XI Std - Maths - Unit-5. Binomial Theorem - EM

Infinite Geometric Series (Binomial Series)

1. $\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+x^3+\cdots$

2) $\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+\cdots$

3) $(1-x)^{-2} = 1+2x+3x^2+4x^3+5x^4+6x^5+\cdots$

4) $(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-6x^5+\cdots$

$|x|<1$

5) $\frac{1}{1-2x} = (1-2x)^{-1} = 1+2x+4x^2+8x^3+\cdots$ $\quad |x|<1/2$

Binomial Theorem for rational exponent

7) $(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

6) $(1+x)^{+n} = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$

8) $(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots$

9) $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots$

10) $(1+x)^{p/q} = 1+\frac{p}{q}x+\frac{p(p-q)}{2!\,q^2}x^2+\frac{p(p-q)(p-2q)}{3!\,q^3}x^3+\cdots$

11) $(1-x)^{p/q} = 1-\frac{p}{q}x+\frac{p(p-q)}{2!\,q^2}x^2+\frac{p(p-q)(p-2q)}{3!\,q^3}x^3+\cdots$

12) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ exponential series.

13) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

14) $e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots$

15) $\frac{1}{e} = e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}$

16) $\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$

17) $\frac{e^x-e^{-x}}{2} = \frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$

18) $\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$

19) $\frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$

20) $e^{2x} = 1+\frac{2x}{1!}+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\cdots$

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21) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ Logarithmic Series.

22) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

23) $\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$

24) $\log\left(\frac{1-x}{1+x}\right) = -2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$

25) $\log(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \cdots$ $|x| < \frac{1}{2}$

26) Infinite Geometric Series $S_n = \frac{1-x^n}{1-x}$ $|x| < 1$.

27) Infinite Arithmetico-Geometric series: $S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

28) Fibonacci Sequence

$1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$x_n = x_{n-1} + x_{n-2}$.

State and prove Binomial Theorem.

If n is any positive integer then $(a+b)^n = a^n + nc_1 a^{n-1} b + \dots + nc_r a^{n-r} b^r + \dots + nc_n a^0 b^n$.

Proof Proof is by mathematical Induction method

Let $P(n) = (a+b)^n = a^n + nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 + \dots + nc_r a^{n-r} b^r + \dots + b^n$.

~~Part~~ $n=1$ $P(1): a+b = a+b$ $\therefore P(1)$ is True.

For $n=k$ $P(k) = (a+b)^k = a^k + kc_1 a^{k-1} b + kc_2 a^{k-2} b^2 + \dots + kc_r a^{k-r} b^r + \dots + b^k$.

For $n=k+1$ TPT $P(k+1) = (a+b)^{k+1} = a^{k+1} + (k+1)c_1 a^k b + (k+1)c_2 a^{k-1} b^2 + \dots$ $kc_r a^{k-r} b^r + (k+1)c_r a^{k+1-r} b^r + \dots + b^{k+1}$

$$\boxed{nc_r + nc_{r-1} = (n+1)c_r}$$

LHS: $(a+b)^{k+1}$

$= (a+b)^k \cdot (a+b)$

$= (a^k + kc_1 a^{k-1} b + kc_2 a^{k-2} b^2 + \dots + kc_r a^{k-r} b^r + \dots + b^k)(a+b)$

$= a^{k+1} + kc_1 a^k b + kc_2 a^{k-1} b^2 + \dots + kc_r a^{k-r+1} b^r + \dots + a b^k$

$+ kc_0 a^k b + kc_1 a^{k-1} b^2 + \dots + kc_{r-1} a^{k-r+1} b^r + \dots + \dots + b^{k+1}$.

$= a^{k+1} + (k+1)c_1 a^k b + (k+1)c_2 a^{k-1} b^2 + \dots (k+1)c_r a^{k+1-r} \cdot b^r + \dots + b^{k+1}$

$\Rightarrow P(k+1)$ is True.

$\therefore P(n) = (a+b)^n = a^n + nc_1 a^{n-1} b + \dots + nc_r a^{n-r} b^r + \dots + b^n$. $\forall n \in Z^+$

Note: 1) In the expansion of $(a+b)^n$ there are $n+1$ terms

2) The power of a decreases by 1 in each term and power of b increases by 1 in each term

3) General Term $T_{r+1} = nc_r a^{n-r} b^r$ $(r+1)^{Th}$ term.

4) The coefficient of $a^{n-r} b^r$ is nc_r. 5) In the expansion $(a+b)^n$ the coefficient of at equidistant from the beginning and from the end are equal $\therefore nc_r = nc_{n-r}$.

5) In the expansion of $(a+b)^n$ $n \in N$, The greatest Co-efficient is $nC_{\frac{n}{2}}$ if n is even and $nC_{\frac{n-1}{2}}$ (or) $nC_{\frac{n+1}{2}}$ if n is odd.

6) In the Expansion of $(a+b)^n$ $n \in N$.
If n is even the middle term $T_{\frac{n}{2}+1} = nC_{\frac{n}{2}} a^{n-\frac{n}{2}} b^{n/2}$
If n is odd the middle term is $T_{\frac{n-1}{2}+1}$ and $T_{\frac{n+1}{2}+1}$

Particular Case of Binomial Theorem

1) $(a-b)^n = nC_0 a^n b^0 - nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \dots + (-1)^r nC_r a^{n-r} b^r + \dots + (-1)^n nC_n a^0 b^n$.

2) Replacing a by 1 and b by x in the binomial Expansion $(a+b)^n$

$(1+x)^n = nC_0 + nC_1 x + nC_2 x^2 + \dots + nC_r x^r + \dots + nC_n x^n$.

When $x=1$ $\quad 2^n = nC_0 + nC_1 + nC_2 + \dots + nC_r + \dots + nC_n$

$(1-x)^n = nC_0 - nC_1 x + nC_2 x^2 + \dots + (-1)^r nC_r x^r + \dots + (-1)^n nC_n x^n$.

when $x=1$: $nC_0 - nC_1 + nC_2 - nC_3 \dots = 0$

$nC_0 + nC_2 + nC_4 + \dots = nC_1 + nC_3 + nC_5 + \dots = \frac{2^n}{2} = 2^{n-1}$

1) $\frac{nC_r}{nC_{r-1}} = \frac{n-r+1}{r}$ $\quad$ 2) $\frac{nC_{r+1}}{nC_r} = \frac{n-r}{r+1}$.

$(1+x)^n + (1-x)^n = 2\left[nC_0 + nC_2 x^2 + nC_4 x^4 + \dots + nC_n x^n\right]$ when n is even

$(1+x)^n - (1-x)^n = 2\left[nC_1 x + nC_3 x^3 + nC_5 x^5 + \dots (-1)^n nC_n x^n\right]$ n is odd

Coefficient of middle term $(x+y)^n$ when n is even $T_{\frac{n}{2}+1} = nC_{\frac{n}{2}}$

when n is odd $T_{\frac{n-1}{2}+1} = nC_{\frac{n-1}{2}}$

(or) $\left(T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}\right)$ $\quad T_{\frac{n+1}{2}+1} = nC_{\frac{n+1}{2}}$

Co-efficient of $(r+1)^{th}$ term in the expansion of $(1+x)^n = nC_r$

" $\quad x^r$ in the expansion of $(1+x)^n = nC_r$

" $\quad x^r$ in the expansion of $(1-x)^n = (-1)^r nC_r$

" $\quad r+1^{th}$ term " $\quad (1-x)^n = (-1)^r nC_r$.

Note: The 11^{th} term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.
There 26 terms; 11^{th} from the end $= (26-11+1)$ from the beginning.

Sums of the Binomial coefficients $C_0+C_1+C_2+\dots+C_n=2^n$

$$C_0+C_2+C_4+\dots = C_1+C_3+C_5+\dots = \frac{2^n}{2} = 2^{n-1}$$

$$\frac{nC_r}{nC_{r-1}} = \frac{n-r+1}{r}, \quad \frac{nC_{r+1}}{nC_r} = \frac{n-r}{r+1}$$

1) $\sum_{k=1}^{n} k^2 = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

2) $\sum_{k=1}^{n} k^3 = 1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

3) $\sum_{k=1}^{n} k = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots+\frac{1}{2^n} = \frac{2^n-1}{2^n}$$

If $|x|<1$ $\frac{1}{1-x} = 1+x+x^2+x^3+\dots$

$|x|<1$ $\frac{1}{1+x} = 1-x+x^2-x^3+\dots$

$|x|<\frac{1}{2}$ $\frac{1}{1-2x} = 1+2x+4x^2+8x^3+\dots$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

$$e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots$$

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

$$e^{-1} = \frac{1}{e} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots$$

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$$

$$\frac{e^x-e^{-x}}{2} = \frac{x^1}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$$

$$\frac{e^1+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots \qquad \frac{e^1-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots$$

$$e^{2x} = 1+\frac{(2x)^1}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\dots$$

$$= 1+\frac{2x}{1!}+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\frac{16x^4}{4!}+\dots$$

If $|x|<1$ $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\dots$

$|x|<1$ $\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\dots$

If $|x|<1$ $\log\left(\frac{1+x}{1-x}\right) = 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]$

$|x|<\frac{1}{2}$ $\log(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$

$= 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \cdots$

1) $AM \geq GM$, $GM \geq HM$, $\Rightarrow AM \geq GM \geq HM$.

2) If $a_1, a_2, a_3 \ldots a_n$ are n positive whole number then its AM $= \frac{a_1+a_2+a_3+\cdots+a_n}{n}$.

3) If $a_1, a_2, a_3 \ldots a_n$ are n positive whole numbers. then GM $= \sqrt[n]{a_1 a_2 a_3 \ldots a_n}$

4) If a, b, c are in AP $= 2b = a+b$.
If a, b, c are in GP $= b^2 = ab$.
If a, b, c are in HP $= b = \frac{2ab}{a+b}$.

5) If the Altitudes of a $\triangle$le are in AP then the sides of a $\triangle$le are in HP.

6) In GP the first term $a \neq 0$

7) $c, c, c, \ldots$ is an AP and also GP except $0, 0, 0, \ldots$ is not GP.

8) When we take logarithm of the elements of GP then the elements become AP.

9) If a vehicle travels at a speed of x KM per hr and travels at a speed of y Km/hr then the average speed is HM: $\frac{2xy}{x+y}$.

10) If all the series are function but all the functions are not series.

11) If all the elements of a series are Constant $c, c, c, \ldots$ then the series is said to be constant series.

12) $GM^2 = AM \times HM$. $\therefore$ AM, GM, HM are in GP

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots)$ if $|x|<1$.

Example: 5.1) Find the expansion of $(2x+3)^5$.

Sol: $a = 2x \quad b = 3 \quad n = 5$

$(a+b)^n = a^n + nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 + \dots + nc_r a^{n-r} b^r + \dots + b^n$

$= (2x)^5 + 5c_1 (2x)^4 \cdot 3 + 5c_2 (2x)^3 \cdot 3^2 + 5c_3 (2x)^2 \cdot 3^3$

$+ 5c_4 (2x)^1 3^4 + 3^5$

$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243.$

5.2) Evaluate 98^4.

Sol: $98^4 = (100-2)^4$

$= 100^4 - 4c_1 100^3 2 + 4c_2 100^2 \cdot 2^2 - 4c_3 100 \cdot 2^3 + 2^4$

$= 100000000 - 8000000 + 240000 - 3200 + 16,$

$= 92236816.$

5.3) Find the middle terms in the expansion of $(x+y)^6$

$n = 6 \quad a = x \quad b = y.$

$\frac{6}{2} + 1 = 4$, T_4 is the middle term. $\quad T_{r+1} = nc_r a^{n-r} b^r$

$r = 3.$

$T_4 = T_{3+1} = 6c_3 x^3 y^3$

$= 20 x^3 y^3$

5.4) Find the middle terms in the expansion of $(x+y)^7$.

$n = 7.$

$\left(\frac{7+1}{2}+1, \ \frac{7-1}{2}+1\right) = 4, 5 \therefore T_4, T_5$ are the middle terms.

$T_{r+1} = nc_r a^{n-r} b^r = 7c_r x^{7-r} y^r$

$T_4 = T_{3+1} = 7c_3 x^4 y^3 = 35 x^4 y^3.$

$T_5 = T_{4+1} = 7c_4 x^3 y^4 = 35 x^3 y^4.$

5.5) Find the Co-efficient of x^6 in the expansion of $(3+2x)^{10}$.

$n = 10, \ a = 3, \ b = 2x.$

$T_{r+1} = nc_r a^{n-r} b^r = 10c_r 3^{10-r} (2x)^r.$

$= 10c_r 3^{10-r} \cdot 2^r \cdot x^r.$

$x^6 = x^r \Rightarrow r = 6.$

$\therefore$ Coeff of $x^6 = 10c_6 \ 3^4 2^6 = 210 \cdot 3^4 \cdot 2^6.$

5.6) Find the co-efficient of x^3 in the expansion of $(2-3x)^7$.

$a = 2 \quad b = -3x \quad n = 7$.

$T_{r+1} = nC_r a^{n-r} b^r$

$= 7C_r a^{7-r} b^r = 7C_r 2^{7-r} (-3x)^r$

$= 7C_r 2^{7-r} (-3)^r \cdot x^r$

$\Rightarrow x^r = x^3 \Rightarrow r = 3$.

$\therefore$ Coefficient of $x^3 = 7C_3 \, 2^4 (-3)^3$

$= -\frac{7 \cdot \cancel{6} \cdot 5}{1 \cdot \cancel{2 \cdot 3}} \times 16 \times 27$

$= -15120$

5.7) The 2^{nd}, 3^{rd} and 4^{th} terms in the binomial Expansion of $(x+a)^n$ are 240, 720, 1080 for a suitable value of x find x, a and n.

$T_2 = nC_1 x^{n-1} a = 240$ — ①

$T_3 = nC_2 x^{n-2} a^2 = 720$ — ②

$T_4 = nC_3 x^{n-3} a^3 = 1080$ — ③

$\frac{nC_r}{nC_{r-1}} = \frac{n-r+1}{r}$

$\frac{②}{①} = \frac{nC_2}{nC_1} \cdot \frac{x^{n-2}}{x^{n-1}} \cdot \frac{a^2}{a} = \frac{720}{240}$

$\frac{nC_{r+1}}{nC_r} = \frac{n-r}{r+1}$

$\frac{n-2+1}{2} \cdot \frac{1}{x} \cdot \frac{a}{} = 3 \Rightarrow \frac{n-1}{2} \frac{a}{x} = 3 \Rightarrow \frac{a}{x} = \frac{6}{n-1}$ — ④

$\frac{③}{②} = \frac{n-3+1}{3} \frac{a}{x} = \frac{\cancel{1080}^{\cancel{18}\,3}}{\cancel{720}_{\cancel{8}\,2}} = \frac{3}{2}$

From ④ and ⑤ $\quad \frac{n-2}{3} \cdot \frac{a}{x} = \frac{3}{2} \Rightarrow \frac{a}{x} = \frac{9}{2(n-2)}$ — ⑤

$\frac{\cancel{6}\,2}{n-1} = \frac{\cancel{9}\,3}{2(n-2)}$

$4(n-2) = 3(n-1) \Rightarrow 4n - 8 = 3n - 3$

$n = 5$

Sub. in ① $\quad \cancel{5} x^4 a = \cancel{240}\,48$ — ⑥

Sub in ④ $\quad \frac{a}{x} = \frac{\cancel{6}}{\cancel{4}} \frac{3}{2}$ — ⑦

$⑥/⑦ = \frac{x^5 \cancel{a}}{\cancel{a}} = \frac{\cancel{48}^{16} \times 2}{\cancel{3}} = 32 = 2^5 \quad \therefore x = 2$

Sub. in ④ $\quad \frac{a}{\cancel{x}} = \frac{\cancel{6}\,3}{\cancel{4x}} \Rightarrow a = 3$.

$\therefore a = 3, x = 2, n = 5$.

Ex: 5.8) Expand $(2x-\frac{1}{2x})^4$.

$$\left(2x-\frac{1}{2x}\right)^4 = 4C_0(2x)^4\left(-\frac{1}{2x}\right)^0 + 4C_1(2x)^3\left(-\frac{1}{2x}\right)^1 + 4C_2(2x)^2\left(-\frac{1}{2x}\right)^2 + 4C_3(2x)^1\left(\frac{-1}{2x}\right)^3 + \left(\frac{-1}{2x}\right)^4$$

$$= 16x^4 - 16x^2 + 6 - \frac{1}{x^2} + \frac{1}{16x^4}$$

Ex 5.9) Expand $(x^2+\sqrt{1-x^2})^5 + (x^2-\sqrt{1-x^2})^5$.

Sol: $(x+y)^5 + (x-y)^5 = 2[nC_0x^5 + nC_2x^3y^2 + nC_4xy^4$

$$(x^2+\sqrt{1-x^2})^5 + (x^2-\sqrt{1-x^2})^5 = 2[(x^2)^5 + 5C_2(x^2)^3(1-x^2) + 5C_4(x^2)(1-x^2)^2]$$

$$= 2[x^{10} + 10x^6(1-x^2) + 5(x^2)(1+x^4-2x^2)]$$

$$= 2[x^{10} + 10x^6 - 10x^8 + 5x^2 + 5x^6 - 10x^4]$$

$$= 2[x^{10} + 15x^6 - 10x^8 - 10x^4 + 5x^2)$$

5.10) using Binomial theorem P.T $6^n - 5n$ always leaves remainder 1 which divided by 25 $\forall n \in Z^+$

Sol: $6^n - 5n - 1$ It is enough to P.T $6^n - 5n - 1$ is divisible by 25.

$$(1+x)^n = nC_0 + nC_1x + nC_2x^2 + \cdots + nC_nx^n$$

$$6^5 = (1+5)^n = nC_0 + nC_1 5 + nC_2 5^2 + \cdots + 5^n$$

$$= 1 + 5n + nC_2 5^2 + nC_3 5^3 + \cdots + 5^n$$

$$6^n - 5n - 1 = 5^2(nC_2 + nC_3 5 + \cdots + nC_n 5^{n-2})$$

$$= 25t$$

$\therefore 6^n - 5n - 1$ is divisible by 25

9/5.1) If n is a positive integer ST $9^{n+1} - 8n - 9$ is always divisible by 64.

Sol: $(1+x)^{n+1} = (n+1)C_0 + (n+1)C_1x + (n+1)C_2x^2 + \cdots + (n+1)C_n x^n$.

$$9^{n+1} = (1+8)^{n+1} = (n+1)C_0 + (n+1)C_1 8 + (n+1)C_2 8^2 + \cdots + (n+1)C_n 8^n$$

$$= 1 + 8(n+1) + 8^2(n+1)C_2 + \cdots + 8^n$$

$$= 1 + 8n + 8 + 8^2((n+1)C_2 + \cdots + 8^{n-2})$$

$$9^{n+1} - 8n - 9 = 8^2((n+1)C_2 + (n+1)C_3 + \cdots + 8^{n-2})$$

$= 64t$ $\therefore$ which is divisible by 64

14) 5.1 - If the binomial coefficients of the three consecutive terms in the expansion of $(a+x)^n$ are in the ratio 1:7:42 find n.

$nC_{r-1} : nC_r : nC_{r+1} = 1 : 7 : 42$

$\frac{nC_r}{nC_{r-1}} = \frac{7}{1} \Rightarrow \frac{n-r+1}{r} = 7 \Rightarrow n-r+1 = 7r$

$n - 8r + 1 = 0$

$\frac{nC_{r+1}}{C_r} = \frac{\cancel{42}^{6}}{\cancel{7}} \Rightarrow \frac{n-r}{r+1} = 6 \qquad n - r = 6r + 6$

$n - 7r = 6$

$n - 8r = -1$

$r = 7$

$\therefore n - 49 = 6$

$n = 55$

15) 5.1 In the binomial coefficients $(1+x)^n$, the coefficient of 5^{th}, 6^{th}, 7^{th} terms are in AP find the values of n.

Sol: Coefficients of T_5, T_6, T_7 are in AP.

Co-eff. of T_5, T_6, T_7 are nC_4, nC_5, nC_6 are in AP.

$nC_5 = \frac{nC_4 + nC_6}{2}$

$2(nC_5) = nC_4 + nC_6 \qquad \frac{nC_{r+1}}{nC_r} = \frac{n-r}{r+1}$

$2 = \frac{nC_4}{nC_5} + \frac{nC_6}{nC_5}$

$2 = \frac{5}{n-4} + \frac{n-5}{6} \Rightarrow 2 = \frac{30 + (n-4)(n-5)}{6(n-4)}$

$12n - 48 = 30 + n^2 - 20n$

$n^2 - 21n + 98 = 0$

$(n-14)(n-7) = 0$

$\therefore n = 7, 14.$

12) 5.1 If a and b are distinct integers P.T $a^n - b^n$ is divisible by $a-b$. $\forall n \in N$.

Sol: $a^n = [(a-b)+b]^n$

$= nC_0 (a-b)^n b^0 + nC_1 (a-b)^{n-1} b + nC_2 (a-b)^{n-2} b^2 + \cdots + nC_n b^n.$

$a^n - b^n = (a-b)^n + nC_1 (a-b)^{n-1} b + nC_2 (a-b)^{n-2} b^2 + \cdots nC_{n-1} (a-b) b^{n-1}$

$= (a-b)\left[(a-b)^{n-1} + nC_1 (a-b)^{n-2} b + \cdots + nC_{n-1} b^{n-1}\right]$

$= (a-b)t \qquad \therefore a^n - b^n$ is divisible by $a-b$.

13) 5.1 In the binomial expansion $(a+b)^n$, The coefficient of 4^{th} and 13^{th} are equal to each other then find n

Coefficient of 4^{th} term is $= nC_3$

" 13^{th} term is : nC_{12}

Given that $nC_3 = nC_{12}$ $\qquad nC_x = nC_y \Rightarrow x+y=n$

$n = 12+3 = 15.$

16) 5.1 P.T $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \frac{2n!}{(n!)^2}$

Sol: $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n.$

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_nx^0$

Equating the co-eff of x^n.

$(1+x)^n(1+x)^n = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2$

$(1+x)^{2n} = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2$

$2nC_n = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2$

$\frac{2n!}{n!\,n!} = C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2$

11) 5.1 If n is the positive integer and r is non-negative integer P.T the Co-eff. of x^r and x^{n-r} are equal.

Co-eff. of $x^r = nC_r$

Co-eff. of $x^{n-r} = nC_{n-r}$

by Property $nC_r = nC_{n-r}$.

10) 5.1 If n is odd positive integer P.T the co-eff of middle terms in the expansion of $(x+y)^n$ are equal.

In this expansion $T_{\frac{n+1}{2}+1}, T_{\frac{n-1}{2}+1}$ are the middle terms

$T_{\frac{n+3}{2}}, T_{\frac{n+1}{2}}$ are the middle terms.

Co-eff of $T_{\frac{n-1}{2}+1} = C_{\frac{n-1}{2}}$ and $T_{\frac{n+1}{2}+1} = C_{\frac{n+1}{2}}$

$C_{\frac{n-1}{2}} = C_{n-\frac{(n+1)}{2}}$

$C_{\frac{n-1}{2}} = C_{\frac{n+1}{2}}$

8) 5.1 Find the last two digits of the number 3^{600}

Sol: $3^{600} = (3^2)^{300} = 9^{300}$

$= (10-1)^{300} = 300C_0 10^{300} - 300C_1 10^{299} + 300C_2 10^{298} + \cdots + 300C_{299} 10 + 300C_{300} 1^n$.

$= 10(300C_0 10^{299} - 300C_1 10^{298} + \cdots + 300C_{299}] + 1$.

= which is multiple of 10 + 1

$\therefore$ Last two digits 0, 1.

7/5.1 Find the constant term of $\left(2x^3 - \frac{1}{3x^2}\right)^5$

$a = 2x^3 \quad b = -\frac{1}{3x^2} \quad n = 5$

$T_{r+1} = nC_r a^{n-r} b^r = 5C_r (2x^3)^{5-r} \left(-\frac{1}{3x^2}\right)^r$

$= 5C_r (2)^{5-r} x^{15-3r} \frac{(-1)^r}{3^r x^{2r}}$

$= 5C_r \frac{(2)^{5-r}}{3^r} (-1)^r \cdot x^{15-5r}$

Here $15 - 5r = 0$

$r = 3$.

$\therefore$ Constant term $= \frac{5C_3 \, 2^2 (-1)^3}{3^3} = -\frac{10 \times 4}{27} = -\frac{40}{27}$

Ex 5.1) 2) $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$.

$(x+y)^4 + (x-y)^4 = 2[x^4 + 4C_2 x^2 y^2 + 4C_4 x^0 y^4]$.

$= 2\left[(2x^2)^4 + 6(2x^2)^2 \cdot 9(1-x^2) + 81(1-x^2)^2\right]$ $\quad \frac{24 \times 9}{216}$

$= 2(16x^8 + 216x^4(1-x^2) + 81(1+x^4-2x^2))$

$= 2[16x^8 + 216x^4 - 216x^6 + 81 + 81x^4 - 162x^2]$

$= 2[16x^8 + 297x^4 - 216x^6 - 162x^2 + 81]$

- Which is larger $(1.01)^{1000000}$ (or) 10,000 ? NCERT

Consider $(1.01)^{1000000} - 10,000$

$(1+0.01)^{10,00,000} - 10,000$

$= {}^{10,00,000}C_0 (0.01)^0 + {}^{10,00,000}C_1 (0.01)^1 + {}^{1,00,000}C_2 (0.01)^2 \dots$

$\dots {}^{1,00,000}C_{1,00,000} (.01)^{1000000} - 10,000$

$= 1 + 10,00,000 \times 0.01 + \text{other +ve terms}) - 10,000$

$= 1 + 10,000 + \text{other +ve terms} > 0$

$\Rightarrow (1.01)^{10,00,000} > 10,000$

P.T AM $\geq$ GM. (Theorem)

Let a, b are two non-negative number.

$AM = \frac{a+b}{2} \quad GM = \sqrt{ab}.$

Consider $= (a+b)^2 - 4ab = (a-b)^2 \geq 0$

$\Rightarrow (a+b)^2 - 4ab \geq 0$

$\Rightarrow (a+b)^2 \geq 4ab.$

$\Rightarrow \left(\frac{a+b}{2}\right)^2 \geq ab. \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$

$\therefore$ AM $\geq$ GM.

. P.T GM $\geq$ HM (Theorem)

Let a, b are any two non-negative numbers.

$GM = \sqrt{ab} \quad HM = \frac{2ab}{a+b}.$

Consider $GM - HM = \sqrt{ab} - \frac{2ab}{a+b}$

$= \frac{\sqrt{ab}\,(a+b) - 2ab}{a+b}$

$= \frac{\sqrt{ab}\,[(a+b) - 2\sqrt{ab})}{a+b}.$

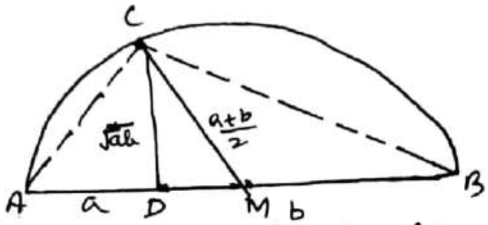
$= \frac{\sqrt{ab}\,[(\sqrt{a}-\sqrt{b})^2]}{a+b} \geq 0$

$\Rightarrow GM - HM \geq 0$

$GM \geq HM.$

Write Geometrical Meaning of AM $\geq$ GM -

Diagram.



Let a, b are non-negative real numbers.

Let $a>0$, $b>0$ Draw a semicircle with $AB = a+b$ as diameter

M is the mid point of AB.

$\therefore AM = \frac{a+b}{2} = MB$.

$\therefore$ Radius of the semicircle is $= \frac{a+b}{2}$

Choose a point on AB s.t $AD = a$ and $DB = b$.

Draw $AD \perp AM$ through D.

Draw CA, CB and CM.

Now $MD = \frac{a+b}{2} - a$.

$\triangle ACD$ and $\triangle CBD$ are similar.

$$\frac{CD}{AD} = \frac{BC}{CD} \Rightarrow CD^2 = AD \cdot BC$$
$$= ab$$
$$CD = \sqrt{ab}.$$

$\therefore CD \leq CM$

$$\sqrt{ab} \leq \frac{a+b}{2} \Rightarrow \frac{a+b}{2} \geq \sqrt{ab}$$
$$AM \geq GM.$$

EXERCISE - 5.2.

4/5.2. The product of the three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to third number the resulting number forming AP Find the numbers in GP.

Let the three numbers in GP are $\frac{a}{r}$, a, ar.

$$\frac{a}{\cancel{r}} \cdot a \cdot a\cancel{r} = 5832$$

$$a^3 = 5832$$

$$a = 18$$

Also $\frac{a}{r}$, $a+6$, $ar+9$ are in AP

(ie) $a+6-\frac{a}{r} = ar+9-a-6$ (common difference)

$$18+6-\frac{a}{r} = 18r+9-18-6$$

$$24-\frac{18}{r} = 18r-15$$

$$\therefore 39 = 18r+\frac{18}{r} \Rightarrow 39 = \frac{18r^2+18}{r}$$

$$\Rightarrow 18r^2-39r+18=0$$

324
$-27-12$

$$18r^2-27r-12r+18=0$$

$$9r(2r-3)-6(2r-3)=0$$

$$(2r-3)(9r-6)=0$$

$$r = 3/2 \text{ or } r = \frac{6}{9} = 2/3$$

When $a=18$, $r=3/2$

$\frac{18}{3/2}$, 18, $18\cdot\frac{3}{2} \Rightarrow 12, 18, 27$ are in GP.

When $a=18$ $r=2/3$ $27, 18, 12$ are in GP.

5/5.2). Write the n^{th} term of the sequence $\frac{3}{1^2\cdot2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}$

Nr are $3, 5, 7$ they are in AP with $a=3$ $d=2$

$$t_n = a+(n-1)d = 3+(n-1)2$$

$$= 2n+1$$

Dr are $1^2 2^2, 2^2 3^2, 3^2 4^2 \ldots$

$$a_n = n^2(n+1)^2$$

$$\therefore t_n = \frac{2n+1}{n^2(n+1)^2} = \frac{2n+1+n^2-n^2}{n^2(n+1)^2} = \frac{(n+1)^2-n^2}{n^2(n+1)^2}$$

$$= \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

6) S.2 If t_k is the k^{th} term of GP. Then S.T t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer.

Sol: $t_n = ar^{n-1}$

$t_k = ar^{k-1}$, $t_{n-k} = ar^{n-k-1}$, $t_{n+k} = ar^{n+k-1}$

Common ratio: $\dfrac{t_n}{t_{n-k}} = \dfrac{\cancel{a}r^{n-1}}{\cancel{a}r^{n-k-1}} = r^k$.

$\dfrac{t_{n+k}}{t_n} = \dfrac{\cancel{a}r^{n+k-1}}{\cancel{a}r^{n-1}} = r^k$.

$\because$ Common ratios are same $\therefore t_{n-k}, t_n, t_{n+k}$ are in GP.

7) 5.2 If a, b, c are in GP and if $a^{1/x} = b^{1/y} = c^{1/z}$. Then P.T x, y, z are in AP.

Sol: $a^{1/x} = b^{1/y} = c^{1/z} = k$.

$a = k^x$, $b = k^y$ $c = k^z$.

$\because$ a, b, c are in GP. $b^2 = ac$

$k^{2y} = k^x \cdot k^z$

$= k^{x+z}$

$\Rightarrow 2y = x+z$ Hence x, y, z are in AP.

8) 5.2 The A.M of two numbers exceeds their GM by 10 and HM by 16 Find the numbers.

Sol: Let the numbers be a, b.

$AM = \dfrac{a+b}{2}$, $GM = \sqrt{ab}$ $HM = \dfrac{2ab}{a+b}$.

Given $A-G=10$ $A-H=16$

$G = A-10$ $H = A-16$.

$G^2 = AH$

$(A-10)^2 = A(A-16)$

$\cancel{A^2} - 20A + 100 = \cancel{A^2} - 16A$

$4A = 100$ $A = 25$.

(ie) $\dfrac{a+b}{2} = 25 \Rightarrow a+b = 50$

$G = A-10 = 25-10 = 15 \Rightarrow \sqrt{ab} = 15$

$ab = 225$

$b = \dfrac{225}{a}$

$a + \dfrac{225}{a} = 50$

$a^2 - 50a + 225 = 0$

HWD.

$(a-45)(a-5)=0 \Rightarrow a=5,45$

when $a=5$ $\quad b=\frac{225}{5}$ 45

$a=45$ $\quad b=5$. $\therefore$ The numbers are $5, 45$.

9) 5.2 If the roots of the equation $(q-r)x^2+(r-p)x+(p-q)=0$ are equal then S.T p, q, r are in AP.

$(q-r)x^2+(r-p)x+(p-q)=0$ $\quad a=q-r$, $b=r-p$, $c=p-q$.

$\therefore$ The roots are equal $b^2-4ac=0$

$(r-p)^2-4(q-r)(p-q)=0$

$(r^2+p^2-2rp)-4(pq-q^2-rp+qr)=0$.

$r^2+p^2+4q^2+2rp-4pq-4rp=0$

$(r+p-2q)^2=0 \Rightarrow r+p-2q=0$

$2q=r+p$.

$\therefore$ p, q, r are in AP.

10) 5.2 If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of GP. S.T $(q-r)\log a+(r-p)\log b+(p-q)\log c=0$.

Given $a=t_p$ $\quad b=t_q$ $\quad c=t_r$. $\qquad$ $\underline{t_n=ar^{n-1}}$

$a=xy^{p-1}$ $\quad b=xy^{q-1}$ $\quad c=xy^{r-1}$

LHS $(q-r)\log a+(r-p)\log b+(p-q)\log c$

$=\log a^{q-r}+\log b^{r-p}+\log c^{p-q}$.

$=\log\left(a^{q-r}\times b^{r-p}\times c^{p-q}\right)$

$=\left(xy^{p-1}\right)^{q-r}\left(x y^{q-1}\right)^{r-p}\times\left(xy^{r-1}\right)^{p-q}$.

$=x^{q-r+r-p+p-q}\cdot y^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$=x^0y^0=1$,

$\log\left(a^{q-r}\times b^{r-p}\times c^{p-q}\right)=\log 1=0$

$\frac{1}{5.2}$ write the first 6 terms and classify they are AP, GP, AGP.

1) $\frac{1}{2^{n+1}} = a_n \quad a_1 = \frac{1}{2^2}, a_2 = \frac{1}{2^3}, a_3 = \frac{1}{2^4}, a_4 = \frac{1}{2^5}, a_6 = \frac{1}{2^6}$

$\therefore$ The 6 terms are $\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}$.

Common Ratio $\frac{1}{2^3} \times \frac{2^2}{1} = \frac{1}{2}$ $\therefore$ They are in GP.

$\frac{1}{2^4} \times \frac{2^3}{1} = \frac{1}{2}$

2) $a_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$

$a_1 = \frac{2\times 3}{4\times 5} = \frac{6}{20} = \frac{3}{10} \quad a_2 = \frac{\cancel{3}\times \cancel{4}2}{5\times \cancel{6}2} = \frac{2}{5}$

$a_3 = \frac{2\cancel{4}\times 5}{3\cancel{6}\times 7} = \frac{10}{21} \quad a_4 = \frac{5\times \cancel{6}3}{7\times \cancel{8}4} = \frac{15}{28}$

$a_5 = \frac{3\cancel{6}\times 7}{4\cancel{8}\times 9} = \frac{\cancel{21}7}{\cancel{36}12} \quad a_6 = \frac{7\times \cancel{8}4}{9\times \cancel{10}5} = \frac{28}{45}$

$\therefore$ Sequence is

$\frac{3}{10}, \frac{2}{5}, \frac{10}{21}, \frac{15}{28}, \frac{7}{12}, \frac{28}{45}$. They are neither AP, GP nor HP and AGP.

3) $a_n = 4\left(\frac{1}{2}\right)^n$

$a_1 = 4\times\frac{1}{2} = 2, a_2 = 4\cdot\frac{1}{4} = 1 \quad a_3 = 4\cdot\frac{1}{8} = \frac{1}{2} \quad a_4 = 4\times\frac{1}{16} = \frac{1}{4}$

$a_5 = \cancel{4}\times\frac{1}{\cancel{32}8} = \frac{1}{8} \quad a_6 = 4\times\frac{1}{64} = \frac{1}{16}$.

$\therefore$ The Sequence is $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

Common ratio $\frac{t_2}{t_1} = \frac{1}{2}$

$\frac{t_3}{t_2} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$ $\therefore$ They are in GP.

4) $a_n = \frac{(-1)^n}{n}$.

$a_1 = \frac{(-1)^1}{1} = -1, \; a_2 = \frac{(-1)^2}{2} = \frac{1}{2}, a_3 = \frac{(-1)^3}{3} = -\frac{1}{3} \; a_4 = \frac{(-1)^4}{4} = \frac{1}{4} \quad a_5 = -\frac{1}{5}$

$a_6 = \frac{1}{6}$

$\therefore$ The sequence is $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$

5) $a_n = \frac{2n+3}{3n+4}$

$a_1 = \frac{5}{7}$, $a_2 = \frac{7}{10}$ $a_3 = \frac{9}{13}$, $a_4 = \frac{11}{16}$, $a_5 = \frac{13}{19}$, $a_6 = \frac{15}{22}$

$\therefore$ The sequence is $5/7, 7/10, 9/13, 11/16, 13/19, 15/22$

This is neither AP, GP nor AGP.

6) $a_n = 2018$

$a_1 = 2018, a_2 = 2018$ - - - - - -

all are same $\therefore$ it is AP, GP and AGP.

7) $a_n = \frac{3n-2}{3^{n-1}}$

$a_1 = \frac{1}{1} = 1$ $a_2 = \frac{4}{3}$ $a_3 = \frac{7}{9}$ $a_4 = \frac{10}{27}$ $a_5 = \frac{13}{81}$ $a_6 = 16/243$.

Sequence is $1, 4/3, 7/9, 10/27, \frac{13}{81}, \frac{16}{243}$.

1, 4, 7, 10, 13, 16 are in AP

1, 3, 9, 27, 81, 243 are in GP $\therefore$ it is AGP.

2) Write the first 6 terms of the sequence whose n^{th} term is

i) $a_n = n+1$ if n is odd
$= n$ if n is even.

$a_1 = 1+1 = 2$ | $a_2 = 2$ | $a_3 = 3+1 = 4$ | $a_4 = 4$ | $a_5 = 5+1 = 6$ | $a_6 = 6$.

$\therefore$ The sequence is 2, 2, 4, 4, 6, 6.

2) $a_n = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n=2 \\ a_{n-1}+a_{n-2} & \text{if } n>2 \end{cases}$

$a_1 = 1$, | $a_2 = 2$ | $a_3 = a_2 + a_1 = 3$ | $a_4 = a_3 + a_2 = 3+2 = 5$ | $a_5 = a_4 + a_3 = 5+3 = 8$ | $a_6 = a_5 + a_4 = 8+5 = 13$.

$\therefore$ The sequence is 1, 2, 3, 5, 8, 13.

3) $a_n = \begin{cases} n & \text{if } n = 1, 2, \text{ or } 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3. \end{cases}$

$a_1 = 1, \ a_2 = 2, \ a_3 = 3$

$a_4 = a_3 + a_2 + a_1 = 6$

$a_5 = a_4 + a_3 + a_2 = \cancel{9}\,11$

$a_6 = a_5 + a_4 + a_3 = \cancel{16}\,20$

$\therefore$ The sequence is $1, 2, 3, 6, 11, 20$

3) write n^{th} term of the following sequence.

1) $2, 2, 4, 4, 6, 6$

odd terms $2, 4, 6$

even terms $2, 4, 6$

$$a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is a even.} \end{cases}$$

2) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}.$

It is clear that $a_n = \frac{n}{n+1}$

3) $1/2, 3/4, 5/6, 7/8, 9/10$

Nr : $1, 3, 5, 7, 9 = 2n-1$

Dr $2, 4, 6, 8, 10, = 2n.$

$$a_n = \frac{2n-1}{2n} = 1 - \frac{1}{2n}.$$

4) $6, 10, 4, 12, 2, 14, 0, 16, -2$

odd terms are $6, 4, 2, 0, -2$

$a = 6, \ d = -2 \quad a_n = a + (n-1)d$

$= 6 + (n-1)(-2)$

$= 6 - 2n + 2 = 8 - 2n.$

Even terms are

$10, 12, 14, 16, \ldots \quad a_n = 8 + 2n$

$$\therefore a_n = \cancel{\frac{8-2n}{8+2n}} \begin{cases} 8 - 2n & \text{if } n \text{ is odd} \\ 8 + 2n & \text{if } n \text{ is even.} \end{cases}$$

5.16. Find the sum of n terms of the series $1+\frac{6}{7}+\frac{11}{49}+\frac{16}{343}+\cdots$

Sol: $a=1,\ d=5,\ r=\frac{1}{7}$

$$S_n = \frac{a-(a+(n-1)d)r^n}{1-r} + dr\left(\frac{1-r^{n-1}}{(1-r)^2}\right)$$

$$= \frac{1-(1+5(n-1))\frac{1}{7^n}}{1-\frac{1}{7}} + 5\cdot\frac{1}{7}\left(\frac{1-\frac{1}{7^{n-1}}}{(1-\frac{1}{7})^2}\right)$$

$$\frac{1-(1+5n-5)\frac{1}{7^n}}{6/7} + \frac{\frac{5}{7}\left(1-\frac{1}{7^{n-1}}\right)}{\frac{36}{72}}$$

$$\frac{7^n-5n+4}{6\cdot 7^{n-1}} + \frac{5\left(7^{n-1}-1\right)}{7^{n-2}\cdot 36}.$$

5.17. Find the sum of first n terms of the series $\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots$

If $t_k = \dfrac{1}{\sqrt{k}+\sqrt{k+1}}$

then $t_k = \dfrac{\sqrt{k}-\sqrt{k+1}}{(\sqrt{k}+\sqrt{k+1})(\sqrt{k}-\sqrt{k+1})} = \dfrac{\sqrt{k}-\sqrt{k+1}}{k-(k+1)} = \dfrac{\sqrt{k}-\sqrt{k+1}}{-1}$

$= \sqrt{k+1}-\sqrt{k}.$

$\therefore t_1+t_2+t_3+\cdots+t_n = \cancel{\sqrt{2}}-\sqrt{1}+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\sqrt{4}-\cancel{\sqrt{3}}+\cdots+\sqrt{n+1}-\cancel{\sqrt{n}}$

$= \sqrt{n+1}-1$

5.18) Find the sum of $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}$

$t_k = \dfrac{1}{k(k+1)} = \dfrac{A}{k}+\dfrac{B}{k+1} \Rightarrow 1 = A(k+1)+B(k)$

$A=1 \quad A+B=0$

$B=-1$

$= \dfrac{1}{k}-\dfrac{1}{k+1}$

$t_1+t_2+t_3+\cdots t_n = \left(1-\cancel{\tfrac{1}{2}}\right)+\left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right)+\left(\cancel{\tfrac{1}{3}}-\tfrac{1}{4}\right)+\cdots\left(\cancel{\tfrac{1}{n}}-\tfrac{1}{n+1}\right)$

$= 1-\dfrac{1}{n+1} = \dfrac{n+1-1}{n+1} = \dfrac{n}{n+1}.$

EXERCISE - 5.3

$\frac{1}{5.3}$ Find the sum of the first 20 terms of the AP having the sum of the first 10 terms as 52 and sum of the first 15 terms is 77.

Sol: $S_n = \frac{n}{2}[2a+(n-1)d]$.

$S_{10} = 5[2a+9d] = 52 \Rightarrow 10a+45d = 52$ —①

$S_{15} = \frac{15}{2}[2a+14d] = 77 \Rightarrow 30a+210d = 154$ —②

①×3 $\quad 30a+135d = 156$

$75d = -2$

$d = -2/75$

$\therefore a = \frac{266}{50} = \frac{133}{25}$

$S_{20} = \frac{20}{2}[2a+19d] = 10\left[\frac{266}{25} + 19\left(-\frac{2}{75}\right)\right]$

$= 10\left[\frac{266}{25} - \frac{38}{75}\right] = 10\left[\frac{798-38}{75}\right] = \frac{10(760)}{75}$

$= \frac{304}{3}$.

$\frac{2}{5.3}$ Find the sum up to 17^{th} term of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$

$T_n = \frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots+(2n-1)} = \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^2} = \frac{\cancel{n^2}(n+1)^2}{4\cancel{n^2}}$

$= \frac{n^2+2n+1}{4}$

$S_n = \sum_{k=1}^{n} \frac{1}{4}(k^2+2k+1)$

$= \frac{1}{4}\left[\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$

$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} + n\right]$

$= \frac{1}{4}\left[\frac{n(2n^2+3n+1)+6n^2+6n+6n}{6}\right]$

$= \frac{1}{24}\left[2n^3+9n^2+12n\right]$

$= \frac{n}{24}\left[2n^2+9n+12\right]$

$S_{17} = \frac{17}{24}\left[2\cdot 17^2 + 9\cdot 17 + 12\right] = \frac{17}{24}[578+153+13]$

$= \frac{17}{24}[744] = 527$

$\frac{3}{5.3}$ Compute the sum of the first n terms of the following series.

1) $8+88+888+\cdots$ 2) $6+66+666+\cdots$

i) $8+88+888+\cdots)$

$= 8(1+11+111+\cdots)$

$= \frac{8}{9}(9+99+999+\cdots)$

$= \frac{8}{9}((10-1)+(100-1)+(1000-1))$

$= \frac{8}{9}\left[(10+10^2+10^3+\cdots)-(1+1+1\cdots n\text{ terms})\right]$

$= \frac{8}{9}\left(\frac{10(10^n-1)}{10-1}-n\right)$

$= \frac{8}{9}\left[\frac{10^{n+1}-10-9n}{9}\right]$

$= \frac{8}{81}\left[10^{n+1}-10-9n\right]$

ii) $6+66+666+\cdots)$

$= 6(1+11+111+\cdots)$

$= \frac{6}{9}(9+99+999+\cdots)$

$= \frac{6}{9}((10-1)+(10^2-1)+(10^3-1)\cdots$

$= \frac{6}{9}\left[(10+10^2+10^3\cdots)-(1+1+1\cdots n\text{ terms})\right]$

$= \frac{6}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$

$= \frac{6}{9}\left[\frac{10^{n+1}-10-9n}{9}\right]$

$= \frac{\cancel{6}2}{\cancel{81}27}\left[10^{n+1}-10-9n\right]$

$= \frac{2}{27}\left[10^{n+1}-10-9n\right].$

$\frac{4}{5.3}$ Compute the sum of the first n terms of

$1+(1+4)+(1+4+4^2)+\cdots$

Sol: $T_K = 1+4+4^2+\cdots+4^{K-1} =$

$S_K = \frac{4^n-1}{4-1} = \frac{4^K-1}{3} = \frac{1}{3}(4^K-1)$

$S_n = \sum_{K=1}^{n}\frac{1}{3}(4^K-1)$

$= \frac{1}{3}\left[\sum_{K=1}^{n}4^K-\sum_{K=1}^{n}1\right]$

$= \frac{1}{3}\left[4+4^2+\cdots+4^n\right]-\frac{n}{3}$

$= \frac{1}{3}\left[\frac{4(4^n-1)}{4-1}\right]-\frac{n}{3}$

$= \frac{4}{9}(4^n-1)-\frac{n}{3}$

$\frac{5}{5.3}$ Find the general term and sum of the n terms of the sequence.

$1, 4/3, 7/9, \frac{10}{27}, \cdots$

Nr: $1, 4, 7, 10 \cdots$ $a=1, d=3.$ $T_n = a+(n-1)d$ $= 1+(n-1)3.$

Dr: $1, 3, 9, 27 \cdots$ $a=1\ r=3$ $T_n = 1\cdot 3^{n-1}$

$T_r = \frac{1+(n-1)3}{3^{n-1}} = \frac{3n-2}{3^{n-1}}$

This is AGP. $a=1,\ d=3,\ r=\frac{1}{3}$.

$$S_n = \frac{a-(a+(n-1)d)r^n}{1-r} + d\cdot r\,\frac{1-r^{n-1}}{(1-r)^2}$$

$$\frac{1-(1+(n-1)3)\frac{1}{3^n}}{1-\frac{1}{3}} + \cancel{3}\cdot\frac{1}{\cancel{3}}\left(\frac{1-\left(\frac{1}{3}\right)^{n-1}}{\left(1-\frac{1}{3}\right)^2}\right)$$

$$=\frac{1-(3n-2)\frac{1}{3^n}}{1-\frac{1}{3}} + \frac{1-\frac{1}{3^{n-1}}}{\left(\frac{2}{3}\right)^2}$$

$$=\frac{3^n-(3n-2)}{3^n\left(\frac{2}{3}\right)} + \frac{3^{n-1}-1)}{3^{n-1}\cdot\frac{4}{3^2}}$$

$$=\frac{3^n-(3n-2)}{2\cdot 3^{n-1}} + \frac{3^{n-1}-1}{3^{n-3}\cdot 4}$$

$$=\frac{3^n-3n+2}{2\cdot 3^{n-1}} + \frac{3^{n-1}-1}{3^{n-3}\cdot 4}.$$

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6 / 53 Find the value of n if the sum to n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\cdots$ is $435\sqrt{3}$.

Sol: $\sqrt{3}+\sqrt{75}+\sqrt{243}+\cdots$

$a=\sqrt{3}\quad d=5\sqrt{3}-\sqrt{3}=4\sqrt{3}.\quad S_n=435\sqrt{3}$

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$435\sqrt{3}=\frac{n}{2}[2\sqrt{3}+(n-1)4\sqrt{3}]$$

$$870\sqrt{3}=n[2\sqrt{3}+4n\sqrt{3}-4\sqrt{3}]$$

$$=n[4n\sqrt{3}-2\sqrt{3}]$$

$$=2[2n^2\sqrt{3}-n\sqrt{3}],$$

$$435\cancel{\sqrt{3}}=2n^2\cancel{\sqrt{3}}-n\cancel{\sqrt{3}}-435\cancel{\sqrt{3}}=0$$

$$=2n^2-n-435=0$$

$$(n-15)(2n+29)=0$$

$$n=15,\ n=-\frac{29}{2}\quad\therefore n=15.$$

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7) S.T the sum of the $(m+n)^{th}$ and $(m-n)^{th}$ term of an AP is equal
5.3 to the twice the m^{th} term.

Sol: $T_n = a + (n-1)d$

$T_{m+n} = a + (m+n-1)d$.

$T_{m-n} = a + (m-n-1)d$.

$T_{m+n} + T_{m-n} = 2a + (m+\cancel{n}-1+m-\cancel{n}-1)d)$

$= 2a + (2m-2)d$

$= 2(a+(m-1)d$.

$= 2T_m$. $\therefore T_{m+n}, T_m, T_{m-n}$ are in AP.

8/5.3 A man repays an amount of Rs 3250 by paying Rs 20. in the first month and then increases the payment by Rs 15 per month. How long it will take him to clear the amount

Sol: $S_n = 3250$ $a = 20$ $d = 15$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$3250 = \frac{n}{2}[40 + (n-1)15]$

$6500 = n[40 + (n-1)15]$

$= n[40 + 15n - 15]$

$= n[15n + 25]$

$= 15n^2 + 25n - 6500 = 0$

$3n^2 + 5n - 1300 = 0$

$(n-20)(3n+65) = 0. \Rightarrow n = 20,\ \cancel{n = \frac{-65}{3}}$

Total number of months required $= 20$.

9) In a race 20 balls are placed in a line of intervals of 4 mts. with the first ball 24 mts away from the starting point. A contestant is required to ~~the~~ bring the ball back to the starting point one at a time. How far would the contestant run to bring back all the balls.

Sol: Distance travelled to bring first ball $= 24 + 24$ mts.

" $= 2(24+4) = 56$

" $= 2(24+8) = 64$.

" $= 2(24+12) = 72$

48, 56, 54, - - - -

$a = 48,\ d = 8\ n = 20$

$S_n : \frac{n}{2}[2a + (n-1)d]$

$= \frac{\cancel{20}^{10}}{\cancel{2}}[96 + 152] = 10(248) = 2480m.$

10) The number of bacterias in certain culture doubles every hr If there were 30 basterias present originally, how many bacteria will be present at the end of the 2^{nd} hr, 4^{th} hr, and n^{th} hr.

It is in GP $a = 30,\ r = 2$

end of 2^{nd} hr $= t_3 = ar^2 = 30 \times 4 = 120$

end of 4^{th} hr $= t_5 = ar^4 = 30 \times 16 = 480$

end of n^{th} hr $= t_{n+1} = ar^{n+1-1} = 30 \times 2^{n}$

11) What will Rs 500 amount to in 10 years after the deposit in a bank which pays annual interest rate of 10% compounded annually.

$P = 500\quad R = 10\%\quad n = 10$

$A = P\left(1 + \frac{R}{100}\right)^n$

$= 500\left(1 + \frac{10}{100}\right)^{10}$

$= 500 \times \left(\frac{11}{10}\right)^{10} = 500 \times (1.1)^{10}$

12) In a certain town, a viral disease caused severe health hazards upon its peoples disturbing their normal life. It was found that on each day the virus which caused the disease in GP. The amount of infections virous particle gets doubled each day, being 5 particles on the first day. Find the day when particles just grow over 1,50,000 units.

Sol: $a = 5,\ r = 2$

$5, 10, 20, 40 \cdots 1{,}50{,}000$

$ar^{n-1} > 1{,}50{,}000$

$5r^{n-1} > 1{,}50{,}000$

$r^{n-1} > 30{,}000$

$2^{n-1} > 30{,}000$

$2^{15} = 32768$

$\because 2^{15} > 30{,}000$

$\Rightarrow n - 1 = 14$

$n = 14 + 1$

$= \underline{\underline{15}}$

10

EXERCISE - 5.4

5.19) Find the sum $1+\frac{4}{5}+\frac{7}{25}+\frac{10}{125}+\cdots$

Sol: $a=1$, $d=3$, and $r=\frac{1}{5}$

$$S_\infty = \frac{a}{1-r}+\frac{d\cdot r}{(1-r)^2} = \frac{1}{1-\frac{1}{5}}+\frac{3\cdot\frac{1}{5}}{(1-\frac{1}{5})^2}$$

$$= \frac{5}{4}+\frac{3}{5}\times\frac{25}{16}$$

$$= \frac{35}{16}.$$

5.20) Find $\sum_{n=1}^{\infty}\frac{1}{n^2+5n+6}$

Let $a_n = \frac{1}{n^2+5n+6} = \frac{1}{n+2}-\frac{1}{n+3}$ (Partial fraction method)

$$S_n = a_1+a_2+a_3+\cdots+a_n$$

$$= \left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$= \frac{1}{3}-\frac{1}{n+3}$$

as $n\to\infty$ $\sum_{n=1}^{\infty}\frac{1}{n^2+5n+6} = \frac{1}{3}.$

5.21) Expand $(1+x)^{2/3}$ up to four terms for $|x|<1$.

$n=\frac{2}{3}$ $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3$

$$(1+x)^{2/3} = 1+\frac{2}{3}x+\frac{\frac{2}{3}(\frac{2}{3}-1)}{2!}x^2+\frac{\frac{2}{3}(\frac{2}{3}-1)(\frac{2}{3}-2)}{3!}x^3$$

$$(1+x)^{2/3} = 1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\cdots$$

5.22) $\frac{1}{(1+3x)^2}$ Expand in powers of x. Find the condition that x for which the expansion is valid.

Sol: Let $y=3x$ $\therefore \frac{1}{(1+3x)^2}=\frac{1}{(1+y)^2}$

when $|y|<1$, the expansion is valid.

$|3x|<1$ "

$|x|<1/3$ "

$$\frac{1}{(1+3x)^2} = (1+3x)^{-2}$$

$$= 1-2(3x)+\frac{2(2+1)}{2!}(3x)^2-\frac{2(2+1)(2+2)}{3!}(3x)^3 + 2\frac{(2+1)(2+2)(2+3)}{3!}(3x)^4+\cdots$$

Hence $\frac{1}{(1+3x)^2} = 1-6x+27x^2-108x^3+405x^4+\cdots \quad |x|<1/3.$

5.23) Expand $\frac{1}{(3+2x)^2}$ in powers of x. Find a condition on x for which the expansion is valid.

$$(1+\cdot)^{-2} = 1-2y+3y^2-4y^3+5y^4-\cdots) \quad |y|<1.$$

$$\frac{1}{(3+2x)^2} = \frac{1}{9\left(1+\frac{2}{3}x\right)^2} = \frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2}$$

$$= \frac{1}{9}\left[1-2\cdot\frac{2x}{3}+3\left(\frac{2x}{3}\right)^2-4\left(\frac{2x}{3}\right)^3+5\left(\frac{2x}{3}\right)^5+\cdots\right) \quad \left|\frac{2x}{3}\right|<1$$

$$= \frac{1}{9}\left[1-\frac{4x}{3}+\frac{4x^2}{3}-\frac{32x^3}{27}+\frac{80x^4}{81}-\cdots\right]$$

$$\frac{1}{(3+2x)^2} = \left[\frac{1}{9}-\frac{4x}{27}+\frac{4x^2}{27}-\frac{32x^3}{243}+\frac{80}{729}x^4-\cdots\right] \quad |x|<\frac{3}{2}$$

5.24) Expand $\sqrt[3]{65}$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots \quad |x|<1$$

$$\sqrt[3]{65} = 65^{1/3} = (1+64)^{1/3}$$

$$= 64^{1/3}\left(1+\frac{1}{64}\right)^{1/3}$$

$$= 4\left(1+\frac{1}{64}\right)^{1/3}.$$

$$= 4\left[1+\frac{1}{3}\cdot\frac{1}{64}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{64^2}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{64^3}+\cdots\right] \quad |x|<1$$

$$= 4+\frac{1}{48}\cancel{+}\frac{1}{36864}$$

$$= 4+0.02 \qquad \frac{1}{36864} \text{ is negligible.}$$

$\sqrt[3]{65} = 4.02$ appr.

5.25) P.T $\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \qquad |x|<1$$

$$\sqrt[3]{x^3+7} = (x^3+7)^{1/3}$$
$$= (x^3)^{1/3}\left(1+\frac{7}{x^3}\right)^{1/3}$$
$$= x\left[1 + \frac{1}{3}\cdot\frac{7}{x^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2 + \cdots\right] \qquad \left|\frac{7}{x^3}\right|<1$$
$$= x\left[1 + \frac{7}{3x^3} - \frac{49}{9}\cdot\frac{1}{x^6} + \cdots\right]$$
$$= x + \frac{7}{3x^2} - \frac{49}{9}\cdot\frac{1}{x^5} + \cdots \quad \text{—— (1)}$$

$$\sqrt[3]{x^3+4} = (x^3+4)^{1/3} = x\left(1+\frac{4}{x^3}\right)^{1/3}$$
$$= x\left[1 + \frac{1}{3}\cdot\frac{4}{x^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{16}{x^6} + \cdots\right]$$
$$= x + \frac{4}{3x^2} - \frac{16}{9}\cdot\frac{1}{x^5} \quad \text{—— (2)}$$

$$\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4} = \left(x + \frac{7}{3x^2} - \cdots\right) - \left(x + \frac{4}{3x^2} - \cdots\right)$$
$$= \frac{7}{3x^2} - \frac{4}{3x^2} = \frac{\cancel{3}}{\cancel{3}x^2} = \frac{1}{x^2}$$

$\frac{3}{5.4}$ P.T $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

$$\sqrt[3]{x^3+6} = (x^3+6)^{1/3} = x\left(1+\frac{6}{x^3}\right)^{1/3} \qquad \left|\frac{6}{x^3}\right|<1 \quad x \text{ is large.}$$
$$= x\left(1 + \frac{1}{\cancel{3}}\cdot\frac{\cancel{6}\,2}{x^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{36}{x^2}\right)\cdots)$$
$$= x + \frac{2\cancel{x}}{x^{\cancel{3}2}} + \cdots \quad \text{—— (1)}$$

$$\sqrt[3]{x^3+3} = (x^3+3)^{1/3} = x\left(1+\frac{3}{x^3}\right)^{1/3} \qquad \left|\frac{3}{x^3}\right|<1 \quad x \text{ is large}$$
$$= x\left(1 + \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{x^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{9}{x^6}\right)$$
$$= x + \frac{1}{x^{2}} + \cdots \quad \text{—— (2)}$$

$$\sqrt[3]{x^3+6}-\sqrt[3]{x^3+3} = \left(x+\frac{2}{x^2}-\cdots\right)-\left(x+\frac{1}{x^2}\cdots\right)$$

$$= \frac{2}{x^2}-\frac{1}{x^2}$$

$$= \frac{1}{x^2} \quad \because x \text{ is large.}$$

$\frac{4}{5.4}$. P.T $\sqrt{\frac{1-x}{1+x}} = 1-x+\frac{x^2}{2}$ (approx) x is large.

$$(1+x)^{-n} = 1-nx+\frac{(-n)(-n-1)}{2!}x^2+\cdots$$

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$\sqrt{\frac{1-x}{1+x}} = (1-x)^{1/2}(1+x)^{-1/2}$$

$$=\left(1-\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}x^2\cdots\right)$$

$$=\left(1-\frac{x}{2}-\frac{x^2}{8}+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$= 1-\frac{x}{2}+\frac{3x^2}{8}-\frac{x}{2}+\frac{x^2}{4}-\frac{3}{16}x^3 \cancel{+} \frac{x^2}{8}\cdots$$

$$= 1-\frac{\cancel{2}x}{\cancel{2}}+\frac{\cancel{3}x^2-2x^2}{8}+\cdots)$$

$$= 1-x+\frac{x^2 4}{8}-\cdots$$

$$= 1-x+\frac{x^2}{2}-\cdots \text{ (approximalely)}$$

$\frac{5}{5.4}$ i) write the first 6 terms of the exponential series i) e^{5x} ii) e^{-2x} iii) $e^{\frac{1}{2}x}$.

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

$$e^{5x} = 1+\frac{5x}{1!}+\frac{(5x)^2}{2!}+\frac{(5x)^3}{3!}+\frac{(5x)^4}{4!}+\frac{(5x)^5}{5!}-\cdots$$

$$=1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\frac{625x^4}{24}+\frac{625x^5}{24}+\cdots$$

$$e^{-2x} = 1+\left(\frac{-2x}{1!}\right)+\frac{(-2x)^2}{2!}+\frac{(-2x)^3}{3!}+\frac{(-2x)^4}{4!}+\frac{(-2x)^5}{5!}+\cdots$$

$$= 1-2x+2x^2-\frac{8x^3}{6}+\frac{16x^4}{24}-\frac{32x^5}{120}+\cdots$$

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$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + \frac{2x^4}{3} - \frac{4x^5}{15} \cdots$$

$|||^{ly}$ $$e^{\frac{1}{2}x} = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \cdots$$

$\frac{6}{5.4}$ Write the first 4 terms of the logarithmic series 1) $\log(1+4x)$ 2) $\log(1-2x)$ 3) $\log\left(\frac{1+3x}{1-3x}\right)$ 4) $\log\left(\frac{1-2x}{1+2x}\right)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \quad (*)$$

$$\log(1+4x) = 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \cdots \quad |4x|<1$$

$$= 4x - 8x^2 + \frac{64}{3}x^3 - 64x^4 \cdots \quad |x| < \frac{1}{4}$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots \quad (*)$$

$$\log(1-2x) = -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} \cdots \quad |2x|<1$$

$$= -2x - 2x^2 - \frac{8x^3}{3} - \frac{16x^4}{4} \cdots \quad |x| < \frac{1}{2}$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \quad (*)$$

$$\log\left(\frac{1+3x}{1-3x}\right) = 2\left[3x + \frac{(3x)^3}{3} + \frac{(3x)^5}{5} + \frac{(3x)^7}{7}\right] \quad |3x|<1, \ |x| < 1/3$$

$$= 6x + 18x^3 + \frac{486}{5}x^5 + \frac{2187\times 2}{7}x^7 \quad |x| < 1/3.$$

$|||^{ly}$ we can P.T $$\log\left(\frac{1-2x}{1+2x}\right) = -2\left[2x + \frac{8x^3}{3} + \frac{32x^5}{5} + \frac{128x^7}{7} \cdots\right] \quad |x| < 1/2$$

$\frac{7}{5.4}$ If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \cdots$ P.T $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots$

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$-y = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$-y = \log(1-x)$$

$$e^{-y} = 1 - x$$

$$x = 1 - e^{-y} = \cancel{1} - \left(\cancel{1} - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots\right)$$

$$= y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots$$

9 / 5.4 Find the co-efficient of x^4 in the expansion of $\frac{3-4x+x^2}{e^{2x}}$.

$$\frac{(3-4x+x^2)}{e^{2x}} = (3-4x+x^2)e^{-2x}$$

$$=(3-4x+x^2)\left(1-\frac{2x}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}-\cdots\right]$$

$$=(3-4x+x^2)\left(1-2x+2x^2-\frac{8x^3}{6}+\frac{16}{24}x^4-\cdots\right]$$

$$= \cancel{3}\cdot\frac{\cancel{16}\,2}{\cancel{24}\,8}+\cancel{4}\,2\cdot\frac{8}{\cancel{6}\,3}+1\times 2 \quad = 2+\frac{16}{3}+2$$

$$=\frac{6+16+6}{3}=\frac{28}{3}$$

10 / 5.4 Find the Value of $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$

$$t_1 = 1\left(1+\frac{1}{9}\right) = 1+\frac{1}{9}.$$

$$t_2 = \frac{1}{2}\left(\frac{1}{9}+\frac{1}{9^3}\right)$$

$$t_3 = \frac{1}{4}\left(\frac{1}{9^2}+\frac{1}{9^5}\right)$$

— — — — — — ~~$t_n =$~~

$$t_1+t_2+t_3+\cdots = 1+\frac{1}{9}+\frac{1}{2\cdot 9}+\frac{1}{2\cdot 9^3}+\frac{1}{4\cdot 9^2}+\frac{1}{4}\cdot\frac{1}{9^5}-\cdots$$

$$=\left(1+\frac{1}{2}\left(\frac{1}{9}\right)+\frac{1}{2^2}\left(\frac{1}{9^2}\right)+\cdots\right)+\left(\frac{1}{9}+\frac{1}{2}\cdot\frac{1}{9^3}+\frac{1}{2^2}\cdot\frac{1}{9^5}+\cdots\right)$$

$$=\left(1+\frac{1}{2}\left(\frac{1}{9}\right)+\frac{1}{2^2}\cdot\frac{1}{9^2}+\cdots\right)+\frac{1}{9}\left(1+\frac{1}{2}\left(\frac{1}{9^2}\right)+\frac{1}{2^2}\left(\frac{1}{9^4}\right)+\cdots\right)$$

$$=\left(1-\frac{1}{1/81}\right)+\frac{1}{9}\left(\frac{1}{1-\frac{1}{162}}\right)$$

$$S_\infty = \frac{a}{1-r}$$

$$=\frac{1}{1-\frac{1}{18}}+\frac{1}{9}\left(\frac{1}{1-\frac{1}{162}}\right)$$

$$=\frac{18}{17}+\frac{1}{9}\cdot\frac{162}{161} = \frac{26082+2754}{17\times 1499}$$

$$=\frac{28836}{24633} = 1.17$$

$\frac{8}{5.4}$ If $p-q$ is small compared to either p or q then the show that

$$\sqrt[n]{\frac{p}{q}} = \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$$ Hence find $\sqrt[8]{\frac{15}{18}}$

$$\sqrt[8]{\frac{15}{18}} = \frac{(8+1)15+(8-1)18}{(8-1)15+(8+1)18}$$

$$= \frac{9\times15+7\times18}{7\times15+9\times18} = \frac{135+112}{105+144} = \frac{247}{249}$$

$= .9919$ (appro)

$\frac{1}{5.4}$ 1) Expand $\frac{1}{5+x} = (5+x)^{-1}$

$$= 5^{-1}\left(1+\frac{5}{x}\right)^{-1} \qquad \left|\frac{x}{5}\right|<1$$

$$= \frac{1}{5}\left(1-\frac{x}{5}+\frac{x^2}{5^2}+\frac{x^3}{5^3}+\cdots\right)$$

$$= \frac{1}{5}\left(1-\frac{x}{5}+\frac{x^2}{25}-\frac{x^3}{125}+\cdots\right) \qquad \left|\frac{x}{5}\right|<1$$

$|x|<5$

2) $$\frac{2}{(3+4x)^2} = 2(3+4x)^{-2}$$

$$= \frac{2}{3^2}\left(1+\frac{4x}{3}\right)^{-2}$$

$$= \frac{2}{9}\left(1-2\left(\frac{4x}{3}\right)+3\left(\frac{4x}{3}\right)^2-\cdots\right)$$

$$= \frac{2}{9}\left(1-\frac{8x}{3}+3\cdot\frac{16x^2}{3^3}-\cdots\right) \qquad \left|\frac{4x}{3}\right|<1$$

$|x|<3/4$

3) $(5+x^2)^{2/3}$

$$(1+x)^{p/q} = 1+\frac{p}{q}x+\frac{p(p-q)}{2!q^2}x^2+\frac{p(p-q)(p-2q)}{3!q^3}x^3\cdots$$

$$5^{2/3}\left(1+\frac{x^2}{5}\right)^{2/3} = 5^{2/3}\left(1+\frac{2}{3}\cdot\frac{x^2}{5}+\frac{2(2-3)}{2!\,3^2}\cdot\left(\frac{x^2}{5}\right)^2+\cdots\right)$$

$$= 5^{2/3}\left(1+\frac{2x^2}{15}-\frac{x^4}{225}+\cdots\right) \qquad \left|\frac{x^2}{5}\right|<1$$

$|x^2|<5$

4) $$(x+2)^{-2/3} = 2^{-2/3}\left(1+\frac{x}{2}\right)^{-2/3}$$

$$= \frac{1}{2^{2/3}}\left(1-\frac{2}{3}\left(\frac{x}{2}\right)+\frac{(-2/3)(-2/3-1)}{2!}\left(\frac{x}{2}\right)^2+\cdots\right) \qquad \left|\frac{x}{2}\right|<1$$

$$= \frac{1}{2^{2/3}}\left(1-\frac{x}{3}+\frac{5x^2}{36}-\cdots\right) \qquad \left|\frac{x}{2}\right|<1$$

$|x|<2$

$\frac{2}{5.4}$ Find $\sqrt[3]{1001}$ approximately

Sol: $\sqrt[3]{1001} = (1000+1)^{1/3} = \left(1000+\frac{1}{}\right)^{1/3}$

$= 1000^{1/3}\left(1+\frac{1}{1000}\right)^{1/3}$

$= 10\left[1+.001\right)^{1/3}$

$= 10\left[1+\frac{1}{3}(0.001)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.001)^2+\cdots\right]$

$= 10\left[1+0.00033+\cdots\right]$

$= 10\times 1.00033$

$\simeq 10.0033$

$\frac{5}{5.4}$ 1) write the first six terms of the exponential series.

i) e^{5x}

$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$e^{5x} = 1+\frac{5x}{1!}+\frac{(5x)^2}{2!}+\frac{(5x)^3}{3!}+\cdots$

$= 1+5x+\frac{25x^2}{2}+\frac{125x^3}{6}+\cdots$

ii) $e^{-2x} = 1-\frac{2x}{1!}+\frac{(2x)^2}{2!}-\frac{(2x)^3}{3!}+\cdots$

$= 1-2x+2x^2-\frac{\cancel{8}^4}{\cancel{6}_3}x^3+\cdots$

iii) $e^{1/2x} = 1+\frac{1}{2}x+\frac{1x^2}{4\,2!}+\frac{1}{8}\frac{x^3}{3!}+\cdots$

$= 1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\cdots$

$\frac{6}{5.4}$ write the first 4 terms of the following series.

1) $\log(1+2x)$

$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \quad |x|<1.$

$\log(1+4x) = 4x-\frac{(4x)^2}{2}+\frac{(4x)^3}{3}-\frac{(4x)^4}{4}+\cdots$

$= 4x-8x^2+\frac{64}{3}x^3-64x^4\cdots \quad |4x|<1$

$x<1/4$

EXERCISE-5.5 one Mark.

1. The value of $2+4+6+\cdots+2n$ is

1) $\frac{n(n-1)}{2}$ 2) $\frac{n(n+1)}{2}$ 3) $\frac{2n(2n+1)}{2}$ 4) $n(n+1)$ ✓

$2+4+6+\cdots+2n = 2(1+2+3+\cdots+n)$

$= \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} = n(n+1)$

2) The coefficient of x^6 in $(2+2x)^{10}$.

1) $10C_6$ 2) 2^6 3) $10C_6 2^6$ 4) $10C_6 2^{10}$ ✓

x^6 is in 7^{th} term $t_{6+1} = 10C_6\, 2^4 (2x)^6 = 10C_6 2^{10}$

$x^6 \Rightarrow r=6$.

3) The coefficient of x^8y^{12} in The expansion of $(2x+3y)^{20}$

1) 0 2) $2^8 3^{12}$ 3) $2^8\cdot3^{12}+2^{12}3^8$ 4) $20C_8\, 2^8 3^{12}$ ✓

y^{12} is in 13^{th} term $t_{13}=t_{12+1} = 20C_{12}(2x)^8(3y)^{12}$

$y^{12} \Rightarrow r=12$ $= 20C_8\, 2^8\cdot3^{12}$

4) $nC_{10} > nC_r$ for all possible value of r then the value of n is

1) 10 2) 21 3) 19 4) 20.

$nC_{10} > nC_{10-r} \Rightarrow 10+10-\cancel{r}=n$ $n=20$

5) If a is the AM and g is the GM of two numbers

1) $a\le g$ 2) $a\ge g$ ✓ 3) $a=g$ 4) $a>g$

Property $a\ge g\ge h$

6) $(1+x^2)^2(1+x)^n = a_0+a_1x+a_2x^2+\cdots+x^{n+4}$ and if $a_0, a_1, a_2\ldots$ are in AP Then n is

1) 1 2) 2 3) 3 ✓ 4) 4

7) If $a, 8, b$ are in AP, $a, 4, b$ are in GP and if a, x, b are in HP. Then x is 1) 2 ✓ 2) 1 3) 4 4) 16.

$a+b=16$ $16=ab$. $x=\frac{2ab}{a+b} = \frac{2\cdot16}{16}$

$= 2\cdot\frac{16}{16} = 2$

8) The sequence $\frac{1}{\sqrt3}, \frac{1}{\sqrt3+\sqrt2}, \frac{1}{\sqrt3+2\sqrt2}, \cdots$ form an

1) AP 2) GP 3) HP ✓ 4) AGP

$\sqrt{3}$, $\sqrt{3}+\sqrt{2}$, $\sqrt{3}+2\sqrt{3}\cdots$

$d=\sqrt{2}$ $\therefore$ This is AP

Hence the given series is HP.

9) The HM of two positive numbers whose AM and GM are 16, 8 resp. is 1) 10 2) 6 3) 5 4) 4.

$a+b=32$ $ab=64$ HM $= \frac{2ab}{a+b} = \frac{2\times 64}{32} = 4$

10) S_n denotes the sum of n terms of AP whose common difference is d The value of $S_n - 2S_{n-1} + S_{n+2}$ is

1) 0 2) 2d 3) 4d 4) d^2

$\frac{n}{2}[2a+(n-1)d] - 2\frac{(n-1)}{2}[2a+(n-2)d] + \frac{n+2}{2}[2a+(n+1)d]$

$na + \frac{n}{2}(n-1)d - 2an + 2a - (n-1)(n-2)d + a(n+2) + \frac{(n+1)(n+2)}{2}d$

$\frac{n}{2}(n-1)d + 4a$

11) The remainder when 38^{15} is divisible by 13

1) 12 2) 1 3) 11 4) 5

$38^{15} = (39-1)^{15} = 39C_{15} - 15C_1(39)^{14} + \cdots + 15C_{14}(39) - 1$.

$\therefore$ The remainder is $= 12$.

12) The n^{th} term of the sequence 1, 2, 4, 7, 11 .... is

1) n^3+3n^2+2n 2) n^3-3n^2+3n 3) $\frac{n(n+1)(n+2)}{3}$ 4) $\frac{n^2-n+2}{2}$

check which one is possible 1) $n=3$ $\neq 4$

2) $n=3$ $\neq 4$ $\therefore$ 4

3) $n=3$ $\frac{3\times4\times5}{3}\neq 4$

13) The sum up to n terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots$

1) $\sqrt{2n+1}$ 2) $\frac{\sqrt{2n+1}}{2}$ 3) $\sqrt{2n+1}-1$ 4) $\frac{\sqrt{2n+1}-1}{2}$

14) The n^{th} term of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$

1) 2^n-n-1 2) $1-2^{-n}$. 3) $2^{-n}+n-1$ 4) 2^{n-1}

$t_n = 1-\frac{1}{2^n} = 1-2^{-n}$

15) The sum up to n terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{32}+\cdots$

1) $\frac{n(n-1)}{2}$ 2) $2n(n+1)$ 3) $\frac{n(n+1)}{2}$ 4) 1.

- $\sqrt{2}+2\sqrt{2}+3\sqrt{2}+4\sqrt{2}\cdots+n\sqrt{2}$

$=\sqrt{2}(1+2+3+\cdots+n) = \cancel{\sqrt{2}}\left(\frac{n(n+1)}{\cancel{2}\sqrt{2}}\right)$

16) The value of the series $\frac{1}{2}+\frac{7}{4}+\frac{13}{8}+\frac{19}{16}+\cdots$ is

1) 14 ✓ 2) 7 3) 4 4) 6

$a=1,\ d=6,\ r=1/2 \quad S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}=\frac{1}{1/2}+\frac{3}{1}\times 4$

$=2+12=14.$

17) The sum of an infinite GP is 18 If the first term is 6 the Common ratio is:

1) $\frac{1}{3}$ 2) $2/3$ ✓ 3) $1/6$ 4) $4/15$

$S_\infty=18=\frac{a}{1-r} \qquad a=6,\ r=?$

$18=\frac{6}{1-r} \Rightarrow 18-18r=6 \Rightarrow 12=18r \quad r=\frac{12}{18}=2/3$

18) The Coefficient of x^5 in the series e^{-2x} is

1) $\frac{2}{3}$ 2) $3/2$ 3) $-\frac{4}{15}$ 4) $4/15$

$\frac{(-2x)^5}{5!}=\frac{-32\,4}{1\cdot 2\cdot 3\cdot 4\cdot 5}=\frac{-4}{15}.$

19) The value of $\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$ is

1) $\frac{e^2+1}{2e}$ 2) $\frac{(e+1)^2}{2a}$ 3) $\frac{(e-1)^2}{2e}$ ✓ 4) $\frac{e^2+1}{2e}$.

$\frac{e^1+e^{-1}}{2}-1=\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$

$\Rightarrow \frac{e+\frac{1}{e}-2}{2}=\frac{e^2-2e+1}{2e}=\frac{(e-1)^2}{2e}$

20) The value of $1-\frac{1}{2}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^2-\frac{1}{4}\left(\frac{2}{3}\right)^3+\cdots$

1) $\log 5/3$ 2) $\frac{3}{2}\log(5/3)$ 3) $5/3\log(5/3)$ 4) $2/3\log 2/3$.

$\log\left(1+\frac{2}{3}\right)=\log(5/3) \qquad S=1-\frac{1}{2}(2/3)$

$\frac{2}{3}S=\frac{2}{3}-\frac{1}{2}(2/3)^2$

$\frac{2}{3}S=\frac{3}{2}\log(5/3)$

XI std Binomial theorem Work Sheet.

1. The co-efficients of $(r-1)^{th}$ r^{th} $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1 : 3 : 5 find n, r. NC $(n=7, r=3)$

2) If the middle term in the binomial expansion of $\left(\frac{1}{x}+\sin x\right)^{10}$ is equal to $\frac{63}{8}$ find the value of x. NC ($x = n\pi + (-1)^n \pi/6$ $n \in z$)

3. The 3^{rd}, 4^{th} and 5^{th} terms in the expansion of $(x+a)^n$ are respectively 84, 280, 560 find the values of x, a, n $(n=7, a=2, x=1)$

4. Find Middle terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ NC

$$\left(\frac{189}{8}x^{17},\ -\frac{21}{16}x^{19}\right)$$

5. $\left(\frac{x}{3}+9y\right)^{10}$

6) $\left(2ax-\frac{b}{x^2}\right)^{12}$

7) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$.

8) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$

9) Find the term independent of x in the expansion of
1) $\left(3x-\frac{2}{x^2}\right)^{15}$ 2) $\left(\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$ 3) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$

10) If the coefficients of 5^{th}, 6^{th}, 7^{th} terms in the expansion $(1+x)^n$ are in AP find n. (NC)

11) If the Co-eff. of 2^{nd} 3^{rd} 4^{th} terms in the expansion $(1+x)^{2n}$ are in AP S.T $2n^2-9n+7=0$ (NC)

12. If the Co-effi. of 2^{nd}, 3^{rd}, 4^{th} terms in the expansion of $(1+x)^n$ are in AP then find the value of n.

13) Find a if the Co.eff. of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal

14) Find a, b, n in the expansion of $(a+b)^n$, if the first three terms in the expansion are 729, 7290, 30375.

15) If the term free from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405 find the value of k.

16) Evaluate $\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$

17) Evaluate $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4$

18) Evaluate $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$.

19) using Binomial Theorem $2^{3n}-7n-1$ is divisible by 49.

20. " " $3^{2n+2}-8n-9$ is divisible by 64 21) $3^{3n}-26n-1$ is divisible by 676.

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