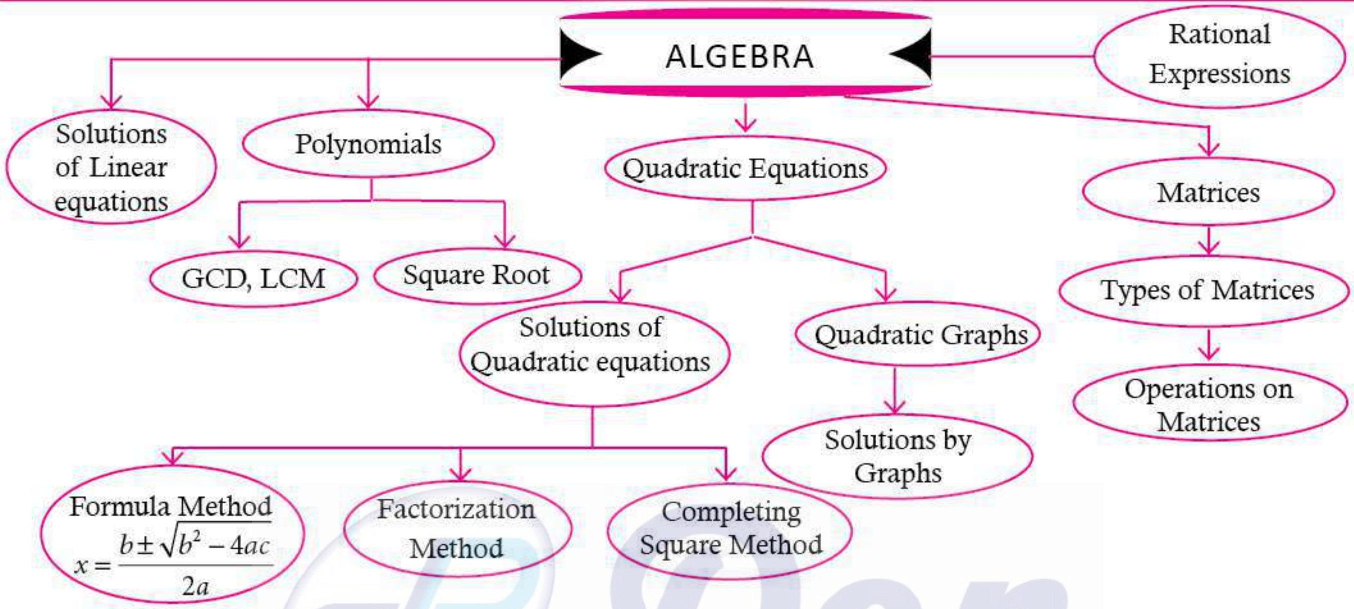
UNIT 3 ALGEBRA

MIND MAP



SIMULTANEOUS LINEAR EQUATIONS IN THREE VARIABLES

Key Points

- Any first degree equation containing two variables x and y is called a linear equation in two variables. The general form of linear equation in two variables x and y is $ax + by + c = 0$ where a, b, c are real numbers.
- A linear equation in two variables represents a straight line in xy-plane.
- A linear equation with three variables x, y and z will be of the form $ax + by + cz + d = 0$ where a, b, c, d are real numbers and atleast one a, b, c is non-zero.
- A linear equation in three variables represents a plane.
- A system of equations can have unique solution (or) infinitely many solutions (or) No solutions.
- The system of equations has no solution if any step comes as $0 = 1$ while solving.
- The system of equations has infinitely many solutions if any step comes as $0 = 0$ while solving.

Worked Examples

3.1 **The father's age is six times his son's age. Six years hence the age of father will be four times his son's age. Find the present ages (in years) of the son and father respectively.**

Sol : Let present age of father be x years and present age of son be y years

Given, $x = 6y$ ... (1)

$x + 6 = 4(y + 6)$ ... (2)

Substituting (1) in (2), $6y + 6 = 4(y + 6)$

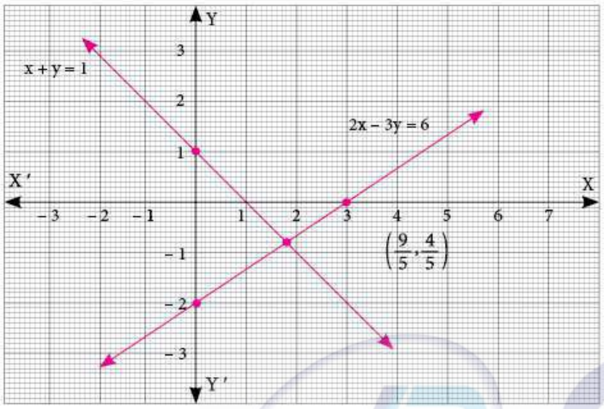
$6y + 6 = 4y + 24 \Rightarrow y = 9$

Therefore, son's age = 9 years and father's age = 54 years.

3.2 Solve $2x - 3y = 6, x + y = 1$

Sol :

$$2x - 3y = 6 \quad \text{... (1)}$$
$$x + y = 1 \quad \text{... (2)}$$
$$(1) \times 1 \Rightarrow 2x - 3y = 6$$
$$(2) \times 2 \Rightarrow 2x + 2y = 2$$
$$-5y = 4 \Rightarrow y = \frac{-4}{5}$$



Substituting $y = \frac{-4}{5}$ in (2),

$$x - \frac{4}{5} = 1 \Rightarrow x = \frac{9}{5}$$

Therefore, $x = \frac{9}{5}, y = \frac{-4}{5}$

3.3 Solve the following system of linear equations in three variables $3x - 2y + z = 2$, $2x + 3y - z = 5$, $x + y + z = 6$

Sol :

$$3x - 2y + z = 2 \quad \text{... (1)}$$
$$2x + 3y - z = 5 \quad \text{... (2)}$$
$$x + y + z = 6 \quad \text{... (3)}$$

Adding (1) and (2), $3x - 2y + z = 2$
$2x + 3y - z = 5$ (+)
$5x + y = 7$ ... (4)

Adding (2) and (3), $2x + 3y - z = 5$
$x + y + z = 6$ (+)
$3x + 4y = 11$ ... (5)

$4 \times (4) - (5)$ $\quad 20x + 4y = 28$
$3x + 4y = 11$ (–)
$17x = 17$
$\Rightarrow x = 1$

Substituting $x = 1$ in (4), $5 + y = 7 \Rightarrow y = 2$

Substituting $x = 1, y = 2$ in (3),
$$1 + 2 + z = 6 \Rightarrow z = 3$$
Therefore, $x = 1, y = 2, z = 3$.

3.4. In an interschool Atheletic meet, with 24 individual-events, securing a total of 56 points, a first place provides 5 points, a second place provides 3 points and a third place secures 1 point. Having as many third place finishers as first and second place finishers, find how many Athelets finished in each place.

Sol :

Let, the number of I, II and III place finishers be x, y & z respectively.

Total number of events = 24 ; Total number of points = 56.

Hence, the linear equations in three variables are

$$x + y + z = 24 \quad \text{... (1)}$$
$$5x + 3y + z = 56 \quad \text{....(2)}$$
$$x + y = z \quad \text{...(3)}$$

Substituting (3) in (1) we get,
$$z + z = 24 \Rightarrow z = 12$$

Therefore, (3) $\Rightarrow$ $\quad x + y = 12$
(2) $\Rightarrow$ $\quad 5x + 3y = 44$
$3 \times (3) \Rightarrow$ $\quad 3x + 3y = 36$ (–)
$2x = 8 \Rightarrow x = 4$

Substituting $x = 4, z = 12$ in (3) we get,
$$y = 12 - 4 = 8$$

Therefore, number of first place finishers is $x = 4$;
number of second place finishers is $y = 8$;
number of third place finishers is $z = 12$

3.5 Solve $x + 2y - z = 5$; $x - y + z = -2$; $-5x - 4y + z = -11$

Sol :

Let, $\quad x + 2y - z = 5$ ... (1)
$$x - y + z = -2 \quad \text{... (2)}$$
$$-5x - 4y + z = -11 \quad \text{...(3)}$$

Adding (1) and (2) $\Rightarrow$
$x + 2y - z = 5$
$x - y + z = -2$ (+)
$2x + y = 3$ ... (4)

Subtracting (2) and (3),
$x - y + z = 2$
$-5x - 4y + z = -11$ (–)
$6x + 3y = 9$

Dividing by 3 $\Rightarrow$ $2x + y = 3$ ... (5)

Subtracting (4) and (5),

$$\begin{array}{r} 2x + y = 3 \\ 2x + y = 3 \\ \hline 0 = 0 \\ \hline \end{array}$$

Here we arrive at an identity $0 = 0$.
Hence the system has an infinite number of solutions.

3.6. Solve $3x + y - 3z = 1$; $-2x - y + 2z = 1$; $-x - y + z = 2$.

Sol :

Let $3x + y - 3z = 1$ ...(1)

$-2x - y + 2z = 1$ ... (2)

$-x - y + z = 2$ ... (3)

Adding (1) and (2),

$$\begin{array}{r} 3x + y - 3z = 1 \\ -2x - y + 2z = 1 \quad (+) \\ \hline x \qquad - z = 2 \\ \hline \end{array} \quad \text{... (4)}$$

Adding (1) and (3),

$$\begin{array}{r} 3x + y - 3z = 1 \\ -x - y + z = 2 \ (+) \\ \hline 2x \qquad - 2z = 3 \\ \hline \end{array} \quad \text{... (5)}$$

$$\begin{array}{r} 2x - 2z = 3 \\ \text{Now, } (5) - 2 \times (4) \Rightarrow 2x - 2z = 4 \quad (-) \\ \hline 0 = -1 \\ \hline \end{array}$$

Here we arrive at a contradiction as $0 = -1$
This means that the system is inconsistent and has no solution.

3.7. Solve $\frac{x}{2} - 1 = \frac{y}{6} + 1 = \frac{z}{7} + 2$; $\frac{y}{3} + \frac{z}{2} = 13$

Sol :

Considering, $\frac{x}{2} - 1 = \frac{y}{6} + 1$

$$\frac{x}{2} - \frac{y}{6} = 1 + 1$$

$$\Rightarrow \frac{6x - 2y}{12} = 2$$

$$\Rightarrow 3x - y = 12 \quad \text{... (1)}$$

Considering, $\frac{x}{2} - 1 = \frac{z}{7} + 2$

$$\frac{x}{2} - \frac{z}{7} = 1 + 2$$

$$\Rightarrow \frac{7x - 2z}{14} = 3$$

$$\Rightarrow 7x - 2z = 42 \quad \text{... (2)}$$

Also, from $\frac{y}{3} + \frac{z}{2} = 13$

$$\Rightarrow \frac{2y + 3z}{6} = 13$$

$$\Rightarrow 2y + 3z = 78 \quad \text{... (3)}$$

Eliminating z from (2) and (3)

$$\begin{array}{rr} (2) \times 3 \Rightarrow & 21x - 6z = 126 \\ (3) \times 2 \Rightarrow & 4y + 6z = 156 \\ \hline & 21x + 4y = 282 \\ (1) \times 4 & 12x - 4y = 48 \\ \hline & 33x \qquad = 330 \Rightarrow x = 10 \\ \hline \end{array}$$

Substituting $x = 10$ in (1), $30 - y = 12 \Rightarrow y = 18$

Substituting $x = 10$ in (2), $70 - 2z = 42 \Rightarrow z = 14$

Therefore, $x = 10$, $y = 18$, $z = 14$.

3.8 Solve :

$$\frac{1}{2x} + \frac{1}{4y} - \frac{1}{3z} = \frac{1}{4}; \frac{1}{x} = \frac{1}{3y}; \frac{1}{x} - \frac{1}{5y} + \frac{4}{z} = 2\frac{2}{15}$$

Sol :

Let $\frac{1}{x} = p$, $\frac{1}{y} = q$, $\frac{1}{z} = r$

$$\Rightarrow \quad \frac{p}{2} + \frac{q}{4} - \frac{r}{3} = \frac{1}{4}$$

$$\Rightarrow \quad p = \frac{q}{3}$$

$$\Rightarrow \quad p - \frac{q}{5} + 4r = 2\frac{2}{15} = \frac{32}{15}$$

$$6p + 3q - 4r = 3 \quad \text{... (1)}$$

$$3p = q \quad \text{... (2)}$$

$$15p - 3q + 60r = 32 \quad \text{... (3)}$$

Substituting (2) in (1) and (3)

we get, $15p - 4r = 3$ ... (4)

(9) $\Rightarrow$ $6p + 60r = 32$

reduces to $\Rightarrow 3p + 30r = 16$ ... (5)

Solving (4) and (5)

$$\begin{array}{r} 15p - \quad 4r = 3 \\ 15p + 150r = 80 \ (-) \\ \hline -154r = -77 \\ \hline \end{array} \Rightarrow r = \frac{1}{2}$$

Substituting $r = \frac{1}{2}$ in (4),

we get $15p - 2 = 3$

$\Rightarrow \quad p = \frac{1}{3}$

From (2), $q = 3p \Rightarrow q = 1$

Therefore, $x = \frac{1}{p} = 3, y = \frac{1}{q} = 1, \ z = \frac{1}{r} = 2.$

That is, $x = 3, y = 1, z = 2$

3.9. The sum of thrice the first number, second number and twice the third number is 5. If thrice the second number is subtracted from the sum of first number and thrice the third we get 2. If the third number is subtracted from the sum of twice the first, thrice the second, we get 1. Find the numbers.

Sol :

Let the three numbers be x, y, z

From the given data we get the following equations,

$$3x + y + 2z = 5 \quad \dots (1)$$
$$x + 3z - 3y = 2 \quad \dots (2)$$
$$2x + 3y - z = 1 \quad \dots (3)$$

$(1) \times 1 \Rightarrow 3x + y + 2z = 5$

$(2) \times 3 \Rightarrow 3x - 9y + 9z = 6 \ (-)$

$$10y - 7z = -1 \quad \dots (4)$$

$(1) \times 2 \Rightarrow 6x + 2y + 4z = 10$

$(3) \times 3 \Rightarrow 6x + 9y - 3z = 3 \ (-)$

$$-7y + 7z = 7 \quad \dots (5)$$

Adding (4) and (5), $10y - 7z = -1$

$-7y + 7z = 7$

$3y = 6$

$\Rightarrow y = 2$

Substituting $y = 2$ in (5), $-14 + 7z = 7 \Rightarrow z = 3$

Substituting $y = 2$ and $z = 3$ in (1),

$3x + 2 + 6 = 5$

$\Rightarrow x = -1$

Therefore, $x = -1, y = 2, z = 3$.

Progress Check

1. **For a system of linear equations with three variables the minimum number of equations required to get unique solution is _____**

   Ans : 3

2. **A system with _____ will reduce to identity.**

   Ans : infinitely many solutions.

3. **A system with ________ will provide absurd equation.**

   Ans : No solution.

Thinking Corner

1. **The number of possible solutions when solving system of linear equations in three variables are _____**

   Ans : 3

2. **If three planes are parallel then the number of possible point(s) of intersection is / are _____**

   Ans : zero.

Exercise 3.1

1. **Solve the following system of linear equations in three variables**

   (i) $x + y + z = 5, \ 2x - y + z = 9, \ x - 2y + 3z = 16$

   (ii) $\frac{1}{x} - \frac{2}{y} + 4 = 0, \ \frac{1}{y} - \frac{1}{z} + 1 = 0, \ \frac{2}{z} + \frac{3}{x} = 14$

   (iii) $x + 20 = \frac{3y}{2} + 10 = 2z + 5 = 110 - (y + z)$

Sol :

(i)

$$x + y + z = 5 \quad \dots (1)$$
$$2x - y + z = 9 \quad \dots (2)$$
$$x - 2y + 3z = 16 \quad \dots (3)$$

Consider (1) and (2)

$x + y + z = 5 \quad \dots (1)$

$2x - y + z = 9 \ (+) \quad \dots (2)$

$(1) + (2) \Rightarrow 3x + 2z = 14 \quad \dots (4)$

Consider (1) and (3)

$x - 2y + 3z = 16 \quad \dots (3)$

$(1) \times (2) \Rightarrow 2x + 2y + 2z = 10 \ (+) \quad \dots (5)$

$(3) + (5) \Rightarrow 3x + 5z = 26 \quad \dots (6)$

Consider (4) and (6)

$$3x + 5z = 26 \quad \dots (6)$$
$$3x + 2z = 14 \quad \dots (4)$$
$$(6) - (4) \Rightarrow \quad 3z = 12$$
$$z = \frac{12}{3} = 4$$

Substituting z = 4 in (4)

$$3x + 2(4) = 14$$
$$3x + 8 = 14$$
$$3x = 14 - 8 = 6$$
$$3x = 6$$
$$x = \frac{6}{3} = 2$$

Substituting x = 2, z = 4 in (1)

$$2 + y + 4 = 5$$
$$y + 6 = 5$$
$$y = 5 - 6$$
$$y = -1$$

∴ Solution : x = 2, y = −1, z = 4

(ii)
$$\frac{1}{x} - \frac{2}{y} + 4 = 0$$
$$\frac{1}{y} - \frac{1}{z} + 1 = 0$$
$$\frac{2}{z} + \frac{3}{x} = 14$$

Let $\frac{1}{x} = a$, $\frac{1}{y} = b$ and $\frac{1}{z} = c$

∴ The equations become

$$a - 2b = -4 \quad \dots (1)$$
$$b - c = -1 \quad \dots (2)$$
$$3a + 2c = 14 \quad \dots (3)$$
$$(1) + (2) + (3) \Rightarrow \quad 4a - b + c = 9 \quad \dots (4)$$

Now, consider (4) and (2)

$$4a - b + c = 9 \quad \dots (4)$$
$$b - c = -1 \quad \dots (2)$$
$$(4) + (2) \Rightarrow \quad 4a = 8$$
$$a = \frac{8}{4} = 2$$

Substituting the value a = 2 in (1)

$$2 - 2b = -4$$
$$-2b = -4 - 2 = -6$$
$$b = \frac{6}{2} = 3$$

Substituting b = 3 in (2)

$$3 - c = -1$$
$$c = 4$$

Now a = 2, b = 3, c = 4

$$\therefore \text{ Solution } x = \frac{1}{a} = \frac{1}{2},$$
$$y = \frac{1}{b} = \frac{1}{3},$$
$$z = \frac{1}{c} = \frac{1}{4}$$

(iii)
$$x + 20 = \frac{3y}{2} + 10$$
$$= 2z + 5$$
$$= 110 - (y + z)$$

Equating
$$x + 20 = \frac{3y}{2} + 10$$
$$2x + 40 = 3y + 20$$

On simplifying
$$2x - 3y = -20 \quad \dots (1)$$

Equating
$$\frac{3y}{2} + 10 = 2z + 5$$
$$3y + 20 = 4z + 10$$

On simplifying
$$3y - 4z = -10 \quad \dots (2)$$

Now equating
$$2z + 5 = 110 - (y + z)$$

Simplifying,
$$3z + y = 105 \quad \dots (3)$$

Considering (2) and (3)

$$(3) \times 3 \Rightarrow \quad 3y + 9z = 315 \quad \dots (4)$$
$$(2) \times 1 \Rightarrow \quad 3y - 4z = -10 \ (+) \quad \dots (2)$$
$$(4) - (2) \Rightarrow \quad 13z = 325$$
$$z = \frac{325}{13} = 25$$

Substituting in (3)

$$3(25) + y = 105$$
$$y = 105 - 75$$
$$y = 30$$

Substituting y = 30 in (1)

$$2x - 3(30) = -20$$
$$2x = -20 + 90$$
$$2x = 70$$
$$x = \frac{70}{2} = 35$$

∴ Solution : x = 35, y = 30, z = 25.

2. Discuss the nature of solutions of the following system of equations

(i) $x + 2y - z = 6$, $-3x - 2y + 5z = -12$, $x - 2z = 3$

(ii) $2y + z = 3(-x + 1)$, $-x + 3y - z = -4$, $3x + 2y + z = -\frac{1}{2}$

(iii) $\frac{y+z}{4} = \frac{z+x}{3} = \frac{x+y}{2}$, $x + y + z = 27$

Sol :

(i)
$$x + 2y - z = 6 \quad \dots (1)$$

$$-3x - 2y + 5z = -12 \quad ...(2)$$
$$x - 2z = 3 \quad ...(3)$$

Consider (1) and (3)

$$x + 2y - z = 6 \quad ...(1)$$
$$x \quad - 2z = 3\ (-) \quad ...(3)$$
$$(1) - (3) \Rightarrow \quad 2y + z = 3 \quad ...(4)$$

Consider (4) and (2)

$$2y + z = 3 \quad ...(4)$$
$$-3x - 2y + 5z = -12\ (+) \quad ...(2)$$
$$(4) + (2) \Rightarrow \quad -3x + 6z = -9 \quad ...(5)$$

Dividing equation (5) by (–3)

We get, $x - 2z = 3 \quad ...(6)$

Now consider (6) and (3)

$$x - 2z = 3 \quad ...(6)$$
$$x - 2z = 3 \quad ...(3)$$
$$(6) - (3) \Rightarrow \quad 0 = 0$$ which is an Identity.

$\therefore$ The system of equations has infinitely many solutions.

(ii) $2y + z = 3\,(-x + 1)$

$$3x + 2y + z = 3 \quad ...(1)$$
$$-x + 3y - z = -4 \quad ...(2)$$
$$3x + 2y + z = -\frac{1}{2} \quad ...(3)$$

Consider (1) and (3)

$$3x + 2y + z = 3 \quad ...(1)$$
$$3x + 2y + z = -\frac{1}{2} \quad ...(2)$$
$$(1) - (2) \Rightarrow \quad 0 = \frac{7}{2} \text{ (or)}$$
$$0 = 7$$

which is a contradiction.

$\therefore$ The system of equations has no solution.

(iii) $\dfrac{y+z}{4} = \dfrac{z+x}{3} = \dfrac{x+y}{2}$,

$$x + y + z = 27 \quad ...(3)$$

Equating $\dfrac{y+z}{4} = \dfrac{z+x}{3}$

We get, $4x - 3y + z = 0 \quad ...(1)$

Equating $\dfrac{y+z}{4} = \dfrac{x+y}{2}$

On simplifying, $2x + y - z = 0 \quad ...(2)$

Consider (1) and (2)

$$4x - 3y + z = 0 \quad ...(1)$$
$$2x + y - z = 0 \quad ...(2)$$
$$(1) + (2) \Rightarrow \quad 6x - 2y = 0 \quad ...(4)$$

Consider (2) and (3)

$$2x + y - z = 0 \quad ...(2)$$
$$x + y + z = 27 \quad ...(3)$$
$$(2) + (3) \Rightarrow \quad 3x + 2y = 27 \quad ...(5)$$

Consider (4) and (5)

$$6x - 2y = 0 \quad ...(4)$$
$$3x + 2y = 27 \quad ...(5)$$
$$9x = 27$$
$$x = \frac{27}{9} = 3$$

Substituting the value x = 2 in (4)

$$6(3) - 2y = 0$$
$$-2y = -18$$
$$y = \frac{18}{2} = 9$$

Substituting x = 3, y = 9 in (3)

$$3 + 9 + z = 27$$
$$z = 27 - 12$$
$$z = 15$$

Solution: x = 3, y = 9, z = 15

3. Vani, her father and her grand father have an average age of 53. One-half of her grand father's age plus one-third of her father's age plus one fourth of Vani's age is 65. If 4 years ago Vani's grandfather was four times as old as Vani then how old are they all now?

Sol :

Let the present ages of Vani, her father and her grand father be x, y, z respectively.

$$\therefore \text{ Given } \frac{x+y+z}{3} = 53$$
$$\therefore x + y + z = 159 \quad ...(1)$$
$$\frac{z}{2} + \frac{y}{3} + \frac{x}{4} = 65$$
$$6z + 4y + 3x = 780 \quad ...(2)$$
$$(z - 4) = 4(x - 4)$$
$$4x - z = 12 \quad ...(3)$$

Consider (1) and (3)

$$x + y + z = 159 \quad ...(1)$$
$$4x \quad - z = 12 \quad ...(3)$$
$$(1) + (3) \Rightarrow \quad 5x \quad + y = 171 \quad ...(4)$$

Consider (2) and (3)

$$3x + 4y + 6z = 780 \quad ...(2)$$
$$(3) \times (6) \Rightarrow \quad 24x \quad - 6z = 72 \quad ...(5)$$
$$(2) + (5) \Rightarrow \quad 27x + 4y = 852 \quad ...(6)$$

Consider (4) and (6)

$$(4)\times 4 \Rightarrow \quad 20x + 4y = 684 \quad \dots (7)$$
$$(6)\times 1 \Rightarrow \quad 27x + 4y = 852 \quad \dots (6)$$
$$(7)-(6) \Rightarrow \quad -7x = -168$$
$$x = \frac{168}{7} = 24$$

Substituting $x = 24$ in (4)

$$5(24) + y = 171$$
$$y = 171 - 120 = 51$$

Substituting $x = 24$, $y = 51$ in (1)

$$24 + 51 + z = 159$$
$$z = 159 - 75 = 84$$

$\therefore$ Present age of Vani = 24 years

Present age of her father = 51 years

Present age of her grandfather = 84 years.

4. The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the old number. If the hundreds digit plus twice the tens digit is equal to the units digit, then find the original three digit number?

Sol :

Let the 100's digit be 'x', 10's be 'y' and Unit digit be 'z'

$\therefore$ The three digit number is $100x + 10y + z$

Now given, $x + y + z = 11 \quad \dots (1)$

$$100z + 10y + x = 5(100x + 10y + z) + 46$$

Simplifying

$$499x + 40y - 95z = -46 \quad \dots (2)$$
$$x + 2y = z$$
$$\Rightarrow \quad x + 2y - z = 0 \quad \dots (3)$$

Consider (1) and (3)

$$x + y + z = 11 \quad \dots (1)$$
$$x + 2y - z = 0 \quad \dots (3)$$
$$(1)+(3) \Rightarrow \quad 2x + 3y = 11 \quad \dots (4)$$

Consider (1)

$$499x + 40y - 95z = -46 \quad \dots (2)$$

$(1)\times 95 \Rightarrow$

$$95x + 95y + 95z = 1045 \quad \dots (5)$$
$$594x + 135y = 999 \quad \dots (6)$$

Consider (6) and (4)

$$594x + 135y = 999 \quad \dots (6)$$
$$(4)\times 45 \Rightarrow \quad 90x + 135y = 495 \quad \dots (7)$$
$$(6)-(7) \Rightarrow \quad 504x = 504$$
$$x = \frac{504}{504} = 1$$

Substituting the value $x = 1$ in (4)

$$2(1) + 3y = 11$$
$$3y = 11 - 2 = 9$$
$$y = \frac{9}{3} = 3$$

Substituting $x = 1$, $y = 3$ in (1)

$$1 + 3 + z = 11$$
$$z = 11 - 4 = 7$$

$\therefore x = 1, \ y = 3, z = 7$

$\therefore$ The original three digit number is 137

i.e., $100(1) + 10(3) + 1(7) = 100 + 30 + 7 = 137$

5. There are 12 pieces of five, ten and twenty rupee currencies whose total value is ₹ 105. But when first 2 sorts are interchanged in their numbers its value will be increased by ₹ 20. Find the number of currencies in each sort.

Sol :

Let the number of five, ten and twenty rupee currencies be x, y and z respectively.

$\therefore$ Given $x + y + z = 12 \quad \dots (1)$

$$5x + 10y + 20z = 105 \quad \dots (2)$$
$$10x + 5y + 20z = 125 \quad \dots (3)$$

Consider (2) and (3)

$$5x + 10y + 20z = 105 \quad \dots (2)$$
$$10x + 5y + 20z = 125 \quad \dots (3)$$
$$(2)-(3) \Rightarrow -5x + 5y = -20 \quad \dots (4)$$

Consider (1) and (2)

$$(1)\times 20 \Rightarrow \ 20x + 20y + 20z = 240 \quad \dots (5)$$
$$(2)\times 1 \Rightarrow \ 5x + 10y + 20z = 105 \quad \dots (2)$$
$$(5)-(2) \Rightarrow 15x + 10y = 135 \quad \dots (6)$$

Consider (4) and (6)

$$(4)\times 3 \Rightarrow \quad -15x + 15y = -60 \quad \dots (7)$$
$$15x + 10y = 135 \quad \dots (6)$$
$$25y = 75$$
$$y = \frac{75}{25} = 3$$

Substituting $y = 3$ in (4)

$$-5x + 5(3) = -20$$
$$-5x = -20 - 15$$
$$-5x = -35$$
$$x = \frac{35}{5} = 7$$

Substituting $x = 7$, $y = 3$ in (1)

$$7 + 3 + z = 12$$
$$z = 12 - 10 = 2$$

$\therefore$ The number of five rupee currencies 7

The number of ten rupee currencies 3

The number of twenty rupee currencies 2

GCD AND LCM OF POLYNOMIALS

Key Points

- GCD can be found out by using long division method.
- If f(x) and g(x) are both polynomials of same degree then the polynomial carrying the highest co efficient will be the dividend.
- Least Common Multiple: The Least Common Multiple of two or more algebraic expressions is the expression of lowest degree (or power) such that the expressions exactly divide it.
- Relationship between LCM and GCD: For two polynomials f(x) and g(x), f(x) × g(x) = [LCM of f(x), g(x)] × [GCD of f(x), g(x)]

Worked Examples

3.10 Find the GCD of the polynomials $x^3 + x^2 - x + 2$ and $2x^3 - 5x^2 + 5x - 3$.

Sol :

Let $f(x) = 2x^3 - 5x^2 + 5x - 3$ and
$g(x) = x^3 + x^2 - x + 2$

$$\begin{array}{r|l} & 2 \\ \hline x^3 + x^2 - x + 2 & 2x^3 - 5x^2 + 5x - 3 \\ & 2x^3 + 2x^2 - 2x + 4 \quad (-) \\ \hline & -7x^2 + 7x - 7 \end{array}$$

$= -7(x^2 - x + 1)$

$-7(x^2 - x + 1) \neq 0$, note that -7 is not a divisor of g(x)

Now dividing $g(x) = x^3 + x^2 - x + 2$ by the new remainder (leaving the constant factor), we get

$$\begin{array}{r|l} & x \\ \hline x^3 - x + 1 & x^3 + x^2 - x + 2 \\ & x^3 - x^2 + x \quad (-) \\ \hline & 2x^2 - 2x + 2 \\ & 2x^2 - 2x + 2 \quad (-) \\ \hline & 0 \end{array}$$

$= 2(x^2 - x + 1)$

Here, we get zero as remainder.

Therefore,

GCD $(2x^3 - 5x^2 + 5x - 3, x^3 + x^2 - x + 2) = x^2 - x + 1$

3.11. Find the GCD of $6x^3 - 30x^2 + 60x - 48$ and $3x^3 - 12x^2 + 21x - 18$.

Sol :

Let, $f(x) = 6x^3 - 30x^2 + 60x - 48$
$= 6(x^2 - 5x^2 + 10x - 8)$ and
$g(x) = 3x^3 - 12x^2 + 21x - 18$
$= 3(x^3 - 4x^2 + 7x - 6)$

Now, we shall find the GCD of $x^3 - 5x^2 + 10x - 8$ and $x^3 - 4x^2 + 7x - 6$

$$\begin{array}{r|l} & 1 \\ \hline x^3 - 5x^2 + 10x - 8 & x^3 - 4x^2 + 7x - 6 \\ & x^3 - 5x^2 + 10x - 8 \quad (-) \\ \hline & x^2 - 3x + 2 \end{array}$$

$$\begin{array}{r|l} & x - 2 \\ \hline x^2 - 3x + 2 & x^3 - 5x^2 + 10x - 8 \\ & x^3 - 3x^2 + 2x \quad (-) \\ \hline & -2x^2 + 8x - 8 \\ & -2x^2 + 6x - 4 \quad (-) \\ \hline & 2x - 4 = 2(x - 2) \end{array}$$

$$\begin{array}{r|l} & x - 1 \\ \hline x - 2 & x^2 - 3x + 2 \\ & x^2 - 2x \quad (-) \\ \hline & -x + 2 \\ & -x + 2 \quad (-) \\ \hline & 0 \end{array}$$

Here, we get zero as remainder.

GCD of leading coefficients 3 and 6 is 3.

Thus, GCD $(6x^3 - 30x^2 + 60x - 48,$
$3x^3 - 12x^2 + 21x - 18) = 3(x - 2)$.

3.12. Find the LCM of the following

(i) $8x^4 y^2, 48x^2 y^4$ (ii) $(5x - 10), (5x^2 - 20)$

(iii) $(x^4 - 1), x^2 - 2x + 1$

(iv) $x^3 - 27, (x - 3)^2, x^2 - 9$

Sol :

(i) $8x^4y^2$, $48x^2y^4$

First let us find the LCM of the numerical coefficients.

That is, LCM (8, 48) $= 2 \times 2 \times 2 \times 6 = 48$

Then find the LCM of the terms involving variables.

That is, LCM $(x^4y^2, x^2y^4) = x^4y^4$

Finally find the LCM of the given expression.

We conclude that the LCM of the given expression is the product of the LCM of the numerical coefficient and the LCM of the terms with variables.

Therefore, LCM $(8x^4y^2, 48x^2y^4) = 48x^4y^4$

(ii) $(5x - 10), (5x^2 - 20)$

$$5x - 10 = 5(x - 2)$$
$$5x^2 - 20 = 5(x^2 - 4) = 5(x + 2)(x - 2)$$

Therefore, LCM

$[(5x - 10), (5x^2 - 20)] = 5(x + 2)(x - 2)$

(iii) $(x^4 - 1), x^2 - 2x + 1$

$$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$$
$$x^2 - 2x + 1 = (x - 1)^2$$

Therefore,

LCM $[(x^4 - 1), (x^2 - 2x + 1)] = (x^2 + 1)(x + 1)(x - 1)^2$

(iv) $x^3 - 27, (x - 3)^2, x^2 - 9$

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9);$$
$$(x - 3)^2 = (x - 3)^2;$$
$$(x^2 - 9) = (x + 3)(x - 3)$$

Therefore, LCM $[(x^3 - 27), (x - 3)^2, (x^2 - 9)]$

$= (x - 3)^2 (x + 3)(x^2 + 3x + 9)$

Progress Check

1. When two polynomials of same degree has to be divided, _____ should be considered to fix the dividend and divisor.

Ans : Highest coefficient

2. If r (x) = 0 when f (x) is divided by g (x) then g (x) is called _____ of the polynomials.

Ans : Factor

3. If f (x) = g (x) q(x) + r(x), _____ must be added to f (x) to make f(x) completely divisible by g(x).

Ans : q (x)

4. If f(x) = g(x) q(x) + r(x), _____ must be subtracted to f(x) to make f(x) completely divisible by g(x).

Ans : r (x)

Thinking Corner

1. Complete the factor tree for the given polynomials f(x) and g(x), Hence find their GCD and LCM.

$f(x) = 2x^3 - 9x^2 - 32x - 21$

$2x + 3$		$x + 1$

$g(x) = 2x^3 - 7x^2 - 43x - 42$

		$x + 2$

GCD [f(x) and g(x)] = _____ ?

LCM [f(x) and g(x)] = _____ ?

Sol :

$f(x) = 2x^3 - 9x^2 - 32x - 21$

$2x + 3$	$x - 7$	$x + 1$

$g(x) = 2x^3 - 7x^2 - 43x - 42$

$2x + 3$	$x - 7$	$x + 2$

GCD [f(x) and g(x)] $= (2x + 3)(x - 7)$

LCM [f(x) and g(x)] $= (2x + 3)(x - 7)(x + 2)(x + 1)$

Exercise 3.2

1. Find the GCD of the given polynomials

(i) $x^4 + 3x^3 - x - 3$, $x^3 + x^2 - 5x + 3$

(ii) $x^4 - 1$, $x^3 - 11x^2 + x - 11$

(iii) $3x^4 + 6x^3 - 12x^2 - 24x$, $4x^4 + 14x^3 + 8x^2 - 8x$

(iv) $3x^3 + 3x^2 + 3x + 3$, $6x^3 + 12x^2 + 6x + 12$

Sol :

(i) $x^4 + 3x^3 - x - 3$, $x^3 + x^2 - 5x + 3$

Let us divide the highest degree polynomial by least degree polynomial and

Let $f(x) = x^4 + 3x^3 - x - 3$

$g(x) = x^3 + x^2 - 5x + 3$

Now, dividing f (x) by g (x).

$$\begin{array}{r|l} & x+2 \\ \hline x^3+x^2-5x+3 & x^4+3x^3+0x^2-x-3 \\ & x^4+x^3-5x^2+3x \quad (-) \\ \hline & 2x^3+5x^2-4x-3 \\ & 2x^3+2x^2-10x+6 \quad (-) \\ \hline & 3x^2+6x-9 \end{array}$$

$= 3(x^2+2x-3) \neq 0$

Since '3' is not the divisor of g (x), let us divide g (x) by x^2+2x-3

$$\begin{array}{r|l} & x-1 \\ \hline x^2+2x-3 & x^3+x^2-5x+3 \\ & x^3+2x^2-3x \quad (-) \\ \hline & -x^2-2x+3 \\ & -x^2-2x+3 \quad (-) \\ \hline & 0 \end{array}$$

Since, the Remainder is zero, the GCD is x^2+2x-3.

(ii) Let $f(x) = x^4-1$, $g(x) = x^3-11x^2+x-11$

Dividing f (x) by g (x)

$$\begin{array}{r|l} & x+11 \\ \hline x^3-11x^2+x-11 & x^4+0x^3+0x^2+0x-1 \\ & x^4-11x^3+x^2-11x \quad (-) \\ \hline & 11x^3-x^2+11x-1 \\ & 11x^3-121x^2+11x-121 \quad (-) \\ \hline & 120x^2+120 \end{array}$$

$= 120(x^2+1) \neq 0$

'120' is not the divisor of g (x).

So, dividing g (x) by x^2+1

$$\begin{array}{r|l} & x-11 \\ \hline x^2+1 & x^3-11x^2+x-11 \\ & x^3 \qquad\quad +x \quad (-) \\ \hline & -11x^2-11 \\ & -11x^2-11 \quad (-) \\ \hline & 0 \end{array}$$

Since, the Remainder is zero, the GCD is x^2+1.

(iii) Let $f(x) = 4x^4+14x^3+8x^2-8x$

$= 2(2x^4+7x^3+4x^2-4x)$

$g(x) = 3x^4+6x^3-12x^2-24x$

$= 3(x^4+2x^3-4x^2-8x)$

Dividing $2x^4+7x^3+4x^2-4x$ by $x^4+2x^3-4x^2-8x$

$$\begin{array}{r|l} & 2 \\ \hline x^4+2x^3-4x^2-8x & 2x^4+7x^3+4x^2-4x \\ & 2x^4+4x^3-8x^2-16x \quad (-) \\ \hline & 3x^3+12x^2+12x \end{array}$$

$= 3(x^3+4x^2+4x)$

Now, dividing $x^4+2x^3-4x^2-8x$ by x^3+4x^2+4x

$$\begin{array}{r|l} & x-2 \\ \hline x^3+4x^2+4x & x^4+2x^3-4x^2-8x \\ & x^4+4x^3+4x^2 \quad (-) \\ \hline & -2x^3-8x^2-8x \\ & -2x^3-8x^2-8x \quad (-) \\ \hline & 0 \end{array}$$

Since, the remainder is zero, the GCD is x^3+4x^2+4x

i.e., $x(x^2+4x+4)$

$= x(x+2)^2$

(iv) Let $f(x) = 6x^3+12x^2+6x+12$

$= 6(x^3+2x^2+x+2)$

$g(x) = 3x^3+3x^2+3x+3$

$= 3(x^3+x^2+x+1)$

For '6' and '3' the GCD is 3

Now, dividing x^3+2x^2+x+2 by x^3+x^2+x+1

$$\begin{array}{r|l} & 1 \\ \hline x^3+x^2+x+1 & x^3+2x^2+x+2 \\ & x^3+x^2+x+1 \quad (-) \\ \hline & x^2+1 \end{array}$$

Now, dividing (x^3+x^2+x+1) by (x^2+1)

$$\begin{array}{r|l} & x+1 \\ \hline x^2+1 & x^3+x^2+x+1 \\ & x^3 \qquad\quad +x \quad (-) \\ \hline & x^2 \qquad +1 \\ & x^2 \qquad +1 \quad (-) \\ \hline & 0 \end{array}$$

Since, the Remainder is zero, the GCD is $3(x^2+1)$

2. Find the LCM of the given expressions

(i) $4x^2y, 8x^3y^2$
(ii) $-9a^3b^2, 12a^2b^2c$
(iii) $16m, -12m^2n^2, 8n^2$
(iv) p^2-3p+2, p^2-4
(v) $2x^2-5x-3, 4x^2-36$
(vi) $(2x^2-3xy)^2, (4x-6y)^3, (8x^3-27y^3)$

Sol :

(i) $4x^2y, 8x^3y^2$

$$\text{LCM of } (4, 8) = 8$$
$$\text{LCM of } (x^2y, x^3y^2) = x^3y^2$$
$$\therefore \text{LCM of } (4x^2y, 8x^3y^2) = 8x^3y^2$$

(ii) $-9a^3b^2, 12a^2b^2c$

$$\text{LCM of } (-9, 12) = -36$$
$$\text{LCM of } (a^3b^2, a^2b^2c) = a^3b^2c$$
$$\therefore \text{LCM of } (-9a^3b^2, 12a^2b^2c) = -36a^3b^2c$$

(iii) $16m, -12m^2n^2, 8n^2$

$$\text{LCM of } (16, -12, 8) = -48$$
$$\text{LCM of } (m, m^2n^2, n^2) = m^2n^2$$
$$\therefore \text{LCM of } 16m, -12m^2n^2, 8n^2 = -48m^2n^2$$

(iv) $p^2-3p+2 = (p-1)(p-2)$

$p^2-4 = (p+2)(p-2)$

$\text{LCM} = (p-1)(p-2)(p+2)$

(v) $2x^2-5x-3 = 2x^2-6x+x-3$

$= 2x(x-3)+1(x-3)$

$= (2x+1)(x-3)$

$4x^2-36 = 4(x^2-9)$

$= 4(x+3)(x-3)$

$\therefore \text{LCM} = 4(x+3)(x-3)(2x+1)$

(vi) $(2x^2-3xy)^2 = [x(2x-3y)]^2$

$= x^2(2x-3y)^2$

$(4x-6y)^3 = [2(2x-3y)]^3$

$= 2^3(2x-3y)^3$

$8x^3-27y^3 = (2x)^3-(3y)^3$

$= (2x-3y)(4x^2+6xy+9y^2)$

$\therefore \text{LCM} = 2^3x^2(2x-3y)^3(4x^2+6xy+9y^2)$

$= 2^3x^2(2x-3y)^3(4x^2+6xy+9y^2)$

RELATIONSHIP BETWEEN LCM AND GCD

Thinking Corner

1. Is $f(x) \times g(x) \times r(x) =$ LCM $[f(x), g(x), r(x)] \times$ GCD $[f(x), g(x), r(x)]$?

Sol :

$f(x) \times g(x) \times r(x) \neq$

LCM of $[f(x), g(x), r(x)] \times$ GCD of $[f(x), g(x), r(x)]$

Exercise 3.3

1. Find the LCM and GCD for the following and verify that $f(x) \times g(x) = \text{LCM} \times \text{GCD}$

(i) $21x^2y, 35xy^2$
(ii) $(x^3-1)(x+1), (x^3+1)$
(iii) $(x^2y+xy^2), (x^2+xy)$

Sol :

(i) Let $f(x) = 21x^2y$

$g(x) = 35xy^2$

$\text{GCD} = 7xy$

$\text{LCM of } 21, 35 = 105$

$\text{LCM of } x^2y, xy^2 = x^2y^2$

$\therefore \text{LCM} = 105x^2y^2$

Now, $f(x) \times g(x) = (21x^2y)(35xy^2)$

$= 735x^3y^3$

$\text{LCM} \times \text{GCD} = (105x^2y^2)(7xy)$

$= 735x^3y^3$

$\therefore f(x) \times g(x) = \text{LCM} \times \text{GCD}$

Hence verified.

(ii) $f(x) = (x^3-1)(x+1)$

$= (x-1)(x^2+x+1)(x+1)$

$g(x) = x^3+1 = (x+1)(x^2-x+1)$

$\text{GCD} = x+1$

$\text{LCM} = (x+1)(x-1)(x^2+x+1)(x^2-x+1)$

$f(x) \times g(x) = (x^3-1)(x+1)(x^3+1)$

$= (x+1)((x^3)^2-(1)^2)$

$= (x+1)(x^6-1)$

$\text{LCM} \times \text{GCD} = (x+1)(x-1)(x^2+x+1)(x^2-x+1)(x+1)$

$= (x+1)(x^2-x+1)(x-1)(x^2+x+1)(x+1)$

$= (x^3+1)(x^3-1)(x+1)$

$= (x^6-1)(x+1)$

$\therefore f(x) \times g(x) = \text{LCM} \times \text{GCD}$

Hence verified.

(iii) Let $f(x) = x^2y+xy^2 = xy(x+y)$

$g(x) = x^2+xy = x(x+y)$

$\text{GCD} = x(x+y)$

$\text{LCM} = xy(x+y)$

$f(x) \times g(x) = (x^2y+xy^2)(x^2+xy)$

$= xy(x+y)\,x(x+y)$

$$= x^2y(x+y)^2$$
$$\text{LCM} \times \text{GCD} = xy(x+y)\,x(x+y)$$
$$= x^2y(x+y)^2$$
$$\therefore \quad f(x) \times g(x) = \text{LCM} \times \text{GCD}$$

Hence verified.

2. Find the LCM of each pair of the following polynomials

(i) $a^2 + 4a - 12$, $a^2 - 5a + 6$ whose GCD is $a - 2$

(ii) $x^4 - 27a^3x$, $(x - 3a)^2$ whose GCD is $(x - 3a)$

Sol :

(i) Let $f(x) = a^2 + 4a - 12$

$g(x) = a^2 - 5a + 6$

Given GCD $= a - 2$

We know $f(x) \times g(x) = \text{LCM} \times \text{GCD}$

$$\therefore \text{LCM} = \frac{f(x) \times g(x)}{GCD}$$
$$= \frac{(a^2+4a-12)(a^2-5a+6)}{a-2}$$
$$= \frac{(a+6)(a-2)(a-3)(a-2)}{a-2}$$
$$\therefore \text{LCM} = (a-2)(a-3)(a+6)$$

(ii) Let $f(x) = x^4 - 27a^3x$

$= x(x^3 - 27a^3)$

$= x(x^3 - (3a)^3)$

$= x(x - 3a)(x^2 + 3ax + 9a^2)$

$g(x) = (x - 3a)^2 = (x - 3a)(x - 3a)$

Given GCD $= (x - 3a)$

We know $f(x) \times g(x) = \text{LCM} \times \text{GCD}$

$$\therefore \text{LCM} = \frac{f(x) \times g(x)}{GCD}$$
$$= \frac{x(x-3a)(x^2+3ax+9a^2)(x-3a)^2}{x-3a}$$
$$\therefore \text{LCM} = x(x^2 + 3ax + 9a^2)(x - 3a)^2$$

3. Find the GCD of each pair of the following polynomials

(i) $12(x^4 - x^3)$, $8(x^4 - 3x^3 + 2x^2)$ whose LCM is $24x^3(x - 1)(x - 2)$

(ii) $(x^3 + y^3)$, $(x^4 + x^2y^2 + y^4)$ whose LCM is $(x^3 + y^3)(x^2 + xy + y^2)$

Sol :

(i) Let $f(x) = 12(x^4 - x^3)$

$= 12x^3(x - 1)$

$g(x) = 8(x^4 - 3x^3 + 2x^2)$

$= 8x^2(x^2 - 3x + 2)$

$= 8x^2(x - 1)(x - 2)$

Given LCM $= 24x^3(x - 1)(x - 2)$

We know $f(x) \times g(x) = \text{LCM} \times \text{GCD}$

$$\therefore \text{GCD} = \frac{f(x) \times g(x)}{LCM}$$
$$= \frac{12x^3(x-1) \times 8x^2(x-1)(x-2)}{24x^3(x-1)(x-2)}$$
$$\text{GCD} = 4x^2(x - 1)$$

(ii) Let $f(x) = x^3 + y^3$

$= (x + y)(x^2 - xy + y^2)$

$g(x) = x^4 + x^2y^2 + y^4$

Given LCM $= (x^3 + y^3)(x^2 + xy + y^2)$

$$\therefore \text{GCD} = \frac{f(x) \times g(x)}{LCM}$$
$$= \frac{(x^3+y^3)(x^4+x^2y^2+y^4)}{(x^3+y^3)(x^2+xy+y^2)}$$
$$= \frac{(x^2-xy+y^2)(x^2+xy+y^2)}{x^2+xy+y^2}$$
$$= x^2 - xy + y^2$$

4. Given the LCM and GCD of the two polynomials, find p (x), q (x) find the unknown polynomial in the following table

	LCM	GCD	p (x)	q (x)
(i)	$a^3 - 10a^2 + 11a + 70$	$a - 7$	$a^2 - 12a + 35$	
(ii)	$(x^2 + y^2)(x^4 + x^2y^2 + y^4)$	$(x^2 - y^2)$		$(x^4 - y^4)(x^2 + y^2 - xy)$

Sol :

(i) Given LCM $= a^3 - 10a^2 + 11a + 70$

GCD $= a - 7$

$p(x) = a^2 - 12a + 35$

Now $p(x) \times q(x) = \text{LCM} \times \text{GCD}$

$$\therefore q(x) = \frac{LCM \times GCD}{p(x)}$$
$$\therefore q(x) = \frac{(a^3-10a^2+11a+70)(a-7)}{a^2-12a+35}$$
$$= \frac{(a-7)(a-5)(a+2)(a-7)}{(a-7)(a-5)}$$
$$= (a + 2)(a - 7)$$

(ii) Given LCM = $(x^2 + y^2)(x^4 + x^2y^2 + y^4)$

GCD = $x^2 - y^2$

$q(x) = (x^4 - y^4)(x^2 + y^2 - xy)$

Now $p(x) \times q(x)$ = LCM × GCD

$$\therefore p(x) = \frac{LCM \times GCD}{q(x)}$$

$$= \frac{(x^2+y^2)(x^4+x^2y^2+y^4)(x^2-y^2)}{(x^4-y^4)(x^2+y^2-xy)}$$

$$= \frac{(x^4-y^4)(x^2+xy+y^2)(x^2-xy+y^2)}{(x^4-y^4)(x^2+y^2-xy)}$$

$$= x^2 + xy + y$$

RATIONAL EXPRESSIONS

Key Points

- A polynomial is called a rational expression if it can be written in the form $\frac{p(x)}{q(x)}, q(x) \neq 0$.
- A rational expression is the ratio of two polynomials.
- A rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest form if GCD [p(x), q(x)] = 1.
- A number that makes a rational expression (in its lowest form) undefined is called an Excluded value.

Worked Examples

3.13. **Reduce the rational expressions to its lowest form.**

(i) $\frac{x-3}{x^2-9}$ (ii) $\frac{x^2-16}{x^2+8x+16}$

Sol :

(i) $\frac{x-3}{x^2-9} = \frac{x-3}{(x+3)(x-3)} = \frac{1}{x+3}$

(ii) $\frac{x^2-16}{x^2+8x+16} = \frac{(x+4)(x-4)}{(x+4)^2} = \frac{x-4}{x+4}$

3.14. **Find the excluded values of the following expressions (if any)**

(i) $\frac{x+10}{8x}$ (ii) $\frac{7p+2}{8p^2+13p+5}$ (iii) $\frac{x}{x^2+1}$

Sol :

(i) $\frac{x+10}{8x}$

The expression $\frac{x+10}{8x}$ is undefined when 8x = 0 or $x = 0$. Hence the excluded value is 0.

(ii) $\frac{7p+2}{8p^2+13p+5}$

The expression $\frac{7p+2}{8p^2+13p+5}$ is undefined when

$8p^2 + 13p + 5 = 0$

that is, $(8p + 5)(p + 1) = 0$

$p = \frac{-5}{8}$, $p = -1$. The excluded values are $\frac{-5}{8}$ and -1.

(iii) $\frac{x}{x^2+1}$

Here $x^2 \geq 0$ for all x. Therefore, $x^2+1 \geq 0+1 = 1$.

Hence, $x^2+1 \neq 0$ for any x. Therefore, there can be no real excluded values for the given rational expression $\frac{x}{x^2+1}$.

Thinking Corner

1. Are $x^2 - 1$ and $\tan x = \frac{\sin x}{\cos x}$ rational expressions?

Ans : No

2. The number of excluded values of $\frac{x^3+x^2-10x+8}{x^4+8x^2-9}$ is _____ .

Ans : 2.

Exercise 3.4

1. Reduce each of the following rational expressions to its lowest form

(i) $\dfrac{x^2-1}{x^2+x}$ (ii) $\dfrac{x^2-11x+18}{x^2-4x+4}$

(iii) $\dfrac{9x^2+81x}{x^3+8x^2-9x}$ (iv) $\dfrac{p^2-3p-40}{2p^3-24p^2+64p}$

Sol :

(i) $$\frac{x^2-1}{x^2+x} = \frac{(x+1)(x-1)}{x(x+1)} = \frac{x-1}{x}$$

(ii) $$\frac{x^2-11x+18}{x^2-4x+4} = \frac{(x-9)(x-2)}{(x-2)^2} = \frac{x-9}{x-2}$$

(iii) $$\frac{9x^2+81x}{x^3+8x^2-9x} = \frac{9x(x+9)}{x(x^2+8x-9)} = \frac{9x(x+9)}{x(x+9)(x-1)} = \frac{9}{x-1}$$

(iv) $$\frac{p^2-3p-40}{2p^3-24p^2+64p} = \frac{(p-8)(p+5)}{2p(p^2-12p+32)} = \frac{(p-8)(p+5)}{2p(p-8)(p-4)} = \frac{p+5}{2p(p-4)}$$

2. Find the excluded values, if any of the following expressions.

(i) $\dfrac{y}{y^2-25}$ (ii) $\dfrac{t}{t^2-5t+6}$

(iii) $\dfrac{x^2+6x+8}{x^2+x-2}$ (iv) $\dfrac{x^3-27}{x^3+x^2-6x}$

Sol :

(i) $\dfrac{y}{y^2-25} = \dfrac{y}{(y+5)(y-5)}$ Which is undefined when $y = -5$ and $y = 5$

$\therefore$ Excluded values $= -5, 5$

(ii) $\dfrac{t}{t^2-5t+6} = \dfrac{t}{(t-2)(t-3)}$ is undefined when $t = 2$ and $t = 3$

$\therefore$ Excluded values are 2 and 3.

(iii) $\dfrac{x^2+6x+8}{x^2+x-2} = \dfrac{(x+2)(x+4)}{(x+2)(x-1)}$

$= \dfrac{x+4}{x-1}$ is undefined when $x = 1$

$\therefore$ Excluded value is 1.

(iv) $\dfrac{x^3-27}{x^3+x^2-6x} = \dfrac{x^3-3^3}{x(x^2+x-6)}$

$= \dfrac{(x-3)(x^2+3x+9)}{x(x+3)(x-2)}$ is undefined when $x = 0$, $x = -3$ and $x = 2$

$\therefore$ Excluded values are $0, -3, 2$.

OPERATIONS OF RATIONAL EXPRESSIONS

Key Points

- If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions, their product is $\dfrac{p(x)}{q(x)} \times \dfrac{r(x)}{s(x)} = \dfrac{p(x)\,r(x)}{q(x)\,s(x)}$
- If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions, their quotient is $\dfrac{p(x)}{q(x)} \div \dfrac{r(x)}{s(x)} = \dfrac{p(x).s(x)}{q(x).r(x)}$
- For addition and subtraction of two rational expressions, by taking the LCM of denominators, we get the simplest form.

Worked Examples

3.15 (i) Multiply $\dfrac{x^3}{9y^2}$ by $\dfrac{27y}{x^5}$

(ii) Multiply $\dfrac{x^4b^2}{x-1}$ by $\dfrac{x^2-1}{a^4b^3}$

Sol :

(i) $$\frac{x^3}{9y^2}\times\frac{27y}{x^5}=\frac{3}{x^2y}$$

(ii) $$\frac{x^4b^2}{x-1}\times\frac{x^2-1}{a^4b^3}=\frac{x^4\times b^2}{x-1}\times\frac{(x+1)(x-1)}{a^4\times b^3}$$

$$=\frac{x^4(x+1)}{a^4b}$$

3.16. Divide:

(i) $\dfrac{14x^4}{y}\div\dfrac{7x}{3y^4}$

(ii) $\dfrac{x^2-16}{x+4}\div\dfrac{x-4}{x+4}$

(iii) $\dfrac{16x^2-2x-3}{3x^2-2x-1}\div\dfrac{8x^2+11x+3}{3x^2-11x-4}$

Sol :

(i) $$\frac{14x^4}{y}\div\frac{7x}{3y^4}=\frac{14x^4}{y}\times\frac{3y^4}{7x}=6x^3y^3$$

(ii) $$\frac{x^2-16}{x+4}\div\frac{x-4}{x+4}=\frac{(x+4)(x-4)}{(x+4)}\times\left(\frac{x+4}{x-4}\right)$$

$$=x+4$$

(iii) $$\frac{16x^2-2x-3}{3x^2-2x-1}\div\frac{8x^2+11x+3}{3x^2-11x-4}$$

$$=\frac{16x^2-2x-3}{3x^2-2x-1}\times\frac{3x^2-11x-4}{8x^2+11x+3}$$

$$=\frac{(8x+3)(2x-1)}{(3x+1)(x-1)}\times\frac{(3x+1)(x-4)}{(8x+3)(x+1)}$$

$$=\frac{(2x-1)(x-4)}{(x-1)(x+1)}=\frac{2x^2-9x+4}{x^2-1}$$

Progress Check

1. Find the unknown expression in the following figures.

$$\boxed{\text{Area} = \frac{(x-4)(x+3)}{3x-12}\,km^2}\quad \text{Breadth} = ?$$

$$\text{Length} = \frac{x-3}{3}\,km$$

Ans :

$$\text{Breadth} = \frac{Area}{length}$$

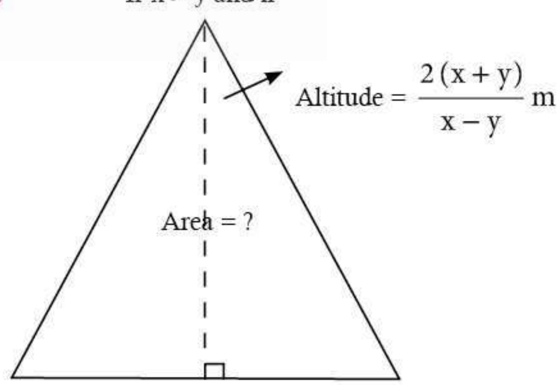
Rectangle

$$=\frac{\dfrac{(x-4)(x+3)}{(3x-12)}}{\dfrac{x-3}{3}}$$

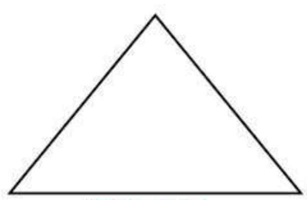
$$=\frac{(x-4)(x+3)}{3(x-4)}\times\frac{3}{(x-3)}$$

$$=\frac{x+3}{x-3}$$

2. If $x > y$ and if



Ans :



Triangle

$$\text{Altitude h} = \frac{2(x+y)}{x-y}\,m$$

Base b = (x + y) (x − y) m

Area = 1/2 bh

$$= \frac{1}{2}(x+y)(x-y)\frac{2(x+y)}{x-y}$$

$$= (x+y)^2 \text{ m}^2$$

Exercise 3.5

1. Simplify

(i) $\frac{4x^2y}{2z^2} \times \frac{6xz^3}{20y^4}$

(ii) $\frac{p^2-10p+21}{p-7} \times \frac{p^2+p-12}{(p-3)^2}$

(iii) $\frac{5t^3}{4t-8} \times \frac{6t-12}{10t}$

Sol :

(i) $\frac{4x^2y}{2z^2} \times \frac{6xz^3}{20y^4} = \frac{3x^3z}{5y^3}$

(ii) $\frac{p^2-10p+21}{p-7} \times \frac{p^2+p-12}{(p-3)^2}$

$$= \frac{(p-7)(p-3)}{(p-7)} \times \frac{(p+4)(p-3)}{(p-3)^2}$$

$$= p+4$$

(iii) $\frac{5t^3}{4t-8} \times \frac{6t-12}{10t} = \frac{5t^3}{4(t-2)} \times \frac{6(t-2)}{10t} = \frac{3}{4}t^2$

2. Simplify

(i) $\frac{x+4}{3x+4y} \times \frac{9x^2-16y^2}{2x^2+3x-20}$

(ii) $\frac{x^3-y^3}{3x^2+9xy+6y^2} \times \frac{x^2+2xy+y^2}{x^2-y^2}$

Sol :

(i) $\frac{x+4}{3x+4y} \times \frac{9x^2-16y^2}{2x^2+3x-20}$

$$= \frac{x+4}{3x+4y} \times \frac{(3x)^2-(4y)^2}{2x^2+8x-5x-20}$$

$$= \frac{x+4}{3x+4y} \times \frac{(3x+4y)(3x-4y)}{(2x-5)(x+4)}$$

$$= \frac{3x-4y}{2x-5}$$

(ii) $\frac{x^3-y^3}{3x^2+9xy+6y^2} \times \frac{x^2+2xy+y^2}{x^2-y^2}$

$$= \frac{(x-y)(x^2+xy+y^2)}{(3x+6y)(x+y)} \times \frac{(x+y)^2}{(x+y)(x-y)}$$

$$= \frac{x^2+xy+y^2}{3x+6y}$$

$$= \frac{x^2+xy+y^2}{3(x+2y)}$$

3. Simplify

(i) $\frac{2a^2+5a+3}{2a^2+7a+6} \div \frac{a^2+6a+5}{-5a^2-35a-50}$

(ii) $\frac{b^2+3b-28}{b^2+4b+4} \div \frac{b^2-49}{b^2-5b-14}$

(iii) $\frac{x+2}{4y} \div \frac{x^2-x-6}{12y^2}$

(iv) $\frac{12t^2-22t+8}{3t} \div \frac{3t^2+2t-8}{2t^2+4t}$

Sol :

(i) $\frac{2a^2+5a+3}{2a^2+7a+6} \div \frac{a^2+6a+5}{-5a^2-35a-50}$

$$= \frac{2a^2+5a+3}{2a^2+7a+6} \times \frac{-5(a^2+7a+10)}{a^2+6a+5}$$

$$= \frac{(2a+3)(a+1)}{(2a+3)(a+2)} \times \frac{-5(a+5)(a+2)}{(a+5)(a+1)}$$

$$= -5$$

(ii) $\frac{b^2+3b-28}{b^2+4b+4} \div \frac{b^2-49}{b^2-5b-14}$

$$= \frac{b^2+3b-28}{b^2+4b+4} \times \frac{b^2-5b-14}{b^2-49}$$

$$= \frac{(b+7)(b-4)}{(b+2)^2} \times \frac{(b-7)(b+2)}{(b-7)(b+7)}$$

$$= \frac{b-4}{b+2}$$

(iii) $\frac{x+2}{4y} \div \frac{x^2-x-6}{12y^2}$

$$= \frac{x+2}{4y} \times \frac{12y^2}{x^2-x-6}$$

$$= \frac{x+2}{4y} \times \frac{12y^2}{(x-3)(x+2)}$$

$$= \frac{3y}{x-3}$$

(iv) $\frac{12t^2-22t+8}{3t} \div \frac{3t^2+2t-8}{2t^2+4t}$

$$= \frac{2(6t^2-11t+4)}{3t} \times \frac{2t(t+2)}{3t^2+2t-8}$$

$$= \frac{2(3t-4)(2t-1)}{3t} \times \frac{2t(t+2)}{(3t-4)(t+2)}$$

$$= \frac{4(2t-1)}{3}$$

4. If x = $\frac{a^2+3a-4}{3a^2-3}$ and y = $\frac{a^2+2a-8}{2a^2-2a-4}$ find the value of x^2y^{-2}.

Sol :

$$\text{Given } x = \frac{a^2+3a-4}{3a^2-3}$$

$$= \frac{(a+4)(a-1)}{3(a^2-1)}$$

$$= \frac{(a+4)(a-1)}{3(a+1)(a-1)} = \frac{a+4}{3(a+1)}$$

$$y = \frac{a^2+2a-8}{2a^2-2a-4} = \frac{(a+4)(a-2)}{2(a^2-a-2)}$$

$$= \frac{(a+4)(a-2)}{2(a-2)(a+1)}$$

$$= \frac{(a+4)}{2(a+1)}$$

$$\text{Now } x^2y^{-2} = \left(\frac{(a+4)}{3(a+1)}\right)^2 \times \left(\frac{a+4}{2(a+1)}\right)^{-2}$$

$$= \left(\frac{a+4}{3(a+1)}\right)^2 \times \left(\frac{2(a+1)}{(a+4)}\right)^2$$

$$= \frac{(a+4)^2}{9(a+1)^2} \times \frac{4(a+1)^2}{(a+4)^2} = \frac{4}{9}.$$

5. If a polynomial p (x) = $x^2 - 5x - 14$ when divided by another polynomial q(x) gets reduced to $\frac{x-7}{x+2}$, find q(x).

Sol :

$$p(x) = x^2 - 5x - 14 \text{ and given}$$

$$p(x) \div q(x) = \frac{x-7}{x+2}$$

$$(x^2-5x-14) \div q(x) = \frac{x-7}{x+2}$$

$$\therefore\ q(x) = (x^2-5x-14) \times \frac{(x+2)}{(x-7)}$$

$$= (x-7)(x+2) \times \frac{(x+2)}{(x-7)}$$

$$= (x+2)^2$$

$$q(x) = x^2 + 4x + 4$$

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

Worked Examples

3.17 Find $\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$

Sol :

$$\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$$

$$= \frac{(x^2+20x+36)-(x^2+12x+4)}{x^2-3x-28}$$

$$= \frac{8x+32}{x^2-3x-28} = \frac{8(x+4)}{(x-7)(x+4)} = \frac{8}{x-7}$$

3.18. Simplify $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$

Sol :

$$\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$$

$$= \frac{1}{(x-2)(x-3)} + \frac{1}{(x-2)(x-1)} - \frac{1}{(x-5)(x-3)}$$

$$= \frac{(x-1)(x-5)+(x-3)(x-5)-(x-1)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x^2-6x+5)+(x^2-8x+15)-(x^2-3x+2)}{(x-1)(x-2)(x-3)(x-5)}$$

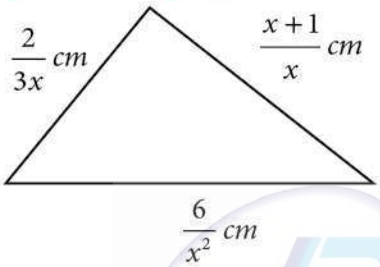
$$= \frac{x^2-11x+18}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)}{(x-1)(x-3)(x-5)}$$

Progress Check

1. Write an expression that represents the perimeter of the figure and simplify.



Ans :

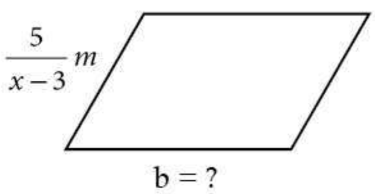
$$\text{Perimeter} = \frac{2}{3x}+\frac{x+1}{x}+\frac{6}{x^2}$$

$$= \frac{2(x)+(x+1)3x+6(3)}{3x^2}$$

$$= \frac{2x+3x^2+3x+18}{3x^2}$$

$$= \frac{3x^2+5x+18}{3x^2}$$

2. Find the base of the given parallelogram whose perimeter is = $\frac{4x^2+10x-50}{(x-3)(x+5)}$



Ans :

$$\text{Perimeter} = \frac{4x^2+10x-50}{(x-3)(x+5)}$$

$$2(l+b) = \frac{4x^2+10x-50}{(x-3)(x+5)}$$

$$2\left(\frac{5}{x-3}+b\right) = \frac{2(2x-5)(x+5)}{(x-3)(x+5)}$$

$$b = \frac{2x-5}{x-3}-\frac{5}{x-3}$$

$$= \frac{2x-5-5}{x-3} = \frac{2x-10}{x-3}$$

Thinking Corner

True or False

1. The sum of two rational expressions is always a rational expression.

Ans : True

2. The product of two rational expressions is always a rational expression.

Ans : True

Exercise 3.6

1. Simplify

(i) $\frac{x(x+1)}{x-2}+\frac{x(1-x)}{x-2}$

(ii) $\frac{x+2}{x+3}+\frac{x-1}{x-2}$

(iii) $\frac{x^3}{x-y}+\frac{y^3}{y-x}$

Sol :

(i) $\frac{x(x+1)}{x-2}+\frac{x(1-x)}{x-2}$

$$= \frac{x(x+1)+x(1-x)}{x-2}$$

$$= \frac{x[x+1+1-x]}{x-2}$$

$$= \frac{x(2)}{x-2} = \frac{2x}{x-2}$$

(ii) $\frac{x+2}{x+3}+\frac{x-1}{x-2}$

$$= \frac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)}$$

$$= \frac{x^2-4+x^2+2x-3}{(x+3)(x-2)}$$

$$= \frac{2x^2+2x-7}{(x+3)(x-2)}$$

(iii) $\frac{x^3}{x-y}+\frac{y^3}{y-x}$

$$= \frac{x^3}{x-y}-\frac{y^3}{x-y}$$

$$= \frac{x^3-y^3}{x-y}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{x-y}$$

$$= x^2+xy+y^2$$

2. Simplify

(i) $\frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$

(ii) $\frac{4x}{x^2-1}-\frac{x+1}{x-1}$

Sol :

(i) $\frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$

$$= \frac{(2x+1)(x-2)-(2x-1)(x-2)}{x-4}$$

$$= \frac{(x-2)(2x+1-2x+1)}{x-4}$$

$$= \frac{(x-2)(2)}{x-4}$$

$$= \frac{2(x-2)}{x-4}$$

(ii) $\frac{4x}{x^2-1}-\frac{x+1}{x-1}$

$$= \frac{4x}{(x+1)(x-1)}-\frac{x+1}{x-1}$$

$$= \frac{4x-(x+1)(x+1)}{(x+1)(x-1)}$$

$$= \frac{4x-(x^2+2x+1)}{(x+1)(x-1)}$$

$$= \frac{4x-x^2-2x-1}{(x+1)(x-1)}$$

$$= \frac{-x^2+2x-1}{(x+1)(x-1)}$$

$$= -\frac{(x^2-2x+1)}{(x+1)(x-1)}$$

$$= -\frac{(x-1)^2}{(x+1)(x-1)}$$

$$= -\frac{(x-1)}{x+1} = \frac{1-x}{1+x}$$

3. Subtract $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

Sol :

$$\frac{2x^3+x^2+3}{(x^2+2)^2}-\frac{1}{x^2+2} = \frac{2x^3+x^2+3-(x^2+2)}{(x^2+2)^2}$$

$$= \frac{2x^3+x^2+3-x^2-2}{(x^2+2)^2}$$

$$= \frac{2x^3+1}{(x^2+2)^2}$$

4. Which rational expression should be subtracted from $\frac{x^2+6x+8}{x^3+8}$ to get $\frac{3}{x^2-2x+4}$.

Sol :

$$\frac{x^2+6x+8}{x^3+8}-f(x) = \frac{3}{x^2-2x+4}$$

$$\therefore\ f(x) = \frac{x^2+6x+8}{x^3+2^3}-\frac{3}{x^2-2x+4}$$

$$= \frac{(x+4)(x+2)}{(x+2)(x^2-2x+4)}-\frac{3}{(x^2-2x+4)}$$

$$= \frac{x+4-3}{(x^2-2x+4)} = \frac{x+1}{x^2-2x+4}$$

5. If A = $\frac{2x+1}{2x-1}$, B = $\frac{2x-1}{2x+1}$ find $\frac{1}{A-B}-\frac{2B}{A^2-B^2}$

Sol :

Given A = $\frac{2x+1}{2x-1}$, B = $\frac{2x-1}{2x+1}$

We need to find $\frac{1}{A-B}-\frac{2B}{A^2-B^2}$

Let us simplify : $\frac{1}{A-B}-\frac{2B}{(A+B)(A-B)}$

$$= \frac{A+B-2B}{(A+B)(A-B)}$$

$$= \frac{A-B}{(A+B)(A-B)} = \frac{1}{A+B}$$

Now, $$A+B = \frac{2x+1}{2x-1}+\frac{2x-1}{2x+1}$$

$$= \frac{(2x+1)^2+(2x-1)^2}{(2x-1)(2x+1)}$$

$$= \frac{4x^2+4x+1+4x^2-4x+1}{4x^2-1}$$

$$A+B = \frac{8x^2+2}{4x^2-1} = \frac{2(4x^2+1)}{4x^2-1}$$

$$\therefore \frac{1}{A-B}-\frac{2B}{A^2-B^2} = \frac{1}{A+B} = \frac{4x^2-1}{2(4x^2+1)}$$

6. If A = $\frac{x}{x+1}$, B = $\frac{1}{x+1}$, prove that

$$\frac{(A+B)^2+(A-B)^2}{\frac{A}{B}} = \frac{2(x^2+1)}{x(x+1)^2}$$

Sol :

Given, $A = \frac{x}{x+1}$, $B = \frac{1}{x+1}$

To prove : $\frac{(A+B)^2+(A-B)^2}{A/B} = \frac{2(x^2+1)}{x(x+1)^2}$

Simplifying $\frac{(A+B)^2+(A-B)^2}{A/B}$

$$= \frac{A^2+2AB+B^2+A^2-2AB+B^2}{A/B}$$

$$= \frac{2A^2+2B^2}{A/B}$$

$$= 2\frac{B}{A}(A^2+B^2)$$

Now $A^2+B^2 = \left(\frac{x}{x+1}\right)^2+\left(\frac{1}{x+1}\right)^2$

$$= \frac{x^2+1}{(x+1)^2}$$

$$\therefore 2\frac{B}{A}(A^2+B^2) = 2\frac{\left(\frac{1}{x+1}\right)}{\left(\frac{x}{x+1}\right)}\left[\frac{x^2+1}{(x+1)^2}\right]$$

$$= 2\frac{(x^2+1)}{x(x+1)^2}$$ Hence proved.

7. Pari needs 4 hours to complete a work. His friend Yuvan needs 6 hours to do the same work. How long will the job take if they work together?

Sol :

Let the work done be 'x'

Pari needs 4 hours and Yuvan needs 6 hours

Work done in 1 hr by Pari = $\frac{1}{4}$ of x = $\frac{x}{4}$

Work done in 1 hr by Yuvan = $\frac{1}{6}$ of x = $\frac{x}{6}$

$\therefore$ Work done by both = $\frac{x}{4}+\frac{x}{6}$

$$= \frac{3x+2x}{12} = \frac{5x}{12} = \frac{5}{12} \text{ of x}$$

$\therefore$ Time needed to complete the work together

$$= \frac{12}{5}hrs = \frac{12}{5}\times 60 = 2 \text{ hrs } 24 \text{ minutes.}$$

8. Iniya bought 50 kg of fruits consisting of apples and bananas. She paid twice as much per kg for the apple as she did for the banana. If Iniya bought ₹ 1800 worth of apples and ₹ 600 worth bananas, then how many kg of each fruit did she buy?

Sol :

Let the weight of Apples be 'x' kg.

Let the weight of bananas be 'y' kg.

Given $x + y = 50$ ... (1)

$$\frac{1800}{x} = 2\left(\frac{600}{y}\right) \Rightarrow \frac{3}{x} = \frac{2}{y}$$

$$3y = 2x$$

$$\Rightarrow x = \frac{3}{2}y \quad \text{... (2)}$$

Substituting in (1)

$$\frac{3}{2}y + y = 50$$

$$5y = 100$$

$$y = \frac{100}{5} = 20$$

Substituting in (2)

$$x = \frac{3}{2}(20) = 30$$

$\therefore$ Iniya bought 30 kg of Apples and 20 kg of bananas.

SQUARE ROOT OF POLYNOMIALS

Key Points

- The square root of a given positive number is another number which when multiplied with itself is the given number.
- The square root of a given expression P(x) is another expression Q(x) which when multiplied by itself gives P(x), that is Q(x).Q(x) = P(x)
- Square root of a given expression can be found out by using Factorization method and Division method.

Worked Examples

3.19. **Find the square root of the following expressions.**

(i) $256(x-a)^8(x-b)^4(x-c)^{16}(x-d)^{20}$

(ii) $\dfrac{144a^8b^{12}c^{16}}{81f^{12}g^4h^{14}}$

Sol :

(i) $\sqrt{256(x-a)^8(x-b)^4(x-c)^{16}(x-d)^{20}}$

$= 16|(x-a)^4(x-b)^2(x-c)^8(x-d)^{10}|$

(ii) $\sqrt{\dfrac{144a^8b^{12}c^{16}}{81f^{12}g^4h^{14}}} = \dfrac{4}{3}\left|\dfrac{a^4b^6c^8}{f^6g^2h^7}\right|$

3.20. **Find the square root of the following expressions**

(i) $16x^2+9y^2-24xy+24x-18y+9$

(ii) $(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)$

(iii) $\left[\sqrt{15}x^2+(\sqrt{3}+\sqrt{10})x+\sqrt{2}\right]\left[\sqrt{5}x^2+(2\sqrt{5}+1)x+2\right]\left[\sqrt{3}x^2+(\sqrt{2}+2\sqrt{3})x+2\sqrt{2}\right]$

Sol :

(i) $\sqrt{16x^2+9y^2-24xy+24x-18y+9}$

$= \sqrt{(4x)^2+(-3y)^2+(3)^2+2(4x)(-3y)+2(-3y)(3)+2(4x)(3)}$

$= \sqrt{(4x-3y+3)^2}$

$= |4x-3y+3|$

(ii) $\sqrt{(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)}$

$= \sqrt{(3x-1)(2x+1)(3x-1)(x+1)(2x+1)(x+1)}$

$= |(3x-1)(2x+1)(x+1)|$

(iii) First let us factorize the polynomials

$\sqrt{15}x^2+(\sqrt{3}+\sqrt{10})x+\sqrt{2}$

$= \sqrt{15}x^2+\sqrt{3}x+\sqrt{10}x+\sqrt{2}$

$= \sqrt{3}x(\sqrt{5}x+1)+\sqrt{2}(\sqrt{5}x+1)$

$= (\sqrt{5}x+1)\times(\sqrt{3}x+\sqrt{2})$

$\sqrt{5}x^2+(2\sqrt{5}+1)x+2 = \sqrt{5}x^2+2\sqrt{5}x+x+2$

$= \sqrt{5}x(x+2)+1(x+2)$

$= (\sqrt{5}x+1)(x+2)$

$\left[\sqrt{3}x^2+(\sqrt{2}+2\sqrt{3})x+2\sqrt{2}\right]$

$= \sqrt{3}x^2+\sqrt{2}x+2\sqrt{3}x+2\sqrt{2}$

$= x(\sqrt{3}x+\sqrt{2})+2(\sqrt{3}x+\sqrt{2})$

$= (x+2)(\sqrt{3}x+\sqrt{2})$

Therefore,

$\sqrt{\left[\sqrt{15}x^2+(\sqrt{3}+\sqrt{10})x+\sqrt{2}\right]\left[\sqrt{5}x^2+(2\sqrt{5}+1)x+2\right]\left[\sqrt{3}x^2+(\sqrt{2}+2\sqrt{3})x+2\sqrt{2}\right]}$

$= \sqrt{(\sqrt{5}x+1)(\sqrt{3}x+\sqrt{2})(\sqrt{5}x+1)(x+2)(\sqrt{3}x+\sqrt{2})(x+2)}$

$= \left|(\sqrt{5}x+1)(\sqrt{3}x+\sqrt{2})(x+2)\right|$

Progress Check

1. Is x^2+4x+4 a perfect square?

Ans : $x^2+4x+4=(x+2)^2$ it is a perfect square.

2. What is the value of x in $3\sqrt{x}=9$?

Ans : $3\sqrt{x}=9$; $\sqrt{x}=9/3=3$; $x=9$

3. The square root of $361x^4y^2$ is ______.

Ans : Square root of $361x^4y^2 = |19x^2y|$

4. $\sqrt{a^2x^2+2abx+b^2}$ = _____.

Ans : $\sqrt{a^2x^2+2abx+b^2} = (ax+b)^2 = |ax+b|$

5. If a polynomial is a perfect square then, its factors will be repeated _____ number of times (odd/even).

Ans : Even

Exercise 3.7

1. Find the square root of the following rational expressions

(i) $\dfrac{400x^4y^{12}z^{16}}{100x^8y^4z^4}$

(ii) $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

(iii) $\dfrac{121(a+b)^8(x+y)^8(b-c)^8}{81(b-c)^4(a-b)^{12}(b-c)^4}$

Sol :

(i) Square root of $\dfrac{400x^4y^{12}z^{16}}{100x^8y^4z^4}$

$$= \sqrt{\frac{400x^4y^{12}z^{16}}{100x^8y^4z^4}} = \sqrt{\frac{4y^8z^{12}}{x^4}}$$

$$= 2\left|\frac{y^4z^6}{x^2}\right|$$

(ii) Square root of $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

$$= \sqrt{\frac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}}$$

$7x^2+2\sqrt{14}x+2$

$$= 7x^2+\sqrt{14}x+\sqrt{14}x+2$$

$$= \sqrt{7}\sqrt{7}x^2+\sqrt{2}\sqrt{7}x+\sqrt{2}\sqrt{7}x+\sqrt{2}\sqrt{2}$$

$$= \sqrt{7}x(\sqrt{7}\,x+\sqrt{2})+\sqrt{2}(\sqrt{7}x+\sqrt{2})$$

$$= \left(\sqrt{7}x+\sqrt{2}\right)\left(\sqrt{7}x+\sqrt{2}\right)$$

$$= \left(\sqrt{7}x+\sqrt{2}\right)^2$$

$x^2-\frac{1}{2}x+\frac{1}{16}$

$$= x^2-\frac{1}{4}x-\frac{1}{4}x+\frac{1}{16}$$

$$= x\left(x-\frac{1}{4}\right)-1/4\left(x-\frac{1}{4}\right)$$

$$= \left(x-\frac{1}{4}\right)^2$$

$$\therefore \sqrt{\frac{7x^2+2\sqrt{14}x+2}{x^2-1/2\,x+\frac{1}{16}}}$$

$$= \sqrt{\frac{\left(\sqrt{7}x+\sqrt{2}\right)^2}{(x-1/4)^2}}$$

$$= \sqrt{16\frac{\left(\sqrt{7}x+\sqrt{2}\right)^2}{(4x-1)^2}}$$

$$= 4\left|\frac{\sqrt{7}x+\sqrt{2}}{4x-1}\right|$$

(iii) Square root of $\dfrac{121\,(a+b)^8\,(x+y)^8\,(b-c)^8}{81\,(b-c)^4\,(a-b)^{12}\,(b-c)^4}$

$$= \sqrt{\frac{121\,(a+b)^8(x+y)^8(b-c)^8}{81\,(b-c)^4(a-b)^{12}(b-c)^4}}$$

$$= \sqrt{\frac{121\,(a+b)^8\,(x+y)^8}{81\,(a-b)^{12}}}$$

$$= \frac{11}{9}\left|\frac{(a+b)^4(x+y)^4}{(a-b)^6}\right|$$

2. Find the square root of the following

(i) $4x^2+20x+25$

(ii) $9x^2-24xy+30xz-40yz+25z^2+16y^2$

(iii) $1+\dfrac{1}{x^6}+\dfrac{2}{x^3}$

(iv) $(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)$

(v) $\left(2x^2+\dfrac{17}{6}x+1\right)\left(\dfrac{3}{2}x^2+4x+2\right)\left(\dfrac{4}{3}x^2+\dfrac{11}{3}x+2\right)$

Sol :

(i) Square root of $4x^2 + 20x + 25$

$$\sqrt{4x^2 + 20x + 25} = \sqrt{4x^2 + 10x + 10x + 25}$$
$$= \sqrt{2x\,(2x+5) + 5\,(2x+5)}$$
$$= \sqrt{(2x+5)\,(2x+5)}$$
$$= \sqrt{(2x+5)^2}$$
$$\therefore \sqrt{4x^2+20x+25} = |2x+5|$$

(ii) Square root of
$9x^2 - 24xy + 30xz - 40yz + 25z^2 + 16y^2$

Rearranging the terms

$9x^2 + 16y^2 + 25z^2 - 24xy - 40yz + 30xz$

$$= (3x)^2 + (-4y)^2 + (5z)^2 + 2\,(3x)\,(-4y) + 2\,(-4y)\,(5z) + 2\,(3x)\,(5z)$$
$$= (3x - 4y + 5z)^2$$
$$\therefore \sqrt{9x^2-24xy+30xz-40yz+25z^2+16y^2}$$
$$= \sqrt{(3x-4y+5z)^2}$$
$$= |3x - 4y + 5z|$$

(iii) Square root of $1+\frac{1}{x^6}+\frac{2}{x^3}$

$$\text{Now } \sqrt{1+\frac{2}{x^3}+\frac{1}{x^6}} = \sqrt{(1)^2+2\left(\frac{1}{x^3}\right)+\left(\frac{1}{x^3}\right)^2}$$
$$\therefore \sqrt{1+\frac{1}{x^6}+\frac{2}{x^3}} = \sqrt{\left(1+\frac{1}{x^3}\right)^2}$$
$$= \left|\left(1+\frac{1}{x^3}\right)\right|$$

(iv) Square Root of
$(4x^2 - 9x + 2)\,(7x^2 - 13x - 2)\,(28x^2 - 3x - 1)$

$$4x^2 - 9x + 2 = 4x^2 - 8x - x + 2$$
$$= 4x\,(x-2) - 1\,(x-2)$$
$$= (4x-1)\,(x-2)$$
$$7x^2 - 13x - 2 = 7x^2 - 14x + x - 2$$
$$= 7x\,(x-2) + 1\,(x-2)$$
$$= (7x+1)\,(x-2)$$
$$28x^2 - 3x - 1 = 28x^2 - 7x + 4x - 1$$
$$= 7x\,(4x-1) + 1\,(4x-1)$$
$$= (7x+1)\,(4x-1)$$
$$\therefore \sqrt{(4x^2-9x+2)\,(7x^2-13x-2)\,(28x^2-3x-1)}$$
$$= \sqrt{(4x-1)\,(x-2)\,(7x+1)(x-2)\,(7x+1)\,(4x-1)}$$
$$= \sqrt{(4x-1)^2(x-2)^2(7x+1)^2}$$
$$= |(4x-1)\,(x-2)\,(7x+1)|$$

(v) Square root of
$$\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$$

$$2x^2+\frac{17}{6}x+1 = \frac{12x^2+17x+6}{6}$$
$$= \frac{12x^2+9x+8x+6}{6}$$
$$= \frac{3x\,(4x+3)+2\,(4x+3)}{6}$$
$$= \frac{(4x+3)\,(3x+2)}{6}$$
$$\frac{3}{2}x^2+4x+2 = \frac{3x^2+8x+4}{2}$$
$$= \frac{3x^2+6x+2x+4}{2}$$
$$= \frac{3x\,(x+2)+2\,(x+2)}{2}$$
$$= \frac{(x+2)\,(3x+2)}{2}$$
$$\frac{4}{3}x^2+\frac{11}{3}x+2 = \frac{4x^2+11x+6}{3}$$
$$= \frac{4x^2+8x+3x+6}{3}$$
$$= \frac{4x\,(x+2)+3\,(x+2)}{3}$$
$$= \frac{(x+2)\,(4x+3)}{3}$$
$$\therefore \sqrt{\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)}$$
$$= \sqrt{\frac{(4x+3)\,(3x+2)}{6}\,\frac{(x+2)\,(3x+2)}{2}\,\frac{(x+2)\,(4x+3)}{3}}$$
$$= \sqrt{\frac{(4x+3)^2\,(x+2)^2\,(3x+2)^2}{36}}$$
$$= \frac{1}{6}\,|(4x+3)\,(x+2)\,(3x+2)|$$

FINDING THE SQUARE ROOT OF POLYNOMIAL BY DIVISION METHOD

Worked Examples

3.21. **Find the square root of $64x^4 - 16x^3 + 17x^2 - 2x + 1$**

Sol :

$$\begin{array}{r|l}
 & 8x^2 - x + 1 \\ \hline
8x^2 & 64x^4 - 16x^3 + 17x^2 - 2x + 1 \\
 & 64x^4 \qquad (-) \\ \hline
16x^2 - x & -16x^3 + 17x^2 \\
 & -16x^3 + x^2 \qquad (-) \\ \hline
16x^2 - 2x + 1 & 16x^2 - 2x + 1 \\
 & 16x^2 - 2x + 1 \ (-) \\ \hline
 & 0
\end{array}$$

Therefore, $\sqrt{64x^4 - 16x^3 + 17x^2 - 2x + 1}$

$= |8x^2 - x + 1|$

3.22. **Find the square root of the expression**

$$4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2}$$

Sol :

$$\begin{array}{r|l}
 & 2\frac{x}{y} + 5 - 3\frac{y}{x} \\ \hline
2\frac{x}{y} & 4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & \frac{4x^2}{y^2} \qquad (-) \\ \hline
4\frac{x}{y} + 5 & 20\frac{x}{y} + 13 \\
 & 20\frac{x}{y} + 25 \qquad (-) \\ \hline
4\frac{x}{y} + 10 - \frac{3y}{x} & -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \ (-) \\ \hline
 & 0
\end{array}$$

Hence, $\sqrt{4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2}}$

$= \left|2\frac{x}{y} + 5 - 3\frac{y}{x}\right|$

3.23. **If $9x^4 + 12x^3 + 28x^2 + ax + b$ is a perfect square, find the values of a and b.**

Sol :

$$\begin{array}{r|l}
 & 3x^2 + 2x + 4 \\ \hline
3x^2 & 9x^4 + 12x^3 + 28x^2 + ax + b \\
 & 9x^4 \qquad (-) \\ \hline
6x^2 + 2x & 12x^3 + 28x^2 \\
 & 12x^3 + 4x^2 \qquad (-) \\ \hline
6x^2 + 4x + 4 & 24x^2 + ax + b \\
 & 24x^2 + 16x + 16 \ (-) \\ \hline
 & 0
\end{array}$$

Because the given polynomial is a perfect square

$a - 16 = 0, b - 16 = 0$

Therefore, $a = 16, b = 16$.

Exercise 3.8

1. Find the square root of the following polynomials by division method

(i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

(ii) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

(iii) $16x^4 + 8x^2 + 1$

(iv) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

Sol :

(i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

$$\begin{array}{r|l}
 & x^2 - 6x + 3 \\ \hline
x^2 & x^4 - 12x^3 + 42x^2 - 36x + 9 \\
 & x^4 \qquad (-) \\ \hline
2x^2 - 6x & -12x^3 + 42x^2 \\
 & -12x^3 + 36x^2 \qquad (-) \\ \hline
2x^2 - 12x + 3 & 6x^2 - 36x + 9 \\
 & 6x^2 - 36x + 9 \ (-) \\ \hline
 & 0
\end{array}$$

$\therefore \quad \sqrt{x^4 - 12x^3 + 42x^2 - 36x + 9} = |x^2 - 6x + 3|$

(ii) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

$$\begin{array}{r|ll}
 & 2x^2 - 7x - 3 & \\
\hline
2x^2 & 4x^4 - 28x^3 + 37x^2 + 42x + 9 & \\
 & 4x^4 & (-) \\
\hline
4x^2 - 7x & -28x^3 + 37x^2 & \\
 & -28x^3 + 49x^2 & (-) \\
\hline
4x^2 - 14x - 3 & -12x^2 + 42x + 9 & \\
 & -12x^2 + 42x + 9 & (-) \\
\hline
 & 0 &
\end{array}$$

$\therefore \sqrt{4x^4 - 28x^3 + 37x^2 + 42x + 9} = |2x^2 - 7x - 3|$

(iii) $16x^4 + 8x^2 + 1$

$$\begin{array}{r|ll}
 & 4x^2 + 1 & \\
\hline
4x^2 & 16x^4 + 8x^2 + 1 & \\
 & 16x^4 & (-) \\
\hline
8x^2 + 1 & 8x^2 + 1 & \\
 & 8x^2 + 1 & (-) \\
\hline
 & 0 &
\end{array}$$

$\therefore \sqrt{16x^4 + 8x^2 + 1} = |4x^2 + 1|$

(iv) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

$$\begin{array}{r|ll}
 & 11x^2 - 9x - 12 & \\
\hline
11x^2 & 121x^4 - 198x^3 - 183x^2 + 216x + 144 & \\
 & 121x^4 & (-) \\
\hline
22x^2 - 9x & -198x^3 - 183x^2 & \\
 & -198x^3 + 81x^2 & (-) \\
\hline
22x^2 - 18x - 12 & -264x^2 + 216x + 144 & \\
 & -264x^2 + 216x + 144 & (-) \\
\hline
 & 0 &
\end{array}$$

$\therefore \sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144} = |11x^2 - 9x - 12|$

2. Find the square root of the expression $\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}$

Sol :

$$\begin{array}{r|ll}
 & \frac{x}{y} - 5 + \frac{y}{x} & \\
\hline
\frac{x}{y} & \frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2} & \\
 & \frac{4x^2}{y^2} & (-) \\
\hline
2\frac{x}{y} - 5 & -10\frac{x}{y} + 27 & \\
 & -10\frac{x}{y} + 25 & (-) \\
\hline
2\frac{x}{y} - 10 + \frac{y}{x} & 2 - 10\frac{y}{x} + \frac{y^2}{x^2} & \\
 & 2 - 10\frac{y}{x} + \frac{y^2}{x^2} & (-) \\
\hline
 & 0 &
\end{array}$$

$\therefore \sqrt{\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2}} = \left|\frac{x}{y} - 5 + \frac{y}{x}\right|$

3. Find the values of a and b if the following polynomials are perfect squares.

(i) $4x^4 - 12x^3 + 37x^2 + bx + a$

(ii) $ax^4 + bx^3 + 361x^2 + 220x + 100$

Sol :

(i)

$$\begin{array}{r|ll}
 & 2x^2 - 3x + 7 & \\
\hline
2x^2 & 4x^4 - 12x^3 + 37x^2 + bx + a & \\
 & 4x^4 & (-) \\
\hline
4x^2 - 3x & -12x^3 + 37x^2 & \\
 & -12x^3 + 9x^2 & (-) \\
\hline
4x^2 - 6x + 7 & 28x^2 + bx + a & \\
 & 28x^2 - 42x + 49 & (-) \\
\hline
 & 0 &
\end{array}$$

Since, the given polynomial is a perfect square

$b + 42 = 0$, $a - 49 = 0$

$\therefore$ $a = 49$, $b = -42$.

(ii) Let us write $ax^4 + bx^3 + 361x^2 + 220x + 100$ in the reverse order as

$100 + 220x + 361x^2 + bx^3 + ax^4$

Now, finding square root

$$
\begin{array}{r|ll}
 & 10 + 11x + 12x^2 & \\
\hline
10 & 100 + 220x + 361x^2 + bx^3 + ax^4 & \\
 & 100 & (-) \\
\hline
20 + 11x & 220x + 361x^2 & \\
 & 220x + 121x^2 & (-) \\
\hline
20 + 22x + 12x^2 & 240x^2 + bx^3 + ax^4 & \\
 & 240x^2 + 264x^3 + 144x^4 & (-) \\
\hline
 & (b - 264)\,x^3 + (a - 144)x^4 &
\end{array}
$$

Since, the given polynomial is a perfect square,

$b - 264 = 0,\ a - 144 = 0$

$\therefore\ a = 144,\ b = 264$

4. Find the values of m and n if the following expressions are perfect squares.

(i) $\frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n$

(ii) $x^4 - 8x^3 + mx^2 + nx + 16$

Sol :

(i)

$$
\begin{array}{r|ll}
 & \frac{1}{x^2} - \frac{3}{x} + 2 & \\
\hline
\frac{1}{x^2} & \frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n & \\
 & \frac{1}{x^4} & (-) \\
\hline
\frac{2}{x^2} - \frac{3}{x} & -\frac{6}{x^3} + \frac{13}{x^2} & \\
 & -\frac{6}{x^3} + \frac{9}{x^2} & (-) \\
\hline
\frac{2}{x^2} - \frac{6}{x} + 2 & \frac{4}{x^2} + \frac{m}{x} + n & \\
 & \frac{4}{x^2} - \frac{12}{x} + 4 & (-) \\
\hline
 & (m + 12)\frac{1}{x} + (n - 4) &
\end{array}
$$

Since, the polynomial is a perfect square

$m + 12 = 0 \quad n - 4 = 0$

$\therefore\ m = -12 \quad n = 4$

(ii)

$$
\begin{array}{r|ll}
 & x^2 - 4x + \left(\frac{m-16}{2}\right) & \\
\hline
x^2 & x^4 - 8x^3 + mx^2 + nx + 16 & \\
 & x^4 & (-) \\
\hline
2x^2 - 4x & -8x^3 + mx^2 & \\
 & -8x^3 + 16x^2 & (-) \\
\hline
2x^2 - 8x + \left(\frac{m-16}{2}\right) & (m - 16)x^2 + nx + 16 & \\
 & (m - 16)x^2 - 4(m - 16)x + \left(\frac{m-16}{2}\right) & (-) \\
\hline
 & [n + 4\,(m - 16)]x + 16 - \left(\frac{m-16}{2}\right)^2 &
\end{array}
$$

Since the polynomial is a perfect square,

$n + 4\,(m - 16) = 0$

and $16 - \frac{(m-16)^2}{4} = 0$

$\therefore\ 64 - (m - 16)^2 = 0$

$(m - 16)^2 = 64$

$m - 16 = 8$

$m = 8 + 16 = 24$

$m = 24$

$\therefore\ n + 4\,(24 - 16) = 0$

$n + 4\,(8) = 0$

$n + 32 = 0$

$n = -32$

$\therefore\ m = 24,\ n = -32$

QUADRATIC EQUATIONS

Key Points

- An expression of degree 2 is called a Quadratic expression which is expressed as $p(x) = ax^2 + bx + c$, $a \neq 0$ and a, b, c are real numbers.
- The values of 'x' such that the expression $ax^2 + bx + c$ becomes zero are called roots of quadratic equation $ax^2 + bx + c = 0$.
- If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$
- x^2 – (sum of the roots) x + Product of the roots = 0 is the general form of the quadratic equation when the roots are given.
- $ax^2 + bx + c = 0$ can equivalently be expressed as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Worked Examples

3.24. Find the zeroes of the quadratic expression $x^2 + 8x + 12$.

Sol :

Let $p(x) = x^2 + 8x + 12 = (x + 2)(x + 6)$

$p(-2) = 4 - 16 + 12 = 0$

$p(-6) = 36 - 48 + 12 = 0$

Therefore,

– 2 and – 6 are zeros of $p(x) = x^2 + 8x + 12$.

3.25. Write down the quadratic equation in general form for which sum and product of the roots are given below.

(i) 9, 14 **(ii)** $-\frac{7}{2}, \frac{5}{2}$

(iii) $-\frac{3}{5}, -\frac{1}{2}$

Sol :

(i) General form of the quadratic equation when the roots are given is

x^2 – (sum of the roots) x + product of the roots = 0

$$x^2 - 9x + 14 = 0$$

(ii) $x^2 - \left(-\frac{7}{2}\right)x + \frac{5}{2} = 0 \Rightarrow 2x^2 + 7x + 5 = 0$

(iii) $$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{10x^2 + 6x - 5}{10} = 0$$

Therefore, $10x^2 + 6x - 5 = 0$

3.26. Find the sum and product of the roots for each of the following quadratic equations:

(i) $x^2 + 8x - 65 = 0$ **(ii)** $2x^2 + 5x + 7 = 0$

(iii) $kx^2 - k^2x - 2k^3 = 0$

Sol :

(i) $x^2 + 8x - 65 = 0$

$\alpha + \beta = -8;$

$\alpha\beta = -65$

(ii) $2x^2 + 5x + 7 = 0$

$\alpha + \beta = -\frac{5}{2};$

$\alpha\beta = \frac{7}{2}$

(iii) $kx^2 - k^2x - 2k^3 = 0$

$a = k,\ b = -k^2\ \ c = -2k^3$

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-k^2)}{k} = k$$

and $\alpha\beta = \frac{c}{a}$

$$= \frac{-2k^3}{k} = -2k^2$$

Exercise 3.9

1. Determine the quadratic equations, whose sum and product of roots are

(i) – 9, 20 **(ii)** $\frac{5}{3}, 4$

(iii) $\frac{-3}{2}, -1$ **(iv)** $-(2 - a)^2, (a + 5)^2$

Sol :

(i) Given sum of the roots = – 9
product of the roots = 20
General form of the Quadratic equation is
x^2 – (sum of the Roots) x + product of the Roots = 0

$x^2 + 9x + 20 = 0$

(ii) Given sum of the roots = 5/3
Product of the roots = 4
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$\therefore$ $x^2 - 5/3\,x + 4 = 0$
Simplifying $3x^2 - 5x + 12 = 0$

(iii) Given sum of the roots = – 3/2
product of the roots = – 1
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$x^2 + 3/2\,x - 1 = 0$
$2x^2 + 3x - 2 = 0$

(iv) Given sum of the roots = $-(2 - a)^2$
product of the roots = $(a + 5)^2$
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$\therefore$ $x^2 + (2 - a)^2 x + (a + 5)^2 = 0$

2. Find the sum and product of the roots for each of the following quadratic equations

(i) $x^2 + 3x - 28 = 0$ **(ii)** $x^2 + 3x = 0$

(iii) $3 + \frac{1}{a} = \frac{10}{a^2}$ **(iv)** $3y^2 - y - 4 = 0$

Sol :

(i) $x^2 + 3x - 28 = 0$
Comparing with $ax^2 + bx + c = 0$
a = 1, b = 3, c = – 28

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{3}{1} = -3$

product of the roots $= \frac{c}{a} = -\frac{28}{1} = -28$

(ii) $x^2 + 3x = 0$
Comparing with $ax^2 + bx + c = 0$
a = 1, b = 3, c = 0

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{3}{1} = -3$

Product of the roots $= \frac{c}{a} = \frac{0}{1} = 0$

(iii) $3 + \frac{1}{a} = \frac{10}{a^2}$

$3 + \frac{1}{a} - \frac{10}{a^2} = 0$

$3a^2 + a - 10 = 0$ is a quadratic equation in 'a'
A = 3, B = 1, C = – 10

$\therefore$ Sum of the roots $= -\frac{B}{A} = -\frac{1}{3}$

product of the roots $= \frac{C}{A} = -\frac{10}{3}$

(iv) $3y^2 - y - 4 = 0$ is a quadratic equation in 'y'
Comparing with $ax^2 + bx + c = 0$
$\therefore$ a = 3, b = – 1, c = – 4

Sum of the roots $= -\frac{b}{a} = -\frac{(-1)}{3} = \frac{1}{3}$

product of the roots $= \frac{c}{a} = -\frac{4}{3}$

SOLVING A QUADRATIC EQUATION

Key Points

- For solving a quadratic equation, we are using different methods, namely
  - (i) Factorization method
  - (ii) Completing the square method and
  - (iii) Formula method.
- Formula for finding roots of a quadratic equation $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked Examples

3.27. Solve $2x^2 - 2\sqrt{6}x + 3 = 0$

Sol :

$$2x^2-2\sqrt{6}x+3 = 2x^2-\sqrt{6}x-\sqrt{6}x+3$$

(by spliting the middle term)

Now, equating the factors to zero we get,

$$(\sqrt{2}x-\sqrt{3})(\sqrt{2}x-\sqrt{3}) = 0$$

$$\sqrt{2}x-\sqrt{3} = 0 \text{ or } \sqrt{2}x-\sqrt{3} = 0$$

$$\sqrt{2}x = \sqrt{3} \text{ or } \sqrt{2}x = \sqrt{3}$$

Therefore the solution is $x = \frac{\sqrt{3}}{\sqrt{2}}$.

3.28. Solve $2m^2 + 19m + 30 = 0$

Sol :

$$\begin{aligned} 2m^2+19m+30 &= 2m^2+4m+15m+30 \\ &= 2m(m+2)+15(m+2) \\ &= (m+2)(2m+15) \end{aligned}$$

Now, equating the factors to zero, we get,

$$m+2 = 0$$

$$\Rightarrow m = -2 \text{ or } 2m+15 = 0$$

We get, $m = \frac{-15}{2}$

Therefore the roots are $-2, \frac{-15}{2}$.

Some equations which are not quadratic can be solved by reducing them to quadratic equations by suitable substitutions. Such examples are illustrated below.

3.29. Solve $x^4 - 13x^2 + 42 = 0$

Sol :

Let $x^2 = a$.

Then, $(x^2)^2 - 13x^2 + 42 = a^2 - 13a + 42$

$= (a-7)(a-6)$

Given, $(a-7)(a-6) = 0 \Rightarrow a = 7$ or 6.

Since $a = x^2$, $x^2 = 7 \Rightarrow x = \pm\sqrt{7}$ or

$x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

Therefore the roots are $x = \pm\sqrt{7}, \pm\sqrt{6}$

3.30. Solve $\frac{x}{x-1}+\frac{x-1}{x} = 2\frac{1}{2}$

Sol :

Let $y = \frac{x}{x-1}$ then $\frac{1}{y} = \frac{x-1}{x}$.

Therefore, $\frac{x}{x-1}+\frac{x-1}{x} = 2\frac{1}{2}$

becomes $y+\frac{1}{y} = \frac{5}{2}$

$\Rightarrow 2y^2 - 5y + 2 = 0 \Rightarrow y = \frac{1}{2}, 2$

$\frac{x}{x-1} = \frac{1}{2} \Rightarrow 2x = x-1 \Rightarrow x = -1$

$\frac{x}{x-1} = 2 \Rightarrow x = 2x-2 \Rightarrow x = 2$

Therefore, the roots are $x = -1, 2$.

Exercise 3.10

1. Solve the following quadratic equations by factorization method

(i) $4x^2 - 7x - 2 = 0$ **(ii)** $3(p^2-6) = p(p+5)$

(iii) $\sqrt{a(a-7)} = 3\sqrt{2}$ **(iv)** $\sqrt{2}x^2+7x+5\sqrt{2} = 0$

(v) $2x^2 - x + \frac{1}{8} = 0$

Sol :

(i)

$$\begin{aligned} 4x^2-7x-2 &= 0 \\ 4x^2-8x+x-2 &= 0 \\ 4x(x-2)+1(x-2) &= 0 \\ (x-2)(4x+1) &= 0 \end{aligned}$$

$x-2 = 0,\quad 4x+1 = 0$

$x = 2,\quad 4x = -1$

$x = -\frac{1}{4}$

$\therefore$ Solution : $x = -\frac{1}{4}, 2$

(ii)

$$\begin{aligned} 3(p^2-6) &= p(p+5) \\ 3p^2-18 &= p^2+5p \\ 3p^2-18-p^2-5p &= 0 \\ 2p^2-5p-18 &= 0 \\ 2p^2+4p-9p-18 &= 0 \\ 2p(p+2)-9(p+2) &= 0 \\ (2p-9)(p+2) &= 0 \end{aligned}$$

$2p-9 = 0,\quad p+2 = 0$

$2p = 9,\quad p = -2$

$\therefore$ Solution $p = -2, 9/2$.

(iii) $\sqrt{a(a-7)} = 3\sqrt{2}$

Squaring on both sides

$$a(a-7) = 9(2)$$
$$a^2 - 7a - 18 = 0$$
$$a^2 - 9a + 2a - 18 = 0$$
$$a(a-9) + 2(a-9) = 0$$
$$(a-9)(a+2) = 0$$

Solution is a = 9, – 2.

(iv) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

$$\sqrt{2}x^2 + 2x + 5x + 5\sqrt{2} = 0$$
$$\sqrt{2}x^2 + \sqrt{2}\,\sqrt{2}x + 5x + 5\sqrt{2} = 0$$
$$\sqrt{2}x(x+\sqrt{2}) + 5(x+\sqrt{2}) = 0$$
$$(x+\sqrt{2})(\sqrt{2}x+5) = 0$$

$x+\sqrt{2} = 0,$ $x = -\sqrt{2}$ | $\sqrt{2}x + 5 = 0$, $\sqrt{2}x = -5$, $x = -\frac{5}{\sqrt{2}}$

$\therefore$ Solution is $x = -\sqrt{2}, -\frac{5}{\sqrt{2}}$

(v) $2x^2 - x + \frac{1}{8} = 0$

$$16x^2 - 8x + 1 = 0$$
$$16x^2 - 4x - 4x + 1 = 0$$
$$4x(4x-1) - 1(4x-1) = 0$$
$$(4x-1)(4x-1) = 0$$
$$4x - 1 = 0, \quad 4x - 1 = 0$$
$$x = \frac{1}{4}, \quad x = \frac{1}{4}$$

$\therefore$ Solution is $x = \frac{1}{4}$ (twice)

2. The number of volleyball games that must be scheduled in a league with n teams is given by $G(n) = \frac{n^2-n}{2}$ where each team plays with every other team exactly once. A league schedules 15 games. How many teams are in the league?

Sol :

Given $G(n) = \frac{n^2-n}{2}$

No. of league schedules = 15

$$\therefore \frac{n^2-n}{2} = 15$$
$$n^2 - n = 30$$
$$n^2 - n - 30 = 0$$
$$(n-6)(n+5) = 0$$
$$n = 6, -5$$

Number of teams can't be negative

$\therefore n = 6.$

SOLUTION OF QUADRATIC EQUATION BY COMPLETING THE SQUARE METHOD

Worked Examples

3.31. Solve $x^2 - 3x - 2 = 0$

Sol :

$$x^2 - 3x - 2 = 0$$
$$x^2 - 3x = 2 \text{ (Shifting the Constant to RHS)}$$
$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 2 + \left(\frac{3}{2}\right)^2$$

(Add $\left[\frac{1}{2}(co-efficient\ of\ x)\right]^2$ to both sides)

$$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4}$$

(writing the LHS as complete square)

$$x - \frac{3}{2} = \pm\frac{\sqrt{17}}{2}$$

(Taking the square root on both sides)

$$x = \frac{3}{2} + \frac{\sqrt{17}}{2} \text{ or}$$
$$x = \frac{3}{2} - \frac{\sqrt{17}}{2} \text{ (Solve for x)}$$

Therefore, $x = \frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}$

3.32. Solve $2x^2 - x - 1 = 0$

Sol :

$$2x^2 - x - 1 = 0$$
$$x^2 - \frac{x}{2} - \frac{1}{2} = 0$$

($\div$ 2 make co-efficient of x^2 as 1)

$$x^2 - \frac{x}{2} = \frac{1}{2}$$

$$x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 = \frac{1}{2} + \left(\frac{1}{4}\right)^2$$

$$\left(x - \frac{1}{4}\right)^2 = \frac{9}{16} = \left(\frac{3}{4}\right)^2$$

$$x - \frac{1}{4} = \pm\frac{3}{4} \Rightarrow x = 1, -\frac{1}{2}$$

3.33. Solve $x^2 + 2x - 2 = 0$ by formula method

Sol :

Compare $x^2 + 2x - 2 = 0$ with the standard form $ax^2 + bx + c = 0$ to get

a, b, c $\Rightarrow$ a = 1, b = 2, c = – 2

Put a, b and c in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$$

Therefore, $x = -1+\sqrt{3}, -1-\sqrt{3}$

3.34. Solve $2x^2 - 3x - 3 = 0$ by formula method.

Sol :

Compare $2x^2 - 3x - 3 = 0$ with the standard form $ax^2 + bx + c = 0$ to get

a, b, c $\Rightarrow$ a = 2, b = – 3, c = – 3

Put a, b and c in the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{33}}{4}$$

Therefore, $x = \dfrac{3+\sqrt{33}}{4}, x = \dfrac{3-\sqrt{33}}{4}$

3.35. Solve $3p^2 + 2\sqrt{5}p - 5 = 0$ by formula method.

Sol :

Compare $3p^2 + 2\sqrt{5}p - 5 = 0$ with the standard form $ax^2 + bx + c = 0$

$$a = 3, b = 2\sqrt{5}, c = -5.$$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$p = \frac{-2\sqrt{5} \pm \sqrt{(2\sqrt{5})^2 - 4(3)(-5)}}{2(3)}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{80}}{6} = \frac{-\sqrt{5} \pm 2\sqrt{5}}{3}$$

Therefore, $x = \dfrac{\sqrt{5}}{3}, -\sqrt{5}$

3.36. Solve $pqx^2 - (p+q)^2 x + (p+q)^2 = 0$

Sol :

Compare the coefficients of the given equation with the standard form

$ax^2 + bx + c = 0$ to get

$$a = pq, b = -(p+q)^2, c = (p+q)^2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting the values of a, b and c in the formula we get,

$$x = \frac{-[-(p+q)^2] \pm \sqrt{[-(p+q)^2]^2 - 4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^4 - 4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2[(p+q)^2 - 4pq]}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2(p^2+q^2+2pq-4pq)}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2(p-q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm (p+q)(p-q)}{2pq}$$

$$= \frac{(p+q)\{(p+q) \pm (p-q)\}}{2pq}$$

$$x = \frac{p+q}{2pq} \times 2p \text{ (or) } \frac{p+q}{2pq} \times 2q \Rightarrow$$

$$x = \frac{p+q}{q}, \frac{p+q}{p}$$

Exercise 3.11

1. Solve the following quadratic equations by completing the square method

(i) $9x^2 - 12x + 4 = 0$

(ii) $\frac{5x+7}{x-1} = 3x + 2$

Sol :

(i) $9x^2 - 12x + 4 = 0$

$9x^2 - 12x = -4$ [Dividing throughout by '9']

$$x^2 - \frac{4}{3}x = -\frac{4}{9}$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = -\frac{4}{9} + \frac{4}{9}$$

$$\left[\because Adding \left[\frac{co\text{-}efficient\ of\ x}{2}\right]^2\right] \text{ on both sides.}$$

$$\left(x - \frac{2}{3}\right)^2 = 0$$

$$x = \frac{2}{3} \text{ (twice)}$$

Solution $x = \frac{2}{3}, \frac{2}{3}$

(ii) $\frac{5x+7}{x-1} = 3x + 2$

$$5x + 7 = (3x + 2)(x - 1)$$
$$5x + 7 = 3x^2 - 3x + 2x - 2$$
$$3x^2 - 6x - 9 = 0$$

Dividing by 3

$$x^2 - 2x - 3 = 0$$
$$x^2 - 2x = 3$$
$$x^2 - 2x + 1 = 3 + 1$$
$$(x - 1)^2 = 4$$
$$\therefore x - 1 = \pm 2$$

$x - 1 = 2$, $\quad x - 1 = -2$

$x = 2 + 1$ $\quad x = -2 + 1$

$x = 3$ $\quad x = -1$

$\therefore$ Solution $x = -1, 3$

2. Solve the following quadratic equations by formula method

(i) $2x^2 - 5x + 2 = 0$

(ii) $\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$

(iii) $3y^2 - 20y - 23 = 0$

(iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

Sol :

(i) $2x^2 - 5x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 2, b = -5, c = 2$

Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{5 \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4}$$

$$= \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$$= \frac{5+3}{4}, \frac{5-3}{4}$$

$$= \frac{8}{4}, \frac{2}{4}$$

$\therefore$ Solution $x = 2, \frac{1}{2}$

(ii) $\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$

$a = \sqrt{2}, b = -6, c = 3\sqrt{2}$

$$\therefore f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{(-6)^2 - 4(\sqrt{2})(3\sqrt{2})}}{2\sqrt{2}}$$

$$= \frac{6 \pm \sqrt{36 - 24}}{2\sqrt{2}}$$

$$= \frac{6 \pm \sqrt{12}}{2\sqrt{2}} = \frac{6 \pm 2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{6 + 2\sqrt{3}}{2\sqrt{2}}, \frac{6 - 2\sqrt{3}}{2\sqrt{2}}$$

$\therefore$ Solution $f = \frac{3+\sqrt{3}}{\sqrt{2}}, \frac{3-\sqrt{3}}{\sqrt{2}}$

(iii) $3y^2 - 20y - 23 = 0$

$a = 3, b = -20, c = -23$

$\therefore$ Solution $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{20 \pm \sqrt{(-20)^2 - 4(3)(-23)}}{2(3)}$$

$$= \frac{20 \pm \sqrt{400+276}}{6}$$

$$= \frac{20 \pm \sqrt{676}}{6}$$

$$= \frac{20 \pm 26}{6}$$

$$= \frac{20+26}{6}, \frac{20-26}{6}$$

$$\therefore y = \frac{46}{6}, -\frac{6}{6}$$

$$= \frac{23}{3}, -1$$

(iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

$A = 36, B = -12a, C = a^2 - b^2$

$$\therefore y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{12a \pm \sqrt{(-12a)^2 - 4(36)(a^2-b^2)}}{2(36)}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144(a^2-b^2)}}{72}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144a^2 + 144b^2}}{72}$$

$$= \frac{12a \pm 12b}{72}$$

$$= \frac{12(a+b)}{72}, \frac{12(a-b)}{72}$$

$$\therefore \text{Solution } y = \frac{a+b}{6}, \frac{a-b}{6}$$

3. A ball rolls down a slope and travels a distance $d = t^2 - 0.75t$ feet in t seconds. Find the time when the distance travelled by the ball is 11.25 feet.

Sol :

Given distance $d = t^2 - 0.75t$ and

$t^2 - 0.75t = 11.25$

$\therefore t^2 - 0.75t - 11.25 = 0$

$a = 1, b = -0.75, c = -11.25$

Using the formula.

$$t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{0.75 \pm \sqrt{(-0.75)^2 - 4(1)(-11.25)}}{2(1)}$$

$$= \frac{0.75 \pm \sqrt{0.5625 + 45}}{2}$$

$$= \frac{0.75 \pm \sqrt{45.5625}}{2}$$

$$= \frac{0.75 \pm 6.75}{2}$$

$$= \frac{0.75+6.75}{2}, \frac{0.75-6.75}{2}$$

$$= \frac{7.5}{2}, -\frac{6}{2}$$

$$= 3.75, -3$$

$\therefore t = 3.75$ seconds.

[$\because$ time cannot be negative $t \neq -3$]

WORD PROBLEMS RELATED TO DAY-TO-DAY LIFE ACTIVITIES

Worked Examples

3.37. The product of Kumaran's age (in years) two years ago and his age four years from now is one more than twice his present age. What is his present age?

Sol :

Let the present age of Kumaran be x years.

Two years ago, his age $= (x-2)$ years.

Four years from now, his age $= (x+4)$ years.

Given, $(x-2)(x+4) = 1 + 2x$

$x^2 + 2x - 8 = 1 + 2x$

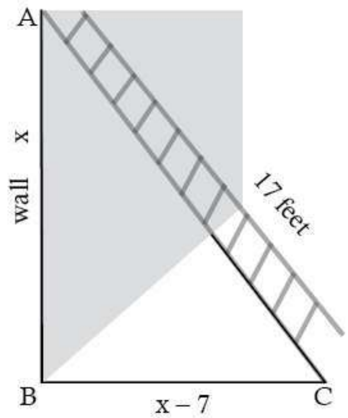
$\Rightarrow (x-3)(x+3) = 0$ then $x = \pm 3$

Therefore, x = 3 (Rejecting – 3 as age cannot be negative)

Kumaran's present age is 3 years.

3.38. A ladder 17 feet long is leaning against a wall. If the ladder, vertical wall and the floor from the bottom of the wall to the ladder form a right triangle, find the height of the wall where the tip of the ladder meets if the distance between bottom of the wall to bottom of the ladder is 7 feet less than the height of the wall?

Sol :



Let the height of the wall AB = x feet.
As per the given data BC = (x – 7) feet.
In ΔABC, AC = 17 feet, BC = (x – 7) feet.
By Pythagoras theorem, $AC^2 = AB^2 + BC^2$

$$(17)^2 = x^2 + (x-7)^2;$$
$$289 = x^2 + x^2 - 14x + 49$$
$$x^2 - 7x - 120 = 0$$
$$\Rightarrow (x-15)(x+8) = 0 \Rightarrow x = 15 \text{ (or) } -8$$

Therefore, height of the wall AB = 15 feet
(Rejecting – 8 as height cannot be negative)

3.39. A flock of swans contained x^2 members. As the clouds gathered, 10x went to a lake and one-eigth of the members flew away to a garden. The remaining three pairs played about in the water. How many swans were there in total?

Sol : As given there are x^2 swans.

As per the given data

$$x^2 - 10x - \frac{1}{8}x^2 = 6 \Rightarrow 7x^2 - 80x - 48 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{80 \pm \sqrt{6400 - 4(7)(-48)}}{14}$$
$$= \frac{80 \pm 88}{14}$$

Therefore, $x = 12, -\frac{4}{7}$.

Here, $x = -\frac{4}{7}$ is not possible as the number of swans cannot be negative.
Hence, x = 12. Therefore total number of swans is $x^2 = 144$.

3.40. A passenger train takes 1 hr more than an express train to travel a distance of 240 km from Chennai to Virudhachalam. The speed of passenger train is less than that of an express train by 20 km per hour. Find the average speed of both the trains.

Sol :
Let the average speed of passenger train be x km/hr.
Then the average speed of express train will be (x + 20) km/hr
Time taken by the passenger train to cover distance of 240 km = $\frac{240}{x}\,hr$
Time taken by express train to cover distance of 240 km = $\frac{240}{x+20}\,hr$

Given, $\frac{240}{x} = \frac{240}{x+20} + 1$

$$240\left[\frac{1}{x} - \frac{1}{x+20}\right] = 1 \Rightarrow$$
$$240\left[\frac{x+20-x}{x(x+20)}\right] = 1 \Rightarrow 4800 = (x^2 + 20x)$$
$$x^2 + 20x - 4800 = 0$$
$$\Rightarrow (x+80)(x-60) = 0 \Rightarrow x = -80 \text{ or } 60.$$

Therefore x = 60 (Rejecting – 80 as speed cannot be negative)
Average speed of the passenger train is 60 km/hr
Average speed of the express train is 80 km/hr.

Exercise 3.12

1. If the difference between a number and its reciprocal is $\frac{24}{5}$, find the number.

Sol :
Let the number be 'x'
$\therefore$ its reciprocal is $\frac{1}{x}$

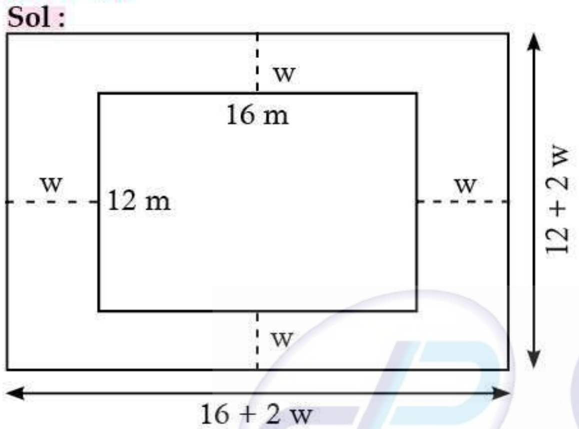
$$\text{Given } x - \frac{1}{x} = \frac{24}{5}$$
$$\frac{x^2-1}{x} = \frac{24}{5}$$
$$5x^2 - 5 = 24x$$
$$5x^2 - 24x - 5 = 0$$
$$5x^2 - 25x + x - 5 = 0$$

$$5x(x-5)+1(x-5) = 0$$
$$(x-5)(5x+1) = 0$$
$$x = 5, -\frac{1}{5}$$

$\therefore$ The number is '5' or '– 1/5'

2. A garden measuring 12 m by 16 m is to have a pedestrian pathway that is '*w*' meters wide installed all the way around so that it increases the total area to 285 m². What is the width of the pathway?

Sol :



Given length of the Garden = 16 m

Breadth of the Garden = 12 m

Width of the path = '*w*' m

$\therefore$ From the figure

$$(16+2w)(12+2w) = 285 \text{ (Given)}$$
$$192+24w+32w+4w^2-285 = 0$$
$$4w^2+56w-93 = 0$$

Using Quadratic formula

$$w = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-56 \pm \sqrt{(56)^2-4(4)(-93)}}{2(4)}$$
$$= \frac{-56 \pm \sqrt{4624}}{8}$$
$$= \frac{-56 \pm 68}{8}$$
$$= \frac{-56+68}{8}, \frac{-56-68}{8}$$

[negative value is not possible]

$$= \frac{12}{8} = 1.5$$

$\therefore$ The width of the pathway is 1.5 m

3. A bus covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more it would have taken 30 minutes less for the journey. Find the original speed of the journey.

Sol :

Let the original speed of the bus be 'x' km/h then increased speed (x + 15) km/h

$$\text{Usual time} = \frac{90}{x} \text{ hours,}$$
$$\text{New time} = \frac{90}{x+15} \text{ hours}$$
$$\left[\because \text{ time} = \frac{Distance}{Speed}\right.$$
$$\therefore \text{ Given } \frac{90}{x} - \frac{90}{x+15} = \frac{1}{2}$$
$$\left[\because \text{ 30 min} = \frac{1}{2} \text{ hr}\right]$$
$$90\left[\frac{1}{x} - \frac{1}{x+15}\right] = \frac{1}{2}$$
$$90\left[\frac{x+15-x}{x(x+15)}\right] = \frac{1}{2}$$
$$\frac{90 \times 15}{x^2+15x} = \frac{1}{2}$$
$$x^2+15x = 2700$$
$$x^2+15x-2700 = 0$$

Factorizing it $(x+60)(x-45) = 0$

$$x = -60 \text{ and } x = 45$$
$$x = -60 \text{ is not possible}$$

$\therefore$ Original speed of the Bus is 45 km/hour.

4. A girl is twice as old as her sister. Five years hence, the product of their ages (in years) will be 375. Find their present ages.

Sol :

Let the present age of her sister be 'x' years.

$\therefore$ Girl's age is 2x

Five years hence their ages will be (x + 5) and (2x + 5)

Given $(x+5)(2x+5) = 375$

Simplifying $2x^2+15x-350 = 0$

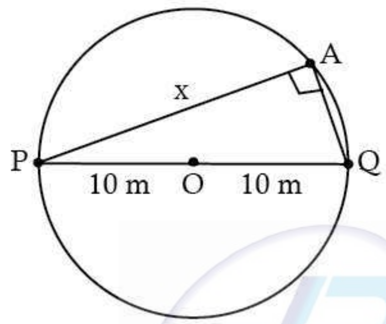
$$2x^2-20x+35x-350 = 0$$
$$2x(x-10)+35(x-10) = 0$$
$$(x-10)(2x+35) = 0$$
$$x = 10, -\frac{35}{2}$$
$$\therefore x = 10$$

$x = \frac{-35}{2}$ is not possible.

$\therefore$ Present age of a girl is 20 years
Present age of her sister is 10 years.

5. A pole has to be erected at a point on the boundary of a circular ground of diameter 20 m in such a way that the difference of its distance from two diametrically opposite fixed gates P and Q on the boundary is 4 m. Is it possible to do so? If answer is yes at what distance from the two gates should the pole be erected?

Sol :



Let 'A' be the point where the pole is erected.
P and Q are two points which are diametrically opposite.

$$\text{Diameter } PQ = 20 \text{ m}$$
$$\therefore PO = OQ = 10 \text{ m}$$
$$\text{Given } PA - QA = 4$$

Let 'PA' be 'x'

$$\therefore QA = x - 4$$

From the figure, $PA^2 + QA^2 = PQ^2$

[ΔPAQ is a right triangle as $\angle A = 90^\circ$ angle in a semi-circle is a right angle.]

$$x^2 + (x-4)^2 = (20)^2$$
$$x^2 + x^2 - 8x + 16 - 400 = 0$$
$$2x^2 - 8x - 384 = 0$$

Divided by 2

$$x^2 - 4x - 192 = 0$$
$$(x-16)(x+12) = 0$$

$x = 16, -12$ [$x = -12$ is not possible]

$$\therefore x = 16$$
$$\therefore \text{If } PA = 16, QA = 16 - 4 = 12$$
$$\therefore PA = 16 \text{ m}, QA = 12 \text{ m}$$

and it is very much possible from the given data.

6. From a group of black bees $2x^2$, square root of half of the group went to a tree. Again eight-ninth of the bees went to the same tree. The remaining two got caught up in a fragrant lotus. How many bees were there in total?

Sol :

Given, total no. of Bees $= 2x^2$

Square root of half of the group $= \left(\frac{2x^2}{2}\right)^{1/2}$

eight-ninth of the bees $= \frac{8}{9}(2x^2)$

From the given data,

$$2x^2 - x - \frac{8}{9}(2x^2) = 2$$
$$18x^2 - 9x - 16x^2 = 18$$
$$2x^2 - 9x - 18 = 0$$
$$2x^2 - 12x + 3x - 18 = 0$$
$$2x(x-6) + 3(x-6) = 0$$
$$(x-6)(2x+3) = 0$$

$x = 6, x = -\frac{3}{2}$ which is not possible.

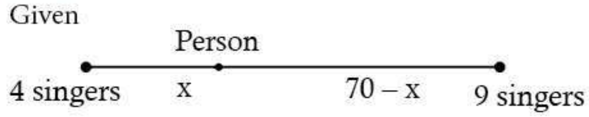
$$\therefore x = 6.$$

$\therefore$ No. of Bees in the group $= 2x^2 = 2(6)^2 = 72$

7. Music is been played in two opposite galleries with certain group of people. In the first gallery a group of 4 singers were singing and in the second gallery 9 singers were singing. The two galleries are separated by the distance of 70 m. Where should a person stand for hearing the same intensity of the singers voice? (Hint: The ratio of the sound intensity is equal to the square of the ratio of their corresponding distance).

Sol :

Given



No. of singers in the first gallery = 4
No. of singers in the second gallery = 9
Let the person is standing 'x' m apart from the first gallery.
$\therefore$ (70 – x) m from the second gallery.
From the given data

$$\frac{4k}{9k} = \frac{x^2}{(70-x)^2}$$

[$\because$ k is the constant]

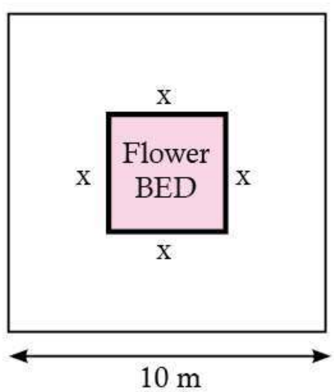
$$4(70-x)^2 = 9x^2$$
$$4(4900-140x+x^2) = 9x^2$$
$$19600-560x+4x^2 = 9x^2$$
$$5x^2+560x-19600 = 0$$

Divided by 5

$$x^2+112x-3920 = 0$$
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-112\pm\sqrt{(112)^2-4(1)(-3920)}}{2(1)}$$
$$= \frac{-112\pm\sqrt{28224}}{2}$$
$$= \frac{-112\pm168}{2}$$
$$= \frac{-112+168}{2}, \frac{-112-168}{2}$$
$$= \frac{56}{2}, -\frac{280}{2}$$
$$= 28, -140 \quad (-140 \text{ is not possible}).$$
$$\therefore x = 28 \text{ m}$$
$$70-x = 70-28 = 42 \text{ m.}$$

8. There is a square field whose side is 10 m. A square flower bed is prepared in its centre leaving a gravel path all round the flower bed. The total cost of laying the flower bed and gravelling the path at ₹ 3 and ₹ 4 per square metre respectively is ₹ 364. Find the width of the gravel path.

Sol :



Given side of the field $= 10$ m

$\therefore$ Area of the field $= (10)^2$

$= 100$ sq. m

Let the side of the flower bed be 'x' m

Area of the flower bed $= x^2$

$\therefore$ Area of the path $= 100 - x^2$

Given total cost for laying the flower bed and gravelling the path at ₹ 3 and ₹ 4 respectively is ₹ 364.

$$\therefore 3x^2 + 4(100-x^2) = 364$$
$$3x^2+400-4x^2 = 364$$
$$x^2-36 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$

[$\because$ $x = -6$ is not possible,]

$$\therefore x = 6$$

$$\therefore \text{Width of the gravel path} = \frac{10-6}{2}$$
$$= \frac{4}{2} = 2 \text{ m.}$$

9. Two women together took 100 eggs to a market, one had more than the other. Both sold them for the same sum of money. The first then said to the second: "If I had your eggs, I would have earned ₹ 15", to which the second replied: "If I had your eggs, I would have earned ₹ $6\frac{2}{3}$". How many eggs did each had in the beginning?

Sol :

Let the no. of eggs with the first woman be 'x'

$\therefore$ Number of eggs with the second woman "(100 – x)"

Now, let 'a' be the cost of an egg sold by the first woman and 'b' be the cost of an egg sold by the second woman.

Given, $x > 100 - x$

and $\quad a(100-x) = 15, \qquad bx = 6\frac{2}{3}$

$$a = \frac{15}{100-x}, \qquad b = \frac{20}{3x}$$

Given that eggs been sold for same sum of money

$$\therefore ax = (100-x)b$$
$$\Rightarrow x = \frac{100b}{a+b}$$

Substituting the values of 'a' and 'b'

$$x = \frac{100\left(\frac{20}{3x}\right)}{\frac{15}{100-x}+\frac{20}{3x}}$$
$$x = \frac{2000(100-x)}{2000+25x}$$
$$25(x+80)x = 25(80)(100-x)$$
$$x^2+80x = 8000-80x$$

$$x^2 + 160x - 8000 = 0$$

Factorizing $\Rightarrow$ $(x + 200)(x - 40) = 0$

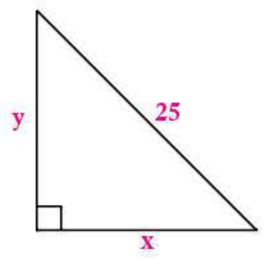
$x = 40$, [$x = -200$ is rejected]

then $100 - x = 100 - 40 = 60$

$\therefore$ The two women had 40 and 60 eggs in the beginning.

10. The hypotenuse of a right angled triangle is 25 cm and its perimeter 56 cm. Find the length of the smallest side.

Sol :



Given hypotenuse = 25 cm

Let the other sides be 'x' and 'y'

$$\therefore x^2 + y^2 = 25^2 \quad \dots (1)$$

Perimeter = 56 cm

$$x + y + 25 = 56$$
$$y = 56 - 25 - x$$
$$\Rightarrow y = 31 - x$$

Substituting in ... (1)

$$x^2 + (31 - x)^2 = 625$$
$$x^2 + 961 - 62x + x^2 - 625 = 0$$
$$2x^2 - 62x + 336 = 0$$

[$\because$ Dividing by 2]

$$x^2 - 31x + 168 = 0$$
$$x^2 - 24x - 7x + 168 = 0$$
$$x(x - 24) - 7(x - 24) = 0$$
$$(x - 24)(x - 7) = 0$$
$$x = 7, 24$$

$\therefore$ The smallest side of the triangle is 7 cm.

NATURE OF ROOTS OF A QUADRATIC EQUATION

Key Points

- Formula for finding roots of a quadratic equation $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The term $b^2 - 4ac$ is called "Discriminant" and it is denoted by Δ.
- If $\Delta > 0$ (i.e., $b^2 - 4ac > 0$), then the roots are real and unequal.
- If $\Delta = 0$ (i.e., $b^2 - 4ac = 0$), then the roots are real and equal.
- If $\Delta < 0$ (i.e., $b^2 - 4ac < 0$), then the roots are unreal.

Worked Examples

3.41. Determine the nature of roots for the following quadratic equations

(i) $x^2 - x - 20 = 0$

(ii) $9x^2 - 24x + 16 = 0$

(iii) $2x^2 - 2x + 9 = 0$

Sol :

(i) $x^2 - x - 20 = 0$

Here, $a = 1, b = -1, c = -20$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-1)^2 - 4(1)(-20) = 81$

Here, $\Delta = 81 > 0$.

So, the equation will have real and unequal roots.

(ii) $9x^2 - 24x + 16 = 0$

Here, $a = 9, b = -24, c = 16$

Now, $\Delta = b^2 - 4ac$

$= (-24)^2 - 4(9)(16) = 0$

Here, $\Delta = 0$.

So, the equation will have real and equal roots.

(iii) $2x^2 - 2x + 9 = 0$

Here, $a = 2, b = -2, c = 9$

Now, $\Delta = b^2 - 4ac$

$= (-2)^2 - 4(2)(9) = -68$

Here, $\Delta = -68 < 0$.

So, the equation will have no real root.

3.42. (i) Find the values of 'k', for which the quadratic equation $kx^2 - (8k + 4) + 81 = 0$ has real and equal roots?

(ii) Find the values of 'k' such that quadratic equation $(k + 9)x^2 + (k + 1)x + 1 = 0$ has no real roots.

Sol :

(i) $kx^2 - (8k + 4) + 81 = 0$

Since the equation has real and equal roots $\Delta = 0$.

That is, $b^2 - 4ac = 0$

Here, $a = k$,

$b = -(8k + 4)$,

$c = 81$

That is, $[-(8k + 4)]^2 - 4(k)(81) = 0$

$64k^2 + 64k + 16 - 324k = 0$

$64k^2 - 260k + 16 = 0$

$\div 4 \Rightarrow 16k^2 - 65k + 4 = 0$

$(16k - 1)(k - 4) = 0$

$\Rightarrow k = \frac{1}{16}$ or $k = 4$

(ii) $(k + 9)x^2 + (k + 1)x + 1 = 0$

Since the equation has no real roots, $\Delta < 0$

That is, $b^2 - 4ac < 0$

Here, $a = k + 9$, $b = k + 1$, $c = 1$

Therefore, $(k + 1)^2 - 4(k + 9)(1) < 0$

$k^2 + 2k + 1 - 4k - 36 < 0$

$k^2 - 2k - 35 < 0$

$(k + 5)(k - 7) < 0$

Therefore, $-5 < k < 7$. {If $\alpha < \beta$ and if $(x-\alpha)(x-\beta) < 0$ then, $\alpha < x < \beta$}.

3.43. Prove that the equation $x^2(p^2 + q^2) + 2x(pr + qs) + r^2 + s^2 = 0$ has no real roots. If ps = qr then show that the roots are real and equal.

Sol : The given quadratic equation is

$x^2(p^2 + q^2) + 2x(pr + qs) + r^2 + s^2 = 0$

Here, $a = p^2 + q^2$, $b = 2(pr + qs)$, $c = r^2 + s^2$

Now, $\Delta = b^2 - 4ac$

$= [2(pr + qs)]^2 - 4(p^2 + q^2)(r^2 + s^2)$

$= 4[p^2r^2 + 2pqrs + q^2s^2 - p^2r^2 - p^2s^2 - q^2r^2 - q^2s^2]$

$= 4[-p^2s^2 + 2pqrs - q^2r^2]$

$= -4[(ps - qr)^2] < 0$ .....**(1)**

$\Delta = b^2 - 4ac < 0$, the roots are not real.

If ps = qr then

$\Delta = -4[ps - qr]^2$

$= -4[qr - qr]^2 = 0$ (using **(1)**).

Thus, $\Delta = 0$ if ps = qr and so the roots will be real and equal.

Thinking Corner

1. Fill up the empty box in each of the given expression so that the resulting quadratic polynomial becomes a perfect square.

i) $x^2 + 14x +$ _____

Ans : 49

ii) $x^2 - 24x +$ _____

Ans : 144

iii) $p^2 + 2pq +$ _____

Ans : q^2

Exercise 3.13

1. Determine the nature of the roots for the following quadratic equations

(i) $15x^2 + 11x + 2 = 0$

(ii) $x^2 - x - 1 = 0$

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

(iv) $9y^2 - 6\sqrt{2}y + 2 = 0$

(v) $9a^2b^2x^2 - 24abc\,dx + 16c^2d^2 = 0$, $a \neq 0$, $b \neq 0$

Sol :

(i) $15x^2 + 11x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 15$, $b = 11$, $c = 2$

Now, $b^2 - 4ac = (11)^2 - 4(15)(2)$

$= 121 - 120 = 1 > 0$

$\therefore$ Roots are real and unequal.

(ii) $x^2 - x - 1 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 1$, $b = -1$, $c = -1$

$b^2 - 4ac = (-1)^2 - 4(1)(-1)$

$= 1 + 4 = 5 > 0$

$\therefore$ Roots are real and unequal.

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

$a = \sqrt{2}$, $b = -3$, $c = 3\sqrt{2}$

$b^2 - 4ac = (-3)^2 - 4(\sqrt{2})(3\sqrt{2})$

$= 9 - 24 = -15 < 0$

$\therefore$ Roots are not real.

(iv) $9y^2 - 6\sqrt{2}y + 2 = 0$

$a = 9$, $b = -6\sqrt{2}$, $c = 2$

$$b^2 - 4ac = (-6\sqrt{2})^2 - 4(9)(2)$$
$$= 72 - 72 = 0$$
$\therefore$ Roots are real and equal.

(v) $9a^2b^2x^2 - 24\,abcd\,x + 16c^2d^2 = 0$

$A = 9a^2b^2,\ B = -24\,abcd,\ C = 16c^2d^2$

$$B^2 - 4AC = (-24\,abcd)^2 - 4(9a^2b^2)(16c^2d^2)$$
$$= 576\,a^2b^2c^2d^2 - 576\,a^2b^2c^2d^2$$
$$= 0$$
$\therefore$ Roots are real and equal.

2. Find the value(s) of 'k' for which the roots of the following equations are real and equal.

(i) $(5k-6)x^2 + 2kx + 1 = 0$

(ii) $kx^2 + (6k+2)x + 16 = 0$

Sol :

(i) $(5k-6)x^2 + 2kx + 1 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 5k - 6,$

$b = 2k,\ c = 1$

Given that roots are real and equal

$$b^2 - 4ac = 0$$
$$(2k)^2 - 4(5k-6)(1) = 0$$
$$4k^2 - 20k + 24 = 0$$

Dividing by 4

$$k^2 - 5k + 6 = 0$$
$$(k-3)(k-2) = 0$$
$$k = 2, 3$$

(ii) $kx^2 + (6k+2)x + 16 = 0$

$a = k,$

$b = 6k + 2,\ c = 16$

Roots are real and equal

$$b^2 - 4ac = 0$$
$$(6k+2)^2 - 4(k)(16) = 0$$
$$36k^2 + 24k + 4 - 64k = 0$$
$$36k^2 - 40k + 4 = 0$$

Dividing by 4

$$9k^2 - 10k + 1 = 0$$
$$9k^2 - 9k - k + 1 = 0$$
$$9k(k-1) - 1(k-1) = 0$$
$$(k-1)(9k-1) = 0$$
$$k - 1 = 0, \qquad 9k - 1 = 0$$
$$k = 1, \qquad k = \frac{1}{9}$$
$$k = 1, \frac{1}{9}$$

3. If the roots of $(a-b)x^2 + (b-c)x + (c-a) = 0$ are real and equal, then prove that b, a, c are in arithmetic progression.

Sol :

$$(a-b)x^2 + (b-c)x + (c-a) = 0$$

$A = a - b,$

$B = b - c,$

$C = c - a$

Given that roots are real and equal

$$B^2 - 4AC = 0$$
$$(b-c)^2 - 4(a-b)(c-a) = 0$$
$$b^2 - 2bc + c^2 - 4(ac - a^2 - bc + ab) = 0$$
$$b^2 - 2bc + c^2 - 4ac + 4a^2 + 4bc - 4ab = 0$$
$$4a^2 + b^2 + c^2 - 4ab - 2bc - 4ac = 0$$
$$(2a)^2 + (-b)^2 + (-c)^2 + 2(2a)(-b) + 2(-b)(-c) + 2(2a)(-c) = 0$$
$$(2a - b - c)^2 = 0$$
$$2a - b - c = 0$$
$$2a = b + c$$
$$a = \frac{b+c}{2}$$

$\therefore$ b, a and c are in arithmetic progression.

4. If a, b are real then show that the roots of the equation $(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$ are real and unequal.

Sol :

$(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$

Comparing with $Ax^2 + Bx + C = 0$

$A = a - b,\ B = -6(a+b),\ C = -9(a-b)$

$$B^2 - 4AC = [-6(a+b)]^2 - 4(a-b)(-9)(a-b)$$
$$= 36(a+b)^2 + 36(a-b)^2$$
$$= 36[(a+b)^2 + (a-b)^2]$$
$$= 36[a^2 + 2ab + b^2 + a^2 - 2ab + b^2]$$
$$= 36(2a^2 + 2b^2) = 36 \times 2(a^2 + b^2)$$
$$= 72(a^2 + b^2)$$

Given that a, b are real

$\therefore\ 72(a^2 + b^2) \geq 0$

$\therefore$ Roots are real and unequal.

5. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$ are real and equal prove that either $a = 0$ (or) $a^3 + b^3 + c^3 = 3abc$.

Sol :

$(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$

Comparing with $Ax^2 + Bx + C = 0$

$A = c^2 - ab,\ B = -2(a^2 - bc),\ C = b^2 - ac$

Given that the roots are real and equal

$\therefore\ B^2 - 4AC = 0$

$$[-2(a^2-bc)]^2 - 4(c^2-ab)(b^2-ac) = 0$$
$$4(a^2-bc)^2 - 4(c^2-ab)(b^2-ac) = 0$$
$$4[a^4 - 2a^2bc + b^2c^2 - c^2b^2 + ac^3 + ab^3 - a^2bc] = 0$$
$$a^4 - 3a^2bc + ac^3 + ab^3 = 0$$

$$a(a^3+b^3+c^3-3abc) = 0$$
$$\therefore\ a = 0 \text{ (or) } a^3+b^3+c^3-3abc = 0$$
$$a^3+b^3+c^3 = 3abc$$

Hence proved.

THE RELATION BETWEEN ROOTS OF THE QUADRATIC EQUATION AND CO-EFFICIENTS

Key Points

- If α and β are the roots of the equation $ax^2 + bx + c = 0$, then
  Sum of the roots $\alpha+\beta = -b/a$
  Product of the roots $\alpha\beta = c/a$

Some More Important Results

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$
$$\alpha^3+\beta^3 = (\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]$$
$$\alpha^3-\beta^3 = (\alpha-\beta)[(\alpha+\beta)^2 - \alpha\beta]$$
$$\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2$$
$$\alpha^4-\beta^4 = (\alpha^2+\beta^2)(\alpha+\beta)(\alpha-\beta)$$

Worked Examples

3.44. If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 17 find k.

Sol :

Comparing $x^2 - 13x + k = 0$

here, $a = 1, b = -13, c = k$

Let α, β be the roots of the equation

$$\alpha+\beta = \frac{-b}{a} = \frac{-(-13)}{1} = 13 \qquad \dots (1)$$

Also $\alpha - \beta = 17 \qquad \dots (2)$

(1) + (2) we get,

$$2\alpha = 30 \Rightarrow \alpha = 15$$

Therefore,

$$15 + \beta = 13 \text{ (from (1))}$$
$$\Rightarrow \beta = -2$$
$$\text{But, } \alpha\beta = \frac{c}{a} = \frac{k}{1}$$
$$\Rightarrow 15 \times (-2) = k \Rightarrow k = -30$$

3.45. If α and β are the roots of $x^2 + 7x + 10 = 0$ find the value of

(i) $(\alpha-\beta)$ **(ii)** $\alpha^2+\beta^2$

(iii) $\alpha^3-\beta^3$ **(iv)** $\alpha^4+\beta^4$

(v) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$ **(vi)** $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$

Sol :

$x^2 + 7x + 10 = 0$, here, $a = 1, b = 7, c = 10$

If α and β are roots of the equation then,

$$\alpha+\beta = \frac{-b}{a} = \frac{-7}{1} = -7;$$
$$\alpha\beta = \frac{c}{a} = \frac{10}{1} = 10$$

(i)
$$\alpha-\beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \sqrt{(-7)^2 - 4\times 10} = \sqrt{9} = 3$$

(ii)
$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (-7)^2 - 2\times 10 = 29$$

(iii) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$
$= (3)^3 + 3(10)(3) = 117$

(iv) $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$
$= 29^2 - 2\times(10)^2 = 641$
(since from **(ii)**, $\alpha^2+\beta^2 = 29$)

(v) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$
$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$
$= \frac{49-20}{10} = \frac{29}{10}$

(vi) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$
$= \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$
$= \frac{(-343)-3(10\times(-7))}{10}$
$= \frac{-343+210}{10} = \frac{-133}{10}$

3.46. If α, β are the roots of the equation $3x^2 + 7x - 2 = 0$, find the values of

(i) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ **(ii)** $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Sol :

$3x^2 + 7x - 2 = 0$
here, $a = 3, b = 7, c = -2$
Since, α, β are the roots of the equation

(i) $\alpha+\beta = \frac{-b}{a} = \frac{-7}{3}$;

$\alpha\beta = \frac{c}{a} = \frac{-2}{3}$

$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$
$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$
$= \frac{\left(\frac{-7}{3}\right)^2 - 2\left(\frac{-2}{3}\right)}{\frac{-2}{3}} = \frac{-61}{6}$

(ii) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$
$= \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$
$= \frac{\left(-\frac{7}{3}\right)^3 - 3\left(-\frac{2}{3}\right)\left(-\frac{7}{3}\right)}{-\frac{7}{3}} = \frac{67}{9}$

3.47. If α, β are the roots of the equation $2x^2 - x - 1 = 0$ then form the equation whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ **(ii)** $\alpha^2\beta, \beta^2\alpha$ **(iii)** $2\alpha+\beta, 2\beta+\alpha$

Sol :

$2x^2 - x - 1 = 0$, here, $a = 2$, $b = -1$, $c = -1$

$\alpha+\beta = \frac{-b}{a} = \frac{-(-1)}{2} = \frac{1}{2}$

$\alpha\beta = \frac{c}{a} = -\frac{1}{2}$

(i) Given roots are $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum of the roots $= \frac{1}{\alpha}+\frac{1}{\beta}$
$= \frac{\alpha+\beta}{\alpha\beta} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$

Product of the roots $= \frac{1}{\alpha}\times\frac{1}{\beta}$
$= \frac{1}{\alpha\beta} = \frac{1}{-\frac{1}{2}} = -2$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0
$x^2 - (-1)x - 2 = 0 \Rightarrow x^2 + x - 2 = 0$

(ii) Given roots are $\alpha^2\beta, \beta^2\alpha$

Sum of the roots $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha+\beta)$
$= -\frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4}$

Product of the roots $(\alpha^2\beta)\times(\beta^2\alpha) = \alpha^3\beta^3$
$= (\alpha\beta)^3 = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0

$$x^2 - \left(-\frac{1}{4}\right)x - \frac{1}{8} = 0 \Rightarrow 8x^2 + 2x - 1 = 0$$

(iii) $2\alpha+\beta, 2\beta+\alpha$

Sum of the roots $2\alpha+\beta+2\beta+\alpha = 3(\alpha+\beta)$

$$= 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

Product of the roots $= (2\alpha+\beta)(2\beta+\alpha)$

$$= 4\alpha\beta+2\alpha^2+2\beta^2+\alpha\beta$$
$$= 5\alpha\beta+2(\alpha^2+\beta^2)$$
$$= 5\alpha\beta+2\left[(\alpha+\beta)^2-2\alpha\beta\right]$$
$$= 5\left[-\frac{1}{2}\right]+2\left[\frac{1}{4}-2\times-\frac{1}{2}\right]$$
$$=0$$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0

$$x^2 - \frac{3}{2}x + 0 = 0 \Rightarrow 2x^2 - 3x = 0$$

Progress Check

1.

Quadratic equation	Roots of quadratic equation α and β	Co-efficients of x^2, x and constants	Sum of Roots $\alpha+\beta$	Product of roots $\alpha\beta$	$-\frac{b}{a}$	$\frac{c}{a}$	Conclusion
$4x^2 - 9x + 2 = 0$							
$\left(x-\frac{4}{5}\right)^2 = 0$							
$2x^2 - 15x - 27 = 0$							

Ans :

Quadratic equation	Roots of quadratic equation α and β	Co-efficients of x^2, x and constants	Sum of Roots $\alpha+\beta$	Product of roots $\alpha\beta$	$-\frac{b}{a}$	$\frac{c}{a}$	Conclusion
$4x^2 - 9x + 2 = 0$	2, 1/4	4, –9, 2	9/4	1/2	9/4	1/2	Roots are real and distinct
$\left(x-\frac{4}{5}\right)^2 = 0$	$\frac{4}{5}, \frac{4}{5}$	$25, -\frac{8}{5}, \frac{16}{25}$	$\frac{8}{5}$	$\frac{16}{25}$	$\frac{8}{5}$	$\frac{16}{25}$	Roots are real and equal
$2x^2 - 15x - 27 = 0$	– 3/2, 9	2, – 15, – 27	15/2	$-\frac{27}{2}$	15/2	– 27/2	Roots are real and distinct

Exercise 3.14

1. Write each of the following expression in terms of $\alpha+\beta$ and $\alpha\beta$.

(i) $\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$ **(ii)** $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$

(iii) $(3\alpha-1)(3\beta-1)$ **(iv)** $\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$

Sol :

(i) $$\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha} = \frac{\alpha^2+\beta^2}{3\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{3\alpha\beta}$$

(ii) $$\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha} = \frac{\beta+\alpha}{\alpha^2\beta^2} = \frac{\alpha+\beta}{(\alpha\beta)^2}$$

(iii) $$(3\alpha-1)(3\beta-1) = 9\alpha\beta-3\alpha-3\beta+1$$
$$= 9\alpha\beta-3(\alpha+\beta)+1$$

(iv) $$\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha} = \frac{\alpha(\alpha+3)+\beta(\beta+3)}{\alpha\beta}$$

$$= \frac{\alpha^2+3\alpha+\beta^2+3\beta}{\alpha\beta}$$

$$= \frac{(\alpha^2+\beta^2)+3(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta+3(\alpha+\beta)}{\alpha\beta}$$

2. The roots of the equation $2x^2-7x+5=0$ are α and β. Without solving for the roots, find

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ **(ii)** $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

(iii) $\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2}$

Sol :

Given equation $2x^2-7x+5=0$

Comparing with $ax^2+bx+c=0$

$a=2,$

$b=-7,$

$c=5$

Roots are α, β

Sum of the roots $= \alpha+\beta$

$$= -\frac{b}{a} = \frac{7}{2}$$

Product of the roots $= \alpha\beta$

$$= \frac{c}{a} = \frac{5}{2}$$

(i) $$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$$

$$= \frac{7/2}{5/2} = \frac{7}{5}$$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$\frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$= \frac{\left(\frac{7}{2}\right)^2-2\left(\frac{5}{2}\right)}{\frac{5}{2}}$$

$$= \frac{\frac{49}{4}-5}{\frac{5}{2}} = \left(\frac{49-20}{4}\right)\times\frac{2}{5}$$

$$= \frac{29}{4}\times\frac{2}{5} = \frac{29}{10}$$

(iii) $$\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2} = \frac{(\alpha+2)^2+(\beta+2)^2}{(\alpha+2)(\beta+2)}$$

$$= \frac{\alpha^2+4\alpha+4+\beta^2+4\beta+4}{\alpha\beta+2\alpha+2\beta+4}$$

$$= \frac{(\alpha^2+\beta^2)+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$$

$$= \frac{\left(\frac{7}{2}\right)^2-2\left(\frac{5}{2}\right)+4\left(\frac{7}{2}\right)+8}{\left(\frac{5}{2}\right)+2\left(\frac{7}{2}\right)+4}$$

$$= \frac{\frac{49}{4}+17}{\frac{5}{2}+11}$$

$$= \frac{117}{4}\times\frac{2}{27} = \frac{117}{54}$$

3. The roots of the equation $x^2+6x-4=0$ are α, β. Find the quadratic equation whose roots are

(i) α^2 **and** β^2 **(ii)** $\frac{2}{\alpha}$ **and** $\frac{2}{\beta}$

(iii) $\alpha^2\beta$ **and** $\beta^2\alpha$

Sol :

Given equation $x^2+6x-4=0$

Comparing with $ax^2+bx+c=0$

$a=1, b=6, c=-4$

α, β are the roots

Sum of the roots $= \alpha+\beta$

$$= -\frac{b}{a} = -\frac{6}{1} = -6$$

Product of the roots $= \alpha\beta$

$$= \frac{c}{a} = -\frac{4}{1} = -4$$

(i) Given Roots are α^2 and β^2

Sum of the roots $= \alpha^2+\beta^2$

$$= (\alpha+\beta)^2-2\alpha\beta$$

$$= (-6)^2-2(-4)$$

$$= 44$$

Product of the roots $= \alpha^2 \beta^2$

$$= (\alpha\beta)^2$$
$$= (-4)^2 = 16$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$\therefore \quad x^2 - 44x + 16 = 0$

(ii) Given roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$

Sum of the roots $= \frac{2}{\alpha} + \frac{2}{\beta}$

$$= \frac{2(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{2(-6)}{-4} = 3$$

Product of the roots $= \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$

$$= \frac{4}{\alpha\beta} = \frac{4}{-4} = -1$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$$x^2 - 3x - 1 = 0$$

(iii) Given roots are $\alpha^2\beta$ and $\beta^2\alpha$

Sum of the roots $= \alpha^2\beta + \beta^2\alpha$

$$= \alpha\beta(\alpha+\beta)$$
$$= (-4)(-6) = 24$$

Product of the roots $= (\alpha^2\beta)(\beta^2\alpha)$

$$= \alpha^3\beta^3 = (\alpha\beta)^3$$
$$= (-4)^3 = -64$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$$x^2 - 24x - 64 = 0$$

4. If α, β are the roots of $7x^2 + ax + 2 = 0$ and if $\beta - \alpha = \frac{-13}{7}$. Find the values of a.

Sol :

Given equation $7x^2 + ax + 2 = 0$

Comparing with $Ax^2 + Bx + C = 0$

$$A = 7, B = a, C = 2$$

α and β are the roots

Sum of the roots $\alpha + \beta = -\frac{B}{A} = -\frac{a}{7}$

Product of the roots $\alpha\beta = \frac{C}{A} = \frac{2}{7}$

and Given $\beta - \alpha = \frac{-13}{7}$

From the data, $(\alpha+\beta)^2 - (\beta-\alpha)^2 = 4\alpha\beta$

$$\left(\frac{-a}{7}\right)^2 - \left(\frac{-13}{7}\right)^2 = 4\left(\frac{2}{7}\right)$$

$$\frac{a^2}{49} - \frac{169}{49} = \frac{8}{7}$$

$$a^2 - 169 = \frac{8 \times 49}{7}$$

$$a^2 - 169 = 56$$
$$a^2 = 56 + 169$$
$$= 225$$
$$a = \pm 15$$

5. If one root of the equation $2y^2 - ay + 64 = 0$ is twice the other then find the values of a.

Sol :

Given equation $2y^2 - ay + 64 = 0$

$A = 2,$
$B = -a,$
$C = 64$

Given that one root is twice the other.

Let the roots be 'α' and '2α'

$\therefore$ Sum of the roots $\alpha + 2\alpha = -\frac{B}{A}$

$$3\alpha = \frac{a}{2}$$

$$\Rightarrow \alpha = \frac{a}{6}$$

Product of the roots $(\alpha)(2\alpha) = \frac{C}{A}$

$$2\alpha^2 = \frac{64}{2}$$
$$= 32$$
$$\alpha^2 = 16$$
$$\left(\frac{a}{6}\right)^2 = 16$$
$$\frac{a}{6} = \pm 4$$
$$\Rightarrow a = 24, -24$$

6. If one root of the equation $3x^2 + kx + 81 = 0$ (having real roots) is the square of the other then find k.

Sol :

Given equation $3x^2 + kx + 81 = 0$

$a = 3,$
$b = k,$
$c = 81$

Given that one root is square of the other.

$\therefore$ Let the root be 'α' and 'α^2'

Sum of the roots $\alpha + \alpha^2 = -b/a$

$$= \frac{-k}{3}$$

Product of the roots $(\alpha)(\alpha^2) = c/a$

$$\alpha^3 = \frac{81}{3}$$

$= 27$

$\therefore \alpha = 3$

Substituting in $\alpha + \alpha^2 = -k/3$

$\Rightarrow (3 + 9)\,3 = -k$

$k = -36$

QUADRATIC GRAPHS

Worked Examples

3.48. Discuss the nature of solutions of the following quadratic equations.

(i) $x^2 + x - 12 = 0$ (ii) $x^2 - 8x + 16 = 0$

(iii) $x^2 + 2x + 5 = 0$

Sol :

(i) $x^2 + x - 12 = 0$

Step 1: Prepare the table of values for the equation $y = x^2 + x - 12$

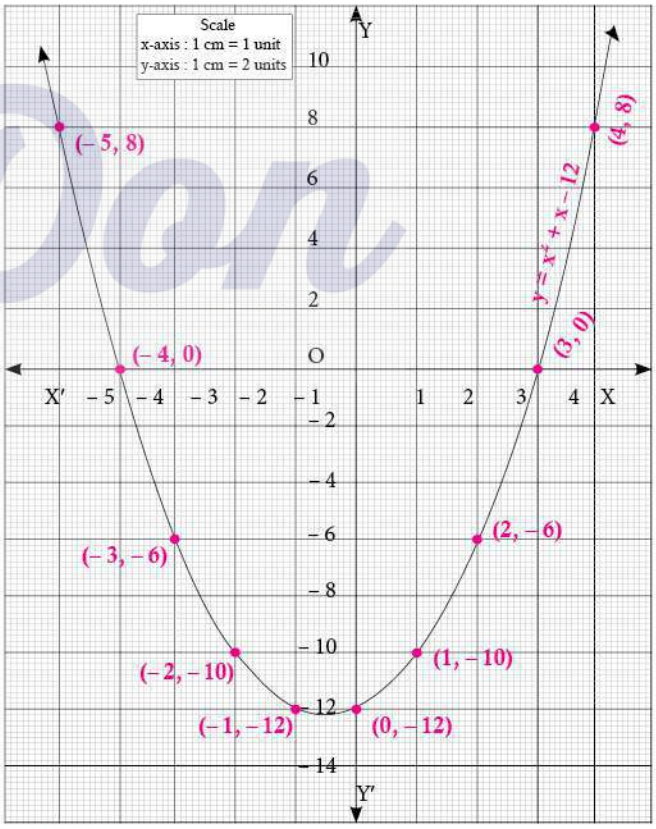
x	–5	–4	–3	–2	–1	0	1	2	3	4
y	8	0	–6	–10	–12	–12	–10	–6	0	8

Step 2: Plot the points for the above ordered pairs (x, y)

Step 3: Draw the parabola and mark the co-ordinates of the parabola which intersect with the X-axis.

Step 4: The roots of the equation are the x-coordinates of the intersecting points of the parabola with the X-axis (– 4, 0) and (3, 0) which are – 4 and 3.

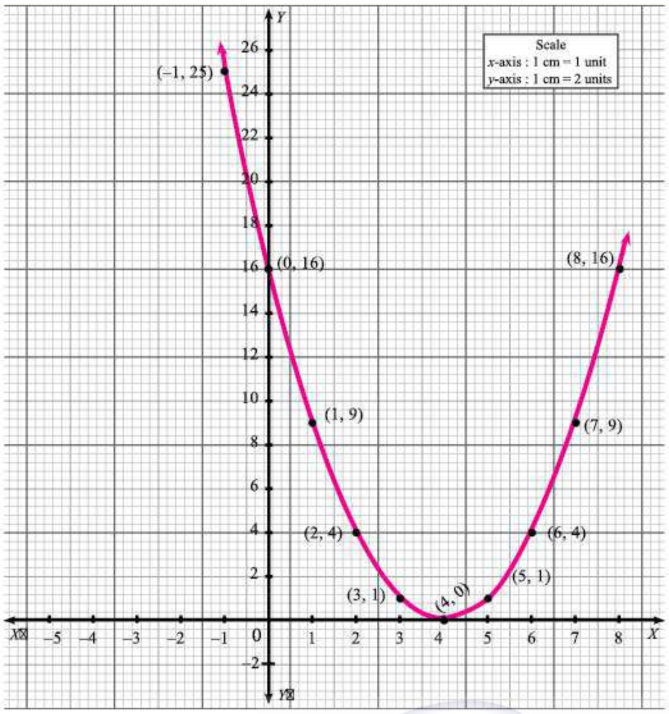
Since there are two points of intersection with the X-axis, the quadratic equation $x^2 + x - 12 = 0$ has real and unequal roots.



(ii) $x^2 - 8x + 16 = 0$

Step 1: Prepare the table of values for the equation $y = x^2 - 8x + 16$

x	–1	0	1	2	3	4	5	6	7	8
y	25	16	9	4	1	0	1	4	9	16



Step 2: Plot the points for the above ordered pairs (x, y)

Step 3: Draw the parabola and mark the co-ordinates of the parabola which intersect with the X-axis.

Step 4: The roots of the equation are the x-coordinates of the intersecting points of the parabola with the X-axis (4, 0) which is 4.

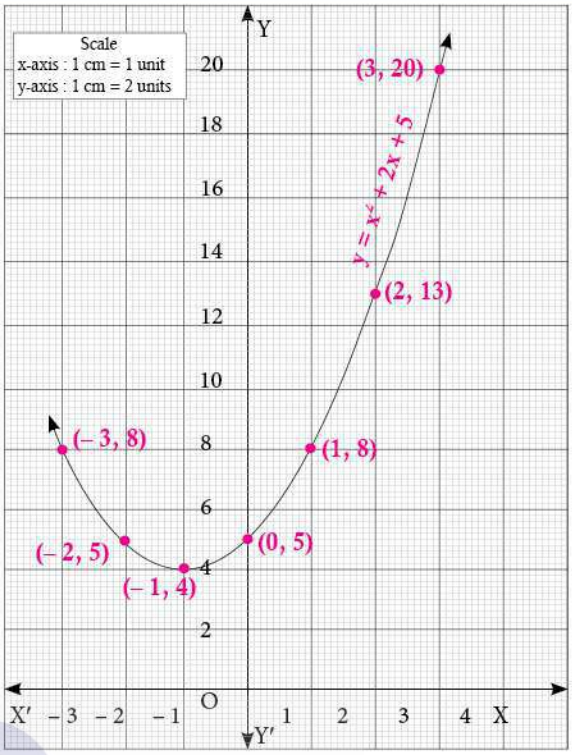
Since there is only one point of intersection with X-axis, the quadratic equation $x^2 - 8x + 16 = 0$ has real and equal roots.

(iii) $x^2 + 2x + 5 = 0$

Let $y = x^2 + 2x + 5$

Step 1: Prepare a table of values for

x	–3	–2	–1	0	1	2	3
y	8	5	4	5	8	13	20



Step 2: Plot the above ordered pairs (x, y) on the graph using suitable scale.

Step 3: Join the points by a free-hand smooth curve this smooth curve is the graph of $y = x^2 + 2x + 5$.

Step 4: The solutions of the given Quadratic equation are the x-coordinates of the intersecting points of the parabola with the X-axis.

Here the parabola doesn't intersect/touch the X-axis.

So, we conclude that there is no real root for the given quadratic equation.

3.49. Draw the graph of $y = 2x^2$ and hence solve $2x^2 - x - 6 = 0$.

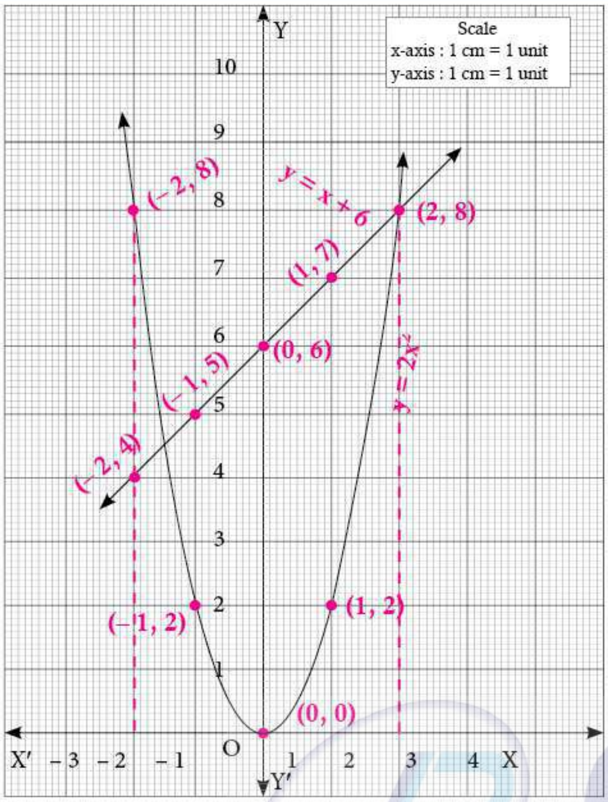
Sol :

Step 1: Draw the graph of $y = 2x^2$ by preparing the table of values as below

x	– 2	– 1	0	1	2
y	8	2	0	2	8

Step 2: To solve $2x^2 - x - 6 = 0$, subtract $2x^2 - x - 6 = 0$ from $y = 2x^2$

that is
$$\begin{aligned} y &= 2x^2 \\ 0 &= 2x^2 - x - 6 \quad (-) \\ \hline y &= x + 6 \end{aligned}$$



The equation $y = x + 6$ represents a straight line. Draw the graph of $y = x + 6$ by forming table of values as below

x	– 2	– 1	0	1	2
y	4	5	6	7	8

Step 3: Mark the points of intersection of the curve $y = 2x^2$ and $y = x + 6$. That is, $(-1.5, 4.5)$ and $(2, 8)$.

Step 4: The x-coordinates of the respective points forms the solution set $\{-1.5, 2\}$ for $2x^2 - x - 6 = 0$.

3.50. Draw the graph of $y = x^2 + 4x + 3$ and hence find the roots of $x^2 + x + 1 = 0$

Sol :

Step 1: Draw the graph of $y = x^2 + 4x + 3$ by preparing the table of values as below

x	– 4	– 3	– 2	– 1	0	1	2
y	3	0	– 1	0	3	8	15

Step 2: To solve $x^2 + x + 1 = 0$, subtract $x^2 + x + 1 = 0$ from $y = x^2 + 4x + 3$ that is,

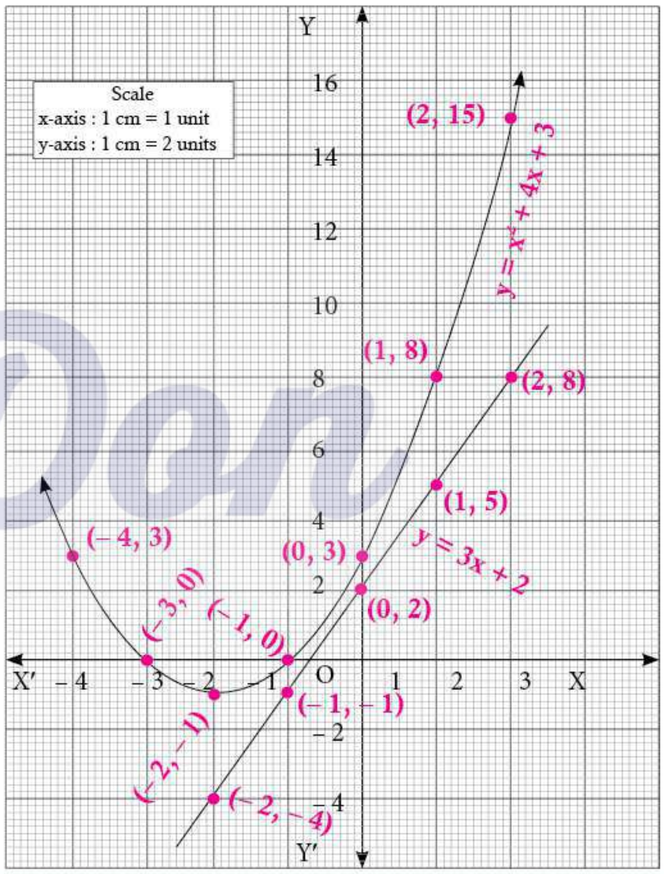
$$\begin{array}{rl} y = & x^2 + 4x + 3 \\ 0 = & x^2 + x + 1 \quad (-) \\ \hline y = & 3x + 2 \end{array}$$

The equation represent a straight line. Draw the graph of $y = 3x + 2$ forming the table of values as below

x	– 2	– 1	0	1	2
y	– 4	– 1	2	5	8

Step 3: Observe that the graph of $y = 3x + 2$ does not intersect/touch the graph of the parabola $y = x^2 + 4x + 3$.

Thus $x^2 + x + 1 = 0$ has no real roots.

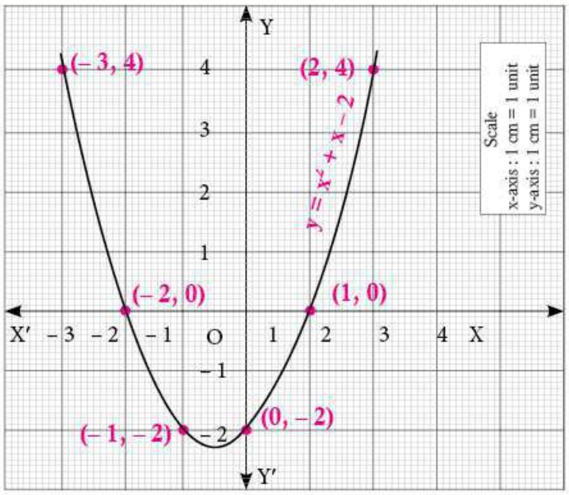


3.51. Draw the graph of $y = x^2 + x - 2$ and hence solve $x^2 + x - 2 = 0$.

Sol :

Step 1: Draw the graph $y = x^2 + x - 2$ by preparing the table of values as below

x	– 3	– 2	– 1	0	1	2
y	4	0	– 2	– 2	0	4



Step 2: To solve $x^2 + x - 2 = 0$ subtract $x^2 + x - 2 = 0$ from $y = x^2 + x - 2$.

$$\begin{array}{rcll} \text{that is } y & = & x^2 + x - 2 & \\ 0 & = & x^2 + x - 2 & (-) \\ \hline y & = & 0 & \\ \hline \end{array}$$

The equation y = 0 represents the X-axis.

Step 3: Mark the point of intersection of the curve $x^2 + x - 2$ with the X-axis. That is (– 2, 0) and (1, 0).

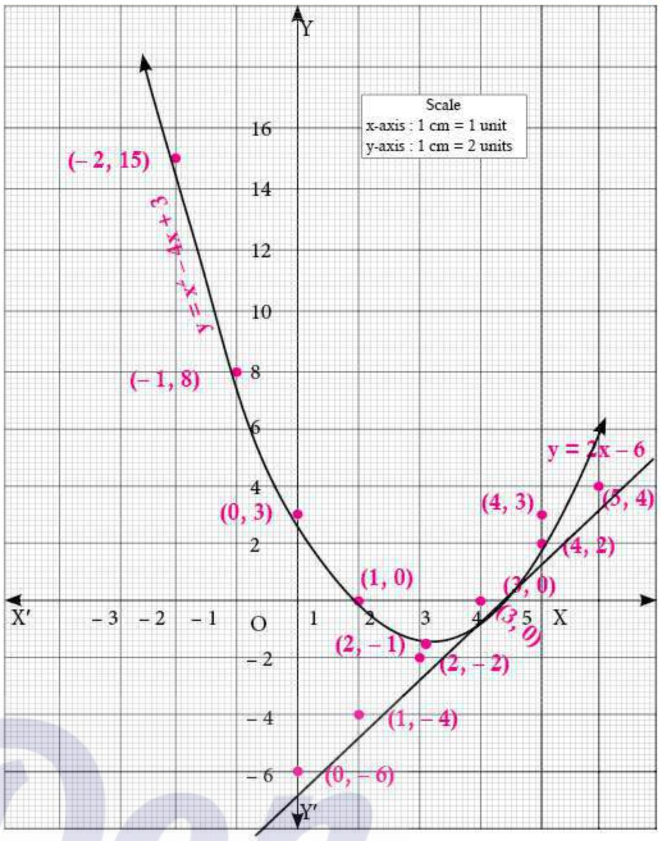
Step 4: The x co-ordinates of the respective points form the solution set {– 2, 1} for $x^2 + x - 2 = 0$

3.52. Draw the graph of $y = x^2 - 4x + 3$ and use it to solve $x^2 - 6x + 9 = 0$.

Sol :

Step 1: Draw the graph of $y = x^2 - 4x + 3$ by preparing the table of values as below

x	– 2	– 1	0	1	2	3	4
y	15	8	3	0	– 1	0	3



Step 2: To solve $x^2 - 6x + 9 = 0$, subtract $x^2 - 6x + 9 = 0$ from $y = x^2 - 4x + 3$

$$\begin{array}{rcll} \text{that is } y & = & x^2 - 4x + 3 & \\ 0 & = & x^2 - 6x + 9 & (-) \\ \hline y & = & 2x - 6 & \\ \hline \end{array}$$

The equation y = 2x – 6 represent a straight line. Draw the graph of y = 2x – 6 forming the table of values as below.

x	0	1	2	3	4	5
y	– 6	– 4	– 2	0	2	4

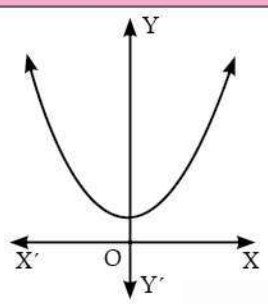
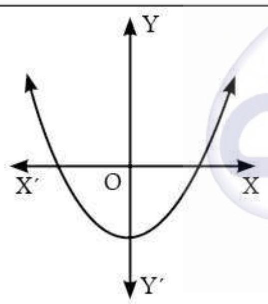
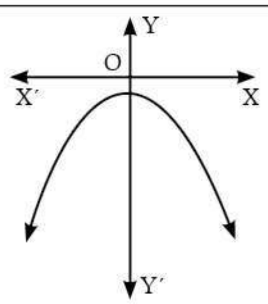
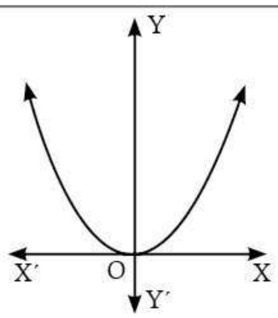
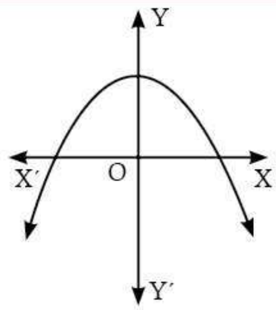
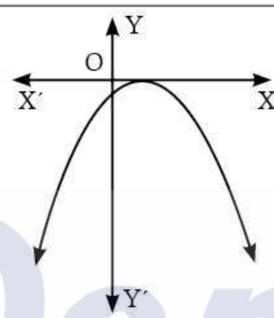
The line y = 2x – 6 intersects $y = x^2 - 4x + 3$ only at one point.

Step 3: Mark the point of intersection of the curve $y = x^2 - 4x + 3$ and y = 2x – 6, that is (3, 0).

Therefore, the x-coordinate 3 is the only solution for the equation $x^2 - 6x + 9 = 0$.

Progress Check

1. **Connect the graphs to its respective number of intersection with X-axis and to its corresponding nature of solutions which is given in the following table.**

	Graphs	Number of points of Inter-section with x-axis	Nature of solutions
1		2	Real and Equal roots
2		1	No Real Roots
3		2	No Real Roots
4		0	Real and Equal Roots
5		0	Real and unequal roots
6		1	Real and unequal Roots

Ans :

	No. of points	Nature of solutions
1	0	No Real roots
2	2	Real and unequal roots
3	0	No Real roots
4	1	Real and equal roots
5	2	Real and unequal roots
6	1	Real and equal roots

Exercise 3.15

1. **Graph the following quadratic equations and state their nature of solutions.**
   (i) $x^2 - 9x + 20 = 0$
   (ii) $x^2 - 4x + 4 = 0$
   (iii) $x^2 + x + 7 = 0$
   (iv) $x^2 - 9 = 0$
   (v) $x^2 - 6x + 9 = 0$
   (vi) $(2x - 3)(x + 2) = 0$

Sol :

(i) $x^2 - 9x + 20 = 0$

Let $y = x^2 - 9x + 20$

Table of values

x	–1	0	1	2	3	4	5	6
x^2	1	0	1	4	9	16	25	36
–9x	9	0	–9	–18	–27	–36	–45	–54
20	20	20	20	20	20	20	20	20
y	30	20	12	6	2	0	0	2

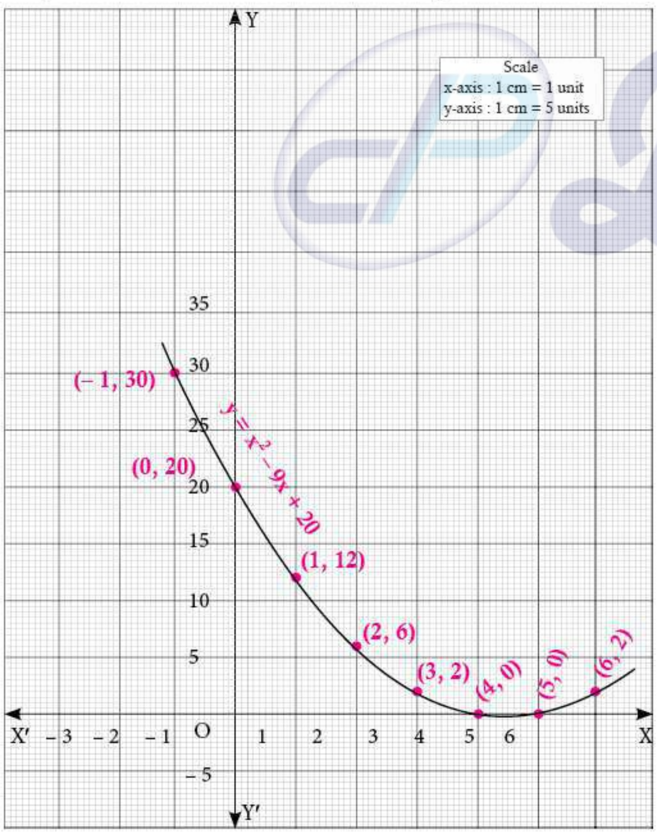
Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 - 9x + 20$$

The parabola intersects X-axis at (4, 0) and (5, 0).

Hence the solution is x = 4, 5.

Since, the parabola intersects X-axis at two different points, the roots are real and unequal.



(ii) Solve $x^2 - 4x + 4 = 0$

Let $y = x^2 - 4x + 4$

Table of values

x	– 2	– 1	0	1	2	3
x^2	4	1	0	1	4	9
– 4x	8	4	0	– 4	– 8	– 12
4	4	4	4	4	4	4
y	16	9	4	1	0	1

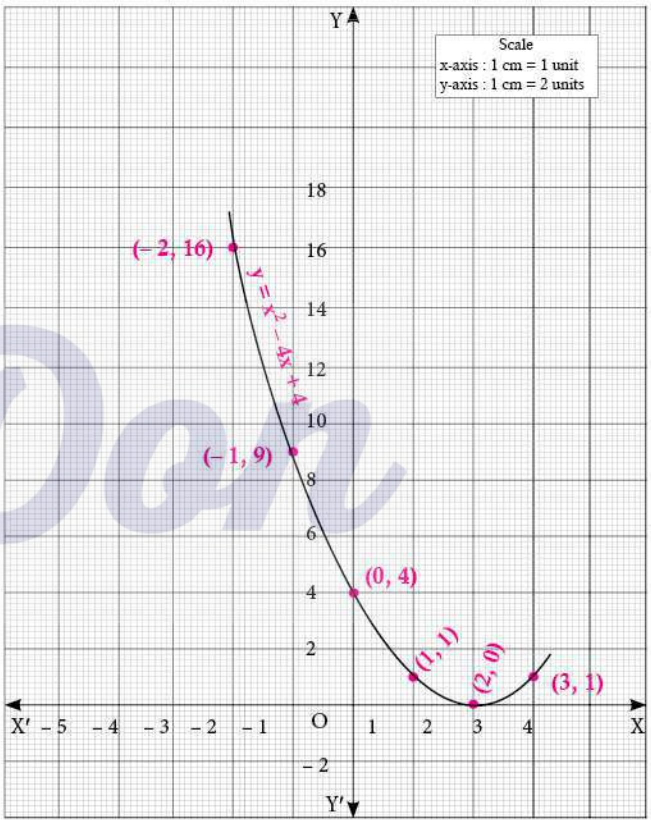
Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 - 4x + 4$$

The parabola intersects the X-axis at (2, 0).

Hence the solution is x = 2

Since, the parabola intersects X-axis at only one point, the roots are real and equal.

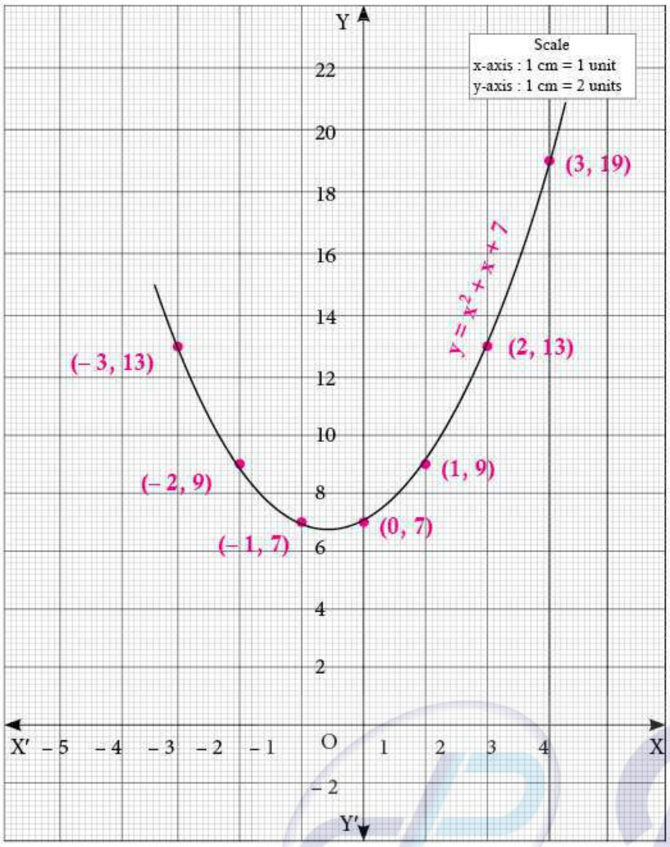


(iii) $x^2 + x + 7 = 0$

Let $y = x^2 + x + 7$

Table of values

x	– 3	– 2	– 1	0	1	2	3
x^2	9	4	1	0	1	4	9
7	7	7	7	7	7	7	7
y	13	9	7	7	9	13	19



Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 + x + 7$$

Here, the parabola doesn't intersect the x - axis

$\therefore$ The given quadratic equation has no real roots.

(iv) $x^2 - 9 = 0$

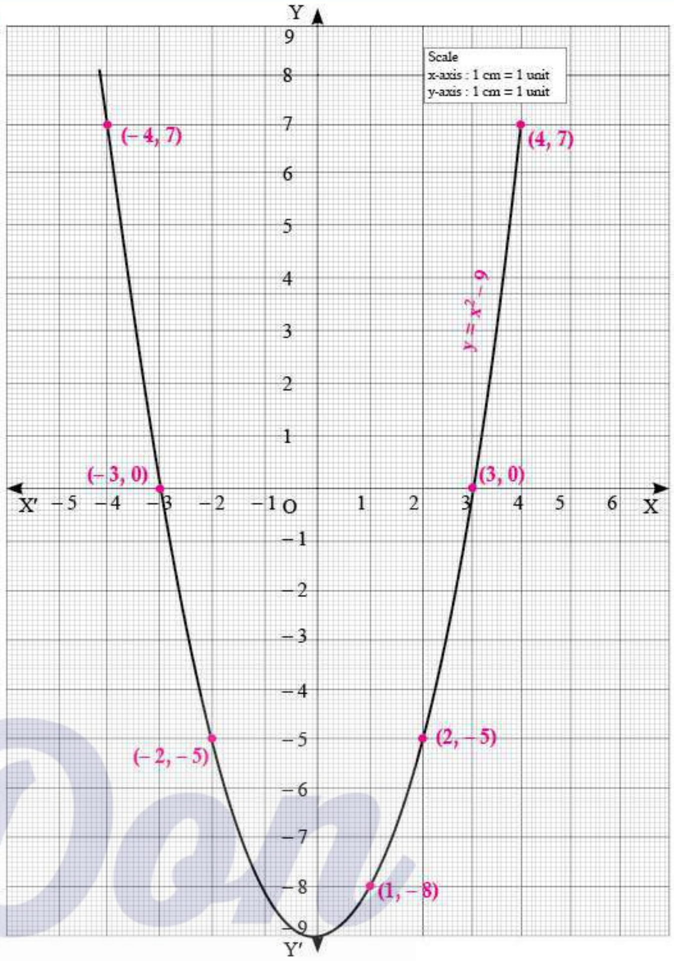
Let $y = x^2 - 9$

Table of Points:

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	3	9	16
–9	–9	–9	–9	–9	–9	–9	–9	–9	–9
y	7	0	–5	–8	–9	–8	–5	0	7

Points to be plotted in the graph are

(– 4, 7), (– 3, 0) (– 2, – 5), (0, – 9), (1, – 8), (2, – 5), (3, 0), (4, 7)



From the graph, we see that the curve $y = x^2 - 9$ intersects X - axis at (– 3, 0) and (3, 0).

$\therefore$ Solution: $x = -3, 3$

Roots are real and unequal.

(v)

$x^2 - 6x + 9 = 0$

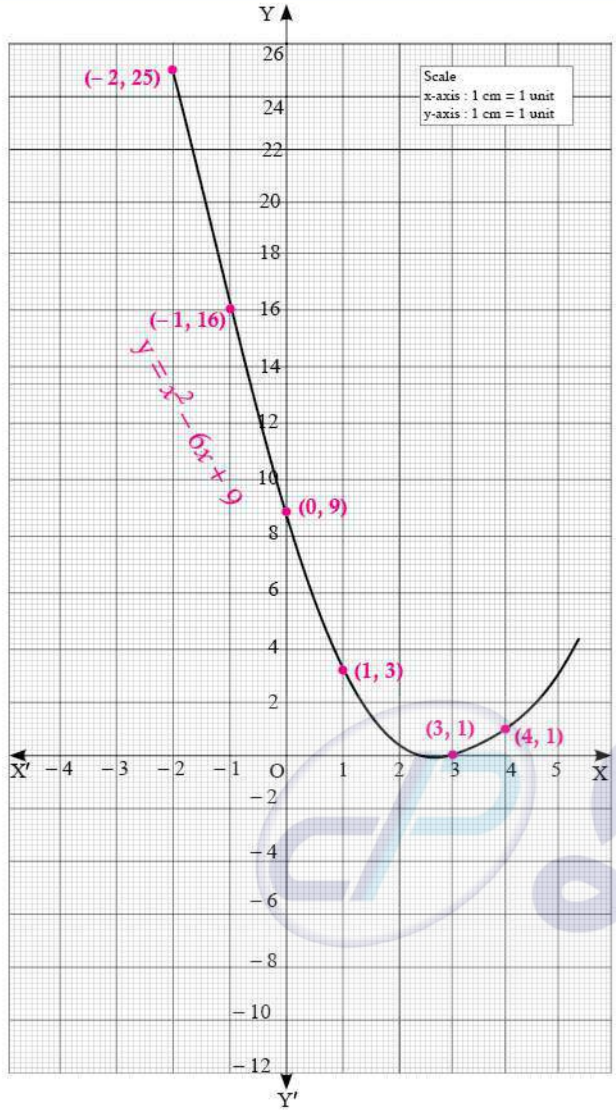
Let $y = x^2 - 6x + 9$

Table of Points:

x	–2	–1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
–6x	12	6	0	–6	–12	–18	–24
9	9	9	9	9	9	9	9
g	25	26	9	4	1	0	1

Points to be plotted in the graph are

(– 2, 25), (– 1, 26) (0, 9), (1, 4), (2, 1), (3, 0), (4, 1)



From the graph, we see that the curve

$y = x^2 - 6x + 9$ intersects

X - axis at (3, 0)

$\therefore$ Solution: $x = 3, 3$

Hence the roots are real and equal.

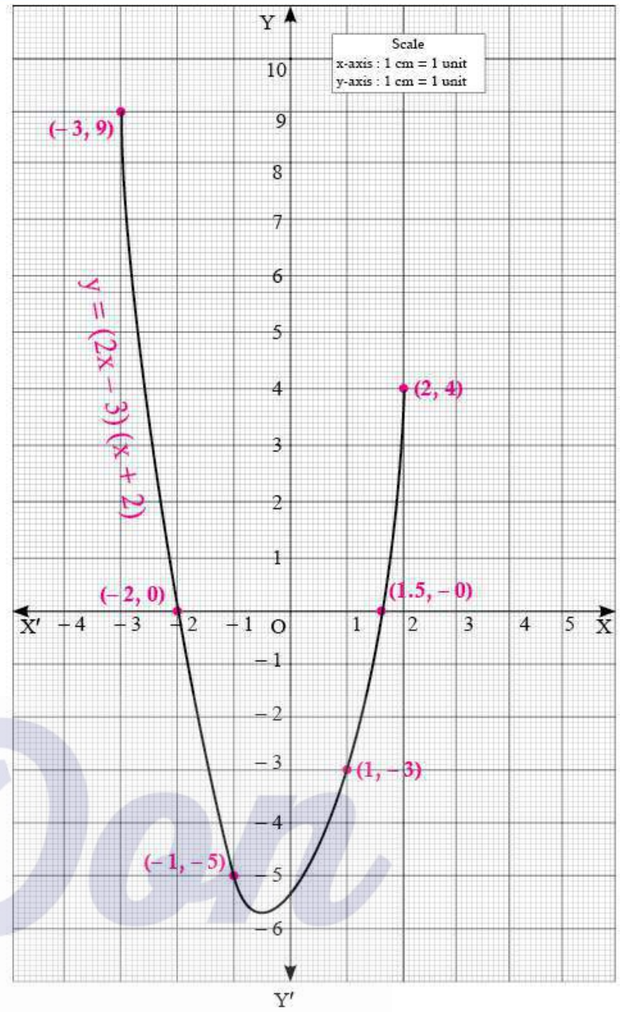
(vi) $(2x - 3)(x + 2) = 0$

Let $y = (2x - 3)(x + 2)$

Table of Points:

x	– 3	– 2	– 1	0	1	1.5	2
2x – 3	– 9	– 7	– 5	– 3	– 1	0	1
x + 2	– 1	0	1	2	3	3.5	4
y	9	0	– 5	– 6	– 3	0	4

Points to be plotted in the graph are (– 3, 9), (– 2, 0) (– 1, – 5), (0, – 6), (1, – 3), (1.5, 0) and (2, 4)



From the graph, we see that the curve

$y = (2x^2 - 3)$ (–2, 0) intersects

X - axis at (–2, 0) and (1.5, 0)

$\therefore$ Solution: $x = -2, 3/2$

Hence the roots are real and unequal.

2. Draw the graph of $y = x^2 - 4$ and hence solve $x^2 - x - 12 = 0$

Sol :

$$y = x^2 - 4$$

Table of points

x	– 4	– 3	– 2	– 1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
– 4	– 4	–4	– 4	– 4	– 4	– 4	– 4	– 4	– 4
y	12	5	0	– 3	– 4	– 3	0	5	12

Now plotting the points on the graph and joining them we get the graph $y = x^2 - 4$

To solve $x^2 - x - 12 = 0$, subtract $x^2 - x - 12 = 0$ from $y = x^2 - 4$, that is

$$y = x^2 + 0x - 4$$
$$0 = x^2 - x - 12 \quad (-)$$
$$y = x + 8$$

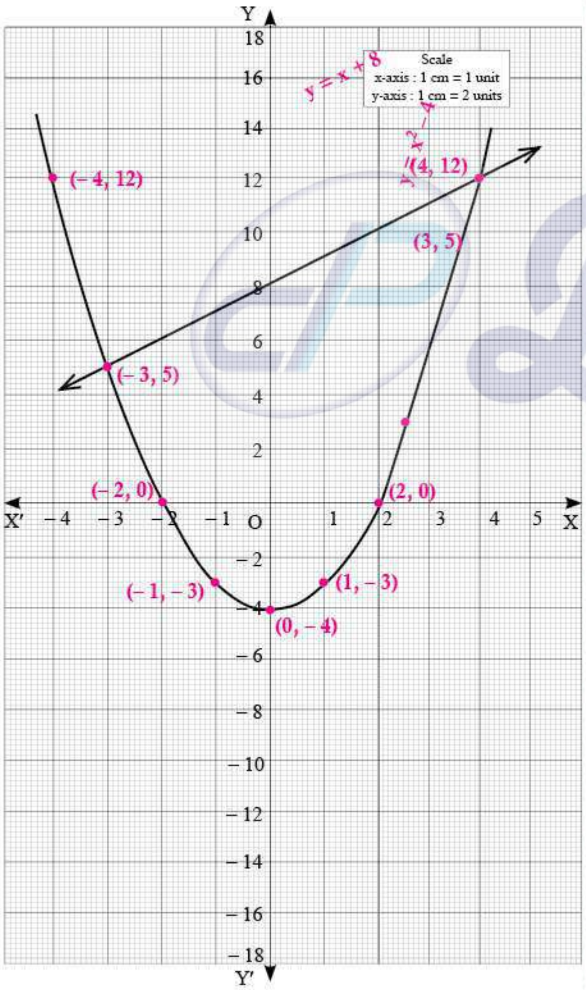
The equation represents a straight line.

Draw the graph of $y = x + 8$ by forming the table of values as below

x	-2	-1	0	1	2
y	6	7	8	9	10

It meets the parabola at (4, 12) and (-3, 5)

The solution x is = 4, -3



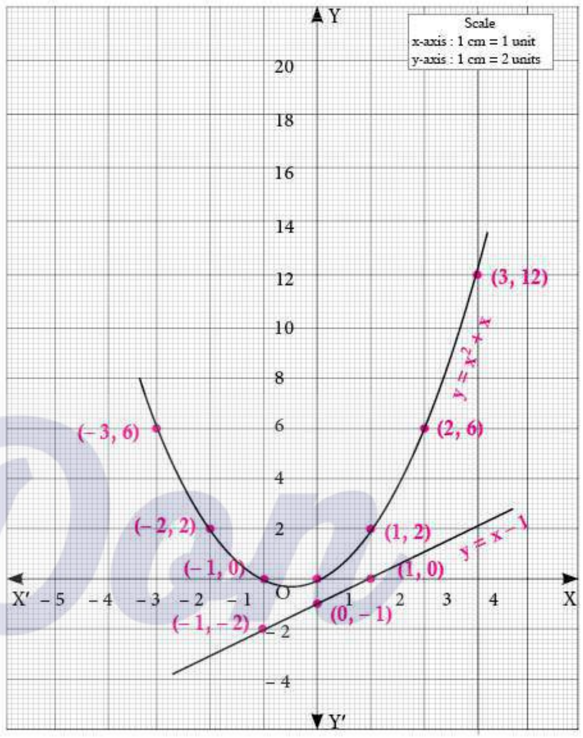
3. Draw the graph of $y = x^2 + x$ and hence solve $x^2 + 1 = 0$

Sol :

$y = x^2 + x$

Table of values

x	− 3	− 2	− 1	0	1	2	3
x^2	9	4	1	0	1	4	9
y	6	2	0	0	2	6	12



Plotting and joining the points, we get the graph of

$y = x^2 + x$.

Now subtracting $x^2 + 1 = 0$ from $y = x^2 + x$

$$\therefore\ y = x^2 + x$$
$$0 = x^2 + 1 \quad (-)$$

$y = x - 1$ is a straight line

Table of values

x	0	1	−1
y	− 1	0	− 2

Let us draw a straight line by joining these points.

From the graph, we see that the straight line does not intersect the parabola.

$\therefore$ The Quadratic equation has no real roots.

4. Draw the graph of $y = x^2 + 3x + 2$ and use it to solve $x^2 + 2x + 1 = 0$

Sol :

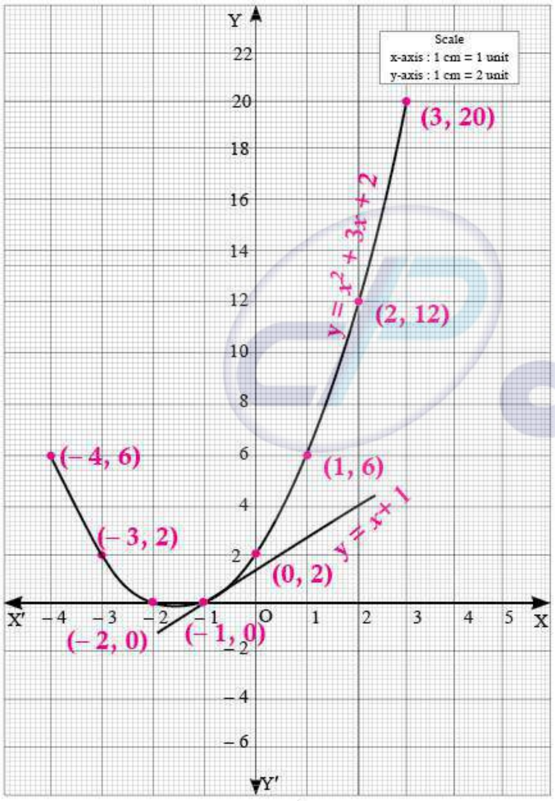
$y = x^2 + 3x + 2$

Table of Points:

x	– 4	– 3	– 2	– 1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
3x	–12	– 9	– 6	– 3	0	3	6	9
2	2	2	2	2	2	2	2	2
y	6	2	0	0	2	6	12	20

Points to be plotted in the graph are

(– 3, 2), (– 2, 0) (– 1, 0), (0, 2), (1, 6), (2, 12) and (3, 20)



Now, Subtracting $y = x^2 + 3x + 2$ and $x^2 + 2x + 1 = 0$, we get

$$\begin{array}{l} y = x^2 + 3x + 2 \\ \underline{0 = x^2 + 2x + 1} \quad (-) \\ \underline{y = x + 1} \end{array}$$ is a straight line.

Table of points

x	0	– 1	1
y	1	0	2

The curve $y = x^2 + 3x + 2$ and the line $y = x + 1$ intersects at (–1, 0)

Hence the solution of $x^2 + 2x + 1 = 0$ is $x = -1, -1$

Roots are real and equal.

5. Draw the graph of $y = x^2 + 3x - 4$ and hence use it to solve $x^2 + 3x - 4 = 0$

Sol :

$$y = x^2 + 3x - 4$$

Table of values

x	–5	–4	–3	–2	–1	0	1	2
x^2	25	16	9	4	1	0	1	4
3x	–15	–12	–9	–6	–3	0	3	6
–4	–4	–4	–4	–4	–4	–4	–4	–4
y	6	0	–4	–6	–6	–4	0	6

Plotting and joining the points on the graph, we get the parabola

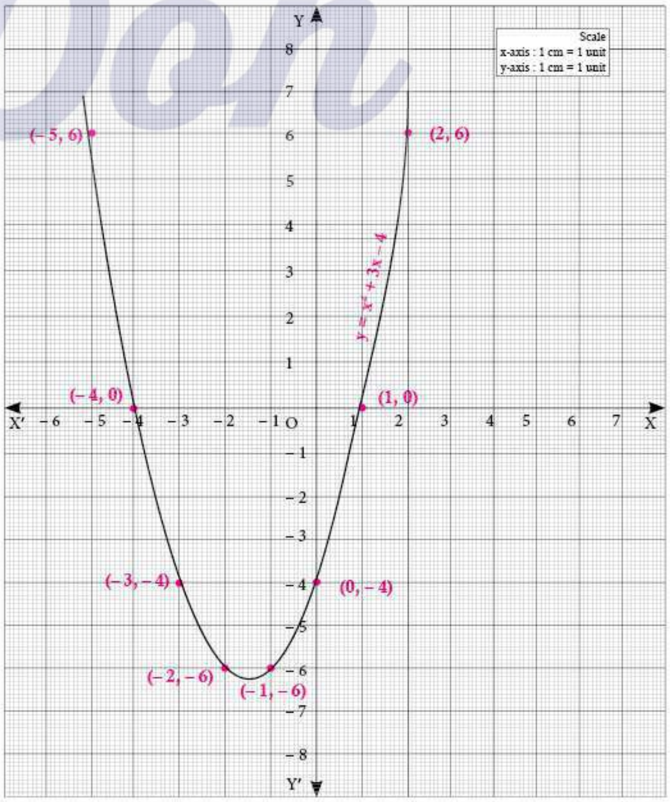
$$y = x^2 + 3x - 4$$

Now, subtracting $x^2 + 3x - 4 = 0$ from $y = x^2 + 3x - 4$

$$\begin{array}{l} y = x^2 + 3x - 4 \\ \underline{0 = x^2 + 3x - 4} \quad (-) \\ y = 0 \quad \Rightarrow \quad \text{X-axis} \end{array}$$

The parabola $y = x^2 + 3x - 4$ intersects X-axis at (–4, 0) and (1, 0)

$\therefore$ Solution $x = -4, 1$



6. Draw the graph of $y = x^2 - 5x - 6$ and hence solve $x^2 - 5x - 14 = 0$

Sol :

$y = x^2 - 5x - 6$

Table of values

x	–2	–1	0	1	2	3	4	5	6	7
x^2	4	1	0	1	4	9	16	25	36	49
–5x	10	5	0	–5	–10	–15	–20	–25	–30	–35
–6	–6	–6	–6	–6	–6	–6	–6	–6	–6	–6
y	8	0	–6	–10	–12	–12	–10	–6	0	8

Plotting and joining these points, we get the parabola

$y = x^2 - 5x - 6$

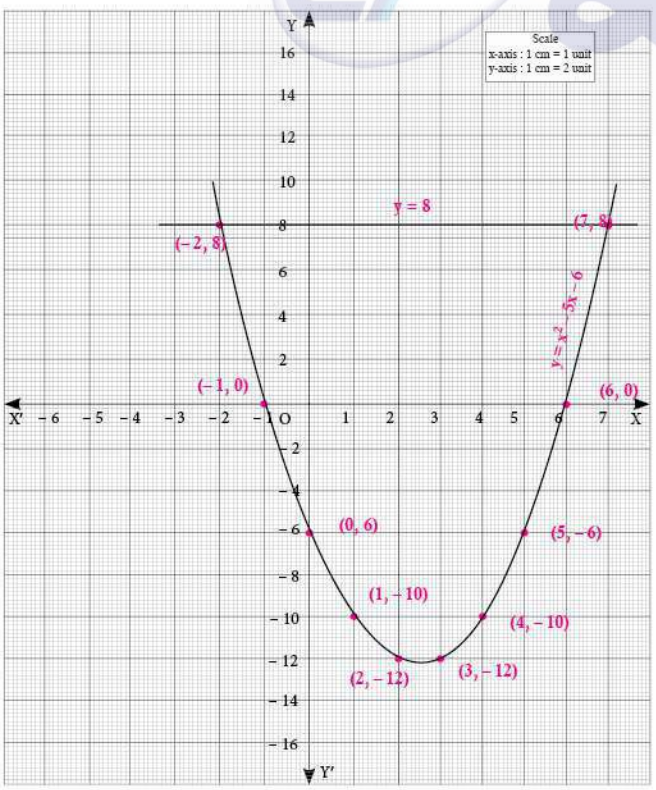
Now, Subtracting $x^2 - 5x - 14 = 0$ from $y = x^2 - 5x - 6$

$$\begin{aligned} y &= x^2 - 5x - 6 \\ 0 &= x^2 - 5x - 14 \ (-) \\ \hline y &= 8 \end{aligned}$$

$y = 8$ is a straight line.

The straight line $y = 8$ and the parabola $y = x^2 - 5x - 6$ intersect at $(-2, 8)$ and $(7, 8)$.

$\therefore$ The solution is $x = -2, 7$.



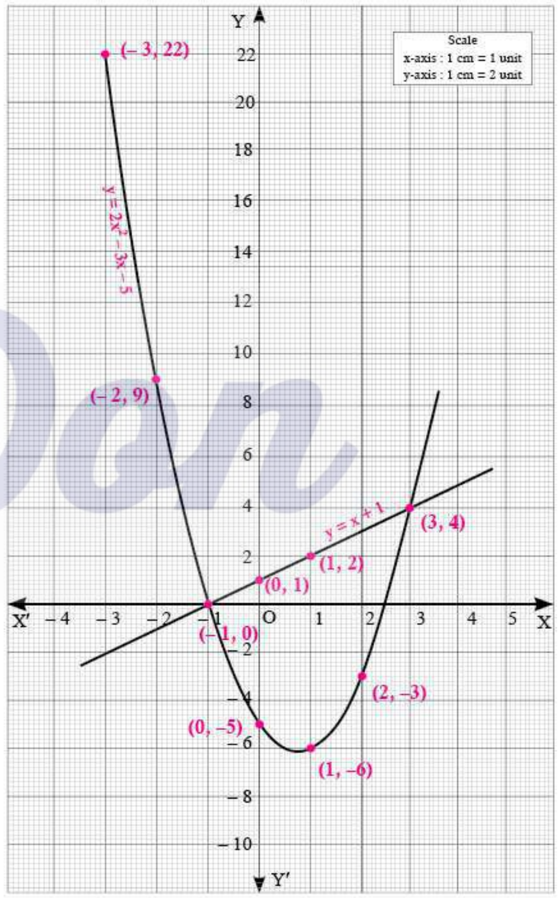
7. Draw the graph of $y = 2x^2 - 3x - 5$ and hence solve $2x^2 - 4x - 6 = 0$

Sol :

Table of Points:

x	– 3	– 2	– 1	0	1	2	3
$2x^2$	18	8	2	0	2	8	18
$-3x$	9	6	3	0	– 3	– 6	– 9
– 5	– 5	– 5	– 5	– 5	– 5	– 5	– 5
y	22	9	0	– 5	– 6	– 3	– 4

Points are $(-3, 22)$, $(-2, 9)$ $(-1, 0)$, $(0, -5)$, $(1, -6)$, $(2, -3)$ and $(3, -4)$



Now, Subtracting $y = 2x^2 - 3x - 5$ and $2x^2 - 4x - 6 = 0$, we get

$$\begin{aligned} y &= 2x^2 - 3x - 5 \\ (-) \quad 0 &= 2x^2 - 4x - 6 \\ \hline y &= x + 1 \end{aligned}$$

Table of points

x	0	– 1	1
y	3	0	6

The curve $y = 2x^2 - 2x - 5$ and the line $y = x + 1$ intersect at $(-1, 0)$ and $(3, 4)$

Hence the solution of $2x^2 - 4x - 6 = 0$ (or) $x^2 - 2x - 3 = 0$ is $x = -1, 3$.

Roots are real and unequal.

8. Draw the graph of y = (x – 1) (x + 3) and hence solve $x^2 - x - 6 = 0$.

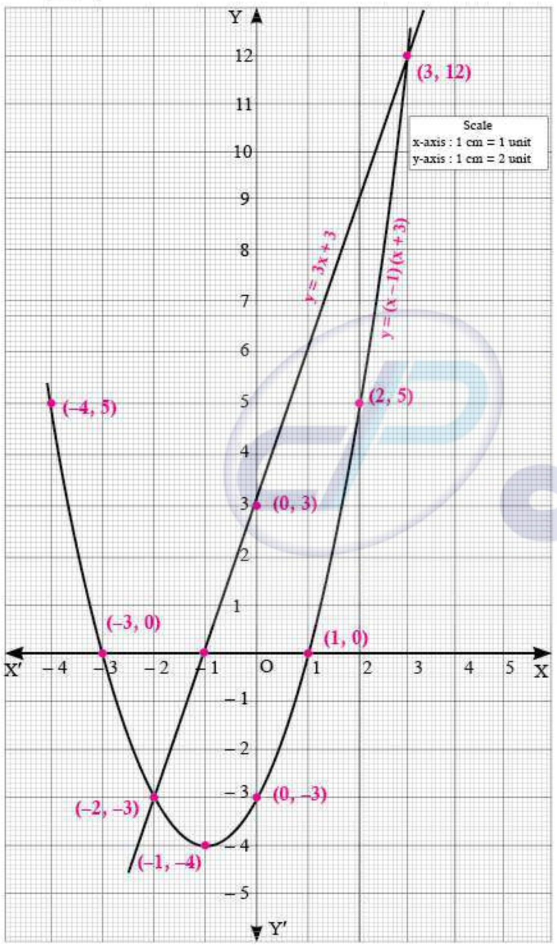
Sol :

Table of Points:

x	– 4	– 3	– 2	– 1	0	1	2	3
$x - 1$	– 5	– 4	– 3	– 2	– 1	0	1	2
$x + 3$	– 1	0	1	2	3	4	5	6
y	5	0	– 3	– 4	– 3	0	5	12

Points to be plotted in the graph are

(– 4, 5), (– 3, 0) (– 2, – 3), (– 1, – 4), (0, – 3), (1, 0) and (2, 5)



Now, Subtracting $y = x^2 + 2x - 3$ and $x^2 - x - 6 = 0$, we get

$$\begin{array}{rl} & y = x^2 + 2x - 3 \\ (-) & 0 = x^2 - x - 6 \\ \hline & y = 3x + 3 \quad \text{is a straight line} \end{array}$$

Table of points

x	0	– 1	1
y	3	0	6

The curve $y = x^2 + 2x - 3$ and the line $y = 3x + 3$ intersect at (–2, –3) and (3, 12)

Hence the solution of $x^2 - x - 6 = 0$ is $x = -2, 3$.

Roots are real and unequal.

MATRICES

Key Points

- A Rectangular array of numbers, variables is called a 'matrix'.
- Horizontal arrangement is 'Row' and vertical arrangement is Column.
- Order of a matrix = No. of Rows × No. of Columns = m × n
- Number of elements in the matrix = mn.
- **Row matrix:** A matrix is having only one row is called a row matrix. Example: $[8 \;\; 4 \;\; -1]$
- **Column matrix:** A matrix is having only one column is called a column matrix. Example: $\begin{pmatrix} 9 \\ 5 \\ 3 \end{pmatrix}$
- **Square matrix:** A matrix in which the number of rows and no. of columns are equal is called the square matrix. Example: $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 3 & -11 \end{bmatrix}$
- **Diagonal matrix:** A square matrix in which all the elements are zero, except the leading diagonal elements is called a Diagonal matrix.

  Example: $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
- **Scalar matrix:** A diagonal matrix in which all the leading elements are equal is called the Scalar matrix. Example: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$
- **Identity (or) Unit matrix:**

  A Scalar matrix in which all the leading diagonal elements are '1', is called the Identity matrix and it is denoted by I. Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- **Zero matrix (or) Null matrix:**

  A matrix is called a zero matrix if all of its elements are zero. Example: $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Triangular matrix:

A square matrix in which all the entries above the leading diagonal are zero is called a lower triangular matrix. Example: $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & -1 & 2 \end{bmatrix}$ If all the entries below the leading diagonal are zero, then it is called upper triangular matrix. Example: $\begin{bmatrix} 1 & 7 & -3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

Transpose of a matrix:

The matrix obtained by interchanging the elements in rows and columns of the matrix A is called transpose of A and it is denoted by A^T. Example: $A = \begin{bmatrix} 1 & 5 \\ 8 & 9 \\ 4 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 8 & 4 \\ 5 & 9 & 3 \end{bmatrix}$ and $(A^T)^T = A$.

Equal matrices:

Two matrices A and B are equal if and only if they have the same order and corresponding elements are same.

Worked Examples

3.53. Consider the following information regarding the number of men and women workers in three factories I, II and III.

	Men	Women
I	23	18
II	47	36
III	15	16

Represent the above information in the form of a matrix. What does the entry in the second row and first column represent?

Sol :

The information is represented in the from of a 3×2 matrix as follows

$$A = \begin{bmatrix} 23 & 18 \\ 47 & 36 \\ 15 & 16 \end{bmatrix}$$

The entry in the second row and first column represent that there are 47 men workers in factory II.

3.54. If a matrix has 16 elements, what are the possible orders it can have?

Sol :

We know that a matrix is of order $m \times n$, has mn elements. Thus to find all possible orders of a matrix with 16 elements, we will find all ordered pairs of natural numbers whose product is 16.

Such ordered pairs are (1, 16), (16, 1), (4, 4), (8, 2), (2, 8)

Hence possible orders are 1×16, 16×1, 4×4, 2×8, 8×2

3.55. Construct a 3×3 matrix whose elements are $a_{ij} = i^2 j^2$.

Sol :

The general 3×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{ij} = i^2 j^2$$

$$\begin{aligned}
a_{11} &= 1^2 \times 1^2 = 1 \times 1 = 1; \\
a_{12} &= 1^2 \times 2^2 = 1 \times 4 = 4; \\
a_{13} &= 1^2 \times 3^2 = 1 \times 9 = 9; \\
a_{21} &= 2^2 \times 1^2 = 4 \times 1 = 4; \\
a_{22} &= 2^2 \times 2^2 = 4 \times 4 = 16; \\
a_{23} &= 2^2 \times 3^2 = 4 \times 9 = 36; \\
a_{31} &= 3^2 \times 1^2 = 9 \times 1 = 9;
\end{aligned}$$

$a_{32} = 3^2 \times 2^2 = 9 \times 4 = 36;$
$a_{33} = 3^2 \times 3^2 = 9 \times 9 = 81;$

Hence the required matrix is

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \\ 9 & 36 & 81 \end{bmatrix}$$

3.56. Find the value of a, b, c, d from the equation $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$

Sol :

The given matrices are equal. Thus all corresponding elements are equal.

Therefore $a - b = 1$ ... (1)

$2a + c = 5$ ... (2)

$2a - b = 0$ ... (3)

$3c + d = 2$ ... (4)

(3) $\Rightarrow$ $2a - b = 0$

$2a = b$ ... (5)

Put $2a = b$ in equation (1), $a - 2a = 1 \Rightarrow a = -1$

Put $a = -1$ in equation (5), $2(-1) = b \Rightarrow b = -2$

Put $a = -1$ in equation (2), $2(-1) + c = 5 \Rightarrow c = 7$

Put $c = 7$ in equation (4), $3(7) + d = 2 \Rightarrow d = -19$

Therefore, $a = -1, b = -2, c = 7, d = -19$

Progress Check

1. Find the element in the second row and third column of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$

Ans : $a_{23} = 5$

2. Find the order of the matrix $\begin{bmatrix} \sin\theta \\ \cos\theta \\ \tan\theta \end{bmatrix}$

Ans : Order = 3×1

3. Determine the entries denoted by $a_{11}, a_{22}, a_{33}, a_{44}$ from the matrix $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 5 & 9 & -4 & \sqrt{7} \\ 3 & 5/2 & 8 & 9 \\ 7 & 0 & 1 & 4 \end{bmatrix}$

Ans : $a_{11} = 2,\quad a_{22} = 9,\quad a_{33} = 8,\quad a_{44} = 4$

4. The number of column (s) in a column matrix are _____

Ans : 1

5. The number of row (s) in a row matrix are _____

Ans : 1

6. The non-diagonal elements in any unit matrix are _____

Ans : 0

7. Does there exist a square matrix with 32 elements?

Ans : No

Exercise 3.16

1. In the matrix $A = \begin{bmatrix} 8 & 9 & 4 & 3 \\ -1 & \sqrt{7} & \frac{\sqrt{3}}{2} & 5 \\ 1 & 4 & 3 & 0 \\ 6 & 8 & -11 & 1 \end{bmatrix}$**, write**

(i) The number of elements

(ii) The order of the matrix

(iii) Write the elements corresponding to $a_{22}, a_{23}, a_{24}, a_{34}, a_{43}, a_{44}$

Sol :

$$A = \begin{bmatrix} 8 & 9 & 4 & 3 \\ -1 & \sqrt{7} & \frac{\sqrt{3}}{2} & 5 \\ 1 & 4 & 3 & 0 \\ 6 & 8 & -11 & 1 \end{bmatrix}$$

(i) The number of elements = 16

(ii) The order of the matrix = No of rows $\times$ No of columns = 4×4

(iii) $a_{22} = \sqrt{7}$, $a_{23} = \frac{\sqrt{3}}{2}$, $a_{24} = 5$, $a_{34} = 0$

$a_{43} = -11$, $a_{44} = 1$

2. If a matrix has 18 elements, what are the possible orders it can have? What if it has 6 elements?

Sol :

Given that a matrix has 18 elements.

$\therefore$ The possible orders are 1×18, 18×1, 9×2, 2×9, 6×3, 3×6.

If a matrix has 6 elements, then the possible orders are 1×6, 6×1, 2×3, 3×2.

3. Construct a 3 × 3 matrix whose elements are given by

(i) $a_{ij} = |i - 2j|$ (ii) $a_{ij} = \frac{(i+j)^3}{3}$

Sol :

(i) $a_{ij} = |i - 2j|$

Since, the matrix is of order 3 × 3, there will be 9 elements.

$$\begin{aligned}
a_{11} &= |1 - 2(1)| = |1 - 2| = |-1| = 1\\
a_{12} &= |1 - 2(2)| = |1 - 4| = |-3| = 3\\
a_{13} &= |1 - 2(3)| = |1 - 6| = |-5| = 5\\
a_{21} &= |2 - 2(1)| = |2 - 2| = 0\\
a_{22} &= |2 - 2(2)| = |2 - 4| = |-2| = 2\\
a_{23} &= |2 - 2(3)| = |2 - 6| = |-4| = 4\\
a_{31} &= |3 - 2(1)| = |3 - 2| = 1\\
a_{32} &= |3 - 2(2)| = |3 - 4| = |-1| = 1\\
a_{33} &= |3 - 2(3)| = |3 - 6| = |-3| = 3
\end{aligned}$$

$$\therefore \text{ The matrix is } \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

(ii) $aij = \frac{(i+j)^3}{3}$

$$\begin{aligned}
a_{11} &= \frac{(1+1)^3}{3} = \frac{(2)^3}{3} = 8/3\\
a_{12} &= \frac{(1+2)^3}{3} = \frac{(3)^3}{3} = \frac{27}{3} = 9\\
a_{13} &= \frac{(1+3)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3}\\
a_{21} &= \frac{(2+1)^3}{3} = \frac{(3)^3}{3} = \frac{27}{3} = 9\\
a_{22} &= \frac{(2+2)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3}\\
a_{23} &= \frac{(2+3)^3}{3} = \frac{(5)^3}{3} = \frac{125}{3}\\
a_{31} &= \frac{(3+1)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3}\\
a_{32} &= \frac{(3+2)^3}{3} = \frac{(5)^3}{3} = \frac{125}{3}\\
a_{33} &= \frac{(3+3)^3}{3} = \frac{(6)^3}{3} = \frac{216}{3} = 72
\end{aligned}$$

$$\therefore \text{ The matrix is } \begin{bmatrix} \frac{8}{3} & 9 & \frac{64}{3} \\ 9 & \frac{64}{3} & \frac{125}{3} \\ \frac{64}{3} & \frac{125}{3} & 72 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{bmatrix}$ then find the transpose of A.

Sol :

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 1 & 3 \\ 4 & -7 & 8 \\ 3 & 9 & 2 \end{bmatrix}$$

5. If $A = \begin{bmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{bmatrix}$ then find the transpose of –A.

Sol :

$$A = \begin{bmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{bmatrix}$$

$$-A = \begin{bmatrix} -\sqrt{7} & 3 \\ \sqrt{5} & -2 \\ -\sqrt{3} & 5 \end{bmatrix}$$

$$(-A)^T = \begin{bmatrix} -\sqrt{7} & \sqrt{5} & -\sqrt{3} \\ 3 & -2 & 5 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & \frac{5}{2} \\ 8 & 3 & 1 \end{bmatrix}$ then verify $(A^T)^T = A$

Sol :

$$A = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & \frac{5}{2} \\ 8 & 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & -\sqrt{17} & 8 \\ 2 & 0.7 & 3 \\ 2 & 5/2 & 1 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & 5/2 \\ 8 & 3 & 1 \end{bmatrix} = A$$

$\therefore \ (A^T)^T = A$

Hence verified.

7. Find the values of x, y and z from the following equations

(i) $\begin{bmatrix} 12 & 3 \\ x & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} y & z \\ 3 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Sol :

(i) $\begin{bmatrix} 12 & 3 \\ x & 3/2 \end{bmatrix} = \begin{bmatrix} y & z \\ 3 & 5 \end{bmatrix}$

Equating the corresponding elements

$x = 3, y = 12, z = 3$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

$x + y = 6, \ xy = 8, \quad 5 + z = 5$

$x + 8/x = 6 \quad y = 8/x, \quad z = 5 - 5 = 0$

$x^2 - 6x + 8 = 0$ when $x = 4, y = 2$

$(x - 4)(x - 2) = 0$ when $x = 2, y = 4, \ z = 0$

$x = 4, 2 \ \therefore \ x = 4, y = 2, z = 0$ (or)

$x = 2, y = 4, z = 0$

(iii) $\begin{bmatrix} x+y+x \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

$x + y + z = 9$ ... (1)

$x + z = 5$ ... (2)

$y + z = 7$ ... (3)

$(1) + (2) + (3) \Rightarrow 2x + 2y + 3z = 21$ ... (4)

$(1) \times 2 \Rightarrow 2x + 2y + 2z = 18$ ... (5)

$(4) - (5) \Rightarrow z = 3$

Substituting in (3)

$y + 3 = 7$

$y = 7 - 3 = 4$

Substituting $z = 3$ in (2)

$x + 3 = 5$

$x = 5 - 3 = 2$

$\therefore \quad x = 2, y = 4, z = 3.$

OPERATIONS ON MATRICES

Key Points

- Two matrices can be added or subtracted if they have the same order and to add or subtract the matrices just add or subtract the corresponding elements.
- When multiplying a matrix by a scalar (constant), we multiply all the elements in the matrix by a scalar.
- Matrix multiplication is possible only if the number of columns in the first matrix is equal to the number of rows in second matrix.
- If the matrix A is of order $m \times n$ and B is of order $p \times q$ then AB is possible only when $n = p$ and BA is possible only when $q = m$.

  Order of AB = $m \times q$

  Order of BA = $p \times n$

Worked Examples

3.57. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$, find A + B.

Sol :

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+7 & 3+0 \\ 4+1 & 5+3 & 6+1 \\ 7+2 & 8+4 & 9+0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 3 \\ 5 & 8 & 7 \\ 9 & 12 & 9 \end{bmatrix}$$

3.58. Two examinations were conducted for 3 groups of students namely group 1, group 2, group 3 and their data on average of marks for the subjects Tamil, English, Science and Mathematics are given below in the form of matrices A and B. Find the total marks of both the examinations for all the 3 groups.

$$A = \begin{array}{r} \\ \textit{Group 1} \\ \textit{Group 2} \\ \textit{Group 3} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tamil} & \textit{English} & \textit{Science} & \textit{Mathematics} \end{array} \\ \begin{bmatrix} 22 & 15 & 14 & 23 \\ 50 & 62 & 21 & 30 \\ 53 & 80 & 32 & 40 \end{bmatrix} \end{array}$$

$$B = \begin{array}{r} \\ \textit{Group 1} \\ \textit{Group 2} \\ \textit{Group 3} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tamil} & \textit{English} & \textit{Science} & \textit{Mathematics} \end{array} \\ \begin{bmatrix} 20 & 38 & 15 & 40 \\ 18 & 12 & 17 & 80 \\ 81 & 47 & 52 & 18 \end{bmatrix} \end{array}$$

Sol :

The total marks in both the examinations for all the 3 groups is the sum of the given matrices.

$$A + B = \begin{bmatrix} 22+20 & 15+38 & 14+15 & 23+40 \\ 50+18 & 62+12 & 21+17 & 30+80 \\ 53+81 & 80+47 & 32+52 & 40+18 \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 53 & 29 & 63 \\ 68 & 74 & 38 & 110 \\ 134 & 127 & 84 & 58 \end{bmatrix}$$

3.59. If $A = \begin{bmatrix} 1 & 3 & -2 \\ 5 & -4 & 6 \\ -3 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 8 \\ 3 & 4 \\ 9 & 6 \end{bmatrix}$, find A + B.

Sol :

It is not possible to add A and B because they have different orders.

3.60. If $A = \begin{bmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$ then Find 2A + B.

Sol :

Since A and B have same order 3×3, 2A + B is defined.

We have

$$2A + B = 2\begin{bmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 16 & 12 \\ 2 & 6 & 18 \\ -8 & 6 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 27 & 9 \\ 1 & 8 & 22 \\ -1 & 11 & -2 \end{bmatrix}$$

3.61. If $A = \begin{bmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{bmatrix}$, find 4A – 3B.

Sol :

Since A, B are of the same order 3×3, subtraction of 4A and 3B is defined.

$$4A - 3B = 4\begin{bmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{bmatrix} - 3\begin{bmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 16 & -8 \\ 2 & 3 & 4\sqrt{2} \\ 4 & 36 & 16 \end{bmatrix} + \begin{bmatrix} 21 & -12 & 9 \\ -\frac{3}{4} & -\frac{21}{2} & -9 \\ -15 & 18 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 4 & 1 \\ \frac{5}{4} & -\frac{15}{2} & 4\sqrt{2}-9 \\ -11 & 54 & -11 \end{bmatrix}$$

3.62. Find the value of a, b, c, d, x, y from the following matrix equation.

$$\begin{pmatrix} d & 8 \\ 3b & a \end{pmatrix} + \begin{pmatrix} 3 & a \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 2a \\ b & 4c \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

Sol :

First, we add the two matrices on both left, right hand sides to get

$$\begin{pmatrix} d+3 & 8+a \\ 3b-2 & a-4 \end{pmatrix} = \begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$$

Equating the corresponding elements of the two matrices, we have

$d + 3 = 2 \Rightarrow d = -1$

$8 + a = 2a + 1 \Rightarrow a = 7$

$3b - 2 = b - 5 \Rightarrow b = \frac{-3}{2}$

Substituting $a = 7$ in $a - 4 = 4c \Rightarrow c = \frac{3}{4}$

Therefore, $a = 7, b = -\frac{3}{2}, c = \frac{3}{4}, d = -1$.

3.63. If $A = \begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}$,

$C = \begin{pmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{pmatrix}$

compute the following :

(i) $3A + 2B - C$ (ii) $\frac{1}{2}A - \frac{3}{2}B$

Sol :

(i) $3A + 2B - C$

$$= 3\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} + 2\begin{bmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 24 & 9 \\ 9 & 15 & 0 \\ 24 & 21 & 18 \end{bmatrix} + \begin{bmatrix} 16 & -12 & -8 \\ 4 & 22 & -6 \\ 0 & 2 & 10 \end{bmatrix} + \begin{bmatrix} -5 & -3 & 0 \\ 1 & 7 & -2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 9 & 1 \\ 14 & 44 & -8 \\ 23 & 19 & 25 \end{bmatrix}$$

(ii) $\frac{1}{2}A - \frac{3}{2}B = \frac{1}{2}[A - 3B]$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} + \begin{bmatrix} -24 & 18 & 12 \\ -6 & -33 & 9 \\ 0 & -3 & -15 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} -23 & 26 & 15 \\ -3 & -28 & 9 \\ 8 & 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{23}{2} & 13 & \frac{15}{2} \\ -\frac{3}{2} & -14 & \frac{9}{2} \\ 4 & 2 & -\frac{9}{2} \end{bmatrix}$$

Exercise 3.17

1. If $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$ then verify that

(i) $A + B = B + A$

(ii) $A + (-A) = (-A) + A = O$.

Sol :

$$A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$$

(i) $A + B = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$

$B + A = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$

$\therefore \quad A + B = B + A$

(ii) $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, -A = \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix}$

$$A + (-A) = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} + \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$(-A) + A = \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$\therefore \quad A + (-A) = (-A) + A = O$

Hence verified

2. If $A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$ then verify that $A + (B + C) = (A + B) + C$.

Sol :

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$$

$$B + C = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix} + \begin{bmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$A + (B + C) = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \quad \text{... (1)}$$

$$A + B = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix}$$

$$(A + B) + C = \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix} + \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \quad \text{... (2)}$$

From (1) and (2)

$A + (B + C) = (A + B) + C$

Hence verified.

3. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

Sol :

$$X + Y = \begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix} \quad \text{... (1)}$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{... (2)}$$

$$(1)+(2) \Rightarrow \quad 2X = \begin{pmatrix} 10 & 0 \\ 3 & 9 \end{pmatrix}$$

$$X = \frac{1}{2}\begin{pmatrix} 10 & 0 \\ 3 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix}$$

Substituting in (1)

$$Y = \begin{pmatrix} 7 & 0 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix},$$

$$Y = \begin{pmatrix} 2 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

4. If $A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$ find the value of (i) B – 5A (ii) 3A – 9B

Sol :

$$A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$$

(i) $$B - 5A = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - 5\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 20 & 45 \\ 40 & 15 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -17 & -37 \\ -39 & -11 & -26 \end{pmatrix}$$

(ii) $$3A - 9B = 3\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix} - 9\begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 12 & 27 \\ 24 & 9 & 21 \end{pmatrix} - \begin{pmatrix} 63 & 27 & 72 \\ 9 & 36 & 81 \end{pmatrix}$$

$$= \begin{pmatrix} -63 & -15 & -45 \\ 15 & -27 & -60 \end{pmatrix}$$

5. Find the values of x, y, z if

(i) $\begin{pmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$

(ii) $\begin{bmatrix} x & y-z & z+3 \end{bmatrix} + \begin{bmatrix} y & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$

Sol :

(i) $$\begin{bmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 6 \end{bmatrix}$$

Equating the corresponding elements

$$\begin{aligned} x-3 &= 1 \\ x &= 1+3 = 4 \\ 3x - z &= 0 \\ 3(4) - z &= 0 \\ 12 &= z \\ x+y+7 &= 1 \\ 4+y+7 &= 1 \\ y &= -10 \end{aligned}$$

$$\therefore x = 4, y = -10, z = 12$$

(ii) $$\begin{bmatrix} x & y-z & z+3 \end{bmatrix} + \begin{bmatrix} y & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} x+y & y-z+4 & z+6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$$

$$\therefore x+y = 4,\ y-z+4 = 8,\ z+6 = 16$$

$$z = 16 - 6 = 10$$

Substituting $z = 10$

$$\begin{aligned} y - 10 + 4 &= 8 \\ y &= 8 + 6 = 14 \\ \therefore x + 14 &= 4 \\ x &= 4 - 14 = -10 \end{aligned}$$

$\therefore$ Solution $x = -10,\ y = 14,\ z = 10$

6. Find x and y if $x\begin{pmatrix} 4 \\ -3 \end{pmatrix} + y\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

Sol :

$$x\begin{pmatrix} 4 \\ -3 \end{pmatrix} + y\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 4x \\ -3x \end{pmatrix} + \begin{pmatrix} -2y \\ 3y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$4x - 2y = 4 \Rightarrow 2x - y = 2 \quad \ldots (1)$$

$$-3x + 3y = 6 \Rightarrow -x + y = 2 \quad \ldots (2)$$

$$(1)+(2) \Rightarrow x = 4$$

Substituting in (2)

$$\begin{aligned} -4 + y &= 2 \\ y &= 2 + 4 = 6 \end{aligned}$$

$$\therefore x = 4, y = 6.$$

7. Find the non-zero values of x satisfying the matrix equation

$$x\begin{bmatrix}2x & 2\\3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\4 & 4x\end{bmatrix}=2\begin{bmatrix}x^2+8 & 24\\10 & 6x\end{bmatrix}$$

Sol :

$$x\begin{bmatrix}2x & 2\\3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\4 & 4x\end{bmatrix}=2\begin{bmatrix}x^2+8 & 24\\10 & 6x\end{bmatrix}$$

$$\begin{bmatrix}2x^2 & 2x\\3x & x^2\end{bmatrix}+\begin{bmatrix}16 & 10x\\8 & 8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\20 & 12x\end{bmatrix}$$

$$\begin{bmatrix}2x^2+16 & 12x\\3x+8 & x^2+8x\end{bmatrix}=\begin{bmatrix}2x^2+16 & 48\\20 & 12x\end{bmatrix}$$

Equating the corresponding elements

$$3x^2+16 = 2x^2+16$$
$$3x^2-2x^2 = 16-16$$
$$x^2 = 0$$
$$x = 0$$

(Rejected as non zero value required).

$$12x = 48$$
$$x = \frac{48}{12}=4$$
$$3x+8 = 20$$
$$3x = 20-8=12$$
$$x = \frac{12}{3}=4$$
$$x^2+8x = 12x$$
$$x^2+8x-12x = 0$$
$$x^2-4x = 0$$
$$x(x-4) = 0$$

$x = 0$ (rejected) and $x = 4$

$\therefore$ The non-zero value of x is 4.

8. Solve for x, y: $\begin{pmatrix}x^2\\y^2\end{pmatrix}+2\begin{pmatrix}-2x\\-y\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$

Sol :

$$\begin{pmatrix}x^2\\y^2\end{pmatrix}+2\begin{pmatrix}-2x\\-y\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$$

$$\begin{pmatrix}x^2\\y^2\end{pmatrix}+\begin{pmatrix}-4x\\-2y\end{pmatrix}=\begin{pmatrix}5\\8\end{pmatrix}$$

$$\begin{pmatrix}x^2-4x\\y^2-2y\end{pmatrix}=\begin{pmatrix}+5\\8\end{pmatrix}$$

Equating the corresponding elements,

$$x^2-4x = 5$$
$$x^2-4x-5 = 0$$
$$(x-5)(x+1) = 0$$
$$x = 5, -1$$
$$y^2-2y = 8$$
$$y^2-2y-8 = 0$$
$$(y-4)(y+2) = 0$$
$$y = 4, -2.$$

MULTIPLICATION OF MATRICES

Worked Example

3.64. If $A=\begin{bmatrix}1 & 2 & 0\\3 & 1 & 5\end{bmatrix}$, $B=\begin{bmatrix}8 & 3 & 1\\2 & 4 & 1\\5 & 3 & 1\end{bmatrix}$ find AB.

Sol :

We observe that A is a 2×3 matrix and B is a 3×3 matrix, hence AB is defined and it will be of the order 2×3.

$$\text{Given } A = \begin{bmatrix}1 & 2 & 0\\3 & 1 & 5\end{bmatrix}_{2\times3}$$

$$B = \begin{bmatrix}8 & 3 & 1\\2 & 4 & 1\\5 & 3 & 1\end{bmatrix}_{3\times3}$$

$$AB = \begin{bmatrix}1 & 2 & 0\\3 & 1 & 5\end{bmatrix}\times\begin{bmatrix}8 & 3 & 1\\2 & 4 & 1\\5 & 3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}(8+4+0) & (3+8+0) & (1+2+0)\\(24+2+25) & (9+4+15) & (3+1+5)\end{bmatrix}$$

$$= \begin{bmatrix}12 & 11 & 3\\51 & 28 & 9\end{bmatrix}$$

3.65. If A $\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 0\\1 & 3\end{bmatrix}$ find AB and BA. Check if AB = BA.

Sol :

We observe that A is a 2×2 matrix and B is a 2×2 matrix, hence AB is defined and it will be of the order 2×2.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 0+3 \\ 2+3 & 0+9 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 5 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 10 \end{bmatrix}$$

Therefore, $AB \neq BA$

3.66. **If A=** $\begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ **and B =** $\begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Show that A and B satisfy commutative property with respect to matrix multiplication.

Sol :

We have to show that AB = BA

$$\text{LHS: } AB = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & 4\sqrt{2}-4\sqrt{2} \\ 2\sqrt{2}-2\sqrt{2} & 4+4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\text{RHS: } BA = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & -4\sqrt{2}+4\sqrt{2} \\ -2\sqrt{2}+2\sqrt{2} & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Hence LHS = RHS i.e., AB = BA

3.67. **Solve** $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Sol :

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}_{2\times 2} \times \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

By matrix multiplication

$$= \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Rewriting $2x + y = 4$ ... (1)

$x + 2y = 5$ ... (2)

(1) – 2 × (2) gives

$$\begin{array}{r} 2x + y = 4 \ (-) \\ 2x + 4y = 10 \\ \hline -3y = -6 \end{array} \quad \Rightarrow y = 2$$

Substituting y = 2 in (1), 2x + 2 = 4 ⇒ x = 1

Therefore x = 1, y = 2.

3.68. **If A = [1 – 1 2], B =** $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$ **and C =** $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

show that (AB) C = A (BC)

Sol :

LHS (AB) C

$$AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1\times 3} \times \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2}$$

$$= [1 - 2 + 2 \quad -1 - 1 + 6]$$

$$= [1 \quad 4]$$

$$(AB)\,C = \begin{bmatrix} 1 & 4 \end{bmatrix}_{1\times 2} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}_{2\times 2}$$

$$= [1 + 8 \quad 2 - 4] = [9 \quad -2] \quad \text{... (1)}$$

RHS A (BC)

$$BC = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1-2 & 2+1 \\ 2+2 & 4-1 \\ 1+6 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1\times 3} \times \begin{bmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} -1-4+14 & 3-3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -2 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2) , (AB) C = A (BC)

3.69. **If** $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}, C = \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix}$ **verify that A (B + C) = AB + AC**

Sol :

LHS A (B + C)

$$B + C = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 8 \\ -1 & 4 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} -6 & 8 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6-1 & 8+4 \\ 6-3 & -8+12 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 12 \\ 3 & 4 \end{bmatrix} \quad \ldots (1)$$

RHS: AB + AC

$$AB = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 2+2 \\ -1-12 & -2+6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 4 \\ -13 & 4 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7+3 & 6+2 \\ 7+9 & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 8 \\ 16 & 0 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -3 & 4 \\ -13 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 8 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 12 \\ 3 & 4 \end{bmatrix}$$

From (1) and (2), A (B + C) = AB + AC.

Hence proved.

3.70. **If** $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$ **show that** $(AB)^T = B^T A^T$

Sol :

LHS: $(AB)^T$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 2-2+0 & -1+8+2 \\ 4+1+0 & -2-4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 \\ 5 & -4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 0 & 9 \\ 5 & -4 \end{bmatrix}^T = \begin{bmatrix} 0 & 5 \\ 9 & -4 \end{bmatrix} \quad \ldots (1)$$

RHS: $(B^T A^T)$

$$B^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix},$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 2-2+0 & 4+1+0 \\ -1+8+2 & -2-4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 \\ 9 & -4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

Hence proved.

Exercise 3.18

1. Find the order of the product matrix AB if

	(i)	(ii)	(iii)	(iv)	(v)
Orders of A	3 × 3	4 × 3	4 × 2	4 × 5	1 × 1
Orders of B	3 × 3	3 × 2	2 × 2	5 × 1	1 × 3

Sol :

(i) Order of A = 3×3, order of B = 3×3
$\therefore$ Order of AB = 3×3

(ii) Order of A = 4×3, Order of B = 3×2
$\therefore$ Order of AB = 4×2

(iii) Order of A = 4×2, Order of B = 2×2
$\therefore$ Order of AB = 4×2

(iv) Order of A = 4×5, Order of B = 5×1
$\therefore$ Order of AB = 4×1

(v) Order of A = 1×1, Order of B = 1×3
$\therefore$ Order of AB = 1×3

2. If A is of order p × q and B is of order q × r what is the order of AB and BA?

Sol :

Order of A = $p \times q$

Order of B = $q \times r$

No.of columns in A = No of Rows in B

$q = q$

$\therefore$ The product 'AB' is possible.

Order of AB = Number of Rows in A × Number of Columns in B

$= p \times r$

No. of columns in B $\neq$ No. of rows in A

i.e., $r \neq q$

$\therefore$ BA is not possible.

3. A has 'a' rows and 'a + 3' columns. B has 'b' rows and '17 – b' columns and if both products AB and BA exist find a, b?

Sol :

Given A has 'a' rows and 'a + 3' columns

B has 'b' rows and '17–b' columns

Since AB and BA exist

For AB, No. of columns in A = No. of rows in B

$\therefore \quad a + 3 = b$

For BA, No. of columns in B = No of rows in A

$$17 - b = a$$
$$\therefore \quad 17 - (a + 3) = a$$
$$17 - a - 3 = a$$
$$2a = 14$$
$$a = 7$$
$$\therefore \quad b = a + 3 = 7 + 3 = 10$$
$$\therefore \quad a = 7, b = 10.$$

4. If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$ find AB, BA and check if AB = BA.

Sol :

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10 & -6+25 \\ 4+6 & -12+15 \end{bmatrix} = \begin{bmatrix} 12 & 19 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & 5-9 \\ 4+20 & 10+15 \end{bmatrix} = \begin{bmatrix} -10 & -4 \\ 24 & 25 \end{bmatrix}$$

$$\therefore AB \neq BA$$

5. Given that

$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$

verify that A (B + C) = AB + AC.

Sol :

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 4 \\ -1 & 6 & 5 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ -1 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 2+18 & 4+15 \\ 10+1 & 10-6 & 20-5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 20 & 19 \\ 11 & 4 & 15 \end{bmatrix} \quad ...(1)$$

$$AB = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & -1+15 & 2+6 \\ 5-3 & -5-5 & 10-2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 & 8 \\ 2 & -10 & 8 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-12 & 3+3 & 2+9 \\ 5+4 & 15-1 & 10-3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 & 11 \\ 9 & 14 & 7 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 10 & 14 & 8 \\ 2 & -10 & 8 \end{bmatrix} + \begin{bmatrix} -11 & 6 & 11 \\ 9 & 14 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 20 & 19 \\ 11 & 4 & 15 \end{bmatrix} \quad \text{... (2)}$$

From (1), (2)

$\therefore$ A (B + C) = AB + AC , Hence verified.

6. Show that the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ satisfy commutative property AB = BA.

Sol :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & -2+2 \\ 3-3 & -6+1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & 2-2 \\ -3+3 & -6+1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$\therefore AB = BA$

Hence proved.

7. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ Show that

(i) A(BC) = (AB)C

(ii) (A – B)C = AC – BC

(iii) $(A - B)^T = A^T - B^T$

Sol :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 8+14 & 0+20 \\ 8+21 & 0+30 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 20 \\ 29 & 30 \end{bmatrix} \quad \text{... (1)}$$

$$(AB) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & 0+10 \\ 4+3 & 0+15 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 7 & 15 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 6 & 10 \\ 7 & 15 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+10 & 0+20 \\ 14+15 & 0+30 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 20 \\ 29 & 30 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2)

$$A(BC) = (AB)C$$

Hence proved.

(ii) $$(A - B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}$$

$$(A - B)C = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6+2 & 0+4 \\ 0-2 & 0-4 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix} \quad \ldots (1)$$

$$AC = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+4 \\ 2+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 6 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix}$$

$$AC - BC = \begin{bmatrix} 4 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2)

$(A - B)C = AC - BC$

Hence proved.

(iii) $$(A - B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}$$

$$(A - B)^T = \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix} \quad \ldots (1)$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$A^T - B^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2)

$$(A - B)^T = A^T - B^T$$

Hence proved.

8. If $A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}, B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}$ then show that $A^2 + B^2 = I$.

Sol :

$$A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix},$$

$$B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + 0 & 0+0 \\ 0+0 & 0+\cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{bmatrix}$$

$$B^2 = B.B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}\begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta + 0 & 0+0 \\ 0+0 & \sin^2\theta \end{bmatrix}$$

$$\text{Now } A^2 + B^2 = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0+0 \\ 0+0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence Proved.

9. If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ prove that $AA^T = I$.

Sol :

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= I \qquad \therefore AA^T = I$$

Hence Proved.

10. Verify that $A^2 = I$ when A = $\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$

Sol :

$$A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24 & -20+20 \\ 30-30 & -24+25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^2 = I$$

Hence proved.

11. If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ show that $A^2 - (a + d) A = (bc - ad)I_2$

Sol :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$(a + d) A = (a + d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

$$A^2 - (a + d) A$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} \qquad ...(1)$$

$$\text{Now } (bc - ad) I_2 = (bc-ad)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} \qquad ...(2)$$

From (1) and (2)

$$A^2 - (a + d) A = (bc - ad) I_2$$

Hence proved.

12. If A = $\begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$

Sol :

$$A = \begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+45 & 35+4-9 \\ 1+2+40 & 7+4-8 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 30 \\ 43 & 3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 52 & 43 \\ 30 & 3 \end{bmatrix} \quad ...(1)$$

$$B^TA^T = \begin{bmatrix} 1 & 1 & 5 \\ 7 & 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 2 \\ 9 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+45 & 1+2+40 \\ 35+4-9 & 7+4-8 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 43 \\ 30 & 3 \end{bmatrix} \quad ...(2)$$

From (1) and (2)

$$(AB)^T = B^TA^T$$

Hence proved.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I_2 = 0$

Sol :

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I_2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Proved.

Exercise 3.19

Choose the correct answer:

1. A system of 3 linear equations in 3 variables is inconsistent if their planes

(1) intersect only at a point
(2) intersect in a line
(3) coincides with each other
(4) do not intersect **[Ans : 4]**

2. The solution of the system $x + y - 3z = -6$, $-7y + 7z = 7$, $3z = 9$ is.

(1) $x = 1, y = 2, z = 3$
(2) $x = -1, y = 2, z = 3$
(3) $x = -1, y = -2, z = 3$
(4) $x = 1, y = 2, z = 3$ **[Ans : 1]**

Sol :

$$x + y - 3z = -6 \quad ...(1)$$
$$-7y + 7z = 7 \quad ...(2)$$
$$3z = 9 \Rightarrow z = 3 \quad ...(3)$$
$$(2) \Rightarrow -7y + 7z = 7 \quad (\div 7)$$
$$-y + z = 1$$
$$-y + 3 = 1$$
$$y = 3 - 1 = 2$$

Substituting in (1)

$$x + 2 - 3(3) = -6$$
$$x + 2 - 9 = -6$$
$$x = -6 + 7 = 1$$

$\therefore$ Solution is $x = 1, y = 2, z = 3$.

3. If $(x - 6)$ is the HCF of $x^2 - 2x - 24$ and $x^2 - kx - 6$ then the value of k is.

(1) 3 (2) 5
(3) 6 (4) 8 **[Ans : 2]**

Sol :

Given $(x - 6)$ is the HCF of $x^2 - kx - 6$ and $x^2 - 2x - 24$.

$\therefore$ $x^2 - kx - 6$ is divisible by $(x - 6)$

$$(6)^2 - k(6) - 6 = 0$$
$$\Rightarrow 36 - 6k - 6 = 0$$
$$6k = 30$$
$$\Rightarrow k = \frac{30}{6} = 5$$

4. $\frac{3y-3}{y} \div \frac{7y-7}{3y^2}$ is

(1) $\frac{9y}{7}$ (2) $\frac{9y^3}{(21y-21)}$

(3) $\frac{21y^2-42y+21}{3y^3}$ (4) $\frac{7(y^2-2y+1)}{y^2}$ **[Ans : 1]**

Sol :

$$\frac{3y-3}{y} \div \frac{7y-7}{3y^2} = \frac{3(y-1)}{y} \times \frac{3y^2}{7(y-1)} = \frac{9y}{7}$$

5. $y^2 + \frac{1}{y^2}$ **is not equal to**

(1) $\frac{y^4+1}{y^2}$ (2) $\left(y+\frac{1}{y}\right)^2$

(3) $\left(y-\frac{1}{y}\right)^2+2$ (4) $\left(y+\frac{1}{y}\right)^2-2$ **[Ans : 2]**

Sol :

$y^2 + \frac{1}{y^2}$ is not equal to $\left(y+\frac{1}{y}\right)^2$

i.e., $y^2 + \frac{1}{y^2} \neq y^2 + \frac{1}{y^2} + 2$

6. $\frac{x}{x^2-25} - \frac{8}{x^2+6x+5}$ **gives**

(1) $\frac{x^2-7x+40}{(x-5)(x+5}$ (2) $\frac{x^2+7x+40}{(x-5)(x+5)(x+1)}$

(3) $\frac{x^2-7x+40}{(x^2-25)(x+1)}$ (4) $\frac{x^2+10}{(x^2-25)(x+1)}$

[Ans : 3]

Sol :

$$\frac{x}{x^2-25} - \frac{8}{x^2+6x+5} = \frac{x}{(x+5)(x-5)} - \frac{8}{(x+5)(x+1)}$$

$$= \frac{x(x+1)-8(x-5)}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2+x-8x+40}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2-7x+40}{(x^2-25)(x+1)}$$

7. The square root of $\frac{256x^8y^4z^{10}}{25x^6y^6z^6}$ **is equal to**

(1) $\frac{16}{5}\left|\frac{x^2z^4}{y^2}\right|$ (2) $16\left|\frac{y^2}{x^2z^4}\right|$

(3) $\frac{16}{5}\left|\frac{y}{xz^2}\right|$ (4) $\frac{16}{5}\left|\frac{xz^2}{y}\right|$ **[Ans : 4]**

Sol :

$$\sqrt{\frac{256\,x^8\,y^4\,z^{10}}{25\,x^6\,y^6\,z^6}} = \frac{16}{5}\left|\frac{x^4\,y^2\,z^5}{x^3\,y^3\,z^3}\right| = \frac{16}{5}\left|\frac{x\,z^2}{y}\right|$$

8. Which of the following should be added to make $x^4 + 64$ a perfect square.

(1) $4x^2$ (2) $16x^2$

(3) $8x^2$ (4) $-8x^2$ **[Ans : 2]**

Sol :

$$\begin{aligned} x^4+64+16x^2 &= x^4+16x^2+64 \\ &= (x^2)^2+2(8)(x^2)+(8)^2 \\ &= (x^2+8)^2 \end{aligned}$$

9. The solution of $(2x-1)^2 = 9$ is equal to

(1) -1 (2) 2

(3) $-1, 2$ (4) None of these

[Ans : 3]

Sol :

$$\begin{aligned} (2x-1)^2 &= 9 \\ 2x-1 &= \pm 3 \end{aligned}$$

$$\begin{aligned} 2x-1 &= 3 & 2x-1 &= -3 \\ 2x &= 4 & 2x &= -2 \\ x &= 2 & x &= -1 \end{aligned}$$

Solution is $x = -1, 2$.

10. The values of a and b if $4x^4 - 24x^3 + 76x^2 + ax + b$ is a perfect square are

(1) 100, 120 (2) 10, 12

(3) -120, 100 (4) 12, 10 **[Ans : 3]**

Sol :

$\sqrt{4x^4-24x^3+76x^2+ax+b}$

	$2x^2-6x+10$
$2x^2$	$4x^4-24x^3+76x^2+ax+b$
	$4x^4$
	$(-)$
$4x^2-6x$	$-24x^3+76x^2$
	$-24x^3+36x^2$
	$(+)\quad(-)$
$4x^2-12x+10$	$40x^2+ax+b$
	$40x^2-120x+100$
	$(-)\quad(+)\quad(-)$
	0

$\therefore$ a = -120, b = 100.

11. If the roots of the equation $q^2x^2 + p^2x + r^2 = 0$ are the squares of the roots of the equation $qx^2 + px + r = 0$ then, q, p, r are in _____

(1) A.P
(2) G.P
(3) Both A.P and G.P
(4) none of these. **[Ans : 2]**

Sol :

Let α, β be the roots of the equation $qx^2 + px + r = 0$

$$\text{Sum of the roots } \alpha+\beta = \frac{-p}{q}$$

$$\text{Product of the roots } \alpha\beta = \frac{r}{q}$$

Given that α^2, β^2 are the roots of $q^2x^2 + p^2x + r^2 = 0$

$$\therefore \text{Sum of the roots } \alpha^2+\beta^2 = \frac{-p^2}{q^2}$$

$$\text{But } \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\therefore \left(\frac{-p}{q}\right)^2 - \frac{2r}{q} = \frac{-p^2}{q^2}$$

$$\frac{p^2}{q^2} + \frac{p^2}{q^2} = \frac{2r}{q}$$

$$\frac{2p^2}{q^2} = \frac{2r}{q} \Rightarrow \frac{p^2}{q^2} = \frac{r}{q}$$

$p^2 = qr \Rightarrow$ p is a Geometric mean and q, p, r are in Geometric Progression.

12. Graph of a linear polynomial is a

(1) Straight line (2) Circle
(3) Parabola (4) Hyperbola **[Ans : 1]**

13. The number of points of intersection of the quadratic polynomial $x^2 + 4x + 4$ with the X -axis is

(1) 0 (2) 1
(3) 0 or 1 (4) 2 **[Ans : 2]**

Sol :

$$x^2 + 4x + 4 = 0$$
$$(x+2)^2 = 0$$
$$x = -2 \text{ (twice)}$$

Roots are real and equal.

$\therefore$ Point of intersection with X-axis is 1.

14. For the given matrix A = $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 9 & 11 & 13 & 15 \end{bmatrix}$ the order of the matrix A^T is

(1) 2×3 (2) 3×2
(3) 3×4 (4) 4×3 **[Ans : 4]**

Sol :

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 9 & 11 & 13 & 15 \end{bmatrix}, \text{ order of A } = 3 \times 4$$

$$A^T = \begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 11 \\ 5 & 6 & 13 \\ 7 & 8 & 15 \end{bmatrix}, \text{ Order of } A^T = 4 \times 3$$

15. If A is a 2×3 matrix and B is a 3×4 matrix, how many columns does AB have

(1) 3 (2) 4
(3) 2 (5) 5 **[Ans : 2]**

Sol :

Order of A = 2×3
Order of B = 3×4
Order of AB = 2×4
No. of columns in AB = 4

16. If number of columns and rows are not equal in a matrix then it is said to be a

(1) Diagonal matrix
(2) Rectangular matrix
(3) Square matrix
(4) Identity matrix **[Ans : 2]**

17. Transpose of a column matrix is

(1) Unit matrix
(2) Diagonal matrix
(3) Column matrix
(4) Row matrix **[Ans : 4]**

18. Find the matrix X if $2X + \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix}$

(1) $\begin{pmatrix} -2 & -2 \\ 2 & -1 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ (4) $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ **[Ans : 2]**

Sol :

Given $2x + \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix}$

$$2x = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

$$x = \frac{1}{2}\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

19. Which of the following can be calculated from the given matrices.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$

(i) A^2 (ii) B^2 (iii) AB (iv) BA

(1) (i) and (ii) only (2) (ii) and (iii) only

(3) (ii) and (iv) only (4) All of these **[Ans : 3]**

Sol :

Order of A = 3×2

Order of B = 3×3

(i) A^2 is not possible as it is not square matrix.

(ii) B^2 is possible as it is a square matrix.

(iii) AB is not possible. No of columns in A $\neq$ No. of Rows in B.

(iv) BA is possible. Since, No. of columns in B = No. of Rows in A.

20. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$ Which of the following statements are correct?

(i) $AB + C = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ (ii) $BC = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ -4 & 10 \end{bmatrix}$

(iii) $BA + C = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$ (iv) $(AB)C = \begin{bmatrix} -8 & 20 \\ -8 & 13 \end{bmatrix}$

(1) (i) and (ii) only

(2) (ii) and (iii) only

(3) (iii) and (iv) only

(3) all of these **[Ans : 1]**

Sol :

$$AB + C = \begin{pmatrix} 5 & 4 \\ 7 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -2 & 5 \end{pmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$$BC = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ -4 & 10 \end{bmatrix}$$

UNIT EXERCISE - 3

1. Solve

$\frac{1}{3}(x + y - 5) = y - z = 2x - 11 = 9 - (x + 2z)$.

Sol :

$$\frac{1}{3}(x + y - 5) = y - z$$
$$x + y - 5 = 3y - 3z$$
$$\Rightarrow x - 2y + 3z = 5 \quad \text{... (1)}$$
$$y - z = 2x - 11$$
$$\Rightarrow 2x - y + z = 11 \quad \text{... (2)}$$
$$2x - 11 = 9 - (x + 2z)$$
$$2x + x + 2z = 9 + 11$$
$$3x + 2z = 20 \quad \text{... (3)}$$

Consider (1) and (2)

$(2) \times 2 \Rightarrow \quad 4x - 2y + 2z = 22 \quad$ ... (4)

$(4) - (1) \Rightarrow \quad x - 2y + 3z = 5$

$3x - z = 17 \quad$ ...(5)

Now, consider (3) and (5)

$3x + 2z = 20 \quad$ ...(3)

$3x - z = 17 \quad$ (5)

$(3)-(5) \Rightarrow \quad 3z = 3$

$z = 3/3 = 1$

Substituting $z = 1$ in (3)

$3x + 2(1) = 20$

$3x = 20 - 2 = 18$

$$x = \frac{18}{3} = 6$$

Substituting $x = 6, z = 1$ is ...(1)

$$6 - 2y + 3(1) = 5$$
$$6 + 3 - 5 = 2y$$
$$4 = 2y$$
$$y = 2y$$
$$y = \frac{4}{2} = 2$$

$\therefore$ Solution : $x = 6, y = 2, z = 1$.

2. One hundred and fifty students are admitted to a school. They are distributed over three sections A, B and C. If 6 students are shifted from section A to section C, the sections will have equal number of students. If 4 times of students of C exceeds the number of students of section A by the number of students in section B, find the number of students in the three sections.

Sol :

Let the number of students in section A, B and C be 'x', 'y' and 'z' respectively

$\therefore$ Given $x + y + z = 150$ ... (1)

$$x - 6 = z + 6$$
$$x - z = 12 \quad \text{... (2)}$$
$$4z = x + y$$
$$x + y - 4z = 0 \quad \text{... (3)}$$

consider (1) and (3)

$$x + y + z = 150 \quad \text{... (1)}$$
$$x + y - 4z = 0 \quad \text{... (3)}$$

(1) – (3) $\Rightarrow$ $5z = 150$

$$z = \frac{150}{5} = 30$$

substituting in (2)

$$x - 30 = 12$$
$$x = 30 + 12$$
$$x = 42$$

substituting $x = 42, z = 30$ in (1)

$$42 + y + 30 = 150$$
$$y = 150 - 72$$
$$y = 78$$

$\therefore$ The number of students in sections A, B and C are 42, 78, 30 respectively.

3. In a three-digit number, when the tens and the hundreds digit are interchanged the new number is 54 more than three times the original number. If 198 is added to the number, the digits are reversed. The tens digit exceeds the hundreds digit by twice as that of the tens digit exceeds the unit digit. Find the original number.

Sol :

Let the 100's digit be 'x'
10's digit be 'y'
Unit's digit be 'z'

Given $100y + 10x + z - 54 = 3(100x + 10y + z)$

Substituting $290x - 70y + 2z = -54 \ (\div 2)$

$$145x - 35y + z = -27 \quad \text{...(1)}$$
$$100x + 10y + z + 198 = 100z + 10y + x$$

Substituting $99x - 99z = -198 \ (\div 99)$

$$x - z = -2 \quad \text{...(2)}$$
$$y = x + 2(y - z)$$
$$\Rightarrow x + y - 2z = 0 \quad \text{...(3)}$$

Consider (1) and (3)

$$145x - 35y + z = -27 \quad \text{...(1)}$$

(3) $\times$ 35 $\Rightarrow$ $35x + 35y - 70z = 0$ ...(4)

(1) + (4) $\Rightarrow$ $180x - 69z = -27$ ...(5)

consider (5) and (2)

$$180x - 69z = -27$$

(2) $\times$ 69 $\Rightarrow$ $69x - 69z = -138$ ...(6)

(5)-(6) $\Rightarrow$ $111x = 111$

$$x = \frac{111}{111} = 1$$

Substituting $x = 1$ in ...(2)

$$1 - z = -2$$
$$z = 1 + 2 = 3$$

Substituting $x = 1, z = 3$ in (3)

$$1 + y - 6 = 0$$
$$y = 5$$

$\therefore$ solution : $x = 1, y = 5, z = 3$.

The number is 153.

4. Find the least common multiple of $xy(k^2 + 1) + k(x^2 + y^2)$ and $xy(k^2 - 1) + k(x^2 - y^2)$

Sol :

$xy(k^2 + 1) + k(x^2 + y^2)$

Factorizing $xyk^2 + xy + kx^2 + ky^2$

$$= xyk^2 + ky^2 + kx^2 + xy$$

$= ky(kx+y) + x(kx+y)$

$= (kx+y)(ky+x)$

and $xy(k^2-1) + k(x^2-y^2)$

$= xyk^2 - xy + kx^2 - ky^2$

$= xyk^2 + kx^2 - xy - ky^2$

$= kx(ky+x) - y(x+ky)$

$= (kx-y)(ky+x)$

$\therefore$ LCM $= (ky+x)(kx+y)(kx-y)$

$= (ky+x)(k^2x^2-y^2)$

5. Find the GCD of the following by division algorithm
$2x^4 + 13x^3 + 27x^2 + 23x + 7$, $x^3 + 3x^2 + 3x + 1$, $x^2 + 2x + 1$

Sol :

Let $f(x) = 2x^4 + 13x^3 + 27x^2 + 23x + 7$

$g(x) = x^3 + 3x^2 + 3x + 1$

$h(x) = x^2 + 2x + 1$ which is the least degree polynomial.

Now Dividing f (x) by h (x)

$$
\begin{array}{r|l}
 & 2x^2 + 9x + 7 \\
x^2+2x+1 & 2x^4 + 13x^3 + 27x^2 + 23x + 7 \\
 & 2x^4 + 4x^3 + 2x^2 \quad (-) \\ \hline
 & 9x^3 + 25x^2 + 23x \\
 & 9x^3 + 18x^2 + 9x \quad (-) \\ \hline
 & 7x^2 + 14x + 7 \\
 & 7x^2 + 14x + 7 \quad (-) \\ \hline
 & 0
\end{array}
$$

Since the Remainder is zero, h (x) is the GCD of f (x) and h (x)

Now, dividing g (x) by h (x)

$$
\begin{array}{r|l}
 & x + 1 \\
x^2+2x+1 & x^3 + 3x^2 + 3x + 1 \\
 & x^3 + 2x^2 + x \quad (-) \\ \hline
 & x^2 + 2x + 1 \\
 & x^2 + 2x + 1 \quad (-) \\ \hline
 & 0
\end{array}
$$

Remainder is zero, $\therefore$ h (x) is the GCD of g (x) and h (x)

$\therefore$ h (x) divides both f(x) and g (x) completely

$\therefore$ GCD $= x^2 + 2x + 1$

6. Reduce the given Rational expressions to its lowest form

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$ **(ii)** $\dfrac{10x^3-25x^2+4x-10}{-4-10x^2}$

Sol :

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$

$$= \frac{(x^a)^3-(2)^3}{x^{2a}+2x^a+4}$$

$$= \frac{(x^a-2)(x^{2a}+2x^a+4)}{x^{2a}+2x^a+4}$$

$[a^3-b^3 = (a-b)(a^2+ab+b^2]$

$= x^a - 2$

(ii) $\dfrac{10x^3-25x^2+4x-10}{-4-10x^2}$

$$= \frac{10x^2(x-5/2)+4(x-5/2)}{-(4+10x^2)}$$

$$= \frac{(x-5/2)(10x^2+4)}{-(4+10x^2)}$$

$= -(x-5/2)$

$= 5/2 - x$

7. Simplify $\dfrac{\dfrac{1}{p}+\dfrac{1}{q+r}}{\dfrac{1}{p}-\dfrac{1}{q+r}} \times \left(1+\dfrac{q^2+r^2-p^2}{2qr}\right)$

Sol :

$$\frac{\dfrac{1}{p}+\dfrac{1}{q+r}}{\dfrac{1}{p}-\dfrac{1}{q+r}} \times \left(1+\frac{q^2+r^2-p^2}{2qr}\right)$$

$$= \frac{\dfrac{q+r+p}{p(q+r)}}{\dfrac{q+r-p}{p(q+r)}} \times \left[\frac{2qr+q^2+r^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \left[\frac{(q+r)^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \frac{[(q+r+p)(q+r-p)}{2qr}$$

$$= \frac{(p+q+r)^2}{2qr} = \frac{1}{2qr}$$

8. Arul, Ravi and Ram working together can clean a store in 6 hours. Working alone Ravi takes twice as long to clean the store as Arul does. Ram needs three times as long as Arul does. How long would it take each if they are working alone?

Sol :

Given that Arul and Ram working together to clean a store in 6 hours.

$\therefore$ Work done by Arul, Ravi and Ram working together in 1 hour is $\frac{1}{6}$

Let the time taken by Arul, Ravi and Ram be x, y and z respectively.

$\therefore$ Time taken by Arul alone $= \frac{1}{x}$

$\therefore$ Time taken by Ravi alone $= \frac{1}{y}$

$\therefore$ Time taken by Ram alone $= \frac{1}{z}$

Given $y = 2x$ and $z = 3x$

$\therefore$ Total work done in 1 hour is

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{6}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{2x}+\frac{1}{3x} = \frac{1}{6}$$

$$\frac{6+3+2}{6x} = \frac{1}{6}$$

$$\frac{11}{6x} = \frac{1}{6}$$

$$\Rightarrow x = 11$$

$$\therefore y = 22,\ z = 33.$$

$\therefore$ Time taken by three persons are 11 hrs, 22 hrs and 33 hrs respectively.

9. Find the square root of $289x^4 - 612x^3 + 970x^2 - 684x + 361$

	$17x^2 - 18x + 19$	
$17x^2$	$289x^4 - 612x^3 + 970x^2 - 684x + 361$	
	$289x^4$	(–)
$34x^2 - 18x$	$-612x^3 + 970x^2$	
	$-612x^3 + 324x^2$	(–)
$34x^2 - 36x + 19$	$646x^2 - 684x + 361$	
	$646x^2 - 684x + 361$	(–)
	0	

$$\therefore \sqrt{289x^4 - 612x^3 + 970x^2 - 684x + 361}$$

$$= |17x^2 - 18x + 19|$$

10. Solve $\sqrt{y+1} + \sqrt{2y-5} = 3$

Sol :

$$\sqrt{2y-5} = 3 - \sqrt{y+1}$$

Squaring on both sides.

$$\left(\sqrt{2y-5}\right)^2 = \left(3-\sqrt{y+1}\right)^2$$

$$2y - 5 = 9 + y + 1 - 2(3)\sqrt{y+1}$$

$$2y - 5 = y + 10 - 6\sqrt{y+1}$$

$$2y - y - 5 - 10 = -6\sqrt{y+1}$$

$$y - 15 = -6\sqrt{y+1}$$

Again, squaring on both rides.

$$(y-15)^2 = \left(-6\sqrt{y+1}\right)^2$$

$$y^2 - 30y + 225 = 36(y+1)$$

$$y^2 - 30y + 225 = 36y + 36$$

$$y^2 - 30y - 36y + 225 - 36 = 0$$

$$y^2 - 66y + 189 = 0$$

Factorizing $\Rightarrow (y-63)(y-3) = 0$

$$\therefore y = 3, 63$$

11. A boat takes 1.6 hours longer to go 36 kms up a river than down the river. If the speed of the water current is 4 km per hr, what is the speed of the boat in still water?

Sol :

Let the speed of the boat in still water be x km/hr.

Speed of the river = 4 km/hr

Speed of the boat upstream = Speed of the boat in still water – Speed of River

$\therefore$ Speed of the boat upstream = $(x - 4)$ km/hr.

Speed of the boat down stream. = (Speed of boat in still water) + (speed of River) = $(x + 4)$ km/hr.

Time of Upstream journey = Time for downstream journey + 1.6 hr.

$$\therefore \frac{\text{Distance covered upstream}}{\text{Speed upstream}} = \frac{\text{Distance covered down stream}}{\text{Speed upstream}} + 1.6\text{ hr}$$

$$\frac{36}{x-4} = \frac{36}{x+4} + 1.6$$

$$\frac{36}{x-4}-\frac{36}{x+4} = 1.6$$

$$36\left[\frac{1}{x-4}-\frac{1}{x+4}\right] = 1.6$$

$$36\left[\frac{x+4-x+4}{(x-4)(x+4)}\right] = 1.6$$

$$\frac{36(8)}{x^2-16} = 1.6$$

$$\Rightarrow \quad x^2-16 = \frac{288}{1.6}$$

$$x^2 - 16 = 180$$
$$x^2 = 180 + 16$$
$$x^2 = 196$$
$$x = 14$$

$\therefore$ Speed of the boat in still water is 14 km/hr.

12. Is it possible to design a rectangular park of perimeter 320 m and area 4800 m^2? If so find its length and breath.

Sol :

Let the length of the rectangular Park be 'x' m breadth of the rectangular Park be 'y' m.

Given Perimeter = 320 m

i.e, $2(x + y) = 320$

$x + y = 160$

$y = 160 - x$

Given, Area = 4800

$\Rightarrow \quad xy = 4800$

$\Rightarrow \quad x(160 - x) = 4800$

$\Rightarrow \quad 160x - x^2 = 4800$

$\Rightarrow \quad x^2 - 160x + 4800 = 0$

Factorizing $\Rightarrow \quad (x - 120)(x - 40) = 0$

$x - 120 = 0,$

$x - 40 = 0$

$x = 120,$

$x = 40$

$\therefore$ length of the park = 120 m

breadth of the park = 40 m.

13. At t minutes past 2 pm. the time needed to 3 pm is 3 minute less than $\frac{t^2}{4}$. Find t.

Sol :

Given that, time needed by the minute hand to show 3 pm is $\frac{t^2}{4} - 3$.

$\therefore$ By the given data.

$$\frac{t^2}{4} - 3 = 60 - t$$

[$\because$ *From 2 pm to 3 pm* = 1 *hr* = 60 min]

$t^2 - 12 = 4(60 - t)$

$t^2 - 12 = 240 - 4t$

$t^2 + 4t - 252 = 0$

Factorizing, $\quad (t + 18)(t - 14) = 0$

$t = -18$ and

$t = 14$

$t = -18$ is rejected as 't' cannot be negative.

$\therefore t = 14$ minutes.

14. The number of seats in a row is equal to the total number of rows in a hall. The total number of seats in the hall will increase by 375 if the number of rows is doubled and the number of seats in each row is reduced by 5. Find the number of rows in the hall at the beginning.

Sol :

Let the number of rows be 'x'

and number of seats in each row is also 'x'

$\therefore$ total no. of seats = $x.x = x^2$

From the given data, we get

$x^2 + 375 = 2x(x - 5)$

$x^2 + 375 = 2x^2 - 10x$

Simplifying, $\quad 2x^2 - x^2 - 10x - 375 = 0$

$x^2 - 10x - 375 = 0$

Factorizing, $\quad (x - 25)(x + 15) = 0$

$x - 25 = 0, \quad x + 15 = 0$

$x = 25, \quad x = -15$ is rejected as 'x cannot be negative.

$\therefore$ No of rows in the hall at the beginning is 25.

15. If α and β are the roots of the polynomial f (x) = $x^2 - 2x + 3$, find the polynomial whose roots are (i) $\alpha+2, \beta+2$ (ii) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$.

Sol :

Given α and β are the zeros of

$f(x) = x^2 - 2x + 3$

$a = 1, b = -2, c = 3.$

Sum of the zeroes $\alpha+\beta = \frac{-b}{a} = \frac{2}{1} = 2$

Product of the zeroes $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$

(i) $\alpha+2, \beta+2$ are the zeroes.

$$\text{sum of the zeroes} = \alpha+2+\beta+2 = \alpha+\beta+4 = 2+4 = 6$$

$$\text{Product of the zeroes} = (\alpha+2)(\beta+2) = \alpha\beta+2(\alpha+\beta)+4 = 3+2(2)+4 = 3+4+4 = 11$$

$\therefore$ the polynomial is

x^2 – (sum of the zeroes) x + (Product of the zeroes) = 0

$$x^2 - 6x + 11 = 0$$

(ii) $\dfrac{\alpha-1}{\alpha+1}, \dfrac{\beta-1}{\beta+1}$ are zeroes

$$\text{Sum of zeroes} = \frac{\alpha-1}{\alpha+1}+\frac{\beta-1}{\beta+1}$$

$$= \frac{(\alpha-1)(\beta+1)+(\beta-1)(\alpha+1)}{(\alpha+1)(\beta+1)}$$

$$= \frac{2\alpha\beta-2}{\alpha\beta+(\alpha+\beta)+1}$$

$$= \frac{2(3)-2}{3+2+1}$$

$$= \frac{6-2}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\text{Product of zeroes} = \left(\frac{\alpha-1}{\alpha+1}\right)\left(\frac{\beta-1}{\beta+1}\right)$$

$$= \frac{\alpha\beta-(\alpha+\beta)+1}{\alpha\beta+(\alpha+\beta)+1}$$

$$= \frac{3-2+1}{3+2+1}$$

$$= \frac{2}{6} = \frac{1}{3}$$

$\therefore$ the polynomial is

x^2 – (sum of the zeroes) x + (Product of the zeroes) = 0

$$x^2 - \frac{2}{3}x + \frac{1}{3} = 0$$

it is simplified as $3x^2 - 2x + 1$

16. If – 4 is a root of the equation $x^2 + px - 4 = 0$ and if the equation $x^2 + px + q = 0$ has equal roots, find the values of p and q.

Sol :

Given that – 4 is a root of the equation.

$$x^2 + px - 4 = 0$$

$$\therefore (-4)^2 + p(-4) - 4 = 0$$

$$16 - 4p - 4 = 0$$

$$\Rightarrow 4p = 12$$

$$p = \frac{12}{4} = 3 \text{ and}$$

$x^2 + px + q = 0$ has equal roots

$\therefore$ Discriminant $b^2 - 4ac = 0$

$[\because a = 1, b = p, c = q]$

$$p^2 - 4(1)(q) = 0$$

$$(3)^2 - 4q = 0$$

$$4q = 9$$

$$q = \frac{9}{4}$$

$$\therefore p = 3, \quad q = \frac{9}{4}$$

17. Two formers Senthil and Ravi cultivates three varieties of grains namely rice, wheat and ragi. If the sale (in ₹) of three varieties of grains by both the formers in the month of April is given by the matrix.

April sale in ₹

$$A = \begin{array}{c} \begin{array}{ccc} \textit{rice} & \textit{wheat} & \textit{ragi} \end{array} \\ \begin{bmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{bmatrix} \end{array} \begin{array}{l} \textit{Senthil} \\ \textit{Ravi} \end{array}$$

and the may sale (in ₹) is exactly twice as that of the April month sale for each variety.

(i) What is the average sales of the months April and May.

(ii) If the sales continues to increase in the same way in the successive months, what will be sales in the month of August?

Sol :

Sale of 'April' month

$$A = \begin{array}{c} \begin{array}{ccc} \textit{rice} & \textit{wheat} & \textit{ragi} \end{array} \\ \begin{bmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{bmatrix} \end{array} \begin{array}{l} \textit{Senthil} \\ \textit{Ravi} \end{array}$$

Given that May month's sale is exactly twice the sale in April month.

$\therefore$ Sale of 'May' month

$$= \begin{bmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{bmatrix}$$

(i) The average sales of April and May

$$= \frac{1}{2}\begin{bmatrix} 500+1000 & 1000+2000 & 1500+3000 \\ 2500+5000 & 1500+3000 & 500+1000 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1500 & 3000 & 4500 \\ 7500 & 4500 & 1500 \end{bmatrix}$$

$$= \begin{bmatrix} 750 & 1500 & 2250 \\ 3750 & 2250 & 750 \end{bmatrix}$$

(ii) Sales in the month of April

$$= \begin{bmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{bmatrix}$$

Sales in the month of May

$$= \begin{bmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{bmatrix}$$

Similarly in the month of August

$$= \begin{bmatrix} 8000 & 16000 & 24000 \\ 40000 & 24000 & 8000 \end{bmatrix}$$

18. If $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} x & -\cos\theta \\ \cos\theta & x \end{bmatrix} = I_2$, find x.

Sol :

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} x\sin\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & x\sin\theta \end{bmatrix} = I_2$$

$$\begin{bmatrix} \cos^2\theta + x\sin\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & x\sin\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x\sin\theta + \cos^2\theta & 0 \\ 0 & x\sin\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x\sin\theta + \cos^2\theta = 1$$

$$x\sin\theta = 1 - \cos^2\theta$$

$$x = \frac{\sin^2\theta}{\sin\theta}$$

$$\therefore x = \sin\theta.$$

19. Given $A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ and if $BA = C^2$. Find p and q.

Sol :

$$A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0-2q \\ p+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix}$$

$$C^2 = C.C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & -4-4 \\ 4+4 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

Given $BA = C^2$

$$\begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

$$\therefore -2q = -8$$

$$q = \frac{8}{2} = 4$$

$$p = 8$$

20. $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 8 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ find the matrix D, such that CD – AB = 0.

Sol :

$$\left.\begin{aligned} &\text{Order of } A = 2\times 2 \\ &\text{Order of } B = 2\times 2 \end{aligned}\right\}$$

$\therefore$ Order of AB = Order of CD

$$\left.\begin{aligned} &\text{Order of } C = 2\times 2 \\ &\text{and } CD - AB = 0 \end{aligned}\right\}$$

$\therefore$ Order of CD = 2×2

$$\text{Let } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given CD – AB = 0

$$\begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3a+6c & 3b+6d \\ a+c & b+d \end{bmatrix} - \begin{bmatrix} 18+0 & 9+0 \\ 24+40 & 12+25 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3a+6c-18 & 3b+6d-9 \\ a+c-64 & b+d-37 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, Equating the Corresponding elements.

$3a + 6c - 18 = 0$

$3a + 6c = 18 \; (\div 3)$

$a + 2c = 6$ ...(1)

$a + c - 64 = 0$

$a + c = 64$ ...(2)

$3b + 6d - 9 = 0$

$3b + 6d = 9 \; (\div 3)$

$b + 2d = 3$ ...(3)

$b + d - 37 = 0$

$b + d = 37$ ...(4)

Consider equation (1) and (2)

$a + 2c = 6$ (1)

$a + c = 64$ (2)

(1) – (2) ⇒ $c = -58$

Substituting in (2)

$a - 58 = 64$

$a = 64 + 58 = 122$

Now, Consider equation (3) and (4)

$b + 2d = 3$ (3)

$b + d = 37$ (4)

(3) – (4) ⇒ $d = -34$

Substituting in (4)

$b - 34 = 37$

$b = 37 + 34 = 71$

$\therefore$ The matrix $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} 122 & 71 \\ -58 & -34 \end{bmatrix}$

CREATIVE QUESTIONS

I. Multiple Choice Questions:

Simultaneous Linear Equations in three variables

1. The value of x in (x + 2) + 2 (x – 1) = 4x – 3

(1) 2 (2) 3

(3) – 2 (4) – 3 **[Ans: (2)]**

Sol :

$(x + 2) + 2(x - 1) = 4x - 3$

$x + 2 + 2x - 2 = 4x - 3$

$3x = 4x - 3$

$4x - 3x = 3$

$x = 3$

2. Solve for x: $\left(x-\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = x + 2$

(1) 4 (2) 8

(3) – 4 (4) – 8 **[Ans: (1)]**

Sol : $\left(x-\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = x + 2$

$x^2 + \frac{1}{4} - x - x^2 - \frac{9}{4} + 3x = x + 2$

Simplifying we get, $2x - 2 = x + 2$

$x = 4$

3. If x men can do a piece of work in y days, in how many days will z men do the same work?

(1) $\frac{xz}{y}$ (2) $\frac{xy}{z}$

(3) $\frac{yz}{x}$ (4) xyz **[Ans : (2)]**

Sol :

x men can do the work in y days

1 man can do the work in xy days.

$\therefore$ z men can do the work in $\frac{xy}{z}$ days.

4. Find the value of x and y if $\frac{5}{y}-\frac{2}{x}=\frac{7}{6}$ and $\frac{36}{x}-\frac{24}{y}=1$

(1) x = 4, y = 3 (2) x = – 4, y = 3
(3) x = – 4, y = – 3 (4) x = 4, y = – 3

[Ans : (1)]

Sol :

$$(1)\times 18 \Rightarrow \quad \frac{-36}{x}+\frac{90}{y} = 21$$

$$(2) \Rightarrow \quad \frac{36}{x}-\frac{24}{y} = 1$$

Adding

$$\frac{66}{y} = 22$$

$$y = 3$$

Substituting in (2) we get x = 4; y = 3.

5. What should be the value of p if 3x + 2y = 8 and 6x + 4y = 9 have infinitely many solutions?

(1) 3 (2) 16
(3) 5 (4) 6 **[Ans : (2)]**

Sol :

If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are the two equations, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ should be true for infinitely many solutions.

$$\therefore \frac{3}{6} = \frac{2}{4} = \frac{8}{p}$$

$$p = 16$$

6. What should be the value of m in the pair of equations 4x + my + 9 = 0 and 3x + 4y + 18 = 0 to have unique solution?

(1) $m \neq 16$ (2) $m \neq 15$
(3) $m \neq \frac{16}{3}$ (4) $m \neq \frac{15}{4}$ **[Ans : (3)]**

Sol :

For unique solution, $a_1b_2 \neq b_1a_2$

$$4 \times 4 \neq 3m$$
$$3m \neq 16$$
$$m \neq \frac{16}{3}$$

7. If the sum of two numbers is 640 and their difference is 280, then the numbers are

(1) 140, 500 (2) 180, 460
(3) 130, 510 (4) 150, 490 **[Ans : (2)]**

Sol :

Let the numbers be x and y

Then, Given x + y = 640 ... (1)
x – y = 280 ... (2)

Adding 2x = 920
⇒ x = 460

Substituting in (1) 460 + y = 640
y = 640 – 460 = 180

8. The total salary of 15 men and 8 women in ₹ 3050. The difference of salaries of 5 women and 3 men is ₹ 50. Find the sum of the salaries of 3 men and 5 women.

(1) ₹ 900 (2) ₹ 850
(3) ₹ 950 (4) ₹ 1000 **[Ans : (3)]**

Sol :

Let the man's salary be 'x'

Let the woman's salary be 'y'

Given 15 x + 8y = 3050 ... (1)
3x – 5y = – 50 ... (2)

(2) × 5 ⇒ 15x – 25y = – 250
15x + 8y = 3050

Subtracting – 33y = – 3300
y = 100

Substituting in (2), 3x – 500 = – 50
3x = 450
x = 150

Total salary = 3(150) + 5(100)
= 450 + 500 = ₹ 950

9. Find the solution to the system x + y + z = 2, 6x – 4y + 5z = 31 and 5x + 2y + 2z = 13

(1) (3, – 2, 1) (2) (2, – 3, 1)
(3) (1, 2, 3) (4) (–1, – 2, – 3)

[Ans : (1)]

Sol :

Substituting the given answers in all the 3 equations, we get the answer.

(3, – 2, 1) satisfying all the three equations.

10. The solution of the system of equations 4x + 2y – 4z = – 18, 8x – 2y – 5z = – 18 and – 16x – 2y – z = – 2

(1) (1, 0, 4) (2) (4, 0, 1)
(3) (0, – 1, – 4) (4) (0, – 1, 4) **[Ans : (4)]**

Sol :

(0, – 1, 4) Satisfying all the three equations.

GCD and LCM

11. The GCD of two numbers is 36 and their LCM is 648. The product of two numbers is

(1) 23328 (2) 648
(3) 3888 (4) 23348 **[Ans : (1)]**

Sol :

$$\text{Product of two numbers} = \text{GCD} \times \text{LCM} = 36 \times 648 = 23328$$

12. The GCD of $10(x^2 + x - 20)$, $15(x^2 - 3x - 4)$ and $20(x^2 + 2x + 1)$ is

(1) $5(x - 4)$ (2) 5
(3) $5(x + 1)$ (4) $5(x + 1)(x - 1)$

[Ans : (2)]

Sol :

GCD of 10, 15 and 20 is 5

Now $x^2 + x - 20 = (x + 5)(x - 4)$

$x^2 - 3x - 4 = (x - 4)(x + 1)$

$x^2 + 2x + 1 = (x + 1)^2$

$\therefore$ GCD $= 5$

13. The LCM of $a^2 + 3a + 2$, $a^2 + 5a + 6$ and $a^2 + 4a + 4$ is

(1) $(a + 2)^2 (a + 3)$
(2) $(a + 2)^2 (a + 1)$
(3) $(a + 2)^2 (a + 3)(a + 1)$
(4) $(a + 3)(a + 2)(a + 1)$ **[Ans : (3)]**

Sol :

$a^2 + 3a + 2 = (a + 2)(a + 1)$

$a^2 + 5a + 6 = (a + 3)(a + 2)$

$a^2 + 4a + 4 = (a + 2)^2$

LCM $= (a + 2)^2 (a + 3)(a + 1)$

14. How many times 5 bells ring together in 1 hour if they start together and ring at intervals of 2, 3, 4, 5 and 6 sec respectively?

(1) 71 times (2) 60 times
(3) 59 times (4) 61 times **[Ans : (2)]**

Sol :

LCM of 2, 3, 4, 5 and 6 is 60 sec = 1 min.

Therefore, they ring together once in a minute and hence, 60 times in an hour.

15. GCD of $x^2 - \frac{1}{x^2}$, $x^2 - 2 + \frac{1}{x^2}$ and $x^3 - \frac{1}{x^3} - 3x - \frac{3}{x}$ is

(1) $x^2 - \frac{1}{x^2}$ (2) $\left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)$
(3) $x - \frac{1}{x}$ (4) $\left(x - \frac{1}{x}\right)^2$ **[Ans : (3)]**

Sol :

$$x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$x^2 - 2 + \frac{1}{x^2} = x^2 - 2x\frac{1}{x^2} + \left(\frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)^2$$

$$x^3 - \frac{1}{x^3} - 3x - \frac{3}{x} = x^3 - \left(\frac{1}{x}\right)^3 - 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 = \left(x - \frac{1}{x}\right)^3$$

$$\therefore \text{GCD} = \left(x - \frac{1}{x}\right)$$

16. If the GCD and LCM of two expressions are $x + 2$ and $(x + 2)^2 (x - 2)$ respectively, then the two expressions are

(1) $(x + 2), (x - 2)$ (2) $(x + 2)^2, (x^2 - 4)$
(3) $(x + 2), (x^2 - 4)$ (4) $(x + 2)^2, (x - 2)$

[Ans : (2)]

Sol :

GCD $= x + 2$,

LCM $= (x + 2)^2 (x - 2)$

$$\frac{LCM}{GCD} = \frac{(x + 2)^2 (x - 2)}{(x + 2)} = (x + 2)(x - 2)$$

The required expressions are $(x + 2)(x + 2)$ and $(x + 2)(x - 2)$

i.e., $(x + 2)^2$ and $(x^2 - 4)$

17. The GCD of $x^2 + 3x + 2$ and $x^3 + 9x^2 + 23x + 15$ is

(1) $x + 1$ (2) $x + 2$
(3) $(x + 1)(x + 2)$ (4) $(x + 1)(x - 1)$

[Ans : (1)]

Sol :

$x^2 + 3x + 2 = (x + 2)(x + 1)$

$(x + 1)$ is a factor of $x^3 + 9x^2 + 23x + 15$ as $x + 1$

divides it completely whereas (x + 2) does not divide it.

$\therefore$ (x + 1) is the common divisor.

18. The biggest length of a tape that can be used for measuring cloth of lengths 3 m, 5 m, 10 m and 90 m is

(1) 30 (2) 50

(3) 90 (4) 100 **[Ans : (3)]**

Sol :

LCM of 3, 5, 10 and 90 is 90

19. The LCM of two numbers is a + b and the GCD is k (a – b). If one of the numbers is k, the other number is

(1) $\frac{ka}{b}$ (2) kab

(3) $a^2 - b^2$ (4) $\frac{ka+b}{ka-b}$ **[Ans : (3)]**

Sol :

$$kx = (a+b)\,k\,(a-b)$$

$$\therefore \quad x = \frac{(a+b)\,k\,(a-b)}{k} = (a+b)(a-b) = a^2 - b^2$$

20. The GCD and LCM of a and b are 27 and 2079 respectively. If a is divided by 9, the quotient is 21. Then b is

(1) 243 (2) 189

(3) 113 (4) 297 **[Ans : (4)]**

Sol :

$$ab = 27 \times 2079$$

$$a = 9 \times 21 = 189$$

$$\therefore \quad b = \frac{27 \times 2079}{189} = 297$$

21. The LCM of $8x^4y^2z^3$, $10xy^3z^5$ and $12x^2y^2z^4$ is

(1) $120x^2y^2z^2$ (2) $120x^4y^3z^5$

(3) $2xy^2z^3$ (4) $120x^3y^3z^5$ **[Ans : (2)]**

Sol :

LCM of 8, 10 and 12 is 120

LCM of $x^4y^2z^3$, xy^3z^5 and $x^2y^2z^4$ is $x^4y^3z^5$

22. The LCM of $x^2 - 3ax + 2a^2$, $x^2 - 4ax + 4a^2$ and $x^2 - ax - 2a^2$ is

(1) $(x-2a)^2(x^2-a^2)$

(2) $(x-a)^2(x-2a)$

(3) $(x-a)(x-2a)(x-3a)$

(4) $(x-2a)^3$ **[Ans : (1)]**

Sol :

$$x^2 - 3ax + 2a^2 = (x-a)(x-2a)$$

$$x^2 - 4xa + 4a^2 = (x-2a)^2$$

$$x^2 - ax - 2a^2 = (x-2a)(x+a)$$

$$\text{LCM} = (x-2a)^2(x+a)(x-a) = (x-2a)^2(x^2-a^2)$$

Rational expressions

23. Simplified form of $\frac{x^3-3x^2}{9x^2-x^4}$ is

(1) $\frac{1}{x+3}$ (2) $-\frac{1}{x+3}$

(3) $\frac{1}{3(x+1)}$ (4) $\frac{1}{3(x-1)}$ **[Ans : (2)]**

Sol :

$$\frac{x^3-3x^2}{9x^2-x^4} = \frac{x^2(x-3)}{x^2(3^2-x^2)} = \frac{(x-3)}{(3+x)(3-x)} = \frac{-1}{x+3}$$

24. Simplest form of $\frac{a^2-b^2}{a^2-3ab+2b^2}$ is

(1) $\frac{(a+b)^2}{a-2b}$ (2) $\frac{a+b}{a-2b}$

(3) $\frac{a-b}{a-2b}$ (4) $\frac{a+b}{a+2b}$ **[Ans : (2)]**

Sol :

$$\frac{a^2-b^2}{a^2-3ab+2ab^2} = \frac{(a+b)(a-b)}{(a-2b)(a-b)} = \frac{a+b}{a-2b}$$

25. Simplest form of $\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \frac{1}{(x+3)(x+1)}$ is

(1) $\frac{1}{(x+1)(x+3)}$ (2) $\frac{2}{(x+1)(x+3)}$

(3) $\frac{3}{(x+2)(x+3)}$ (4) $\frac{3}{(x+1)(x+3)}$ **[Ans : (4)]**

Sol :

$$\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+1)}$$

$$= \frac{(x+3)+(x+1)+(x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{3x+6}{(x+1)(x+2)(x+3)}$$

$$= \frac{3(x+2)}{(x+1)(x+2)(x+3)} = \frac{3}{(x+1)(x+3)}$$

26. If x = 2 $\left(t+\frac{1}{t}\right)$ and y = 3 $\left(t-\frac{1}{t}\right)$ and then $\frac{x^2}{4}-\frac{y^2}{9}$ is

(1) 3 (2) -4
(3) 4 (4) -3 **[Ans : (3)]**

Sol :

$$\frac{x^2}{4}-\frac{y^2}{9} = \frac{2^2\left[t+\frac{1}{t}\right]^2}{4}-\frac{3^2\left[t-\frac{1}{t}\right]^2}{9}$$

$$= \left[t+\frac{1}{t}\right]^2-\left[t-\frac{1}{t}\right]^2$$

$$= 2+2 = 4$$

27. Simplified form of $\dfrac{p+p^2+p^3+p^4+p^5+p^6+p^7}{p^{-3}+p^{-4}+p^{-5}+p^{-6}+p^{-7}+p^{-8}+p^{-9}}$

(1) p^{10} (2) p^{-10}
(3) p^9 (4) p^{-9} **[Ans : (1)]**

Sol :

$$\frac{p+p^2+p^3+p^4+p^5+p^6+p^7}{p^{-3}+p^{-4}+p^{-5}+p^{-6}+p^{-7}+p^{-8}+p^{-9}}$$

$$= \frac{p(1+p+p^2+p^3+p^4+p^5+p^6)}{p^{-9}(p^6+p^5+p^4+p^3+p^2+p+1)}$$

$$= p^1 . p^9 = p^{1+9} = p^{10}$$

28. Simplest form of $\dfrac{x^7+2x^6+x^5}{x^3(x+1)^8}$ is

(1) $\frac{x^2}{(x^6+1)}$ (2) $\frac{x^2}{(x+1)^6}$
(3) $\frac{x^3}{x+1}$ (4) $\frac{x^4}{x+2}$ **[Ans : (2)]**

Sol :

$$\frac{x^7+2x^6+x^5}{x^3(x+1)^8} = \frac{x^5(x^2+2x+1)}{x^3(x+1)^8}$$

$$= \frac{x^2(x+1)^2}{(x+1)^8} = \frac{x^2}{(x+1)^6}$$

29. $\dfrac{x^2-5x-14}{x^2-3x+2}\times\dfrac{x^2-4}{x^2-14x+49}$ =

(1) $\frac{x+2}{x+7}$ (2) $\frac{(x+2)^2}{x-7}$
(3) $\frac{(x+2)^2}{(x-1)(x-7)}$ (4) $\frac{x-2}{(x-1)(x-7)}$
[Ans : (3)]

Sol :

$$\frac{x^2-5x-14}{x^2-3x+2}\times\frac{x^2-4}{x^2-14x+49}$$

$$= \frac{(x-7)(x+2)}{(x-2)(x-1)}.\frac{(x+2)(x-2)}{(x-7)^2}$$

$$= \frac{(x+2)^2}{(x-1)(x-7)}$$

30. $\dfrac{m^2-9}{m^2+5m+6}\div\dfrac{3-m}{m+2}$ = ?

(1) 1 (2) 3
(3) -3 (4) -1 **[Ans : (4)]**

Sol :

$$\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}$$

$$= \frac{(m+3)(m-3)}{(m+3)(m+2)}\times\frac{(m+2)}{-(m-3)}$$

$$= \frac{m-3}{-(m-3)} = -1$$

31. Simplify : $\dfrac{(y^2+5y+4)}{\left(\dfrac{y^2-1}{y+5}\right)}$

(1) $\frac{y-1}{y-4}$ (2) $\frac{y+5}{y-1}$
(3) $\frac{(y+4)(y+3)}{y-1}$ (4) $\frac{y^2+9y+20}{y-1}$ **[Ans : (4)]**

Sol :

$$\frac{(y^2+5y+4)}{\left(\frac{y^2-1}{y+5}\right)} = \frac{(y+4)(y+1)}{(y+1)(y-1)} \times (y+5)$$

$$= \frac{(y+4)(y+5)}{y-1} = \frac{y^2+9y+20}{y-1}$$

32. $\frac{x^2}{x+3} + \frac{11x+24}{x+3} =$

(1) x + 8 (2) x – 8

(3) 8 – x (4) x + 3 **[Ans : (1)]**

Sol :

$$\frac{x^2}{x+3} + \frac{11x+24}{x+3} = \frac{x^2+11x+24}{x+3}$$

$$= \frac{(x+3)(x+8)}{x+3} = (x+8)$$

33. What is the result in simplest form when $\frac{4x-5}{x^2-64}$ **is subtracted from** $\frac{5x+3}{x^2-64}$

(1) x – 8 (2) $(x-8)^{-1}$

(3) $(x-8)^{-2}$ (4) $(x-8)^{-3}$ **[Ans : (2)]**

Sol :

$$\frac{5x+3}{x^2-64} - \frac{4x-5}{x^2-64} = \frac{5x+3-4x+5}{x^2-64}$$

$$= \frac{x+8}{(x+8)(x-8)}$$

$$= \frac{1}{x-8} = (x-8)^{-1}$$

34. Excluded values of $\frac{2x+1}{x^2-x-6}$ **are**

(1) 1, – 2 (2) – 2, 3

(3) 2, – 3 (4) 2, 3 **Ans : (2)]**

Sol :

$$\frac{2x+1}{x^2-x-6} = \frac{2x+1}{(x-3)(x+2)}$$

When x = – 2 and x = 3, denominator becomes zero.

$\therefore$ Excluded values are – 2, 3.

35. Excluded values of $\frac{4x-2}{2x^2+x-1}$ **is / are**

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$

(3) $-\frac{1}{2}$ (4) – 1 **[Ans : (4)]**

Sol :

$$\frac{4x-2}{2x^2+x-1} = \frac{2(2x-1)}{(2x-1)(x+1)} = \frac{2}{x+1}$$

When x = – 1, the denominator becomes zero.

$\therefore$ – 1 is the excluded value.

36. If a + b + c = 0, then the value of $\frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca}$ **is**

(1) 0 (2) 1

(3) 2 (4) 3 **[Ans : (4)]**

Sol :

$$\frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca}$$

Given a + b + c = 0 $\Rightarrow$ a + b = – c, b + c = – a, c + a = – b

$$\therefore \frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} =$$

$$\frac{c^2}{ab} + \frac{a^2}{bc} + \frac{b^2}{ca} = \frac{a^3+b^3+c^3}{abc}$$

If a + b + c = 0, then $a^3 + b^3 + c^3 = 3abc$ (Algebraic Identity)

$$\therefore = \frac{3abc}{abc} = 3$$

Square root of Rational Expression and Polynomials

37. Number of methods to find square root of an algebraic expression are

(1) 3 (2) 4

(3) 5 (4) 2 **[Ans : (4)]**

38. The square root of (x + 1) (x + 2) (x + 3) (x + 4) + 1 is

(1) $x^2 + 2x + 3$ (2) $x^2 + 5x + 5$

(3) $x^2 + 3x + 2$ (4) $x^2 + 2x + 1$ **[Ans : (2)]**

Sol :

$(x + 1)(x + 2)(x + 3)(x + 4) + 1$

Rearranging $= (x + 1)(x + 4)(x + 2)(x + 3) + 1$

$= (x^2 + 5x + 4)(x^2 + 5x + 6) + 1$

Substituting $x^2 + 5x = a$

Then

$(a + 4)(a + 6) + 1 = a^2 + 10a + 24 + 1$

$= a^2 + 10a + 25 = (a + 5)^2$

$\therefore$ Square root of

$(a + 5)^2 = |a + 5|$

i.e., $= |x^2 + 5x + 5|$

39. If the polynomial $16x^4 - 24x^3 + 41x^2 - mx + 16$ be a perfect square, then the value of 'm' is

(1) 12 (2) – 12

(3) 24 (4) – 24 **[Ans : (3)]**

40. $\sqrt{\frac{x^{-7}y^{14}}{x^{14}y^{-28}}} \div \sqrt{\frac{x^{-15}y^{25}}{x^{10}y^{-15}}}$

(1) $\sqrt{x^3y^5}$ (2) xy

(3) x^2y (4) xy^2 **[Ans : (3)]**

Sol :

$$\sqrt{\frac{x^{-7}y^{14}}{x^{14}y^{-28}}} \div \sqrt{\frac{x^{-15}y^{25}}{x^{10}y^{-15}}} = \sqrt{\frac{x^{-7}y^{14}}{x^{14}y^{-28}} \times \frac{x^{10}y^{-15}}{x^{-15}y^{25}}}$$

$$= \sqrt{\frac{y^{-1}}{x^{-1}} \times \frac{x^3}{y^{-3}}}$$

$$= \sqrt{x^4y^2} = x^2y$$

41. $\sqrt{(4a^2)(6b^2)(3a^2b^2)}$ **=**

(1) a^2b^2 (2) $6\sqrt{2}\,a^2b^2$

(3) $72a^4b^4$ (4) a^4b^4 **[Ans : (2)]**

Quadratic equation

42. Which of the following is a quadratic equation?

(1) $x^{1/2} + 2x + 3 = 0$

(2) $(x - 1)(x + 4) = x^2 + 1$

(3) $x^2 - 3x + 5 = 0$

(4) $(2x + 1)(3x - 4) = 6x^2 + 3$ **[Ans : (3)]**

43. The quadratic equation whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$ is

(1) $x^2 - 4x + 2 = 0$ (2) $x^2 - 2x + 2 = 0$

(3) $x^2 + 2x - 4 = 0$ (4) $x^2 - 2x + 4 = 0$

[Ans : (1)]

Sol :

Sum of the roots $= 2 + \sqrt{2} + 2 - \sqrt{2} = 4$

Product of the roots $= (2 + \sqrt{2})(2 - \sqrt{2})$

$= 4 - 2 = 2$

Quadratic Equation : $x^2 - (S \text{ o } R)x + P \text{ o } R = 0$

$\Rightarrow x^2 - 4x + 2 = 0$

44. The Quadratic equation whose roots are $\frac{p}{q}, \frac{-q}{p}$ is

(1) $qx^2 - (q^2 + p^2)x - pq = 0$

(2) $pqx^2 - (p^2 - q^2)x - pq = 0$

(3) $px^2 - (p^2 + 1)x + p = 0$

(4) $p^2x^2 - (p^2 - q^2)x - pq = 0$ **[Ans : (2)]**

Sol :

Sum of the roots $= \frac{p}{q} - \frac{q}{p} = \frac{p^2 - q^2}{pq}$

Product of the roots $= \left(\frac{p}{q}\right)\left(\frac{-q}{p}\right) = -1$

Quadratic Equation: x^2 – (Sum) x + Product = 0

$$x^2 - \left(\frac{p^2 - q^2}{pq}\right)x - 1 = 0$$

$pq\,x^2 - (p^2 - q^2)x - pq = 0$

45. If $ax^2 + bx + c$ is a perfect square, then b^2 =

(1) 2ac (2) ac

(3) 4ac (4) $\sqrt{2ac}$ **[Ans : (3)]**

Sol :

$b^2 = 4ac$

46. One root of $px^2 + qx + r = 0$ is r, then the second root is

(1) p (2) q

(3) $\frac{1}{q}$ (4) $\frac{1}{p}$ **[Ans : (4)]**

Sol :

Product of roots $\alpha\beta = \frac{r}{p}$

$$r(\beta) = \frac{r}{p}$$

$$\beta = \frac{1}{p}$$

47. The condition for $px^2 + qx + r = 0$ to be a pure quadratic equation is

(1) $p = 0$ (2) $q = 0$

(3) $r = 0$ (4) $p = q = 0$ **[Ans : (2)]**

48. Common root of $x^2 + x - 6 = 0$ and $x^2 + 3x - 10 = 0$ is

(1) -2 (2) 2

(3) -3 (4) -5 **[Ans : (2)]**

Sol :

Let the common factor be $x - k$

then, $k^2 + k - 6 = k^2 + 3k - 10$

$4 = 2k$

$k = 2$ (common root)

49. Ratio of the sum of the roots of $x^2 - 9x + 18 = 0$ to the product of the roots is

(1) $1 : 2$ (2) $2 : 1$

(3) $-1 : 2$ (4) $-2 : 1$ **[Ans : (1)]**

Sol :

$$x^2 - 9x + 18 = 0$$

$$\text{Sum of the roots} = -\left(\frac{-9}{1}\right) = 9$$

$$\text{Product of roots} = \frac{18}{1} = 18$$

Ratio $9 : 18 = 1 : 2$

50. If the discriminant of $3x^2 - 14x + k = 0$ is 100, then k =

(1) 8 (2) 32

(3) 16 (4) 24 **[Ans : (1)]**

Sol :

$$\begin{aligned} \text{Discriminant} &= 100 \\ b^2 - 4ac &= 100 \\ (-14)^2 - 4(3)(k) &= 100 \\ 196 - 12k &= 100 \\ k &= 8 \end{aligned}$$

51. The roots of the equation $4x^2 - 2x + 8 = 0$ are

(1) Real and equal

(2) Rational and not equal

(3) Irrational

(4) Not real **[Ans : (4)]**

Sol :

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(4)(8) \\ &= 4 - 128 = -124 < 0 \end{aligned}$$

52. The roots of the equation $(x - a)(x - b) = b^2$ are

(1) Real and equal (2) Real and unequal

(3) Imaginary (4) equal **[Ans : (2)]**

Sol :

$$\begin{aligned} (x - a)(x - b) &= b^2 \\ x^2 - (a + b)x + ab - b^2 &= 0 \\ \text{Discriminant } B^2 - 4AC &= (a + b)^2 - 4(1)(ab - b^2) \\ &= a^2 + 2ab + b^2 - 4ab + 4b^2 \\ &= a^2 - 2ab + b^2 + 4b^2 \\ &= (a - b)^2 + 4b^2 > 0 \end{aligned}$$

$\therefore$ Roots are real and unequal.

53. The Discriminant of $\sqrt{x^2 + x + 1} = 2$ is

(1) -3 (2) 13

(3) 11 (4) 12 **[Ans : (2)]**

Sol :

$$\sqrt{x^2 + x + 1} = 2$$

$$\begin{aligned} \text{Squaring} \Rightarrow \quad x^2 + x + 1 &= 4 \\ x^2 + x - 3 &= 0 \\ \text{Discriminant} \quad b^2 - 4ac &= (1)^2 - 4(1)(-3) \\ &= 1 + 12 = 13. \end{aligned}$$

54. If a and b are the roots of the equation $x^2 - 6x + 6 = 0$, then the value of $a^2 + b^2$ is

(1) 36 (2) 24

(3) 12 (4) 6 **[Ans : (2)]**

Sol :

Sum of the roots $= a + b = 6$

Product of roots $= ab = 6$

$\therefore a^2 + b^2 = (a + b)^2 - 2ab = 36 - 12 = 24$

55. The roots of the equation $x^2 + kx + 12 = 0$ will differ by unity only when

(1) $k = \pm\sqrt{12}$ (2) $k = \pm\sqrt{48}$

(3) $k = \pm\sqrt{47}$ (4) $k = \pm\sqrt{49}$ **[Ans : (4)]**

Sol :

Roots are $\alpha, \alpha + 1$

$$\text{Sum of the roots} = \alpha + \alpha + 1 = -k$$

$$2\alpha + 1 = -k$$

$$\alpha = \frac{-k-1}{2}$$

$$\text{Product of roots} = \alpha(\alpha + 1) = 12$$

$$\alpha^2 + \alpha = 12$$

$$\left(\frac{-k-1}{2}\right)^2 + \left(\frac{-k-1}{2}\right) = 12$$

$$k^2 + 2k + 1 - 2k - 2 = 48$$

$$k^2 - 49 = 0$$

simplifying and factorizing

$$(k+7)(k-7) = 0$$

56. Ajay and Vijay Solved an equation. In solving it, Ajay made a mistake in the constant term only and got the roots as 8 and 2, while Vijay made a mistake in the coefficient of x only and obtained the roots as – 9 and – 1. The correct roots of the equation are

(1) 8, – 1 (2) – 9, 2
(3) – 8, – 2 (4) 9, 1 **[Ans : (4)]**

Sol :

Let the equation be $x^2 + ax + b = 0$

Ajay made a mistake in 'b' and roots are 8, 2,

Hence the equation is $x^2 - 10x + 16 = 0$

Vijay made a mistake in the coefficient of 'x' and roots are – 9, – 1.

Hence the equation is $x^2 + 10x + 16 = 0$

Vijay made a mistake in the coefficient of 'x' and roots are – 9, –1.

Hence the equation $x^2 + 10x + 9 = 0$

$\therefore$ The real equation is $x^2 - 10x + 9 = (x-9)(x-1) = 0$

$\therefore$ The roots are 9 and 1.

57. If the sum of the squares of two consecutive even numbers is 100, then the numbers are

(1) 6, 8 or – 8, –6
(2) – 6, 8 or – 8, –6
(3) – 6, – 8 or 6, – 8
(4) –6, – 8 or 7, 8 **[Ans : (1)]**

Sol :

Let the numbers be x and x + 2

Then, $x^2 + (x+2)^2 = 100$

$$2x^2 + 4x - 96 = 0$$

$$x^2 + 2x - 48 = 0$$

$$(x+8)(x-6) = 0$$

$$x = 6 \text{ or } x = -8$$

$\therefore$ The numbers are 6, 8 and – 8, –6.

58. What is the smallest integral value of k such that $2x(kx-4) - x^2 + 6 = 0$ has no real roots?

(1) 3 (2) 1
(3) 2 (4) 4 **[Ans : (3)]**

Sol :

Equation is $2x(kx-4) - x^2 + 6 = 0$

$$(2k-1)x^2 - 8x + 6 = 0$$

$$\begin{aligned}\text{Discriminant } \Delta &= b^2 - 4ac \\ &= 64 - 4(2k-1)(6) \\ &= 88 - 48k \\ &= 8(11 - 6k)\end{aligned}$$

A quadratic equation has no real roots if Δ is negative. Δ is negative if $11 - 6k < 0$ i.e., when $k > \frac{11}{6}$, the smallest integral value of k for which the equation has no real roots is 2.

59. If α and β are the roots of $ax^2 + 2bx + c = 0$ then $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} =$

(1) $\frac{2b}{ac}$ (2) $\frac{2b}{\sqrt{ac}}$
(3) $\frac{-2b}{\sqrt{ac}}$ (4) $\frac{-b}{\sqrt{ac}}$ **[Ans : (3)]**

Sol :

$$ax^2 + 2bx + c = 0$$

α and β are the roots

$$\text{Sum of roots} = \alpha + \beta = \frac{-2b}{a}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$\therefore \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \frac{\frac{-2b}{a}}{\sqrt{\frac{c}{a}}} = \frac{-2b}{\sqrt{ac}}$$

60. If α, β are the roots of the equation $x^2 + kx + 12 = 0$ such that $\alpha - \beta = 1$, then the value of k is.

(1) 0 (2) ±5
(3) ±1 (4) ±7 **[Ans : (4)]**

Sol :

α, β are the roots of $x^2 + kx + 12 = 0$

$$\therefore \alpha + \beta = -k \text{ and}$$

$$\alpha\beta = 12 \text{ and}$$

$$\text{given } \alpha - \beta = 1$$

$$\text{Now } (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$

$$(1)^2 = k^2 - 48$$

$$k^2 = 49; \quad k = \pm 7$$

61. If α and β are the roots of the equation $ax^2 + bx + c = 0$, identify the quadratic equation whose roots are $\alpha+\beta$ and $\alpha\beta$

(1) $a^2 x^2 + a(b-c)x + bc = 0$
(2) $a^2 x^2 + a(b-c)x - bc = 0$
(3) $ax^2 + (b+c)x + bc = 0$
(4) $ax^2 - (b+c)x - bc = 0$ **[Ans : (2)]**

Sol :

$$\alpha+\beta = \frac{-b}{a} \quad \text{----- 1}^{\text{st}} \text{ root}$$

$$\alpha\beta = \frac{c}{a} \quad \text{----- 2}^{\text{nd}} \text{ root}$$

$\therefore$ Required equation is $x^2 - (\text{sum})\,x + \text{product} = 0$

$$\text{i.e., } x^2 - (\alpha+\beta+\alpha\beta)x + (\alpha+\beta)(\alpha\beta) = 0$$

$$x^2 - \left(\frac{-b}{a}+\frac{c}{a}\right)x + \left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = 0$$

$$a^2x^2 + a(b-c)x - bc = 0$$

Matrices

62. The order of the matrix A is 3 × 5 and that of B is 2 × 3. The order of the matrix BA is

(1) 2 × 3 (2) 3 × 2
(3) 2 × 5 (4) 5 × 2 **[Ans : (3)]**

63. If A $\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and k A = $\begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, then the values of k, a, b are respectively

(1) – 6, – 12, – 18 (2) – 6, 4, 9
(3) – 6, – 4, – 9 (4) – 6, 12, 18 **[Ans : (3)]**

Sol :

$$k\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$-4k = 24 \Rightarrow k = -6$$
$$2k = 3a \Rightarrow a = -4$$
$$3k = 2b \Rightarrow b = -9$$

64. If $m\begin{bmatrix} -3 & 4 \end{bmatrix} + n\begin{bmatrix} 4 & -3 \end{bmatrix}$ = then find 3m + 7n

(1) 3 (2) 5
(3) 10 (4) 1 **[Ans : (4)]**

Sol :

$$-3m + 4n = 10$$
$$4m - 3n = -11$$

Solving them, we get m = – 2 and n = 1

$$\therefore 3m + 7n = -6 + 7 = 1$$

65. If A = $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and $A^2 = I$, then x =

(1) 0 (2) 1
(3) – 1 (4) 2 **[Ans : (1)]**

Sol :

$$A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

Given $A^2 = I$

$$\therefore \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x = 0$$

66. If A = $\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$ then A + A^T =

(1) $\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & -4 \\ 10 & 6 \end{bmatrix}$
(3) $\begin{bmatrix} 2 & 4 \\ -10 & 6 \end{bmatrix}$ (4) None of these **[Ans : (1)]**

Sol :

$$A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

67. If U = $\begin{bmatrix} 2 & -3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$ and Y = $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then UV + XY =

(1) 20 (2) [– 20]
(3) – 20 (4) [20] **[Ans : (4)]**

Sol :

$$UV = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= [6 - 6 + 4] = [4]$$

$$XY = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$= [0 + 4 + 12] = [16]$$

$$UV + XY = [4] + [16] = [20]$$

68. If A $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = kA$, then k =

(1) 4 (2) 5

(3) 6 (4) 7 **[Ans : (3)]**

Sol :

$$A^2 = A\,A$$

$$= \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+9 & -9-9 \\ -9-9 & 9+9 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

$$kA = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}$$

$$\text{Given } A^2 = kA$$

$$\begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}$$

$$3k = 18 \Rightarrow k = 6$$

69. If $A + B = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$ and $A - B = \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$, then A =

(1) $\begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & 2 \\ 4 & 6 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & 6 \\ 4 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ **[Ans : (1)]**

Sol :

$$A + B = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix} \quad \ldots (1)$$

$$A - B = \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix} \quad \ldots (2)$$

$$(1) + (2) \Rightarrow 2A = \begin{bmatrix} 12 & 4 \\ 8 & 10 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$$

70. If $A = \begin{bmatrix} 5 & x \\ y & 6 \end{bmatrix}$, $B = \begin{bmatrix} -4 & y \\ -4 & -5 \end{bmatrix}$ and A + B = I, then the values of x and y respectively are

(1) – 4, 4 (2) – 4, – 4

(3) 4, 4 (4) 4, – 4 **[Ans : (1)]**

Sol :

$$A + B = I$$

$$\begin{bmatrix} 5 & x \\ y & 6 \end{bmatrix} + \begin{bmatrix} -4 & y \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x+y \\ y-4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x + y = 0, \quad y - 4 = 0$$

$$x + 4 = 0 \qquad y = 4$$

$$x = -4$$

71. Number of matrices obtained with 36 elements is

(1) 10 (2) 9

(3) 8 (4) 7 **[Ans : (2)]**

Sol :

With 36 elements, The Possible orders are 6×6, 9×4, 4×9, 12×3, 3×12, 2×18, 18×2, 36×1 and 1×36.

72. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $f(x) = x^2 - 5x + 4I$, then f(A) =

(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (4) $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ **[Ans : (4)]**

Sol :

$$f(x) = x^2 - 5x + 4I$$

$$\therefore f(A) = A^2 - 5A + 4I$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

73. If order of A, B, C are 3 × 4, 5 × 4 and 5 × 8, then the order of $(AB^T C)$ is

(1) 8 × 3 (2) 3 × 8

(3) 3 × 4 (4) 4 × 5 **[Ans : (2)]**

Sol :

Order of A = 3×4, Order of B^T = 4×5

Order of AB^T = 3×5

Order of C = 5×8

$\therefore$ Order of AB^TC is 3×8

74. Given $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A^3 - A^2 =$

(1) 2A (2) 2I
(3) A (4) I **[Ans : (1)]**

Sol :

$$A^2 = AA$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$A^3 = A^2A$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$$
$$A^3 - A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= 2\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = 2A$$

75. If AB = A, BA = B then $A^2 + B^2 =$

(1) A + B (2) A – B
(3) AB (4) 0 **[Ans : (1)]**

Sol :

$$A^2 + B^2 = A \times A + B \times B$$
$$= (AB)(AB) + (BA)(BA)$$
$$= A(BA)B + B(AB)A$$
$$= (AB)B + (BA)A \quad [\because AB = A, BA = B]$$
$$= AB + BA$$
$$= A + B$$

II. Very Short Answer Questions

1. Solve: 5x + 2y = 3, 3x + 2y = 5

Sol :

$$5x + 2y = 3 \quad \dots (1)$$
$$3x + 2y = 5 \quad \dots (2)$$
$$(1) - (2) \Rightarrow \quad 2x = -2$$
$$x = \frac{-2}{2} = -1$$

Substituting in (1)

$$5(-1) + 2y = 3$$
$$2y = 3 + 5$$
$$2y = \frac{8}{2} = 4$$

$\therefore$ Solution: $x = -1, y = 4$.

2. Find the GCD of $x^3 + 8y^3$ and $x + 2y$.

Sol :

$$x^3 + 8y^3 = x^3 + (2y)^3$$
$$= (x + 2y))(x^2 - 2xy + 4y^2)$$

by using $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$x + 2y = x + 2y.$$
$$\therefore \text{GCD} = (x + 2y)$$

3. $m^2 - 5m - 14$ is an expression. Find out another similar expression such that their HCF is $(m - 7)$ and LCM is $m^3 - 10m^2 + 11m + 70$.

Sol :

We know $P(x) \times Q(x) = \text{GCD} \times \text{LCM}$

$$\therefore Q(x) = \frac{GCD \times LCM}{P(x)}$$
$$Q(x) = \frac{(m-7)(m^3 - 10m^2 + 11m + 70)}{m^2 - 5m - 14}$$
$$= \frac{(m-7)(m-5)(m^2 - 5m - 14)}{m^2 - 5m - 14}$$
$$= m^2 - 12m + 35$$

4. Find the GCD of $10(x^2 + x - 20)$, $15(x^2 - 3x - 4)$ and $20(x^2 + 2x + 1)$

Sol :

GCD of 10, 15 and 20 is 5

$$x^2 + x - 20 = (x + 5)(x - 4)$$
$$x^2 - 3x - 4 = (x - 4)(x + 1)$$
$$x^2 + 2x + 1 = (x + 1)(x + 1)$$
$$\therefore \text{GCD} = 5$$

5. Find the LCM of $a^2 + 3a + 2$, $a^2 + 5a + 6$ and $a^2 + 4a + 4$

Sol :

$$a^2 + 3a + 2 = (a + 2)(a + 1)$$
$$a^2 + 5a + 6 = (a + 2)(a + 3)$$
$$a^2 + 4a + 4 = (a + 2)^2$$
$$\text{LCM} = (a + 2)^2(a + 1)(a + 3)$$

6. Simplify: $\dfrac{3x^2 - 3x}{3x^3 - 6x^2 + 3x}$

Sol :

$$\frac{3x^2 - 3x}{3x^3 - 6x^2 + 3x} = \frac{3x(x-1)}{3x(x^2 - 2x + 1)}$$
$$= \frac{3x(x-1)}{3x(x-1)^2} = \frac{1}{x-1}$$

7. Simplify: $\frac{x^2+3x}{x^2-4x-21}$

Sol :

$$\frac{x^2+3x}{x^2-4x-21} = \frac{x(x+3)}{(x-7)(x+3)}$$
$$= \frac{x}{x-7}$$

8. Simplify: $\frac{2x^2+26x+84}{2x^2+12x-14}$

Sol :

$$\frac{2x^2+26x+84}{2x^2+12x-14} = \frac{2(x^2+13x+42)}{2(x^2+6x-7)}$$
$$= \frac{(x+6)(x+7)}{(x+7)(x-1)}$$
$$= \frac{x+6}{x-1}$$

9. Simplify: $\frac{1}{4x^2}+\frac{5}{6xy^2}$

Sol :

$$\frac{1}{4x^2}+\frac{5}{6xy^2} = \frac{1(3y^2)+5(2x)}{12x^2y^2}$$
$$= \frac{3y^2+10x}{12x^2y^2}$$

10. Simplify: $\frac{3x}{x^2+3x-10}-\frac{6}{x^2+3x-10}$

Sol :

$$\frac{3x}{x^2+3x-10}-\frac{6}{x^2+3x-10} = \frac{3x-6}{x^2+3x-10}$$
$$= \frac{3(x-2)}{(x+5)(x-2)}$$
$$= \frac{3}{x+5}$$

11. Simplify: $\frac{x+4}{2x}-\frac{x-1}{x^2}$

Sol :

$$\frac{x+4}{2x}-\frac{x-1}{x^2} = \frac{x(x+4)-2(x-1)}{2x^2}$$
$$= \frac{x^2+4x-2x+2}{2x^2}$$
$$= \frac{x^2+2x+2}{2x^2}$$

12. Find the Quadratic equation whose sum and product of the roots are (i) – 12, 32 (ii) 2a, a^2-b^2.

Sol :

(i) Sum = – 12,

Product = 32.

Quadratic equation:

x^2 – (Sum of Roots) x + Product of Roots = 0

$x^2 + 12x + 32 = 0$

(ii) Sum of the roots '2a'

Product of the roots '$a^2 - b^2$'

Quadratic equation:

x^2 – (Sum of Roots) x + Product of Roots = 0

$x^2 - 2a\,x + a^2 - b^2 = 0.$

13. Find the sum and product of the quadratic equation

(i) $x^2 + 2x - 360 = 0$

(ii) $\frac{a^2d^2}{b}x^2+2acdx+c^2b = 0.$

Sol :

(i) $x^2 + 2x - 360 = 0,$

$a = 1, b = 2, c = -360$

Sum of the Roots $= \frac{-b}{a} = \frac{-2}{1} = -2.$

Product of the Roots $= \frac{c}{a} = \frac{-360}{1} = -360.$

(ii) $\frac{a^2d^2}{b}x^2+2ac\,dx+c^2b = 0$

Sum of the roots $= \frac{-B}{A}$

$$= \frac{-2acd}{\left(\frac{a^2d^2}{b}\right)} = \frac{-2bc}{ad}$$

Product of the roots $= \frac{C}{A}$

$$= \frac{c^2b}{\left(\frac{a^2d^2}{b}\right)} = \frac{c^2b^2}{a^2d^2}$$

14. Solve: $25p^2 - 49 = 0$ by factorization method.

Sol :

$$25p^2 - 49 = 0$$
$$(5p)^2 - (7)^2 = 0 \quad [\because a^2 - b^2 = (a+b)(a-b)]$$
$$(5p+7)(5p-7) = 0$$
$$5p + 7 = 0, \quad 5p - 7 = 0$$
$$p = -7/5, \quad p = 7/5$$
$$P = -7/5, 7/5$$

15. Solve: $8x^2 - 22x - 21 = 0$ by factorization method.

Sol :

$$8x^2 - 22x - 21 = 0$$
$$8x^2 - 28x + 6x - 21 = 0$$
$$4x(2x-7) + 3(2x-7) = 0$$
$$(2x-7)(4x+3) = 0$$
$$2x - 7 = 0, \quad 4x + 3 = 0$$
$$x = 7/2, \quad x = -3/4$$
$$\text{Solution : } x = -3/4, 7/2.$$

16. Solve: $x^2 - (1+\sqrt{2})x + \sqrt{2} = 0$ by factorization method.

Sol :

$$x^2 - (1+\sqrt{2})x + \sqrt{2} = 0$$
$$x^2 - x - \sqrt{2}\,x + \sqrt{2} = 0$$
$$x(x-1) - \sqrt{2}(x-1) = 0$$
$$(x-1)(x-\sqrt{2}) = 0$$
$$x - 1 = 0 \text{ or } x - \sqrt{2} = 0$$
$$x = 1 \text{ or } x = \sqrt{2}$$
$$\text{Solution : } x = 1, \sqrt{2}$$

17. Solve: $21x^2 - 2x + \frac{1}{21} = 0$ by factorization method.

Sol :

$$21x^2 - 2x + \frac{1}{21} = 0$$
$$441x^2 - 42x + 1 = 0$$
$$441x^2 - 21x - 21x + 1 = 0$$
$$21x(21x-1) - 1(21x-1) = 0$$
$$(21x-1)(21x-1) = 0$$
$$x = \frac{1}{21} \text{ (twice)}$$
$$\text{Solution: } x = \frac{1}{21}, \frac{1}{21}$$

18. Solve the following Quadratic equation by the method of completing the square.

(i) $x^2 - 4x + 1 = 0$,

(ii) $x^2 + 3x - 5 = 0$,

(iii) $4x^2 + 4\sqrt{3}x + 3 = 0$,

(iv) $2x^2 + x - 4 = 0$

Sol :

(i)
$$x^2 - 4x + 1 = 0$$
$$x^2 - 4x = -1$$
$$x^2 - 4x + 4 = -1 + 4$$
$$(x-2)^2 = 3$$
$$x - 2 = \pm\sqrt{3}$$
$$x - 2 = +\sqrt{3} \text{ and } x - 2 = -\sqrt{3}$$
$$\text{Solution is } x = 2 \pm \sqrt{3}$$

(ii)
$$x^2 + 3x - 5 = 0$$
$$x^2 + 3x = 5$$
$$x^2 + 3x + \frac{9}{4} = 5 + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{29}{4}$$
$$x + \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$$
$$\therefore x + \frac{3}{2} = \frac{\sqrt{29}}{2}, \quad x + \frac{3}{2} = \frac{-\sqrt{29}}{2}$$
$$\therefore \text{Solution is } x = \frac{3}{2} + \frac{\sqrt{29}}{2}, \quad x = -\frac{3}{2} - \frac{\sqrt{29}}{2}$$

(iii)
$$4x^2 + 4\sqrt{3}x + 3 = 0$$
$$4x^2 + 4\sqrt{3}x = -3$$
$$\text{Dividing by 4} \quad x^2 + \sqrt{3}x = \frac{-3}{4}$$
$$x^2 + \sqrt{3}x + \frac{3}{4} = \frac{-3}{4} + \frac{3}{4}$$
$$\left(x + \frac{\sqrt{3}}{2}\right)^2 = 0$$
$$x = -\frac{\sqrt{3}}{2} \text{ (twice)}$$

(iv)
$$2x^2 + x - 4 = 0$$
$$2x^2 + x = 4$$
$$\text{Dividing by 2} \quad x^2 + \frac{x}{2} = 2$$
$$x^2 + \frac{x}{2} + \frac{1}{16} = 2 + \frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2 = \frac{33}{16}$$

$$x+\frac{1}{4} = \pm\frac{\sqrt{33}}{4}$$

$$\therefore \quad x+\frac{1}{4} = \frac{\sqrt{33}}{4} \text{ and}$$

$$x+\frac{1}{4} = \frac{-\sqrt{33}}{4}$$

$$x = \frac{\sqrt{33}-1}{4},\ x = \frac{\sqrt{33}+1}{4}$$

Hence the solution.

19. Solve the following Quadratic equation by using formula method.

(i) $9x^2 - 15x + 6 = 0$,
(ii) $4x^2 - 3 = 2x$,
(iii) $2t^2 - 5t + 3 = 0$,
(iv) $3x^2 = -7x - 2$

Sol :

(i) $9x^2 - 15x + 6 = 0$

Comparing this with $ax^2 + bx + c = 0$

$$a = 9, b = -15, c = 6$$

$$\therefore x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{15\pm\sqrt{225-216}}{18}$$

$$= \frac{15\pm\sqrt{9}}{18} = \frac{15\pm 3}{18}$$

$$= \frac{15+3}{18}, \frac{15-3}{18}$$

$$x = \frac{18}{18}, \frac{12}{18}$$

Solution: $x = 1, \frac{2}{3}$

(ii) $4x^2 - 3 = 2x$

$4x^2 - 2x - 3 = 0$

comparing with $ax^2 + bx + c = 0$, we get

$$a = 4, b = -2, c = -3$$

Now, $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2\pm\sqrt{4+48}}{8}$$

$$= \frac{2\pm\sqrt{52}}{8}$$

$$= \frac{2\pm\sqrt{4\times 13}}{8}$$

$$= \frac{2\pm 2\sqrt{13}}{8}$$

$$= \frac{1\pm\sqrt{13}}{4}$$

$$\therefore x = \frac{1+\sqrt{13}}{4}, \frac{1-\sqrt{13}}{4}$$

(iii) $2t^2 - 5t + 3 = 0$

comparing with $ax^2 + bx + c = 0$, we get

$$a = 2, b = -5, c = 3$$

Now, $$t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{5\pm\sqrt{25-24}}{4} = \frac{5\pm 1}{4}$$

$$= \frac{5+1}{4}, \frac{5-1}{4} = \frac{6}{4}, \frac{4}{4}$$

$$\therefore t = \frac{3}{2}, 1$$

(iv) $3x^2 = -7x - 2$

$\Rightarrow 3x^2 + 7x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$, we get

$a = 3, b = 7, c = 2$

Now, $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-7\pm\sqrt{49-24}}{6}$$

$$= \frac{-7\pm\sqrt{25}}{6}$$

$$= \frac{-7\pm 5}{6}$$

$$= \frac{-7+5}{6}, \frac{-7-5}{6}$$

$$= -\frac{2}{6}, -\frac{12}{6}$$

$$= \frac{-1}{3}, -2$$

20. Determine the nature of the roots of

(i) $3x^2 - 5x + 2 = 0$,

(ii) $x^2 - 2\sqrt{2}x - 6 = 0$,

(iii) $2x^2 - 4x + 3 = 0$,

(iv) $x^2 - 4x + 4 = 0$

Sol :

(i) $3x^2 - 5x + 2 = 0$

Comparing this with $ax^2 + bx + c = 0$,

We get $a = 3, b = -5, c = 2$.

$$\begin{aligned}\text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(2) \\ &= 25 - 24 \\ &= 1 > 0\end{aligned}$$

$\therefore$ The roots are real and distinct.

(ii) $x^2 - 2\sqrt{2}x - 6 = 0$

Comparing this with $ax^2 + bx + c = 0$, we get

$a = 1, b = -2\sqrt{2}$,

$c = -6$

$$\begin{aligned}\text{Discriminant } \Delta &= b^2 - 4ac \\ &= \left(-2\sqrt{2}\right)^2 - 4(1)(-6) \\ &= 8 + 24 = 32 > 0.\end{aligned}$$

Hence the roots are real and distinct.

(iii) $2x^2 - 4x + 3 = 0$

Comparing this with $ax^2 + bx + c = 0$,

We get $a = 2, b = -4, c = 3$.

$$\begin{aligned}\text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(2)(3) \\ &= 16 - 24 \\ &= -8 < 0.\end{aligned}$$

Hence the roots are unreal.

(iv) $x^2 - 4x + 4 = 0$

Comparing with $ax^2 + bx + c = 0$,

We get $a = 1, b = -4, c = 4$.

$$\begin{aligned}\text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(1)(4) \\ &= 16 - 16 = 0\end{aligned}$$

Hence, the roots are real and equal.

21. If one root of the equation $5x^2 + 13x + k = 0$ is reciprocal of the other, then what is the value of k?

Sol :

Let the roots be α and $\frac{1}{\alpha}$

Equation $5x^2 + 13x + k = 0$

Comparing with $ax^2 + bx + c = 0$,

we get $a = 5, b = 13, c = k$

$$\alpha + \beta = \frac{-b}{a}$$

$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \frac{-13}{5}$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad \alpha\left(\frac{1}{\alpha}\right) = \frac{k}{5}$$

$$1 = \frac{k}{5} \Rightarrow k = 5.$$

22. Construct a 2×2 matrix where $a_{ij} = -2i + 3j$.

Sol :

$$\begin{aligned}a_{ij} &= -2i + 3j \\ a_{11} &= -2(1) + 3(1) = 1 \\ a_{12} &= -2(1) + 3(2) = 4 \\ a_{21} &= -2(2) + 3(1) = -1 \\ a_{22} &= -2(2) + 3(2) = 2\end{aligned}$$

Hence the matrix of order 2×2 is $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}$

23. Find the transpose of $A = \begin{bmatrix} 0 & 0 \\ 3 & 8 \\ 8 & 7 \end{bmatrix}$

Sol :

$$A = \begin{bmatrix} 0 & 0 \\ 3 & 8 \\ 8 & 7 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 0 & 3 & 8 \\ 0 & 8 & 7 \end{bmatrix}$$

24. If $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$. Find $(A^T)^T$.

Sol :

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 5/2 & 4 \\ 1/2 & 2 & 7/2 \\ 0 & 3/2 & 3 \\ 1/2 & 1 & 5/2 \end{bmatrix}$$

$$(A^{T})^T = A$$

$$= \begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 5/2 & 2 & 3/2 & 1 \\ 4 & 7/2 & 3 & 5/2 \end{bmatrix}$$

25. If $\begin{bmatrix} x-y & 2y \\ 2y+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$, then find the value of x + y + z.

Sol :

Since the matrices are equal,

Let us equate the corresponding elements.

Now, $x - y = 1, \quad 2y = 4$

$2y + z = 9, \quad x + y = 5$

$2y = 4 \quad x + y = 5 \quad 2y + z = 9$

$y = 2 \quad x + 2 = 5 \quad 2(2) + z = 9$

$x = 3 \quad z = 9 - 4 = 5$

$\therefore x + y + z = 3 + 2 + 5 = 10.$

26. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find 5A – 3B + 2C.

Sol :

$$5A - 3B + 2C$$

$$= 5\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -10 \\ 15 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 12 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -20 \\ 7 & -9 \end{bmatrix}$$

27. If $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, verify that A + (B + C) = (A + B) + C.

Sol :

$$B + C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ -3 & 3 \end{bmatrix}$$

$$\text{LHS} : A + (B + C) = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 \\ 1 & 5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\text{RHS} : (A + B) + C = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 \\ 1 & 5 \end{bmatrix}$$

$\therefore$ LHS = RHS

$A + (B + C) = (A + B) + C$

Hence verified.

28. Simplify:

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Sol :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\cos\theta\sin\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

29. Find a matrix 'X' such that 2A + B + X = 0, where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$

Sol :

Given $2A + B + X = 0$

$$2\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$$

30. Given A = $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. Find AB and BA.

Sol :

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$$

31. If A = $\begin{bmatrix} 3 & 2 \\ 12 & 8 \end{bmatrix}$ and B = $\begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix}$. Show that AB = 0.

Sol :

$$AB = \begin{bmatrix} 3 & 2 \\ 12 & 8 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} 24-24 & 12-12 \\ 96-96 & 48-48 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence proved.

32. If A = $\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$, find $-A^2 + 6A$.

Sol :

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$A^2 = A.A$$
$$= \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 4+6 & -4-8 \\ -6-12 & 6+16 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & -12 \\ -18 & 22 \end{bmatrix}$$
$$\therefore \quad -A^2 + 6A = \begin{bmatrix} -10 & 12 \\ 18 & -22 \end{bmatrix} + 6\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} -10 & 12 \\ 18 & -22 \end{bmatrix} + \begin{bmatrix} 12 & -12 \\ -18 & 24 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

33. Solve for 'x' if $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$.

Sol :

$$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} x-2 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Equating the corresponding elements

$$x - 2 = 0$$
$$x = 2$$

III. Short Answer Questions

1. If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes 1/2 if we only add 1 to the denominator. What is the fraction?

Sol :

Let the fraction be $\frac{'x'}{y}$

$$\text{Given}, \frac{x+1}{y-1} = 1$$
$$x - y + 2 = 0$$
$$x - y = -2 \quad \dots (1)$$
$$\text{and} \quad \frac{x}{y+1} = \frac{1}{2}$$
$$\Rightarrow \quad 2x - y - 1 = 0 \quad \dots (2)$$
$$\Rightarrow \quad 2x - y = 1$$
$$x - y = -2$$
$$(1) - (2) \Rightarrow \quad -x = -3$$
$$x = 3$$

Substituting in (1)

$$3 - y = -2 \quad \Rightarrow y = 5$$

Hence, the required fraction is 3/5.

2. Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

Sol :

Let the present age of Nuri be 'x' years and that of Sonu be 'y' years.

Given, $x - 5 = 3(y - 5)$

$\Rightarrow \quad x - 3y = -10 \quad \ldots (1)$

and $x + 10 = 2(y + 10)$

$\Rightarrow \quad x - 2y = 10 \quad \ldots (2)$

Solving (1) and (2)

$$x - 3y = -10$$
$$x - 2y = 10$$

$(1) - (2) \Rightarrow \quad -y = -20$

$$y = 20$$

Substituting in (2)

$$x - 2(20) = 10$$
$$\Rightarrow x = 10 + 40 = 50$$

$\therefore$ Present age of Nuri is 50 years

Present age of Sonu is 20 years.

3. Find the LCM of the polynomial $f(x) = 18x^4 - 36x^3 + 18x^2$ and $g(x) = 45x^6 - 45x^3$.

Sol :

$$f(x) = 18x^4 - 36x^3 + 18x^2$$
$$= 18x^2(x^2 - 2x + 1)$$
$$= 18x^2(x - 1)^2$$
$$g(x) = 45x^6 - 45x^3 = 45x^3(x^3 - 1)$$
$$= 45x^3(x - 1)(x^2 + x + 1)$$
$$\therefore \text{LCM} = 90x^3(x - 1)^2(x^2 + x + 1)$$

4. Find the GCD of the following polynomials using division algorithm

Sol :

$$f(x) = x^3 - 9x^2 + 23x - 15,$$
$$g(x) = 4x^2 - 16x + 12$$
$$= 4(x^2 - 4x + 3)$$

Now dividing $x^3 - 9x^2 + 23x - 15$ by $x^2 - 4x + 3$

$$\begin{array}{r|l} & x - 5 \\ \hline x^2 - 4x + 3 & x^3 - 9x^2 + 23x - 15 \\ & x^3 - 4x^2 + 3x \quad (-) \\ \hline & -5x^2 + 20x - 15 \\ & -5x^2 + 20x - 15 \quad (-) \\ \hline & 0 \end{array}$$

$\therefore$ GCD $= x^2 - 4x + 3$

5. Find the excluded values, if any of the expression $\dfrac{6x^3 + 57x^2 + 72x}{10x^3 + 85x^2 + 40x}$

Sol :

$$\frac{6x^3 + 57x^2 + 72x}{10x^3 + 85x^2 + 40x} = \frac{3x(2x^2 + 19x + 24)}{5x(2x^2 + 17x + 8)}$$
$$= \frac{3(x + 8)(2x + 3)}{5(x + 8)(2x + 1)}$$
$$= \frac{3(2x + 3)}{5(2x + 1)}$$

when $x = -1/2$, $2x + 1 = 0$, then the fraction becomes undefined.

$\therefore$ The excluded value is $-1/2$

6. Simplify: $\dfrac{x^3 + 27}{x^2 + 12x + 27} \times \dfrac{x^2 + 3x}{x^2 - 4x - 21}$

Sol :

$$\frac{x^3 + 27}{x^2 + 12x + 27} \times \frac{x^2 + 3x}{x^2 - 4x - 21}$$
$$= \frac{x^3 + 3^3}{(x + 3)(x + 9)} \times \frac{x(x + 3)}{(x - 7)(x + 3)}$$
$$= \frac{(x + 3)(x^2 - 3x + 9)}{(x + 3)(x + 9)} \times \frac{x(x + 3)}{(x - 7)(x + 3)}$$
$$= \frac{x(x^2 - 3x + 9)}{(x + 9)(x - 7)} = \frac{x(x^2 - 3x + 9)}{x^2 + 2x - 63}$$

7. Simplify: $\dfrac{x^2 - 25}{5x + x^2} \div \dfrac{x^2 - 10x + 25}{x^2 + 8x + 15}$

Sol :

$$\frac{x^2 - 25}{5x + x^2} \div \frac{x^2 - 10x + 25}{x^2 + 8x + 15}$$
$$= \frac{x^2 - 5^2}{x(5 + x)} \times \frac{x^2 + 8x + 15}{x^2 - 10x + 25}$$
$$= \frac{(x + 5)(x - 5)}{x(5 + x)} \times \frac{(x + 3)(x + 5)}{(x - 5)^2}$$
$$= \frac{(x + 3)(x + 5)}{x(x - 5)}$$

8. Find the Square Root of $x^4 - 4x^3 + 10x^2 - 12x + 9$ by division method.

Sol :

$$\begin{array}{r|l}
 & x^2 - 2x + 3 \\
\hline
x^2 & x^4 - 4x^3 + 10x^2 - 12x + 9 \\
 & x^4 \\
 & (-) \\
\hline
2x^2 - 2x & -4x^3 + 10x^2 \\
 & -4x^3 + 4x^2 \\
 & (+) \quad (-) \\
\hline
2x^2 - 4x + 3 & 6x^2 - 12x + 9 \\
 & 6x^2 - 12x + 9 \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 0 \\
\hline
\end{array}$$

$\therefore \sqrt{x^4 - 4x^3 + 10x^2 - 12x + 9} = |x^2 - 2x + 3|$

9. Find the square root of $x^4 - 2x^3 + 3x^2 - 2x + 1$ by division method.

$$\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x^2 & x^4 - 2x^3 + 3x^2 - 2x + 1 \\
 & x^4 \\
 & (-) \\
\hline
2x^2 - x & -2x^3 + 3x^2 \\
 & -2x^3 + x^2 \\
 & (+) \quad (-) \\
\hline
2x^2 - 2x + 1 & 2x^2 - 2x + 1 \\
 & 2x^2 - 2x + 1 \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 0 \\
\end{array}$$

$\therefore \sqrt{x^4 - 2x^3 + 3x^2 - 2x + 1} = |x^2 - x + 1|$

10. The sum of the reciprocals of Raman's ages (in years) 3 years ago and 5 years hence is $\frac{1}{3}$. Find his present age.

Sol :

Let the present age of Raman be 'x' years.

$\therefore$ 3 years ago, his age was $(x - 3)$ and 5 years hence, age will be $(x + 5)$

By the Given data,

$$\frac{1}{x-3} + \frac{1}{x+5} = \frac{1}{3}$$

$$\frac{x+5+x-3}{(x-3)(x+5)} = \frac{1}{3}$$

$$3(2x + 2) = (x - 3)(x + 5)$$

$$6x + 6 = x^2 + 2x - 15$$

$$x^2 - 4x - 21 = 0$$

$$(x - 7)(x + 3) = 0$$

$x - 7 = 0$ and

$x + 3 = 0,$

$x = -3$ is not possible.

$\therefore$ Present age of Raman is 7 years.

11. The sum of two natural numbers is 8. Determine the numbers if the sum of their reciprocals is $\frac{8}{15}$.

Sol :

Let the first number be 'x'

$\therefore$ Other number is $8 - x$ [$\because$ Sum is 8]

Given, $\frac{1}{x} + \frac{1}{8-x} = \frac{8}{15}$

$$\frac{8-x+x}{x(8-x)} = \frac{8}{15}$$

$$\frac{8}{x(8-x)} = \frac{8}{15}$$

$$x^2 - 8x + 15 = 0$$

$$(x - 5)(x - 3) = 0$$

$$x = 3, 5$$

$\therefore$ The two natural numbers are 3, 5

12. For what value of k, the roots of the Quadratic equation $(k + 1)x^2 - 2(k - 1)x + 1 = 0$ are real and equal?

Sol :

$$(k + 1)x^2 - 2(k - 1)x + 1 = 0$$

Comparing this with $ax^2 + bx + c = 0$

We get $a = k + 1, b = -2(k - 1), c = 1$

For real and equal roots $\Delta = 0$

i.e., $b^2 - 4ac = 0$

$$[-2(k - 1)]^2 - 4(k + 1)(1) = 0$$

$$4(k - 1)^2 - 4(k + 1) = 0$$

$$(k - 1)^2 - (k + 1) = 0$$

$$k^2 - 2k + 1 - k - 1 = 0$$

$$k^2 - 3k = 0$$

$$k(k - 3) = 0$$

$k = 0; \quad k - 3 = 0$

$k = 0$ (or) $k = 3$

13. Show that the equation $(x-a)(x-b) = h^2$ has real roots.

Sol :

The given equation is

$$(x-a)(x-b) = h^2$$

$$x^2-(a+b)x+ab-h^2 = 0$$

$$\text{Discriminant } \Delta = B^2-4AC$$

$$= [-(a+b)]^2-4(1)(ab-h^2)$$

$$= (a+b)^2-4ab+4h^2$$

$$= (a-b)^2+4h^2 > 0$$

Hence the roots are real.

14. If the sum of the roots of $ax^2+bx+c=0$ is equal to the sum of the squares of the roots. Find the condition.

Sol :

Let 'α' and 'β' are the roots of $ax^2+bx+c = 0$

then $\alpha+\beta = -b/a$ and $\alpha\beta = c/a$

Given that $\alpha+\beta = \alpha^2+\beta^2$

$$\alpha+\beta = (\alpha+\beta)^2-2\alpha\beta$$

$$\frac{-b}{a} = \left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)$$

$$\frac{-b}{a} = \frac{-b^2}{a^2}-\frac{2c}{a}$$

$$\frac{-b}{a} = \frac{b^2-2ac}{a^2}$$

$$-ab = b^2-2ac$$

$2ac = b^2+ab$ is the required condition.

15. Given that α,β are the roots of the equation $2x^2+3x+7=0$, then find

(i) $\alpha^2+\beta^2$, (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$, (iii) $\alpha^3+\beta^3$

Sol :

$$2x^2+3x+7 = 0$$

Comparing with $ax^2+bx+c = 0$,

we get $a = 2, b = 3, c = 7$

$$\alpha+\beta = \frac{-b}{a} = \frac{-3}{2},$$

$$\alpha\beta = \frac{c}{a} = \frac{7}{2}$$

(i) $$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$$

$$= \left(\frac{-3}{2}\right)^2-2\left(\frac{7}{2}\right) = \frac{-19}{4}$$

(ii) $$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{-3}{2}\times\frac{2}{7} = \frac{-3}{7}$$

(iii) $$\alpha^3+\beta^3 = (\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$$

$$= \left(\frac{-3}{2}\right)\left[\left(\frac{-3}{2}\right)^2-3\left(\frac{7}{2}\right)\right]$$

$$= -\frac{3}{2}\left[\frac{9}{4}-\frac{21}{2}\right]$$

$$= \frac{-3}{2}\left[\frac{9-42}{4}\right]$$

$$= \frac{-3}{2}\left(\frac{-33}{4}\right) = \frac{99}{8}$$

16. α,β are the roots of the equation $x^2-3ax+a^2 = 0$ such that $\alpha^2+\beta^2 = 1.75$. Find the value of 'a'.

Sol :

$$\text{Equation is } x^2-3ax+a^2 = 0$$

$$\text{Comparing with } ax^2+bx+c = 0,$$

we get $a = 1$, $b = -3a$, $c = a^2$

$$\alpha+\beta = \frac{-b}{a}$$

$$\Rightarrow \quad = \frac{-(-3a)}{1} = 3a$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad \frac{a^2}{1} = a^2$$

Given $\alpha^2+\beta^2 = 1.75$

$$(\alpha+\beta)^2-2\alpha\beta = 1.75$$

$$(3a)^2-2a^2 = 1.75$$

$$9a^2-2a^2 = 1.75$$

$$7a^2 = 1.75$$

$$a^2 = \frac{1.75}{7} = 0.25$$

$$a = \pm 0.5$$

17. If α, β are the roots of the equation $ax^2 + bx + b = 0$, then find the value of $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}}$.

Sol :

Given equation : $ax^2 + bx + b = 0$

$$\alpha + \beta = \frac{-b}{a},$$

$$\alpha\beta = \frac{b}{a}$$

Now, $$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\left(\frac{-b}{a}\right)}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}}$$

$$= -\sqrt{\frac{b}{a}} + \sqrt{\frac{b}{a}} = 0$$

18. In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 \\ 0 & 5 & y \end{bmatrix}$, Find

(i) the order of the matrix 'A'

(ii) the number of elements in 'A'

(iii) a_{23}, a_{31}, a_{12}

Sol :

$$A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 \\ 0 & 5 & y \end{bmatrix}$$

(i) Order of the matrix A $= 3 \times 3$

(ii) Number of elements $= 3 \times 3 = 9$

(iii) $a_{23} = x^2, a_{31} = 0, a_{12} = 1$

19. Find the values of a, b, c and d from the following equation $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Sol :

Equating the corresponding elements.

$a - b = -1$ ... (1)

$2a + c = 5$ ... (2)

$2a - b = 0$ ... (3)

$3c + d = 13$ ... (4)

Solving (1) and (3)

$$a - b = -1$$
$$2a - b = 0$$

(1) – (2) $\Rightarrow$ $-a = -1$

$a = 1$

Substituting in (1) $1 - b = -1$

$b = 2$

Substituting $a = 1$ in (2) $\Rightarrow 2(1) + c = 5$

$c = 3$

Substituting in (4) $\Rightarrow 3(3) + d = 13$

$d = 4$

$\therefore a = 1, b = 2, c = 3, d = 4$

20. Find A and B if $2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix}$

Sol :

$$2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \quad \text{... (1)}$$

$$A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} \quad \text{... (2)}$$

$$(2) \times (2) \Rightarrow \quad 2A + 4B = \begin{bmatrix} 10 & 0 & 6 \\ 2 & 12 & 4 \end{bmatrix} \quad \text{... (3)}$$

$$2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \quad \text{... (1)}$$

$$(3) - (1) \Rightarrow \quad B = \begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix}$$

Substituting in (2)

$$A + 2\begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 16 & 2 & 4 \\ -2 & 20 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -2 & -1 \\ 3 & -14 & 4 \end{bmatrix}$$

21. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$, $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, find the values of k, a and b.

Sol :

$$A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$$

$$kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$k\begin{bmatrix}0 & 2\\3 & -4\end{bmatrix} = \begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$$

$$\begin{bmatrix}0 & 2k\\3k & -4k\end{bmatrix} = \begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$$

Equating the corresponding elements.

$$2k = 3a, 3k = 2b, -4k = 24,$$

$$k = \frac{24}{-4} = -6.$$

$$\therefore \quad a = \frac{2k}{3} = \frac{2(-6)}{3} = -4$$

$$b = \frac{3k}{2} = \frac{3(-6)}{2} = -9$$

$$\therefore \; k = -6, \quad a = -4, \quad b = -9$$

22. If $2\begin{bmatrix}x & 5\\7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$ **find the values of x and y.**

Sol :

$$2\begin{bmatrix}x & 5\\7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

$$\begin{bmatrix}2x+3 & 10+4\\14+1 & 2y-6+2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

$$\begin{bmatrix}2x+3 & 14\\15 & 2y-4\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

Equating the corresponding elements

$2x + 3 = 7 \qquad 2y - 4 = 14$

$2x = 4 \qquad 2y = 18$

$x = \frac{4}{2} = 2 \qquad y = 9$

$\therefore x = 2 \qquad y = 9$

23. If $\begin{bmatrix}4 & 1 & 2\\0 & 5 & 3\end{bmatrix}\begin{bmatrix}3 & 4 & 5\\-1 & 0 & -2\\3 & 4 & 7\end{bmatrix} = \begin{bmatrix}8x+3y & 6z & 32\\4 & 12 & 26x-5y\end{bmatrix}$, **find the values of x, y and z.**

Sol :

Now, LHS

$$= \begin{bmatrix}4 & 1 & 2\\0 & 5 & 3\end{bmatrix}\begin{bmatrix}3 & 4 & 5\\-1 & 0 & -2\\3 & 4 & 7\end{bmatrix}$$

$$= \begin{bmatrix}12-1+6 & 16+0+8 & 20-2+14\\0-5+9 & 0+0+12 & 0-10+21\end{bmatrix}$$

$$= \begin{bmatrix}17 & 24 & 32\\4 & 12 & 11\end{bmatrix} = \begin{bmatrix}8x+3y & 6z & 32\\4 & 12 & 26x-5y\end{bmatrix}$$

Equating corresponding elements:

$8x + 3y = 17 \qquad$ ... (1)

$6z = 24 \Rightarrow z = 4 \qquad$ ... (2)

$26x - 5y = 11 \qquad$ ... (3)

Solving (1) and (3)

We get $x = 1, y = 3$

$\therefore x = 1, y = 3, z = 4$

24. If $A = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$, **find (A – 2I) (A – 3I)**

Sol :

$$A - 2I = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix} - 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix} - \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2\\-1 & -1\end{bmatrix}$$

$$A - 3I = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix} - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2\\-1 & -2\end{bmatrix}$$

Now, (A – 2I) (A – 3I)

$$= \begin{bmatrix}2 & 2\\-1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}2-2 & 4-4\\-1+1 & -2+2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$$

25. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Prove that $A^3 - 4A^2 + A = 0$

Sol :

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+12 & 21+24 \\ 8+7 & 12+14 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix}$$

Now, $A^3 - 4A^2 + A$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - 4\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - \begin{bmatrix} 28 & 48 \\ 16 & 28 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence proved.

IV. Long Answer Questions:

1. Solve $4x - 2y + 3z = 1$, $x + 3y - 4z = -7$, $3x + y + 2z = 5$

Sol :

$4x - 2y + 3z = 1$ ... (1)

$x + 3y - 4z = -7$ ... (2)

$3x + y + 2z = 5$ ... (3)

Consider (2) and (3)

$(3) \times (3) \Rightarrow$

$9x + 3y + 6z = 15$ ... (4)

$x + 3y - 4z = -7$ ... (2)

(–)

$(4) - (2) \Rightarrow \quad 8x + 10z = 22$ ... (5)

Now, consider (1) and (3)

$(3) \times 2 \Rightarrow \quad 6x + 2y + 4z = 10$ ... (6)

$4x - 2y + 3z = 1$ ... (1)

$(6) + (1) \Rightarrow \quad 10x + 7z = 11$ ... (7)

Consider (5) and (7)

$(5) \times 7 \Rightarrow \quad 56x + 70z = 154$ ... (8)

$(7) \times 10 \Rightarrow \quad 100x + 70z = 110$ ... (9)

(–)

$(8) - (9) \Rightarrow \quad -44x = 44$

$x = \frac{44}{44} = -1$

Substituting, $x = -1$ is in (7)

$10(-1) + 7z = 11$

$-10 + 7z = 11$

$7z = 11 + 10$

$z = \frac{21}{7} = 3$

Substituting, $x = -1, z = 3$ is in (3)

$3(-1) + y + 2(3) = 5$

$-3 + y + 6 = 5$

$y = 5 - 3 = 2$

$\therefore$ solution $x = -1, y = 2, z = 3$

2. Solve $x = 3z - 5$, $2x + 2z = y + 16$, $7x - 5z = 3y + 19$

Sol :

Standard form of the given equations

$x - 3z = -5$ ... (1)

$2x - y + 2z = 16$ ... (2)

$7x - 3y - 5z = 19$ ... (3)

Consider (2) and (3)

$(2) \times 3 \Rightarrow \quad 6x - 3y + 6z = 48$ ... (4)

$7x - 3y - 5z = 19$ ... (3)

(–)

$(4) - 3 \Rightarrow \quad -x + 11z = 29$ ... (5)

Consider (1) and (5)

$x - 3z = -5$ ... (1)

$-x + 11z = 29$ ... (5)

$(1) + (5) \Rightarrow \quad 8z = 24$

$z = \frac{24}{8} = 3$

Substituting, $z = 3$ in (1)

$$x = 3(3) - 5 = 9 - 5 = 4$$

Substituting, $x = 4, z = 3$ in (2)

$$2(4) - y + 2(3) = 16$$
$$8 + 6 - 16 = y$$
$$\Rightarrow y = 14 - 16 = -2$$

$\therefore$ solution is $x = 4, y = -2, z = 3$

3. Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, find the sides of the two squares.

Sol :

Let the sides of two squares be 'x' and 'y' respectively.

Sum of areas $x^2 + y^2 = 468$ ... (1)

Difference of perimeters $4x - 4y = 24$ [$\because x > y$]

$$x - y = 6 \quad \text{... (2)}$$
$$y = x - 6$$

Substituting in (1)

$$x^2 + (x-6)^2 = 468$$
$$x^2 + x^2 - 12x + 36 - 468 = 0$$
$$2x^2 - 12x - 432 = 0$$
$$x^2 - 6x - 216 = 0$$
$$(x - 18)(x + 12) = 0$$
$$x = 18$$

When $x = 18, y = 18 - 6 = 12$

$\therefore$ The sides of two squares are 18 m and 12 m respectively.

4. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

Sol :

Let the larger tap fill the tank in 'x' hours.

$\therefore$ Smaller tap fill the tank in $(x + 10)$ hours (given)

Portion of tank filled by larger tap in 1 hour $= \frac{1}{x}$

Portion of tank filled by smaller tap in 1 hour $= \frac{1}{x+10}$

Given, Both the taps can fill the tank in $9\frac{3}{8} = \frac{75}{8}$ hours.

[Time and work done are reciprocals of each other]

$$\frac{1}{x} + \frac{1}{x+10} = \frac{8}{75}$$
$$\frac{x+10+x}{x(x+10)} = \frac{8}{75}$$
$$75(2x+10) = 8(x^2+10x)$$
$$8x^2 - 70x - 750 = 0$$
$$4x^2 - 35x - 375 = 0$$
$$4x^2 - 60x + 25x - 375 = 0$$
$$4x(x - 15) + 25(x - 15) = 0$$
$$(4x + 25)(x - 15) = 0$$

$x = 15$, $\quad x = \frac{-25}{7}$ is not possible.

$\therefore$ Time required to fill the tank by the 1st tap is 15 hrs and by the 2nd tap is 25 hours.

5. A plane left 30 minutes later than the scheduled time and in order to reach its destination 1500 km away in time it has to increase its speed by 250 km/hr from its usual speed. Find its usual speed.

Sol :

Let the usual speed of the plane be 'x' km/hr and

usual time to cover 1500 km $= \frac{1500}{x}$ hrs

Given that speed increased by 250 km/hr.

Time taken to cover the distance with new speed $= \frac{1500}{x+150}$

$$\therefore \frac{1500}{x+250} - \frac{1500}{x} = \frac{1}{2}$$

On Simplifying we get, $x^2 + 250x - 750000 = 0$

$$x(x + 1000) - 750(x + 1000) = 0$$
$$(x - 750)(x + 1000) = 0$$

$x = 750, x = -1000$ is not possible.

$\therefore$ The usual speed of the plane is 750 km/hr.

6. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, verify that $(A + B)^2 \neq A^2 + 2AB + B^2$

Sol :

$$A + B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

$$(A + B)^2 = (A + B)(A + B)$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0 & 0+0 \\ 9+9 & 0+9 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 18 & 9 \end{bmatrix} \quad \text{... (1)}$$

$$\text{Now } A^2 = A.A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & -1-3 \\ 2+6 & -2+9 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+0 \\ 4+3 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 2 \end{bmatrix}$$

$$B^2 = B.B$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 2+0 \\ 2+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Now $A^2 + 2AB + B^2$

$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 14 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 24 & 12 \end{bmatrix} \quad \text{... (2)}$$

from (1), (2), $(A + B)^2 \neq A^2 + 2AB + B^2$.

7. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then prove that $A^2 - 8A + 7I = 0$.

Sol :

$$A^2 = A.A$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 8A + 7I = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7+7 & 0 \\ 0+0 & -7+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence proved.

8. Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, show that (AB) C = A (BC).

Sol :

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-1 & 3+2-4 \\ 2+0+3 & 6+0+12 \\ 3+0+2 & 9-2+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 18 \\ 5 & 15 \end{bmatrix}$$

LHS = (AB) C

$$= \begin{bmatrix} 0 & 1 \\ 5 & 18 \\ 5 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 0+0 & 0-2 & 0+1 \\ 5+36 & 10+0 & 15-36 & -20+18 \\ 5+30 & 10+0 & 15-30 & -20+15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 & 1 \\ 41 & 10 & -21 & -2 \\ 35 & 10 & -15 & -5 \end{bmatrix}$$

$$\text{Now, BC} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ 1+8 & 2+0 & 3-8 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 9 & 2 & -5 & 0 \end{bmatrix}$$

$$\text{A (BC)} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 9 & 2 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7+4-9 & 2+0-2 & -3-4+5 & -1+2+0 \\ 14+0+27 & 4+0+6 & -6+0-15 & -2+0+0 \\ 21-4+18 & 6+0+4 & -9+4-10 & -3-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 & 1 \\ 41 & 10 & -21 & -2 \\ 35 & 10 & -15 & -5 \end{bmatrix}$$

LHS = RHS

$\therefore (AB)\,C$ = A (BC)

Hence proved.