7. Application of Differential Calculus

G. Karthikeyan
Thiruvarur DT

Exercise 7.1

① A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ m

(i) Find the average velocity of the points between $t=3$ and $t=6$ seconds.

(ii) Find the instantaneous velocities at $t=3$ and $t=6$ seconds.

$$s(t) = 2t^2 + 3t$$

(i) $a=3 \quad b=6$

$$s(3) = 2(3)^2 + 3(3) \qquad s(6) = 2(6)^2 + 3(6)$$
$$= 18 + 9 \qquad = 72 + 18$$
$$s(3) = 27 \qquad s(6) = 90$$

$$\text{average velocity} = \frac{s(b) - s(a)}{b - a}$$

G.KARTHIKEYAN
9715634957

$$= \frac{90 - 27}{6 - 3}$$
$$= \frac{63}{3}$$
$$= 21 \text{ m/s}$$

(ii)
$$s(t) = 2t^2 + 3t$$
$$s'(t) = 4t + 3$$

instantaneous rate of change at $t=3$
(velocity)

$$= s'(3)$$
$$= 4(3) + 3 = 12 + 3$$
$$= 15 \text{ m/s}$$

instantaneous (rate of change) velocity at $t=6$

$$= s'(6)$$
$$= 4(6) + 3 = 24 + 3$$
$$= 27 \text{ m/s}$$

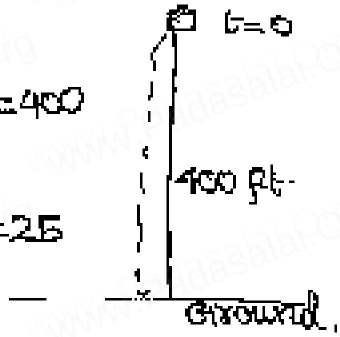
② A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s=16t^2$ in t seconds

(i) How long does the camera fall before it hits the ground

(ii) what is the average velocity with which the camera falls during the last 2 seconds?

(iii) What is the instantaneous velocity of the camera when it hits the ground.

$$s(t)=16t^2$$



(i) when hits the ground $s=400$

$$16t^2=400$$
$$t^2=\frac{400}{16}=\frac{100}{4}=25$$
$$t^2=25$$
$$t=\pm 5$$

$t=5$ sec

(ii) average velocity in last two seconds $t=3$, $t=5$

$$\text{average velocity}=\frac{s(5)-s(3)}{5-3}$$
$$=\frac{16(5^2)-16(3)^2}{2}=\frac{16(25-9)}{2}$$
$$=8\times 16$$
$$=128 \text{ ft/s}$$

(iii) Instantaneous velocity $=s'(t)$

$$=16(2t)$$
$$=32t$$

when it hits the ground. $t=5$

$$s'(5)=32(5)=160 \text{ ft/s}$$

③ A particle moves along a line according to the law $s(t)=2t^3-9t^2+12t-4$, where $t\geq 0$

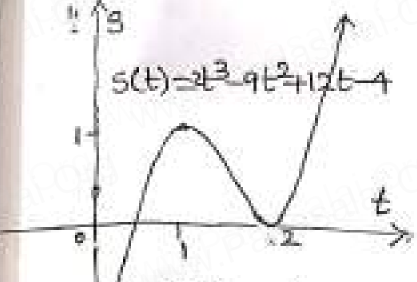
(i) At what time the particle changes direction?

(ii) Find the total distance travelled by the particle in the first 4 seconds.

(iii) Find the particles acceleration each time the velocity is zero.

$$s(t)=2t^3-9t^2+12t-4$$
$$s'(t)=6t^2-18t+12$$

(i) when changes of direction $s'(t)=0$



$$6t^2-18t+12=0$$
$$6(t^2-3t+2)=0$$
$$(t-1)(t-2)=0 \qquad \boxed{t=1,2 \text{ sec}}$$
$$t=1,\ t=2$$

(ii) total distance travelled in first 4 seconds

$S(0)=-4$
$S(1)=1$
$S(2)=0$
$S(4)=28$

$$=|S(1)-S(0)|+|S(2)-S(1)|+|S(4)-S(2)|$$
$$=|1+4|+|0-1|+|28-0|$$
$$=5+1+28$$
$$=34\text{ m}$$

(iii)
$$S'(t)=6t^2-18t+12$$
$$S''(t)=12t-18$$

when velocity is zero $t=1,2$

$$S''(1)=12(1)-18=-6 \text{ m/s}^2$$
$$S''(2)=12(2)-18=6 \text{ m/s}^2$$

(4) If the volume of a cube of side length x is $V=x^3$, Find the rate of change of the volume with respect to x when $x=5$ units.

$$V=x^3$$
diff. w. r. to. x
$$\frac{dV}{dx}=3x^2$$
$$\left[\frac{dV}{dx}\right]_{x=5}=3(5)^2=3\times25=75 \text{ units.}$$

(5) If the mass $m(x)$ (in kgs) of a thin rod of length x (in metres) is given by, $m(x)=\sqrt{3x}$ then what is the rate of change of mass w.r.to the length when it is $x=3$ and $x=27$ mts.

$$m(x)=\sqrt{3x}=\sqrt{3}\sqrt{x}$$

$a=3 \quad b=27$

$$\text{rate of change}=m'(x)=\sqrt{3}\cdot\frac{1}{2\sqrt{x}}$$

(i) $x=3$ $\quad m'(3)=\sqrt{3}\frac{1}{2\sqrt{3}}=\frac{1}{2}$ kg/m

(ii) $x=27$ $\quad m'(27)=\sqrt{3}\frac{1}{2\sqrt{27}}=\sqrt{3}\frac{1}{2(3\sqrt{3})}=\frac{1}{6}$ kg/m

6) A stone is dropped into a pond causing ripple in the concentric circles. The radius r of the outer ripple is increasing at 2 cm/s. When the radius is 5cm find the rate of changing of the total area of the disturbed water?

Let A be the area at time t.

Area of the circle $A = \pi r^2$

diff w.r.to t

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt} \quad \text{——} (1)$$

when $r = 5$ $\quad \frac{dr}{dt} = 2$

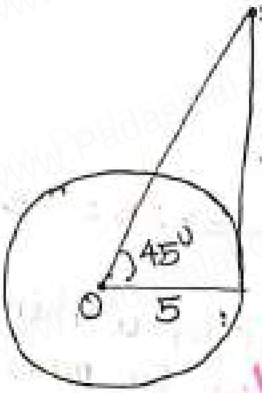
$$(1) \Rightarrow \frac{dA}{dt} = \pi 2(5)2$$

$$= 20\pi$$

Area changing rate $= 20\pi$ sq cm/s

7) A beacon makes one revolution every 10 sec. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam (beacon) moving along the shore line when it makes an angle of 45° with the shore.

Let the angular velocity $\frac{d\theta}{dt}$



revolution in 10 sec $= 1 = 2\pi$

1 sec $= \frac{2\pi}{10}$

$$\frac{d\theta}{dt} = \frac{\pi}{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan\theta = \frac{x}{5} \qquad x = 5\tan\theta$$

diff. w.r.to t

$$\frac{dx}{dt} = 5\sec^2\theta \frac{d\theta}{dt}$$

$\theta = 45^\circ \quad x = 5$

$$\frac{dx}{dt} = \cancel{5}\sec^2 45^\circ \left(\frac{\pi}{\cancel{5}}\right)$$

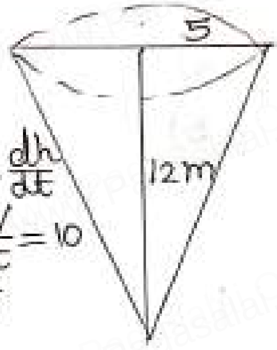
$$= (\sqrt{2})^2 \pi$$

$$\frac{dx}{dt} = 2\pi \text{ km/sec.}$$

beacon moving rate 2π km/s.

⑧ A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Let v be the volume of tank at any time t



$r = 5 \quad h = 12$

to find $\frac{dh}{dt}$ when $\frac{dv}{dt} = 10$, $h = 8$

$$\frac{r}{h} = \frac{5}{12}$$

$$r = \frac{5h}{12}$$

volume of cone $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi \frac{25h^2}{144} h$$

$$V = \frac{25\pi}{3 \times 144} h^3$$

diff. w.r to t

$$\frac{dv}{dt} = \frac{25\pi}{\cancel{3} \times 144} \cancel{3} h^2 \frac{dh}{dt}$$

$$10 = \frac{25\pi}{\underset{\substack{18 \\ 9}}{\cancel{144}}} \times \cancel{8} \times \overset{4}{\cancel{8}} \frac{dh}{dt}$$

G. Karthikeyan
PG.T GGHSS
Thirumakkottai
Thiruvarur DT

$$\frac{10 \times 9}{25\pi \times 4} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9\cancel{0}}{10\cancel{0}\pi} \qquad \frac{dh}{dt} = \frac{9}{10\pi} \text{ m/min}$$

depth increase rate $\frac{9}{10\pi}$ m/min.

9) A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. when the base of the ladder is 8 m. from the wall,

(i) How fast is the top of the ladder moving down the wall?

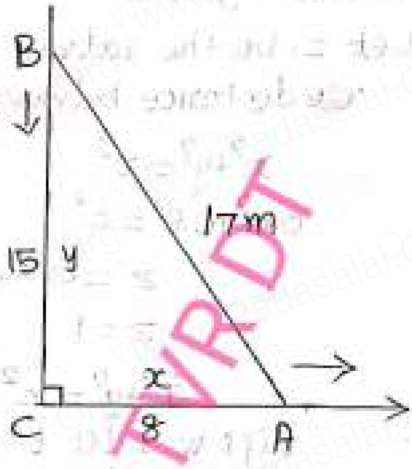
(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

(9) AB is the ladder

$AB = 17\,m$

Let x, y be the horizondal and vertical movements of ladder

$\frac{dx}{dt} = 5$ when $x = 8$



$AB^2 = BC^2 + AC^2$

$17^2 = y^2 + 8^2$

$289 - 64 = y^2$

$225 = y^2$

$y = 15$

(i) $x^2 + y^2 = 17^2$

diff w.r.to. t $\quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2\div \quad 8(5) + (15)\frac{dy}{dt} = 0$

$15\frac{dy}{dt} = -40$

$\frac{dy}{dt} = -\frac{40}{15}\frac{8}{3} \qquad \frac{dy}{dt} = -8/3 \text{ m/s}$

height decreasing rate $= -\frac{8}{3}$ m/s

(ii) Area of $\triangle ABC \quad \Delta = \frac{1}{2}xy$

diff w.r to t $\quad \frac{d\Delta}{dt} = \frac{1}{2}\left(\frac{dx}{dt}(y) + x\frac{dy}{dt}\right)$

$= \frac{1}{2}\left(5(15) + 8\left(-\frac{8}{3}\right)\right)$

G. Karthikeyan
Thiruvarur DT

$= \frac{1}{2}\left(\frac{3\times 75 - 64}{3}\right)$

$= \frac{225-64}{6} = \frac{161}{6}$

Area of triangle changing rate $= 26.83$ sq.m/s. $\quad \frac{d\Delta}{dt} = 26.83$

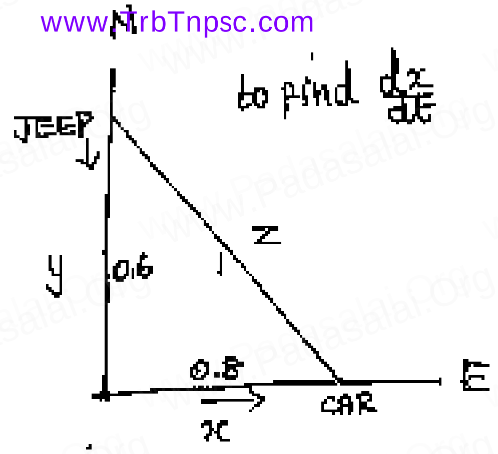
10) A police jeep, approaching an orthogonal intersection from the northern direction is chasing a speeding car that has turned and moving straight east. when the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Let x, y be the distance moving by car and jeep at any time t

speed of the jeep $\frac{dy}{dt} = -60$ km/hr (take - for approaching)

$x = 0.6 \quad y = 0.8$

Let z be the rate of change of
$\therefore$ distance between them.



$$x^2+y^2=z^2 \qquad \frac{dz}{dt}=20$$

$$0.6^2+0.8^2=z^2$$

$$z^2=0.36+0.64=1$$

$$z=1$$

$$x^2+y^2=z^2$$

diff w.r to t

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$$

$$2\div \quad 0.8\frac{dx}{dt}+0.6(60)=1(20)$$

$$0.8\frac{dx}{dt}-36=20$$

$$0.8\frac{dx}{dt}=20+36$$

$$0.8\frac{dx}{dt}=56$$

$$\frac{dx}{dt}=\frac{56}{0.8}\times\frac{10}{10}$$

$$=\frac{560}{8}$$

Speed of car $\frac{dx}{dt}=70$ km/s

—x—

Equations of tangent and Normal

① Equation of tangent

$$y-y_1=\left(\frac{dy}{dx}\right)_{(a,b)}(x-x_1)$$

② Equation of Normal.

$$y-y_1=\frac{-1}{\left(\frac{dy}{dx}\right)_{(a,b)}}(x-x_1)$$

or $\left(\frac{dy}{dx}\right)_{(a,b)}(y-y_1)=-(x-x_1)$

③ angle between the two curves

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

④ two curves are orthogonal if $m_1 \times m_2 = -1$

Exercise 7.2

① Find the slope of the tangent to the curves at the respective given points,

(i) $y=x^4+2x^2-x$ at $x=1$

diff. w.r to x

$$\frac{dy}{dx}=4x^3+4x-1$$

$$\text{slope}=\left[\frac{dy}{dx}\right]_{x=1}=4(1)+4(1)-1=8-1=7$$

② (ii) $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \pi/2$

$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ $\frac{dy}{dt} = 3b\sin^2 t(\cos t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{b}{a}\frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = -\frac{b}{a}\tan t$$

$$\text{slope} = \left(\frac{dy}{dx}\right)_{t=\pi/2} = -\frac{b}{a}\tan\pi/2 = \infty$$

② Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$

$y = x^2 - 5x + 4$ —①

$3x + y = 7$

$\frac{dy}{dx} = 2x - 5$

slope $= -\frac{3}{1}$

Let the tangential point be (x_1, y_1)

$$\text{slope} = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 2x_1 - 5$$

$$2x_1 - 5 = -3$$
$$2x_1 = 5 - 3$$
$$2x_1 = 2$$
$$x_1 = 1$$

at (x_1, y_1) ① ⇒ $y_1 = x_1^2 - 5x_1 + 4$
$y_1 = 1 - 5 + 4$
$y_1 = 0$

∴ Required point is $(1, 0)$

3) Find the point on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.

slope of normal $= -\frac{1}{1}$

slope of tangent $= \frac{-1}{-1}$

$m = 1$

Let the tangential point be (x_1, y_1) $y_1 = x_1^3 - 6x_1^2 + x_1 + 3$ —①

$y = x^3 - 6x^2 + x + 3$

$\frac{dy}{dx} = 3x^2 - 12x + 1$

$$\text{slope} = \left(\frac{dy}{dx}\right)_{x_1,y_1} = 3x_1^2 - 12x_1 + 1 = 1$$

$$3x_1^2 - 12x_1 = 1 - 1$$

$$3x_1(x_1 - 4) = 0 \qquad x_1 = 0 \quad x_1 = 4$$

$x_1 = 0$ ① ⇒ $y_1 = 3$ $(0, 3)$

$x_1 = 4$ ① ⇒ $y_1 = 4^3 - 6(4)^2 + 4 + 3 = 64 - 96 + 4 + 3 = 71 - 96 = -25$

The points are $(0, 3), (4, -25)$

④ Find the point on the curve $y^2-4xy=x^2+5$ for which the tangent is horizontal.

Let (x_1,y_1) be the point

$y_1^2-4x_1y_1=x_1^2+5$ —①

$$y^2-4xy=x^2+5$$

diff w.r to x

$$2y\frac{dy}{dx}-4\left(x\frac{dy}{dx}+y(1)\right)=2x+0$$

$$2y\frac{dy}{dx}-4x\frac{dy}{dx}-4y=2x$$

$$2\frac{dy}{dx}(y-2x)=2x+4y$$

$$\frac{dy}{dx}=\frac{\cancel{2}(x+2y)}{\cancel{2}(y-2x)}$$

$$\text{slope at } x_1,y_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{x_1+2y_1}{y_1-2x_1}$$

given tangent is horizondal slope $=0$

$$\frac{x_1+2y_1}{y_1-2x_1}=0 \qquad x_1+2y_1=0$$

$x_1=-2y_1$ —②

$$① \Rightarrow y_1^2-4(-2y_1)y_1=4y_1^2+5$$

$$y_1^2+8y_1^2=4y_1^2+5$$

$$9y_1^2-4y_1^2=5$$

$$5y_1^2=5 \qquad y_1^2=1$$

$$y_1=\pm1$$

$y_1=+1$ $x_1=-2(1)=-2$

$y_1=-1$ $x_1=-2(-1)=2$

The required points are $(2,-1),(-2,1)$

⑤ Find the tangent and normal to the following curves at the given points on the curve.

(i) $y=x^2-x^4$ at $(1,0)$ [x_1,y_1]

$$\frac{dy}{dx}=2x-4x^3$$

$$m=\left(\frac{dy}{dx}\right)_{(1,0)}=2(1)-4(1)$$

$$=-2$$

$$m=-2$$

Equation of tangent

$$y-y_1=m(x-x_1)$$

$$y-0=-2(x-1)$$

$$y=-2x+2$$

$$2x+y-2=0$$

Equation of Normal.

$x-2y+k=0$ —① This passing through $(1,0)$

$$1-2(0)+k=0$$

$$k=-1$$

Equation of normal

$$x-2y-1=0$$

(ii) $y=x^4+2e^x$ at $(0,2)$ [x_1,y_1]

$$\frac{dy}{dx}=4x^3+2e^x$$

$$m=\left(\frac{dy}{dx}\right)_{(0,2)}=4(0)+2e^0$$

$$=0+2(1)$$

$$m=2$$

Equation of tangent

$$y-y_1=m(x-x_1)$$

$$y-2=2(x-0)$$

$$y-2=2x$$

$$2x-y+2=0$$

Equation of normal $x+2y+k=0$ —①

This passing through $(0,2)$

$0+2(2)+k=0$

$k=-4$

$① \Rightarrow$ Equation of normal $x+2y-4=0$

(iii) $y=x\sin x$ at $(\pi/2, \pi/2)$

$\frac{dy}{dx} = x\cos x + \sin x(1)$

$m = \left(\frac{dy}{dx}\right)_{(\pi/2,\pi/2)} = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$

$= \frac{\pi}{2}(0)+1$

$m=1$

Equation of tangent

$y-y_1 = m(x-x_1)$

$y-\pi/2 = 1(x-\pi/2)$

$y-\cancel{\pi/2} = x-\cancel{\pi/2}$

$x-y=0$

Equation of normal

$x+y+k=0$ —①

This normal passing through $(\pi/2, \pi/2)$

$\frac{\pi}{2}+\frac{\pi}{2}+k=0 \quad k=-\pi$

$① \Rightarrow x+y-\pi=0$

(iv) $x=\cos t,\ y=2\sin^2 t$ at $t=\pi/3$

$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = 2\cdot 2\sin t\cos t$

G. Karthikeyan
Thiruvarur DT

$\frac{dy}{dx} = \frac{4\sin t\cos t}{-\sin t} = -4\cos t$

slope $m = \left(\frac{dy}{dx}\right)_{t=\pi/3} = -4\cos\frac{\pi}{3} = -4\left(\frac{1}{2}\right) = -2$

$\boxed{m=-2}$

$(x_1,y_1) = \left(\cos\frac{\pi}{3}, 2\sin^2\frac{\pi}{3}\right) = \left(\frac{1}{2}, 2\left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{1}{2}, \frac{2\times 3}{4}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$

Equation of tangent $y-y_1 = m(x-x_1)$

$y-\frac{3}{2} = -2\left(x-\frac{1}{2}\right)$

$\frac{2y-3}{2} = -2x+\frac{2}{2}$

$2y-3 = -4x+2$

$4x+2y=5$

Equation of normal is

$2x-4y=k$ —①

This passing through $\left(\frac{1}{2}, \frac{3}{2}\right)$

$2\left(\frac{1}{2}\right)-4\left(\frac{3}{2}\right)=k$

$1-6=k$

$k=-5$

$① \Rightarrow 2x-4y=-5$

⑥ Find the equations of the tangents to the curve $y=1+x^3$ for which the tangent is orthogonal with the line $x+12y=12$

$y=1+x^3$

Let the tangential point be (x_1,y_1)

$y_1 = 1+x_1^3$ —①

$y=1+x^3$

$\frac{dy}{dx} = 0+3x^2$

tangent slope at $(x_1,y_1) = 3x_1^2$

$m_1 = 3x_1^2$

$x+12y=12$

slope $= -\frac{1}{12}$

$m_2 = -\frac{1}{12}$

$m_1 \times m_2 = -1$

$3x_1^2\left(\frac{-1}{12}\right) = -1$

$x_1^2 = 4 \qquad \boxed{x_1 = \pm 2}$

(i) $x_1 = +2$ ① ⇒ $y_1 = 1 + 2^3 = 1 + 8$

$y_1 = 9$

point $(x_1, y_1) = (2, 9)$ slope $m_1 = 3(2)^2$

$m_1 = 12$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 9 = 12(x - 2)$

$y - 9 = 12x - 24$

$12x - y - 24 + 9 = 0$

$12x - y - 15 = 0$

(ii) $x_1 = -2$ ① ⇒

$y_1 = 1 - 8 = -7$

point $(x_1, y_1) = (-2, -7)$

slope $m_2 = 3(-2)^2$

$m_2 = 12$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y + 7 = 12(x + 2)$

$y + 7 = 12x + 24$

$12x - y + 17 = 0$

⑦ Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$

Let the tangential point be (x_1, y_1)

$y_1 = \frac{x_1+1}{x_1-1}$ —①

$y = \frac{x+1}{x-1}$

$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$

$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2}{(x_1-1)^2}$

$x + 2y = 6$

slope $m_2 = -\frac{1}{2}$

$m_1 = m_2$

$\frac{\cancel{-}2}{(x_1-1)^2} = \frac{\cancel{-}1}{2}$

$(x_1 - 1)^2 = 4$

$x_1 - 1 = \pm 2$

$x_1 = 1 \pm 2$

$x_1 = 3, -1$

$x_1 = 3$ ① ⇒ $y_1 = \frac{3+1}{3-1} = \frac{4}{2} = 2$ $(3, 2)$

$x_1 = -1$ ① ⇒ $y_1 = \frac{-1+1}{-1-1} = 0$ $(-1, 0)$

Equation of tangents $x + 2y = k$ —①

at $(3, 2)$ ① ⇒ $3 + 2(2) = k_1$

$k_1 = 7$

$x + 2y = 7$ //

at $(-1, 0)$ ① ⇒ $-1 + 0 = k_2$

$k = -1$

$x + 2y = -1$ //

⑧ Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in R$ at any point on the curve.

$\frac{dx}{dt} = -7\sin t$ $\frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx} = \frac{2\cos t}{-7\sin t}$ Equation of tangent

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$y - 2\sin t = \frac{2\cos t}{-7\sin t}(x - 7\cos t)$$

$$-7y\sin t + 14\sin^2 t = 2x\cos t - 14\cos^2 t$$

$$14\sin^2 t + 14\cos^2 t = 2x\cos t + 7y\sin t$$

$$2x\cos t + 7y\sin t = 14(1)$$

$$2x\cos t + 7y\sin t = 14$$

Equation of normal

$$7\sin t\, x - 2y\cos t = k \quad \text{---} (1)$$

passing through $(7\cos t, 2\sin t)$ $t \in R$

$$7\sin t\, 7\cos t - 2\cos t\, 2\sin t = k$$

$$49\sin t\cos t - 4\sin t\cos t = k$$

$$k = 45\sin t\cos t$$

$$(1) \Rightarrow 7\sin t\, x - 2y\cos t = 45\sin t\cos t$$

(4) Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$

$xy = 2$ --- (1) $\quad x^2 = -4y$ --- (2)

$y = \frac{2}{x}$ $\quad x^2 = -4\left(\frac{2}{x}\right)$

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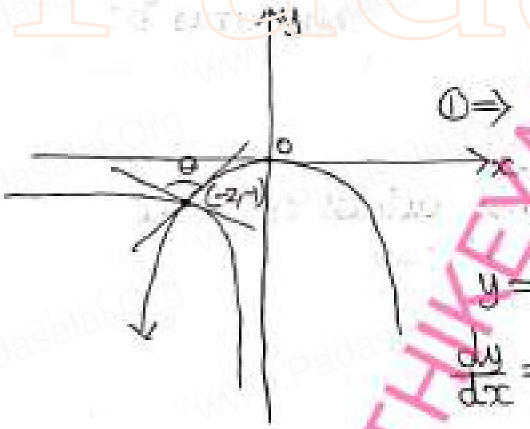
$$x^3 = -8$$

$$\boxed{x = -2}$$



$(1) \Rightarrow -2(y) = 2$

$y = -1$ point of intersection is $(-2, -1)$

$y = \frac{2}{x}$ $\quad 4y = -x^2$

$\frac{dy}{dx} = \frac{-2}{x^2}$ $\quad 4\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{4}$

$m_1 = \left[\frac{dy}{dx}\right]_{(-2,-1)} = \frac{-2}{(-2)^2} = \frac{-2}{4} = -\frac{1}{2}$

$m_2 = \left[\frac{dy}{dx}\right]_{(-2,-1)} = \frac{+2}{2} = 1$

$m_1 = -\frac{1}{2}$ $\quad m_2 = 1$

angle between the curves $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{-\frac{1}{2} - 1}{1 + (-\frac{1}{2})(1)}\right|$$

$\tan\theta = \left|\frac{-3/2}{1/2}\right|$ $\quad \tan\theta = 3$

$\theta = \tan^{-1}(3)$

10) Show that the two curves $x^2-y^2=r^2$ and $xy=c^2$ where c,r are constants, cut orthogonally.

$$x^2-y^2=r^2 \qquad xy=c^2$$

Let the point of intersection be (x_1,y_1)

$$x_1^2-y_1^2=r^2 \text{ --- (1)} \qquad x_1y_1=c^2 \text{ --- (2)}$$

$x^2-y^2=r^2$

diff w.r.to x

$$2x-2y\frac{dy}{dx}=0$$

$$\cancel{2}y\frac{dy}{dx}=\cancel{2}x$$

$$\frac{dy}{dx}=\frac{x}{y}$$

$$m_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{x_1}{y_1}$$

$y_1=\frac{c^2}{x_1}$

diff w.r.to x

$$\frac{dy}{dx}=\frac{-c^2}{x^2}$$

$$\therefore m_2=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-c^2}{x_1^2}$$

$$m_1m_2=\frac{\cancel{x_1}}{y_1}\times\left(\frac{-c^2}{x_1^{\cancel{2}}}\right)$$

$$=\frac{-c^2}{x_1y_1}=\frac{-\cancel{c^2}}{\cancel{c^2}}$$

$$m_1m_2=-1$$

$\therefore$ The curves are cut orthogonally.

——x——

G. Karthikeyan
Thiruvarur DT

Mean value Theorem

① Rolle's Theorem

Let $f(x)$ be a function

(i) $f(x)$ is continuous on $[a,b]$

(ii) $f(x)$ is differentiable on (a,b)

(iii) $f(a)=f(b)$, then there exist atleast one point $c\in(a,b)$ s.t $f'(c)=0$.

② Lagrange's mean value Theorem.

Let $f(x)$ be a function

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

there exist atleast one point $c\in(a,b)$ s.t

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

③ If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) and if $f'(x)>0$ $\forall x\in(a,b)$ then for $x_1,x_2\in[a,b]$ s.t. $x_1<x_2$ we have $f(x_1)<f(x_2)$

Exercise 7.3

1) Explain why Rolle's theorem is not applicable to the following function in the respective intervals.

(i) $f(x) = \left|\frac{1}{x}\right|$, $x \in [-1,1]$

$f(x)$ is not continuous on $[-1,1]$, ($f(x)$ is discontinuous at $x=0$)

$\therefore$ Rolle's Theorem cannot be applicable.

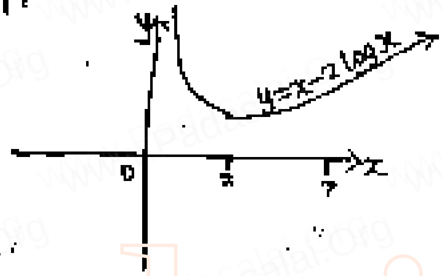
(ii) $f(x) = \tan x$, $x \in [0,\pi]$

$f(x)$ is not continuous on $[0,\pi]$

$f(x)$ is discontinuous at $x = \pi/2$

Rolle's theorem cannot be applicable.

(iii) $f(x) = x - 2\log x$, $x \in [2,7]$



(i) $f(x)$ is continuous on $[2,7]$

(ii) $f(x)$ is differentiable on $(2,7)$

$f(a) = f(2) = 2 - 2\log 2$

$f(b) = f(7) = 7 - 7\log 7$

$2 - 2\log 2 \neq 7 - 7\log 7$

$f(a) \neq f(b)$

$2-2\log 2 = 7 - 7\log 7$

$7\log 7 - 2\log 2 = 7-2$

$\log \frac{7^7}{2^2} = 5$

$f(x)$ does not satisfy all the 3 conditions

$\therefore$ Rolle's theorem cannot be applicable.

2) Using the Rolle's theorem, determine the value of x at which the tangent is parallel to the x axis for the following functions.

(i) $f(x) = x^2 - x$, $x \in [0,1]$

(i) $f(x)$ is continuous on $[0,1]$

(ii) $f(x)$ is differentiable on $(0,1)$

(iii) $f(a) = f(0) = 0 - 0 = 0$

$f(b) = f(1) = 1 - 1 = 0$ $\qquad f(a) = f(b)$

$\therefore$ By Rolle's theorem there exist $c \in (0,1)$ such that $f'(c) = 0$ at which the tangent is parallel to the x axis

$f(x) = x^2 - x$

$f'(x) = 2x - 1$

$f'(c) = 0 \Rightarrow 2c - 1 = 0$

$c = \frac{1}{2} \in (0,1)$

$x = \frac{1}{2}$ the tangent is parallel to x axis.

(ii) $f(x)=\frac{x^2-2x}{x+2}, x\in[-1,6]$

$f(x)$ is continuous on $[-1,6]$ and differentiable on $(-1,6)$

$f(a)=f(-1)=\frac{1+2}{-1+2}=3$

$f(b)=f(6)=\frac{36-12}{6+2}=\frac{24}{8}=3$ $\qquad f(a)=f(b)$

$\therefore$ By Rolle's Theorem, $\exists\ c\in(-1,6)$ such that $f'(c)=0$ at which the tangent is parallel to x axis

$$f(x)=\frac{x^2-2x}{x+2}$$

$$f'(x)=\frac{(x+2)(2x-2)-(x^2-2x)(1)}{(x+2)^2}=\frac{2x^2-\cancel{2x}+4x-4-x^2+\cancel{2x}}{(x+2)^2}$$

$$f'(x)=\frac{x^2+4x-4}{(x+2)^2}$$

$f'(c)=0 \Rightarrow c^2+4c-4=0$

$c^2+4c+4=+8$

$(c+2)^2=(2\sqrt{2})^2$

$c+2=\pm 2\sqrt{2}$

$c=-2\pm2\sqrt{2}$ $\qquad c=-2+2\sqrt{2}\in(-1,6)$

at $x=-2+2\sqrt{2}$ the tangent is parallel to x axis.

(iii) $f(x)=\sqrt{x}-\frac{x}{3}, x\in[0,9]$

$f(x)$ is continuous on $[0,9]$, and differentiable on $(0,9)$

$f(a)=f(0)=0-\frac{0}{3}=0$

$f(b)=f(9)=\sqrt{9}-\frac{9}{3}=3-3=0$ $\qquad f(a)=f(b)$

$\therefore$ By Rolle's Theorem $\exists\ c\in(0,9)$ such that $f'(c)=0$ at which the tangent is parallel to x axis

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{3}$$

$f'(c)=0 \qquad \frac{1}{2\sqrt{c}}-\frac{1}{3}=0$

G. Karthikeyan
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$\frac{1}{2\sqrt{c}}=\frac{1}{3} \qquad 2\sqrt{c}=3$

$\sqrt{c}=3/2$

$c=\frac{9}{4}\in(0,9)$

$x=\frac{9}{4}$ the tangent parallel to x axis.

③ Explain why Lagrange's mean value theorem is not applicable.

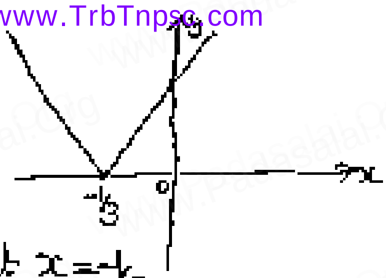
(i) $f(x)=\frac{x+1}{x}, x\in[-1,2]$

$f(x)$ is not continuous in $[-1,2]$

When $x=0$ $f(x)=\infty$ not defined

$\therefore$ Lagrange's mean value Theorem cannot be applicable.

(ii) $f(x) = |3x+1|$, $x \in [-1,3]$



$f(x)$ is continuous on $[-1,3]$

but $f(x)$ is not differentiable on $(-1,3)$

because $f(x)$ is not different at $x = -1/3$

Lagrange's mean value theorem cannot be applicable.

4) Using the Lagrange's mean value theorem determine the value of x at which the tangent is parallel to the secant line at the end point of the given interval.

(i) $f(x) = x^3 - 3x + 2$, $x \in [-2,2]$

$f(x)$ is continuous on $[-2,2]$

$f(x)$ is differentiable on $(-2,2)$

$\therefore$ By MVT there exist one point $c \in (-2,2)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ —(1)

$f'(x) = 3x^2 - 3$

$f(a) = f(-2) = -8+6+2 = 8-8 = 0$

$f(b) = f(2) = 8-6+2 = 4$

$\therefore (1) \Rightarrow 3c^2 - 3 = \frac{4-0}{2+2}$

$3c^2 - 3 = 1$

$3c^2 = 1+3$

$3c^2 = 4$ $\quad c = \pm\frac{2}{\sqrt{3}} \in (-2,2)$

tangent at $x = \pm\frac{2}{\sqrt{3}}$ the secant parallel to secant line

(ii) $f(x) = (x-2)(x-7)$ $\quad x \in [3,11]$

$f(x) = x^2 - 9x + 14$

$f(x)$ is continuous on $[3,11]$

$f(x)$ is differentiable on $(3,11)$

$\therefore$ By MVT $\exists$ one point $c \in (3,11)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ —(1)

$f'(c) = 2c - 9$

$f(b) = f(11) = 121 - 99 + 14 = 36$

$f(a) = f(3) = 9 - 27 + 14 = -4$

$(1) \Rightarrow 2c - 9 = \frac{36+4}{11-3}$ $\quad c = 7 \in (3,11)$

$2c - 9 = \frac{40}{8} = 5$

$2c = 5+9$

$2c = 14$

$x = 7$ the tangent line lie to secant.

5) Show that the value in the conclusion of MVT for

(i) $f(x)=\frac{1}{x}$ on a closed interval of positive number $[a,b]$ is $\sqrt{ab}$

$\rightarrow$

$f(x)$ is continuous on $[a,b]$ $\quad (a,b)$ are positive

$f(x)$ is continuous on (a,b)

$\therefore$ By MVT $\exists$ $c\in(a,b)$ such that

$f(x)=\frac{1}{x}$

$f'(x)=-\frac{1}{x^2}$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$-\frac{1}{c^2}=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$

$$-\frac{1}{c^2}=\frac{a-b}{ab}\times\frac{1}{-(a-b)}$$

$$\cancel{-}\frac{1}{c^2}=\cancel{-}\frac{1}{ab}$$

$$c^2=ab \quad c=\sqrt{ab}$$

$c=\sqrt{ab}\in(a,b)$

(ii) $f(x)=Ax^2+Bx+C$ on any interval $[a,b]$ is $\frac{a+b}{2}$

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

$\therefore$ By MVT $\exists$ $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$2c+B+0=\frac{Ab^2+Bb+\cancel{C}-Aa^2-Ba-\cancel{C}}{b-a}$$

$$2cA+B=\frac{A(b^2-a^2)+B(b-a)}{b-a}$$

$$=\frac{A(b+a)(b-a)+B(b-a)}{b-a}$$

G. Karthikeyan
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$$2cA+B=\frac{\cancel{(b-a)}[A(a+b)+B]}{\cancel{(b-a)}}$$

$$2cA+\cancel{B}=A(a+b)+\cancel{B}$$

$$c=\frac{\cancel{A}(a+b)}{2\cancel{A}} \qquad \boxed{c=\frac{a+b}{2}\in(a,b)}$$

(6) A race car driver is racing at 20^{th} km If his speed never exceeds 150 km/hr what is the maximum distance he can cover in the next two hours.

Let $s(t)$ be the distance at time t

$s(t_1)=20 \quad s(t_2)=?$

$t_1=0 \quad t_2=2$

$s(t)$ is continuous $[0,2]$ and continuous on $(0,2)$

$\therefore$ By MVT $\exists\, t \in (0,2)$ $\quad s'(t) = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ $\quad s'(t) = \text{speed} \le 150$

$$\frac{s(t_2) - 20}{2 - 0} = s'(t) \le 150$$

$$\therefore \frac{s(t_2) - 20}{2} \le 150$$

$$s(t_2) - 20 \le 300$$

$$s(t_2) \le 300 + 20$$

$$s(t_2) \le 320$$

maximum distance can cover $= 320$ km.

(7) suppose that for a function $f(x)$, $f'(x) \le 1$ for all $1 \le x \le 4$ show that $f(4) - f(1) \le 3$

given that $f'(x) \le 1$

$\therefore f(x)$ is differentiable on $(1,4)$

$\Rightarrow f(x)$ is continuous on $[1,4]$

$\therefore$ By MVT $\quad f'(x) = \frac{f(b) - f(a)}{b - a}$

$a = 1$
$b = 4$

$$\frac{f(4) - f(1)}{4 - 1} = f'(x) \le 1$$

$$\frac{f(4) - f(1)}{3} \le 1$$

$$f(4) - f(1) \le 3$$

(8) Does there exist a diff. function $f(x)$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \le 2$ for all x. Justify your answer.

given that $f'(x) \le 2$

$\therefore f(x)$ is differentiable on $(0,2)$

$\Rightarrow f(x)$ is continuous on $[0,2]$

$\therefore$ By MVT $\exists$ one point $x \in (0,2)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$a = 0$
$b = 2$

$$f'(x) = \frac{f(2) - f(0)}{2 - 0}$$

$$\frac{f(2) - f(0)}{2} = f'(x) \le 2$$

$$\frac{f(2) - f(0)}{2} \le 2$$

$$f(2)\ \frac{4+1}{2} \le 2$$

$$2.5 \not\le 2$$

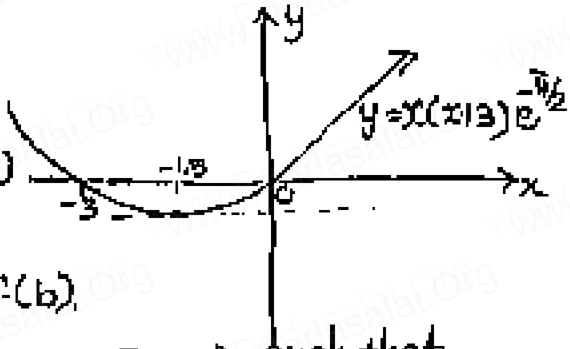
contradiction. $f(x)$ does not have given values

$f'(x) = \frac{4+1}{2-0}$

$f'(x) = \frac{5}{2} \not\le 2$

NO, $f'(x)$ cannot be 2.5

9) Show that there lies a point on the curve $f(x) = x(x+3)e^{-\pi/2}$, $-3 \le x \le 0$ where tangent drawn is parallel to the x-axis.



f(x) is continuous on [-3,0]
f(x) is differentiable on (-3,0)

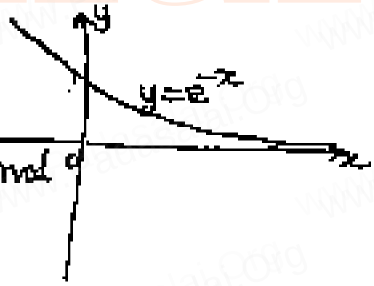
$f(a) = f(-3) = 0$
$f(b) = f(0) = 0$ $\quad f(a) = f(b)$

∴ By Rolle's Theorem ∃ $c \in (-3,0)$ such that $f'(c) = 0$ and that $x = c$ the tangent line parallel to x-axis

$$f(x) = (x^2+3x)e^{-\pi/2}$$
$$f'(x) = (2x+3)e^{-\pi/2}$$
$$f'(c) = 0 \Rightarrow (2c+3)e^{-\pi/2} = 0$$
$$2c+3 = 0$$
$$2c = 3$$
$$c = -1.5 \in (-3,0)$$

$x = -1.5$ the tangent line parallel to x-axis

10) Using mean value theorem prove that for $a, b > 0$
$$|e^{-a} - e^{-b}| < |a-b|$$



$f(x) = e^{-x}$
f(x) is continuous on the intervals
f(x) is differentiable on the open interval

∴ By MVT
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$
$$\frac{f(a)-f(b)}{a-b} = -e^{-c}$$
$$\frac{e^{-a}-e^{-b}}{a-b} = -e^{-c}$$
taking modulus on both sides
$$\frac{|e^{-a}-e^{-b}|}{|a-b|} = |e^{-c}| \le 1 \qquad e^{-x} \le 1 \text{ for}$$
$$\frac{|e^{-a}-e^{-b}|}{|a-b|} \le 1$$
$$|e^{-a}-e^{-b}| \le |a-b|$$

Series Expansions

(a) Taylor's Series

$f(x)$ is infinitely differentiable at $x=a$

$$f(x) = f(a) + \frac{f^{1}(a)}{1!}(x-a) + \dots \frac{f^{n}(a)}{n!}(x-a)^{n} + \dots$$

b) Maclaurin's series

If $a=0$, the expansion takes the form,

$$f(x) = f(0) . \frac{f^{1}(0)}{1!}x + \dots \frac{f^{n}(0)}{n!}x^{n} + \dots$$

Exercise 7.4

1) Write the Maclaurin's series expansion

(i) e^{x}

Function	e^{x} and derivatives	value at $x=0$
$f(x)=e^{x}$	e^{x}	$e^{0}=1$
$f^{1}(x)$	e^{x}	$e^{0}=1$
$f^{11}(x)$	e^{x}	$e^{0}=1$

Maclaurin's series is

$$f(x) = f(0) + \frac{f^{1}(0)}{1!}x + \frac{f^{11}(0)}{2!}x^{2} + \dots$$

$$e^{x} = 1 + \frac{(1)}{1!}x + \frac{1}{2!}x^{2} + \dots$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \dots$$

ii) $\sin x$

Function	$\sin x$ and derivatives	value at $x=0$
$f(x)=\sin x$	$\sin x$	$\sin 0=0$
$f^{1}(x)$	$\cos x$	$\cos 0=1$
$f^{11}(x)$	$-\sin x$	$-\sin 0=0$
$f^{111}(x)$	$-\cos x$	$-\cos 0=-1$
$f^{(IV)}(x)$	$+\sin x$	$\sin 0=0$
$f^{V}(x)$	$\cos x$	$\cos 0=1$

Maclaurin's Series expansion

$$f(x) = f(0) + \frac{f^{1}(0)}{1!}x + \frac{f^{11}(0)}{2!}x^{2} + \frac{f^{111}(0)}{3!}x^{3} + \dots$$

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^{2} + \frac{-1}{3!}x^{3} + \frac{0}{4!}x^{4} + \frac{1}{5!}x^{5} \dots$$

$$\sin x = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \dots$$

(iii) $\cos x$

function	$\cos x$ and derivatives	value at $x=0$
$f(x)=\cos x$	$\cos x$	$\cos 0 = 1$
$f'(x)$	$-\sin x$	$-\sin 0 = 0$
$f''(x)$	$-\cos x$	$-\cos 0 = -1$
$f'''(x)$	$\sin x$	$\sin 0 = 0$
$f^{IV}(x)$	$\cos x$	$\cos 0 = 1$
$f^{V}(x)$	$-\sin x$	$-\sin 0 = 0$
$f^{VI}(x)$	$-\cos x$	$-\cos 0 = -1$

Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + 0\frac{x^5}{5!} - \frac{1}{6!}x^6 + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

(iv) $\log(1-x)$; $-1 \le x \le 1$

function	$\log(1-x)$ and derivatives	value at $x=0$
$f(x)=\log(1-x)$	$\log(1-x)$	$\log 1 = 0$
$f'(x)$	$\frac{-1}{1-x}$	$\frac{-1}{1-0} = -1$
$f''(x)$	$\frac{-1}{(1-x)^2}$	$\frac{-1}{(1-0)^2} = -1$
$f'''(x)$	$\frac{-2}{(1-x)^3}$	$\frac{-2}{(1-0)^3} = -2$
$f^{IV}(x)$	$\frac{-6}{(1-x)^4}$	$\frac{-6}{(1-0)^4} = -6$

G. Karthikeyan
Thiruvarur DT

Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\log(1-x) = 0 - \frac{1}{1!}x - \frac{1}{2}x^2 - \frac{2}{1\times2\times3}x^3 - \frac{6}{1\times2\times3\times4}x^4 - \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

(v) $\tan^{-1}(x)$; $-1 \le x \le 1$

function	$\tan^{-1}x$ and derivatives	value at $x=0$
$f(x)=\tan^{-1}x$	$\tan^{-1}x$	$\tan^{-1}0 = 0$
$f'(x)$	$\frac{1}{1+x^2}$	$\frac{1}{1+0} = 1$

$f'(x)=\frac{1}{1+x^2}=(1+x^2)^{-1}=1-x^2+x^4-x^6+\cdots$

$f''(x)$	$0-2x+4x^3-6x^5+\cdots$	0
$f'''(x)$	$-2+12x^2-30x^4+\cdots$	-2
$f^{iv}(x)$	$0+12x-120x^3+\cdots$	0
$f^{v}(x)$	$24-360x^2+\cdots$	$24=4!$

Maclaurin's series expansion is

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots$$

$$\tan^{-1}x=0+\frac{1}{1}x+\frac{0}{2!}x^2-\frac{2}{1\times2\times3}x^3+\frac{0}{4!}x^4+\frac{4!}{4!5}x^5\cdots$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}\cdots$$

(vi) $\cos^2x$

Function	$\cos^2x$ and derivatives	value at $x=0$
$f(x)=\cos^2x$	$f(x)=\cos^2x$	$\cos^2 0=1^2=1$
$f'(x)$	$2\cos x(-\sin x)$ $=-\sin 2x$	$-\sin 0=0$
$f''(x)$	$-\cos 2x\,(2)$	$-2\cos 0=-2(1)=-2$
$f'''(x)$	$-2(-\sin 2x)2$ $=4\sin 2x$	$4\sin 0=0$
$f^{iv}(x)$	$4\cos 2x\,(2)$	$8\cos 2(0)=8$

Maclaurin's series expansion

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$f(x)=1+\frac{0}{1!}x-\frac{2}{2!}x^2+\frac{0}{3!}x^3+\frac{8x^4}{4!}+\cdots$$

$$\cos^2x=1-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}-\cdots$$

2) write down the Taylor series expansion of the function $\log x$ about $x=1$ upto three non-zero terms for $x>0$

Function	$\log x$ and derivatives	value at $x=1$
$f(x)=\log x$	$\log x$	$\log 1=0$
$f'(x)$	$\frac{1}{x}$	$\frac{1}{1}=1$
$f''(x)$	$-\frac{1}{x^2}$	$\frac{-1}{1}=-1$

$f'''(x)$	$\frac{2}{x^3}$	$\frac{2}{1^3} = 2$
$f^{IV}(x)$	$\frac{-6}{x^4}$	$\frac{-6}{1^4} = -6$
$f^{V}(x)$	$\frac{24}{x^5}$	$\frac{24}{1} = 24$

Taylor's series expansion is

$a=1$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\log x = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$= 0 + \frac{(x-1)}{1!} + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{1\times2\times3}(x-1)^3 + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

③ Expand $\sin x$ in ascending powers $x-\frac{\pi}{4}$ upto 3 non-zero terms

function	$\sin x$ and derivatives	value at $x=\pi/4$
$f(x)=\sin x$	$\sin x$	$\sin \pi/4 = \frac{1}{\sqrt{2}}$
$f'(x)$	$\cos x$	$\cos \pi/4 = \frac{1}{\sqrt{2}}$
$f''(x)$	$-\sin x$	$-\sin \pi/4 = -1/\sqrt{2}$

Taylor's series expansion at $x=\pi/4$

$$f(x) = f(\pi/4) + \frac{f'(\pi/4)}{1!}(x-\pi/4) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1/\sqrt{2}}{1!}(x-\pi/4) + \frac{-1/\sqrt{2}}{2!}(x-\pi/4)^2 + \cdots$$

$$= \frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}}\left[1 + \frac{(x-\pi/4)}{1!} - \frac{(x-\pi/4)^2}{2!} + \cdots\right]$$

$$= \frac{\sqrt{2}}{2}\left[1 + \frac{(x-\pi/4)}{1!} - \frac{(x-\pi/4)^2}{2!} + \cdots\right]$$

4) Expand the polynomial $f(x) = x^2-3x+2$ in powers of $(x-1)$

function	x^2-3x+2 and derivatives	value at $x=1$
$f(x)=x^2-3x+2$	x^2-3x+2	$1-3+2=0$
$f'(x)$	$2x-3$	$2-3=-1$
$f''(x)$	2	2
$f'''(x)$	0	0

Taylor's series expansion $\therefore$ at $x=1$ is

$$f(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\cdots$$

$$x^2-3x+2=0+\frac{(-1)(x-1)}{1}+\frac{2}{2}(x-1)^2+0$$

$$x^2-3x+2=(x-1)^2-(x-1)$$

—x—

Indeterminate forms

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}$$

The L'Hopital's Rule.

$f(x)$ and $g(x)$ are diff. and $g'(x)\neq 0$

$\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$ then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

$\lim\limits_{x\to a} f(x)=\pm\infty=\lim\limits_{x\to a} g(x)$ then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$.

Exercise 7.5

Evaluate the following limits, if necessary use L'Hopital's Rule.

1) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1-\cos 0}{0}=\frac{1-1}{0}=\frac{0}{0}$$

This is in Indeterminate form. $\left(\frac{0}{0}\right)$

Use L'Hopitals Rule

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{0+\sin x}{2x}=\frac{+\sin 0}{2(0)}=\frac{0}{0}$$

again applying L'Hopital's Rule.

$$\lim_{x\to 0}\frac{+\sin x}{2x}=\lim_{x\to 0}\frac{+\cos x}{2}=\frac{+\cos 0}{2}=\frac{1}{2}$$

② $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{x^2-5x+3} = \lim\limits_{x\to\infty} \dfrac{x^2\left(2-\frac{3}{x^2}\right)}{x^2\left(1-\frac{5}{x}+\frac{3}{x^2}\right)}$

$= \dfrac{2-\frac{3}{\infty}}{1-\frac{5}{\infty}+\frac{3}{\infty}} = \dfrac{2-0}{1-0} = 2$

II $\lim\limits_{x\to\infty} \dfrac{2x^2-3}{x^2-5x+3} = \dfrac{\infty}{\infty}$

This is in ID form

applying L'Hopitals Rule two times

$\lim\limits_{x\to\infty} \dfrac{2x^2-3}{x^2-5x+3} = \lim\limits_{x\to\infty} \dfrac{4x-0}{2x-5}$

$= \lim\limits_{x\to\infty} \dfrac{4}{2} = \dfrac{4}{2} = 2$

③ $\lim\limits_{x\to\infty} \dfrac{x}{\log x}$ $\left(= \dfrac{\infty}{\log\infty} = \dfrac{\infty}{\infty}\right)$

This is in ID form

Applying L'Hopitals Rule.

G. Karthikeyan
Thiruvarur DT

$\lim\limits_{x\to\infty} \dfrac{x}{\log x} = \lim\limits_{x\to\infty} \dfrac{1}{1/x}$

$= \lim\limits_{x\to\infty} x = \infty$

④ $\lim\limits_{x\to\pi/2} \dfrac{\sec x}{\tan x}$ $\left(= \dfrac{\sec\pi/2}{\tan\pi/2} = \dfrac{1/0}{\infty} = \dfrac{\infty}{\infty}\right)$

This is in ID form

Applying L'Hopitals Rule

$\lim\limits_{x\to\pi/2^-} \dfrac{\sec x}{\tan x} = \lim\limits_{x\to\pi/2^-} \dfrac{\sec x\tan x}{\sec^2 x}$

$= \lim\limits_{x\to\pi/2^-} \dfrac{\frac{\sin x}{\cos x}}{1/\cos x} = \lim\limits_{x\to\pi/2^-} \sin x$

$= \sin\pi/2 = 1$

M-II $\lim\limits_{x\to\pi/2^-} \dfrac{\sec x}{\tan x} = \lim\limits_{x\to\pi/2^-} \dfrac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{1}{\cos x}\times\dfrac{\cos x}{\sin x}$

$= \dfrac{1}{\sin\pi/2} = \dfrac{1}{1} = 1$

⑤ $\lim\limits_{x\to\infty} e^{-x}\sqrt{x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{e^x}$ $\left(= \dfrac{\infty}{e^\infty} = \dfrac{\infty}{\infty}\right)$

This is in ID form

Applying L'Hopitals Rule

$\lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{e^x} = \lim\limits_{x\to\infty} \dfrac{1/2\sqrt{x}}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{2\sqrt{x}}\times\dfrac{1}{e^x}$

$= \dfrac{1}{\infty}\cdot\dfrac{1}{e^\infty} = \dfrac{1}{\infty} = 0$

6) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \quad \left(=\frac{0-0}{0}=\frac{0}{0}\right)$

This is in ID form
apply L'Hopital Rule

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} = \left(\frac{1-1}{0+0}=\frac{0}{0}\right)$$

again Apply L'Hopital's Rule.

$$\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} = \lim_{x\to 0}\frac{0+\sin x}{x(-\sin x)+\cos x+\cos x}$$

$$= \frac{\sin 0}{0+\cos 0+\cos 0} = \frac{0}{2}$$

$$=0$$

7) $\lim_{x\to 1^+}\left(\frac{2}{x^2-1}-\frac{x}{x-1}\right)$

$$=\lim_{x\to 1^+}\left(\frac{2}{(x+1)(x-1)}-\frac{x}{x-1}\right)$$

$$=\lim_{x\to 1^+}\left(\frac{2-x(x+1)}{(x+1)(x-1)}\right) = \lim_{x\to 1^+}\left(\frac{2-x^2-x}{x^2-1}\right)$$

This is in I.D. form $\left(=\frac{2-1-1}{1-1}=\frac{0}{0}\right)$

applying L'Hopital's Rule

$$\lim_{x\to 1^+}\left(\frac{2-x-x^2}{x^2-1}\right) = \lim_{x\to 1^+}\frac{0-1-2x}{2x}$$

$$= \frac{-1-2}{2} = \frac{-3}{2}$$

8) $\lim_{x\to 0^+} x^x$

$(=0^0)$

This is in ID form

> composite function Theorem
> $\lim_{x\to a}\log f(x) = \log\left(\lim_{x\to a} f(x)\right)$

Let $y=x^x$

$\log y = \log x^x$

$\log y = x\log x$

taking limit

$$\lim_{x\to 0^+}\log y = \lim_{x\to 0^+}\frac{\log x}{1/x} \quad \left(=\frac{\log 0}{1/0}=\frac{-\infty}{\infty}\right)$$

This is in ID form
applying L'Hopital Rule

$$=\lim_{x\to 0^+}\frac{\log x}{1/x} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-1/x^2} = \lim_{x\to 0^+}\frac{1}{\cancel{x}}\times\frac{-x^{\cancel{2}}}{1}$$

$$=\lim_{x\to 0^+} -x = 0$$

$$\lim_{x\to 0^+}\log y = 0$$

Using composite function Rule

$$\log\left(\lim_{x\to 0} y\right)=0$$

taking antilog

$$\left(\lim_{x\to 0} y\right)=e^0$$

$$\lim_{x\to 0} x^x=1$$

9) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \left(=\left(1+\frac{1}{\infty}\right)^\infty=1^\infty\right)$

This is in ID form

G. Karthikeyan.
PGT GGHSS
Thirumakkottai
Thiruvarur DT

Let $y=\left(1+\frac{1}{x}\right)^x$

$$\log y=\log\left(1+\frac{1}{x}\right)^x$$

$$\log y=x\log\left(1+\frac{1}{x}\right)$$

taking limit

$$\lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log(1+1/x)}{1/x} \quad \left(=\frac{\log(1+0)}{1/\infty}=\frac{0}{0}\right)$$

Applying L'Hopital Rule

$$\lim_{x\to\infty}\frac{\log(1+1/x)}{1/x}=\lim_{x\to\infty}\frac{\frac{1}{1+1/x}\left(\cancel{\frac{-1}{x^2}}\right)}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$=\frac{1}{1+\frac{1}{\infty}}=\frac{1}{1+0}=1$$

$$\therefore \lim_{x\to 0}\log y=1$$

Apply composite function Rule $\log\left(\lim_{x\to\infty} y\right)=1$

taking antilog $\lim_{x\to\infty} y=e^1$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$

10) $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$

This is in ID form 0^∞

Let $y=\sin x^{\tan x}$

$$\log y=\log\sin x^{\tan x}$$

$$\log y=\tan x\log\sin x$$

$$\log y=\frac{\log\sin x}{\cot x}$$

taking Limit $\lim_{x\to\pi/2}\log y=\lim_{x\to\pi/2}\frac{\log\sin x}{\cot x} \quad =\left(\frac{\log 1}{\cot\pi/2}=\frac{0}{0}\right)$

Applying L'Hopital's Rule

$$\lim_{x\to\pi/2}\frac{\log\sin x}{\cot x}=\lim_{x\to\pi/2}\frac{\frac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to\pi/2} \frac{\cos x}{\frac{\sin x}{-\frac{1}{\sin^2 x}}} - \lim_{x\to\pi/2} \cos x \cdot \frac{\sin^2 x}{1}$$

$$= \cos\frac{\pi}{2} \left(-\sin\frac{\pi}{2}\right)$$

$$\lim_{x\to\pi/2} \log y = 0$$

$$\log \left(\lim_{x\to\pi/2} y\right) = 0$$

taking antilog $\lim_{x\to\pi/2} y = e^0$

$$\lim_{x\to\pi/2} (\sin x)^{\tan x} = 1$$

11) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

This is in 1^{∞} form (growth) 1^{∞}

Let $y = (\cos x)^{\frac{1}{x^2}}$

$$\log y = \log \cos x^{\frac{1}{x^2}}$$

$$\log y = \frac{\log \cos x}{x^2}$$

taking limit

$$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\log \cos x}{x^2} \quad \left(\because \frac{\log 1}{0} = \frac{0}{0}\right)$$

Applying L'Hopital's Rule

$$\lim_{x\to 0^+} \frac{\log \cos x}{x^2} = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}\cdot -\sin x}{2x}$$

$$= \lim_{x\to 0^+} \frac{-1}{2\cos x} \cdot \frac{\sin x}{x}$$

G. Karthikeyan
Thiruvarur DT

$$= -\frac{1}{2} \cdot \frac{1}{\cos 0} \times \lim_{x\to 0^+} \frac{\sin x}{x}$$

$$\lim_{x\to 0^+} \log y = -\frac{1}{2}$$

$$\log \left(\lim_{x\to 0^+} y\right) = -\frac{1}{2}$$

taking antilog

$$\lim_{x\to 0^+} y = e^{-1/2} = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}}$$

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}.$$

(12) If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously (that is $n\to\infty$) show that the amount after t years is $A = A_0 e^{rt}$.

$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0 \left(1+\frac{r}{n}\right)^{nt}$$

W.K.T $\boxed{\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k}$ [∵ From 11th std]

$$\lim_{n\to\infty} A = A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right)^t$$

$$= A_0\left(e^r\right)^t$$

$$\lim_{n\to\infty} A = A_0 e^{rt}$$

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$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0\left(1+\frac{r}{\infty}\right)^{\infty} = A_0\left(1^{\infty}\right)$$

This is in ID form

Let $y = \left(1+\frac{r}{n}\right)^{nt}$

$$\log y = \log\left(1+\frac{r}{n}\right)^{nt}$$

$$= nt\log\left(1+\frac{r}{n}\right)$$

$$\log y = t\,\frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

taking limits $\lim_{n\to\infty}\log y = \lim_{n\to\infty} t\,\frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{n}} \quad \left(=\frac{0}{0}\right)$

Applying L'Hopital's Rule

$$\lim_{n\to\infty} t\,\frac{\log'\left(1+\frac{r}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} t\,\frac{\left(\frac{1}{1+\frac{r}{n}}\right)\left(\frac{-r}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= rt\lim_{n\to\infty}\left(\frac{1}{1+\frac{r}{n}}\right)$$

$$= rt\,\frac{1}{1+0}$$

$$\lim_{n\to\infty}\log y = rt$$

$$\log\left(\lim_{n\to\infty} y\right) = rt$$

taking antilog $\lim_{n\to\infty} y = e^{rt}$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

$$\therefore \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

i.e $\lim_{n\to\infty} A = A_0 e^{rt}$

Monotonicity of functions.

Increasing function

A function $f(x)$ is increasing function in an interval I

if $a<b \Rightarrow f(a) \le f(b) \quad \forall a,b \in I$

Decreasing function

$f(x)$ is Decreasing function in I

if $a<b \Rightarrow f(a) \ge f(b) \quad \forall a,b \in I$

* $f(x)$ is differentiable in an open interval (a,b)

If (i) $f'(x) \ge 0$ then $f(x)$ is increasing in (a,b)

(ii) $f'(x) > 0$ then $f(x)$ is strictly increasing in (a,b)

(iii) $f'(x) \le 0$ then $f(x)$ is decreasing in (a,b)

(iv) $f'(x) < 0$ then $f(x)$ is strictly decreasing in (a,b)

$\forall x \in (a,b)$

Stationary point.

A stationary point $(x_0, f(x_0))$ of a diff. function $f(x)$ is where $f'(x_0) = 0$

Critical Number and critical point

A critical point $(x_0, f(x_0))$ of a function $f(x)$ is where $f'(x_0) = 0$ or does not exist (that x_0 is called critical number)

Absolute maximum & minimum

$f(x_0)$ is called

absolute max of $f(x)$ $\quad f(x_0) \ge f(x) \quad \forall x \in D$

absolute min of $f(x)$ $\quad f(x_0) \le f(x) \quad \forall x \in D$

(where x_0 is in the Domain D)

A procedure for finding the absolute extrema

1) find critical number of $f(x)$
2) Evaluate value of $f(x)$ at that critical no. and also find $f(a), f(b)$
3) Largest value in step 2 is absolute max, smallest value in step 2 is absolute min.

Fermat Theorem

If c is a critical number $f(x)$ has relative extremum at $x=c$. Invariably there will be critical numbers of $f(x)$ obtained by $f'(x)=0$ or $f'(x)$ doesnot exist values.

First Derivative test.

① If $f'(x)$ changes its sign +ve to −ve then $f(x)$ has local max $f(c)$

② If $f'(x)$ changes from −ve to +ve then $f(x)$ has local min $f(c)$.

③ $f'(x)$ is +ve on both sides of c or −ve on both sides of c then $f(c)$ is neither local max nor min.

—x—

Exercise 7.6

① Find the absolute extrema.

(i) $f(x) = x^2-12x+10$; $[1,2]$.

$f'(x) = 2x-12$

$f'(x)=0 \Rightarrow 2x-12=0$,

$2x=12$

$x=\frac{12}{2}=6$

critical numbers is $x=6 \notin [1,2]$ we cannot take $x=6$

$\cancel{f(6)=6^2-12(6)+10=36-72+10=-26}$

$f(1) = 1-12+10 = -1$

$f(2)=2^2-12(2)+10 = 4-24+10 = -10$

absolute maximum $=-1$, absolute minimum $= \cancel{-26}\ -10$

(ii) $f(x)=3x^4-4x^3$; $[-1,2]$ [change the interval as $[1,7]$ we get book answer]

$f'(x)=12x^3-12x^2$

$f'(x)=0 \Rightarrow 12x^2(x-1)=0$

$x^2=0 \quad x-1=0$

$x=0 \quad x=1$

critical numbers $x=0,1$

$f(0)=3(0)-4(0)=0$

$f(1)=3(1)-4(1)=3-4=-1$

$f(-1)=3(-1)^4-4(-1)^3=3+4=7$

$f(2)=3(2)^4-4(2)^3=48-32=16$

absolute maximum $=16$ absolute minimum $=-1$

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$ $[-1, 1]$

$f'(x) = \overset{2}{\cancel{6}}\left(\frac{4}{\cancel{3}}\right)x^{4/3-1} - \cancel{3}\frac{1}{\cancel{3}}x^{1/3-1}$

$= 8x^{1/3} - x^{1/3}x^{-1}$

$= x^{1/3}\left(8 - \frac{1}{x}\right) = \frac{x^{1/3}(8x-1)}{x} = \frac{8x-1}{x^{2/3}}$

$f'(x) = 0 \Rightarrow 8x - 1 = 0$

$8x = 1$

$x = 1/8$

at $x = 0$ $f'(x)$ does not exist critical numbers $x = 0, 1/8$

$f(0) = 0$

$f(1/8) = 6\left(\frac{1}{8}\right)^{4/3} - 3\left(\frac{1}{8}\right)^{1/3}$

$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right) = \frac{\cancel{6}3}{\cancel{16}8} - \frac{3}{2} = \frac{3-12}{8} = \frac{-9}{8}$

$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 6+3 = 9$

$f(1) = 6 - 3 = 3$

absolute maximum $= 9$ absolute minimum $= -\frac{9}{8}$

(iv) $f(x) = 2\cos x + \sin 2x$; $[0, \pi/2]$

$f'(x) = 2(-\sin x) + 2\cos 2x$

$f'(x) = 2(-\sin x + \cos 2x)$

$f'(x) = 0 \Rightarrow$

$-\sin x + 1 - 2\sin^2 x = 0$

$2\sin^2 x + \sin x - 1 = 0$

$\left(\sin x + \frac{2}{2}\right)\left(\sin x - \frac{1}{2}\right) = 0$

$\sin x + 1 = 0$ $\sin x = 1/2$

$\sin x = -1$ | $\sin x = \sin \pi/6$

$\sin x = \sin -\pi/2$ | $x = n\pi + (-1)^n \pi/6$

$x = n\pi + (-1)^n(-\pi/2)$

take one value

Critical number $x = 3\pi/2$, $x = \pi/6$

$f(3\pi/2) = 2\cos 3\pi/2 + \sin 2\left(\frac{3\pi}{2}\right) = 0 + 0 = 0$

$f(\pi/6) = 2\cos \pi/6 + \sin 2\pi/6_3 = 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(0) = 2\cos(0) + \sin 0 = 2(1) = 2$

$f(\pi/2) = 2\cos \pi/2 + \sin 2\pi/2 = 0 + 0 = 0$

absolute maximum $= 3\sqrt{3}/2$ absolute minimum $= 0$

② Find the interval of monotonicities and hence find the local extremum.

(i) $f(x) = 2x^3+3x^2-12x$

$f'(x) = \cancel{3}6x^2+6x-12$

$f'(x)=0 \Rightarrow 6x^2+6x-12=0$

$6(x^2+x-2)=0$

$(x+2)(x-1)=0$

$x+2=0 \quad x-1=0$

$x=-2, \quad x=1$

critical numbers are $-2, 1$

intervals are $(-\infty,-2), (-2,1), (1,\infty)$

Interval	value x	$f'(x)=6(x+2)(x-1)$	sign of $f'(x)$	monotonicity
$(-\infty,-2)$	-3	$6(-3+2)(-3-1)$ $6(-1)(-4)=24$	$+$	strictly increasing
$(-2,1)$	0	$6(0+2)(0-1)$ $=-12$	$-$	strictly decreasing
$(1,\infty)$	2	$6(4)(1)=24$	$+$	strictly increasing

(i) $f'(x)$ changes from $+$ve to $-$ve through $x=-2$

$f(x)$ has local max at $x=-2$

max value is $f(-2) = 2(-8)+3(-2)^2-12(-2)$

$= -16+12+24$

$= 20$

(ii) $f'(x)$ changes from $-$ve to $+$ve through $x=1$

$f(x)$ has local minimum at $x=1$

minimum value $= f(1)$

$= 2+3-12$

$= -7$

Local maximum $=20$, Local minimum $=-7$

(ii) $f(x) = \frac{x}{x-5}$

$$f'(x) = \frac{(x-5)(1)-x(1)}{(x-5)^2} = \frac{\cancel{x}-5-\cancel{x}}{(x-5)^2} = \frac{-5}{(x-5)^2}$$

$f'(x) = \frac{-5}{(x-5)^2} \neq 0 \quad \forall x \in R$, and

$f'(x)$ does not exist at $x=5$

no critical number at $x=5$

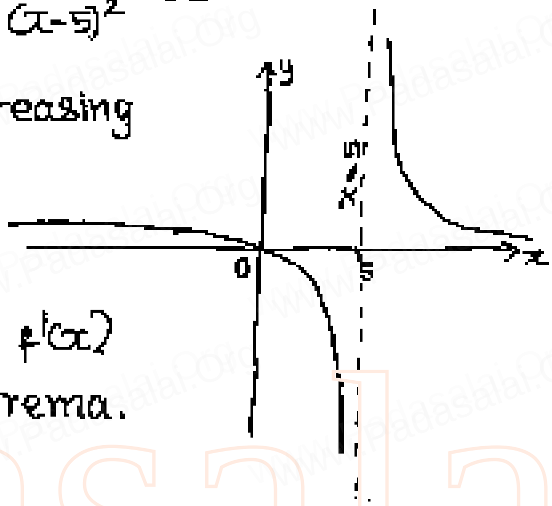
because at $x=5$ $f(x)$ also does not exist

$\therefore$ domain of $f(x)$ is $\boxed{R-\{5\}}$

$\therefore x=5$ is not in domain. ($\therefore$ No need to take $x=5$ as a critical No)

$\therefore$ In $R-\{5\}$ $f'(x) = \frac{-5}{(x-5)^2} < 0$

$\therefore f(x)$ is strictly decreasing on $R-\{5\}$



No critical number

$\therefore$ no change of sign of $f'(x)$

$\therefore$ There is no local extrema.

(iii) $f(x) = \frac{e^x}{1-e^x}$

$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} \Rightarrow f'(x) = \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$

$f'(x) = \frac{e^x}{(1-e^x)^2} \neq 0 \quad \forall x \in R-\{0\}$

when $x=0$ $f(x)$ does not exist $\therefore$ domain is $R-\{0\}$

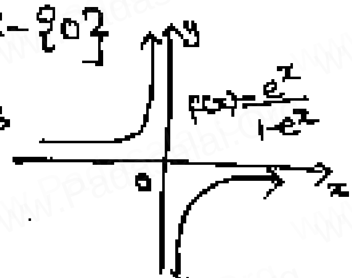
$\therefore x=0$ is not in domain of $f(x)$ ($\therefore$ No need to take $x=0$ as a critical no.)

$\therefore$ In $R-\{0\}$ $f'(x) = \frac{e^x}{(1-e^x)^2} > 0$

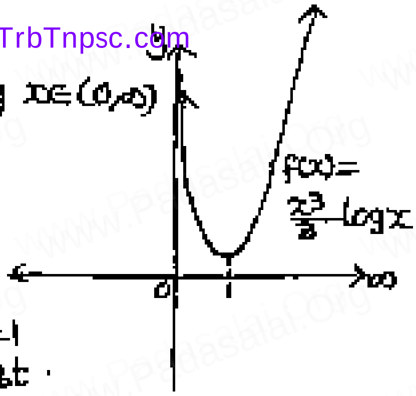
$f(x)$ is strictly increasing on $R-\{0\}$

$f'(x)$ does not changes its sign

$\therefore$ There is no local extrema,



(iv) $f(x)=\frac{x^3}{3}-\log x$ $\quad$ $f(x)$ exist only $x\in(0,\infty)$



$$f'(x)=\frac{\cancel{3}x^2}{\cancel{3}}-\frac{1}{x}=\frac{x^3-1}{x}$$

$$f'(x)=0 \Rightarrow \frac{x^3-1}{x}=0$$

$$x^3=1 \quad x=1$$

when $x=0$, $f'(x)$ does not exist.

critical numbers are $x=0,1$

Intervals are ~~$(-\infty,0)$~~, $(0,1)$, $(1,\infty)$

Intervals	value x	$f'(x)=\frac{x^3-1}{x}$	sign of $f'(x)$	Monotonicity
~~$(-\infty,0)$~~	~~-1~~	~~$\frac{-1-1}{-1}=2$~~	~~+~~	~~strictly increasing~~
$(0,1)$	$\frac{1}{2}$	$\frac{\frac{1}{8}-1}{\frac{1}{2}}$	$-$	strictly decreasing
$(1,\infty)$	2	$\frac{8-1}{2}=\frac{7}{2}$	$+$	strictly increasing

(i) $(-\infty,0)$ $f(x)$ is not defined.
also at $x=0$ $f(x)$ ($=-\infty$) not defined.
$\therefore f(x)$ lies only on $(0,\infty)$

From the beginning we take $x\in(0,\infty)$ is better. $(0,1)$, $(1,\infty)$

(ii) $f'(x)$ changes its sign from $-$ve to $+$ve at $x=1$

$\therefore f(x)$ has a local minimum at $x=1$

local minimum value $=f(1)$

$$=\frac{1}{3}-\log 1$$

local minimum $=\frac{1}{3}$

(v) $f(x)=\sin x\cos x+5$ $\quad$ $x\in(0,2\pi)$

$$f(x)=\frac{1}{2}\,2\sin x\cos x+5$$

$$f(x)=\frac{1}{2}\sin 2x+5$$

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Thiruvarur DT

$$f'(x)=\frac{1}{\cancel{2}}\cos 2x(\cancel{2})=\cos 2x$$

$$f'(x)=0 \Rightarrow \cos 2x=0$$

$$\cos 2x=\pi/2,\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2},\ldots$$

critical numbers $x=\pi/4,\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4},\ldots$

intervals are $(0,\pi/4)$, $(\pi/4,\frac{3\pi}{4})$, $(\frac{3\pi}{4},5\pi/4)$, $(5\pi/4,\frac{7\pi}{4})$, $(\frac{7\pi}{4},2\pi)$

Intervals	value	$f'(x)=\cos 2x$	sign of $f'(x)$	monotonicity
$(0,\pi/4)$	$\pi/6$	$\cos\pi/3=\frac{1}{2}$	+	strictly increasing
$(\pi/4,\frac{3\pi}{4})$	$\pi/2$	$\cos\pi=-1$	−	strictly decreasing
$(\frac{3\pi}{4},\frac{5\pi}{4})$	π	$\cos 2\pi=1$	+	strictly increasing
$(\frac{5\pi}{4},\frac{7\pi}{4})$	$\frac{3\pi}{2}$	$\cos 3\pi=-1$	−	strictly decreasing
$(\frac{7\pi}{4},2\pi)$	$\frac{11\pi}{6}$	$\cos 11\pi/3$ $\cos 660^\circ=\cos-60^\circ=\frac{1}{2}$	+	strictly Increasing

$f(x)$ is strictly increasing on $(0,\pi/4),(\frac{3\pi}{4},\frac{5\pi}{4}),(\frac{7\pi}{4},2\pi)$

and strictly decreasing on $(\pi/4,\frac{3\pi}{4}),(\frac{5\pi}{4},\frac{7\pi}{4})$

(i) $f'(x)$ changes from +ve to −ve at $x=\frac{\pi}{4},\frac{5\pi}{4}$

$f(x)$ has local maximum at $x=\pi/4, x=\frac{5\pi}{4}$

$$f(\pi/4)=\frac{1}{2}\sin 2\frac{\pi}{4}+5=\frac{1}{2}+5=11/2$$

$$f(5\pi/4)=\frac{1}{2}\sin 2(\frac{5\pi}{4})+5$$

$$=\frac{1}{2}\sin\pi/2+5=\frac{1}{2}+5=\frac{11}{2}$$

Local max $=11/2$

(ii) $f'(x)$ changes from −ve to +ve through $x=\frac{3\pi}{4},\frac{7\pi}{4}$

$f(x)$ has local minimum at $x=\frac{3\pi}{4},\frac{7\pi}{4}$

Local minimum values are

$$f(\frac{3\pi}{4})=\frac{1}{2}\sin 2(\frac{3\pi}{4})+5=\frac{1}{2}(-1)+5=\frac{9}{2}$$

$$f(\frac{7\pi}{4})=\frac{1}{2}\sin 2(\frac{7\pi}{4})+5=\frac{1}{2}\sin(630^\circ)+5$$

$$=\frac{1}{2}\sin(720-90)+5$$

$$=-\frac{1}{2}+5=9/2$$

Local maximum $=11/2$

Local minimum $=9/2$

—x—

Applications of second derivatives

Test of concavity

(i) If $f''(x)>0$ on an open interval I, then $f(x)$ is concave up on I

If $f''(x)<0$ on I, then $f(x)$ is concave down on I

point of Inflection

(i) If $f''(c)$ exists and $f''(c)$ changes sign (+ve to −ve or −ve to +ve) through $x=c$ then $(c, f(c))$ is called point of inflection.

(ii) If $f''(c)$ exists at point of inflection, then $f''(c)=0$.

The second Derivative Test

(i) $f'(x)$ exists and give the critical point $x=c$

(ii) find $f''(x)$ then.

$f''(x)<0$ at $x=c$ then $f(x)$ has relative maximum

$f''(x)>0$ at $x=c$ then $f(x)$ has relative minimum.

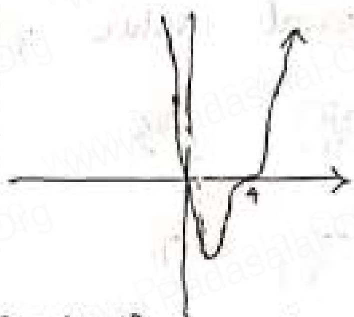
$f''(x)=0$ at $x=c$ The test is not informative.

Exercise 7.7

1) Find the intervals of concavity and points of inflection.

(i) $f(x)=x(x-4)^3$

$$f'(x) = x3(x-4)^2+(x-4)^3(1)$$
$$= (x-4)^2(3x+x-4)$$
$$= (x-4)^2(4x-4)$$
$$= 4(x-4)^2(x-1)$$



$$f''(x) = 4(x-4)^2(1)+4(x-1)\,2(x-4)$$
$$= 4(x-4)[x-4+2(x-1)]$$
$$= 4(x-4)(x-4+2x-2) = 4(x-4)(3x-6)$$
$$f''(x) = 12(x-4)(x-2)$$

$f''(x)=0 \Rightarrow 12(x-4)(x-2)=0$

$x=2,4$

intervals are $(-\infty,2),(2,4),(4,\infty)$

Interval	value	$f''(x)=12(x-4)(x-2)$	sign of $f''(x)$	concavity
$(-\infty,2)$	0	$12(-4)(-2)=96$	+	concave up
$(2,4)$	3	$12(-1)(1)=-12$	−	concave down
$(4,\infty)$	5	$12(1)(3)=36$	+	concave up

The curve is concave up on $(-\infty,2),(4,\infty)$.
concave downward on $(2,4)$.

$f''(x)$ changes its sign through $x=2$ and $x=4$

$f(2)=2(-2)^3=-16$ point of inflection
$f(4)=4(0)=0$ $(2,-16)\ (4,0)$.

(ii) $f(x)=\sin x+\cos x\ ,\ 0<x<2\pi$

$f'(x)=\cos x-\sin x$

$f''(x)=-\sin x-\cos x$.

$f''(x)=0 \Rightarrow -\sin x-\cos x=0$

$\sin x=-\cos x$.

$\tan x=-1=\tan -45^\circ$ $\quad \alpha=-\pi/4$.

$x=n\pi+\alpha$

$x=n\pi-\pi/4 \quad n\in z$

$x=\frac{3\pi}{4},\frac{7\pi}{4} \in(0,2\pi)$

intervals are $(0,\frac{3\pi}{4}),(\frac{3\pi}{4},\frac{7\pi}{4}),(\frac{7\pi}{4},2\pi)$

Interval	value	$f''(x)=-(\sin x+\cos x)$	$f''(x)$ sign	concavity
$(0,\frac{3\pi}{4})$	$\pi/2$	$-(\sin\pi/2+\cos\pi/2)$ $=-(1+0)=-1$	−	concave down
$(\frac{3\pi}{4},\frac{7\pi}{4})$	π	$-(0+(-1))=1$	+	concave up
$(\frac{7\pi}{4},2\pi)$	$11\pi/6=330^\circ$	$-(-\sin 30^\circ+\cos 30^\circ)$ $-(-1/2+\frac{\sqrt3}{2})=\frac{1-\sqrt3}{2}$	−	concave down

$f(x)$ is concave upward on $(\frac{3\pi}{4},\frac{7\pi}{4})$
concave downward on $(0,\frac{3\pi}{4}),(\frac{7\pi}{4},2\pi)$

$f''(x)$ changes its sign through $x=\frac{3\pi}{4},\frac{7\pi}{4}$

$f\left(\frac{3\pi}{4}\right) = \sin 135^\circ + \cos 135^\circ = \sin(180-45) + \cos(180-45)$

$= \sin 45 - \cos 45^\circ$

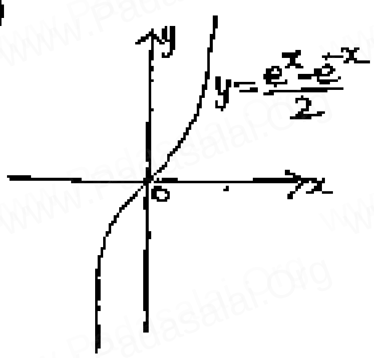
$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$f\left(\frac{3\pi}{4}\right) = 0$

$f\left(\frac{7\pi}{4}\right) = \sin 315^\circ + \cos 315^\circ = -\sin 45 + \cos 45^\circ = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

point of inflection $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$



$f'(x) = \frac{1}{2}(e^x + e^{-x})$

$f''(x) = \frac{1}{2}(e^x - e^{-x})$

$f''(x) = 0 \Rightarrow e^x - e^{-x} = 0$

$e^x = e^{-x}$

$\boxed{x = 0}$

Intervals $(-\infty, 0), (0, \infty)$

Interval	value	$f''(x) = \frac{e^x - e^{-x}}{2}$	sign of $f''(x)$	concavity
$(-\infty, 0)$	-1	$\frac{e^{-1} - e^{1}}{2}$	$-$	concave downward
$(0, \infty)$	1	$\frac{e^{1} - e^{-1}}{2}$	$+$	concave upward.

$f(x)$ is concave upward on $\therefore (0, \infty)$

concave downward on $(-\infty, 0)$

$f''(x)$ changes its sign through $x = 0$

$f(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$ $\therefore$ point of Inflection $= (0, 0)$

2) Find the local extrema for the following functions using second derivative test..

(i) $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

$= 15x^2(x^2 - 1)$

$f'(x) = 15x^2(x+1)(x-1)$

$f'(x) = 0 \Rightarrow 15x^2(x+1)(x-1) = 0 \Rightarrow x = 0, 1, -1$

The critical numbers are $x = 0, 1, -1$

$f''(x) = -60x^3 + 30x$

(i) $x = -1 \Rightarrow f''(-1) = +60 - 30 = 30 > 0$

$f(x)$ has local minimum at $x = -1$.

Local minimum value $f(-1) = 3 - 5$

(ii) $x=1$ $f''(1)=-60+30=-30<0$

$f(x)$ has local maximum at $x=1$

local maximum value $=f(1)$
$=-3+5$
$=2$

(iii) $x=1$ $f''(0)=0$ second derivative test is not informative of Local extremum of $f(x)$.

Local min $=-2$ Local max $=2$.

② (ii) $f(x)=x\log x$

$f'(x)=x\left(\frac{1}{x}\right)+\log x$ $f'(x)=1+\log x$

$f'(x)=0 \Rightarrow$ $1+\log x=0$
$\log x=-1$
$x=e^{-1}$ $\boxed{x=\frac{1}{e}}$

critical number is $x=1/e$

$f''(x)=0+\frac{1}{x}$

$f''(x)=\frac{1}{x}$

when $x=1/e$ $f''(1/e)=\frac{1}{1/e}=e>0$

$f(x)$ has local minimum at $x=1/e$.

local minimum value $=f(1/e)$
$=1/e\log 1/e$
$=\frac{1}{e}(\log 1-\log e)$
$=\frac{1}{e}(0-1)=-\frac{1}{e}$

Local minimum $=-1/e$

2(iii) $f(x)=x^2e^{-2x}$

$f'(x)=e^{-2x}(2x)+x^2e^{-2x}(-2)$
$=2e^{-2x}(x-x^2)$

$f'(x)=0 \Rightarrow$ $2e^{-2x}(x-x^2)=0$
$e^{-2x}\neq 0$, $x-x^2=0$
$-x(x-1)=0$
$x=0$ $x=1$

critical numbers are $x=0,1$

$f''(x)=2e^{-2x}(1-2x)+(x-x^2)2e^{-2x}(-2)$
$=2e^{-2x}(1-2x+(x-x^2)(-2))$
$=2e^{-2x}(1-2x-2x+2x^2)$

$f''(x)=2e^{-2x}(2x^2-4x+1)$

(i) when $x=0$ $f''(x)=2e^0(1)=2>0$

$f(x)$ has local minimum at $x=0$

Local minimum value $=f(0)$
$=0$

(ii) when $x=1$ $f''(1)=2e^{-2}(2-4+1)$
$=2e^{-2}(-1)<0$

$f(x)$ has local maximum at $x=1$

local maximum value $=f(1)$
$=1e^{-2}=\frac{1}{e^2}$

Local minimum $=0$
Local maximum $=1/e^2$

③ For the function $f(x)=4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extrema intervals of concavity and point of inflection.

$f'(x)=12x^2+6x-6$

$f'(x)=6(2x^2+x-1)=6(x+1)(2x-1)$

$f'(x)=0 \Rightarrow 6(2x^2+x-1)=0$ $\boxed{f'(x)=6(x+1)(2x-1)}$ —(1)

$(x+\frac{2}{2})(x-\frac{1}{2})=0$

$x=-1, 1/2$

critical numbers are $x=-1, 1/2$

$-\infty \quad -1 \quad 1/2 \quad \infty$

Intervals are $(-\infty,-1), (-1,1/2), (1/2,\infty)$

Interval	value x	$f'(x)=6(x+1)(2x-1)$	sign of $f'(x)$	monotonicity
$(-\infty,-1)$	-2	$6(-1)(-5)=30$	+	strictly increasing
$(-1,1/2)$	0	$6(1)(-1)=-6$	−	strictly decreasing
$(1/2,\infty)$	1	$6(2)(1)=12$	+	strictly increasing

$f(x)$ is strictly increasing on $(-\infty,-1), (1/2,\infty)$
strictly decreasing on $(-1,1/2)$

$f'(x)$ changes from +ve to −ve through $x=-1$

$f(x)$ has local maximum at $x=-1$

Local maximum value $= f(-1)$

$= -4+3+6+1 = 6$

$f'(x)$ changes from $-ve$ to $+ve$ through $x = \frac{1}{2}$

$f(x)$ has local minimum at $x = \frac{1}{2}$

Local minimum value $= f(\frac{1}{2})$

$= 4\frac{1}{8} + 3\frac{1}{4} - \frac{6}{2} + 1$

$= \frac{2+3-12+4}{4} = \frac{9-12}{4} = \frac{-3}{4}$

$\therefore$ Local maximum $= 6$

Local minimum $= -\frac{3}{4}$.

(ii)

$f'(x) = 6(2x^2+x-1)$

$f''(x) = 6(4x+1)$

$f''(x) = 0 \Rightarrow 4x+1 = 0$

$4x = -1$ $\boxed{x = \frac{-1}{4}}$

$-\infty \quad -\frac{1}{4} \quad \infty$

Interval	value x	$f''(x) = 6(4x+1)$	sign of $f''(x)$	concavity
$(-\infty, -\frac{1}{4})$	-1	$6(-4+1) = -18$	$-$	concave downward
$(-\frac{1}{4}, \infty)$	0	$6(1) = 6$	$+$	concave upward

$f(x)$ is concave ~~upward~~ down on $(-\infty, -\frac{1}{4})$

concave upward on $(-\frac{1}{4}, \infty)$

$f''(x)$ changes its sign through $x = -\frac{1}{4}$

$f(-\frac{1}{4}) = 4(\frac{-1}{64}) + 3\frac{1}{16} - 6(\frac{-1}{4}) + 1$

$= \frac{-1}{16} + \frac{3}{16} + \frac{6}{4} + 1 = \frac{2}{16} + \frac{3}{2} + 1 = \frac{1+12+8}{8} = \frac{21}{8}$

point of inflection is $(-\frac{1}{4}, \frac{21}{8})$

——x——

Applications in optimization.

Exercise 7.8

① Find the two $+ve$ numbers whose sum is 12 and their product is maximum.

Let the two numbers be x, y

$x+y = 12$

$y = 12-x$ —— (1)

product $P = xy$

$P = x(12-x)$

$P(x) = 12x - x^2$

$P'(x) = 12 - 2x$

$P'(x) = 0 \Rightarrow 12 - 2x = 0$

$\cancel{-}2x = \cancel{-}12$

$x = \frac{12}{2}$ $\boxed{x=6}$

$P''(x) = -2$

when $x = 6$ $P''(x) = -2 < 0$

$P(x)$ has Local maximum at $x = 6$

when $x = 6$ The product is maximum

$① \Rightarrow y = 12 - 6 = 6$

The two numbers are 6, 6.

② Find the two +ve numbers whose product is 20 and their sum is minimum.

Let the numbers be x, y

product $= 20$

$xy = 20$

$y = \frac{20}{x}$ —①

Sum $S = x + y$

$S = x + \frac{20}{x}$

$S'(x) = 1 - \frac{20}{x^2}$

$S'(x) = 0 \Rightarrow 1 - \frac{20}{x^2} = 0$ $1 = \frac{20}{x^2}$

$x^2 = 20$

$x = \pm\sqrt{20}$

$\boxed{x = 2\sqrt{5}}$

$S''(x) = \frac{+20(2)}{x^3}$

when $x = 2\sqrt{5}$ $S''(x) = \frac{20 \times 2}{(2\sqrt{5})^3} > 0$

when $x = 2\sqrt{5}$ $S(x)$ has Local minimum

$x = 2\sqrt{5}$ sum is minimum

$① \Rightarrow y = \frac{20}{2\sqrt{5}}$ $y = 2\sqrt{5}$

The numbers are $2\sqrt{5}, 2\sqrt{5}$

3) Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

$x + y = 10$

$y = 10 - x$

$f(x) = x^2 + (10-x)^2$

$f'(x) = 2x + 2(10-x)(-1)$

$f'(x) = 2(x+x-10)$

$= 2(2x-10)$

$f'(x) = 4(x-5)$

$f'(x) = 0 \Rightarrow 4(x-5) = 0$

$\boxed{x=5}$

$f''(x) = 4$

when $x=5$ $f''(x) = 4 > 0$

$f(x)$ has local minimum at $x=5$

at $x=5$ $f(x)$ has minimum value

$f(5) = 5^2 + (10-5)^2$

$= 25+25$

smallest value $= 50$

④ A garden is to be laid out in a rectangular area and protected by wire fence, what is the largest possible area of the fenced garden with metres of wire,

Let x, y be the length and breadth of the Rectangular garden

given total fencing $= 40$

$2x+2y = 40$

$x+y = 20$

$\boxed{y = 20-x}$ ——①

Area $= xy$

$A(x) = x(20-x)$

$A(x) = 20x - x^2$

$A'(x) = 20-2x$

$A'(x) = 0 \Rightarrow$ $20-2x=0$

$-2x = -20$

$\boxed{x=10}$

$A''(x) = -2$

when $x=10$, $A''(x) = -2 < 0$

when $x=10$ $A(x)$ is local maximum

$x=10$ Area is maximum $x=10 \Rightarrow y = 20-10$

$y=10$

Largest possible area

$= xy = 10 \times 10$

$= 100\ m^2$

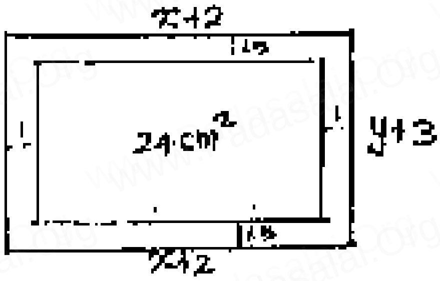
⑤ A rectangular page is contain $24\ cm^2$ of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. what should be the dimensions of the page so that the area of the paper is minimum.

Let x, y be the length and breadth of the printed area. Area $= 24$

$xy = 24$

$y = \frac{24}{x}$ ——①

Length of the page is $x+2$
breadth $= y+3$ $\Big\}$ —①



Area of paper $= (x+2)(y+3)$

$A = xy + 2y + 3x + 6$

$= 24 + 3x + 2y + 6$

$A(x) = 3x + 2\left(\frac{24}{x}\right) + 30$

$A'(x) = 3 + 48\left(\frac{-1}{x^2}\right) + 0$

$A'(x) = 0 \Rightarrow$

$3 - \frac{48}{x^2} = 0$

$3 = \frac{48}{x^2}$

$x^2 = \frac{48}{3} = 16 = 4^2$

$\boxed{x=4}$

$A''(x) = 48 \times \frac{2}{x^3}$

when $x = 4 \Rightarrow A''(x) = \frac{96}{4^3} > 0$

$= \frac{96}{16 \times 4} = \frac{3}{2} > 0$

when $x = 4$ $A(x)$ is minimum

$x = 4$ Area of the page is minimum

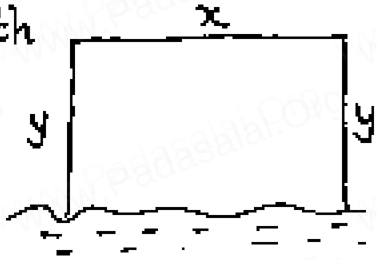
① $\Rightarrow$ $y = \frac{24}{4} = 6$

Length of the page $= x+2 = 4+2 = 6$ cm

breadth of the page $= y+3 = 6+3 = 9$ cm

6) A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq mtrs in order to provide enough grass for herds. No fencing is needed along the river. what is the length of the minimum needed fencing material?

Let x, y be the length and breadth of the pasture



given Area $= 1,80,000$

$xy = 1,80000$

$y = \frac{180000}{x}$ —①

Length of fencing $= x + 2y$

$L(x) = x + \frac{360000}{x}$

$L'(x) = 1 - \frac{360000}{x^2}$

$L'(x) = 0 \Rightarrow 1 - \frac{360000}{x^2} = 0$

$1 = \frac{360000}{x^2}$

$x^2 = 360000$ $\boxed{x = 600 \text{ m}}$

$L''(x) = \frac{720000}{x^3}$

when $x=600$ $L''(x) = \frac{720000}{(600)^3} > 0$

$x=600$ $L(x)$ is minimum (Local)

$x=600$ length of fencing is minimum

$① \Rightarrow y = \frac{180000}{600}$ $y = 300$ m

minimum length of fencing $= x+2y$

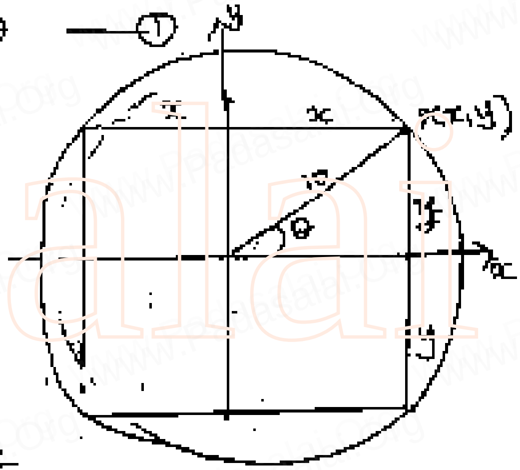
$= 600+600 = 1200$ m

7) Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm

Let $2x, 2y$ be the length and breadth of the rectangle. $r=10$

$2x = 2(10\cos\theta)$ $2y = 2(10\sin\theta)$

$2x = 20\cos\theta$, $2y = 20\sin\theta$ — ①



Area $= (2x)(2y)$

$= (20\cos\theta)(20\sin\theta)$

$= 200 \times 2\cos\theta\sin\theta$

$A(\theta) = 200\sin 2\theta$

$A'(\theta) = 200(\cos 2\theta)2$

$A'(\theta) = 0 \Rightarrow 400\cos 2\theta = 0$

$2\theta = \pi/2$

$\boxed{\theta = \pi/4}$

$A''(\theta) = 400(-\sin 2\theta)2$

when $\theta = \pi/4$ $A''(\theta) = -800\sin 2(\frac{\pi}{4})$

$= -800(1)$

$= -800 < 0$

when $\theta = \pi/4$ $A(\theta)$ is maximum (Local)

$\theta = \pi/4$, Area of rectangle attains maximum.

$① \Rightarrow$ $2x = 20\cos\pi/4$ | $2y = 20\sin\pi/4$

$2x = 20\frac{1}{\sqrt{2}} = 10\sqrt{2}$ | $2y = 10\sqrt{2}$

dimension is $10\sqrt{2}, 10\sqrt{2}$

8) Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Let x, y be the length and breadth of the rectangle. Let L be the given perimeter

$$2x+2y=L$$
$$2y=L-2x$$
$$y=\frac{L}{2}-x \quad \text{—} (1)$$

rectangle. Area $= xy$

$$A(x)=x\left(\frac{L}{2}-x\right)$$
$$A(x)=\frac{Lx}{2}-x^2$$

$$A'(x)=\frac{L}{2}-2x \qquad A'(x)=0 \quad \frac{L}{2}-2x=0$$
$$\frac{L}{2}=2x$$
$$\boxed{x=\frac{L}{4}}$$

$$A''(x)=-2<0$$

when $x=\frac{L}{4}$ $A(x)$ is maximum (local)

when $x=L/4$ Area of rectangle is maximum

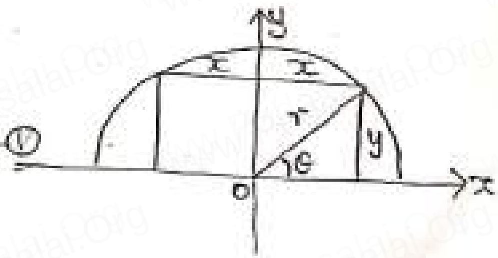
$$(1) \Rightarrow y=\frac{L}{2}-\frac{L}{4} \Rightarrow y=\frac{2L-L}{4}=\frac{L}{4}$$

$x=y=\frac{L}{4}$ It is a square.

Square has maximum Area.

9) Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r cm.

Let $2x, y$ be the length and breadth of the rectangle.



$$2x=2r\cos\theta, \quad y=r\sin\theta \quad \text{—} (1)$$

$$\text{Area} = 2x(y)$$
$$=2r\cos\theta \, r\sin\theta$$
$$A(\theta)=r^2\sin 2\theta$$
$$A'(\theta)=r^2\cos 2\theta(2) \qquad A'(\theta)=0 \Rightarrow$$
$$2r^2\cos 2\theta=0$$
$$\cos 2\theta=0$$
$$2\theta=\pi/2$$
$$\boxed{\theta=\pi/4}$$

$$A''(\theta)=-4r^2\sin 2\theta$$

when $\theta=\pi/4$ $\quad A''(\theta)=-4r^2\sin\frac{2\pi}{4\,2}$

$$A''(\theta)=-4r^2(1)<0$$

when $\theta=\pi/4$ $A(\theta)$ is (Local) maximum

$\theta=\pi/4$ Area of rectangle is maximum.

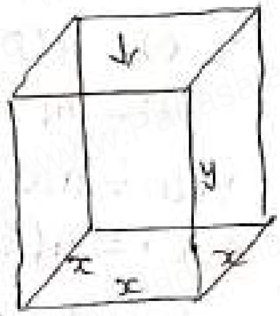
$(1) \Rightarrow 2x=2r\cos\pi/4=\sqrt{2}\sqrt{2}r\frac{1}{\sqrt{2}}=\sqrt{2}r \qquad y=r\sin\pi/4=r\frac{1}{\sqrt{2}}$

length $=\sqrt{2}r$ breadth $=\frac{r}{\sqrt{2}}$

10) A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.

Let x, x, y be the dimensions of the box.

$l=x$, $b=x$, $h=y$



base area $=x^2$

sides area $=4xy$

Surface Area $=108$

$x^2+4xy=108$

$4xy=108-x^2$

$y=\frac{108-x^2}{4x}$ —①

volume of box (cuboid) $=lbh$

$V=x^2y$

(31.07.19)

G. Karthikeyan.

THIRUVARUR DT

$V(x)=x^2\frac{(108-x^2)}{4x}$

$V(x)=\frac{1}{4}(108x-x^3)$

$V'(x)=\frac{1}{4}(108-3x^2)$ $\quad V'(x)=0 \Rightarrow$

$108-3x^2=0$

$3x^2=108$

$x^2=\frac{108}{3}=36$

$x^2=36$

$\boxed{x=6}$

$V''(x)=\frac{1}{4}(-6x)=-\frac{3}{2}x$

when $x=6$ $\quad V''(x)=-\frac{3}{2}(6)=-9<0$

when $x=6$ $V(x)$ is maximum (local)

when $x=6$ volume of box is maximum

①$\Rightarrow$ $y=\frac{108-6^2}{4(6)}=\frac{108-36}{24}=\frac{72}{24}=3$

dimension of box x, x, y $=6$cm, 6cm, 3cm

⑪ The volume of a cylinder is given by the formula $V=\pi r^2h$. Find the greatest and least values of V if $r+h=6$

$r+h=6$

$h=6-r$ —①

volume $V=\pi r^2h$

$V(r)=\pi(r^2)(6-r)$

$=\pi(6r^2-r^3)$

$V(r)=\pi(6r^2-r^3)$

$V'(r)=\pi(12r-3r^2)$

$V'(r)=0$

$12r-3r^2=0$

$-3r(r-4)=0$

$r=0,\ r-4=0$

$r=0,\ r=4$

$h, r,$ are +ve and maximum value 6

$0\le h, r\le 6$ and $h+r=6$

$V(0)=\pi(0)=0$

$V(4)=\pi(6\times16-64)=\pi(96-64)=32\pi$

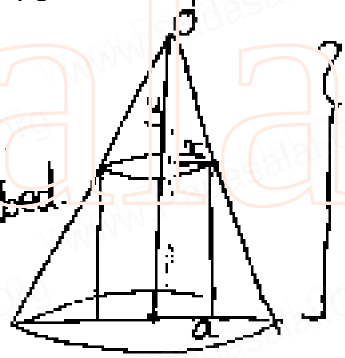
$V(0)=\pi(0)=0$

$V(6)=\pi(6(6^2)-6^3)=\pi(0)=0$

absolute maximum $=32\pi$

absolute minimum $=0$

12) A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

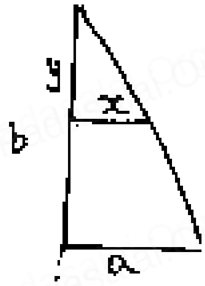


Let x be the radius of the cylinder and y be the distance from the top of cone to the inscribed cylinder.

given height of cone $=b$

height of cylinder $=b-y$

volume of cylinder $V=\pi x^2(b-y)$



$\frac{y}{x}=\frac{b}{a}$

$y=\frac{bx}{a}$

$V(x)=\pi x^2\left(b-\frac{bx}{a}\right)$ —— ①

$V(x)=\pi b\left(x^2-\frac{x^3}{a}\right)$

$V'(x)=\pi b\left(2x-\frac{3x^2}{a}\right)$

$V'(x)=0 \Rightarrow$

$2x-\frac{3x^2}{a}=0$

$x\left(2-\frac{3x}{a}\right)=0$

$x=0$ or $2-\frac{3x}{a}=0$

$\frac{\cancel{+}3x}{a}=\cancel{+}2$

$\boxed{x=\frac{2a}{3}}$

$v''(x)=\pi b\left(2-\frac{6x}{a}\right)$

when $x=\frac{2a}{3} \Rightarrow v''(x)=\pi b\left(2-\frac{6}{a}\left(\frac{2a}{3}\right)\right)=\pi b(2-4)$

$=-2\pi b<0$

when $x=\frac{2a}{3}$, $v''(x)<0$

$\therefore x=\frac{2a}{3}$ $v(x)$ is maximum

$x=\frac{2a}{3}$ volume of cylinder is maximum.

(1) $\Rightarrow$ volume of cylinder $v(x)=\pi x^2\left(b-\frac{bx}{a}\right)$

$=\pi\left(\frac{2a}{3}\right)^2\left(b-\frac{b}{a}\frac{2a}{3}\right)$

$=\pi\frac{4a^2}{9}\left(b\left(1-\frac{2}{3}\right)\right)$

$=\frac{4\pi}{9}a^2b\left(\frac{1}{3}\right)$

$=\frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$

max volume of cylinder $=\frac{4}{9}$ volume of cone.

—x—

Symmetry:

(1) symmetric w.r to y axis (f(x) have all x has x^2 terms)

$f(-x,y)=f(x,y)$

(2) symmetric w.r to. x axis

$f(x,-y)=f(x,y)$ (all y has y^2 terms)

(3) symmetric w.r to origin, if

$f(-x,-y)=f(x,y)$ (all x, y has x^2, y^2 terms)

Asymptote:

(i) Horizontal asymptote

$\lim_{x\to\pm\infty} f(x)=L$

(ii) vertical asymptote: $\lim_{x\to a} f(x)=\pm\infty$

③ <u>Slant asymptote</u>.

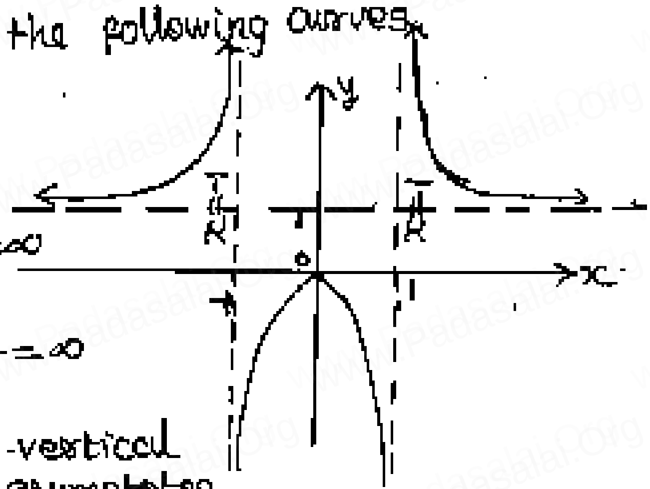
numerator degree > denominator degree

divide the numerator by the denominator we get the slant asymptote.

<u>Exercise 7.9</u>

① Find the asymptotes of the following curves.

(i) $f(x)=\frac{x^2}{x^2-1}$



① $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\frac{x^2}{x^2-1}=\frac{1}{0}=\infty$

$\lim\limits_{x\to -1} f(x)=\lim\limits_{x\to -1}\frac{x^2}{x^2-1}=\frac{1}{0}=\infty$

$\therefore x=1, x=-1$ are vertical asymptotes.

② $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{x^2}{x^2-1}=\lim\limits_{x\to\infty}\frac{x^2}{\left(1-\frac{1}{x^2}\right)x^2}=\frac{1}{1-0}=1$

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\frac{x^2}{x^2-1}=\frac{1}{1-0}=1$

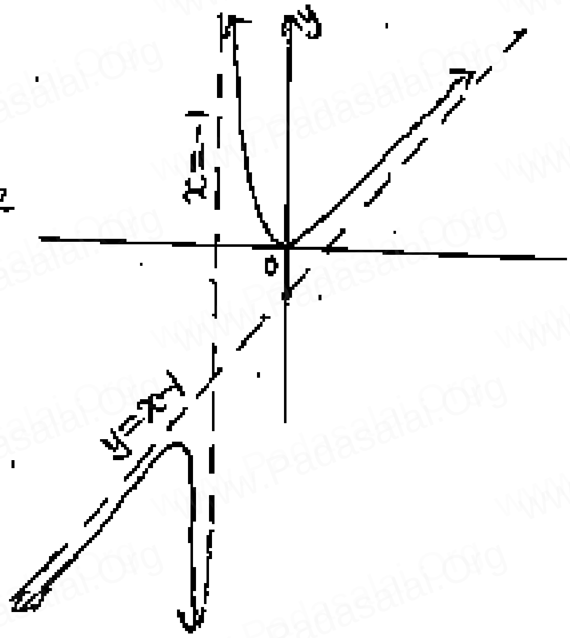
$\therefore y=1$ is a horizontal asymptotes.

(ii) $f(x)=\frac{x^2}{x+1}$

$\lim\limits_{x\to -1} f(x)=\lim\limits_{x\to -1}\frac{x^2}{x+1}=\frac{1}{0}=\infty$

$x=-1$ is a vertical asymptote

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & x^2+x \\ \hline & -x \\ & -x-1 \\ \hline & 1 \end{array}$$



Slant asymptote is $y=x-1$

(iii) $f(x)=\frac{3x}{\sqrt{x^2+2}}$

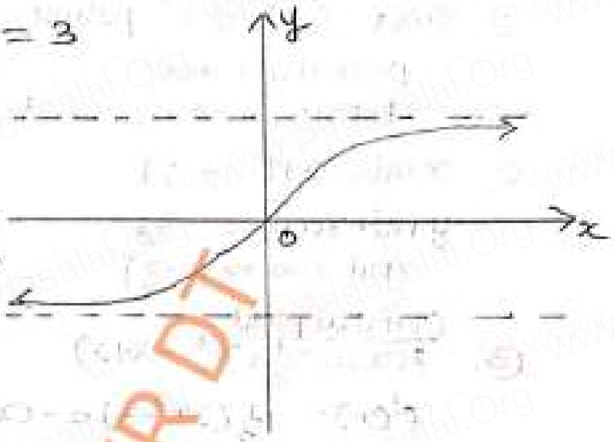
$x^2+2\neq 0$ $\therefore$ no vertical asymptotes

$x\to\infty \Rightarrow \sqrt{x^2}=x$

$x\to-\infty \Rightarrow \sqrt{x^2}=-x$

$\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\frac{3x}{x\sqrt{1+\frac{2}{x^2}}}=\frac{3}{\sqrt{1+0}}=3$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3x}{x\sqrt{1+\frac{2}{x^2}}} = 3$



$x=3$, $x=-3$ are horizondal asymptotes.

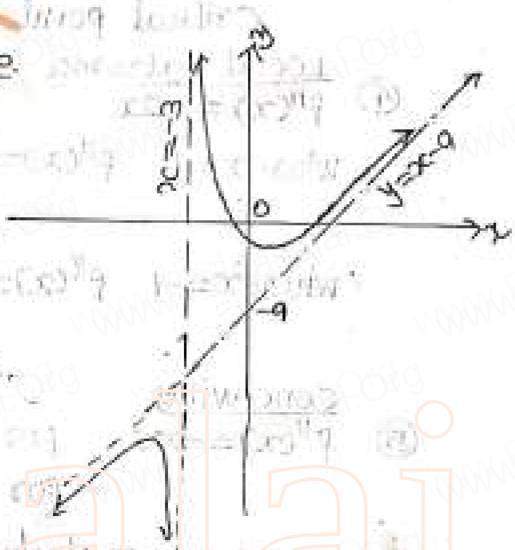
(iv) $f(x) = \frac{x^2-6x-1}{x+3}$

$\lim_{x\to-3} f(x) = \lim_{x\to-3} \frac{9+18-1}{0} = \infty$

$x=-3$ is a only vertical asymptote

$$\begin{array}{r} x-9 \\ x+3 \overline{\big) x^2-6x-1} \\ \underline{x^2+3x} \\ -9x-1 \\ \underline{-9x-27} \\ 26 \end{array}$$

$y=x-9$ is a slant asymptote.



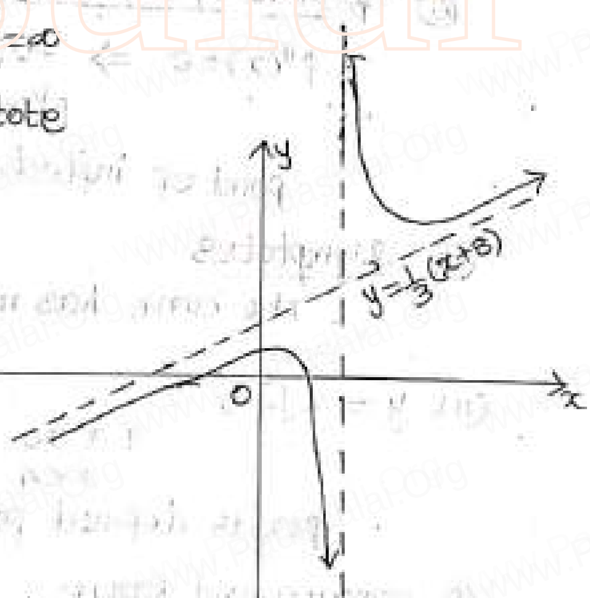
(v) $f(x) = \frac{x^2+6x-4}{3x-6}$

$\lim_{x\to2} f(x) = \lim_{x\to2} \frac{x^2+6x-4}{3x-6} = \frac{4+12-4}{0} = \infty$

$x=2$ is a only vertical asymptote

$$\begin{array}{r} \frac{1}{3}x+\frac{8}{3} \\ 3x-6 \overline{\big) x^2+6x-4} \\ \underline{x^2-2x} \\ 8x-4 \\ \underline{8x-16} \\ 12 \end{array}$$

$y=\frac{1}{3}x+\frac{8}{3}$ (or) $y=\frac{1}{3}(x+8)$ is a slant asymptote.



(2) Sketch the graphs of the following functions.

(i) $y = -\frac{1}{3}(x^3-3x+2)$

$x^3-3x+2 = (x+2)(x-1)(x-1)$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline 1 & 1 & 1 & -2 & 0 \\ & 0 & 1 & 2 & \\ \hline & 1 & 2 & 0 & \end{array}$$

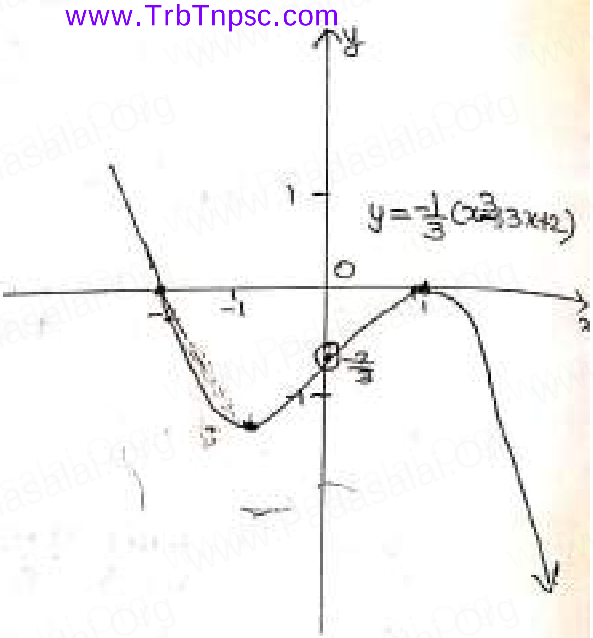
when $y=0$ $x=-2, 1, 1$.

Domain and Range

① Since $f(x)$ is a polynomial

Domain $(-\infty,\infty)$

Range $(-\infty,\infty)$



② x intercept $=-2,1$

y intercept $=-2/3$

(put $x=0 \Rightarrow y=-\frac{2}{3}$)

③ Critical point

$f(x)=-\frac{1}{3}(x^3-3x+2)$

$f'(x)=-\frac{1}{3}(3x^2-3)=-(x^2-1)$

$f'(x)=0 \Rightarrow x^2-1=0$

$x=\pm 1$

Critical points are $1,-1$

④ Local extrema

$f''(x)=-2x$

when $x=1$ $f''(x)=-2<0$ $f(x)$ has local maximum

maximum value is $f(1)=0$

when $x=-1$ $f''(x)=2>0$ $f(x)$ has local minimum

minimum value is $f(-1)=-\frac{1}{3}(-1+3+2)$

$=-\frac{4}{3}$

⑤ Concavity

$f''(x)=-2x$ for $x>0$ concave downward

for $x<0$ concave upward

⑥ point of Inflection

$f''(x)=0 \Rightarrow -2x=0$, $x=0$

$f''(x)$ changes its sign through $x=0$

point of inflection is $=(0,f(0))=(0,-\frac{2}{3})$

⑦ asymptotes

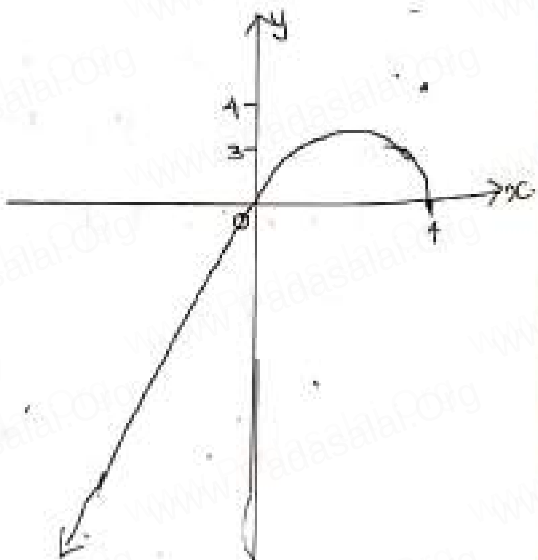
The curve has no asymptotes

(ii) $y=x\sqrt{4-x}$

$4-x\geq 0$

$x\leq 4$

$f(x)$ is defined for $x\leq 4$



① Domain and Range

Domain $=(-\infty,4]$

Range

② intercepts

x intercepts $=0,4$

y intercepts $=0$

3) critical numbers

$$f(x) = x(4-x)^{1/2}$$

$$f'(x) = x\frac{1}{2}(4-x)^{1/2-1}(-1) + (4-x)^{1/2}(1)$$

$$= (4-x)^{-1/2}\left[-\frac{x}{2} + (4-x)\right]$$

$$= (4-x)^{-1/2}\left[\frac{-x+8-2x}{2}\right]$$

$$f'(x) = \frac{8-3x}{2(4-x)^{1/2}}$$

$f'(x)=0 \Rightarrow \quad 8-3x=0$

$x = 8/3$

when $x=4$, $f'(x)$ does not exist

critical numbers. $x = 8/3, 4$

4) Local Extrema

$$f'(x) = \frac{8-3x}{2(4-x)^{1/2}}$$

$$f''(x) = \frac{1}{2}\,\frac{(4-x)^{1/2}(-3) - (8-3x)\frac{1}{2}(4-x)^{-1/2}(-1)}{(4-x)}$$

$$= \frac{(4-x)^{-1/2}}{2(4-x)}\left[(4-x)(-3) + \frac{1}{2}(8-3x)\right]$$

$$= \frac{1}{2(4-x)^{3/2}}\left[-6(4-x) + 8-3x\right]$$

$$= \frac{1}{2(4-x)^{3/2}}\left[-24+6x+8-3x\right] = \frac{1}{4(4-x)^{3/2}}(3x-16)$$

i) when $x = 8/3$

$$f''(x) = \frac{1}{4(4-8/3)^{3/2}}\left(3\tfrac{8}{3}-16\right) = \frac{-8}{4(4/3)^{3/2}} < 0$$

when $x = 8/3$ $f''(x) < 0$ $f(x)$ has local maximum

max value $= f(8/3) = \frac{8}{3}\sqrt{4-8/3}$

$= \frac{8}{3}\sqrt{\frac{4}{3}} = \frac{16}{3\sqrt{3}}$

ii) concavity

$$f''(x) = 0 \Rightarrow \frac{3x-16}{4(4-x)^{3/2}} = 0$$

$3x-16=0 \qquad x = \frac{16}{3}$ But

domain of $f(x)$ is $(x \le 4)$ $(-\infty, 4]$

we test only on the domain.

In $(-\infty,4]$ $\quad f''(x) < 0$ $\therefore f(x)$ is concave downward only.

$f(x)$ is doesnot changes the sign at $x = 16/3$

$\therefore$ No point of inflection. (No curve exist at $16/3$)

6). asymptotes

No asymptotes.

(iii) $y = \frac{x^2+1}{x^2-4}$ $\quad f(x) = \frac{x^2+1}{x^2-4}$

$f(x)$ is not defined for $x = \pm 2$

① Domain ~~and range~~ (Range $(-\infty, -\frac{1}{4}] \cup (1, \infty)$)

$x = -2, 2$ $f(x)$ not defined

Domain is $R - \{-2, 2\}$

② symmetry

$f(-x, y) = f(x, y)$

$\therefore f(x)$ is symmetric about y axis

③ Intercepts

put $x = 0$ $\quad y = -\frac{1}{4}$

$y = 0 \Rightarrow x^2+1 = 0$

y intercept $= -1/4$

no x intercept

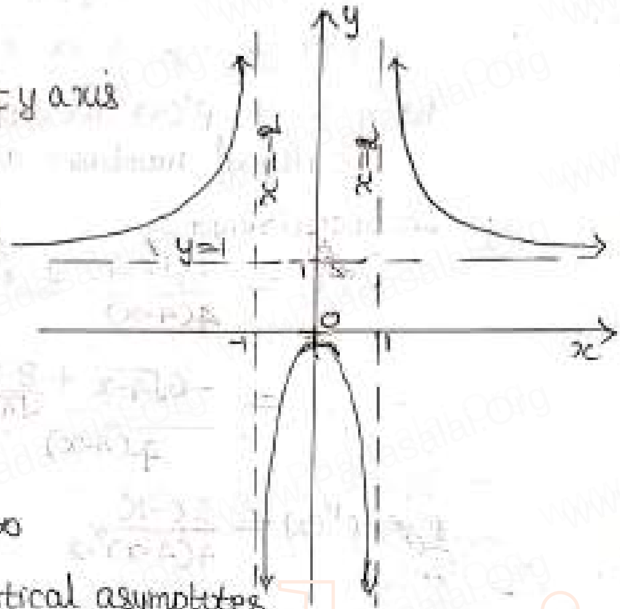
④ asymptotes

$\lim_{x\to -2} f(x) = \infty$, $\lim_{x \to 2} f(x) = \infty$

$x = -2$, $x = 2$ are the vertical asymptotes

$\lim_{x\to\infty} f(x) = 1$

$\therefore y = 1$ is a horizontal asymptotes.



⑤ (monotonicity) critical numbers

$f(x) = \frac{x^2+1}{x^2-4}$

$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)2x}{(x^2-4)^2} = \frac{2x(x^2-4-x^2-1)}{(x^2-4)^2} = \frac{2x(-5)}{(x^2-4)^2}$

$f'(x) = \frac{-10x}{(x^2-4)^2}$

$f'(x) = 0 \Rightarrow -10x = 0 \quad x = 0$

$f'(x)$ does not exist at $x = -2, 2$

critical numbers are $x = 0, -2, 2$.

⑥ $f''(x) = \frac{(x^2-4)^2(-10) + 10x(2(x^2-4)2x)}{(x^2-4)^4}$

$= \frac{(x^2-4)(-10x^2+40+40x^2)}{(x^2-4)^{4\,3}} = \frac{30x^2+40}{(x^2-4)^3}$

(i) $x = 0$ $\quad f''(x) = \frac{40}{(-4)^3} < 0$

$f(x)$ has Local maximum at $x = 0$

Local maximum value $f(0) = -1/4$.

$x = -2, 2$ is not in the domain and $f'(x) \neq 0 \;\forall x \in R - \{-2, 2\}$

⑦ concavity

$f''(x) \neq 0$ but $f''(x) = \frac{30x^2+40}{(x^2-4)^3}$

does not exist at $x = -2, 2$

Interval	value	$f''(x) = \frac{30x^2+40}{(x^2-4)^3}$	sign of $f''(x)$	Concavity
$(-\infty,-2)$	-3	$\frac{30\times9+40}{5^3} = \frac{310}{125}$	+	upward
$(-2,2)$	0	$\frac{40}{-64}$	−	downward
$(2,\infty)$	3	$\frac{310}{125}$	+	upward

$f(x)$ is concave upward on $(-\infty,-2),(2,\infty)$
downward on $(-2,2)$

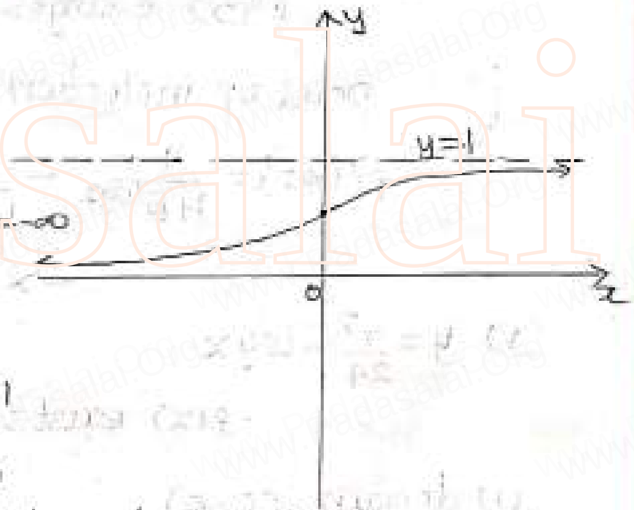
⑧ No point of Inflection.

(iv) $y = \frac{1}{1+e^{-x}}$

$y=f(x)$ is defined for all $x \in R$

① Domain $(-\infty,\infty)$
Range $(0,1)$.

② Intercept
no x intercept (Put $y=0$ only at $x \to -\infty$)
y intercept $= \frac{1}{2}$



③ asymptotes

$\lim_{x\to\infty} f(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

$\lim_{x\to-\infty} f(x) = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$

horizondal asymptotes at $y=1, y=0$

④ critical numbers,

$f'(x) = \frac{-1}{(1+e^{-x})^2}(-e^{-x}) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$ no critical number

⑤ no critical numbers ∴ no sign changes of $f'(x)$
no Local extremum

⑥ concavity

$f''(x) = \frac{(1+e^{-x})^2(-e^{-x}) - e^{-x}(2(1+e^{-x})(-e^{-x}))}{(1+e^{-x})^4}$

$= \frac{(1+e^{-x})(-e^{-x}-e^{-2x}+2e^{-2x})}{(1+e^{-x})^{\cancel{4}\,3}} = \frac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$

$f''(x)=0 \Rightarrow e^{-2x}-e^{-x}=0$

$e^{-x}(e^{-x}-1)=0$

$e^{-x}=1$

$-x=\log 1$ $\boxed{x=0}$

Intervals $(-\infty,0),(0,\infty)$

when $x\in(-\infty,0)$ $f''(x)>0$ concave upward

when $x\in(0,\infty)$, $f''(x)<0$ concave downward.

(7) Point of Inflection

$f''(x)$ changes its sign through $x=0$

point of inflection at $x=0$

i.e $(0,f(0))=(0,\frac{1}{1+1})=(0,\frac{1}{2})$

$\therefore$ point of inflection is $(0,\frac{1}{2})$

—x—

(v) $y=\frac{x^3}{24}-\log x$ (This sum deleted in Tamil medium book)

$f(x)$ exists only for $x>0$

(1) domain $(0,\infty)$.

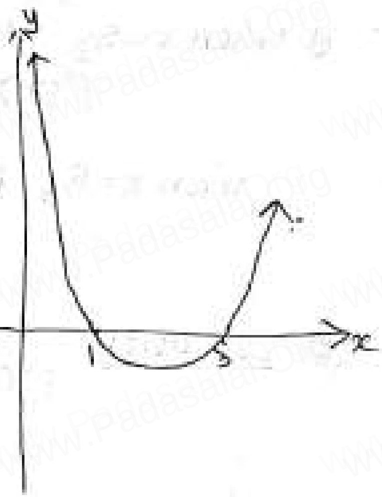
2) Intercept

when $x=0$ $\log x \to -\infty$ no y intercept.

$y=0 \Rightarrow \frac{x^3}{24}=\log x$

$x=1$

$x=3$



3) asymptotes

$\lim_{x\to 0} f(x)=\infty$

$x=0$ is a vertical asymptotes.

4) critical numbers

$f'(x)=\frac{3x^2}{24}-\frac{1}{x}=\frac{x^2-8}{8x}$

$f'(x)=0 \Rightarrow \frac{x^2-8}{8x}=0$

$x^2-8=0$

$x=\pm 2\sqrt{2}$

$x=2\sqrt{2},-2\sqrt{2}$ $x>0$ $\therefore \boxed{x=2\sqrt{2}}$

when $x=0$, $f'(x)$ does no exist (also $f(x)$ not exist)

$\therefore x=0$ is not a critical number

$x=2\sqrt{2}$ is only critical number.

5) local extrema

$$f'(x) = \frac{x^2-8}{8x}$$

$$f''(x) = \frac{8x(2x)-(x^2-8)(8)}{(8x)^2}$$

$$= \frac{16x^2-8x^2+64}{64x^2} = \frac{8x^2+8\times8}{64x^2}$$

$$f''(x) = \frac{8(x^2+8)}{8\,64x^2} = \frac{x^2+8}{8x^2} > 0$$

when

$x = 2\sqrt{2}$ $\quad f''(x) = \frac{8+8}{8(8)} = \frac{16}{64} > 0$

f(x) has a local minimum at $x = 2\sqrt{2}$

minimum value $= f(2\sqrt{2})$

$$= \frac{(2\sqrt{2})^3}{24} - \log 2\sqrt{2}$$

$$= \frac{2\cdot 8\sqrt{2}}{3\cdot 24} - \log 2\sqrt{2}$$

$$= \frac{2\sqrt{2}}{3} - \log 2\sqrt{2}$$

6) $f''(x) > 0$ $\therefore$ f(x) is always concave upward on $(0, \infty)$

7) no change of sign of $f''(x)$.

$\therefore$ No point of inflection.

—— x ——

G. karthikeyan
9715634957 Need suggestions (whatsapp only).