11. PROBABILITY DISTRIBUTIONS

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Random Variable

A random variable X is a function defined on a sample space S into the real numbers R such that the inverse image of points or subset or interval of R is an event in S, for which probability is assigned.

Exercise 11.1

1) Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

Sample space $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

given that $n(S) = 8$

X be the random variable denotes no of tails

$$X = 0, 1, 2, 3$$

0 tail: $X(HHH) = 0$

1 tail: $X(HHT) = 1$ $X(HTH) = 1$ $X(THH) = 1$

2 tail: $X(HTT) = 2$ $X(THT) = 2$ $X(TTH) = 2$

3 tail: $X(TTT) = 3$

$X(\omega)$ denotes the number of tails

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = HHH \\ 1 & \text{if } \omega = HHT, THH, HTH \\ 2 & \text{if } \omega = HTT, THT, TTH \\ 3 & \text{if } \omega = TTT \end{cases}$$

Values of Random Variable	0	1	2	3	Total
Number of points in inverse image	1	3	3	1	8

Examples 11.1

Suppose two coins are tossed once. If x denote the number of tails, (i) write down the sample space. (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

(i) sample space $S=\{HH, HT, TH, TT\}$ $n(S)=4$

(ii) x be the number of tails

$X(TT)=2$ $X(TH)=1$ $X(HT)=1$
$X(HH)=0$

$X=0,1,2$

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$$X(w)=\begin{cases} 2 & \text{if } w=TT \\ 1 & \text{if } w=TH, HT \\ 0 & \text{if } w=HH \end{cases}$$

∴ inverse image of 1 is $\{TH, HT\}$

(iii)

value of Random variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

Example 11.2

Suppose a pair of unbiased dice is rolled once. If x denotes the total score of two dice, write down. (i) the sample space (ii) the values taken by the random variable x (iii) The inverse image of 10, (iv) the number of elements in inverse image of X.

(i) sample space $S=\{$ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) $\}$

$n(S)=36$

$X(\alpha,\beta)=\alpha+\beta$

$X(1,1)=2$
$X(1,2)=X(2,1)=3$
$X(1,3)=X(2,2)=X(3,1)=4$
$X(1,4)=X(2,3)=X(3,2)=X(4,1)=5$
$X(1,5)=X(2,4)=X(3,3)=X(4,2)=X(5,1)=6$
$X(1,6)=X(2,5)=X(3,4)=X(4,3)=X(5,2)=X(6,1)=7$
$X(2,6)=X(3,5)=X(4,4)=X(5,3)=X(6,2)=8$

$X(3,6) = X(4,5) = X(5,4) = X(6,3) = 9$

$X(4,6) = X(5,5) = X(6,4) = 10$

$X(5,6) = X(6,5) = 11$

$X(6,6) = 12$

(ii) $X = 2,3,4,5,6,7,8,9,10,11,12$

(iii) Inverse image of 10 is $\{(4,6),(5,5),(6,4)\}$

Value of random variable	2	3	4	5	6	7	8	9	10	11	12	Total
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

Example 11.3

An urn contains 2 white and 3 red balls. A sample of 3 balls chosen. If X denotes the number of red balls, Find the value of random variable X and its number of inverse images

W	R	Total
2	3	5

$n(S) = 5C_3 = 5C_2 = \frac{5\times4}{1\times2} = 10$

X denote no.of red balls

$X = 1,2,3$ (0 red not possible

$X(\text{one red}) = 2C_2 \times 3C_1 = 1\times3 = 3$

$X(\text{two red}) = 2C_1 \times 3C_2 = 2\times3 = 6$

$X(\text{three red}) = 2C_0 \times 3C_3 = 1\times1 = 1$

value of random variable X	1	2	3	TOTAL
Number of elements in inverse images	3	6	1	10

Example 11.4

Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹30 for each black ball selected and we lose ₹20 for each white ball selected. If X denotes the winning amount, then find the values of X and number of points in its inverse images

W	B	Total
6	4	10

$n(S) = 10C_2 = \frac{10\times9}{1\times2} = 45$

X denote winning amount

$X(2\text{ black}) = 30+30 = 60$

$X(1\text{ black}, 1\text{ white}) = 30-20 = 10$

$X(2\text{ white}) = 2(-20) = -40$

3A

$X = -40, 10, 60$

$X(-40) = X(2\text{ white}) = 6C_2 \times 4C_0$

$= \frac{\overset{3}{\cancel{6}} \times 5}{1 \times \cancel{2}} \times 1 = 15$

$X(10) = X(1\text{ white } 1\text{ black}) = 6C_1 \times 4C_1 = 6 \times 4 = 24$

$X(60) = X(2\text{ black}) = 6C_0\ 4C_2 = 1 \times \frac{\overset{2}{\cancel{4}} \times 3}{1 \times \cancel{2}} = 6$

Values of the Random variable	60	10	-40	Total
Number of elements in inverse image	6	24	15	45

2) In a pack of 52 playing cards, two cards are drawn at random simultaneously, If the number of black cards drawn is a random variable, find the values of the random variable and numbers of points in its inverse images.

BLACK	OTHER	TOTAL
26	26	52

$$n(S) = 52C_2$$
$$= \frac{\overset{26}{\cancel{52}} \times 51}{1 \times \cancel{2}}$$
$$n(S) = 1326$$

Let X be the random variable denote no. of Black cards drawn.

$$X = 0, 1, 2$$

$X(\omega)$ denotes no. of Black cards

$$X(\omega) = \begin{cases} 0 & \text{if no black card} \\ 1 & \text{if one is black card} \\ 2 & \text{if two black} \end{cases}$$

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$$X(\text{no black}) = X(0) = 26C_0 \times 26C_2$$
$$= 1 \times \frac{\overset{13}{\cancel{26}} \times 25}{1 \times \cancel{2}} = 325$$

$$X(\text{one black}) = X(1) = 26C_1 \times 26C_1$$
$$= 26 \times 26 = 676$$

$$X(2\text{ black}) = X(2) = 26C_2 \times 26C_0$$
$$= \frac{\overset{13}{\cancel{26}} \times 25}{1 \times \cancel{2}} = 325$$

value of random variable	0	1	2	TOTAL
No. of points in Inverse Image	325	676	325	1326

3) An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, Then find the values of the random variable and number of points in the inverse images.

Total Fruits $= 5 + 4 = 9$

(Without replacing) $n(S) = 9C_3 = \frac{\overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}} \times 7}{1 \times \cancel{2} \times \cancel{3}} = 84$

Let X be the random variable denotes the no. of apples taken.

$$X = 0, 1, 2, 3$$

$X(\omega)$ denots no. of apples

$$X(\omega) = \begin{cases} 0 & \text{if no apple (3 mangoes) draw} \\ 1 & \text{if one apple (2 mangoes) taken} \\ 2 & \text{if two apples (1 mango) taken} \\ 3 & \text{if 3 apples (no mango) taken} \end{cases}$$

$$X(\text{no apple}) = X(0) = 4C_0\, 5C_3 = 1 \times \frac{5\times4\times3}{1\times2\times3}$$

$$X(0) = 10$$

$$X(\text{one apple}) = X(1) = 4C_1 \times 5C_2 = 4 \times \frac{5\times4}{1\times2} = 40$$

$$X(\text{two apple}) = X(2) = 4C_2 \times 5C_1 = \frac{4\times3}{1\times2} \times 5 = 30$$

$$X(3\text{ apples}) = X(3) = 4C_3 \times 5C_0 = 4C_1 \times 1 = 4$$

Value of Random variable X	0	1	2	3	Total
Number of elements in Inverse image	10	40	30	4	84

Book Answer wrong

4) Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win ₹15 for each red ball selected and we lose ₹10 for 10 black ball selected. X denotes the winning amount, then find the values of X and number of points in its inverse images.

$$\text{Total balls} = 6 + 8 = 14$$

$$n(S) = 14C_2 = \frac{14\times13}{1\times2} = 91$$

Let X be the amount won

$$X = -10\times2,\ -10+15,\ 15+15$$

$$X = -20, 5, 30$$

$$X(\omega) = \begin{cases} -20 & \text{if two black balls.} \\ 5 & \text{if one black one red.} \\ 30 & \text{if two red balls.} \end{cases}$$

$$X(\text{two black ball}) = X(-20) = 6C_0 \times 8C_2 = 1 \times \frac{8\times7}{1\times2} = 28$$

$$X(1\text{ black }1\text{ red}) = X(5) = 6C_1 \times 8C_1 = 6\times8 = 48$$

$$X(2\text{ red balls}) = X(30) = 6C_2 \times 8C_0 = \frac{6\times5}{1\times2} \times 1 = 15$$

Value of random variable	-20	5	30	Total
No. of points in Inverse Image	28	48	15	91

5) A six sided die is marked '2' on one face '3' on two of its faces and 4 on remaining three faces. The die is thrown twice. If X denote the total scores in two throws, find the values of the random variable and number of points in its inverse images.

II \ I	2	3	3	4	4	4
2	4	5	5	6	6	6
3	5	6	6	7	7	7
3	5	6	6	7	7	7
4	6	7	7	8	8	8
4	6	7	7	8	8	8
4	6	7	7	8	8	8

$n(S) = 36$

X is assigned to each point (α, β) the sum on the faces

$X(\alpha, \beta) = \alpha + \beta$

$X(2,2) = 4$

$X(2,3) = X(3,2) = 5$

$X(3,4) = X(3,3) = X(4,2) = 6$

$X(4,3) = X(3,4) = 7$

$X(4,4) = 8$

$X(2,2)$ onetimes
$X(2,3)$ 2 times
$X(3,2)$ 2 times

X takes the values 4, 5, 6, 7, 8

values of random variable	4	5	6	7	8	Total
Number of elements in inverse image	1	4	10	12	9	36

A random variable X is said to be a discrete random variable if the range of X is countable.

<u>Probability Mass function</u> (PMF)

If X is a discrete random variable with discrete values $x_1, x_2, \ldots x_n, \ldots$ then the function f or p defined by

$$f(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, \ldots n.$$

It is called PMF.

* $f(x_k) \geq 0$ for $k = 1, 2, \ldots n$ and * $\sum_k f(x_k) = 1$

Cumulative Distribution Function (or) Distribution Function

The cumulative distribution function $F(x)$ of a discrete random variable X, taking values $x_1, x_2, \ldots$ such that $x_1 < x_2 < x_3 < \ldots$ with p.m.f $f(x_i)$

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), \quad x \in R$$

$$* \; f(x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \ldots$$

Exercise 11.2

1) Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$n(S) = 8$

Let X be random variable denotes no. of heads. $X = 0, 1, 2, 3$.

Values of Random variable	0	1	2	3	Total
Number of elements in inverse images	1	3	3	1	8

$P(X=0) = 1/8$ $\quad P(X=1) = \frac{3}{8}$ $\quad P(X=2) = \frac{3}{8}$ $\quad P(X=3) = \frac{1}{8}$

probability mass function is

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

2) A six sided die is marked '1' on one face, '3' on two of its faces, and 5 on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

(i) the probability mass function (ii) the cumulative distribution function (iii) $P(4 \leq X < 10)$ (iv) $P(X \geq 6)$

I\II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

X denote the total score in two throws

it takes 2, 4, 6, 8, 10

values of R.V X	2	4	6	8	10	TOTAL
Number of elements in Inverse images	1	4	10	12	9	36

$P(X=2)=\frac{1}{36}$, $P(X=4)=\frac{4}{36}$, $P(X=6)=\frac{10}{36}$, $P(X=8)=\frac{12}{36}$

$P(X=10)=\frac{9}{36}$.

(i) The probability mass function is

x	2	4	6	8	10
f(x)	1/36	4/36	10/36	12/36	9/36

(ii) The cumulative distribution function

x	2	4	6	8	10
F(x)	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

(iii) $P(4 \le X < 10) = P(X=4)+P(X=6)+P(X=8)$

$= \frac{4}{36}+\frac{10}{36}+\frac{12}{36} = \frac{26}{36}$

$= \frac{13}{18}$

(iv) $P(X \geq 6) = P(X=6)+P(X=8)+P(X=10)$

$= \frac{10}{36}+\frac{12}{36}+\frac{9}{36} = \frac{31}{36}$.

3) Find the probability mass function and cumulative distribution function of number of girls child in families with 4 children, assuming equal probabilities for boys and girls.

$n(S)=2^4=16$

Let x be the random variable denotes no.of girls child.

$x = 0, 1, 2, 3, 4$

Value of R.V X	0	1	2	3	4
Number of elements in inverse image	$4C_0$	$4C_1$	$4C_2$	$4C_3$	$4C_4$
	1	4	6	4	1

$P(X=0) = 1/16 = P(X=4)$

$P(X=1) = \frac{4}{16}\frac{1}{4} = P(X=3)$

$P(X=2) = \frac{6}{16}\frac{3}{8}$

Probability mass function is $f(x) = \begin{cases} \frac{1}{16} & \text{for } x=0,4 \\ \frac{1}{4} & \text{for } x=1,3 \\ 3/8 & \text{for } x=2 \end{cases}$

x	0	1	2	3	4
$f(x)$	1/16	4/16	6/16	4/16	1/16

$F(x) = P(X \le x)$

$F(0) = P(X \le 0) = 1/16$

$F(1) = P(X \le 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

$F(2) = P(X \le 2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$

$F(3) = P(X \le 3) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$

$F(4) = P(X \le 4) = \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$

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cumulative distribution function is

x	0	1	2	3	4
$F(x)$	1/16	5/16	11/16	15/16	1

4) Suppose a discrete random variable can only take the values 0, 1, and 2. The probability mass function is defined by.

$$f(x) = \begin{cases} \frac{x^2+1}{k}, & \text{for } x = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find the (i) value of k, (ii). cumulative distribution function (iii) $P(X \ge 1)$.

given $f(x)$ is a P.M.F

$\therefore \sum f(x) = 1$

$$\frac{0^2+1}{k} + \frac{1^2+1}{k} + \frac{2^2+1}{k} = 1$$

$\frac{1}{k}+\frac{2}{k}+\frac{5}{k}=1$

$\frac{8}{k}=1$ $\boxed{k=8}$

$\therefore f(x)=\begin{cases} \frac{x^2+1}{8} & \text{for } x=0,1,2 \\ 0 & \text{otherwise} \end{cases}$ or

x	0	1	2
$f(x)$	$1/8$	$2/8$	$5/8$

(i) $F(x)=P(X\le x)$

$F(x)=\begin{cases} 1/8 & \text{for } x\le 0 \\ 3/8 & \text{for } x\le 1 \\ 8/8=1 & \text{for } x\le 2 \end{cases}$

$F(0)=P(x=0)=1/8$

$F(1)=P(x=0)+P(x=1)=\frac{3}{8}$

$F(2)=P(x=0)+P(x=1)+P(x=2)=\frac{8}{8}=1$

(iii) $P(X\ge 1)=P(x=1)+P(x=2)$

$=\frac{2}{8}+\frac{5}{8}$

$=\frac{7}{8}$

⑤ The cumulative distribution function of a discrete random variable is given by

$F(x)=\begin{cases} 0 & -\infty<x<-1 \\ 0.15 & -1\le x<0 \\ 0.35 & 0\le x<1 \\ 0.60 & 1\le x<2 \\ 0.85 & 2\le x<3 \\ 1 & 3\le x<\infty \end{cases}$

Find (i) Probability mass function
(ii) $P(X<1)$
(iii) $P(X\ge 2)$

$P(X=-1)=F(-1)=0.15$

$P(X=0)=F(0)-F(-1)=0.35-0.15$

$P(X=0)=0.20$

$P(X=1)=F(1)-F(0)=0.60-0.35$

$P(X=1)=0.25$

$P(X=2)=F(2)-F(1)=0.85-0.60$

$=0.25$

$P(X=3)=F(3)-F(2)=1-0.85$

$=0.15$

(i) Probability mass function is

x	-1	0	1	2	3
$P(X=x)$	0.15	0.20	0.25	0.25	0.15

(ii) $P(X<1)=P(X\le 0)=f(0)+f(-1)$

$=P(X=-1)+P(X=0)=0.20+0.15=0.35$

(iii) $P(X\ge 2)=P(2\le X\le 3)=P(X=2)+P(X=3)$

$=0.25+0.15$

$=0.40$

(6) A random variable x has the following probability mass function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$.

$f(x)$ is a p.m.f.

$\therefore \sum f(x) = 1$

$k^2+2k^2+3k^2+2k+3k = 1$

$6k^2+5k-1=0$

$(k+\cancel{6}/\cancel{6})(k-\frac{1}{6}) = 0$

$k+1=0 \quad k-\frac{1}{6}=0$

$k=-1 \quad k=\frac{1}{6}$

$\boxed{k=\frac{1}{6}}$

p.m.f. is

x	1	2	3	4	5
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{6}$	$\frac{3}{6}$

(ii) $P(2 \le X < 5) = P(X=2)+P(X=3)+P(X=4)$

$= 2k^2+3k^2+2k = \frac{2}{36}+\frac{3}{36}+\frac{2}{6} = \frac{2+3+12}{36}$

$= \frac{17}{36}$

(iii) $P(3<X) = P(X=4)+P(X=5) = 2k+3k = 5k = \frac{5}{6}$

(7) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \frac{1}{2} & \text{for } 0 \le x < 1 \\ \frac{3}{5} & \text{for } 1 \le x < 2 \\ \frac{4}{5} & \text{for } 2 \le x < 3 \\ \frac{9}{10} & \text{for } 3 \le x < 4 \\ 1 & \text{for } 4 \le x < \infty \end{cases}$$

Find (i) The probability mass function (ii) $P(X<3)$ and (iii) $P(X \ge 2)$

$f(0) = P(X=0) = F(0) = \frac{1}{2}$

$f(1) = P(X=1) = F(1)-F(0) = \frac{3}{5}-\frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$

$f(2) = P(X=2) = F(2)-F(1) = \frac{4}{5}-\frac{3}{5} = \frac{1}{5}$

$f(3) = P(X=3) = F(3)-F(2) = \frac{9}{10}-\frac{4}{5} = \frac{1}{10}$

$f(4) = P(X=4) = F(4) - F(3) = 1 - \frac{9}{10} = \frac{1}{10}$

The probability mass function is

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{5}$	$1/10$	$\frac{1}{5}$	$\frac{1}{10}$	$1/10$

(ii) $P(X<3) = (P(X=0) + P(X=1) + P(X=2))$

or

$= P(X\le 2) = F(2) = \frac{4}{5}$

(iii) $P(X\ge 2) = 1 - P(X<2) = 1 - P(X\le 1) = 1 - F(1)$

$= 1 - \frac{3}{5} = \frac{2}{5}$

—— x ——

Continuous Distributions

A random variable takes any value in a set I of R.

Probability Density Function (p.d.f)

$f(x)$ is said to be p.d.f. If $x \in [a,b]$ of a continuous random variable X

$$P(a\le x\le b) = \int_a^b f(x)\,dx$$

* $f(x)$ is a p.d.f $\iff \int_{-\infty}^{\infty} f(x)\,dx = 1$.

Cumulative distribution function

$$F(x) = P(X\le x) = \int_{-\infty}^{x} f(u)\,du \qquad -\infty < u < \infty$$

* $f(x) = \frac{dF}{dx}$ or $F'(x)$

Exercise 11.3

1) The probability density function of X is given by

$f(x) = \begin{cases} kxe^{-2x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$ Find the value of k.

$f(x)$ is a p.d.f $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} kx^1 e^{-2x}\,dx = 1$$

$\boxed{\int_0^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}}$ $a=2$, $n=1$

$k\left[\frac{1!}{2^{1+1}}\right] = 1$

$k = 2^2$

$\boxed{k=4}$

② The probability density function of x is

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) $P(0.2 \le x < 0.6)$

(ii) $P(1.2 \le x \le 1.8)$

(iii) $P(0.5 \le x < 1.5)$

(i) $P(0.2 \le x < 0.6)$

$$= \int_{0.2}^{0.6} f(x)\,dx = \int_{0.2}^{0.6} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0.2}^{0.6} = \frac{1}{2}\left[0.6^2 - 0.2^2\right]$$

$$= \frac{1}{2}\left[0.36 - 0.04\right] = \frac{1}{2}\left[0.32\right]$$

$$= 0.16$$

(ii) $P(1.2 \le x < 1.8) = \int_{1.2}^{1.8} f(x)\,dx = \int_{1.2}^{1.8} (2-x)\,dx$

$$= -\left[\frac{(2-x)^2}{2}\right]_{1.2}^{1.8} = -\frac{1}{2}\left[(2-1.8)^2 - (2-1.2)^2\right]$$

$$= -\frac{1}{2}\left[0.2^2 - 0.8^2\right] = -\frac{1}{2}\left[0.04 - 0.64\right]$$

$$= -\frac{1}{2}(-0.60) = 0.30$$

(iii) $P(0.5 \le x < 1.5) = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$

$$= \int_{0.5}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[-\frac{(2-x)^2}{2}\right]_{1}^{1.5}$$

$$= \frac{1}{2}\left[(1^2 - 0.5^2) - \left((2-1.5)^2 - (2-1)^2\right)\right]$$

$$= \frac{1}{2}\left[(1 - 0.25) - (0.5^2 - 1^2)\right]$$

$$= \frac{1}{2}\left[0.75 + 0.75\right] = 0.75$$

(3) Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function $f(x) = \begin{cases} k & 200 \le x \le 600 \\ 0 & \text{otherwise} \end{cases}$

Find (i) The value of k (ii) The distribution function (iii) The probability that daily sales will fall between 300 litres and 500 litres.

(i) given $f(x)$ is a p.d.f $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{200}^{600} k\,dx = 1 \Rightarrow k\left[x\right]_{200}^{600} = 1$$

$$k(600-200) = 1$$

$$400k = 1 \qquad \boxed{k = \frac{1}{400}}$$

(ii) distribution function $F(x) = \int_{-\infty}^{x} f(t)\,dt$

a) when $x \in (-\infty, 200)$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{200} 0\,dx = 0$$

b) when $x \in [200, 600]$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{200} f(t)dt + \int_{200}^{x} f(t)dt$$

$$= 0 + \int_{200}^{x} \frac{1}{400}\,dt = \frac{1}{400}\left[t\right]_{200}^{x}$$

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$$= \frac{x}{400} - \frac{200}{400}$$

$$= \frac{x}{400} - \frac{1}{2}$$

c) when $x \in (600, \infty)$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{200} f(t)dt + \int_{200}^{600} f(t)dt + \int_{600}^{x} f(t)dt$$

$$= 0 + 1 + 0$$

$$= 1$$

$$F(x) = \begin{cases} 0 & x < 200 \\ \frac{x}{400} - \frac{1}{2} & x \in [200, 600] \\ 1 & x \in (600, \infty) \end{cases}$$

(iii) $P(300 < x < 500) = F(500) - F(300)$

$$= \left(\frac{500}{400} - \frac{1}{2}\right) - \left(\frac{300}{400} - \frac{1}{2}\right)$$

$$= \frac{500-300}{400} \cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}}$$

$$= \frac{200}{400} = \frac{1}{2}$$

④ The probability density function of X is given by $f(x) = \begin{cases} ke^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$

Find (i) the value of k (ii) the distribution function (iii) $P(X<3)$ (iv) $P(5 \le x)$ (v) $P(X \le 4)$

Given $f(x)$ is a p.d.f $\therefore \int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^{\infty} ke^{-x/3}\, dx = 1$

$$\frac{k}{-1/3}\left[e^{-x/3}\right]_0^{\infty} = 1$$

$$\Rightarrow -3k\left[e^{-\infty} - e^{0}\right] = 1$$

$$-3k[0-1] = 1$$

$$3k = 1 \qquad k = \frac{1}{3}$$

(ii) distribution function

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

a) $x \in (-\infty, 0]$ $\quad F(x) = \int_{-\infty}^{x} f(t)dt = 0$

b) when $x \in (0, \infty)$ $\quad F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t) + \int_0^x f(t)dt$

$$= 0 + \int_0^x \frac{1}{3} e^{-t/3}\, dt$$

$$= \frac{1}{3}\cdot\frac{1}{-1/3}\left[e^{-t/3}\right]_0^x = -e^{-x/3} + e^{0}$$

$$= 1 - e^{-x/3}$$

$$F(x) = \begin{cases} 1 - e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

(iii) $P(X<3) = P(X \le 3) = F(3) = 1-e^{-3/3} = 1-e^{-1}$

(iv) $P(5 \le X) = P(X \ge 5) = 1-P(X<5) = 1-F(5)$

$= 1-(1-e^{-5/3})$

$= e^{-5/3}$

(v) $P(X \le 4) = F(4) = 1-e^{-4/3}$

(5) If X is the random variable with probability density function $f(x)$ given by
$$f(x) = \begin{cases} x+1 & -1 \le x < 0 \\ -x+1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the distribution function $F(x)$ (ii) $P(-0.5 \le X \le 0.5)$

(i) distribution function

a) when $x \in (-\infty, -1)$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

b) when $x \in [-1, 0)$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{x} f(t)\,dt$$
$$= 0 + \int_{-1}^{x} (t+1)\,dt$$
$$= \left[\frac{t^2}{2}+t\right]_{-1}^{x} = \frac{x^2}{2}+x-\frac{1}{2}+1$$
$$\therefore F(x) = \frac{x^2}{2}+x+\frac{1}{2}$$

c) when $x \in [0, 1)$
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$
$$= 0 + \int_{-1}^{0} (t+1)\,dt + \int_{0}^{x} (-t+1)\,dt$$
$$= \left[\frac{t^2}{2}+t\right]_{-1}^{0} + \left[-\frac{t^2}{2}+t\right]_{0}^{x}$$
$$= 0-\left(\frac{1}{2}-1\right) + \left(-\frac{x^2}{2}+x\right) - (-0+0)$$
$$= +\frac{1}{2} - \frac{x^2}{2} + x$$

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d) when $x \in [1,\infty)$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt$$

$$= 0 + \frac{1}{2} + \int_0^1 (-t+1)dt + 0$$

$$= \frac{1}{2} + \left(-\frac{t^2}{2}+t\right)_0^1$$

$$= \frac{1}{2} + \left(-\frac{1}{2}+1\right) - 0$$

$$= 1$$

$$F(x) = \begin{cases} 0 & x \le -1 \\ \frac{x^2}{2}+x+\frac{1}{2} & -1 \le x < 0 \\ -\frac{x^2}{2}+x+\frac{1}{2} & 0 \le x < 1 \\ 1 & 1 \le x. \end{cases}$$

(ii) $P(-0.5 \le x \le 0.5)$

$$= F(0.5) - F(-0.5)$$

$$= F(1/2) - F(-1/2) = \left(\frac{-(1/2)^2}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(\frac{(-1/2)^2}{2} - \frac{1}{2} + \frac{1}{2}\right)$$

$$= \left(-\frac{1}{8}+1\right) - \left(\frac{1}{8} - \frac{1}{2} + \frac{1}{2}\right)$$

$$= 1 - \frac{2}{8} = \frac{6}{8} = 0.75$$

---

6) If x is the random variable with distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(x^2+x) & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

then find the (i) The p.d.f, $f(x)$

(ii) $P(0.3 \le x \le 0.6)$

$$f(x) = \frac{d}{dx}(F(x))$$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(2x+1) & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2}(2x+1) & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) $P(0.3 \le X \le 0.6) = F(0.6) - F(0.3)$

$$= \frac{1}{2}(0.6^2 + 0.6) - \frac{1}{2}(0.3^2 + 0.3)$$

$$= \frac{1}{2}[0.36 + 0.6 - 0.09 - 0.3]$$

$$= \frac{1}{2}[0.96 - 0.39]$$

$$= \frac{1}{2}[0.57]$$

$$= 0.285$$

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$$P(0.3 \le X \le 0.6) = \int_{0.3}^{0.6} \frac{1}{2}(2x+1)\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(\frac{[2x+1]^2}{2}\right)_{0.3}^{0.6}$$

(Book answer wrong)

$$= \frac{1}{4\times 2}\left[2.2^2 - 1.6^2\right]$$

$$= \frac{1}{8}[4.84 - 2.56] = \frac{1}{8}[2.28]$$

$$= 0.285$$

Mathematical Expectation

Mean

The expected value or mean or mathematical Expectation of x, denoted by $E(x)$ or μ is

$$E(x) = \begin{cases} \sum x f(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } x \text{ is continuous} \end{cases}$$

$$* \quad E(g(x)) = \begin{cases} \sum_x g(x) f(x) & \text{if } g(x) \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\,dx & \text{if } g(x) \text{ is continuous} \end{cases}$$

$$* \quad E(1) = \begin{cases} \sum_x f(x) = 1 & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} f(x)\,dx = 1 & \text{if } x \text{ is continuous.} \end{cases}$$

variance

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = E(x - E(x))^2 = E(x-\mu)^2$$

properties

(1) $E(ax+b) = aE(x) + b$

(2) $var(ax+b) = a^2 var(x)$

Exercise 11.4

(1) For the random variable x with the given p.m.f. as below, find the mean and variance.

(i) $f(x) = \begin{cases} \frac{1}{10} & x = 2, 5 \\ \frac{1}{5} & x = 0, 1, 3, 4 \end{cases}$

x	0	1	2	3	4	5
$f(x)$	$1/5$	$1/5$	$1/10$	$1/5$	$1/5$	$1/10$
$xf(x)$	0	$1/5$	$\frac{2}{10}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{10}$
$x^2f(x)$	0	$1/5$	$\frac{4}{10}$	$\frac{9}{5}$	$\frac{16}{5}$	$\frac{25}{10}$

$$\text{mean} = \sum_x x f(x) = 0 + \frac{1}{5} + \frac{2}{10} + \frac{3}{5} + \frac{4}{5} + \frac{5}{10}$$

$$E(x) = \frac{8}{5} + \frac{7}{10} = \frac{16+7}{10} = \frac{23}{10}$$ $\boxed{\text{mean} = 2.3}$

$$E(x^2) = \sum_x x^2 f(x) = 0 + \frac{1}{5} + \frac{4}{10} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10}$$

$$= \frac{29}{10} + \frac{26}{5} = \frac{29+52}{10} = \frac{81}{10} = 8.1$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= 8.1 - 2.3^2$$
$$= 8.1 - 5.29$$
$$= 2.81$$

mean $= 2.3$
variance $= 2.81$

(ii) $f(x) = \left\{ \frac{4-x}{6} \quad x = 1, 2, 3 \right.$

x	1	2	3
$f(x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$xf(x)$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{3}{6}$
$x^2f(x)$	$\frac{3}{6}$	$\frac{8}{6}$	$\frac{9}{6}$

$$E(x) = \Sigma x\, f(x) = \frac{3}{6}+\frac{4}{6}+\frac{3}{6} = \frac{\cancel{10}\,5}{\cancel{6}\,3} = 1.667$$

$\boxed{\text{mean} = 1.67}$

$$E(x^2) = \Sigma x^2 f(x) = \frac{3}{6}+\frac{8}{6}+\frac{9}{6} = \frac{20}{6}$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= \frac{20}{6} - \left(\frac{5}{3}\right)^2 = \frac{20}{6} - \frac{25}{9} = \frac{60-50}{18} = \frac{\cancel{10}\,5}{\cancel{18}\,9}$$

$$= 0.56$$

$\boxed{\text{variance} = 0.56}$

mean $= 1.67$, variance $= 0.56$

(14)
$$f(x) = \begin{cases} 2(x-1) & 1<x<2 \\ 0 & \text{otherwise} \end{cases}$$

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$$E(x) = \int_1^2 x f(x)\, dx$$

$$= 2\int_1^2 x(x-1)\, dx = 2\int_1^2 (x^2-x)\, dx$$

$$= 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 = 2\left[\left(\frac{8}{3}-\frac{4}{2}\right)-\left(\frac{1}{3}-\frac{1}{2}\right)\right]$$

$$= 2\left[\frac{8}{3}-\frac{4}{2}-\frac{1}{3}+\frac{1}{2}\right] = 2\left[\frac{7}{3}-\frac{3}{2}\right] = \cancel{2}\left[\frac{14-9}{\cancel{6}\,3}\right]$$

$$E(x) = \frac{5}{3}$$

$$E(x^2) = \int_1^2 x^2 f(x)\, dx$$

$$= \int_1^2 x^2\, 2(x-1)\, dx = 2\int_1^2 (x^3-x^2)\, dx$$

$$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = 2\left[\left(\frac{16}{4}-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right]$$

$$= 2\left[\frac{16}{4}-\frac{8}{3}-\frac{1}{4}+\frac{1}{3}\right] = 2\left[\frac{15}{4}-\frac{7}{3}\right] = \cancel{2}\left[\frac{45-28}{\cancel{12}\,6}\right]$$

$$E(x^2) = \frac{17}{6}$$

$$var(x) = E(x^2) - (E(x))^2$$

$$= \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \frac{1}{18}$$

mean $=\frac{5}{3}$ variance $=\frac{1}{18}$

(iv) $f(x)=\begin{cases}\frac{1}{2}e^{-x/2} & \text{for } x>0\\ 0 & \text{otherwise.}\end{cases}$

change $\left[e^{+x/2}\rightarrow e^{-x/2}\right]$

$$E(x)=\int_0^\infty x f(x)dx$$

$$=\int_0^\infty x^1 \frac{1}{2}e^{-x/2}dx$$

$$\boxed{\int_0^\infty x^n e^{-ax}dx=\frac{n!}{a^{n+1}}}$$

$$=\frac{1}{2}\frac{1!}{(\frac{1}{2})^{1+1}}=\frac{1}{2}2^2 \qquad n=1,\ a=\frac{1}{2}$$

$E(x)=2$

$$E(x^2)=\int_0^\infty x^2 f(x)dx=\int_0^\infty x^2\frac{1}{2}e^{-x/2}dx \qquad n=2 \quad a=\frac{1}{2}$$

$$=\frac{1}{2}\frac{2!}{(\frac{1}{2})^{2+1}}=\frac{1}{\cancel{2}}\cancel{2}\times 2^3$$

$E(x^2)=8$

$$\text{var}(x)=E(x^2)-(E(x))^2$$
$$=8-2^2=8-4$$
$$=4$$

$$\boxed{\begin{gathered}\text{mean}=2\\ \text{variance}=4\end{gathered}}$$

2) Two balls are drawn in succession without replacement from an urn containing four red balls and 3 black balls. Let x be the possible outcomes drawing red balls. Find the probability mass function and mean of x.

RED	BLACK	TOTAL
4	3	7

Let x be the random variable denotes the number of red balls

$x=0,1,2$

$$f(0)=P(x=0)=\frac{4C_0\,3C_2}{7C_2}=\frac{1\times 3}{\frac{7\times 6}{1\times 2}}=\frac{3}{21}=\frac{1}{7}$$

$$f(1)=P(x=1)=\frac{4C_1\times 3C_1}{7C_2}=\frac{4\times 3}{21}=\frac{4}{7}$$

$$f(2)=P(x=2)=\frac{4C_2\times 3C_0}{7C_2}=\frac{6\times 1}{21}=\frac{2}{7}$$

probability mass function is

x	0	1	2
$P(X=x)$ $f(x)$	$1/7$	$4/7$	$2/7$

$E(X) = \sum_x x\, f(x)$

$= 0\left(\frac{1}{7}\right) + 1\left(\frac{4}{7}\right) + 2\left(\frac{2}{7}\right)$

$= 0 + \frac{4}{7} + \frac{4}{7}$

mean $= \frac{8}{7}$

3) If μ and σ^2 are the mean and variance of the discrete random variable X and $E(X+3) = 10$ and $E(X+3)^2 = 116$ find μ and σ^2

$E(X+3) = 10$ $\quad E(aX+b) = aE(X)+b$

$E(X)+3 = 10$

$E(X) = 10-3$ $\quad \boxed{\mu = 7}$

$E(X) = 7$

$E(X+3)^2 = 116$

$E(X^2+6X+9) = 116$ G. Karthikeyan Thiruvarur DT

$E(X^2)+6E(X)+9 = 116$

$E(X^2)+6(7)+9 = 116$

$E(X^2)+51 = 116$

$E(X^2) = 116-51$

$E(X^2) = 65$

$Var(X) = E(X^2) - (E(X))^2 = 65 - 7^2 = 65 - 49 = 16$

$\boxed{\mu = 7 \quad \sigma^2 = 16}$

4) Four fair coins are tossed once, Find the probability mass function, mean and variance for number of heads occurred.

$n(S) = 2^4 = 16$

Let X be the random variable denotes number of heads $\quad X = 0, 1, 2, 3, 4$

$P(X=0) = \frac{4C_0}{16} = \frac{1}{16}$, $\quad P(X=1) = \frac{4C_1}{16} = \frac{4}{16}$

$P(X=2) = \frac{4C_2}{16} = \frac{6}{16}$, $\quad P(X=3) = \frac{4C_3}{16} = \frac{4}{16}$, $\quad P(X=4) = \frac{4C_4}{16} = \frac{1}{16}$

p.m.f is

x	0	1	2	3	4
$f(x)$ or $P(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$x f(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$
$x^2 f(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$

$$E(x) = \sum_x x f(x) = 0 + \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2$$

$\boxed{\text{mean} = 2}$

$$E(x^2) = \sum_x x^2 f(x) = 0 + \frac{4}{16} + \frac{24}{16} + \frac{36}{16} + \frac{16}{16}$$

$$= \frac{80}{16} = 5$$

$$\text{var}(x) = E(x^2) - (E(x))^2 = 5 - 2^2 = 5 - 4 = 1$$

$\boxed{\begin{array}{l}\text{mean} = 2 \\ \text{variance} = 1\end{array}}$

5) A commuter train punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let x denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of x is $f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$

obtain and interpret the expected value of the random variable x.

given x be the random variable denotes the waiting time. x is continuous on $(0, 30)$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{30} x \frac{1}{30}\,dx$$

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$$= \frac{1}{30}\left[\frac{x^2}{2}\right]_0^{30} = \frac{1}{30 \times 2}\left[30^2 - 0\right]$$

$$= \frac{30 \times 30}{30 \times 2} = 15$$

$E(x) = 15$

expected value of waiting time $= 15$ minutes.

6) The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function $f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

Find the expected life of this electronic equipment.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x^1\, 3e^{-3x}\,dx$$

$\boxed{\int_0^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}}$

$n = 1$
$a = 3$

$$= 3\,\frac{1!}{3^{1+1}} = \frac{3^1}{9 \cdot 3}$$

$E(x) = 1/3$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$=\int_0^{\infty} x^2 3e^{-3x}\,dx \qquad \begin{matrix} n=2 \\ a=3 \end{matrix}$$

$$=3\,\frac{2!}{3^{2+1}}=\frac{\cancel{3}\times 2}{3^{\cancel{3}2}}=\frac{2}{9}$$ (not asked in question but answer is given)

7) The probability density function of the random variable X is given by $f(x)=\begin{cases} 16xe^{-4x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$

find the mean and variance of X.

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx \qquad \boxed{\int_0^{\infty} x^n e^{-ax}\,dx=\frac{n!}{a^{n+1}}}$$

$$=\int_0^{\infty} x\,16xe^{-4x}\,dx=16\int_0^{\infty} x^2 e^{-4x}\,dx \qquad n=2 \quad a=4$$

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$$=16\,\frac{2!}{4^{2+1}}=\frac{16\times 2}{4\times 4\times 4}=\frac{1}{2} \qquad \boxed{\text{mean}=\frac{1}{2}}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=16\int_0^{\infty} x^3 e^{-4x}\,dx \qquad \begin{matrix} n=3 \\ a=4 \end{matrix}$$

$$=16\,\frac{3!}{4^4}=\frac{16\times 6}{16\times 16}=\frac{3}{8}$$

$$\text{var}(X)=E(X^2)-(E(X))^2=\frac{3}{8}-\left(\frac{1}{2}\right)^2=\frac{3}{8}-\frac{1}{4}=\frac{3-2}{8}=\frac{1}{8}$$

$$\boxed{\text{variance}=\frac{1}{8}}$$

8) A lottery with 600 tickets gives one prize of ₹200 four prizes of ₹100, and six prizes of ₹50. If the ticket costs is ₹2. find the expected winning amount of ticket.

$n(S)=600$

Let X be the random variable denote the amount of winning

$$X=200, 100, 50, 0$$

$P(X=200)=\frac{1}{600}$ $\quad P(X=100)=\frac{4}{600}$ $\quad P(X=50)=\frac{6}{600}$

$P(X=0)=\frac{589}{600}$

x	200	100	50	0
$P(X=x)$ or $f(x)$	$\frac{1}{600}$	$\frac{4}{600}$	$\frac{6}{600}$	$\frac{589}{600}$

$E(x) = \sum_x x f(x)$

$= 200\left(\frac{1}{600}\right) + 100 \times \frac{4}{600} + \frac{50 \times 6}{600} + 0\ \frac{589}{600}$

$= \frac{200+400+300}{600} = \frac{900}{600} = \frac{3}{2} = 1.50$

Expected amount winning $= ₹1.50$

one Ticket cost $= ₹2.00$

profit (diffence) $= 1.50 - 2.00 = -0.50$

i.e. Loss $= 0.50$ Rupees.

The Bernoulli distribution

Let x be a random variable follows Bernoullis trial $x(\text{success}) = 1$, $x(\text{failure}) = 0$ such that

$$f(x) = \begin{cases} p & x=1 \\ q=1-p & x=0 \end{cases} \quad \text{where } 0<p<1$$

x is called Bernoulli R.V.

$f(x)$ is called Bernoulli's distribution.

* mean $= p$ variance $= pq$

Binomial distribution

The binomial random variable x, equals no. of success with probability p, $q = 1-p$ for a failure, p.m.f is

$$f(x) = nC_x\, p^x (1-p)^{n-x}, \quad x = 0,1,2,\cdots n.$$

$$\boxed{f(x) = nC_x\, p^x q^{n-x}},\ x = 0,1,2,\cdots n$$

$p+q=1$

mean $= np$ variance $= npq$

Exercise 11.5

1) Compute $P(X=k)$ for the binomial distribution $B(n,p)$ where (i) $n=6$, $p=\frac{1}{3}$, $k=3$

$P(X=x) = nC_x\, p^x (1-p)^{n-x}$

$P(X=k) = 6C_k \left(\frac{1}{3}\right)^k \left(1-\frac{1}{3}\right)^{6-k}$

$k=3$ $P(X=3) = 6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{6-3}$

$= \frac{6\times5\times4}{1\times2\times3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = \frac{20 \times 2^3}{3^6}$

$$p(x=3) = \frac{20\times 8}{9\times 9\times 9} = \frac{160}{729}$$

(ii) $n=10$, $p=\frac{1}{5}$, $k=4$

$$p(x=k) = nC_k\, p^k (1-p)^{n-k}$$

$$p(x=4) = 10C_4 \left(\tfrac{1}{5}\right)^4 \left(1-\tfrac{1}{5}\right)^{10-4}$$

$$= \frac{10\times 9\times 8\times 7}{1\times 2\times 3\times 4} \left(\tfrac{1}{5}\right)^4 \left(\tfrac{4}{5}\right)^6$$

$$p(x=4) = 210 \left(\tfrac{1}{5}\right)^4 \left(\tfrac{4}{5}\right)^6$$

(iii) $n=9$, $p=\frac{1}{2}$, $k=7$

$$p(x=k) = nC_k\, p^k (1-p)^{n-k}$$

$$p(x=7) = 9C_7 \left(\tfrac{1}{2}\right)^7 \left(1-\tfrac{1}{2}\right)^{9-7}$$

$$= 9C_2 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^2$$

$$= \frac{9\times 8}{1\times 2} \left(\tfrac{1}{2^9}\right) = \frac{9}{2^7} = \frac{9}{128}$$

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2) The probability that Mr. G hits a target at any trial is $\frac{1}{4}$. suppose he tries at the target 10 times. Find the probability that he hits the target (i) exactly 4 times (ii) atleast one time.

$n=10$ Let x be the random variable denote number of hits $x = 0, 1, 2, \dots 10$.

x follows the Binomial distribution.

$p =$ probability of success $= \frac{1}{4}$

$p = \frac{1}{4}$ $\quad q = 1-p = 1-\frac{1}{4} = \frac{3}{4}$

$f(x)$ or $p(x=x) = nC_x\, p^x q^{n-x}$. $\quad x = 0, 1, \dots 10$

$p(x=x)$ or $f(x) = 10C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$.

(i) exactly 4 times

$$f(4) = 10C_4 \left(\tfrac{1}{4}\right)^4 \left(\tfrac{3}{4}\right)^{10-4}$$

$$= 10C_4 \left(\tfrac{1}{4}\right)^4 \left(\tfrac{3}{4}\right)^6$$

(ii) atleast one time $= P(x \geq 1)$

$$= 1 - P(x<1)$$

$$= 1 - P(x=0)$$

$$= 1 - 10C_0 \left(\tfrac{1}{4}\right)^0 \left(\tfrac{3}{4}\right)^{10-0}$$

$$= 1 - \left(\tfrac{3}{4}\right)^{10}$$

3) Using binomial distribution find the mean and variance of x for the following experiments.

(i) A pair coin is tossed 100 times, and x denote the number of heads.

$n=100$ $\quad p=$ probability of getting head

$p=\frac{1}{2}$ $\quad q=1-p=\frac{1}{2}$

mean $= np = \overset{50}{\cancel{100}} \times \frac{1}{\cancel{2}} = 50$

variance $= npq = \overset{50}{\cancel{100}} \frac{1}{\cancel{2}} \cdot \frac{1}{2} = 25$

(ii) A pair die is tossed 240 times, and x denotes the number of times that 4 appeared.

$n=240$ $\quad p=$ probability of getting 4

$p=\frac{1}{6}$ $\quad q=1-p=1-\frac{1}{6}=\frac{5}{6}$

mean $= np = \overset{40}{\cancel{240}} \times \frac{1}{\cancel{6}} = 40$

variance $= npq = \overset{40}{\cancel{240}} \times \frac{1}{\cancel{6}} \times \frac{5}{6} = \frac{200}{6} = \frac{100}{3}$

④ The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

$n=5$
$p=\frac{3}{4}$

Let x be the random variable denotes no. of survive components.

$x=0,1,2,3,4,5$

$p=$ probability of a components survive after test $\qquad X \sim B(5, \frac{3}{4})$

$p=\frac{3}{4}$ $\quad q=1-p$ $\quad q=1-\frac{3}{4}=\frac{1}{4}$

$P(X=x) = nC_x \, p^x q^{n-x}$, $\quad x=0,1,\ldots,5$

$P(X=x) = 5C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{5-x}$

P(exactly 3 survive) $= P(X=3)$

$= 5C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^{5-3}$

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$= \frac{5\times \overset{2}{\cancel{4}}}{1\times \cancel{2}} \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2$

$= 10 \, \frac{3^3}{4^3} \, \frac{1}{4^2} = \frac{270}{1024}$

⑤ A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective device rate is 5%.

The inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be (i) atleast one defective item (ii) exactly 2 defective items.

$n=10$ $\quad$ x be the random variable denotes no. of defective items $\quad x \sim B(10, 0.05)$

p = probability of a defective item

$p = 5\%$

$p = 0.05$ $\qquad p(X=x) = nC_x \, p^x q^{n-x}$

$q = 1-p = 0.95$ $\qquad \boxed{P(X=x) = 10C_x \,(0.05)^x (0.95)^{10-x}}$

(i) atleast one defective

$= P(X \geq 1)$

$= 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - 10C_0 \,(0.05)^0 (0.95)^{10-0}$

$= 1 - (0.95)^{10}$

(ii) exactly 2 defective

$= P(X=2)$

$= 10C_2 \,(0.05)^2 (0.95)^{10-2}$

$= 10C_2 \,(0.05)^2 . (0.95)^8$

8) If the probability that a fluoresent light has a useful life of atleast 600 hours is 0.9, find the probability that among 12 such lights will have a useful life of atleast 600 hours.

(i) exactly 10 " " "

(ii) atleast 11 " " "

(iii) atleast 2 will not " "

$n=12$ $\quad$ p = probability of useful life

$p = 0.9$ $\qquad q = 1-p = 1-0.9 = 0.1$

x be the random variable denotes useful life of atleast 600 hours of a light.

$X \sim B(12, 0.9)$

$P(X=x) = nC_x \, p^x q^{n-x}$

$\boxed{P(X=x) = 12C_x \,(0.9)^x (0.1)^{12-x}}$ $\qquad x = 0, 1, \dots 12.$

(i) exactly 10.

$P(X=10) = 12C_{10} \,(0.9)^{10} (0.1)^{12-10}$

$= 12C_{10} \,(0.9)^{10} (0.1)^2$

(ii) atleast 11

$P(X \geq 11) = P(X=11) + P(X=12)$

$= 12C_{11} \,(0.9)^{11} (0.1)^{12-11} + 12C_{12} \,(0.9)^{12} (0.1)^{12-12}$

$= 12 \,(0.9)^{11} (0.1) + 1 (0.9)^{12} (1)$

$= (0.9)^{11}[12\times 0.1 + 0.9]$

$= (0.9)^{11}[1.2+0.9]$

$P(X\geq 11) = 2.1\,(0.9)^{11}$

(iii) atleast 2 will not have a useful life of atleast 600 hours

$= P(X<11)$

$P(\text{atleast 2 will not}) = 1-P(X\geq 11)$ [each probability having 2 or more depending]

$= 1-(2.1)(0.9)^{11}$

⑦ The mean and standard deviation of a binomial variate X are respectively 6 and 2.

Find (i) The probability mass function (ii) $P(X=3)$ (iii) $P(X\geq 2)$

mean = 6 $\quad$ S.D = 2 $\quad$ $S.D^2 = 4$

$np = 6$ — ① $\quad$ variance = 4

$npq = 4$ — ②

$\frac{②}{①} \Rightarrow \frac{npq}{np} = \frac{4}{6}$ $\quad$ $\boxed{q=\frac{2}{3}}$ $\quad$ $p = 1-q = 1-\frac{2}{3}$ $\quad$ $\boxed{p=\frac{1}{3}}$

$① \Rightarrow n\frac{1}{3} = 6$

$\boxed{n=18}$

(i) probability mass function

$P(X=x) = nC_x\, p^x q^{n-x}$

$$\boxed{P(X=x) = 18C_x \left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{18-x}}$$

$x = 0,1,2\ldots 18$

(ii) $P(X=3) = 18C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{18-3}$

$= 18C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{15}$

(iii) $P(X\geq 2) = 1-P(X<2)$

$= 1-[P(X=0)+P(X=1)]$

$= 1-\left[18C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{18-0} + 18C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{18-1}\right]$

$= 1-\left[1\,(1)\left(\frac{2}{3}\right)^{18} + \cancel{18}^{6}\,\frac{1}{\cancel{3}}\left(\frac{2}{3}\right)^{17}\right]$

$= 1-\left(\frac{2}{3}\right)^{17}\left[\frac{2}{3}+6\right]$

$= 1-\frac{20}{3}\left(\frac{2}{3}\right)^{17}$

8) If $X \sim B(n,p)$ such that $4P(X=4) = P(X=2)$ and $n=6$. Find the distribution, mean and S.D.

$\boxed{P(X=x) = nC_x p^x q^{n-x}}$

$n=6$ $\quad 4P(X=4) = P(X=2)$

$$4(6C_4 (p)^4 (q)^{6-4}) = 6C_2 \, p^2 (q)^{6-2}$$

$$4 \times 6C_2 \, p^4 q^2 = 6C_2 \, p^2 q^4$$

$$4 \frac{p^4}{p^2} = \frac{q^4}{q^2}$$

$$4(p^2) = q^2$$

$$4p^2 = (1-p)^2$$

$$4p^2 = 1 + p^2 - 2p$$

$$4p^2 - 1 - p^2 + 2p = 0$$

$$3p^2 + 2p - 1 = 0$$

$$(p+1)(3p-1) = 0$$

$(p+\frac{3}{3})(p-\frac{1}{3})=0$

$p = -1 \quad 3p = 1$

$p \neq -1 \quad \boxed{p = 1/3}$ $\quad q = 1-p = 1-\frac{1}{3}$ $\quad \boxed{q = \frac{2}{3}}$

The distribution is

$$P(X=x) = nC_x \, p^x q^{n-x}$$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} \quad x = 0, 1, \ldots 6$$

mean $= np = 6 \times \frac{1}{3} = 2$

variance $= npq = 6 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{3}$

standard deviation $= \sqrt{npq} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$

9) In a Binomial distribution consisting of 5 independent trails, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance.

$n=5$ given $P(X=1) = 0.4096$

$P(X=2) = 0.2048$

$p(X=x) = nC_x p^x q^{n-x}$

$P(X=1) = 5C_1 \, p^1 q^{5-1} = 0.4096 \Rightarrow 5pq^4 = 0.4096$ — ①

$P(X=2) = 5C_2 \, p^2 q^{5-2} = 0.2048 \Rightarrow 10p^2q^3 = 0.2048$ — ②

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$$① \div ② \Rightarrow \frac{5pq^4}{10p^2q^3} = \frac{0.4096}{0.2048}$$

$\frac{q}{2p} = 2$

$1-p = 4p$

$1 = 5p$

$p = 1/5$

$q = 1-p = 1-\frac{1}{5} = \frac{4}{5}$

mean $= np = 5 \times \frac{1}{5} = 1$

variance $= npq = 1 \times \frac{4}{5} = \frac{4}{5}$

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(Book answer wrong)

(checking) $n=5 \quad p=\frac{1}{5} \quad q=\frac{4}{5}$

$$P(X=1) = 5C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{5-1}$$

$$= \cancel{5}\,\frac{1}{\cancel{5}}\left(\frac{4}{5}\right)^4 = \frac{4^4}{5^4} = \frac{256}{25\times25}\times\frac{16}{4\times4}$$

$$P(X=1) = \frac{4096}{10000} = 0.4096$$

correct answer

mean $= 1$ variance $= \frac{4}{5}$

Need suggestions

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