt{x^2+y^2}=Cx^2$

Question 7:

$\left\{x\cos\left(\dfrac{y}{x}\right)+y\sin\left(\dfrac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\dfrac{y}{x}\right)-x\cos\left(\dfrac{y}{x}\right)\right\}xdy$

Solution:

$\left\{x\cos\left(\dfrac{y}{x}\right)+y\sin\left(\dfrac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\dfrac{y}{x}\right)-x\cos\left(\dfrac{y}{x}\right)\right\}xdy$

$\dfrac{dy}{dx}=\dfrac{\left\{x\cos\left(\dfrac{y}{x}\right)+y\sin\left(\dfrac{y}{x}\right)\right\}y}{\left\{y\sin\left(\dfrac{y}{x}\right)-x\cos\left(\dfrac{y}{x}\right)\right\}x}$

$$\text{Let } F(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\therefore F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}=\lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

$$\text{Let } y=vx$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=\frac{(x\cos v+vx\sin v).vx}{(vx\sin v-x\cos v)x}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v}{v\sin v-\cos v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v\cos v+v^2\sin v-v^2\sin v+v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v\cos v}{v\sin v-\cos v}$$

$$\Rightarrow \left[\frac{v\sin v-\cos v}{v\cos v}\right]dv=\frac{2dx}{x}$$

$$\Rightarrow \left(\tan v-\frac{1}{v}\right)dv=\frac{2dx}{x}$$

$$\log(\sec v)-\log v=2\log x+\log C$$

$$\Rightarrow \log\left(\frac{\sec v}{v}\right)=\log\left(Cx^2\right)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right)=Cx^2$$

$$\Rightarrow \sec v=Cx^2v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right)=C.x^2.\frac{y}{x}$$

$$\Rightarrow \sec\left(\frac{y}{x}\right)=Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right)=\frac{1}{Cxy}=\frac{1}{C}.\frac{1}{xy}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right)=k \qquad \left(k=\frac{1}{C}\right)$$

Question 8:

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$$

Solution:

$$x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x} \quad \text{....(1)}$$

Let $F(x,y)=\dfrac{y-x\sin\left(\frac{y}{x}\right)}{x}$

$$\therefore F(\lambda x,\lambda y)=\frac{\lambda y-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}=\lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \frac{dv}{\sin v} = -\frac{dx}{x}$$

$$\Rightarrow \cos ec v dv = -\frac{dx}{x}$$

$$\log\left|\cos ec v - \cot v\right| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow \cos ec\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x}$$

$$\Rightarrow x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

Question 9:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Solution:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

$$\Rightarrow ydx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

Let $F(x, y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \lambda^0 F(x, y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$

$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - v}{2 - \log v}$

$\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$

$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$

$\Rightarrow \left[\frac{1}{v(\log v - 1)} - \frac{1}{v}\right]dv = \frac{dx}{x}$

$\int \frac{1}{v(\log v - 1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$

$\Rightarrow \int \frac{dv}{v(\log v - 1)} - \log v = \log x + \log C$

Let, $\log v - 1 = t$

$\Rightarrow \frac{d}{dv}(\log v - 1) = \frac{dt}{dv}$

$\Rightarrow \frac{1}{v} = \frac{dt}{dv}$

$\Rightarrow \frac{dv}{v} = dt$

$\Rightarrow \int \frac{dt}{t} - \log v = \log x + \log C$

$$\Rightarrow \log t - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right) - 1\right] - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}}\right] = \log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right) - 1\right] = Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right) - 1 = Cy$$

Required solution of the given differential equation.

Question 10:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

Solution:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}}$$

$$\text{Let } F(x, y) = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}}$$

$$\therefore F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{\lambda y}}\left(1 - \frac{\lambda x}{\lambda y}\right)}{1 + e^{\frac{\lambda x}{\lambda y}}} = \frac{-e^{\frac{\lambda x}{\lambda y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{\lambda}}} = \lambda^0 F(x, y)$$

Given differential equation is a homogeneous equation.

Let $x = vy$

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = -\left[\frac{v+e^v}{1+e^v}\right]$$

$$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right]dv = -\frac{dy}{y}$$

$$\Rightarrow \log\left(v+e^v\right) = -\log y + \log C$$

$$\Rightarrow \log\left(v+e^v\right) = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y} + e^{\frac{x}{y}}\right] = \frac{C}{y}$$

$$\Rightarrow x + ye^{\frac{x}{y}} = C$$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

Question 11:

$(x+y)dy + (x-y)dx = 0$; $y = 1$ when $x = 1$

Solution:

$$(x+y)dy + (x-y)dx = 0$$

$$\Rightarrow (x+y)dy = -(x-y)dx$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x-y)}{x+y}$$

Let $F(x,y) = \dfrac{-(x-y)}{x+y}$

$$\therefore F(\lambda x, \lambda y) = \frac{-(\lambda x - \lambda y)}{\lambda x + \lambda y} = \frac{-(x-y)}{x+y} = \lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-(x - vx)}{x + vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v(v+1)}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1} = \frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{v+1}{1+v^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \left[\frac{v}{1+v^2} + \frac{1}{1+v^2}\right]dv = -\frac{dx}{x}$$

$$\frac{1}{2}\log\left(1+v^2\right) + \tan^{-1}v = -\log x + k$$

$$\Rightarrow \log\left(1+v^2\right) + 2\tan^{-1}v = -2\log x + 2k$$

$$\Rightarrow \log\left[\left(1+v^2\right)x^2\right] + 2\tan^{-1}v = 2k$$

$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)x^2\right] + 2\tan^{-1}\frac{y}{x} = 2k$$

$$\Rightarrow \log\left(x^2+y^2\right) + 2\tan^{-1}\frac{y}{x} = 2k$$

Now, $y = 1$ at $x = 1$

$$\Rightarrow \log 2 + 2\tan^{-1}1 = 2k$$

$$\Rightarrow \log 2 + 2\times\frac{\pi}{4} = 2k$$

$$\Rightarrow \frac{\pi}{2} + \log 2 = 2k$$

$$\log\left(x^2+y^2\right) + 2\tan^{-1}\frac{y}{x} = \frac{\pi}{2} + \log 2$$

Question 12:

$x^2dy + \left(xy + y^2\right)dx = 0;\ y = 1$ when $x = 1$

Solution:

$x^2dy + \left(xy + y^2\right)dx = 0$

$\Rightarrow x^2dy = -\left(xy + y^2\right)dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-\left(xy + y^2\right)}{x^2}$

$F\left(x, y\right) = \dfrac{-\left(xy + y^2\right)}{x^2}$

$\therefore F\left(\lambda x, \lambda y\right) = \dfrac{\left[\lambda x.\lambda y + \left(\lambda y\right)^2\right]}{\left(\lambda x\right)^2} = \dfrac{-\left(xy + y^2\right)}{\left(\lambda x\right)^2} = \lambda^0 F\left(x, y\right)$

Given differential equation is a homogeneous equation.

$$y = vx$$
$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$
$$v + x\frac{dv}{dx} = \frac{-\left[x.vx + (vx)^2\right]}{x^2} = -v - v^2$$
$$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v = -v(v+2)$$
$$\Rightarrow \frac{dv}{v(v+2)} = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\frac{2}{v(v+2)}\right]dv = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right]dv = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right]dv = -\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2}\left[\log v - \log(v+2)\right] = -\log x + \log C$$
$$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right) = \log\frac{C}{x}$$
$$\Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$$
$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$$
$$\Rightarrow \frac{y}{y+2x} = \frac{C^2}{x^2}$$
$$\Rightarrow \frac{x^2 y}{y+2x} = C^2$$
$y = 1$ at $x = 1$
$$\Rightarrow \frac{1}{1+2} = C^2$$
$$\Rightarrow C^2 = \frac{1}{3}$$
$$\frac{x^2 y}{y+2x} = \frac{1}{3}$$
$$\Rightarrow y + 2x = 3x^2 y$$

Question 13:

$$\left[x\sin^2\left(\frac{y}{x}-y\right)\right]dx+xdy=0;\ y=\frac{\pi}{4}\text{ when }x=1$$

Solution:

$$\left[x\sin^2\left(\frac{y}{x}-y\right)\right]dx+xdy=0$$

$$\Rightarrow\frac{dy}{dx}=\frac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x}$$

$$\text{Let }F(x,y)=\frac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x}$$

$$\therefore F(\lambda x,\lambda y)=\frac{-\left[\lambda x\sin^2\left(\frac{\lambda y}{\lambda x}\right)-\lambda y\right]}{x}=\frac{-\left[x\sin^2\left(\frac{y}{x}\right)-y\right]}{x}=\lambda^0F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{-\left[x\sin^2 v - vx\right]}{x}$

$\Rightarrow v + x\frac{dv}{dx} = -\left[\sin^2 v - v\right] = v - \sin^2 v$

$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$

$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{x}$

$\Rightarrow \cos ec^2 dv = -\frac{dx}{x}$

$\Rightarrow -\cot v = -\log|x| - \log C$

$\Rightarrow \cot v = \log|x| + \log C$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$

$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx|$

$y = \frac{\pi}{4}$ at $x = 1$

$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$

$\Rightarrow 1 = \log C$

$\Rightarrow C = e^1 = e$

$\cot\left(\frac{y}{x}\right) = \log|ex|$

Question 14:

$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$; $y = 0$ when $x = 1$

Solution:

$$\frac{dy}{dx}-\frac{y}{x}+\text{cosec}\left(\frac{y}{x}\right)=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$$

Let $F(x,y)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)$

$$\therefore F(\lambda x,\lambda y)=\frac{\lambda y}{\lambda x}-\cos ec\left(\frac{\lambda y}{\lambda x}\right)=\frac{y}{x}-\cos ec\left(\frac{y}{x}\right)=\lambda^0 F(x,y)$$

Given differential equation is a homogeneous equation.

Let $y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=v-\cos ecv$$

$$\Rightarrow -\frac{dv}{\cos ecv}=\frac{dx}{x}$$

$$\Rightarrow -\sin v dv=\frac{dx}{x}$$

$$\Rightarrow \cos v=\log x+\log C=\log|Cx|$$

$$\Rightarrow \cos\left(\frac{y}{x}\right)=\log|Cx|$$

$y=0$ at $x=1$

$$\Rightarrow \cos(0)=\log C$$

$$\Rightarrow C=e^1=e$$

$$\cos\left(\frac{y}{x}\right)=\log|(ex)|$$

Required solution of the given differential equation.

Question 15:

$2xy+y^2-2x^2\frac{dy}{dx}=0; y=2$ when $x=1$

Solution:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2\frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$$

Let $F(x, y) = \frac{2xy + y^2}{2x^2}$

$$\therefore F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 F(x, y)$$

Given differential equation is a homogeneous equation.

Let $y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2x(vx) + (vx)^2}{2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v + v^2}{2}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$$

$$\Rightarrow 2.\frac{v^{-2+1}}{-2+1} = \log|x| + C$$

$$\Rightarrow -\frac{2}{v} = \log|x| + C$$

$$\Rightarrow -\frac{2}{\left(\frac{y}{x}\right)} = \log|x| + C$$

$$\Rightarrow -\frac{2x}{y} = \log|x| + C$$

$y = 2$ at $x = 1$

$$\Rightarrow -1 = \log(1) + C$$

$$\Rightarrow C = -1$$

$$\Rightarrow -\frac{2x}{y} = \log|x| - 1$$

$$\Rightarrow \frac{2x}{y} = 1 - \log|x|$$

$$\Rightarrow y = \frac{2x}{1 - \log|x|}, (x \neq 0, x \neq e)$$

Question 16:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution

(A) $y = vx$

(B) $v = yx$

(C) $x = vy$

(D) $x = v$

Solution:

For solving homogeneous equation of form $\frac{dx}{dy}=h\left(\frac{x}{y}\right)$, we need to make substitution as $x=vy$
.

Thus, the correct option is C.

Question 17:
Which of the following is a homogeneous differential equation?

(A) $(4x+6y+5)dy-(3y+2x+4)dx=0$

(B) $(xy)dx-(x^3+y^3)dy=0$

(C) $(x^3+2y^2)dx+2xydy=0$

(D) $y^2dx+(x^2-xy^2-y^2)dy=0$

Solution:

$F(x,y)$ is homogeneous function of degree n, if $F(\lambda x,\lambda v)=\lambda' F(x,y)$ for non-zero constant (λ).

Consider equation given in D:

$$y^2dx+(x^2-xy^2-y^2)dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y^2}{x^2-xy^2-y^2}=\frac{y^2}{y^2+xy^2-x^2}$$

$$F(x,y)=\frac{y^2}{y^2+xy^2-x^2}$$

$$F(\lambda x,\lambda y)=\frac{(\lambda y)^2}{(\lambda y)^2+(\lambda x)(\lambda y)^2-(\lambda x)^2}$$

$$=\frac{\lambda^2 y^2}{\lambda^2(y^2+xy^2-x^2)}$$

$$=\lambda^0\left(\frac{y^2}{y^2+xy^2-x^2}\right)$$

$$=\lambda^0 F(x,y)$$

Differential equation given in D is a homogeneous equation.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

Question 1:

$$\frac{dy}{dx}+2y=\sin x$$

Solution:

Differential equation is $\frac{dy}{dx}+2y=\sin x$

This is in the form $\frac{dy}{dx}+py=Q$ where $p=2$ and $Q=\sin x$

$$I.F.=e^{\int pdx}=e^{\int 2dx}=e^{2x}$$

$$y(I.F)=\int(Q\times I.F).dx+C$$

$$\Rightarrow ye^{2x}=\int \sin xe^{2x}dx+C$$

Let, $I=\int \sin x.e^{2x}$

$$\Rightarrow I=\int \sin x.\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x).\int e^{2x}dx\right)dx$$

$$\Rightarrow I=\sin x.\frac{e^{2x}}{2}-\int\left(\cos x.\frac{e^{2x}}{2}\right)dx$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\int e^{2x}-\int\left(\frac{d}{dx}(\cos x).\int e^{2x}dx\right)dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\int\frac{e^{2x}}{2}-\int\left[(-\sin x).\frac{e^{2x}}{2}\right]dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}=\frac{1}{4}\int(\sin x.e^{2x})dx$$

$$\Rightarrow I=\frac{e^{2x}}{4}(2\sin x-\cos x)-\frac{1}{4}t$$

$$\Rightarrow\frac{5}{4}I=\frac{e^{2x}}{4}(2\sin x-\cos x)$$

$$\Rightarrow I=\frac{e^{2x}}{5}(2\sin x-\cos x)$$

$$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+C$$

$$\Rightarrow y=\frac{1}{5}(2\sin x-\cos x)+Ce^{-2x}$$

Question 2:

$$\frac{dy}{dx}+3y=e^{-2x}$$

Solution:

Differential equation is $\frac{dy}{dx}+py=Q$

$\left(\text{where, } p=3 \text{ and } Q=e^{-2x}\right)$

$$I.F.=e^{\int pdx}=e^{\int 3dx}=e^{3x}$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow ye^{3x}=\int\left(e^{-2x}\times e^{3x}\right)+C$$

$$\Rightarrow ye^{3x}=\int e^{x}dx+C$$

$$\Rightarrow ye^{3x}=e^{x}+C$$

$$\Rightarrow y=e^{-2x}+Ce^{-3x}$$

Question 3:

$$\frac{dy}{dx}+\frac{y}{x}=x^2$$

Solution:

$$\frac{dy}{dx}+py=Q$$

$\left(\text{where, } p=\frac{1}{x} \text{ and } Q=x^2\right)$

$$I.F.=e^{\int pdx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow yx=\int\left(x^2.x\right)dx+C$$

$$\Rightarrow yx=\int x^3dx+C$$

$$\Rightarrow yx=\frac{x^4}{4}+C$$

$$\Rightarrow xy=\frac{x^4}{4}+C$$

Question 4:

$$\frac{dy}{dx}+(\sec x)y=\tan x\left(0\le x\le\frac{\pi}{2}\right)$$

Solution:

$$\frac{dy}{dx}+py=Q$$

$(\text{where, } p=\sec x \text{ and } Q=\tan x)$

$$I.F.=e^{\int pdx}=e^{\int \sec xdx}=e^{\log(\sec x+\tan x)}=\sec x+\tan x$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\int \sec x\tan xdx+\int \tan^2 dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\int(\sec^2 x-1)dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\tan x-x+C$$

Question 5:

$$\cos^2 x\frac{dy}{dx}+y=\tan x\left(0\le x<\frac{\pi}{2}\right)$$

Solution:

$$\cos^2 x\frac{dy}{dx}+y=\tan x\left(0\le x<\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{\cos^2 x}=\frac{\tan x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx}+(\sec^2 x)y=\sec^2 x.\tan x$$

$$\frac{dy}{dx}+py=Q$$

$(\text{where, } p=\sec^2 x \text{ and } Q=\sec^2 x.\tan x)$

$$I.F. = e^{\int pdx} = e^{\int \sec^2 xdx} = e^{\tan x}$$

$$y(I.F) = \int (Q \times I.F.)\,dx + C$$

$$\Rightarrow ye^{\tan x} = \int \left(\sec^2 x.\tan x.e^{\tan x}\right)dx + C$$

$$\Rightarrow ye^{\tan x} = e^{\tan x}(\tan x - 1) + C$$

$$\Rightarrow y = \tan x - 1 + Ce^{\tan x}$$

Question 6:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Solution:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

$$\frac{dy}{dx} + py = Q$$

$$\left(\text{where, } p = \frac{2}{x} \text{ and } Q = x\log x\right)$$

$$I.F = e^{\int pdx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

$$y(I.F) = \int (Q \times I.F)\,dx + C$$

$$\Rightarrow y.x^2 = \int \left(x\log x.x^2\right)dx + C$$

$$\Rightarrow x^2 y = \int \left(x^3 \log x\right)dx + C$$

$$\Rightarrow x^2 y = \log x \int x^3 dx - \int \left[\frac{d}{dx}(\log x).\int x^3 dx\right]dx + C$$

$$\Rightarrow x^2 y = \log x.\frac{x^4}{4} - \int \left(\frac{1}{x}.\frac{x^4}{4}\right)dx + C$$

$$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4}\int x^3 dx + C$$

$$\Rightarrow x^2 y = \frac{x^4 \log x}{4} - \frac{1}{4}\frac{x^4}{4} + C$$

$$\Rightarrow x^2 y = \frac{1}{16}x^4(4\log x - 1) + C$$

$$\Rightarrow y = \frac{1}{16}x^2(4\log x - 1) + Cx^{-2}$$

Question 7:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

Solution:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

$$\Rightarrow\frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

$$\frac{dy}{dx}+py=Q$$

$$\left(\text{where, } p=\frac{1}{x\log x}\text{ and } Q=\frac{2}{x^2}\right)$$

$$I.F.=e^{\int pdx}=e^{\int\frac{1}{x\log x}dx}=e^{\log(\log x)}=\log x$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow y\log x=\int\left(\frac{2}{x^2}\log x\right)dx+C$$

$$\int\left(\frac{2}{x^2}\log x\right)dx=2\int\left(\log x.\frac{1}{x^2}\right)dx$$

$$=2\left[\log x.\int\frac{1}{x^2}dx-\int\left\{\frac{d}{dx}(\log x).\int\frac{1}{x^2}dx\right\}dx\right]$$

$$=2\left[\log x\left(-\frac{1}{x}\right)-\int\left(\frac{1}{x}\left(-\frac{1}{x}\right)\right)dx\right]$$

$$=2\left[-\frac{\log x}{x}+\int\frac{1}{x^2}dx\right]$$

$$=2\left[-\frac{\log x}{x}-\frac{1}{x}\right]$$

$$=-\frac{2}{x}(1+\log x)$$

$$y\log x=-\frac{2}{x}(1+\log x)+C$$

Question 8:

$$\left(1+x^2\right)dy+2xydx=\cot xdx\left(x\neq 0\right)$$

Solution:

$$\left(1+x^2\right)dy+2xydx=\cot xdx$$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{\left(1+x^2\right)}=\frac{\cot x}{\left(1+x^2\right)}$$

$$\frac{dy}{dx}+py=Q\quad\left(\text{where, } p=\frac{2xy}{\left(1+x^2\right)}\text{ and } Q=\frac{\cot x}{\left(1+x^2\right)}\right)$$

$$I.F.=e^{\int pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log\left(1+x^2\right)}=1+x^2$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int\left(\frac{\cot x}{1+x^2}\times\left(1+x^2\right)\right)dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int\cot xdx+C$$

$$\Rightarrow y\left(1+x^2\right)=\log|\sin x|+C$$

Question 9:

$$x\frac{dy}{dx}+y-x+xy\cot x=0\,(x\neq 0)$$

Solution:

$$x\frac{dy}{dx}+y-x+xy\cot x=0\,(x\neq 0)$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}-1+y\cot x=0$$

$$\Rightarrow \frac{dy}{dx}+y\left(\frac{1}{x}+\cot x\right)-1=0$$

$$\Rightarrow \frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

$$\frac{dy}{dx}+py=Q\quad\left(\text{where, } p=\left(\frac{1}{x}+\cot x\right)\text{ and } Q=1\right)$$

$$I.F. = e^{\int p dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx} = e^{\log+\log(\sin x)} = e^{\log(x\sin x)} = x\sin x$$

$$y(I.F) = \int (Q \times I.F.)\,dx + C$$

$$\Rightarrow y(x\sin x) = \int (1 \times x\sin x)\,dx + C$$

$$\Rightarrow y(x\sin x) = \int (x\sin x)\,dx + C$$

$$\Rightarrow y(x\sin x) = x\int \sin x dx - \int\left[\frac{d}{dx}(x).\int \sin x dx\right] + C$$

$$\Rightarrow y(x\sin x) = x(-\cos x) - \int 1.(-\cos x)\,dx + C$$

$$\Rightarrow y(x\sin x) = -x\cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$$

$$\Rightarrow y = \frac{1}{x} - \cot x + \frac{C}{x\sin x}$$

Question 10:

$$\left(x + y\frac{dy}{dx}\right) = 1$$

Solution:

$$(x+y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$$

$$\Rightarrow \frac{dx}{dy} = x + y$$

$$\Rightarrow \frac{dx}{dy} - x = y$$

Put in form $\frac{dx}{dy} + p_1 x = Q_1$

$(\text{where, } p_1 = -1 \text{ and } Q_1 = y)$

$$I.F. = e^{\int p_1 dy} = \int e^{-dy} = e^{-y}$$

$$y(I.F) = \int (Q_1 \times I.F.)dx + C$$

$$\Rightarrow xe^{-y} = \int (y.e^{-y})dy + C$$

$$\Rightarrow xe^{-y} = y.\int e^{-y}dy - \int \left[\frac{d}{dy}(y).\int e^{-y}dy \right] dy + C$$

$$\Rightarrow xe^{-y} = y\left(\frac{e^{-y}}{-1}\right) - \int \left(\frac{e^{-y}}{-1}\right) dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$\Rightarrow x = -y - 1 + Ce^{y}$$

$$\Rightarrow x + y + 1 = Ce^{y}$$

Question 11:

$$ydx + (x - y^2)dy = 0$$

Solution:

$$ydx + (x - y^2)dy = 0$$

$$\Rightarrow ydx - (y^2 - x)dy = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = y$$

Put in form $\frac{dx}{dy} + p_1 x = Q_1$

$$\left(\text{where, } p_1 = \frac{1}{y} \text{ and } Q_1 = y \right)$$

$$I.F. = e^{\int p_1 dy} = \int e^{\frac{1}{y}dy} = e^{\log y} = y$$

$$y(I.F) = \int (Q_1 \times I.F.)dx + C$$

$$\Rightarrow xy = \int (y.y)dy + C$$

$$\Rightarrow xy = \int y^2 dy + C$$

$$\Rightarrow xy = \frac{y^3}{3} + C$$

$$\Rightarrow x = \frac{y^3}{3} + \frac{C}{y}$$

Question 12:

$$\left(x + 3y^3\right)\frac{dy}{dx} = y\left(y > 0\right)$$

Solution:

$$\left(x + 3y^3\right)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x + 3y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 3y^2}{y} = \frac{x}{y} + 3y$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 3y$$

$$\frac{dx}{dy} + p_1 x = Q_1$$

$$\left(\text{where, } p_1 = -\frac{1}{y} \text{ and } Q_1 = 3y\right)$$

$$I.F. = e^{\int p_1 dy} = e^{-\int \frac{dy}{y}} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$$

$$y\left(I.F\right) = \int \left(Q_1 \times I.F.\right) dx + C$$

$$\Rightarrow x \times \frac{1}{y} = \int \left(3y \times \frac{1}{y}\right) dy + C$$

$$\Rightarrow \frac{x}{y} = 3y + C$$

$$\Rightarrow x = 3y^2 + Cy$$

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

Question 13:

$$\frac{dy}{dx}+2y\tan x=\sin x;\ y=0 \text{ when } x=\frac{\pi}{3}$$

Solution:

$$\frac{dy}{dx}+2y\tan x=\sin x$$

$$\frac{dy}{dx}+py=Q$$

$$(\text{where, } p=2\tan x \text{ and } Q=\sin x)$$

$$I.F.=e^{\int pdx}=e^{\int 2\tan x dx}=e^{2\log|\sec x|}=e^{\log(\sec^2 x)}=\sec^2 x$$

$$y(I.F)=\int(Q\times I.F.)dx+C$$

$$\Rightarrow y(\sec^2 x)=\int(\sin x.\sec^2 x)dx+C$$

$$\Rightarrow y\sec^2 x=\int(\sec x.\tan x)dx+C$$

$$\Rightarrow y\sec^2 x=\sec x+C$$

$$y=0 \text{ at } x=\frac{\pi}{3}$$

$$0\times\sec^2\frac{\pi}{3}=\sec\frac{\pi}{3}+C$$

$$\Rightarrow 0=2+C$$

$$\Rightarrow C=-2$$

$$y\sec^2 x=\sec x-2$$

$$\Rightarrow y=\cos x-2\cos^2 x$$

Question 14:

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0 \text{ when } x=1$$

Solution:

$$\left(1+x^2\right)\frac{dy}{dx}+2xy=\frac{1}{1+x^2}$$

$$\Rightarrow\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{\left(1+x^2\right)^2}$$

$$\frac{dy}{dx}+py=Q$$

$$\left(\text{where, } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{1}{\left(1+x^2\right)^2}\right)$$

$$I.F. = e^{\int p dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

$$y(I.F) = \int (Q \times I.F.)\,dx + C$$

$$\Rightarrow y(1+x^2) = \int \left[\frac{1}{(1+x^2)^2}.(1+x^2) \right] dx + C$$

$$\Rightarrow y(1+x^2) = \int \frac{1}{1+x^2} dx + C$$

$$\Rightarrow y(1+x^2) = \tan^{-1} x + C \qquad ....(1)$$

$y = 0$ at $x = 1$

$$\Rightarrow C = -\frac{\pi}{4}$$

$$y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$$

Question 15:

$$\frac{dy}{dx} - 3y\cot x = \sin 2x;\ y = 2 \text{ when } x = \frac{\pi}{2}$$

Solution:

$$\frac{dy}{dx} - 3y\cot x = \sin 2x$$

$$\frac{dy}{dx} + py = Q$$

$$(\text{where, } p = -3\cot x \text{ and } Q = \sin 2x)$$

$$I.F. = e^{\int p dx} = e^{-3\int \cot x dx} = e^{-3\log|\sin x|} = e^{\log|\sin x|^{-3}} = e^{\log\left|\frac{1}{\sin^3 x}\right|} = \frac{1}{\sin^3 x}$$

$$y(I.F) = \int (Q \times I.F.)\,dx + C$$

$$\Rightarrow y.\frac{1}{\sin^3 x} = \int \left[\sin 2x.\frac{1}{\sin^3 x} \right] dx + C$$

$$\Rightarrow y\cos ec^3 x = 2\int (\cot x \cos ecx)\,dx + C$$

$$\Rightarrow y\cos ec^3 x = -2\cos ecx + C$$

$$\Rightarrow y = -\frac{2}{\cos ec^2 x} + \frac{C}{\cos ec^3 x}$$

$$\Rightarrow y = -2\sin^2 x + C\sin^3 x$$

$y = 2$ at $x = \frac{\pi}{2}$

$2 = -2 + C$

$\Rightarrow C = 4$

$y = -2\sin^2 x + 4\sin^3 x$

$\Rightarrow y = 4\sin^3 x - 2\sin^2 x$

Question 16:

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution:

Let $F(x, y)$ be the curve passing through origin.

At (x, y), slope of curve will be $\frac{dy}{dx}$

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} - y = x$$

$$\frac{dy}{dx} + py = Q$$

$(\text{where, } p = -1 \text{ and } Q = x)$

$$I.F. = e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$$

$$y(I.F) = \int (Q \times I.F.) dx + C$$

$$\Rightarrow ye^{-x} = \int xe^{-x} dx + C$$

$$\Rightarrow ye^{-x} = x\int e^{-x} dx - \int \left[\frac{d}{dx}(x). \int e^{-x} dx \right] dx + C$$

$$\Rightarrow ye^{-x} = -xe^{-x} + \int e^{-x} dx + C$$

$$\Rightarrow ye^{-x} = -xe^{-x} + \left(-e^{-x}\right) + C$$

$$\Rightarrow ye^{-x} = -e^{-x}(x+1) + C$$

$$\Rightarrow y = -(x+1) + Ce^{x}$$

$$\Rightarrow x + y + 1 = Ce^{x}$$

Curve passes through origin.

$1 = C$
$\Rightarrow x + y + 1 = e^x$

Question 17:

Find the equation of a curve passing through the point $(0,2)$ given that the sum of the coordinates of any point in the curve exceeds the magnitude of the slope of the tangent to the curve at any point by 5.

Solution:

$F(x,y)$ be curve and let (x,y) be a point on the curve. Slope of the tangent to curve at (x,y) is

$$\frac{dy}{dx}$$

$$\frac{dy}{dx} + 5 = x + y$$
$$\Rightarrow \frac{dy}{dx} - y = x - 5$$
$$\frac{dy}{dx} + py = Q$$

$(\text{where, } p = -1 \text{ and } Q = x - 5)$

$$I.F. = e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$$
$$y(I.F) = \int (Q \times I.F.) dx + C$$
$$\Rightarrow ye^{-x} = \int (x-5) e^{-x} dx + C$$
$$\int (x-5)e^{-x} dx = (x-5)\int e^{-x} dx - \int \left[\frac{d}{dx}(x-5).\int e^{-x} dx\right] dx$$
$$= (x-5)(-e^{-x}) - \int (-e^{-x}) dx$$
$$= (5-x)e^{-x} - (-e^{-x})$$
$$= (4-x)e^{-x}$$
$$\Rightarrow ye^{-x} = (4-x)e^{-x} + C$$

Curve passes through $(0,2)$

$0+2-4=C.0$
$\Rightarrow -2=C$
$\Rightarrow C=-2$
$x+y-4=-2e^x$
$\Rightarrow y=4-x-2e^x$

Question 18:

The integrating factor of the differential equation $x\frac{dy}{dx}-y=2x^2$

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

Solution:

$x\frac{dy}{dx}-y=2x^2$
$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=2x$

$\frac{dy}{dx}+py=Q$
$\left(\text{where, } p=-\frac{1}{x} \text{ and } Q=2x\right)$
$\therefore I.F.=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log\left(x^{-1}\right)}=x^{-1}=\frac{1}{x}$

Thus, the correct option is C.

Question 19:

The integrating factor of the differential equation $\left(1-y^2\right)\frac{dx}{dy}+yx=ay\left(-1>y<1\right)$

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Solution:

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay$$

$$\Rightarrow \frac{dx}{dy}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$$

$$\frac{dx}{dy}+p_1x=Q_1$$

$$\left(\text{where, } p_1=\frac{y}{1-y^2} \text{ and } Q_1=\frac{ay}{1-y^2}\right)$$

$$\therefore I.F.=e^{\int p_1 dy}=e^{\int \frac{y}{1-y^2}dy}=e^{-\frac{1}{2}\log\left(1-y^2\right)}=e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]}=\frac{1}{\sqrt{1-y^2}}$$

Thus, the correct option is D.

MISCELLANEOUS EXERCISE

Question 1:

For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

(iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution:

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

$\Rightarrow \frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x=0$

Highest order derivative present in differential equation is $\frac{d^2y}{dx^2}$. Its order is two.

Highest power raised to $\frac{d^2y}{dx^2}$ is one. Its degree is one.

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

$\Rightarrow \left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$

Highest order derivative in differential equation is $\frac{dy}{dx}$. its order is one.

Highest power raised to $\frac{dy}{dx}$ is three. Its degree is three.

(iii) $\frac{d^4y}{dx^4}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Highest order derivative in differential equation is $\frac{d^4y}{dx^4}$. Order is four.

The given differential equation is not a polynomial equation. Degree is not defined.

Question 2:

For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $xy = ae^x + be^{-x} + x^2$: $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y = e^x(a\cos x + b\sin x)$: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

(iii) $y = x\sin 3x$: $\dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2\log y$: $(x^2 + y^2)\dfrac{dy}{dx} - xy = 0$

Solution:

(i) $xy = ae^x + be^{-x} + x^2$: $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

$$xy = ae^x + be^{-x} + x^2 \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get:

$$x\frac{dy}{dx} + y.1 = a\frac{d}{dx}(e^x) + b\frac{d}{dx}(e^{-x}) + \frac{d}{dx}(x^2)$$
$$\Rightarrow x\frac{dy}{dx} + y = ae^x + be^{-x} + 2x$$

Again, differentiating both sides with respect to x, we get:

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$$
$$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2 \qquad \ldots(2)$$

Now, we have $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$

$$\begin{aligned}
LHS &= x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 \\
&= ae^x + be^{-x} + 2 - (ae^x + be^{-x} + x^2) + x^2 - 2 \qquad [\text{using } (1) \text{ and } (2)] \\
&= ae^x + be^{-x} + 2 - ae^x - be^{-x} - x^2 + x^2 - 2 \\
&= 0 \\
&= RHS
\end{aligned}$$

Thus, the given function is a solution of the corresponding differential equation.

(ii) $y = e^x(a\cos x + b\sin x)$: $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$

$$y = e^x\left(a\cos x + b\sin x\right) \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get:

$$y = e^x\left(a\cos x + b\sin x\right) = ae^x\cos x + be^x\sin x$$

$$\Rightarrow \frac{dy}{dx} = a.\frac{d}{dx}\left(e^x\cos x\right) + b.\frac{d}{dx}\left(e^x\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = a\left(e^x\cos x - e^x\sin x\right) + b\left(e^x\sin x + e^x\cos x\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(a+b\right)e^x\cos x + \left(b-a\right)e^x\sin x \qquad \ldots(2)$$

Again, differentiating both sides with respect to x, we get:

$$\Rightarrow \frac{d^2y}{dx^2} = \left(a+b\right)\frac{d}{dx}\left(e^x\cos x\right) + \left(b-a\right)\frac{d}{dx}\left(e^x\sin x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = (a+b).\left(e^x\cos x - e^x\sin x\right) + \left(b-a\right)\left(e^x\sin x + e^x\cos x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\left[\left(a+b\right)\left(\cos x - \sin x\right) + \left(b-a\right)\left(\sin x + \cos x\right)\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x\left[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^x\left(b\cos x - a\sin x\right) \qquad \ldots(3)$$

Now, we have $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

$$\begin{aligned}
LHS &= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y \\
&= 2e^x\left(b\cos x - a\sin x\right) - 2\left[\left(a+b\right)e^x\cos x + \left(b-a\right)e^x\sin x\right] + 2e^x\left(a\cos x + b\sin x\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left[\text{using } (1), (2) \text{ and } (3)\right] \\
&= e^x\left[\left(2b\cos x - 2a\sin x\right) - \left(2a\cos x + 2b\cos x\right) - \left(2b\sin x - 2a\sin x\right) + \left(2a\cos x + 2b\sin x\right)\right] \\
&= e^x\left[2b\cos x - 2a\sin x - 2a\cos x - 2b\cos x - 2b\sin x + 2a\sin x + 2a\cos x + 2b\sin x\right] \\
&= e^x\left[0\right] \\
&= 0 \\
&= RHS
\end{aligned}$$

Thus, the given function is a solution of the corresponding differential equation.

(iii) $y = x\sin 3x$: $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

$y = x\sin 3x \qquad \ldots(1)$

Differentiating both sides with respect to x, we get:

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x\sin 3x) = \sin 3x + x.\cos 3x.3$$
$$\Rightarrow \frac{dy}{dx} = \sin 3x + 3x\cos 3x$$

Again, differentiating both sides with respect to x, we get:

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(\sin 3x) + 3\frac{d}{dx}(x\cos 3x)$$
$$\Rightarrow \frac{d^2y}{dx^2} = 3\cos 3x + 3\left[\cos 3x + x(-\sin 3x).3\right]$$
$$\Rightarrow \frac{d^2y}{dx^2} = 6\cos 3x - 9x\sin 3x \qquad \ldots(2)$$

Now, we have $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

$$\begin{aligned} LHS &= \frac{d^2y}{dx^2} + 9y - 6\cos 3x \\ &= (6\cos 3x - 9x\sin 3x) + 9x\sin 3x - 6\cos 3x \qquad \left[\text{using } (1) \text{ and } (2)\right] \\ &= 0 \\ &= RHS \end{aligned}$$

Thus, the given function is a solution of the corresponding differential equation.

(iv) $x^2 = 2y^2\log y$: $(x^2 + y^2)\frac{dy}{dx} - xy = 0$

$x^2 = 2y^2\log y \qquad \ldots(1)$

Differentiating both sides with respect to x, we get:

$$\Rightarrow 2x = 2\frac{d}{dx}\left[y^2 \log y\right]$$

$$\Rightarrow x = \left[2y.\log y.\frac{dy}{dx} + y^2.\frac{1}{y}.\frac{dy}{dx}\right]$$

$$\Rightarrow x = \frac{dy}{dx}(2y\log y + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1+2\log y)} \qquad \ldots(2)$$

Now, we have $\left(x^2+y^2\right)\frac{dy}{dx} - xy = 0$

$$LHS = \left(x^2+y^2\right)\frac{dy}{dx} - xy$$

$$= \left(2y^2\log y + y^2\right).\frac{x}{y(1+2\log y)} - xy \qquad [\text{using } (1) \text{ and } (2)]$$

$$= y^2(1+2\log y).\frac{x}{y(1+2\log y)} - xy$$

$$= xy - xy$$

$$= 0$$

$$= RHS$$

Thus, the given function is a solution of the corresponding differential equation.

Question 3:

Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$ where a is an arbitrary constant.

Solution:

$$(x-a)^2 + 2y^2 = a^2$$

$$\Rightarrow x^2 + a^2 - 2ax + 2y^2 = a^2$$

$$\Rightarrow 2y^2 = 2ax - x^2 \qquad \ldots(1)$$

Differentiating both sides with respect to x, we get:

$$\Rightarrow 4y\frac{dy}{dx}=2a-2x$$

$$\Rightarrow \frac{dy}{dx}=\frac{2a-2x}{4y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2ax-2x^2}{4xy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2+x^2-2x^2}{4xy} \qquad \left[\text{from}(1),\ 2ax=2y^2+x^2\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

Thus, the differential equation of the family of curves is given as $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$.

Question 4:

Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general solution of differential equation $\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$, where c is a parameter.

Solution:

$$\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y} \qquad \ldots(1)$$

This is a homogeneous equation, to simplify it, let $y=vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \ldots(2)$$

Using (1) and (2)

$$\Rightarrow v+x\frac{dv}{dx}=\frac{x^3-3x(vx)^2}{(vx)^3-3x^2(vx)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v(v^3-3v)}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

$$\Rightarrow \frac{v^3-3v}{1-v^4}dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\Rightarrow \int\left(\frac{v^3-3v}{1-v^4}\right)dv=\log x+\log C' \qquad \ldots(3)$$

$$\Rightarrow \int\left(\frac{v^3-3v}{1-v^4}\right)dv=\int\frac{v^3dv}{1-v^4}-3\int\frac{vdv}{1-v^4}$$

$$\Rightarrow \int\left(\frac{v^3-3v}{1-v^4}\right)dv=I_1-3I_2 \qquad \ldots(4)$$

$$\left(\text{where, } I_1=\int\frac{v^3dv}{1-v^4} \text{ and } I_2=\int\frac{vdv}{1-v^4}\right)$$

Let $1-v^4=t$

Therefore,

$$\Rightarrow \frac{d}{dv}(1-v^4)=\frac{dt}{dv}$$

$$\Rightarrow -4v^3=\frac{dt}{dv}$$

$$\Rightarrow v^3dv=-\frac{dt}{4}$$

Now,

$$I_1=\int-\frac{dt}{4t}$$

$$=-\frac{1}{4}\log t$$

$$=-\frac{1}{4}\log(1-v^4) \qquad \ldots(5)$$

And

$$I_2 = \int \frac{v dv}{1-v^4}$$

$$= \int \frac{v dv}{1-\left(v^2\right)^2}$$

Let $v^2 = p$

Therefore,

$$\Rightarrow \frac{d}{dv}\left(v^2\right) = \frac{dp}{dv}$$

$$\Rightarrow 2v = \frac{dp}{dv}$$

$$\Rightarrow v dv = \frac{dp}{2}$$

Now,

$$I_2 = \frac{1}{2}\int \frac{dp}{1-p^2}$$

$$= \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right|$$

$$= \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right| \qquad \ldots(6)$$

Using (4), (5) and (6)

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}\log\left(1-v^4\right) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| \qquad \ldots(7)$$

Using (2) and (7)

$$-\frac{1}{4}\log\left(1-v^4\right) - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log x + \log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[\left(1-v^4\right)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$$

$$\Rightarrow -\frac{1}{4}\log\left[\left(1-v^2\right)\left(1+v^2\right)\left(\frac{1+v^2}{1-v^2}\right)^3\right]=\log C'x$$

$$\Rightarrow \frac{\left(1+v^2\right)^4}{\left(1-v^2\right)^2}=\left(C'x\right)^{-4}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2}=\frac{1}{C'^4x^4}$$

$$\Rightarrow \frac{\left(x^2+y^2\right)^4}{x^4\left(x^2-y^2\right)^2}=\frac{1}{C'^4x^4}$$

$$\Rightarrow \left(x^2-y^2\right)^2=C'^4\left(x^2+y^2\right)^4$$

Taking square root on both sides

$$\Rightarrow \left(x^2-y^2\right)=C'^2\left(x^2+y^2\right)^2$$

$$\Rightarrow \left(x^2-y^2\right)=C\left(x^2+y^2\right)^2 \qquad \left(where,\ C=C'^2\right)$$

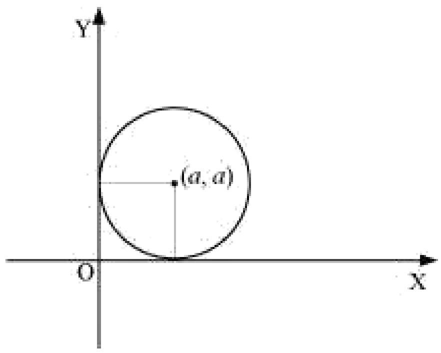
Question 5:

Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Solution:

Equation of a circle in first quadrant with centre (a,a) and radius (a) which touches coordinate axes is:

$$(x-a)^2+(y-a)^2=a^2 \qquad \ldots(1)$$



Differentiating both sides with respect to x, we get:

$$\Rightarrow 2(x-a)+2(y-a)\frac{dy}{dx}=0$$
$$\Rightarrow (x-a)+(y-a)y'=0$$
$$\Rightarrow (x-a)+yy'-ay'=0$$
$$\Rightarrow x+yy'-a(1+y')=0$$
$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Substituting this value in equation (1), we get:

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$
$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$
$$\Rightarrow (x-y)^2.y'^2+(x-y)^2=(x+yy')^2$$
$$\Rightarrow (x-y)^2\left[1+(y')^2\right]=(x+yy')^2$$

Hence, the differential equation of the family of circles is $(x-y)^2\left[1+(y')^2\right]=(x+yy')^2$

Question 6:

Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Solution:

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$
$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=-\frac{dx}{\sqrt{1-x^2}}$$

Integrating both sides, we get:

$$\Rightarrow \sin^{-1}y=-\sin^{-1}x+C$$
$$\Rightarrow \sin^{-1}x+\sin^{-1}y=C$$

Question 7:

Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by $(x+y+1)=A(1-x-y-2xy)$, where A is parameter.

Solution:

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$

$$\Rightarrow \frac{dy}{dx}=-\left(\frac{y^2+y+1}{x^2+x+1}\right)$$

$$\Rightarrow \frac{dy}{y^2+y+1}=-\frac{dx}{x^2+x+1}$$

Integrating both sides, we get:

$$\int\frac{dy}{y^2+y+1}=-\int\frac{dx}{x^2+x+1}$$

$$\Rightarrow \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=-\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=-\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+C$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right]+\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{(2y+1)}{\sqrt{3}}\frac{(2x+1)}{\sqrt{3}}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\frac{(4xy+2x+2y+1)}{3}}\right]=\frac{\sqrt{3}C}{2}$$

$$\Rightarrow\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1-\frac{(4xy+2x+2y+1)}{3}}\right]=\tan\left(\frac{\sqrt{3}C}{2}\right)$$

$$\Rightarrow\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{\frac{3-(4xy+2x+2y+1)}{3}}\right]=C_1 \qquad \left[\text{where, } C_1=\tan\left(\frac{\sqrt{3}C}{2}\right)\right]$$

$$\Rightarrow\frac{\sqrt{3}(2x+2y+2)}{3-(4xy+2x+2y+1)}=C_1$$

$$\Rightarrow 2\sqrt{3}(x+y+1)=C_1(3-4xy-2x-2y-1)$$

$$\Rightarrow 2\sqrt{3}(x+y+1)=C_1(2-4xy-2x-2y)$$

$$\Rightarrow 2\sqrt{3}(x+y+1)=C_1\times 2(1-2xy-x-y)$$

$$\Rightarrow \sqrt{3}(x+y+1)=C_1(1-x-y-2xy)$$

$$\Rightarrow (x+y+1)=\frac{C_1}{\sqrt{3}}(1-x-y-2xy)$$

$$\Rightarrow (x+y+1)=A(1-x-y-2xy) \qquad \left[\text{where } A=\frac{C_1}{\sqrt{3}}\right]$$

Question 8:

Find the equation of the curve passing through the point $\left(0,\frac{\pi}{4}\right)$ whose differential equation is $\sin x\cos y dx+\cos x\sin y dy=0$.

Solution:

$$\sin x\cos y dx+\cos x\sin y dy=0$$

$$\Rightarrow\frac{\sin x\cos y dx+\cos x\sin y dy}{\cos x\cos y}=0$$

$$\Rightarrow \tan x dx+\tan y dy=0$$

$$\Rightarrow \log(\sec x)+\log(\sec y)=\log C$$

$$\Rightarrow \log(\sec x.\sec y)=\log C$$

$$\Rightarrow \sec x.\sec y=C$$

The curve passes through the point $\left(0,\frac{\pi}{4}\right)$

Therefore,

$$\Rightarrow 1\times\sqrt{2}=C$$
$$\Rightarrow C=\sqrt{2}$$
$$\sec x.\sec y=\sqrt{2}$$
$$\Rightarrow \sec x.\frac{1}{\cos y}=\sqrt{2}$$
$$\Rightarrow \cos y=\frac{\sec x}{\sqrt{2}}$$

Question 9:

Find the particular solution of the differential equation $\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0$, given that $y=1$ when $x=0$.

Solution:

$$\left(1+e^{2x}\right)dy+\left(1+y^2\right)e^x dx=0$$
$$\Rightarrow \frac{dy}{1+y^2}+\frac{e^x}{1+e^{2x}}=0$$

Integrating both sides, we get:

$$\Rightarrow \tan^{-1} y+\int\frac{e^x dx}{1+e^{2x}}=C \qquad \ldots(1)$$

Let $e^x=t\Rightarrow e^{2x}=t^2$

$$\frac{d}{dx}\left(e^x\right)=\frac{dt}{dx}$$
$$\Rightarrow e^x=\frac{dt}{dx}$$
$$\Rightarrow e^x dx=dt$$

Substituting this value in equation (1), we get:

$$\tan^{-1} y+\int\frac{dt}{1+t^2}=C$$
$$\Rightarrow \tan^{-1} y+\tan^{-1} t=C$$
$$\Rightarrow \tan^{-1} y+\tan^{-1}\left(e^x\right)=C$$

When $x=0;\ \ y=1$

Hence,

$$\tan^{-1}1+\tan^{-1}1=C$$

$$\Rightarrow \frac{\pi}{4}+\frac{\pi}{4}=C$$

$$\Rightarrow C=\frac{\pi}{2}$$

Thus, $\tan^{-1}y+\tan^{-1}\left(e^x\right)=\frac{\pi}{2}$

Question 10:

Solve the differential equation $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy\ \left(y\neq 0\right)$.

Solution:

$$ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y^2\right)dy$$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy}=xe^{\frac{x}{y}}+y^2$$

$$\Rightarrow e^{\frac{x}{y}}\left[y.\frac{dx}{dy}-x\right]=y^2$$

$$\Rightarrow e^{\frac{x}{y}}\frac{\left[y.\frac{dx}{dy}-x\right]}{y^2}=1 \qquad \ldots(1)$$

Let $e^{\frac{x}{y}}=z$

Differentiating it with respect to y, we get:

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right)=\frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}.\frac{d}{dy}\left(\frac{x}{y}\right)=\frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}\left[\frac{y.\frac{dx}{dy}-x}{y^2}\right]=\frac{dz}{dy}$$

$$\Rightarrow \frac{dz}{dy}=1 \qquad \left[from\ (1)\right]$$

$$\Rightarrow dz=dy$$

Integrating both sides, we get

$\Rightarrow z = y + C$

$\Rightarrow e^{\frac{x}{y}} = y + C$

Question 11:

Find a particular solution of the differential equation $(x-y)(dx+dy)=(dx-dy)$, given that $y=-1$, when $x=0$. (Hint: put $x-y=t$)

Solution:

$(x-y)(dx+dy)=(dx-dy)$

$\Rightarrow (x-y+1)dy=(1-x+y)dx$

$\Rightarrow \frac{dy}{dx}=\frac{1-x+y}{x-y+1}$

$\Rightarrow \frac{dy}{dx}=\frac{1-(x-y)}{1+(x-y)} \qquad \ldots(1)$

Let $x-y=t \qquad \ldots(2)$

$\Rightarrow \frac{d}{dx}(x-y)=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow 1-\frac{dt}{dx}=\frac{dy}{dx} \qquad \ldots(3)$

Using (1), (2) and (3)

$$\Rightarrow 1-\frac{dt}{dx}=\frac{1-t}{1+t}$$
$$\Rightarrow \frac{dt}{dx}=1-\left(\frac{1-t}{1+t}\right)$$
$$\Rightarrow \frac{dt}{dx}=\frac{(1+t)-(1-t)}{1+t}$$
$$\Rightarrow \frac{dt}{dx}=\frac{2t}{1+t}$$
$$\Rightarrow \left(\frac{1+t}{t}\right)dt=2dx$$
$$\Rightarrow \left(1+\frac{1}{t}\right)dt=2dx$$

Integrating both sides, we get:

$$\Rightarrow t+\log|t|=2x+C$$
$$\Rightarrow (x-y)+\log|x-y|=2x+C$$
$$\Rightarrow \log|x-y|=x+y+C$$

When $x=0;\ \ y=-1$

$$\log 1=0-1+C$$
$$\Rightarrow C=1$$
$$\log|x-y|=x+y+1$$

Question 12:

Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1\ (x\neq 0)$

Solution:

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1$$
$$\Rightarrow \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}$$
$$\Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This equation is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$ where $P=\frac{1}{\sqrt{x}}$ and $Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now, $I.F.=e^{\int Pdx}=e^{\int\frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$

The general solution of the given differential equation is given by,

$$y(I.F.)=\int(Q\times I.F.)dx+C$$
$$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{2\sqrt{x}}\right)dx+C$$
$$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{\sqrt{x}}dx+C$$
$$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+C$$

Question 13:

Find a particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\cos ecx\left(x\neq 0\right)$, given that $y=0$ when $x=\frac{\pi}{2}$

Solution:

$$\frac{dy}{dx}+y\cot x=4x\cos ecx$$

This equation is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$ where $P=\cot x$ and $Q=4x\text{cosec}x$

Now, $I.F.=e^{\int Pdx}=e^{\int\cot xdx}=e^{\log|\sin x|}=\sin x$

The general solution of the given differential equation is given by,

$$y(I.F.)=\int(Q\times I.F.)dx+C$$
$$\Rightarrow y\sin x=\int(4x\text{cosec}x.\sin x)dx+C$$
$$\Rightarrow y\sin x=4\int xdx+C$$
$$\Rightarrow y\sin x=4.\frac{x^2}{2}+C$$
$$\Rightarrow y\sin x=2x^2+C$$

When $x=\frac{\pi}{2};\ y=0$

Therefore,

$$\Rightarrow 0=2\times\frac{\pi^2}{4}+C$$
$$\Rightarrow C=-\frac{\pi^2}{2}$$

Thus, $y\sin x=2x^2-\frac{\pi^2}{2}$

Question 14:

Find a particular solution of the differential equation $(x+1)\frac{dy}{dx}=2e^{-y}-1$, given that $y=0$ when $x=0$.

Solution:

$$(x+1)\frac{dy}{dx}=2e^{-y}-1$$
$$\Rightarrow \frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$$
$$\Rightarrow \frac{e^y dy}{2-e^y}=\frac{dx}{x+1}$$

Integrating both sides, we get:

$$\int\frac{e^y dy}{2-e^y}=\log|x+1|+\log C \qquad \ldots(1)$$

Let $2-e^y=t$

$$\Rightarrow \frac{d}{dy}\left(2-e^y\right)=\frac{dt}{dy}$$
$$\Rightarrow -e^y=\frac{dt}{dy}$$
$$\Rightarrow -e^y dy=dt$$

Substituting this value in equation (1), we get:

$$\Rightarrow \int \frac{-dt}{t} = \log|x+1| + \log C$$

$$\Rightarrow -\log|t| = \log|C(x+1)|$$

$$\Rightarrow -\log|2-e^y| = \log|C(x+1)|$$

$$\Rightarrow \frac{1}{2-e^y} = C(x+1)$$

$$\Rightarrow 2-e^y = \frac{1}{C(x+1)}$$

When $x=0;\ y=0$

Therefore,

$$\Rightarrow 2-1 = \frac{1}{C}$$

$$\Rightarrow C = 1$$

Hence,

$$2-e^y = \frac{1}{x+1}$$

$$\Rightarrow e^y = 2-\frac{1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+2-1}{x+1}$$

$$\Rightarrow e^y = \frac{2x+1}{x+1}$$

$$\Rightarrow y = \log\left|\frac{2x+1}{x+1}\right|,\quad (x \neq -1)$$

Question 15:
The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20,000 in 1999 and 25,000 in the year 2004, what will be the population of the village in 2009?

Solution:
Let the population at any instant (t) be y.
It is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$$\Rightarrow \frac{dy}{dx} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky$$

$$\Rightarrow \frac{dy}{y} = kdt$$

$$\log y = kt + C$$

In 1999, $t = 0$ and $y = 20000$

$$\log 20000 = C$$

In 2004, $t = 5$ and $y = 25000$

$$\log 25000 = k.5 + C$$

$$\Rightarrow \log 25000 = 5k + \log 20000$$

$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$$

$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$$

In 2009, $t = 10$ years

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$$

$$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$$

$$\Rightarrow y = 31250$$

Therefore, population of village in 2009 is 31250.

Question 16:

The general solution of the differential equation $\frac{ydx - xdy}{y} = 0$ is

(A) $xy = C$ (B) $x = Cy^2$ (C) $y = Cx$ (D) $y = Cx^2$

Solution:

$$\frac{ydx-xdy}{y}=0$$

$$\Rightarrow \frac{ydx-xdy}{xy}=0$$

$$\Rightarrow \frac{1}{x}dx-\frac{1}{y}dy=0$$

$$\log|x|-\log|y|=\log k$$

$$\Rightarrow \log\left|\frac{x}{y}\right|=\log k$$

$$\Rightarrow \frac{x}{y}=k$$

$$\Rightarrow y=\frac{1}{k}x$$

$$\Rightarrow y=Cx \qquad \left(\text{where, } C=\frac{1}{k}\right)$$

Thus, the correct option is C.

Question 17:

The general solution of a differential equation of the type $\frac{dx}{dy}+P_1x=Q_1$ is

(A) $y.e^{\int P_1dy}=\int\left(Q_1e^{\int P_1dy}\right)dy+C$

(B) $y.e^{\int P_1dx}=\int\left(Q_1e^{\int P_1dx}\right)dx+C$

(C) $x.e^{\int P_1dy}=\int\left(Q_1e^{\int P_1dy}\right)dy+C$

(D) $x.e^{\int P_1dx}=\int\left(Q_1e^{\int P_1dx}\right)dx+C$

Solution:

I.F. for $\frac{dx}{dy}+P_1x=Q_1$ is $e^{\int P_1dy}$

$$x(I.F.)=\left(\int Q_1\times I.F.\right)dy+C$$

$$x.e^{\int P_1dy}=\int\left(Q_1e^{\int P_1dy}\right)dy+C$$

Thus, the correct option is C.

Question 18:

The general solution of the differential equation $e^x dy + (ye^x + 2x)dx = 0$ is

(A) $xe^y + x^2 = C$ (B) $xe^y + y^2 = C$

(C) $ye^x + x^2 = C$ (D) $ye^y + x^2 = C$

Solution:

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$ where, $P = 1$ and $Q = -2xe^{-x}$

Now,

$$I.F. = e^{\int P dx} = e^{\int dx} = e^x$$

$$y\left(I.F.\right) = \int \left(Q \times I.F.\right)dx + C$$

$$\Rightarrow ye^x = \int \left(-2xe^{-x}.e^x\right)dx + C$$

$$\Rightarrow ye^x = -\int 2x dx + C$$

$$\Rightarrow ye^x = -x^2 + C$$

$$\Rightarrow ye^x + x^2 = C$$

Thus, the correct option is C.