Chapter 5 Continuity and Differentiability

EXERCISE 5.1

Question 1:

Prove that the function $f(x)=5x-3$ is continuous at $x=0$, $x=-3$ and at $x=5$.

Solution:

The given function is $f(x)=5x-3$

At $x=0, f(0)=5(0)-3=-3$

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5(0)-3=-3$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, f is continous at $x=0$.

At $x=-3, f(-3)=5(-3)-3=-18$

$\lim_{x\to -3} f(x)=\lim_{x\to -3}(5x-3)=5(-3)-3=-18$

$\therefore \lim_{x\to -3} f(x)=f(-3)$

Therefore, f is continous at $x=-3$.

At $x=5, f(5)=5(5)-3=22$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5(5)-3=22$

$\therefore \lim_{x\to 5} f(x)=f(5)$

Therefore, f is continous at $x=5$.

Question 2:

Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Solution:

The given function is $f(x)=2x^2-1$

At $x=3, f(3)=2(3)^2-1=17$

$\lim_{x\to 3} f(x)=\lim_{x\to 3}(2x^2-1)=2(3^2)-1=17$

$\therefore \lim_{x\to 3} f(x)=f(3)$

Therefore, f is continous at $x=3$.

Question 3:

Examine the following functions for continuity.

(i) $f(x)=x-5$

(ii) $f(x)=\dfrac{1}{x-5}, x\neq 5$

(iii) $f(x)=\dfrac{x^2-25}{x+5}, x\neq -5$

(iv) $f(x)=|x-5|, x\neq 5$

Solution:

(i) The given function is $f(x)=x-5$

It is evident that f is defined at every real number k and its value at k is $k-5$.
It is also observed that

$$\lim_{x\to k} f(x)=\lim_{x\to k}(x-5)=k-5=f(k)$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

(ii) The given function is $f(x)=\dfrac{1}{x-5}, x\neq 5$

For any real number $k\neq 5$, we obtain

$$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$$

Also,

$$f(k)=\frac{1}{k-5} \qquad (\text{As } k\neq 5)$$

$$\therefore \lim_{x\to k} f(x)=f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iii) The given function is $f(x)=\dfrac{x^2-25}{x+5}, x\neq -5$

For any real number $c\neq -5$, we obtain

$$\lim_{x\to c} f(x)=\lim_{x\to c}\frac{x^2-25}{x+5}=\lim_{x\to c}\frac{(x+5)(x-5)}{x+5}=\lim_{x\to c}(x-5)=(c-5)$$

Also,

$$f(c)=\frac{(c+5)(c-5)}{c+5}=(c-5)$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iv) The given function is $f(x)=|x-5|=\begin{cases}5-x, & \text{if } x<5\\ x-5, & \text{if } x\geq 5\end{cases}$

This function f is defined at all points of the real line. Let c be a point on a real line. Then, $c<5$, $c=5$ or $c>5$

Case I: $c<5$

Then, $f(c)=5-c$

$\lim_{x\to c} f(x)=\lim_{x\to c}(5-x)=5-c$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all real numbers less than 5.

Case II: $c=5$

Then, $f(c)=f(5)=(5-5)=0$

$\lim_{x\to 5^-} f(x)=\lim_{x\to 5}(5-x)=(5-5)=0$

$\lim_{x\to 5^+} f(x)=\lim_{x\to 5}(x-5)=0$

$\therefore \lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)=f(c)$

Therefore, f is continuous at $x=5$

Case III: $c>5$

Then, $f(c)=f(5)=c-5$

$\lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x)=x^n$ is continuous at $x=n$, where n is a positive integer.

Solution:

The given function is $f(x)=x^n$

It is observed that f is defined at all positive integers, n, and its value at n is n^n.

Then,

$$\lim_{x\to n} f(n)=\lim_{x\to n}\left(x^n\right)=x^n$$

$$\therefore \lim_{x\to n} f(x)=f(n)$$

Therefore, f is continuous at n, where n is a positive integer.

Question 5:

Is the function f defined by $f(x)=\begin{cases} x, \text{if } x\le 1 \\ 5, \text{if } x>1 \end{cases}$ continuous at $x=0$? At $x=1$? At $x=2$?

Solution:

The given function is $f(x)=\begin{cases} x, \text{if } x\le 1 \\ 5, \text{if } x>1 \end{cases}$

At $x=0$,

It is evident that f is defined at 0 and its value at 0 is 0.

Then,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(x)=0$$

$$\therefore \lim_{x\to 0} f(x)=f(0)$$

Therefore, f is continuous at $x=0$.

At $x=1$,

It is evident that f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x=1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x)=1$$

The right hand limit of f at $x=1$ is,

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(5)=5$$

$$\therefore \lim_{x\to 1^-} f(x)\ne \lim_{x\to 1^+} f(x)$$

Therefore, f is not continuous at $x=1$.

At $x=2$,

It is evident that f is defined at 2 and its value at 2 is 5.

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}(5)=5$$

$$\therefore \lim_{x\to 1} f(x)=f(2)$$

Therefore, f is continuous at $x=2$.

Question 6:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$.

Solution:

The given function is $f(x)=\begin{cases}2x+3, & \text{if } x\le 2\\ 2x-3, & \text{if } x>2\end{cases}$

It is evident that the given function f is defined at all the points of the real line.
Let c be a point on the real line. Then, three cases arise.

$c<2$
$c>2$
$c=2$

Case I: $c<2$

$f(c)=2c+3$

Then,

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x+3)=2c+3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<2$.

Case II: $c>2$

Then,

$f(c)=2c-3$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(2x-3)=2c-3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>2$

Case III: $c=2$

Then, the left hand limit of f at $x=2$ is,

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^-}(2x+3)=2(2)+3=7$

The right hand limit of f at $x=2$ is,

$\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(2x-3)=2(2)-3=1$

It is observed that the left and right hand limit of f at $x=2$ do not coincide.

Therefore, f is not continuous at $x=2$.

Hence, $x=2$ is the only point of discontinuity of f.

Question 7:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$

Solution:

The given function is $f(x)=\begin{cases}|x|+3, & \text{if } x\le -3\\ -2x, & \text{if } -3<x<3\\ 6x+2, & \text{if } x\ge 3\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<-3$, then $f(c)=-c+3$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-x+3)=-c+3$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<-3$.

Case II:

If $c=-3$, then $f(-3)=-(-3)+3=6$

$\lim\limits_{x\to-3^-} f(x)=\lim\limits_{x\to-3^-}(-x+3)=-(-3)+3=6$

$\lim\limits_{x\to-3^+} f(x)=\lim\limits_{x\to-3^+}(-2x)=-2(-3)=6$

$\therefore \lim\limits_{x\to-3} f(x)=f(-3)$

Therefore, f is continuous at $x=-3$.

Case III:

If $-3<c<3$, then $f(c)=-2c$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(-2x)=-2c$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous in $(-3,3)$.

Case IV:

If $c=3$, then the left hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-}(-2x)=-2(3)=-6$

The right hand limit of f at $x=3$ is,

$\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(6x+2)=6(3)+2=20$

It is observed that the left and right hand limit of f at $x=3$ do not coincide.

Therefore, f is not continuous at $x=3$.

Case V:

If $c>3$, then $f(c)=6c+2$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(6x+2)=6c+2$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>3$.

Hence, $x=3$ is the only point of discontinuity of f.

Question 8:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}\frac{|x|}{x}, \text{if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$

Solution:

The given function is $f(x)=\begin{cases}\dfrac{|x|}{x}, \text{if } x\neq 0\\ 0, \text{ if } x=0\end{cases}$

It is known that, $x<0\Rightarrow |x|=-x$ and $x>0\Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\dfrac{|x|}{x}=\dfrac{-x}{x}=-1, \text{ if } x<0\\ 0, \text{ if } x=0\\ \dfrac{|x|}{x}=\dfrac{x}{x}=1, \text{ if } x>0\end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<0$, then $f(c)=-1$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(-1)=-1$$
$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points $x<0$.

Case II:

If $c=0$, then the left hand limit of f at $x=0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-1)=-1$$

The right hand limit of f at $x=0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(1)=1$$

It is observed that the left and right hand limit of f at $x=0$ do not coincide.

Therefore, f is not continuous at $x=0$.

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1$$
$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 0$.

Hence, $x = 0$ is the only point of discontinuity of f.

Question 9:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

It is known that $x < 0 \Rightarrow |x| = -x$
Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$$\Rightarrow f(x) = -1 \, \forall \, x \in R$$

Let c be any real number.

Then, $\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

Also, $f(c) = -1 = \lim_{x \to c} f(x)$

Therefore, the given function is a continuous function.
Hence, the given function has no point of discontinuity.

Question 10:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

Solution:

The given function is $f(x)=\begin{cases}x+1, \text{ if } x\geq 1\\ x^2+1, \text{ if } x<1\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c<1$, then $f(c)=c^2+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2+1\right)=c^2+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<1$.

Case II:

If $c=1$, then $f(c)=f(1)=1+1=2$

The left hand limit of f at $x=1$ is,

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^2+1\right)=1^2+1=2$

The right hand limit of f at $x=1$ is,

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x+1)=1+1=2$

$\therefore \lim\limits_{x\to 1} f(x)=f(1)$

Therefore, f is continuous at $x=1$.

Case III:

If $c>1$, then $f(c)=c+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+1)=c+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>1$.

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f, where f is defined by $f(x)=\begin{cases}x^3-3, \text{ if } x\leq 2\\ x^2+1, \text{ if } x>2\end{cases}$

Solution:

The given function is $f(x)=\begin{cases}x^3-3, \text{ if } x\leq 2\\ x^2+1, \text{ if } x>2\end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.
Case I:

If $c < 2$, then $f(c) = c^3 - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x^3 - 3) = c^3 - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 2$.
Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-}(x^3 - 3) = 2^3 - 3 = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, f is continuous at $x = 2$.
Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x > 2$.

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x^{10} - 1) = c^{10} - 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x<1$.

Case II:
If $c=1$, then the left hand limit of f at $x=1$ is,
$\lim\limits_{x\to1^-} f(x)=\lim\limits_{x\to1^-}\left(x^{10}-1\right)=1^{10}-1=1-1=0$
The right hand limit of f at $x=1$ is,

$\lim\limits_{x\to1^+} f(x)=\lim\limits_{x\to1^+}\left(x^2\right)=1^2=1$
It is observed that the left and right hand limit of f at $x=1$ do not coincide.
Therefore, f is not continuous at $x=1$.

Case III:
If $c>1$, then $f(c)=c^2$
$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(x^2\right)=c^2$
$\therefore \lim\limits_{x\to c} f(x)=f(c)$
Therefore, f is continuous at all points x, such that $x>1$.
Thus from the above observation, it can be concluded that $x=1$ is the only point of discontinuity of f.

Question 13:

Is the function defined by $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$ a continous function?

Solution:

The given function is $f(x)=\begin{cases} x+5, & \text{if } x\le 1 \\ x-5, & \text{if } x>1 \end{cases}$
The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:
If $c<1$, then $f(c)=c+5$
$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+5)=c+5$
$\therefore \lim\limits_{x\to c} f(x)=f(c)$
Therefore, f is continuous at all points x, such that $x<1$.

Case II:
If $c=1$, then $f(1)=1+5=6$

The left hand limit of f at $x=1$ is,

$$\lim_{x\to1^-} f(x)=\lim_{x\to1^-}(x+5)=1+5=6$$

The right hand limit of f at $x=1$ is,

$$\lim_{x\to1^+} f(x)=\lim_{x\to1^+}(x-5)=1-5=-4$$

It is observed that the left and right hand limit of f at $x=1$ do not coincide.

Therefore, f is not continuous at $x=1$.

Case III:

If $c>1$, then $f(c)=c-5$

$$\lim_{x\to c} f(x)=\lim_{x\to c}(x-5)=c-5$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x>1$.

From the above observation it can be concluded that, $x=1$ is the only point of discontinuity of f.

Question 14:

Discuss the continuity of the function f, where f is defined by $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

Solution:

The given function is $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

The given function f is defined at all the points of the interval $[0,10]$.

Let c be a point in the interval $[0,10]$.

Case I:

If $0 \le c < 1$, then $f(c) = 3$

$\lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous in the interval $[0,1)$.

Case II:

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$

The right hand limit of f at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(4) = 4$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $1 < c < 3$, then $f(c) = 4$

$\lim_{x \to c} f(x) = \lim_{x \to c}(4) = 4$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at in the interval $(1,3)$.

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(4) = 4$

The right hand limit of f at $x = 3$ is,

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(5) = 5$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is discontinuous at $x = 3$.

Case V:

If $3 < c \le 10$, then $f(c) = 5$

$\lim_{x \to c} f(x) = \lim_{x \to c}(5) = 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points of the interval $(3,10]$.

Hence, f is discontinuous at $x=1$ and $x=3$.

Question 15:

Discuss the continuity of the function f , where f is defined by $f(x) = \begin{cases} 2x, \text{ if } x < 0 \\ 0, \text{ if } 0 \le x \le 1 \\ 4x, \text{ if } x > 1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} 2x, \text{ if } x < 0 \\ 0, \text{ if } 0 \le x \le 1 \\ 4x, \text{ if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c}(2x) = 2c$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, f is continuous at all points x, such that $x < 0$.

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x=0$ is,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}(2x) = 2(0) = 0$

The right hand limit of f at $x=0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(0) = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Therefore, f is continuous at $x=0$

Case III:

If $0 < c < 1$, then $f(x) = 0$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(0) = 0$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous in the interval $(0,1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x) = 4(1) = 4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case V:

If $c < 1$, then $f(c) = 4c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(4x) = 4c$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.

Hence, f is not continuous only at $x = 1$.

Question 16:

Discuss the continuity of the function f, where f is defined by $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(-2) = -2$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$
Therefore, f is continuous at all points x, such that $x < -1$.

Case II:

If $c = -1$, then $f(c) = f(-1) = -2$

The left hand limit of f at $x = -1$ is,
$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(-2) = -2$$
The right hand limit of f at $x = -1$ is,
$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(2x) = 2(-1) = -2$$
$$\therefore \lim_{x\to -1} f(x) = f(-1)$$
Therefore, f is continuous at $x = -1$

Case III:

If $-1 < c < 1$, then $f(c) = 2c$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x) = 2c$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$
Therefore, f is continuous in the interval $(-1,1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 2(1) = 2$

The left hand limit of f at $x = 1$ is,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x) = 2(1) = 2$$
The right hand limit of f at $x = 1$ is,
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2) = 2$$
$$\therefore \lim_{x\to 1} f(x) = f(c)$$
Therefore, f is continuous at $x = 2$.

Case V:

If $c > 1$, then $f(c) = 2$
$$\lim_{x\to c} f(x) = \lim_{x\to c}(2) = 2$$
$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$.

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by $f(x)=\begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continous at $x=3$.

Solution:

The given function is $f(x)=\begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$

For f to be continuous at $x=3$, then

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax+1) = 3a+1$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b+\frac{2}{3}$$

Therefore, the required relationship is given by, $a = b+\frac{2}{3}$.

Question 18:

For what value of λ is the function defined by $f(x)=\begin{cases} \lambda(x^2-2x), & \text{if } x \le 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$ is continous at $x=0$? What about continuity at $x=1$?

Solution:

The given function is $f(x)=\begin{cases}\lambda\left(x^2-2x\right), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If f is continuous at $x=0$, then

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0^-}\lambda\left(x^2-2x\right)=\lim_{x\to 0^+}(4x+1)=\lambda\left(0^2-2\times 0\right)$$

$$\Rightarrow \lambda\left(0^2-2\times 0\right)=4(0)+1=0$$

$$\Rightarrow 0=1=0 \qquad \text{[which is not possible]}$$

Therefore, there is no value of λ for which f is continuous at $x=0$.

At $x=1$

$$f(1)=4x+1=4(1)+1=5$$

$$\lim_{x\to 1}(4x+1)=4(1)+1=5$$

$$\therefore \lim_{x\to 1} f(x)=f(1)$$

Therefore, for any values of λ, f is continuous at $x=1$.

Question 19:

Show that the function defined by $g(x)=x-[x]$ is discontinuous at all integral point. Here $[x]$ denotes the greatest integer less than or equal to x.

Solution:

The given function is $g(x)=x-[x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$g(n) = n - [n] = n - n = 0$

The left hand limit of g at $x = n$ is,

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} (x - [x]) = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x] = n - (n-1) = 1$$

The right hand limit of g at $x = n$ is,

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} (x - [x]) = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0$$

It is observed that the left and right hand limit of g at $x = n$ do not coincide.
Therefore, g is not continuous at $x = n$.
Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution:

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = \pi$.

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5)$

Put $x = \pi + h$, it is evident that if $x \to \pi$, then $h \to 0$

$$\begin{aligned}
\therefore \lim_{x \to \pi} f(x) &= \lim_{x \to \pi} (x^2 - \sin x) + 5 \\
&= \lim_{h \to 0} \left[(\pi + h)^2 - \sin(\pi + h) + 5 \right] \\
&= \lim_{h \to 0} (\pi + h)^2 - \lim_{h \to 0} \sin(\pi + h) + \lim_{h \to 0} 5 \\
&= (\pi + 0)^2 - \lim_{h \to 0} [\sin \pi \cos h + \cos \pi \sin h] + 5 \\
&= \pi^2 - \lim_{h \to 0} \sin \pi \cos h - \lim_{h \to 0} \cos \pi \sin h + 5 \\
&= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5 \\
&= \pi^2 - 0(1) - (-1)0 + 5 \\
&= \pi^2 + 5 \\
&= f(\pi)
\end{aligned}$$

Therefore, the given function f is continuous at $x = \pi$.

Question 21:

Discuss the continuity of the following functions.

(i) $f(x)=\sin x+\cos x$

(ii) $f(x)=\sin x-\cos x$

(iii) $f(x)=\sin x\times\cos x$

Solution:

It is known that if g and h are two continuous functions, then $g+h, g-h$ and g,h are also continuous.

Let $g(x)=\sin x$ and $h(x)=\cos x$ are continuous functions.

It is evident that $g(x)=\sin x$ is defined for every real number.

Let c be a real number. Put $x=c+h$

If $x\to c$, then $h\to 0$

$$g(c)=\sin c$$

$$\begin{aligned}\lim_{x\to c} g(x) &= \lim_{x\to c}\sin x\\ &= \lim_{h\to 0}\sin(c+h)\\ &= \lim_{h\to 0}[\sin c\cos h+\cos c\sin h]\\ &= \lim_{h\to 0}(\sin c\cos h)+\lim_{h\to 0}(\cos c\sin h)\\ &= \sin c\cos 0+\cos c\sin 0\\ &= \sin c(1)+\cos c(0)\\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x\to c} g(x)=g(c)$$

Therefore, $g(x)=\sin x$ is a continuous function.

Let $h(x)=\cos x$

It is evident that $h(x)=\cos x$ is defined for every real number.

Let c be a real number. Put $x=c+h$

If $x\to c$, then $h\to 0$

$$h(c)=\cos c$$

$$\begin{aligned}\lim_{x\to c} h(x) &= \lim_{x\to c} \cos x \\ &= \lim_{h\to 0} \cos(c+h) \\ &= \lim_{h\to 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h\to 0} (\cos c \cos h) - \lim_{h\to 0} (\sin c \sin h) \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c (1) - \sin c (0) \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x\to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.
Therefore, it can be concluded that,

(i) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function.

(ii) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function.

(iii) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function.

Question 22:
Discuss the continuity of the cosine, cosecant, secant, and cotangent functions.

Solution:
It is known that if g and h are two continuous functions, then

(i) $\frac{h(x)}{g(x)}, g(x) \neq 0$ is continuous.

(ii) $\frac{1}{g(x)}, g(x) \neq 0$ is continuous.

(iii) $\frac{1}{h(x)}, h(x) \neq 0$ is continuous.

Let $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

It is evident that $g(x) = \sin x$ is defined for every real number.
Let c be a real number. Put $x = c + h$
If $x \to c$, then $h \to 0$

$$g(c) = \sin c$$

$$\begin{aligned}
\lim_{x \to c} g(x) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c+h) \\
&= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
&= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h) \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c (1) + \cos c (0) \\
&= \sin c
\end{aligned}$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, $g(x) = \sin x$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$h(c) = \cos c$$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c+h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c (1) - \sin c (0) \\
&= \cos c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

Therefore, it can be concluded that,

$\operatorname{cosec} x = \dfrac{1}{\sin x}, \sin x \neq 0$ is continuous.

$\Rightarrow \operatorname{cosec} x, x \neq n\pi \ (n \in Z)$ is continuous.

Therefore, cosecant is continuous except at $x = n\pi \ (n \in Z)$

$\sec x = \dfrac{1}{\cos x}, \cos x \neq 0$ is continuous.

$\Rightarrow \sec x, x \neq (2n+1)\dfrac{\pi}{2} (n \in Z)$ is continuous.

Therefore, secant is continuous except at $x=(2n+1)\frac{\pi}{2}(n\in Z)$

$\cot x=\frac{\cos x}{\sin x},\sin x\neq 0$ is continuous.

$\Rightarrow \cot x, x\neq n\pi(n\in Z)$ is continuous.

Therefore, cotangent is continuous except at $x=n\pi(n\in Z)$.

Question 23:

Find the points of discontinuity of f, where $f(x)=\begin{cases}\frac{\sin x}{x}, \text{if } x<0\\ x+1, \text{ if } x\geq 0\end{cases}$

Solution:

The given function is $f(x)=\begin{cases}\frac{\sin x}{x}, \text{if } x<0\\ x+1, \text{ if } x\geq 0\end{cases}$

The given function f is defined at all the points of the real line.
Let c be a point on the real line.

Case I:

If $c<0$, then $f(c)=\frac{\sin c}{c}$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}\left(\frac{\sin x}{x}\right)=\frac{\sin c}{c}$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x<0$.

Case II:

If $c>0$, then $f(c)=c+1$

$\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}(x+1)=c+1$

$\therefore \lim\limits_{x\to c} f(x)=f(c)$

Therefore, f is continuous at all points x, such that $x>0$.

Case III:

If $c=0$, then $f(c)=f(0)=0+1=1$

The left hand limit of f at $x=0$ is,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(\frac{\sin x}{x}\right)=1$$

The right hand limit of f at $x=0$ is,

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x+1)=1$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

Therefore, f is continuous at $x=0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

Determine if f defined by $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$ is a continuous function?

Solution:

The given function is $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c\neq 0$, then $f(c)=c^2\sin\frac{1}{c}$

$$\lim_{x\to c} f(x)=\lim_{x\to c}\left(x^2\sin\frac{1}{x}\right)=\left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right)=c^2\sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x)=f(c)$$

Therefore, f is continuous at all points x, such that $x\neq 0$.

Case II:

If $c=0$, then $f(0)=0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \le \sin\frac{1}{x} \le 1, x \ne 0$

$$\Rightarrow -x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \le \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \le 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f, where f is defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1, & \text{if } x = 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \ne 0 \\ -1, & \text{if } x = 0 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c \ne 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x \ne 0$.

Case II:

If $c=0$, then $f(0)=-1$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0}(\sin x-\cos x)=\sin 0-\cos 0=0-1=-1$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$

Therefore, f is continuous at $x=0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq \dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ at $x=\dfrac{\pi}{2}$

Solution:

The given function is $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq \dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$

The given function f is continuous at $x=\dfrac{\pi}{2}$, if f is defined at $x=\dfrac{\pi}{2}$ and if the value of the f at $x=\dfrac{\pi}{2}$ equals the limit of f at $x=\dfrac{\pi}{2}$.

It is evident that f is defined at $x=\dfrac{\pi}{2}$ and $f\left(\dfrac{\pi}{2}\right)=3$

$\lim\limits_{x\to\frac{\pi}{2}} f(x)=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{k\cos x}{\pi-2x}$

Put $x=\dfrac{\pi}{2}+h$

Then $x\to\dfrac{\pi}{2}\Rightarrow h\to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi-2x}=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$=k\lim_{h\to 0}\frac{-\sin h}{-2h}=\frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}=\frac{k}{2}.1=\frac{k}{2}$$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2}=3$$

$$\Rightarrow k=6$$

Therefore, the value of $k=6$.

Question 27:

Find the values of k so that the function f is continuous at the indicated point. $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$ at $x=2$

Solution:

The given function is $f(x)=\begin{cases} kx^2, & \text{if } x\le 2 \\ 3, & \text{if } x>2 \end{cases}$

The given function f is continuous at $x=2$, if f is defined at $x=2$ and if the value of the f at $x=2$ equals the limit of f at $x=2$.

It is evident that f is defined at $x=2$ and $f(2)=k(2)^2=4k$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}\left(kx^2\right)=\lim_{x\to 2^+}(3)=4k$$

$$\Rightarrow k\times 2^2=3=4k$$

$$\Rightarrow 4k=3$$

$$\Rightarrow k=\frac{3}{4}$$

Therefore, the value of $k=\frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point $f(x)=\begin{cases}kx+1, \text{ if } x\le\pi \\ \cos x, \text{ if } x>\pi\end{cases}$ at $x=\pi$

Solution:

The given function is $f(x)=\begin{cases}kx+1, \text{ if } x\le\pi \\ \cos x, \text{ if } x>\pi\end{cases}$

The given function f is continuous at $x=\pi$, if f is defined at $x=\pi$ and if the value of the f at $x=\pi$ equals the limit of f at $x=\pi$.

It is evident that f is defined at $x=\pi$ and $f(\pi)=k\pi+1$

$$\lim_{x\to\pi^-} f(x)=\lim_{x\to\pi^+} f(x)=f(\pi)$$

$$\Rightarrow \lim_{x\to\pi^-}(kx+1)=\lim_{x\to\pi^+}(\cos x)=k\pi+1$$

$$\Rightarrow k\pi+1=\cos\pi=k\pi+1$$

$$\Rightarrow k\pi+1=-1=k\pi+1$$

$$\Rightarrow k=-\frac{2}{\pi}$$

Therefore, the value of $k=-\dfrac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point $f(x)=\begin{cases}kx+1, \text{ if } x\le 5 \\ 3x-5, \text{ if } x>5\end{cases}$ at $x=5$.

Solution:

The given function is $f(x)=\begin{cases}kx+1, \text{ if } x\le 5 \\ 3x-5, \text{ if } x>5\end{cases}$

The given function f is continuous at $x=5$, if f is defined at $x=5$ and if the value of the f at $x=5$ equals the limit of f at $x=5$.

It is evident that f is defined at $x=5$ and $f(5)=kx+1=5k+1$

$$\lim_{x\to5^-} f(x)=\lim_{x\to5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{x\to5^-}(kx+1)=\lim_{x\to5^+}(3x-5)=5k+1$$
$$\Rightarrow 5k+1=3(5)-5=5k+1$$
$$\Rightarrow 5k+1=15-5=5k+1$$
$$\Rightarrow 5k+1=10=5k+1$$
$$\Rightarrow 5k+1=10$$
$$\Rightarrow 5k=9$$
$$\Rightarrow k=\frac{9}{5}$$

Therefore, the value of $k=\frac{9}{5}$.

Question 30:

Find the values of a & b such that the function defined by $f(x)=\begin{cases}5, \text{ if } x\le 2\\ ax+b, \text{ if } 2<x<10\\ 21, \text{ if } x\ge 10\end{cases}$, is a continuous function.

Solution:

The given function is $f(x)=\begin{cases}5, \text{ if } x\le 2\\ ax+b, \text{ if } 2<x<10\\ 21, \text{ if } x\ge 10\end{cases}$

It is evident that f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x=2$ and $x=10$

Since f is continuous at $x=2$, we obtain

$$\lim_{x\to2^-} f(x)=\lim_{x\to2^+} f(x)=f(2)$$
$$\Rightarrow \lim_{x\to2^-}(5)=\lim_{x\to2^+}(ax+b)=5$$
$$\Rightarrow 5=2a+b=5$$
$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since f is continuous at $x=10$, we obtain

$$\lim_{x\to10^-} f(x)=\lim_{x\to10^+} f(x)=f(10)$$
$$\Rightarrow \lim_{x\to10^-}(ax+b)=\lim_{x\to10^+}(21)=21$$
$$\Rightarrow 10a+b=21=21$$
$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting $a = 2$ in equation (1), we obtain

$2(2) + b = 5$

$\Rightarrow 4 + b = 5$

$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Solution:

The given function is $f(x) = \cos(x^2)$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x) = \cos x$ and $h(x) = x^2$

$\left[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)\right]$

It has to be proved first that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Let $g(c) = \cos c$. Put $x = c + h$

If $x \to c$, then $h \to 0$

$$\begin{aligned}
\lim_{x \to c} g(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c(1) - \sin c(0) \\
&= \cos c
\end{aligned}$$

$\therefore \lim_{x \to c} g(x) = g(c)$

Therefore, $g(x) = \cos x$ is a continuous function.

Let $h(x)=x^2$

It is evident that h is defined for every real number.

Let k be a real number, then $h(k)=k^2$

$\lim\limits_{x \to k} h(x)=\lim\limits_{x \to k} x^2=k^2$

$\therefore \lim\limits_{x \to k} h(x)=h(k)$

Therefore, h is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x)=(goh)(x)=\cos\left(x^2\right)$ is a continuous function.

Question 32:

Show that the function defined by $f(x)=|\cos x|$ is a continuous function.

Solution:

The given function is $f(x)=|\cos x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f=goh$, where $g(x)=|x|$ and $h(x)=\cos x$

$\left[\because (goh)(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\cos x$ are continuous functions.

$g(x)=|x|$ can be written as $g(x)=\begin{cases}-x, & \text{if } x<0 \\ x, & \text{if } x \geq 0\end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$

$\lim\limits_{x \to c} g(x)=\lim\limits_{x \to c}(-x)=-c$

$\therefore \lim\limits_{x \to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x<0$.

Case II:

If $c>0$, then $g(c)=c$

$$\lim_{x \to c} g(x) = \lim_{x \to c}(x) = c$$
$$\therefore \lim_{x \to c} g(x) = g(c)$$
Therefore, g is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$
$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+}(x) = g(0)$$

Therefore, g is continuous at all $x = 0$.
From the above three observations, it can be concluded that g is continuous at all points.
Let $h(x) = \cos x$
It is evident that $h(x) = \cos x$ is defined for every real number.
Let c be a real number. Put $x = c + h$
If $x \to c$, then $h \to 0$
$$h(c) = \cos c$$
$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \cos x \\
&= \lim_{h \to 0} \cos(c + h) \\
&= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \to 0} (\cos c \cos h) - \lim_{h \to 0} (\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c (1) - \sin c (0) \\
&= \cos c
\end{aligned}$$
$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Show that the function defined by $f(x) = |\sin x|$ is a continuous function.

Solution:

The given function is $f(x)=|\sin x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x)=|x|$ and $h(x)=\sin x$

$\left[\because (goh)(x)=g(h(x))=g(\sin x)=|\sin x|=f(x)\right]$

It has to be proved first that $g(x)=|x|$ and $h(x)=\sin x$ are continuous functions.

$g(x)=|x|$ can be written as $g(x)=\begin{cases}-x, \text{ if } x<0 \\ x, \text{ if } x\geq 0\end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c<0$, then $g(c)=-c$

$\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x<0$.

Case II:

If $c>0$, then $g(c)=c$

$\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(x)=c$

$\therefore \lim\limits_{x\to c} g(x)=g(c)$

Therefore, g is continuous at all points x, such that $x>0$.

Case III:

If $c=0$, then $g(c)=g(0)=0$

$\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=0$

$\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}(x)=0$

$\therefore \lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^+}(x)=g(0)$

Therefore, g is continuous at all $x=0$.

From the above three observations, it can be concluded that g is continuous at all points.

Let $h(x)=\sin x$

It is evident that $h(x)=\sin x$ is defined for every real number.

Let c be a real number. Put $x=c+k$

If $x\to c$, then $k\to 0$

$$h(c) = \sin c$$

$$\begin{aligned}
\lim_{x \to c} h(x) &= \lim_{x \to c} \sin x \\
&= \lim_{k \to 0} \sin(c+k) \\
&= \lim_{k \to 0} [\sin c \cos k + \cos c \sin k] \\
&= \lim_{k \to 0} (\sin c \cos k) + \lim_{k \to 0} (\cos c \sin k) \\
&= \sin c \cos 0 + \cos c \sin 0 \\
&= \sin c (1) + \cos c (0) \\
&= \sin c
\end{aligned}$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x) = \sin x$ is a continuous function.

It is known that for real valued functions g and h, such that (goh) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Solution:

The given function is $f(x) = |x| - |x+1|$.

The two functions, g and h are defined as $g(x) = |x|$ and $h(x) = |x+1|$.

Then, $f = g - h$

The continuity of g and h are examined first.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \text{ if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$$\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$$

$$\therefore \lim_{x \to c} g(x) = g(c)$$

Therefore, g is continuous at all points x, such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c}(x) = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-}(-x) = 0$

$\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+}(x) = 0$

$\therefore \lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^+}(x) = g(0)$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as $h(x) = \begin{cases} -(x+1), & \text{if } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$

It is evident that h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$

$\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c}\left[-(x+1)\right] = -(c+1)$

$\therefore \lim\limits_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x < -1$.

Case II:

If $c > -1$, then $h(c) = c+1$

$\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c}(x+1) = c+1$

$\therefore \lim\limits_{x \to c} h(x) = h(c)$

Therefore, h is continuous at all points x, such that $x > -1$.

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$\lim\limits_{x \to -1^-} h(x) = \lim\limits_{x \to -1^-}\left[-(x+1)\right] = -(-1+1) = 0$

$\lim\limits_{x \to -1^+} h(x) = \lim\limits_{x \to -1^+}(x+1) = (-1+1) = 0$

$\therefore \lim\limits_{x \to -1^-} h(x) = \lim\limits_{x \to -1^+} h(x) = h(-1)$

Therefore, h is continuous at $x=-1$.

From the above three observations, it can be concluded that h is continuous at all points.
It concludes that g and h are continuous functions. Therefore, $f=g-h$ is also a continuous function.

Therefore, f has no point of discontinuity.

EXERCISE 5.2

Question 1:

Differentiate the function with respect to x.

$\sin\left(x^2+5\right)$

Solution:

Let $f(x)=\sin\left(x^2+5\right)$, $u(x)=x^2+5$ and $v(t)=\sin t$

Then, $(vou)(x)=v(u(x))=v\left(x^2+5\right)=\tan\left(x^2+5\right)=f(x)$

Thus, f is a composite of two functions.

Put $t=u(x)=x^2+5$

Then, we get

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos\left(x^2+5\right)$$

$$\frac{dt}{dx}=\frac{d}{dx}\left(x^2+5\right)=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)=2x+0=2x$$

By chain rule of derivative,

$$\frac{df}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos\left(x^2+5\right)\times 2x=2x\cos\left(x^2+5\right)$$

Alternate method:

$$\begin{aligned}\frac{d}{dx}\left[\sin\left(x^2+5\right)\right]&=\cos\left(x^2+5\right).\frac{d}{dx}\left(x^2+5\right)\\&=\cos\left(x^2+5\right).\left[\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(5)\right]\\&=\cos\left(x^2+5\right).[2x+0]\\&=2x\cos\left(x^2+5\right)\end{aligned}$$

Question 2:

Differentiate the function with respect to x

$\cos(\sin x)$

Solution:

Let $f(x)=\cos(\sin x), u(x)=\sin x$ and $v(t)=\cos t$

Then, $(vou)(x)=v(u(x))=v(\sin x)=\cos(\sin x)=f(x)$

Here, f is a composite function of two functions.

Put $t=u(x)=\sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

By chain rule,

$$\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx} = -\sin(\sin x).\cos x = -\cos x \sin(\sin x)$$

Alternate method:

$$\frac{d}{dx}[\cos(\sin x)] = -\sin(\sin x).\frac{d}{dx}(\sin x)$$
$$= -\sin(\sin x) \times \cos x$$
$$= -\cos x \sin(\sin x)$$

Question 3:

Differentiate the function with respect to x

$\sin(ax+b)$

Solution:

Let $f(x) = \sin(ax+b), u(x) = ax+b$ and $v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = f(x)$

Here, f is a composite function of two functions u and v.

Put, $t = u(x) = ax+b$

Thus,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we get

$$\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx} = \cos(ax+b).a = a\cos(ax+b)$$

Alternate method:

$$\frac{d}{dx}\left[\sin(ax+b)\right]=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$=\cos(ax+b).\left[\frac{d}{dx}(ax)+\frac{d}{dx}(b)\right]$$
$$=\cos(ax+b).(a+0)$$
$$=a\cos(ax+b)$$

Question 4:

Differentiate the function with respect to *x*

$\sec\left(\tan\left(\sqrt{x}\right)\right)$

Solution:

Let $f(x)=\sec\left(\tan\left(\sqrt{x}\right)\right), u(x)=\sqrt{x}, v(t)=\tan t$ and $w(s)=\sec s$

Then, $(wovou)(x)=w\left[v(u(x))\right]=w\left[v\left(\sqrt{x}\right)\right]=w\left(\tan\sqrt{x}\right)=\sec\left(\tan\sqrt{x}\right)=f(x)$

Here, f is a composite function of three functions *u, v* and *w.*

Put, $s=v(t)=\tan t$ and $t=u(x)=\sqrt{x}$

Then,

$$\frac{dw}{ds}=\frac{d}{ds}(\sec s)$$
$$=\sec s\tan s$$
$$=\sec(\tan t).\tan(\tan t) \qquad [s=\tan t]$$
$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right) \qquad \left[t=\sqrt{x}\right]$$

Now,

$$\frac{ds}{dt}=\frac{d}{dt}(\tan t)=\sec^2 t=\sec^2\sqrt{x}$$
$$\frac{dt}{dx}=\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}.x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

Hence, by chain rule, we get

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\frac{dw}{ds}.\frac{ds}{dt}.\frac{dt}{dx}$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\sqrt{x}.\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)$$

$$=\frac{\sec^2\sqrt{x}\sec\left(\tan\sqrt{x}\right)\tan\left(\tan\sqrt{x}\right)}{2\sqrt{x}}$$

Alternate method:

$$\frac{d}{dx}\left[\sec\left(\tan\sqrt{x}\right)\right]=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{d}{dx}\left(\sqrt{x}\right)$$

$$=\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec\left(\tan\sqrt{x}\right).\tan\left(\tan\sqrt{x}\right).\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$$

Question 5:

Differentiate the function with respect to x

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Solution:

Given, $f(x)=\dfrac{\sin(ax+b)}{\cos(cx+d)}$, where $g(x)=\sin(ax+b)$ and $h(x)=\cos(cx+d)$

$\therefore f=\dfrac{g'h-gh'}{h^2}$

Consider $g(x)=\sin(ax+b)$

Let $u(x)=ax+b, v(t)=\sin t$

Then $(vou)(x)=v(u(x))=v(ax+b)=\sin(ax+b)=g(x)$

$\therefore g$ is a composite function of two functions, u and v.

Put, $t=u(x)=ax+b$

$$\frac{dv}{dt}=\frac{d}{dt}(\sin t)=\cos t=\cos(ax+b)$$
$$\frac{dt}{dx}=\frac{d}{dx}(ax+b)=\frac{d}{dx}(ax)+\frac{d}{dx}(b)=a+0=a$$

Thus, by chain rule, we get

$$g'=\frac{dg}{dx}=\frac{dv}{dt}.\frac{dt}{dx}=\cos(ax+b).a=a\cos(ax+b)$$

Consider $h(x)=\cos(cx+d)$

Let $p(x)=cx+d, q(y)=\cos y$

Then, $(qop)(x)=q(p(x))=q(cx+d)=\cos(cx+d)=h(x)$

$\therefore h$ is a composite function of two functions, p and q.

Put, $y=p(x)=cx+d$

$$\frac{dq}{dy}=\frac{d}{dy}(\cos y)=-\sin y=-\sin(cx+d)$$
$$\frac{dy}{dx}=\frac{d}{dx}(cx+d)=\frac{d}{dx}(cx)+\frac{d}{dx}(d)=c$$

Using chain rule, we get

$$\begin{aligned}h'&=\frac{dh}{dx}=\frac{dq}{dy}.\frac{dy}{dx}\\&=-\sin(cx+d)\times c\\&=-c\sin(cx+d)\end{aligned}$$

Therefore,

$$\begin{aligned}f'&=\frac{a\cos(ax+b).\cos(cx+d)-\sin(ax+b)\{-c\sin(cx+d)\}}{[\cos(cx+d)]^2}\\&=\frac{a\cos(ax+b)}{\cos(cx+d)}+c\sin(ax+b).\frac{\sin(cx+d)}{\cos(cx+d)}\times\frac{1}{\cos(cx+d)}\\&=a\cos(ax+b)\sec(cx+d)+c\sin(ax+b)\tan(cx+d)\sec(cx+d)\end{aligned}$$

Question 6:

Differentiate the function with respect to x

$\cos x^3.\sin^2(x^5)$

Solution:

Given, $\cos x^3.\sin^2(x^5)$

$$\frac{d}{dx}\left[\cos x^3.\sin^2\left(x^5\right)\right]=\sin^2\left(x^5\right)\times\frac{d}{dx}\left(\cos x^3\right)+\cos x^3\times\frac{d}{dx}\left[\sin^2\left(x^5\right)\right]$$

$$=\sin^2\left(x^5\right)\times\left(-\sin x^3\right)\times\frac{d}{dx}\left(x^3\right)+\cos x^3\times 2\sin\left(x^5\right).\frac{d}{dx}\left[\sin x^5\right]$$

$$=-\sin x^3\sin^2\left(x^5\right)\times 3x^2+2\sin x^5\cos x^3.\cos x^5\times\frac{d}{dx}\left(x^5\right)$$

$$=-3x^2\sin x^3.\sin^2\left(x^5\right)+2\sin x^5\cos x^5\cos x^3\times 5x^4$$

$$=10x^4\sin x^5\cos x^5\cos x^3-3x^2\sin x^3\sin^2\left(x^5\right)$$

Question 7:

Differentiate the function with respect to x

$$2\sqrt{\cot\left(x^2\right)}$$

Solution:

$$\frac{d}{dx}\left[2\sqrt{\cot\left(x^2\right)}\right]=2.\frac{1}{2\sqrt{\cot\left(x^2\right)}}\times\frac{d}{dx}\left[\cot\left(x^2\right)\right]$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times-\operatorname{cosec}^2\left(x^2\right)\times\frac{d}{dx}\left(x^2\right)$$

$$=\sqrt{\frac{\sin\left(x^2\right)}{\cos\left(x^2\right)}}\times\frac{-1}{\sin^2\left(x^2\right)}\times(2x)$$

$$=\frac{-2x}{\sin x^2\sqrt{\cos x^2\sin x^2}}$$

$$=\frac{-2\sqrt{2}x}{\sin x^2\sqrt{2\sin x^2\cos x^2}}$$

$$=\frac{-2\sqrt{2}x}{\sin x^2\sqrt{\sin 2x^2}}$$

Question 8:

Differentiate the function with respect to x

$$\cos\left(\sqrt{x}\right)$$

Solution:

Let $f(x)=\cos\left(\sqrt{x}\right)$

Also, let $u(x)=\sqrt{x}$ and, $v(t)=\cos t$

Then,

$$\begin{aligned}(vou)(x) &= v(u(x)) \\ &= v\left(\sqrt{x}\right) \\ &= \cos\sqrt{x} \\ &= f(x)\end{aligned}$$

Since, f is a composite function of *u* and *v*.

$$t = u(x) = \sqrt{x}$$

Then,

$$\frac{dt}{dx} = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{-1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\frac{dv}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

And, $= -\sin\left(\sqrt{x}\right)$

Using chain rule, we get

$$\begin{aligned}\frac{dt}{dx} &= \frac{dv}{dt}.\frac{dt}{dx} \\ &= -\sin\left(\sqrt{x}\right).\frac{1}{2\sqrt{x}} \\ &= -\frac{1}{2\sqrt{x}}\sin\left(\sqrt{x}\right) \\ &= -\frac{\sin\left(\sqrt{x}\right)}{2\sqrt{x}}\end{aligned}$$

Alternate method:

$$\begin{aligned}\frac{d}{dx}\left[\cos\left(\sqrt{x}\right)\right] &= -\sin\left(\sqrt{x}\right).\frac{d}{dx}\left(\sqrt{x}\right) \\ &= -\sin\left(\sqrt{x}\right)\times\frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= -\sin\sqrt{x}\times\frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{-\sin\sqrt{x}}{2\sqrt{x}}\end{aligned}$$

Question 9:

Prove that the function f given by

$f(x)=|x-1|, x \in \mathbf{R}$ is not differentiable at $x=1$.

Solution:

Given, $f(x)=|x-1|, x \in \mathbf{R}$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim_{h \to 0^-} \frac{f(c)-f(c-h)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x=1$,

Consider LHD at $x=1$

$$\begin{aligned}
\lim_{h \to 0^-} \frac{f(1)-f(1-h)}{h} &= \lim_{h \to 0^-} \frac{f|1-1|-|1-h-1|}{h} \\
&= \lim_{h \to 0^-} \frac{0-|h|}{h} \\
&= \lim_{h \to 0^-} \frac{-h}{h} \qquad (h<0 \Rightarrow |h|=-h) \\
&= -1
\end{aligned}$$

Consider RHD at $x=1$

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} &= \lim_{h \to 0^+} \frac{f|1+h-1|-|1-1|}{h} \\
&= \lim_{h \to 0^+} \frac{|h|-0}{h} \\
&= \lim_{h \to 0^+} \frac{h}{h} \qquad (h>0 \Rightarrow |h|=h) \\
&= 1
\end{aligned}$$

Since LHD and RHD at $x=1$ are not equal,

Therefore, f is not differentiable at $x=1$.

Question 10:

Prove that the greatest integer function defined by $f(x)=[x], 0<x<3$ is not differentiable at $x=1$ and $x=2$.

Solution:

Given, $f(x)=[x], 0<x<3$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim_{h\to 0^-}\frac{f(c)-f(c-h)}{h}$ and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$ are finite and equal.

At $x=1$,

Consider the LHD at $x=1$

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(1)-f(1-h)}{h} &= \lim_{h\to 0^-}\frac{[1]-[1-h]}{h}\\ &= \lim_{h\to 0^-}\frac{1-0}{h}\\ &= \lim_{h\to 0^-}\frac{1}{h}\\ &= \infty\end{aligned}$$

Consider RHD at $x=1$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+}\frac{[1+h]-[1]}{h}\\ &= \lim_{h\to 0^+}\frac{1-1}{h}\\ &= \lim_{h\to 0^+}0\\ &= 0\end{aligned}$$

Since LHD and RHD at $x=1$ are not equal,

Hence, f is not differentiable at $x=1$.

To check the differentiability of the given function at $x=2$,

Consider LHD at $x=2$

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(2)-f(2-h)}{h} &= \lim_{h\to 0^-}\frac{[2]-[2-h]}{h}\\ &= \lim_{h\to 0^-}\frac{2-1}{h}\\ &= \lim_{h\to 0^-}\frac{1}{h}\\ &= \infty\end{aligned}$$

Now, consider RHD at $x=2$

$$\begin{aligned}\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0^+}\frac{[2+h]-[2]}{h}\\ &= \lim_{h\to 0^+}\frac{2-2}{h}\\ &= \lim_{h\to 0^+} 0\\ &= 0\end{aligned}$$

Since, LHD and RHD at $x=2$ are not equal.

Hence, f is not differentiable at $x=2$.

EXERCISE 5.3

Question 1:

Find $\frac{dy}{dx}: 2x+3y=\sin x$

Solution:

Given, $2x+3y=\sin x$

Differentiating with respect to x, we get

$$\frac{d}{dy}(2x+3y)=\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\cos x$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos x$$

$$\Rightarrow 3\frac{dy}{dx}=\cos x-2$$

$$\therefore \frac{dx}{dy}=\frac{\cos x-2}{3}$$

Question 2:

Find $\frac{dy}{dx}$: $2x+3y=\sin y$

Solution:

Given, $2x+3y=\sin y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(2x)+\frac{d}{dx}(3y)=\frac{d}{dx}(\sin y)$$

$$\Rightarrow 2+3\frac{dy}{dx}=\cos y\frac{dy}{dx} \qquad \text{[By using chain rule]}$$

$$\Rightarrow 2=(\cos y-3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{2}{\cos y-3}$$

Question 3:

Find $\frac{dy}{dx}$: $ax+by^2=\cos y$

Solution:

Given, $ax+by^2=\cos y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(ax)+\frac{d}{dx}(by^2)=\frac{d}{dx}(\cos y)$$

$$\Rightarrow a+b\frac{d}{dx}(y^2)=\frac{d}{dx}(\cos y) \qquad \ldots(1)$$

$$\frac{d}{dx}(y^2)=2y\frac{dy}{dx} \text{ and } \frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$a+b\times 2y\frac{dy}{dx}=-\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by+\sin y)\frac{dy}{dx}=-a$$

$$\therefore \frac{dy}{dx}=\frac{-a}{2by+\sin y}$$

Question 4:

Find $\frac{dy}{dx}$: $xy+y^2=\tan x+y$

Solution:

Given, $xy+y^2=\tan x+y$

Differentiating with respect to x, we get

$$\frac{d}{dx}(xy+y^2)=\frac{d}{dx}(\tan x+y)$$

$$\Rightarrow \frac{d}{dx}(xy)+\frac{d}{dx}(y^2)=\frac{d}{dx}(\tan x)+\frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \qquad [\text{using product rule and chain rule}]$$

$$\Rightarrow y.1+x\frac{dy}{dx}+2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx} \Rightarrow (x+2y-1)\frac{dy}{dx}=\sec^2 x-y$$

$$\therefore \frac{dy}{dx}=\frac{\sec^2 x-y}{(x+2y-1)}$$

Question 5:

Find $\frac{dy}{dx}$: $x^2+xy+y^2=100$

Solution:

Given, $x^2+xy+y^2=100$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=0$$

$$\Rightarrow 2x+\left[y.\frac{d}{dx}(x)+x.\frac{dy}{dx}\right]+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y.1+x.\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$\Rightarrow 2x+y+(x+2y)\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{2x+y}{x+2y}$$

Question 6:

Find $\frac{dy}{dx}$: $x^3+x^2y+xy^2+y^3=81$

Solution:

Given, $x^3+x^2y+xy^2+y^3=81$

Differentiating with respect to x, we get

$$\frac{d}{dx}\left(x^3+x^2y+xy^2+y^3\right)=\frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(x^2y\right)+\frac{d}{dx}\left(xy^2\right)+\frac{d}{dx}\left(y^3\right)=0$$

$$\Rightarrow 3x^2+\left[y\frac{d}{dx}\left(x^2\right)+x^2\frac{dy}{dx}\right]+\left[y^2\frac{d}{dx}(x)+x\frac{d}{dx}\left(y^2\right)\right]+3y^2\frac{dy}{dx}=0$$

$$\Rightarrow 3x^2+\left[y.2x+x^2\frac{dy}{dx}\right]+\left[y^2.1+x.2y.\frac{dy}{dx}\right]+3y^2\frac{dx}{dy}=0$$

$$\Rightarrow \left(x^2+2xy+3y^2\right)\frac{dy}{dx}+\left(3x^2+2xy+y^2\right)=0$$

$$\therefore \frac{dy}{dx}=\frac{-\left(3x^2+2xy+y^2\right)}{\left(x^2+2xy+3y^2\right)}$$

Question 7:

Find $\frac{dx}{dy}$: $\sin^2 y + \cos xy = \pi$

Solution:

Given, $\sin^2 y + \cos xy = \pi$
Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\sin^2 y + \cos xy\right) = \frac{d}{dx}(\pi)$$
$$\Rightarrow \frac{d}{dx}\left(\sin^2 y\right) + \frac{d}{dx}(\cos xy) = 0 \qquad \ldots(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}\left(\sin^2 y\right) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad \ldots(2)$$
$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$
$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy\frac{dy}{dx} \qquad \ldots(3)$$

From (1), (2) and (3), we obtain

$$2\sin y\cos y\frac{dy}{dx} + \left(-y\sin xy - x\sin xy\frac{dy}{dx}\right) = 0$$
$$\Rightarrow (2\sin y\cos y - x\sin xy)\frac{dy}{dx} = y\sin xy$$
$$\Rightarrow (\sin 2y - x\sin xy)\frac{dx}{dy} = y\sin xy$$
$$\therefore \frac{dx}{dy} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

Question 8:

Find $\frac{dy}{dx}$: $\sin^2 x + \cos^2 y = 1$

Solution:

Given, $\sin^2 x + \cos^2 y = 1$
Differentiating with respect to x, we get

$$\frac{d}{dx}\left(\sin^2 x+\cos^2 y\right)=\frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}\left(\sin^2 x\right)+\frac{d}{dx}\left(\cos^2 y\right)=0$$

$$\Rightarrow 2\sin x.\frac{d}{dx}(\sin x)+2\cos y.\frac{d}{dx}(\cos y)=0$$

$$\Rightarrow 2\sin x\cos x+2\cos y(-\sin y).\frac{dy}{dx}=0$$

$$\Rightarrow \sin 2x-\sin 2y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=\frac{\sin 2x}{\sin 2y}$$

Question 9:

Find $\frac{dy}{dx}: y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Given,

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y=\frac{2x}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \qquad \ldots(1)$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

The function $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$

By quotient rule, we get

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)=\frac{\left(1+x^2\right)\frac{d}{dx}(2x)-2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$=\frac{\left(1+x^2\right).2-2x.\left[0+2x\right]}{\left(1+x^2\right)^2}$$

$$=\frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}$$

$$=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

Also, $\sin y=\frac{2x}{1+x^2}$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}$$

$$=\sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}$$

$$=\frac{1-x^2}{1+x^2}$$

From (1), (2) and (3), we get

$$\frac{1-x^2}{1+x^2}\times\frac{dy}{dx}=\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{1+x^2}$$

Question 10:

Find $\frac{dy}{dx}: y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

Solution:

Given, $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$$\Rightarrow \tan y=\left(\frac{3x-x^3}{1-3x^2}\right) \qquad \ldots(1)$$

Since, we know that

$$\Rightarrow \tan y=\left(\frac{3\tan\frac{y}{3}-\tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}\right) \qquad (2)$$

Comparing (1) and (2) we get,

$$x=\tan\frac{y}{3}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x)=\frac{d}{dx}\left(\tan\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}.\frac{d}{dx}\left(\frac{y}{3}\right)$$

$$\Rightarrow 1=\sec^2\frac{y}{3}.\frac{1}{3}.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3}{\sec^2\frac{y}{3}}=\frac{3}{1+\tan^2\frac{y}{3}}$$

$$\therefore \frac{dy}{dx}=\frac{3}{1+x^2}$$

Question 11:

Find $\frac{dy}{dx}$: $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0<x<1$

Solution:

Given, $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \cos y = \left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

Comparing LHS and RHS, we get

$$\tan\frac{y}{2} = x$$

Differentiating with respect to x, we get

$$\sec^2\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \sec^2\frac{y}{2}\times\frac{1}{2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+\tan^2\frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx} : y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Solution:

Given, $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) \qquad \ldots(1)$$

Using chain rule, we get

$$\frac{d}{dx}(\sin y)=\cos y.\frac{dy}{dx}$$

$$\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$=\sqrt{\frac{\left(1+x^2\right)^2-\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

Therefore,

$$\frac{d}{dx}(\sin y)=\frac{2x}{1+x^2}\frac{dy}{dx} \qquad \ldots(2)$$

$$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{\left(1+x^2\right).\frac{d}{dx}\left(1-x^2\right)-\left(1-x^2\right).\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} \qquad [\text{using quotient rule}]$$

$$=\frac{\left(1+x^2\right)(-2x)-\left(1-x^2\right)(2x)}{\left(1+x^2\right)^2}$$

$$=\frac{-2x-2x^3-2x+2x^3}{\left(1+x^2\right)^2}$$

$$=\frac{-4x}{\left(1+x^2\right)^2} \qquad \ldots(3)$$

From equation (1), (2) and (3), we get

$$\frac{2x}{1+x^2}\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 13:

Find $\frac{dy}{dx}: y=\cos^{-1}\left(\frac{2x}{1+x^2}\right), -1<x<1$

Solution:

Given, $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\cos y=\left(\frac{2x}{1+x^2}\right)$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y.\frac{dy}{dx}=\frac{\left(1+x^2\right).\frac{d}{dx}(2x)-2x.\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\,\frac{dy}{dx}=\frac{\left(1+x^2\right)\times 2-2x\times 2x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx}=-\left[\frac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}\right]$$

$$\Rightarrow \sqrt{\frac{\left(1+x^2\right)^2-4x^2}{\left(1+x^2\right)^2}}.\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)}$$

$$\Rightarrow \sqrt{\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2}}\,\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1-x^2\right)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}.\frac{dy}{dx}=\frac{-2\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^2}$$

Question 14:

Find $\frac{dy}{dx}: y=\sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$

Solution:

Given, $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin y=\left(2x\sqrt{1-x^2}\right)$$

Differentiating with respect to x, we get

$$\cos y.\frac{dy}{dx}=2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$\Rightarrow\sqrt{1-\sin^2 y}\frac{dy}{dx}=2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}}+\sqrt{1-x^2}\right]$$

$$\Rightarrow\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}.\frac{dy}{dx}=2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{1-4x^2\left(1-x^2\right)}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\sqrt{\left(1-2x^2\right)^2}\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\left(1-2x^2\right)\frac{dy}{dx}=2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{2}{\sqrt{1-x^2}}$$

Question 15:

Find $\frac{dy}{dx}: y=\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0<x<\frac{1}{\sqrt{2}}$

Solution:

Given, $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$\Rightarrow y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow \sec y = \left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = \sin \frac{y}{2} . \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

EXERCISE 5.4

Question 1:

Differentiating the following wrt x: $\frac{e^x}{\sin x}$

Solution:

Let $y=\frac{e^x}{\sin x}$

By using the quotient rule, we get

$$\frac{dy}{dx}=\frac{\sin x\frac{d}{dx}\left(e^x\right)-e^x\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$=\frac{\sin x.\left(e^x\right)-e^x.(\cos x)}{\sin^2 x}$$

$$=\frac{e^x(\sin x-\cos x)}{\sin^2 x}$$

Question 2:

Differentiating the following $e^{\sin^{-1}x}$

Solution:

Let $y=e^{\sin^{-1}x}$

By using the quotient rule, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\sin^{-1}x}\right)$$

$$=e^{\sin^{-1}x}.\frac{d}{dx}\left(\sin^{-1}x\right)$$

$$=e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}},x\in(-1,1)$$

Question 3:

Differentiating the following wrt $x:e^{x^3}$

Solution:

Let $y = e^{x^3}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right)$$
$$= e^{x^3}.\frac{d}{dx}\left(x^3\right)$$
$$= e^{x^3}.3x^2$$
$$= 3x^2 e^{x^3}$$

Question 4:

Differentiate the following wrt $x : \sin\left(\tan^{-1} e^{-x}\right)$

Solution:

Let $y = \sin\left(\tan^{-1} e^{-x}\right)$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\tan^{-1} e^{-x}\right)\right]$$
$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{d}{dx}\left(\tan^{-1} e^{-x}\right)$$
$$= \cos\left(\tan^{-1} e^{-x}\right).\frac{1}{1+\left(e^{-x}\right)^2}.\frac{d}{dx}\left(e^{-x}\right)$$
$$= \frac{\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}.e^{-x}.\frac{d}{dx}(-x)$$
$$= \frac{e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}} \times (-1)$$
$$= \frac{-e^{-x}\cos\left(\tan^{-1} e^{-x}\right)}{1+e^{-2x}}$$

Question 5:

Differentiate the following wrt $x : \log\left(\cos e^{x}\right)$

Solution:

Let $y = \log\left(\cos e^{x}\right)$

By using the chain rule, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\cos e^x\right)\right]$$
$$=\frac{1}{\cos e^x}\cdot\frac{d}{dx}\left(\cos e^x\right)$$
$$=\frac{1}{\cos e^x}\cdot\left(-\sin e^x\right)\cdot\frac{d}{dx}\left(e^x\right)$$
$$=\frac{-\sin e^x}{\cos e^x}\cdot e^x$$
$$=-e^x\tan e^x, e^x\neq(2n+1)\frac{\pi}{2}, n\in\mathbf{N}$$

Question 6:

Differentiate the following wrt $x: e^x+e^{x^2}+\ldots+e^{x^5}$

Solution:

$$\frac{d}{dx}\left(e^x+e^{x^2}+\ldots+e^{x^5}\right)$$

Differentiating wrt *x*, we get

$$\frac{d}{dx}\left(e^x+e^{x^2}+\ldots+e^{x^5}\right)=\frac{d}{dx}\left(e^x\right)+\frac{d}{dx}\left(e^{x^2}\right)+\frac{d}{dx}\left(e^{x^3}\right)+\frac{d}{dx}\left(e^{x^4}\right)+\frac{d}{dx}\left(e^{x^5}\right)$$
$$=e^x+\left[e^{x^2}\times\frac{d}{dx}\left(x^2\right)\right]+\left[e^{x^3}\times\frac{d}{dx}\left(x^3\right)\right]+\left[e^{x^4}\times\frac{d}{dx}\left(x^4\right)\right]+\left[e^{x^5}\times\frac{d}{dx}\left(x^5\right)\right]$$
$$=e^x+\left(e^{x^2}\times 2x\right)+\left(e^{x^3}\times 3x^2\right)+\left(e^{x^4}\times 4x^3\right)+\left(e^{x^5}\times 5x^4\right)$$
$$=e^x+2xe^{x^2}+3x^2e^{x^3}+4x^3e^{x^4}+5x^4e^{x^5}$$

Question 7:

Differentiating the following wrt $x:\sqrt{e^{\sqrt{x}}}, x>0$

Solution:

Let $y=\sqrt{e^{\sqrt{x}}}$

Then, $y^2=e^{\sqrt{x}}$

Differentiating wrt *x*, we get

$$y^2=e^{\sqrt{x}}$$

$$\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}\left(e^{\sqrt{x}}\right)$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow 2y\frac{dy}{dx}=e^{\sqrt{x}}\frac{1}{2}.\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4y\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}},x>0$$

Question 8:

Differentiating the following wrt $x:\log\left(\log x\right),x>1$

Solution:

Let $y=\log\left(\log x\right)$

By using the chain rule, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\log x\right)\right]$$

$$=\frac{1}{\log x}.\frac{d}{dx}\left(\log x\right)$$

$$=\frac{1}{\log x}.\frac{1}{x}$$

$$=\frac{1}{x\log x},x>1$$

Question 9:

Differentiating the following wrt $x:\frac{\cos x}{\log x},x>0$

Solution:

Let $y=\frac{\cos x}{\log x}$

By using the quotient rule, we get

$$\frac{dy}{dx}=\frac{\frac{d}{dx}(\cos x).\log x-\cos x.\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{-\sin x\log x-\cos x.\frac{1}{x}}{(\log x)^2}$$

$$=-\left[\frac{x\log x.\sin x+\cos x}{x(\log x)^2}\right],x>0$$

Question 10:

Differentiate the following wrt $x:\cos\left(\log x+e^x\right),x>0$

Solution:

Let $y=\cos\left(\log x+e^x\right)$

By using the chain rule, we get

$$\frac{dy}{dx}=-\sin\left[\log x+e^x\right].\frac{d}{dx}\left(\log x+e^x\right)$$

$$=-\sin\left(\log x+e^x\right).\left[\frac{d}{dx}(\log x)+\frac{d}{dx}\left(e^x\right)\right]$$

$$=-\sin\left(\log x+e^x\right).\left(\frac{1}{x}+e^x\right)$$

$$=-\left(\frac{1}{x}+e^x\right)\sin\left(\log x+e^x\right),x>0$$

EXERCISE 5.5

Question 1:

Differentiate the function with respect to x: $\cos x.\cos 2x.\cos 3x$

Solution:

Let $y=\cos x.\cos 2x.\cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y=\log(\cos x.\cos 2x.\cos 3x)$$
$$\Rightarrow \log y=\log(\cos x)+\log(\cos 2x)+\log(\cos 3x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{\cos x}.\frac{d}{dx}(\cos x)+\frac{1}{\cos 2x}.\frac{d}{dx}(\cos 2x)+\frac{1}{\cos 3x}.\frac{d}{dx}(\cos 3x)$$
$$\Rightarrow \frac{dy}{dx}=y\left[-\frac{\sin x}{\cos x}-\frac{\sin 2x}{\cos 2x}.\frac{d}{dx}(2x)-\frac{\sin 3x}{\cos 3x}.\frac{d}{dx}(3x)\right]$$
$$\therefore \frac{dy}{dx}=-\cos x.\cos 2x.\cos 3x[\tan x+2\tan 2x+3\tan 3x]$$

Question 2:

Differentiate the function with respect to x: $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution:

Let $y=\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking logarithm on both the sides, we obtain

$$\log y=\log\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$
$$\Rightarrow \log y=\frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$
$$\Rightarrow \log y=\frac{1}{2}\left[\log\{(x-1)(x-2)\}-\log\{(x-3)(x-4)(x-5)\}\right]$$
$$\Rightarrow \log y=\frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\begin{array}{l}\frac{1}{x-1}.\frac{d}{dx}(x-1)+\frac{1}{x-2}.\frac{d}{dx}(x-2)-\frac{1}{x-3}.\frac{d}{dx}(x-3)\\ -\frac{1}{x-4}.\frac{d}{dx}(x-4)-\frac{1}{x-5}.\frac{d}{dx}(x-5)\end{array}\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

$$\therefore\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

Question 3:

Differentiate the function with respect to x: $(\log x)^{\cos x}$

Solution:

Let $y=(\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$\log y=\cos x.\log(\log x)$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}(\cos x).\log(\log x)+\cos x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=-\sin x\log(\log x)+\cos x.\frac{1}{\log x}.\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{d}{dx}=y\left[-\sin x\log(\log x)+\frac{\cos x}{\log x}.\frac{1}{x}\right]$$

$$\therefore\frac{dy}{dx}=(\log x)^{\cos x}\left[\frac{\cos x}{x\log x}-\sin x\log(\log x)\right]$$

Question 4:

Differentiate the function with respect to x: $x^x-2^{\sin x}$

Solution:

Let $y = x^x - 2^{\sin x}$

Also, let $x^x = u$ and $2^{\sin x} = v$

$$\therefore y = u - v$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$
$$u = x^x$$

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$
$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$
$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x)$$

$$v = 2^{\sin x}$$

Taking logarithm on both the sides, we obtain

$$\log v = \sin x . \log 2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}.\frac{dv}{dx} = \log 2.\frac{d}{dx}(\sin x)$$
$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$
$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$
$$\therefore \frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to x: $(x+3)^2.(x+4)^3.(x+5)^4$

Solution:

Let $y=(x+3)^2.(x+4)^3.(x+5)^4$

Taking logarithm on both the sides, we obtain

$$\log y=\log(x+3)^2+\log(x+4)^3+\log(x+5)^4$$
$$\Rightarrow \log y=2\log(x+3)+3\log(x+4)+4\log(x+5)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=2.\frac{1}{x+3}.\frac{d}{dx}(x+3)+3.\frac{1}{x+4}.\frac{d}{dx}(x+4)+4.\frac{1}{x+5}.\frac{d}{dx}(x+5)$$
$$\Rightarrow \frac{dy}{dx}=y\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4.\left[\frac{2}{x+3}+\frac{3}{x+4}+\frac{4}{x+5}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)^2(x+4)^3(x+5)^4.\left[\frac{2(x+4)(x+5)+3(x+3)(x+5)+4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$
$$\Rightarrow \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3.\left[2(x^2+9x+20)+3(x^2+8x+15)+4(x^2+7x+12)\right]$$
$$\therefore \frac{dy}{dx}=(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$$

Question 6:

Differentiate the function with respect to x: $\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

Solution:

Let $y=\left(x+\frac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$

Also, let $u=\left(x+\frac{1}{x}\right)^x$ and $v=x^{\left(1+\frac{1}{x}\right)}$

$$\therefore y=u+v$$
$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

Then, $u=\left(x+\frac{1}{x}\right)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u=\log\left(x+\frac{1}{x}\right)^x$$
$$\Rightarrow \log u=x\log\left(x+\frac{1}{x}\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx}=\frac{d}{dx}(x)\times\log\left(x+\frac{1}{x}\right)+x\times\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right]$$
$$\Rightarrow \frac{1}{u}.\frac{du}{dx}=1\times\log\left(x+\frac{1}{x}\right)+x\times\frac{1}{\left(x+\frac{1}{x}\right)}.\frac{d}{dx}\left(x+\frac{1}{x}\right)$$
$$\Rightarrow \frac{du}{dx}=u\left[\log\left(x+\frac{1}{x}\right)+\frac{x}{\left(x+\frac{1}{x}\right)}\times\left(1-\frac{1}{x^2}\right)\right]$$
$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{\left(x-\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)}\right]$$
$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\log\left(x+\frac{1}{x}\right)+\frac{x^2-1}{x^2+1}\right]$$
$$\Rightarrow \frac{du}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right] \qquad \ldots(2)$$

Now, $v=x^{\left(1+\frac{1}{x}\right)}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v=\log\left[x^{\left(1+\frac{1}{x}\right)}\right]$$
$$\Rightarrow \log v=\left(1+\frac{1}{x}\right)\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}.\frac{dv}{dx}=\left[\frac{d}{dx}\left(1+\frac{1}{x}\right)\right]\times\log x+\left(1+\frac{1}{x}\right).\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=\left(-\frac{1}{x^2}\right)\log x+\left(1+\frac{1}{x}\right).\frac{1}{x}$$

$$\Rightarrow\frac{1}{v}.\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]$$

$$\Rightarrow\frac{dv}{dx}=x^{\left(1+\frac{1}{x}\right)}\left[\frac{-\log x+x+1}{x^2}\right]\qquad\ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(1+\frac{1}{x}\right)}\left[\frac{x+1-\log x}{x^2}\right]$$

Question 7:

Differentiate the function with respect to x: $(\log x)^x+x^{\log x}$

Solution:

Let $y=(\log x)^x+x^{\log x}$

Also, let $u=(\log x)^x$ and $v=x^{\log x}$

$$\therefore y=u+v$$

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad\ldots(1)$$

Then, $u=(\log x)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow\log u=\log\left[(\log x)^x\right]$$

$$\Rightarrow\log u=x\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\log x)+x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=u\left[1\times\log(\log x)+x.\frac{1}{(\log x)}\cdot\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{x}{(\log x)}\cdot\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\log(\log x)+\frac{1}{(\log x)}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^x\left[\frac{\log(\log x).\log x+1}{\log x}\right]$$

$$\Rightarrow\frac{du}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]\qquad\ldots(2)$$

$$v=x^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow\log v=\log\left(x^{\log x}\right)$$

$$\Rightarrow\log v=\log x\log x=(\log x)^2$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}\left[(\log x)^2\right]$$

$$\Rightarrow\frac{1}{v}\cdot\frac{dv}{dx}=2(\log x).\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{dv}{dx}=2v(\log x).\frac{1}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow\frac{dv}{dx}=2x^{\log x-1}.\log x\qquad\ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx}=(\log x)^{x-1}\left[1+\log x.\log(\log x)\right]+2x^{\log x-1}.\log x$$

Question 8:

Differentiate the function with respect to x: $(\sin x)^x+\sin^{-1}\sqrt{x}$

Solution:

Let $y=(\sin x)^{x}+\sin^{-1}\sqrt{x}$

Also, let $u=(\sin x)^{x}$ and $v=\sin^{-1}\sqrt{x}$

$\therefore y=u+v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

Then, $u=(\sin x)^{x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u=\log(\sin x)^{x}$$

$$\Rightarrow \log u=x\log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx}=\frac{d}{dx}(x)\times\log(\sin x)+x.\frac{d}{dx}\left[\log(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=u\left[1\times\log(\sin x)+x.\frac{1}{(\sin x)}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^{x}\left[\log(\sin x)+\frac{x}{(\sin x)}.\cos x\right]$$

$$\Rightarrow \frac{du}{dx}=(\sin x)^{x}\left[x\cot x+\log\sin x\right] \qquad \ldots(2)$$

$v=\sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-\left(\sqrt{x}\right)^{2}}}.\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{x-x^{2}}} \qquad \ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx}=(\sin x)^{x}\left[x\cot x+\log\sin x\right]+\frac{1}{2\sqrt{x-x^{2}}}$$

Question 9:

Differentiate the function with respect to x: $x^{\sin x}+(\sin x)^{\cos x}$

Solution:

Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Also, let $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$

$$\therefore y=u+v$$
$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \qquad \ldots(1)$$

Then, $u=x^{\sin x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u=\log\left(x^{\sin x}\right)$$
$$\Rightarrow \log u=\sin x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx}=\frac{d}{dx}(\sin x).\log x+\sin x.\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx}=u\left[\cos x\log x+\sin x.\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx}=x^{\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right] \qquad \ldots(2)$$

$$v=(\sin x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}(\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x.\log(\sin x) + \cos x.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[-\sin x \log(\sin x) + \frac{\cos x}{\sin x}\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}[-\sin x \log(\sin x) + \cot x \cos x]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)] \qquad \ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] + (\sin x)^{\cos x}[\cot x \cos x - \sin x \log(\sin x)]$$

Question 10:

Differentiate the function with respect to x: $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Solution:

Let $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Also, let $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

Then, $u = x^{x\cos x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log\left(x^{x\cos x}\right)$$

$$\Rightarrow \log u = x \cos x \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x).\cos x.\log x+x.\frac{d}{dx}(\cos x).\log x+x\cos x.\frac{d}{dx}(\log x)$$

$$\Rightarrow\frac{du}{dx}=u\left[1.\cos x.\log x+x.(-\sin x)\log x+x\cos x.\frac{1}{x}\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x\log x-x.\sin x\log x+\cos x\right]$$

$$\Rightarrow\frac{du}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x.\sin x\log x\right]\qquad\ldots(2)$$

$$v=\frac{x^2+1}{x^2-1}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow\log v=\log\left(x^2+1\right)-\log\left(x^2-1\right)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx}=\frac{2x}{x^2+1}-\frac{2x}{x^2-1}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{2x\left(x^2-1\right)-2x\left(x^2+1\right)}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{x^2+1}{x^2-1}\times\left[\frac{-4x}{\left(x^2+1\right)\left(x^2-1\right)}\right]$$

$$\Rightarrow\frac{dv}{dx}=\frac{-4x}{\left(x^2-1\right)^2}\qquad\ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx}=x^{x\cos x}\left[\cos x(1+\log x)-x.\sin x\log x\right]-\frac{4x}{\left(x^2-1\right)^2}$$

Question 11:

Differentiate the function with respect to x: $(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Solution:

Let $y=(x\cos x)^x+(x\sin x)^{\frac{1}{x}}$

Also, let $u=(x\cos x)^x$ and $v=(x\sin x)^{\frac{1}{x}}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

Then, $u = (x\cos x)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = (x\cos x)^x$$
$$\Rightarrow \log u = x\log(x\cos x)$$
$$\Rightarrow \log u = x[\log x + \log\cos x]$$
$$\Rightarrow \log u = x\log x + x\log\cos x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx} = \frac{d}{dx}(x\log x) + \frac{d}{dx}(x\log\cos x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log x)\right\} + \left\{\log\cos x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log\cos x)\right\}\right]$$
$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left(\log x.1 + x.\frac{1}{x}\right) + \left\{\log\cos x.1 + x.\frac{1}{\cos x}.\frac{d}{dx}(\cos x)\right\}\right]$$
$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(\log x + 1) + \left\{\log\cos x + \frac{x}{\cos x}.(-\sin x)\right\}\right]$$
$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(1 + \log x) + (\log\cos x - x\tan x)\right]$$
$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[(1 - x\tan x) + (\log x + \log\cos x)\right]$$
$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] \qquad \ldots(2)$$

$$v = (x\sin x)^{\frac{1}{x}}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(x\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x}\log(x\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}(\log x + \log\sin x)$$

$$\Rightarrow \log v = \frac{1}{x}\log x + \frac{1}{x}\log\sin x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{x}\log x\right) + \frac{d}{dx}\left[\frac{1}{x}\log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}(\log x)\right] + \left[\log(\sin x).\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}.\frac{d}{dx}\{\log(\sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \left[\log x.\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{x}\right] + \left[\log(\sin x).\left(-\frac{1}{x^2}\right) + \frac{1}{x}.\frac{1}{\sin x}.\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1-\log x) + \left[-\frac{\log(\sin x)}{x^2} + \frac{1}{x\sin x}.\cos x\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x}{x^2} + \frac{-\log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log x - \log(\sin x) + x\cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1-\log(x\sin x) + x\cot x}{x^2}\right] \qquad \ldots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = (x\cos x)^x\left[1 - x\tan x + \log(x\cos x)\right] + (x\sin x)^{\frac{1}{x}}\left[\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right]$$

Question 12:

Find $\frac{dy}{dx}$ of the function $x^y + y^x = 1$

Solution:

The given function is $x^y + y^x = 1$

Let, $x^y = u$ and $y^x = v$

$\therefore u+v=1$

$$\Rightarrow \frac{du}{dx}+\frac{dv}{dx}=0 \qquad \ldots(1)$$

Then, $u = x^y$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log\left(x^y\right)$$
$$\Rightarrow \log u = y\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}.\frac{du}{dx} = \log x\frac{dy}{dx} + y.\frac{d}{dx}\left(\log x\right)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x\frac{dy}{dx} + y.\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^y\left[\log x\frac{dy}{dx} + \frac{y}{x}\right] \qquad \ldots(2)$$

Now, $v = y^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log\left(y^x\right)$$
$$\Rightarrow \log v = x\log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log y.\frac{d}{dx}\left(x\right) + x.\frac{d}{dx}\left(\log y\right)$$
$$\Rightarrow \frac{dv}{dx} = v\left[\log y.1 + x.\frac{1}{y}.\frac{dy}{dx}\right]$$
$$\Rightarrow \frac{dv}{dx} = y^x\left[\log y + \frac{x}{y}.\frac{dy}{dx}\right] \qquad \ldots(3)$$

Therefore, from (1), (2) and (3);

$$x^y\left[\log x\frac{dy}{dx}+\frac{y}{x}\right]+y^x\left[\log y+\frac{x}{y}.\frac{dy}{dx}\right]=0$$

$$\Rightarrow\left(x^y\log x+xy^{x-1}\right)\frac{dy}{dx}=-\left(yx^{y-1}+y^x\log y\right)$$

$$\therefore\frac{dy}{dx}=\frac{-\left(yx^{y-1}+y^x\log y\right)}{\left(x^y\log x+xy^{x-1}\right)}$$

Question 13:

Find $\frac{dy}{dx}$ of the function $y^x=x^y$

Solution:

The given function is $y^x=x^y$

Taking logarithm on both the sides, we obtain

$$x\log y=y\log x$$

Differentiating both sides with respect to x, we obtain

$$\log y.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log y)=\log x.\frac{d}{dx}(y)+y.\frac{d}{dx}(\log x)$$

$$\Rightarrow\log y.1+x.\frac{1}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+y.\frac{1}{x}$$

$$\Rightarrow\log y+\frac{x}{y}.\frac{dy}{dx}=\log x.\frac{dy}{dx}+\frac{y}{x}$$

$$\Rightarrow\left(\frac{x}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}-\log y$$

$$\Rightarrow\left(\frac{x-y\log x}{y}\right)\frac{dy}{dx}=\frac{y-x\log y}{x}$$

$$\therefore\frac{dy}{dx}=\frac{y}{x}\left(\frac{y-x\log y}{x-y\log x}\right)$$

Question 14:

Find $\frac{dy}{dx}$ of the function $(\cos x)^y=(\cos y)^x$

Solution:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides with respect to x, we obtain

$$\log \cos x . \frac{dy}{dx} + y . \frac{d}{dx}(\log \cos x) = \log \cos y . \frac{d}{dx}(x) + x . \frac{d}{dx}(\log \cos y)$$
$$\Rightarrow \log \cos x . \frac{dy}{dx} + y . \frac{1}{\cos x} . \frac{d}{dx}(\cos x) = \log \cos y . 1 + x . \frac{1}{\cos y} . \frac{d}{dx}(\cos y)$$
$$\Rightarrow \log \cos x . \frac{dy}{dx} + \frac{y}{\cos x} . (-\sin x) = \log \cos y + \frac{x}{\cos y} . (-\sin y) . \frac{dy}{dx}$$
$$\Rightarrow \log \cos x . \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$
$$\Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} = y \tan x + \log \cos y$$
$$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

Question 15:

Find $\frac{dy}{dx}$ of the function $xy = e^{(x-y)}$

Solution:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\log(xy) = \log\left(e^{x-y}\right)$$
$$\Rightarrow \log x + \log y = (x - y) \log e$$
$$\Rightarrow \log x + \log y = (x - y) \times 1$$
$$\Rightarrow \log x + \log y = (x - y)$$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\Rightarrow \left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Solution:

The given function is $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x)=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{f(x)}\cdot\frac{d}{dx}[f(x)]=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)$$
$$+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$
$$\Rightarrow\frac{1}{f(x)}\cdot f'(x)=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)+\frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$
$$+\frac{1}{1+x^4}\cdot\frac{d}{dx}(1+x^4)+\frac{1}{1+x^8}\cdot\frac{d}{dx}(1+x^8)$$
$$\Rightarrow f'(x)=f(x)\left[\frac{1}{1+x}+\frac{1}{1+x^2}.2x+\frac{1}{1+x^4}.4x^3+\frac{1}{1+x^8}.8x^7\right]$$
$$\therefore f'(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right]$$

Hence,

$$f'(1)=(1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1}+\frac{2(1)}{1+1^2}+\frac{4(1)^3}{1+1^4}+\frac{8(1)^7}{1+1^8}\right]$$
$$=2\times2\times2\times2\left[\frac{1}{2}+\frac{2}{2}+\frac{4}{2}+\frac{8}{2}\right]$$
$$=16\left(\frac{15}{2}\right)$$
$$=120$$

Question 17:

Differentiate $(x^2-5x+8)(x^3+7x+9)$ in three ways mentioned below.

(i) By using product rule
(ii) By expanding the product to obtain a single polynomial.
(iii) By logarithmic differentiation.

Do they all give the same answer?

Solution:

Let $y=(x^2-5x+8)(x^3+7x+9)$

(i) By using product rule

Let $u=(x^2-5x+8)$ and $v=x^3+7x+9$

$\therefore y=uv$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}.v+u.\frac{dv}{dx} \qquad \text{(product rule)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^2-5x+8).(x^3+7x+9)+(x^2-5x+8).\frac{d}{dx}(x^3+7x+9)$$

$$\Rightarrow \frac{dy}{dx}=(2x-5)(x^3+7x+9)+(x^2-5x+8)(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=2x(x^3+7x+9)-5(x^3+7x+9)+x^2(3x^2+7)$$

$$-5x(3x^2+7)+8(3x^2+7)$$

$$\Rightarrow \frac{dy}{dx}=(2x^4+14x^2+18x)-5x^3-35x-45+(3x^4+7x^2)$$

$$\Rightarrow \frac{dy}{dx}=-15x^3-35x+24x^2+56$$

$$\therefore \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(ii) By expanding the product to obtain a single polynomial.

$$y=(x^2-5x+8)(x^3+7x+9)$$
$$=x^2(x^3+7x+9)-5x(x^3+7x+9)+8(x^3+7x+9)$$
$$=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$
$$=x^5-5x^4+15x^3-26x^2+11x+72$$

Therefore,

$$\frac{dy}{dx}=\frac{d}{dx}(x^5-5x^4+15x^3-26x^2+11x+72)$$
$$=\frac{d}{dx}(x^5)-5\frac{d}{dx}(x^4)+15\frac{d}{dx}(x^3)-26\frac{d}{dx}(x^2)+11\frac{d}{dx}(x)+\frac{d}{dx}(72)$$
$$=5x^4-5(4x^3)+15(3x^2)-26(2x)+11(1)+0$$
$$=5x^4-20x^3+45x^2-52x+11$$

(iii) By logarithmic differentiation.

$$y=(x^2-5x+8)(x^3+7x+9)$$

Taking logarithm on both the sides, we obtain

$$\log y=\log(x^2-5x+8)+\log(x^3+7x+9)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx}=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$
$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{1}{x^2-5x+8}.\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}.\frac{d}{dx}\left(x^3+7x+9\right)$$
$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x^2-5x+8}.(2x-5)+\frac{1}{x^3+7x+9}.\left(3x^2+7\right)\right]$$
$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$
$$\Rightarrow \frac{dy}{dx}=\left(x^2-5x+8\right)\left(x^3+7x+9\right)\left[\frac{(2x-5)\left(x^3+7x+9\right)+\left(3x^2+7\right)\left(x^2-5x+8\right)}{\left(x^2-5x+8\right)\left(x^3+7x+9\right)}\right]$$
$$\Rightarrow \frac{dy}{dx}=2x\left(x^3+7x+9\right)-5\left(x^3+7x+9\right)+3x^2\left(x^2-5x+8\right)+7\left(x^2-5x+8\right)$$
$$\Rightarrow \frac{dy}{dx}=2x^4+14x^2+18x-5x^3-35x-45+3x^5-15x^3+24x^2+7x^2-35x+56$$
$$\Rightarrow \frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If *u, v* and *w* are functions of *x*, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Let $y=u.v.w=u.(v.w)$

By applying product rule, we get

$$\frac{dy}{dx}=\frac{du}{dx}.(v.w)+u.\frac{d}{dx}(v.w)$$
$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}.(v.w)+u\left[\frac{dv}{dx}.w+v.\frac{dw}{dx}\right] \qquad \text{(Again applying product rule)}$$
$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}.v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$$

Taking logarithm on both the sides of the equation $y = u.v.w$, we obtain

$\log y = \log u + \log v + \log w$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$$

EXERCISE 5.6

Question 1:

If x and y are connected parametrically by the equations $x = 2at^2, y = at^4$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = 2at^2, y = at^4$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}\left(2at^2\right) = 2a.\frac{d}{dt}\left(t^2\right) = 2a.2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(at^4\right) = a.\frac{d}{dt}\left(t^4\right) = a.4.t^3 = 4at^3$$

$$\therefore \frac{dy}{dt} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equations $x = a\cos\theta, y = b\cos\theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a\cos\theta, y = b\cos\theta$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\cos\theta\right) = a\left(-\sin\theta\right) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(b\cos\theta\right) = b\left(-\sin\theta\right) = -b\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equations $x = \sin t, y = \cos 2t$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = \sin t, y = \cos 2t$

Then, $\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t.\frac{d}{dt}(2t) = -2\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{\cos t} = \frac{-2.2\sin t\cos t}{\cos t} = -4\sin t$$

Question 4:

If x and y are connected parametrically by the equations $x = 4t, y = \frac{4}{t}$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = 4t, y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4.\frac{d}{dt}\left(\frac{1}{t}\right) = 4.\left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equations $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x=\cos\theta-\cos 2\theta, y=\sin\theta-\sin 2\theta$

Then,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\cos\theta-\cos 2\theta)=\frac{d}{d\theta}(\cos\theta)-\frac{d}{d\theta}(\cos 2\theta)$$
$$=-\sin\theta-(-2\sin 2\theta)=2\sin 2\theta-\sin\theta$$
$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta-\sin 2\theta)=\frac{d}{d\theta}(\sin\theta)-\frac{d}{d\theta}(\sin 2\theta)$$
$$=\cos\theta-2\cos 2\theta$$
$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

Question 6:

If x and y are connected parametrically by the equations $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$

Then, $\frac{dx}{d\theta}=a\left[\frac{d}{d\theta}(\theta)-\frac{d}{d\theta}(\sin\theta)\right]=a(1-\cos\theta)$

$$\frac{dy}{d\theta}=a\left[\frac{d}{d\theta}(1)+\frac{d}{d\theta}(\cos\theta)\right]=a\left[0+(-\sin\theta)\right]=-a\sin\theta$$
$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}=-\cot\frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equations $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}, y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}, y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then,

$$\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\sin^3 t\right)-\sin^3 t.\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\sin^2 t.\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\sin^2 t.\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{3\cos 2t.\sin^2 t\cos t+\sin^3 t.\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}.\frac{d}{dt}\left(\cos^3 t\right)-\cos^3 t.\frac{d}{dt}\left(\sqrt{\cos 2t}\right)}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}.3\cos^2 t.\frac{d}{dt}(\cos t)-\cos^3 t.\frac{1}{2\sqrt{\cos 2t}}.\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}.\cos^2 t(-\sin t)-\cos^3 t.\frac{1}{\sqrt{\cos 2t}}.(-2\sin 2t)}{\cos 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t.\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{\cos 2t.\sqrt{\cos 2t}}}{\frac{3\cos 2t.\sin^2 t.\cos t+\sin^3 t\sin 2t}{\cos 2t.\sqrt{\cos 2t}}}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t\sin 2t}{3\cos 2t.\sin^2 t.\cos t+\sin^3 t\sin 2t}$$

$$=\frac{-3\cos 2t.\cos^2 t.\sin t+\cos^3 t(2\sin t\cos t)}{3\cos 2t.\sin^2 t.\cos t+\sin^3 t(2\sin t\cos t)}$$

$$=\frac{\sin t\cos t\left[-3\cos 2t.\cos t+2\cos^3 t\right]}{\sin t\cos t\left[3\cos 2t\sin t+2\sin^3 t\right]}$$

$$=\frac{\left[-3\left(2\cos^2 t-1\right)\cos t+2\cos^3 t\right]}{\left[3\left(1-2\sin^2 t\right)\sin t+2\sin^3 t\right]} \qquad \left[\begin{array}{l}\cos 2t=\left(2\cos^2 t-1\right)\\ \cos 2t=\left(1-2\sin^2 t\right)\end{array}\right]$$

$$=\frac{-4\cos^3 t+3\cos t}{3\sin t-4\sin^3 t} \qquad \left[\begin{array}{l}\cos 3t=4\cos^3 t-3\cos t\\ \sin 3t=3\sin t-4\sin^2 t\end{array}\right]$$

$$=\frac{-\cos 3t}{\sin 3t}=-\cot 3t$$

Question 8:

If x and y are connected parametrically by the equations $x=a\left(\cos t+\log\tan\frac{t}{2}\right), y=a\sin t$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x=a\left(\cos t+\log\tan\frac{t}{2}\right), y=a\sin t$

Then,

$$\frac{dx}{dt}=a.\left[\frac{d}{dt}(\cos t)+\frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}.\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\cot\frac{t}{2}.\sec^2\frac{t}{2}.\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$=a\left[-\sin t+\frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\times\frac{1}{\cos^2\frac{t}{2}}\times\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=a\left(\frac{-\sin^2 t+1}{\sin t}\right)$$

$$=a\left(\frac{\cos^2 t}{\sin t}\right)$$

$$\frac{dy}{dt}=a\frac{d}{dt}(\sin t)=a\cos t$$

Therefore,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{a\cos t}{\left(a\frac{\cos^2 t}{\sin t}\right)}=\frac{\sin t}{\cos t}=\tan t$$

Question 9:

If x and y are connected parametrically by the equations $x=a\sec\theta, y=b\tan\theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x=a\sec\theta, y=b\tan\theta$

Then,

$$\frac{dx}{d\theta} = a.\frac{d}{d\theta}(\sec\theta) = a\sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = b.\frac{d}{d\theta}(\tan\theta) = b\sec^2\theta$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$= \frac{b}{a}\sec\theta\cot\theta$$

$$= \frac{b\cos\theta}{a\cos\theta\sin\theta}$$

$$= \frac{b}{a}\times\frac{1}{\sin\theta}$$

$$= \frac{b}{a}\operatorname{cosec}\theta$$

Question 10:

If x and y are connected parametrically by the equations $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

Then,

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right]$$

$$= a\left[-\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\theta + \theta\cos\theta + \sin\theta]$$

$$= a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right] = a\left[\cos\theta - \left\{\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta.\frac{d}{d\theta}(\theta)\right\}\right]$$

$$= a[\cos\theta + \theta\sin\theta - \cos\theta]$$

$$= a\theta\sin\theta$$

Therefore,

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$
$$=\frac{a\theta\sin\theta}{a\theta\cos\theta}$$
$$=\tan\theta$$

Question 11:

If $x=\sqrt{a^{\sin^{-1}t}}, y=\sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx}=-\frac{y}{x}$

Solution:

Given, $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

Hence,

$x=\sqrt{a^{\sin^{-1}t}}=\left(a^{\sin^{-1}t}\right)^{\frac{1}{2}}=a^{\frac{1}{2}\sin^{-1}t}$ and $y=\sqrt{a^{\cos^{-1}t}}=\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}=a^{\frac{1}{2}\cos^{-1}t}$

Consider $x=a^{\frac{1}{2}\sin^{-1}t}$

Taking log on both sides, we get

$$\log x=\frac{1}{2}\sin^{-1}t\log a$$

Therefore,

$$\Rightarrow\frac{1}{x}.\frac{dx}{dt}=\frac{1}{2}\log a.\frac{d}{dt}\left(\sin^{-1}t\right)$$
$$\Rightarrow\frac{dx}{dt}=\frac{x}{2}\log a.\frac{1}{\sqrt{1-t^2}}$$
$$\Rightarrow\frac{dx}{dt}=\frac{x\log a}{2\sqrt{1-t^2}}$$

Now, $y=a^{\frac{1}{2}\cos^{-1}t}$

Taking log on both sides, we get

$$\log x=\frac{1}{2}\cos^{-1}t\log a$$

Therefore,

$$\Rightarrow \frac{1}{y}.\frac{dy}{dt} = \frac{1}{2}\log a.\frac{d}{dt}\left(\cos^{-1} t\right)$$

$$\Rightarrow \frac{dy}{dt} = \frac{y}{2}\log a.\frac{-1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-y\log a}{2\sqrt{1-t^2}}$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y\log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}$$

EXERCISE 5.7

Question 1:

Find the second order derivative of the function x^2+3x+2

Solution:

Consider, $y=x^2+3x+2$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(3x)+\frac{d}{dx}(2)$$
$$=2x+3+0$$
$$=2x+3$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(2x+3)$$
$$=\frac{d}{dx}(2x)+\frac{d}{dx}(3)$$
$$=2+0$$
$$=2$$

Question 2:

Find the second order derivative of the function x^{20}

Solution:

Consider, $y=x^{20}$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^{20}\right)$$
$$=20x^{19}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(20x^{19}\right)$$
$$=20\frac{d}{dx}\left(x^{19}\right)$$
$$=20.19.x^{18}$$
$$=380x^{18}$$

Question 3:

Find the second order derivative of the function $x\cos x$

Solution:

Consider, $y = x\cos x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x.\cos x)\\ &= \cos x.\frac{d}{dx}(x) + x\frac{d}{dx}(\cos x)\\ &= \cos x.1 + x(-\sin x)\\ &= \cos x - x\sin x\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}[\cos x - x\sin x]\\ &= \frac{d}{dx}(\cos x) - \frac{d}{dx}(x\sin x)\\ &= -\sin x - \left[\sin x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\sin x)\right]\\ &= -\sin x - (\sin x + x\cos x)\\ &= -(x\cos x + 2\sin x)\end{aligned}$$

Question 4:

Find the second order derivative of the function $\log x$

Solution:

Let $y = \log x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log x)\\ &= \frac{1}{x}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{x}\right)\\ &= \frac{-1}{x^2}\end{aligned}$$

Question 5:

Find the second order derivative of the function $x^3 \log x$

Solution:

Let $y = x^3 \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^3 \log x\right]$$
$$= \log x.\frac{d}{dx}\left(x^3\right) + x^3.\frac{d}{dx}\left(\log x\right)$$
$$= \log x.3x^2 + x^3.\frac{1}{x} = \log x.3x^2 + x^2$$
$$= x^2\left(1 + 3\log x\right)$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[x^2\left(1 + 3\log x\right)\right]$$
$$= \left(1 + 3\log x\right).\frac{d}{dx}\left(x^2\right) + x^2\frac{d}{dx}\left(1 + 3\log x\right)$$
$$= \left(1 + 3\log x\right).2x + x^2.\frac{3}{x}$$
$$= 2x + 6\log x + 3x$$
$$= 5x + 6x\log x$$
$$= x\left(5 + 6\log x\right)$$

Question 6:

Find the second order derivative of the function $e^x \sin 5x$

Solution:

Let $y = e^x \sin 5x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^x\sin 5x\right)$$
$$=\sin 5x\times\frac{d}{dx}\left(e^x\right)+e^x\frac{d}{dx}\left(\sin 5x\right)$$
$$=\sin 5x.e^x+e^x.\cos 5x.\frac{d}{dx}\left(5x\right)$$
$$=e^x\sin 5x+e^x\cos 5x.5$$
$$=e^x\left(\sin 5x+5\cos 5x\right)$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[e^x\left(\sin 5x+5\cos 5x\right)\right]$$
$$=\left(\sin 5x+5\cos 5x\right).\frac{d}{dx}\left(e^x\right)+e^x.\frac{d}{dx}\left(\sin 5x+5\cos 5x\right)$$
$$=\left(\sin 5x+5\cos 5x\right)e^x+e^x\left[\cos 5x.\frac{d}{dx}\left(5x\right)+5\left(-\sin 5x\right).\frac{d}{dx}\left(5x\right)\right]$$
$$=e^x\left(\sin 5x+5\cos 5x\right)+e^x\left(5\cos 5x-25\sin 5x\right)$$
$$=e^x\left(10\cos 5x-24\sin 5x\right)$$
$$=2e^x\left(5\cos 5x-12\sin 5x\right)$$

Question 7:

Find the second order derivative of the function $e^{6x}\cos 3x$

Solution:

Let $y=e^{6x}\cos 3x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{6x}\cos 3x\right)=\cos 3x.\frac{d}{dx}\left(e^{6x}\right)+e^{6x}.\frac{d}{dx}\left(\cos 3x\right)$$
$$=\cos 3x.e^{6x}.\frac{d}{dx}\left(6x\right)+e^{6x}.\left(-\sin 3x\right).\frac{d}{dx}\left(3x\right)$$
$$=6e^{6x}\cos 3x-3e^{6x}\sin 3x \qquad \ldots(1)$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(6e^{6x}\cos 3x-3e^{6x}\sin 3x\right)=6.\frac{d}{dx}\left(e^{6x}\cos 3x\right)-3.\frac{d}{dx}\left(e^{6x}\sin 3x\right)$$

$$=6.\left[6e^{6x}\cos 3x-3e^{6x}\sin 3x\right]-3.\left[\sin 3x.\frac{d}{dx}\left(e^{6x}\right)+e^{6x}.\frac{d}{dx}(\sin 3x)\right] \qquad \left[\text{using } (1)\right]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-3\left[\sin 3x.e^{6x}.6+e^{6x}.\cos 3x.3\right]$$

$$=36e^{6x}\cos 3x-18e^{6x}\sin 3x-18e^{6x}\sin 3x-9e^{6x}\cos 3x$$

$$=27e^{6x}\cos 3x-36e^{6x}\sin 3x$$

$$=9e^{6x}\left(3\cos 3x-4\sin 3x\right)$$

Question 8:

Find the second order derivative of the function $\tan^{-1}x$

Solution:

Let $y=\tan^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$=\frac{1}{1+x^2}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{d}{dx}\left(1+x^2\right)^{-1}$$

$$=(-1).\left(1+x^2\right)^{-2}.\frac{d}{dx}\left(1+x^2\right)=\frac{-1}{\left(1+x^2\right)^2}\times 2x$$

$$=\frac{-2x}{\left(1+x^2\right)^2}$$

Question 9:

Find the second order derivative of the function $\log(\log x)$

Solution:

Consider, $y=\log(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log(\log x)\right]$$
$$=\frac{1}{\log x}.\frac{d}{dx}(\log x)$$
$$=\frac{1}{\log x}.\frac{1}{x}=(x\log x)^{-1}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[(x\log x)^{-1}\right]$$
$$=(-1).(x\log x)^{-2}\frac{d}{dx}(x\log x)$$
$$=\frac{-1}{(x\log x)^2}.\left[\log x.\frac{d}{dx}(x)+x.\frac{d}{dx}(\log x)\right]$$
$$=\frac{-1}{(x\log x)^2}.\left[\log x.1+x.\frac{1}{x}\right]$$
$$=\frac{-(1+\log x)}{(x\log x)^2}$$

Question 10:

Find the second order derivative of the function $\sin(\log x)$

Solution:

Let $y=\sin(\log x)$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin x(\log x)\right]$$
$$=\cos(\log x).\frac{d}{dx}(\log x)$$
$$=\frac{\cos(\log x)}{x}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$
$$=\frac{x.\frac{d}{dx}\left[\cos(\log x)\right]-\cos(\log x).\frac{d}{dx}(x)}{x^2}$$

$$\frac{d^2y}{dx^2}=\frac{x\left[-\sin(\log x).\frac{d}{dx}(\log x)\right]-\cos(\log x).1}{x^2}$$

$$=\frac{-x\sin(\log x).\frac{1}{x}-\cos(\log x)}{x^2}$$

$$=\frac{-[\sin(\log x)+\cos(\log x)]}{x^2}$$

Question 11:

If $y=5\cos x-3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$

Solution:

Given, $y=5\cos x-3\sin x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}(5\cos x)-\frac{d}{dx}(3\sin x)$$

$$=5\frac{d}{dx}(\cos x)-3\frac{d}{dx}(\sin x)$$

$$=5(-\sin x)-3\cos x$$

$$=-(5\sin x+3\cos x)$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[-(5\sin x+3\cos x)]$$

$$=-\left[5.\frac{d}{dx}(\sin x)+3.\frac{d}{dx}(\cos x)\right]$$

$$=-[5\cos x+3(-\sin x)]$$

$$=-[5\cos x-3\sin x]$$

$$=-y$$

Thus, $\frac{d^2y}{dx^2}+y=0$

Hence proved.

Question 12:

If $y=\cos^{-1}x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Solution:

Given, $y=\cos^{-1}x$

Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos^{-1}x\right)$$
$$=\frac{-1}{\sqrt{1-x^2}}$$
$$=-\left(1-x^2\right)^{\frac{-1}{2}}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[-\left(1-x^2\right)^{\frac{-1}{2}}\right]$$
$$=-\left(-\frac{1}{2}\right).\left(1-x^2\right)^{\frac{-3}{2}}.\frac{d}{dx}\left(1-x^2\right)$$
$$=\frac{1}{2\sqrt{\left(1-x^2\right)^3}}\times(-2x)$$
$$\frac{d^2y}{dx^2}=\frac{-x}{\sqrt{\left(1-x^2\right)^3}} \qquad \ldots(1)$$

But we need to calculate $\frac{d^2y}{dx^2}$ in terms of y

$\Rightarrow y=\cos^{-1}x$

$\Rightarrow x=\cos y$

Putting $x=\cos y$ in equation (1), we get

$$\frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{\left(1-\cos^2 y\right)^3}}$$

$$= \frac{-\cos y}{\sqrt{\left(\sin^2 y\right)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$= -\cot y.\text{cosec}^2 y$$

Question 13:

If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

Solution:

Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Then,

$$y_1 = 3.\frac{d}{dx}\left[\cos(\log x)\right] + 4.\frac{d}{dx}\left[\sin(\log x)\right]$$

$$= 3.\left[-\sin(\log x).\frac{d}{dx}(\log x)\right] + 4.\left[\cos(\log x).\frac{d}{dx}(\log x)\right]$$

$$= \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$= \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

Therefore,

$$y_2=\frac{d}{dx}\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)$$
$$=\frac{x.\{4\cos(\log x)-3\sin(\log x)\}'-\{4\cos(\log x)-3\sin(\log x)\}\{x\}'}{x^2}$$
$$=\frac{x.\left[4\{\cos(\log x)\}'-\{3\sin(\log x)\}'\right]-\{4\cos(\log x)-3\sin(\log x)\}.1}{x^2}$$
$$=\frac{x.\left[-4\sin(\log x).(\log x)'-3\cos(\log x).(\log x)'\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$
$$=\frac{x.\left[-4\sin(\log x)\frac{1}{x}-3\cos(\log x)\frac{1}{x}\right]-4\cos(\log x)+3\sin(\log x)}{x^2}$$
$$=\frac{-4\sin(\log x)-3\cos(\log x)-4\cos(\log x)+3\sin(\log x)}{x^2}$$
$$=\frac{-\sin(\log x)-7\cos(\log x)}{x^2}$$

Thus,

$$x^2y_2+xy_1+y=\left[\begin{array}{r}x^2\left(\frac{-\sin(\log x)-7\cos(\log x)}{x^2}\right)+x\left(\frac{4\cos(\log x)-3\sin(\log x)}{x}\right)\\+3\cos(\log x)+4\sin(\log x)\end{array}\right]$$
$$=\left[\begin{array}{r}-\sin(\log x)-7\cos(\log x)+4\cos(\log x)-3\sin(\log x)\\+3\cos(\log x)+4\sin(\log x)\end{array}\right]$$
$$=0$$

Hence proved.

Question 14:

If $y=Ae^{mx}+Be^{nx}$, show that $\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=0$.

Solution:

Given, $y=Ae^{mx}+Be^{nx}$

Then,

$$\frac{dy}{dx}=A.\frac{d}{dx}\left(e^{mx}\right)+B.\frac{d}{dx}\left(e^{nx}\right)$$
$$=A.e^{mx}.\frac{d}{dx}(mx)+B.e^{nx}.\frac{d}{dx}(nx)$$
$$=Ame^{mx}+Bne^{nx}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(Ame^{mx}+Bne^{nx}\right)$$
$$=Am.\frac{d}{dx}\left(e^{mx}\right)+Bn.\frac{d}{dx}\left(e^{nx}\right)$$
$$=Am.e^{mx}.\frac{d}{dx}(mx)+Bn.e^{nx}.\frac{d}{dx}(nx)$$
$$=Am^2e^{mx}+Bn^2e^{nx}$$

Thus,

$$\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny=Am^2e^{mx}+Bn^2e^{nx}-(m+n).\left(Ame^{mx}+Bne^{nx}\right)+mn\left(Ae^{mx}+Be^{nx}\right)$$
$$=Am^2e^{mx}+Bn^2e^{nx}-Am^2e^{mx}-Bmne^{nx}-Amne^{mx}-Bn^2e^{nx}+Amne^{mx}+Bmne^{nx}$$
$$=0$$

Hence proved.

Question 15:

If $y=500e^{7x}+600e^{-7x}$, show that $\frac{d^2y}{dx^2}=49y$

Solution:

Given, $y=500e^{7x}+600e^{-7x}$

Then,

$$\frac{dy}{dx}=500.\frac{d}{dx}\left(e^{7x}\right)+600.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=500.e^{7x}.\frac{d}{dx}(7x)+600.e^{-7x}.\frac{d}{dx}(-7x)$$
$$=3500e^{7x}-4200e^{-7x}$$

Therefore,

$$\frac{d^2y}{dx^2}=3500e^{7x}.\frac{d}{dx}\left(e^{7x}\right)-4200.\frac{d}{dx}\left(e^{-7x}\right)$$
$$=3500.e^{7x}.\frac{d}{dx}(7x)-4200.e^{-7x}.\frac{d}{dx}(-7x)$$
$$=7\times3500.e^{7x}+7\times4200.e^{-7x}$$
$$=49\times500.e^{7x}+49\times600e^{-7x}$$
$$=49\left(500e^{7x}+600e^{-7x}\right)$$
$$=49y$$

Hence proved.

Question 16:

If $e^{y}(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Solution:

Given, $e^{y}(x+1)=1$

$$\Rightarrow e^{y}(x+1)=1$$

$$\Rightarrow e^{y}=\frac{1}{x+1}$$

Taking log on both sides, we get

$$y=\log\frac{1}{(x+1)}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=(x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right)$$

$$=(x+1).\frac{-1}{(x+1)^2}$$

$$=\frac{-1}{x+1}$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{-1}{x+1}\right)=-\left(\frac{-1}{(x+1)^2}\right)$$

$$=\frac{1}{(x+1)^2}=\left(\frac{-1}{x+1}\right)^2$$

$$=\left(\frac{dy}{dx}\right)^2$$

Hence proved.

Question 17:

If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2y_2+2x\left(x^2+1\right)y_1=2$

Solution:

Given, $y=\left(\tan^{-1}x\right)^2$

Then,

$$\Rightarrow y_1 = 2\tan^{-1} x \frac{d}{dx}\left(\tan^{-1} x\right)$$

$$\Rightarrow y_1 = 2\tan^{-1} x.\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right) y_1 = 2\tan^{-1} x$$

Again, differentiating with respect to x, we get

$$\Rightarrow \left(1+x^2\right) y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow \left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right) y_1 = 2$$

Hence proved.

EXERCISE 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x)=x^2+2x-8, x\in[-4,2]$

Solution:

Given, $f(x)=x^2+2x-8$, being polynomial function is continuous in $[-4,2]$ and also differentiable in $(-4,2)$.

$$\begin{aligned}f(-4)&=(-4)^2+2.(-4)-8\\&=16-8-8\\&=0\end{aligned}$$

$$\begin{aligned}f(2)&=(2)^2+2\times2-8\\&=4+4-8\\&=0\end{aligned}$$

Therefore, $f(-4)=f(2)=0$

The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c\in(-4,2)$ such that $f'(c)=0$

$f(x)=x^2+2x-8$

Therefore, $f'(x)=2x+2$

Hence,

$$f'(c)=0$$
$$2c+2=0$$
$$c=-1$$

Thus, $c=-1\in(-4,2)$

Hence, Rolle's Theorem is verified.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,9]$

(ii) $f(x)=[x]$ for $x\in[-2,2]$

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Solution:

By Rolle's Theorem, $f:[a,b]\to\mathbf{R}$,

If

(a) f is continuous on $[a,b]$

(b) f is continuous on (a,b)

(c) $f(a)=f(b)$

Then, there exists some $c\in(a,b)$ such that $f'(c)=0$

Thus, Rolle's Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x=5$ and $x=9$

Therefore, $f(x)$ is not continuous in $[5,9]$

Also, $f(5)=[5]=5$ and $f(9)=[9]=9$

Thus, $f(5)\neq f(9)$

The differentiability of f in $(5,9)$ is checked as follows.

Let n be an integer such that $n\in(5,9)$

The LHD of f at $x=n$ is

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The RHD of f at $x=n$ is

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since LHD and RHD of f at $x=n$ are not equal, f is not differentiable at $x=n$

Therefore, f is not differentiable in $(5,9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,9]$.

(ii) $f(x)=[x]$ for $x\in[-2,2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x=-2$ and $x=2$

Therefore, $f(x)$ is not continuous in $[-2,2]$

Also, $f(-2)=[-2]=-2$ and $f(2)=[2]=2$

Thus, $f(-2)\neq f(2)$

The differentiability of f in $(-2,2)$ is checked as follows.

Let n be an integer such that $n\in(-2,2)$

The LHD of f at $x=n$ is

$$\lim_{h\to 0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^-}\frac{n-1-n}{h}=\lim_{h\to 0^-}\frac{-1}{h}=\infty$$

The RHD of f at $x=n$ is

$$\lim_{h\to 0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to 0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to 0^+}\frac{n-1-n}{h}=\lim_{h\to 0^+}0=0$$

Since LHD and RHD of f at $x=n$ are not equal, f is not differentiable at $x=n$

Therefore, f is not differentiable in $(-2,2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,2]$

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Since, f being a polynomial function is continuous in $[1,2]$ and is differentiable in $(1,2)$
.

Thus,

$f(1)=(1)^2-1=0$

$f(2)=(2)^2-1=3$

Therefore, $f(1)\neq f(2)$

Since, f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=x^2-1 \text{ for } x\in[1,2]$.

Question 3:

If $f:[-5,5]\to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5)\neq f(5)$.

Solution:

Given, $f:[-5,5]\to \mathbf{R}$ is a differentiable function.
Since every differentiable function is a continuous function, we obtain

(i) f is continuous on $[-5,5]$

(ii) f is continuous on $(-5,5)$

Thus, by the Mean Value Theorem, there exists $c\in(-5,5)$ such that

$$\Rightarrow f'(c)=\frac{f(5)-f(-5)}{5-(-5)}$$

$$\Rightarrow 10f'(c)=f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

Therefore, $f'(c)\neq 0$
Thus,

$$\Rightarrow 10f'(c)\neq 0$$

$$\Rightarrow f(5)-f(-5)\neq 0$$

$$\Rightarrow f(5)\neq f(-5)$$

Hence proved.

Question 4:

Verify Mean Value Theorem, if $f(x)=x^2-4x-3$ in the integral $[a,b]$, where $a=1$ and $b=4$.

Solution:

Given, $f(x)=x^2-4x-3$

f, being a polynomial function, is continuous in $[1,4]$ and is differentiable in $(1,4)$, whose derivative is $2x-4$.

Thus,

$f(1)=1^2-4\times1-3=-6$

$f(4)=4^2-4\times4-3=-3$

Therefore,

$$\frac{f(b)-f(a)}{b-a}=\frac{f(4)-f(1)}{4-1}$$
$$=\frac{-3-(-6)}{3}$$
$$=\frac{3}{3}$$
$$=1$$

Mean Value Theorem states that there is a point $c\in(1,4)$ such that $f'(c)=1$

Hence,

$\Rightarrow f'(c)=1$

$\Rightarrow 2c-4=1$

$\Rightarrow c=\frac{5}{2} \qquad \left[\text{where } c=\frac{5}{2}\in(1,4)\right]$

Thus, mean value theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x)=x^3-5x^2-3x$ in the interval $[a,b]$ where $a=1$ and $b=3$. Find all $c\in(1,3)$ for which $f'(c)=0$.

Solution:

Given, *f* is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in $[1,3]$ and is differentiable in $(1,3)$, whose derivative is $3x^2-10x-3$

Thus,

$f(1)=1^3-5\times1^2-3\times1=-7$

$f(3)=3^3-5\times3^2-3\times3=-27$

Therefore,

$$\frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}$$
$$=\frac{-27-(-7)}{3-1}$$
$$=-10$$

Mean Value Theorem states that there exists a point $c \in (1,3)$ such that $f'(c)=-10$
Hence,

$$\Rightarrow f'(c)=-10$$
$$\Rightarrow 3c^2-10c-3=-10$$
$$\Rightarrow 3c^2-10c+7=0$$
$$\Rightarrow 3c^2-3c-7c+7=0$$
$$\Rightarrow 3c(c-1)-7(c-1)=0$$
$$\Rightarrow (c-1)(3c-7)=0$$
$$\Rightarrow c=1,\frac{7}{3} \qquad \left[\text{where } c=\frac{7}{3}\in(1,3)\right]$$

Thus, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,3)$ is the only point for which $f'(c)=0$.

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given

(i) $f(x)=[x]$ for $x\in[5,9]$

(ii) $f(x)=[x]$ for $x\in[-2,2]$

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Solution:

Mean Value Theorem states that for a function $f:[a,b]\to \mathbf{R}$, if

(a) f is continuous on $[a,b]$

(b) f is continuous on (a,b)

Then there exists some $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Thus, Mean Value Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x=5$ and $x=9$

Therefore, $f(x)$ is not continuous in $[5,9]$

The differentiability of f in $(5,9)$ is checked as follows.

Let n be an integer such that $n \in (5,9)$

The LHD of f at $x=n$ is

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x=n$ is

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since LHD and RHD of f at $x=n$ are not equal, f is not differentiable at $x=n$

Therefore, f is not differentiable in $(5,9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x \in [5,9]$

(ii) $f(x)=[x]$ for $x \in [-2,2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x=-2$ and $x=2$

Therefore, $f(x)$ is not continuous in $[-2,2]$

The differentiability of f in $(-2,2)$ is checked as follows.

Let n be an integer such that $n \in (-2,2)$

The LHD of f at $x=n$ is

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{x\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(-2,2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,2]$

Since, f being a polynomial function is continuous in $[1,2]$ and is differentiable in $(1,2)$
It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x)=x^2-1$ for $x\in[1,2]$.

It can be proved as follows.

We have, $f(x)=x^2-1$
Then,

$f(1)=(1)^2-1=0,$

$f(2)=(2)^2-1=3$

Therefore,

$$\frac{f(b)-f(a)}{b-a}=\frac{f(2)-f(1)}{2-1}=\frac{3-0}{1}$$
$$=3$$

Hence, $f'(x)=2x$

Thus,

$\Rightarrow f'(c)=3$

$\Rightarrow 2c=3$

$\Rightarrow c=\dfrac{3}{2}$

$\Rightarrow c=1.5 \qquad [\text{where } 1.5\in[1,2]]$

MISCELLANEOUS EXERCISE

Question 1:

Differentiate with respect to x the function $\left(3x^2-9x+5\right)^9$.

Solution:

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we get

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8.\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8.(6x-9)\\&=9\left(3x^2-9x+5\right)^8.3(2x-3)\\&=27\left(3x^2-9x+5\right)^8(2x-3)\end{aligned}$$

Question 2:

Differentiate with respect to x the function $\sin^3 x+\cos^6 x$.

Solution:

Let $y=\sin^3 x+\cos^6 x$

Using chain rule, we get

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&=3\sin^2 x.\frac{d}{dx}(\sin x)+6\cos^5 x.\frac{d}{dx}(\cos x)\\&=3\sin^2 x.\cos x+6\cos^5 x.(-\sin x)\\&=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Question 3:

Differentiate with respect to x the function $(5x)^{3\cos 2x}$.

Solution:

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$$\log y = 3\cos 2x \log 5x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5x.\frac{d}{dx}(\cos 2x)+\cos 2x.\frac{d}{dx}(\log 5x)\right]$$

$$\frac{dy}{dx} = 3y\left[\log 5x.(-\sin 2x).\frac{d}{dx}(2x)+\cos 2x.\frac{1}{5x}.\frac{d}{dx}(5x)\right]$$

$$= 3y\left[-2\sin 2x.\log 5x+\frac{\cos 2x}{x}\right]$$

$$= y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

$$= (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]$$

Question 4:

Differentiate with respect to x the function $\sin^{-1}\left(x\sqrt{x}\right),\ 0\le x\le 1$.

Solution:

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we get

$$\frac{dy}{dx}=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\times\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\cdot\frac{3}{2}\cdot x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Question 5:

Differentiate with respect to x the function $\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$.

Solution:

Let $y=\frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

Using quotient rule, we get

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}.\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right).\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}.\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right).\frac{1}{2\sqrt{2x+7}}.\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}.\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right).\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\left(\sqrt{4-x^2}\right).(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Question 6:

Differentiate with respect to x the function $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0<x<\frac{\pi}{2}$.

Solution:

Let $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(1)

Then,

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}=\frac{1+\cos x}{\sin x}$$

$$=\frac{1+2\cos^2\frac{x}{2}-1}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes,

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

Thus,

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$=\frac{1}{2}$$

Question 7:

Differentiate with respect to x the function $(\log x)^{\log x}, x>1$.

Solution:

Let $y=(\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y=\log x.\log(\log x)$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[\log x.\log(\log x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log(\log x).\frac{d}{dx}(\log x)+\log x.\frac{d}{dx}\left[\log(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx}=y\left[\log(\log x).\frac{1}{x}+\log x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{1}{x}.\log(\log x)+\frac{1}{x}\right]$$

$$\Rightarrow \frac{dy}{dx}=(\log x)^{\log x}\left[\frac{1}{x}+\frac{\log(\log x)}{x}\right]$$

Question 8:

Differentiate with respect to x the function $\cos(a\cos x+b\sin x)$, for some constant a and b.

Solution:

Let $y=\cos(a\cos x+b\sin x)$

Using chain rule, we get

$$\frac{dy}{dx}=\frac{d}{dx}\cos(a\cos x+b\sin x)$$

$$=-\sin(a\cos x+b\sin x).\frac{d}{dx}(a\cos x+b\sin x)$$

$$=-\sin(a\cos x+b\sin x).\left[a(-\sin x)+b\cos x\right]$$

$$=(a\sin x-b\cos x).\sin(a\cos x+b\sin x)$$

Question 9:

Differentiate with respect to x the function $(\sin x-\cos x)^{(\sin x-\cos x)}, \frac{\pi}{4}<x<\frac{3\pi}{4}$

Solution:

Let $y=(\sin x-\cos x)^{(\sin x-\cos x)}$

Taking log on both the sides, we obtain

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$
$$= (\sin x - \cos x)\log(\sin x - \cos x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\log(\sin x - \cos x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x).\frac{d}{dx}\log(\sin x - \cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x).(\cos x + \sin x) + (\sin x - \cos x).\frac{1}{(\sin x - \cos x)}.\frac{d}{dx}(\sin x - \cos x)$$
$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x).\log(\sin x - \cos x) + (\cos x + \sin x)\right]$$
$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$$

Question 10:

Differentiate with respect to x the function $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Solution:

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u,\ x^a = v,\ a^x = w$ and $a^a = s$

Therefore,

$$\Rightarrow y = u + v + w + s$$
$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad \ldots(1)$$

Now, $u = x^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log x^x$$
$$\Rightarrow \log u = x\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x) + x.\frac{d}{dx}(\log x)$$

$$\frac{du}{dx} = u\left[\log x.1 + x.\frac{1}{x}\right]$$

$$= x^x[\log x + 1] = x^x(1+\log x) \qquad \ldots(2)$$

Now, $v = x^a$
Hence,

$$\frac{dv}{dx} = \frac{d}{dx}(x^a)$$

$$= ax^{a-1} \qquad \ldots(3)$$

Now, $w = a^x$
Taking logarithm on both the sides, we obtain

$$\Rightarrow \log w = \log a^x$$

$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{w}\frac{dw}{dx} = \log a.\frac{d}{dx}(x)$$

$$\frac{dw}{dx} = w \log a$$

$$= a^x \log a \qquad \ldots(4)$$

Now, $s = a^a$
Since a is constant, a^a is also a constant.
Hence,

$$\frac{ds}{dx} = 0 \qquad \ldots(5)$$

From (1), (2), (3), (4) and (5), we obtain

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a + 0$$

$$= x^x(1+\log x) + ax^{a-1} + a^x \log a$$

Question 11:

Differentiate with respect to x the function $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

Therefore,

$$y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(1)$$

Now, $u = x^{x^2-3}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\log u &= \log\left(x^{x^2-3}\right)\\ &= \left(x^2-3\right)\log x\end{aligned}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}\left(x^2-3\right) + \left(x^2-3\right).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x.2x + \left(x^2-3\right).\frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] \qquad \ldots(2)$$

Now, $v = (x-3)^{x^2}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\log v &= \log(x-3)^{x^2}\\ &= x^2\log(x-3)\end{aligned}$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \log(x-3).\frac{d}{dx}\left(x^2\right) + \left(x^2\right).\frac{d}{dx}\left[\log(x-3)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \log(x-3).2x + x^2.\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v\left[2x\log(x-3) + \frac{x^2}{x-3}.1\right]$$

$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right] \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx}=x^{x^2-3}\left[\frac{x^2-3}{x}+2x\log x\right]+(x-3)^{x^2}\left[\frac{x^2}{x-3}+2x\log(x-3)\right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y=12(1-\cos t), x=10(t-\sin t), \frac{-\pi}{2}<t<\frac{\pi}{2}$

Solution:

The given function is $y=12(1-\cos t), x=10(t-\sin t)$
Hence,

$$\frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]$$
$$=10.\frac{d}{dt}(t-\sin t)$$
$$=10(1-\cos t)$$
$$\frac{dy}{dt}=\frac{d}{dt}\left[12(1-\cos t)\right]$$
$$=12.\frac{d}{dt}(1-\cos t)$$
$$=12.\left[0-(-\sin t)\right]$$
$$=12\sin t$$

Therefore,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{12\sin t}{10(1-\cos t)}$$
$$=\frac{12.2\sin\frac{t}{2}.\cos\frac{t}{2}}{10.2\sin^2\frac{t}{2}}$$
$$=\frac{6}{5}\cot\frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}, -1\le x\le 1$.

Solution:

The given function is $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

Hence,

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$=0$$

Question 14:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$ for $-1<x<1$, prove that $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$.

Solution:

The given function is $x\sqrt{1+y}+y\sqrt{1+x}=0$

$$\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow x^2+x^2y=y^2+xy^2$$

$$\Rightarrow x^2-y^2=xy^2-x^2y$$

$$\Rightarrow x^2-y^2=xy(y-x)$$

$$\Rightarrow (x+y)(x-y)=xy(y-x)$$

$$\Rightarrow x+y=-xy$$

$$\Rightarrow (1+x)y=-x$$

$$\Rightarrow y=\frac{-x}{(1+x)}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx}=-\left[\frac{(1+x)\frac{d}{dx}(x)-(x).\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$=-\frac{(1+x)-x}{(1+x)^2}$$

$$=-\frac{1}{(1+x)^2}$$

Hence proved.

Question 15:

If $(x-a)^2+(y-b)^2=c^2$ for $c>0$, prove that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Solution:

The given function is $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a).\frac{d}{dx}(x-a)+2(y-b).\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a).1+2(y-b).\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad \ldots(1)$$

Therefore,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\left[\frac{(y-b).\frac{d}{dx}(x-a)-(x-a).\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a).\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a).\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$=-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

Hence,

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$$=-c$$

$-c$ is a constant and is independent of a and b.

Hence proved.

Question 16:

If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Solution:

The given function is $\cos y = x\cos(a+y)$
Therefore,

$$\Rightarrow \frac{d}{dx}[\cos y] = \frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y).\frac{d}{dx}(x) + x.\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x.[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \qquad \ldots(1)$$

Since, $\cos y = x\cos(a+y)$ $\Rightarrow x = \frac{\cos y}{\cos(a+y)}$

Then, equation (1) becomes,

$$\left[\frac{\cos y}{\cos(a+y)}.\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow [\cos y.\sin(a+y) - \sin y.\cos(a+y)].\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Hence proved.

Question 17:

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Solution:

The given function is $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

Therefore,

$$\begin{aligned}\frac{dx}{dt}&=a.\frac{d}{dt}(\cos t+t\sin t)\\&=a\left[-\sin t+\sin t.\frac{d}{dx}(t)+t.\frac{d}{dt}(\sin t)\right]\\&=a[-\sin t+\sin t+t\cos t]\\&=at\cos t\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt}&=a.\frac{d}{dt}(\sin t-t\cos t)\\&=a\left[\cos t-\left\{\cos t.\frac{d}{dt}(t)+t.\frac{d}{dt}(\cos t)\right\}\right]\\&=a[\cos t-\{\cos t-t\sin t\}]\\&=at\sin t\end{aligned}$$

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{at\sin t}{at\cos t}=\tan t$$

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\tan t)=\sec^2 t.\frac{dt}{dx}\\&=\sec^2 t.\frac{1}{at\cos t} \qquad \left[\frac{dx}{dt}=at\cos t\Rightarrow\frac{dt}{dx}=\frac{1}{at\cos t}\right]\\&=\frac{\sec^3 t}{at},0<t<\frac{\pi}{2}\end{aligned}$$

Question 18:

If $f(x)=|x|^3$, show that $f''(x)$ exists for all real x, and find it.

Solution:

It is known that $|x|=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$

Therefore, when $x\geq 0,\ f(x)=|x|^3=x^3$

In this case, $f'(x)=3x^2$ and hence, $f''(x)=6x$

When $x<0,\ f(x)=|x|^3=(-x)^3=-x^3$

In this case, $f'(x)=-3x^2$ and hence, $f''(x)=-6x$

Thus, for $f(x)=|x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x)=\begin{cases}6x, & \text{if } x\geq 0\\ -6x, & \text{if } x<0\end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n.

Solution:

To prove: $P(n):\frac{d}{dx}(x^n)=nx^{n-1}$ for all positive integers n.

For $n=1$,

$$P(1):\frac{d}{dx}(x)=1=1\cdot x^{1-1}$$

Therefore, $P(n)$ is true for $n=1$.

Let $P(k)$ is true for some positive integer k.

That is, $P(k):\frac{d}{dx}(x^k)=kx^{k-1}$

It has to be proved that $P(k+1)$ is also true.

Consider

$$\begin{aligned}\frac{d}{dx}(x^{k+1})&=\frac{d}{dx}(x\cdot x^k)\\ &=x^k.\frac{d}{dx}(x)+x.\frac{d}{dx}(x^k) \qquad [\text{By applying product rule}]\\ &=x^k.1+x\cdot k\cdot x^{k-1}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^{k+1})&=x^k+kx^k\\ &=(k+1)\cdot x^k\\ &=(k+1)\cdot x^{(k+1)-1}\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer n.

Hence, proved.

Question 20:

Using the fact that $\sin(A+B)=\sin A\cos B+\cos A\sin B$ and the differentiation, obtain the sum formula for cosines.

Solution:

Given, $\sin(A+B)=\sin A\cos B+\cos A\sin B$

Differentiating both sides with respect to x, we obtain

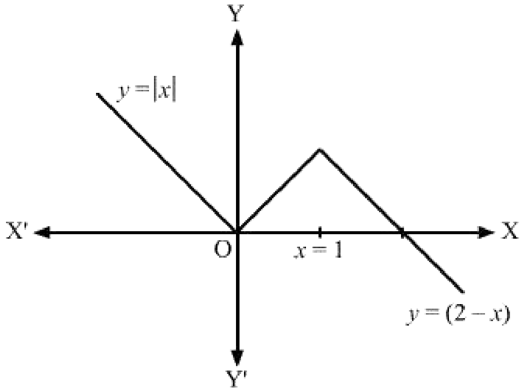
$$\frac{d}{dx}\left[\sin(A+B)\right]=\frac{d}{dx}(\sin A\cos B)+\frac{d}{dx}(\cos A\sin B)$$
$$\Rightarrow \cos(A+B).\frac{d}{dx}(A+B)=\cos B.\frac{d}{dx}(\sin A)+\sin A.\frac{d}{dx}(\cos B)+\sin B.\frac{d}{dx}(\cos A)+\cos A.\frac{d}{dx}(\sin B)$$
$$\Rightarrow \cos(A+B).\frac{d}{dx}(A+B)=\cos B.\cos A\frac{dA}{dx}+\sin A(-\sin B)\frac{dB}{dx}+\sin B(-\sin A).\frac{dA}{dx}+\cos A\cos B\frac{dB}{dx}$$
$$\Rightarrow \cos(A+B).\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B).\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$
$$\Rightarrow \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Question 21:

Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer?

Solution:

Consider, $y=\begin{cases}|x| & -\infty<x\le 1\\ 2-x & 1\le x\le\infty\end{cases}$



It can be seen from the above graph that the given function is continuous everywhere but not differentiable at exactly two points which are 0 and 1.

Question 22:

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Solution:

Given, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$\Rightarrow y = (mc - nb) f(x) - (lc - na) g(x) + (lb - ma) h(x)$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[(mc-nb)f(x)\right] - \frac{d}{dx}\left[(lc-na)g(x)\right] + \frac{d}{dx}\left[(lb-ma)h(x)\right] \\ &= (mc-nb)f'(x) - (lc-na)g'(x) + (lb-ma)h'(x) \\ &= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}\end{aligned}$$

Thus, $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ proved.

Question 23:

If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Solution:

The given function is $y = e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log y = a\cos^{-1}x \log e$$

$$\Rightarrow \log y = a\cos^{-1}x$$

Differentiating both sides with respect to x, we obtain

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow \left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again, differentiating both sides with respect to x, we obtain

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 \frac{d}{dx}\left(1-x^2\right) + \left(1-x^2\right) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2\frac{d}{dx}\left(y^2\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + \left(1-x^2\right) \times 2\frac{dy}{dx}.\frac{d^2y}{dx^2} = a^2.2y.\frac{dy}{dx}$$

$$\Rightarrow -x\frac{dy}{dx} + \left(1-x^2\right)\frac{d^2y}{dx^2} = a^2.y \qquad \left[\frac{dy}{dx} \ne 0\right]$$

$$\Rightarrow \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

Hence proved.