# 9. APPLICATIONS OF INTEGRATION

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## Riemann Integral

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = f(\xi_1)(x_1-x_0) + f(\xi_2)(x_2-x_1) + \cdots + f(\xi_n)(x_n - x_{n-1}) \quad \text{--- ①}$$

is called Riemann sum of $f(x)$.

* $\int_a^b f(x)dx = \lim_{n\to\infty}$ and $\max(x_i - x_{i-1}) \to 0$ $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$

 is known as left-end Rule

* $\int_a^b f(x)dx = \lim_{n\to\infty}$ and $\max(x_i - x_{i-1}) \to 0$ $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$

 is known as Right-end Rule.

$\int_a^b f(x)dx = \lim_{n\to\infty}$ and $\max(x_i - x_{i-1}) \to 0$ $\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1})$

is known as mid-point Rule.

## Exercise 9.1

1) Find an approximate value of $\int_1^{1.5} x\,dx$ by applying the left end Rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$\boxed{n=5}$

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, $x_4 = 1.4$

$x_5 = 1.5$ $\qquad \Delta x = 1.1 - 1 = 0.1$

left end Rule $\int_a^b f(x)\,dx = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$$\int_1^{1.5} x\,dx = (f(1) + f(1.1) + f(1.2) + f(1.3) + f(1.4))\,0.1$$

$$= (1 + 1.1 + 1.2 + 1.3 + 1.4)\,0.1$$

$$= 6 \times 0.1$$

$$\int_1^{1.5} x\,dx \approx 0.6$$

$$\int_1^{1.5} x\,dx = \left[\frac{x^2}{2}\right]_1^{1.5} = \frac{2.25-1}{2} = \frac{1.25}{2}$$

  

2) Find an approximate value of $\int_1^{1.5} x^2\,dx$ by applying the right-end rule with the partitions $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$\boxed{n=5}$

$x_0 = 1, \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3 \quad x_4 = 1.4 \quad x_5 = 1.5$

$$\Delta x = 1.1 - 1.0 = 0.1$$

Right end Rule

$$\text{i.e.}\ \int_a^b f(x)\,dx = \big(f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$

$$\int_1^{1.5} x^2\,dx \simeq S = \big(f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5)\big)0.1$$

$$= \left(1.1^2 + 1.2^2 + 1.3^2 + 1.4^2 + 1.5^2\right) 0.1$$

$$= (1.21 + 1.44 + 1.69 + 1.96 + 2.25)\, 0.1$$

$$= 8.55 \times 0.1$$

$$\int_1^{1.5} x^2\,dx \simeq S = 0.855.$$

$$\int_1^{1.5} x^2\,dx = \left[\frac{x^3}{3}\right]_1^{1.5} = \frac{1.5^3 - 1}{3} = \frac{3.375 - 1}{3} = \frac{2.375}{3} = 0.7916.$$

3) Find an approximate value of $\int_1^{1.5} (2-x)\,dx$ by applying the mid point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$$n = 5$$

$x_0 = 1, \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3 \quad x_4 = 1.4 \quad x_5 = 1.5$

$$\Delta x = 1.1 - 1.0 = 0.1.$$

$$f(x) = 2 - x.$$

mid point rule

$$\int_a^b f(x)\,dx = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\Delta x$$

$$S = \left[f\left(\frac{1+1.1}{2}\right) + f\left(\frac{1.1+1.2}{2}\right) + f\left(\frac{1.2+1.3}{2}\right) + f\left(\frac{1.3+1.4}{2}\right) + f\left(\frac{1.4+1.5}{2}\right)\right](0.1)$$

$$S = \left[f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45)\right](0.1)$$

$S = ((2-1.05)+(2-1.15)+(2-1.25)+(2-1.35)+(2-1.45))(0.1)$

$= (0.95+0.85+0.75+0.65+0.55)\,0.1$

$= (3.75)\,0.1$

$S = 0.375$ $\quad \int_1^{1.5}(2-x)\,dx \simeq 0.375$

$$= \left[\frac{-(2-x)^2}{2}\right]_1^{1.5}$$

$$= -\frac{0.25}{2}+\frac{1}{2} = \frac{0.75}{2}$$

$$= 0.375$$

---x---

. limit formula to Evaluate $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$$

$a=0$, $b=1$

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

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## Exercise 9.2

1) Evaluate the following integrals as the limit of sums.

(i) $\int_0^1 (5x+4)\,dx$

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

$f(x) = 5x+4$

$f\left(\frac{r}{n}\right) = \frac{5r}{n}+4$

$$\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \sum_{r=1}^{n} \frac{5r}{n}+4$$

$$= \sum_{r=1}^{n}\frac{5r}{n} + \sum_{r=1}^{n} 4$$

$$\boxed{\sum_{r=1}^{n} r = \frac{n(n+1)}{2}}$$

$$= \frac{5}{n}\sum_{r=1}^{n} r + 4\sum_{r=1}^{n}(1)$$

$$= \frac{5}{n}(1+2+3+\cdots+n) + 4(1+1+\cdots 1)$$

$$= \frac{5}{n}\,\frac{n(n+1)}{2} + 4n$$

$$\therefore \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \frac{5}{2}(n+1) + 4n$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{5}{2}(n+1) + 4n\right)$$

$$= \lim_{n\to\infty} \frac{5}{2}\left(1+\frac{1}{n}\right) + 4$$

$$= \frac{5}{2}(1+0) + 4$$

$$= \frac{5}{2} + 4 = \frac{13}{2}$$

$$\therefore \int_0^1 (5x+4)\,dx = \frac{13}{2}$$

Checking

$$= \left(\frac{5x^2}{2} + 4\right)_0^1$$

$$= \frac{5}{2} + 4 = \frac{13}{2}$$

(ii) $\int_1^2 (4x^2-1)\,dx.$

$a = 1 \quad b = 2 \qquad f(x) = 4x^2 - 1$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + (b-a)\frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2-1}{n} \sum_{r=1}^{n} f\left(1 + (2-1)\frac{r}{n}\right)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(1+\frac{r}{n}\right)$$

$$f(x) = 4x^2 - 1$$

$$f\left(1+\frac{r}{n}\right) = 4\left(1+\frac{r}{n}\right)^2 - 1$$

$$= 4\left(1^2 + \frac{r^2}{n^2} + \frac{2r}{n}\right) - 1.$$

$$\sum_{r=1}^{n} f\left(1+\frac{r}{n}\right) = 4\sum_{r=1}^{n}\left(1+\frac{r^2}{n^2}+\frac{2r}{n}\right) - \sum_{r=1}^{n} 1$$

$$= 4\sum_{r=1}^{n} 1 + \frac{4}{n^2}\sum_{r=1}^{n} r^2 + \frac{8}{n}\sum_{r=1}^{n} r - n.$$

$$= 4n + \frac{\cancel{4}^{2}}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{3\cancel{6}} + \frac{\cancel{8}^{4}}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} - n$$

$$= 3n + \frac{2}{3}\frac{1}{n}(n+1)(2n+1) + 4(n+1)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(1+\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left(3n + \frac{2}{3n}(n+1)(2n+1) + 4(n+1)\right)$$

$$= \lim_{n\to\infty}\left(3+\frac{2}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)\right)$$

$$= 3+\frac{2}{3}(1+0)(2+0)+4(1+0))$$

$$= 3+\frac{4}{3}+4 = 7+\frac{4}{3}$$

$$\int_1^2 (4x^2-1)dx = \frac{25}{3}$$

checking

$$\int_1^2 (4x^2-1)dx = \left[\frac{4x^3}{3}-x\right]_1^2$$

$$=\left(\frac{4}{3}8-2\right)-\left(\frac{4}{3}-1\right)$$

$$= \frac{28}{3}-1 = \frac{25}{3}$$

## Fundamental Theorems of Integral calculus and their Applications

### * First Fundamental Theorem of Integral calculus

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and $F(x) = \int_a^x f(u)du$, $a<x<b$ then $\frac{d}{dx}F(x) = f(x)$.

### * Second Fundamental Theorem of Integral calculus

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$. Then $\int_a^b f(x)dx = F(b)-F(a)$.

### Properties

1) $\int_a^b f(x)dx = \int_a^b f(u)du$, $a<b$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $a<c<b$

4) $\int_a^b (\alpha f(x)+\beta g(x))dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx$.

where $\alpha,\beta$ are constants

5) $\int_a^b f(x)dx = \int_c^d f(g(u))\frac{dg(u)}{du}du$ where $g(c)=a$ $g(d)=b$ $x=g(u)$

6) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

7) $\int_0^{2a} f(x)dx = \int_0^a (f(x)+f(2a-x))dx$

8) $f(x)$ is an even function $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

9). $f(x)$ is an odd function $\int_{-a}^{a} f(x)dx = 0$

10). If $f(2a-x)=f(x)$ then $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$.

11) If $f(2a-x)=-f(x)$ then $\int_0^{2a} f(x)dx = 0$

12). $\int_0^a x f(x)dx = \frac{a}{2}\int_0^a f(x)dx$ if $f(a-x)=f(x)$

Exercise 9.3.

1) Evaluate the following definite Integrals.

(i) $\int_3^4 \frac{dx}{x^2-4}$

$$I = \int_3^4 \frac{dx}{x^2-2^2}$$

$$\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c}$$

$a=2$

$$= \left[\frac{1}{2(2)}\log\left|\frac{x-2}{x+2}\right|\right]_3^4$$

$$= \frac{1}{4}\left[\log\left|\frac{4-2}{4+2}\right| - \log\left|\frac{3-2}{3+2}\right|\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{2}{6}\right) - \log\left(\frac{1}{5}\right)\right] = \frac{1}{4}\log\left(\frac{2}{6_3}\times\frac{5}{1}\right)$$

$$= \frac{1}{4}\log\frac{5}{3}$$

(ii) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$$I = \int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

$$x^2+2x+5 = x^2+2x+1^2+4 = (x+1)^2+4 = (x+1)^2+2^2$$

$$I = \int_{-1}^{1} \frac{dx}{2^2+(x+1)^2}$$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$a=2 \quad x \to x+1$

$$I = \left[\frac{1}{2}\tan^{-1}\frac{x+1}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{1+1}{2} - \tan^{-1}\frac{-1+1}{2}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{2}{2} - \tan^{-1}\frac{0}{2}\right] = \frac{1}{2}\left[\pi/4 - 0\right]$$

$$= \frac{\pi}{8}$$

(iii) $\int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\, dx$

$$I = \int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\, dx$$

put $x = \sin t$ $\quad x=0 \quad t=0$

$\frac{dx}{dt} = \cos t$ $\quad x=1 \quad t=\pi/2$

| $x$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | $\pi/2$ |

$dx = \cos t\, dt$

$$I = \int_{0}^{\pi/2} \sqrt{\frac{1-\sin t}{1+\sin t} \times \frac{(1-\sin t)}{(1-\sin t)}}\, dt(\cos t)$$

$$= \int_{0}^{\pi/2} \sqrt{\frac{(1-\sin t)^2}{1-\sin^2 t}}\, \cos t\, dt$$

$$= \int_{0}^{\pi/2} \sqrt{\frac{(1-\sin t)^2}{\cos^2 t}}\, \cos t\, dt$$

$$= \int_{0}^{\pi/2} \frac{1-\sin t}{\cos t}\, \cos t\, dt$$

$$= (t+\cos t)_{0}^{\pi/2}$$

$$= \left(\pi/2 + \cos\pi/2\right) - (0+\cos 0)$$

$$= \pi/2 - 1$$

(iv) $\int_0^{\pi/2}\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$I=\int_0^{\pi/2}\left(\frac{1+\sin x}{1+\cos x}\right)dx$$

$$=\int_0^{\pi/2}\frac{1+2\sin x/2\cos x/2}{2\cos^2 x/2}\,dx$$

$$=\int_0^{\pi/2}\left(\frac{1}{2\cos^2 x/2}+\frac{\cancel{2}\sin x/2\cos x/2}{\cancel{2}\cos^2 x/2}\right)dx$$

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$$=\int_0^{\pi/2}\left(\frac{1}{2}\sec^2 x/2+\tan x/2\right)dx$$

$$=\frac{1}{2}\left[\frac{1}{1/2}\tan x/2+\frac{1}{1/2}\log(\sec x/2)\right]_0^{\pi/2}$$

$$=\frac{2}{2}\left[\left(\tan\pi/4+\log\sec\pi/4\right)-\left(\tan 0+\log 1\right)\right]$$

$$=\left[1+\log\sqrt{2}\right]$$ (Book answer wrong)

(v) $\int_0^{\pi/2}\sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

$$I=\int_0^{\pi/2}\sqrt{\cos\theta}\,\sin^3\theta\,d\theta$$

$$=\int_0^{\pi/2}\sqrt{\cos\theta}\,\sin^2\theta\,\sin\theta\,d\theta$$

$$=\int_0^{\pi/2}\sqrt{\cos\theta}\,(1-\cos^2\theta)\,\sin\theta\,d\theta$$

put $\cos\theta=t$

$-\sin\theta=\frac{dt}{d\theta}$

$\sin\theta\,d\theta=-dt$

$\theta=0\Rightarrow t=\cos 0=1$

$\theta=\pi/2\Rightarrow t=\cos\pi/2=0$

| $\theta$ | 0 | $\pi/2$ |
|---|---|---|
| $t$ | 1 | 0 |

$$I=\int_1^0\sqrt{t}\,(1-t^2)(-dt)$$

$$=\int_0^1\left(\sqrt{t}-\sqrt{t}\,t^2\right)dt=\int_0^1\left(t^{1/2}-t^{5/2}\right)dt$$

$$=\left[\frac{t^{3/2}}{3/2}-\frac{t^{7/2}}{7/2}\right]_0^1=\left(\frac{2}{3}-\frac{2}{7}\right)-0=\frac{14-6}{21}=\frac{8}{21}$$

(vi) $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

$$I = \int_0^1 \frac{(1-x^2)}{(1+x^2)^2}\,dx$$

put $x = \tan t$ $\quad x=0 \quad \tan t = 0 \Rightarrow t = 0$

$\frac{dx}{dt} = \sec^2 t \quad x=1 \quad \tan t = 1 \Rightarrow t = \pi/4$

$dx = \sec^2 t\,dt$

| $x$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | $\pi/4$ |

$$I = \int_0^{\pi/4} \frac{1-\tan^2 t}{(1+\tan^2 t)^2}\sec^2 t\,dt$$

$$= \int_0^{\pi/4} \frac{1-\tan^2 t}{(\sec^2 t)^{\cancel{2}}}\cancel{\sec^2 t}\,dt$$

$$= \int_0^{\pi/4}\left(\frac{1}{\sec^2 t} - \frac{\sin^2 t}{\cos^2 t}\times\frac{1}{\sec^2 t}\right)dt$$

$$= \int_0^{\pi/4}\left(\cos^2 t - \frac{\sin^2 t}{\cancel{\cos^2 t}}\times\cancel{\cos^2 t}\right)dt$$

$$= \int_0^{\pi/4}\cos 2t\,dt = \left[\frac{1}{2}\sin 2t\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sin 2\tfrac{\pi}{4} - \sin 0\right]$$

$$= \frac{1}{2}[1-0]$$

$$= 1/2$$

② Evaluate the following integrals using properties of integration.

(i) $I = \int_{-\pi}^{\pi} x\cos\left(\frac{e^x-1}{e^x+1}\right)dx$

$$f(x) = x\cos\left(\frac{e^x-1}{e^x+1}\right)$$

$$f(-x) = -x\cos\left(\frac{e^{-x}-1}{e^{-x}+1}\right)$$

$$= -x\cos\left(\frac{1-e^x/e^x}{1+e^x/e^x}\right) \qquad \boxed{\cos(-\theta) = \cos\theta}$$

$$= -x\cos\left(-\frac{(e^x-1)}{e^x+1}\right)$$

$$f(-x) = -x\cos\left(\frac{e^x-1}{e^x+1}\right)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function

$$\therefore \int_{-a}^{a} f(x)\,dx = 0$$

$$\therefore I = 0$$

(ii) $\displaystyle\int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x + 1)\,dx$

$$I = \int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x + 1)\,dx \quad \text{--- ①}$$

$f(x) = x^5 + x\cos x + \tan^3 x + 1$

$a = \pi/2 \quad b = -\pi/2 \qquad a+b-x = \cancel{\pi/2} - \cancel{\pi/2} - x = -x$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$I = \int_{-\pi/2}^{\pi/2} f(-x)\,dx$$

$$I = \int_{-\pi/2}^{\pi/2} (-x^5 - x\cos x - \tan^3 x + 1)\,dx \quad \text{--- ②}$$

$$① + ② \Rightarrow \quad I + I = \int_{-\pi/2}^{\pi/2} (\cancel{x^5} + \cancel{x\cos x} + \cancel{\tan^3 x} + 1 - \cancel{x^5} - \cancel{x\cos x} - \cancel{\tan^3 x} + 1)\,dx$$

$$2I = 2\int_{-\pi/2}^{\pi/2} dx$$

$$2I = 2\left[x\right]_{-\pi/2}^{\pi/2} = 2\left[\pi/2 + \pi/2\right]$$

$$2I = 2\pi$$

$$I = \pi$$

(iii) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$

$$I = \int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$$

$$I=\int_{-\pi/4}^{\pi/4}\left[\frac{1-\cos 2x}{2}\right]dx$$

$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{4}-\frac{1}{2}\sin 2\pi/4\right)-\left(-\pi/4-\frac{1}{2}\sin 2(-\pi/4)\right)\right]$$

$$=\frac{1}{2}\left[\left(\pi/4-\frac{1}{2}\right)-\left(-\pi/4+\frac{1}{2}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}-\frac{1}{2}-\frac{1}{2}\right]=\frac{1}{2}\left[\frac{2\pi}{4}-1\right]$$

$$=\pi/4-1/2$$

(iv) $\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$

$$I=\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$f(x)=\log\left(\frac{3+\cos x}{3-\cos x}\right)$$

$$f(2\pi-x)=\log\left(\frac{3+\cos(2\pi-x)}{3-\cos(2\pi-x)}\right)=\log\left(\frac{3+\cos x}{3-\cos x}\right)$$

$$f(2\pi-x)=f(x)$$

$$\boxed{\int_0^a x f(x)dx=\frac{a}{2}\int_0^a f(x)dx \quad \text{if } f(a-x)=f(x)}$$

$a=2\pi$

$$I=\frac{2\pi}{2}\int_0^{2\pi}\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$I=\pi\, 2\int_0^{\pi}\log\left(\frac{3+\cos x}{3-\cos x}\right)dx \quad \text{--- (1)}$$

$$f(\pi-x)=\log\left(\frac{3+\cos(\pi-x)}{3-\cos(\pi-x)}\right)=\log\left(\frac{3-\cos x}{3+\cos x}\right)$$

$$=-\log\left(\frac{3+\cos x}{3-\cos x}\right)$$

$$f(\pi-x)=-f(x)$$

$$I = 2\pi \int_0^{\pi} \log\left(\frac{3-\cos x}{3+\cos x}\right) dx \quad \text{--- (2)}$$

(1)+(2) ⇒

$$I+I = 2\pi \int_0^{\pi} \left(\log\left(\frac{3+\cos x}{3-\cos x}\right) + \log\left(\frac{3-\cos x}{3+\cos x}\right)\right) dx$$

$$2I = 2\pi \int_0^{\pi} \log\left(\frac{\cancel{3+\cos x}}{\cancel{3-\cos x}} \times \frac{\cancel{3-\cos x}}{\cancel{3+\cos x}}\right) dx$$

$$= 2\pi \int_0^{\pi} \log 1 \, dx$$

$$2I = 2\pi(0)$$

$$I = 0.$$

(v) $\int_0^{2\pi} \sin^4 x \cos^3 x \, dx$

$$I = \int_0^{2\pi} \sin^4 x \cos^3 x \, dx$$

$$I = \int_0^{2\pi} (1-\cos^2 x)^2 \cos^3 x \, dx$$

$$= \int_0^{2\pi} (1+\cos^4 x - 2\cos^2 x) \cos^3 x \, dx$$

$$= \int_0^{2\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) \, dx.$$

$$I = 2\int_0^{\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) \, dx \quad \text{--- (1)}$$

$$\boxed{\int_0^{2\pi} g(\cos x)dx = 2\int_0^{\pi} g(\cos x) dx}$$

$a = \pi$ $\quad f(x) = 2(\cos^3 x + \cos^7 x - 2\cos^5 x)$

$$f(a-x) = f(\pi - x) = 2(-\cos^3 x - \cos^7 x + 2\cos^5 x)$$

$$I = 2\int_0^{\pi} (-\cos^3 x - \cos^7 x + 2\cos^5 x) dx \quad \text{--- (2)}$$

$$\boxed{\int_0^a f(x)dx = \int_0^a f(a-x)dx}$$

(1)+(2) ⇒

$$I+I = 2\int_0^{\pi} (\cancel{\cos^3 x} + \cancel{\cos^7 x} - \cancel{2\cos^5 x} \cancel{-\cos^3 x} \cancel{-\cos^7 x} \cancel{+2\cos^5 x}) dx$$

$$2I = 0$$

(vi) $\int_0^1 |5x-3|\,dx$.

$$|(5x-3)| = \begin{cases} (5x-3) & (5x-3)>0 \\ -(5x-3) & (5x-3)<0 \end{cases}$$

$$= \begin{cases} 5x-3 & x > 3/5 \\ -(5x-3) & x < \frac{3}{5} \end{cases}$$

$$= \begin{cases} (5x-3) & 3/5 < x < 1 \\ -(5x-3) & 0 < x < 3/5 \end{cases}$$

$$I = \int_0^1 |5x-3|\,dx$$

$$= \int_0^{3/5} -(5x-3)\,dx + \int_{3/5}^1 (5x-3)\,dx$$

$$= -\frac{1}{5}\left[\frac{(5x-3)^2}{2}\right]_0^{3/5} + \frac{1}{5}\left[\frac{(5x-3)^2}{2}\right]_{3/5}^1$$

$$= -\frac{1}{10}\left[0-(-3)^2\right] + \frac{1}{10}\left[(5-3)^2-(0)\right]$$

$$= -\frac{1}{10}(-9) + \frac{1}{10}(4) = \frac{9}{10}+\frac{4}{10}$$

$$= \frac{13}{10}$$

vii $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$

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$$I = I_1 + I_2$$

$$I_1 = \int_0^{\sin^2 x} \underset{u}{\sin^{-1}\sqrt{t}}\,\underset{dv}{dt}$$

$u = \sin^{-1}\sqrt{t}$ $\quad dv = dt$

$du = \frac{1}{\sqrt{1-t}}\,\frac{1}{2\sqrt{t}}$ $\quad v = \int dt$

$v = t$

$\int u\,dv = uv - \int v\,du$

$$I = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

$u = \sin^{-1}\sqrt{t}$ | $dv = dt$ $\quad$ $u = \cos^{-1}\sqrt{t}$, $dv = dt$

$du = \frac{1}{\sqrt{1-t}}\frac{1}{2\sqrt{t}}$ | $v = t$ $\quad$ $du = \frac{-1}{\sqrt{1-t}}\frac{1}{2\sqrt{t}}$, $v = t$

$$I = \left[\sin^{-1}\sqrt{t}\,(t)\right]_0^{\sin^2 x} - \int_0^{\sin^2 x} \sqrt{t}\cancel{t}\,\frac{1}{\sqrt{1-t}}\cdot\frac{1}{2\cancel{\sqrt{t}}}\,dt + \left[\cos^{-1}\sqrt{t}\,(t)\right]_0^{\cos^2 x}$$

$$-\int_0^{\cos^2 x} \sqrt{t}\cancel{t}\left(\frac{-1}{\sqrt{1-t}}\right)\frac{1}{2\cancel{\sqrt{t}}}\,dt$$

$$I = \left[\sin^{-1}(\sin x)\sin^2 x - 0\right] - \frac{1}{2}\int_0^{\sin^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt + \left[\cos^{-1}(\cos x)\cos^2 x - 0\right]$$

$$+\frac{1}{2}\int_0^{\cos^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$I = x\sin^2 x + x\cos^2 x - \frac{1}{2}\int_0^{\sin^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt + \frac{1}{2}\int_0^{\cos^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x(\sin^2 x + \cos^2 x) + \frac{1}{2}\int_{\sin^2 x}^{0}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt + \frac{1}{2}\int_0^{\cos^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x(1) + \frac{1}{2}\int_{\sin^2 x}^{\cos^2 x}\frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x + \frac{1}{2}\int_x^{\pi/2 - x}\frac{\sin\theta}{\sqrt{\cos^2\theta}}\,2\sin\theta\cos\theta\,d\theta$$

put $t = \sin^2\theta$ $\quad$ $\sqrt{t} = \sin\theta$

$dt = 2\sin\theta\cos\theta\,d\theta$

$t = \sin^2 x \Rightarrow \sin^2\theta = \sin^2 x$

$\theta = x$

$t = \cos^2 x \Rightarrow \sin^2\theta = \sin^2(\pi/2 - x)$

$\theta = \pi/2 - x$

$$= x + \frac{2}{2}\int_x^{\pi/2 - x}\sin^2\theta\,d\theta$$

$$= x + \frac{2}{2}\int_x^{\pi/2 - x}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

| $t$ | $\sin^2 x$ | $\cos^2 x$ |
|---|---|---|
| $\theta$ | $x$ | $\pi/2 - x$ |

$$= x + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)_{x}^{\pi/2 - x}$$

$$= x + \frac{1}{2}\left(\left(\frac{\pi}{2} - x - x\right) - \frac{1}{2}\left(\sin 2\left(\frac{\pi}{2} - x\right) - \sin 2x\right)\right)$$

$$= x + \frac{1}{2}\left(\frac{\pi}{2} - 2x - \frac{1}{2}\left(\sin(\pi - 2x) - \sin 2x\right)\right)$$

$$= x + \frac{\pi}{4} - x - \frac{1}{4}\left[\sin 2x - \sin 2x\right]$$

$$= \cancel{x} + \frac{\pi}{4} - \cancel{x}$$

$$I = \pi/4$$

(viii) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

put: $x = \tan t$

$dx = \sec^2 t\,dt$

$x = 0 \Rightarrow \tan t = 0, \quad t = 0$

$x = 1 \quad \tan t = 1 \quad t = \pi/4$

| $x$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | $\pi/4$ |

$$I = \int_0^{\pi/4} \frac{\log(1+\tan t)}{1+\tan^2 t}\sec^2 t\,dt$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan t)}{\cancel{\sec^2 t}}\cancel{\sec^2 t}\,dt$$

$$I = \int_0^{\pi/4} \log(1+\tan t)\,dt \quad \text{——} (1)$$

$f(t) = \log(1+\tan t)$

$a = \pi/4$

$$f(\pi/4 - t) = \log\left(1 + \tan(\pi/4 - t)\right)$$

$$= \log\left(1 + \frac{\tan\pi/4 - \tan t}{1 + \tan\pi/4\tan t}\right)$$

$$= \log\left(1 + \frac{1-\tan t}{1+\tan t}\right) = \log\left(\frac{1 + \cancel{\tan t} + 1 - \cancel{\tan t}}{1+\tan t}\right)$$

$$f(\pi/4 - t) = \log\left(\frac{2}{1+\tan t}\right)$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan t}\right) dt \quad \text{———} (2)$$

(1)+(2) $\Rightarrow$

$$I + I = \int_0^{\pi/4} \left(\log(1+\tan t) + \log\left(\frac{2}{1+\tan t}\right)\right) dt$$

$$2I = \int_0^{\pi/4} \log\left(\cancel{(1+\tan t)} \times \frac{2}{\cancel{(1+\tan t)}}\right) dt$$

$$2I = \int_0^{\pi/4} \log 2 \, dt$$

$$2I = \log 2 \,[t]_0^{\pi/4}$$

$$2I = \log 2 \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4}\log 2$$

$$I = \frac{\pi}{8}\log 2$$

(ix) $\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$

$$I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$$

$$f(x) = \frac{\sin x}{1+\sin x}$$

$a = \pi$

$$f(\pi - x) = \frac{\sin(\pi - x)}{1+\sin(\pi - x)} = \frac{\sin x}{1+\sin x} = f(x)$$

$$f(x) = f(\pi - x)$$

$$\Rightarrow \int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx.$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \left(\frac{\sin x}{1+\sin x}\right)\frac{1-\sin x}{(1-\sin x)}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x - \sin^2 x}{1-\sin^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi}\left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right)dx$$

$$=\frac{\pi}{2}\int_0^{\pi}(\sec x\tan x-\tan^2 x)\,dx$$

$$=\pi/_2\int_0^{\pi}(\sec x\tan x-(\sec^2 x-1))\,dx$$

$$=\frac{\pi}{2}\int_0^{\pi}(\sec x\tan x-\sec^2 x+1)\,dx$$

$$=\frac{\pi}{2}\left[\sec x-\tan x+x\right]_0^{\pi}$$

$$=\frac{\pi}{2}\left[(\sec\pi-\tan\pi+\pi)-(\sec 0-\tan 0+0)\right]$$

$$=\frac{\pi}{2}\left[-1-0+\pi-1\right]$$

$$I=\frac{\pi}{2}\left[\pi-2\right]$$

(x) $\displaystyle\int_{\pi/8}^{3\pi/8}\frac{1}{1+\sqrt{\tan x}}\,dx$

$$I=\int_{\pi/8}^{3\pi/8}\frac{1}{1+\sqrt{\tan x}}\,dx$$

$$=\int_{\pi/8}^{3\pi/8}\frac{1}{1+\sqrt{\frac{\sin x}{\cos x}}}\,dx$$

$$I=\int_{\pi/8}^{3\pi/8}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{——} (1)$$

$a=\pi/_8$ $\quad b=\frac{3\pi}{8}$ $\quad a+b=\frac{\pi}{8}+\frac{3\pi}{8}=\frac{4\pi}{8}=\pi/_2$

$f(x)=\dfrac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$ $\quad f(a+b-x)=f(\pi/_2-x)=\dfrac{\sqrt{\cos(\pi/_2-x)}}{\sqrt{\cos\pi/_2-x}+\sqrt{\sin(\pi/_2-x)}}$

$$f\left(\frac{\pi}{2}-x\right)=\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$$

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$I=\int_{\pi/8}^{3\pi/8}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{——} (2)$$

add (1),(2)

$$2I+I=\int_{\pi/8}^{3\pi/8}\cancel{\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}}\,dx$$

$$2I=\int_{\pi/8}^{3\pi/8}dx$$

$$2I = \left[x\right]_{\pi/8}^{3\pi/8}$$

$$2I = \frac{3\pi}{8} - \frac{\pi}{8}$$

$$2I = \frac{2\pi}{8_4} \qquad 2I = \frac{\pi}{4} \qquad I = \frac{\pi}{4\times 2} \qquad I = \frac{\pi}{8}$$

(xi) $I = \int_0^{\pi} x\left[\sin^2(\sin x) + \cos^2(\cos x)\right] dx$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$a = \pi$ $\quad f(a-x) = f(\pi - x)$

$$= \sin^2 \sin(\pi - x) + \cos^2 \cos(\pi - x))$$

$$= \sin^2 \sin x + \cos^2(\cos x)$$

$$= \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(a-x) = f(x)$$

$$\therefore \int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$$

$$I = \frac{\pi}{2}\int_0^{\pi}\left[\sin^2(\sin x) + \cos^2(\cos x)\right] dx \quad \text{---(1)}$$

$$\int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\pi/2} g(\sin x)\,dx$$

$$f(x) = \cos^2(\cos x)$$

$$f(\pi + x) = \cos^2(\cos(\pi + x))$$

$$= \cos^2(-\cos x)$$

$$= \cos^2 \cos x$$

$$f(\pi + x) = f(x)$$

$$\Rightarrow \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

$$\int_0^{\pi}\cos^2(\cos x)\,dx = 2\int_0^{\pi/2}\cos^2(\cos x)\,dx$$

$$I = 2\frac{\pi}{2}\int_0^{\pi/2}\sin^2(\sin x) + \cos^2(\cos x)\,dx \quad \text{---(1)}$$

$$I = \pi\int_0^{\pi/2}\sin^2(\cos x) + \cos^2(\sin x)\,dx \quad \text{---(2)}$$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(a-x) = f(\tfrac{\pi}{2} - x) = \sin^2(\cos x) + \cos^2(\sin x)$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(1) + (2) $\Rightarrow$

$$I + I = \pi\int_0^{\pi/2}\left(\begin{matrix}\sin^2(\sin x) + \cos^2(\cos x) \\ \cos^2(\sin x) + \sin^2(\cos x)\end{matrix}\right) dx$$

$$2I = \pi\int_0^{\pi/2}(1+1)\,dx$$

$$2I = 2\pi\left[x\right]_0^{\pi/2}$$

$$I = \pi\left[\tfrac{\pi}{2} - 0\right]$$

$$I = \frac{\pi^2}{2} - 0$$

$$I = \frac{\pi^2}{2}$$

## Bernoulli's Formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

### Exercise 9.4

Evaluate the following

1) $\int_0^1 x^3 e^{-2x}\,dx$

$$I = \int_0^1 x^3 e^{-2x}\,dx$$

$u = x^3$ $\quad dv = e^{-2x},\ v = \frac{1}{-2}e^{-2x}$

$u' = 3x^2$ $\quad v_1 = \frac{1}{4}e^{-2x}$

$u'' = 6x$ $\quad v_2 = -\frac{1}{8}e^{-2x}$

$u''' = 6$ $\quad v_3 = +\frac{1}{16}e^{-2x}$

$u^{iv} = 0$

$$I = \int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$I = \left[x^3\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\frac{1}{4}e^{-2x} + 6x\left(-\frac{1}{8}\right)e^{-2x} - 6\frac{1}{16}e^{-2x}\right]_0^1$$

$$= \left[\left(-\frac{e^{-2}}{2} - \frac{3}{4}e^{-2} - \frac{3}{4}e^{-2} - \frac{3}{8}e^{-2}\right) - \left(0+0+0+0 - \frac{3}{8}e^0\right)\right]$$

$$= -e^{-2}\left[\frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{8}\right] + \frac{3}{8}$$

$$= -e^{-2}\left[\frac{4+6+6+3}{8}\right] + \frac{3}{8}$$

$$= \frac{3}{8} - e^{-2}\frac{19}{8} = \frac{1}{8}\left[3 - 19e^{-2}\right]$$

2) $\int_0^1 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}\,dx$

$$I = \int_0^1 \frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}\,dx$$

put $\tan^{-1}x = t$ $\qquad x=0 \quad t = \tan^{-1}(0) = 0$

$\therefore \frac{1\,dx}{1+x^2} = dt$ $\qquad x=1 \quad t = \tan^{-1}(1) = \pi/4$

| $x$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | $\pi/4$ |

$$I=\int_0^{\frac{\pi}{4}} \sin 3t \; t \; dt$$

$$I=\int_0^{\pi/4} \underset{u}{t} \; \underset{dv}{\sin 3t dt}$$

$u=t$ $\qquad dv=\sin 3t dt$

$\qquad V=-\frac{1}{3}\cos 3t$

$u'=1$ $\qquad V_1=-\frac{1}{9}\sin 3t$

$u''=0$ $\qquad V_2=+\frac{1}{27}\cos 3t$

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$I=\left[t\left(-\frac{1}{3}\cos 3t\right) - 1\left(-\frac{1}{9}\sin 3t\right) + 0\right]_0^{\pi/4}$$

$$=\left(-\frac{\pi}{4}\frac{1}{3}\cos\frac{3\pi}{4}+\frac{1}{9}\sin\frac{3\pi}{4}\right)-\left(0+0\right)$$

$\cos\frac{3\pi}{4}=\cos 135$

$=\cos(180-45)$

$=-\cos 45$

$=-\frac{1}{\sqrt{2}}$

$\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$

$$=\cancel{-}\frac{\pi}{12}\left(\cancel{-}\frac{1}{\sqrt{2}}\right)+\frac{1}{9}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{12}+\frac{1}{9}\right)$$

$$I=\frac{1}{\sqrt{2}}\left(\frac{\pi}{12}+\frac{1}{9}\right)$$

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3) $$I=\int_0^{\frac{1}{\sqrt{2}}} \frac{e^{a\sin^{-1}x}\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

put $\sin^{-1}x = t$

$\frac{1}{\sqrt{1-x^2}}dx = dt$

$x=0 \Rightarrow t=\sin^{-1}(0)=0$

$x=1/\sqrt{2} \Rightarrow t=\sin^{-1}(1/\sqrt{2})=\pi/4$

| x | 0 | $1/\sqrt{2}$ |
|---|---|---|
| t | 0 | $\pi/4$ |

$$I=\int_0^{\pi/4} e^{at}\, t \, dt$$

$$I=\int_0^{\pi/4} \underset{u}{t} \; \underset{dv}{e^{at}dt}$$

$u=t$ $\qquad dv=e^{at}dt$

$u'=1$ $\qquad V=\frac{1}{a}e^{at}$

$u''=0$ $\qquad V_1=\frac{1}{a^2}e^{at}$

$$\int u\,dv = uv - \int v\,du$$

$$I = \left[ t\frac{1}{a}e^{at} - 1\frac{1}{a^2}e^{at} \right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4a}e^{a\pi/4} - \frac{1}{a^2}e^{a\pi/4}\right) - \left(0 - \frac{1}{a^2}\right)$$

$$I = e^{a\pi/4}\left(\frac{\pi}{4a} - \frac{1}{a^2}\right) + \frac{1}{a^2}$$

put $a=1$ $\quad I = e^{\pi/4}\left(\frac{\pi}{4} - 1\right) + 1$ (Book answer)

4) $$I = \int_0^{\pi/2} \underset{u}{x^2}\,\underset{dv}{\cos 2x\,dx}$$

$u = x^2 \qquad dv = \cos 2x\,dx$

$\qquad\qquad v = \frac{1}{2}\sin 2x$

$u' = 2x \qquad v_1 = -\frac{1}{4}\cos 2x$

$u'' = 2 \qquad v_2 = -\frac{1}{8}\sin 2x$

$u''' = 0$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$I = \left[ x^2\frac{1}{2}\sin 2x + \cancel{2}x\left(\frac{+1}{\cancel{4}_2}\cos 2x\right) + \cancel{2}\left(\frac{-1}{\cancel{8}_4}\sin 2x\right) \right]_0^{\pi/2}$$

$$= \left(\frac{\pi^2}{4}\frac{1}{2}\sin 2\frac{\pi}{2} + \frac{\pi}{2}\frac{1}{2}\cos 2\frac{\pi}{2} - \frac{1}{4}\sin 2\frac{\pi}{2}\right) - (0+0+0)$$

$$= 0 + \frac{\pi}{4}(-1) - 0$$

$$I = -\pi/4$$

———×———

## Improper Integrals

Exercise 9.5

1) Evaluate (i) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$.

$$I=\int_0^{\pi/2}\frac{dx}{1+5\cos^2x}$$

$$=\int_0^{\pi/2}\frac{dx}{\cos^2x\left(\frac{1}{\cos^2x}+5\right)}$$

$$=\int_0^{\pi/2}\frac{\sec^2x}{\sec^2x+5}\,dx.$$

$$=\int_0^{\pi/2}\frac{\sec^2x}{\sec^2x-1+6}\,dx=\int_0^{\pi/2}\frac{\sec^2x\,dx}{\tan^2x+6}$$

put $\tan x=t$

$\sec^2x\,dx=dt$

$x=0 \quad t=\tan 0=0$

$x=\pi/2 \quad t=\tan\pi/2=\infty$

| $x$ | $0$ | $\pi/2$ |
|---|---|---|
| $t$ | $0$ | $\infty$ |

$$I=\int_0^{\infty}\frac{dt}{6+t^2}$$

This is an improper integral

$a^2=6$

$a=\sqrt6$

$$=\frac{1}{\sqrt6}\left[\tan^{-1}\frac{t}{\sqrt6}\right]_0^{\infty}$$

$$\int_0^{\infty}\frac{dx}{a^2+x^2}=\frac{\pi}{2a}=\frac{\pi}{2\sqrt6}$$

$$=\frac{1}{\sqrt6}\left[\tan^{-1}\infty-\tan^{-1}(0)\right]=\frac{1}{\sqrt6}\left(\frac{\pi}{2}-0\right)$$

$$=\frac{\pi}{2\sqrt6}$$

(ii) $$\int_0^{\pi/2}\frac{dx}{5+4\sin^2x}$$

$$I=\int_0^{\pi/2}\frac{dx}{5+4\sin^2x}$$

$$= \int_0^{\pi/2} \frac{dx}{5+4(1-\cos^2 x)}$$

$$= \int_0^{\pi/2} \frac{dx}{5+4-4\cos^2 x} = \int_0^{\pi/2} \frac{dx}{9-4\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{dx}{\cos^2 x\left(\frac{9}{\cos^2 x}-4\right)}$$

$$= \int_0^{\pi/2} \frac{dx \sec^2 x}{9\sec^2 x-4}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{9(1+\tan^2 x)-4} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{9+9\tan^2 x-4}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\, dx}{5+9\tan^2 x} = \frac{1}{9}\int_0^{\pi/2} \frac{\sec^2 x\, dx}{\left(\frac{5}{9}\right)+\tan^2 x}$$

put $\tan x = t$

$\sec^2 x\, dx = dt$ $\quad x=0 \quad t=0$; $\quad x=\pi/2 \quad t=\infty$

$$I = \frac{1}{9}\int_0^{\infty} \frac{dt}{\frac{\sqrt{5}^2}{9}+t^2} \qquad a=\frac{\sqrt{5}}{3} \qquad \boxed{\int_0^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{2a}}$$

$$= \frac{1}{9}\cdot\frac{3\pi}{2\sqrt{5}}$$

$$= \frac{\pi}{6\sqrt{5}} \qquad \text{(Book answer wrong)}$$

---

## Reduction Formula.

① (i) $n$ is even

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\cdots\frac{1}{2}\,\frac{\pi}{2}$$

(ii) $n$ is odd

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}$$

(2)(3) If $I_{m,n} = \int_0^1 x^m (1-x)^n dx$, then $I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}$, $n \geq 1$

(3) (i) n is even and m is even.

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

(ii) n is odd. m is any (even or odd)

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

(iii) n is even m is odd

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{n+3} \cdot \frac{1}{n+1}$$

Exercise 9.6

1) Evaluate the following :

(i) $\int_0^{\pi/2} \sin^{10} x\, dx$

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n = 10 even

$$I = \int_0^{\pi/2} \sin^{10} x\, dx$$

$$= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{63\pi}{512}$$

(Book answer $\pi$ missing).

(ii) $I = \int_0^{\pi/2} \cos^7 x\, dx$ n = 7 odd

$$I = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} = \frac{16}{35}$$ (Book answer wrong)

(iii) $I = \int_0^{\pi/4} \sin^6 2x\, dx$

put $2x = t$
$2dx = dt$
$dx = \frac{1}{2} dt$

$x = 0$ $t = 0$
$x = \pi/4$ $t = 2\pi/4 = \pi/2$

| x | 0 | $\pi/4$ |
|---|---|---|
| t | 0 | $\pi/2$ |

$$I = \frac{1}{2} \int_0^{\pi/2} \sin^6 t\, dt$$

n = 6 even

$$\boxed{\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{64}$$

(iv) $I = \int_0^{\pi/6} \sin^5 3x \, dx$

put $3x = t$

$3dx = dt$

$dx = dt/3$

when $x=0$ $t=0$

$x = \pi/6$ $t = \frac{3\pi}{6} = \pi/2$

| $x$ | 0 | $\pi/6$ |
|---|---|---|
| $t$ | 0 | $\pi/2$ |

$$I = \frac{1}{3}\int_0^{\pi/2} \sin^5 t \, dt$$

$n = 5$

$n$ is odd.

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3}$$

$$= \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{45}$$

(v) $I = \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$.

$m = 2$ $n = 4$

both are even.

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \frac{\pi}{2}$$

$$I = \frac{4-1}{2+4} \cdot \frac{4-3}{2+4-2} \cdots \frac{2-1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{32}$$

(v) Alitor

$$I = \int_0^{\pi/2} (1-\cos^2 x)\cos^4 x \, dx$$

$$= \int_0^{\pi/2} \cos^4 x \, dx - \int_0^{\pi/2} \cos^6 x \, dx$$

$$= \frac{3}{4} \frac{1}{2} \frac{\pi}{2} - \frac{5}{6} \frac{3}{4} \frac{1}{2} \frac{\pi}{2} = \frac{3\pi}{16}\left[1 - \frac{5}{6}\right]$$

$$= \frac{3\pi}{16} \times \frac{1}{6} = \frac{\pi}{32}$$

(vi) $I = \int_0^{2\pi} \sin^7 \frac{x}{4} \, dx$

put $\frac{x}{4} = t$

$x = 4t$

$dx = 4dt$

$x = 0$, $t = 0$

$x = 2\pi$ $t = \frac{2\pi}{4} = \pi/2$

| $x$ | 0 | $2\pi$ |
|---|---|---|
| $t$ | 0 | $\pi/2$ |

$$I = 4\int_0^{\pi/2} \sin^7 t \, dt$$

$I = 4\ \frac{\cancel{6}^{2}}{7} \times \frac{4}{5}\ \frac{2}{\cancel{3}\ 1}$

$n=7$ odd

$$\boxed{\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}}$$

$I = \frac{64}{35}$

(vii) $I = \int_0^{\pi/2} \sin^3\theta \cos^5\theta\, d\theta$

$m=3 \quad n=5$ (both are odd)

$$I = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdots\frac{2}{m+3}\cdot\frac{1}{m+1}$$

$$= \frac{5-1}{3+5}\cdot\frac{5-3}{3+5-2}\cdot\frac{1}{3+1}$$

$I = \frac{\cancel{4}}{8}\cdot\frac{\cancel{2}^{1}}{\cancel{6}_3}\cdot\frac{1}{\cancel{4}} = \frac{1}{24}$

(viii) $I = \int_0^1 x^2 (1-x)^3\, dx$

$$I = \int_0^1 x^m (1-x)^n\, dx = \frac{m!\times n!}{(m+n+1)!}$$

$m=2 \quad n=3$

$$= \frac{2!\times 3!}{(2+3+1)!} = \frac{2\times 6}{6!} = \frac{\cancel{2}\times\cancel{6}}{\cancel{6}\times 5\times 4\times 3\times\cancel{2}\times 1}$$

$= \frac{1}{60}$

---

## Gamma Integral

① $\int_0^\infty e^{-ax} x^n\, dx = \frac{n!}{a^{n+1}}$

$a=1$

② $\int_0^\infty e^{-x} x^n\, dx = n!$

(Gamma n) $\Gamma n = (n-1)!$

③ $\int_0^\infty e^{-x} x^{n-1}\, dx = (n-1)! = \Gamma n$

## Exercise 9.7

Evaluate 1) (i) $\int_0^\infty x^5 e^{-3x}\, dx$

$n=5 \quad a=3$

$$\boxed{\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}}$$

$$I = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$$

(ii) $\displaystyle\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x}dx$

$$I = \int_0^{\pi/2} e^{-\tan x} \sec^6 x\, dx$$

put $\tan x = t$ $\quad x=0 \quad t=\tan 0=0$

$\sec^2 x\,dx = dt$ $\quad x=\pi/2 \quad t=\tan\pi/2=\infty$

$$I = \int_0^{\pi/2\ \infty} e^{-\tan x}\sec^4 x\, \sec^2 x\, dx$$

| $x$ | 0 | $\pi/2$ |
|---|---|---|
| $t$ | 0 | $\infty$ |

$$I = \int_0^\infty e^{-t} (\sec^2 x)^2 dt$$

$$= \int_0^\infty e^{-t}(1+\tan^2 x)^2 dt = \int_0^\infty e^{-t}(1+t^2)^2 dt$$

$$= \int_0^\infty t^0 e^{-t}dt + \int_0^\infty 2t^2 e^{-t}dt + \int_0^\infty t^4 e^{-t}dt$$

$n=0 \qquad n=2 \qquad n=4$

$= 0! + 2(2!) + 4!$ $\qquad \boxed{\int_0^\infty x^n e^{-x}dx = n!}$

$= 1+4+24$

$= 29.$

② If $\displaystyle\int_0^\infty e^{-\alpha x^2} x^3 dx = 32$, $\alpha > 0$ find $\alpha$.

put $\alpha x^2 = t \quad x^2 = \frac{t}{\alpha}$ $\qquad x=0 \quad t=0$

$\alpha 2x\,dx = dt$ $\qquad x=\infty \quad t=\infty$

$x\,dx = \frac{dt}{2\alpha}$

| $x$ | 0 | $\infty$ |
|---|---|---|
| $t$ | 0 | $\infty$ |

$$I = \int_0^\infty e^{-\alpha x^2} x^2\, x\,dx$$

$$I = \int_0^\infty e^{-t} \frac{t}{\alpha} \frac{dt}{2\alpha}$$

$$= \frac{1}{2\alpha^2} \int_0^\infty t e^{-t} dt \qquad \boxed{\int_0^\infty e^{-x} x^n \, dx = n!}$$

$$I = \frac{1}{2\alpha^2}(1!)$$

given $I = 32$

$$\frac{1}{2\alpha^2} = 32 \qquad \frac{1}{\alpha^2} = 64$$

$$\alpha^2 = \frac{1}{64} \qquad \alpha > 0$$

$$\boxed{\alpha = \frac{1}{8}}$$

———x———

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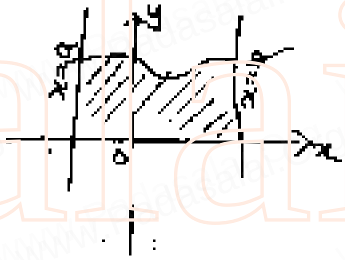
## Area

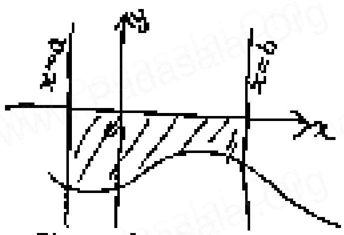
① Area of the region bounded by the curve $y=f(x)$ and $x=a$, $x=b$

(i) lies above the x axis




$$\text{Area} = \int_a^b y \, dx$$

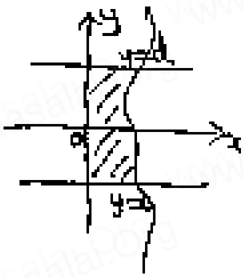
(ii) lies below the x axis




$$\text{Area} = -\int_a^b y \, dx$$

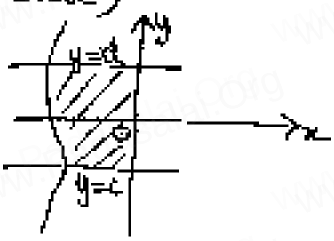
② Area of the region bounded by the curve $x=f(y)$ and $y=c$, $y=d$.

(i) lies +ve direction of x axis (Right side)




$$\text{Area} = \int_c^d x \, dy$$

(ii) lies −ve direction of x axis (left side)




$$\text{Area} = -\int_c^d x \, dy$$

③ Area of the region bounded between two curves

(i) $y=f(x)$, $y=g(x)$ $\quad f(x)\geq g(x)\ \forall x\in[a,b]$

Area bounded by two curves (using vertical strips) $\quad A=\int_a^b (y_U-y_L)\,dx$

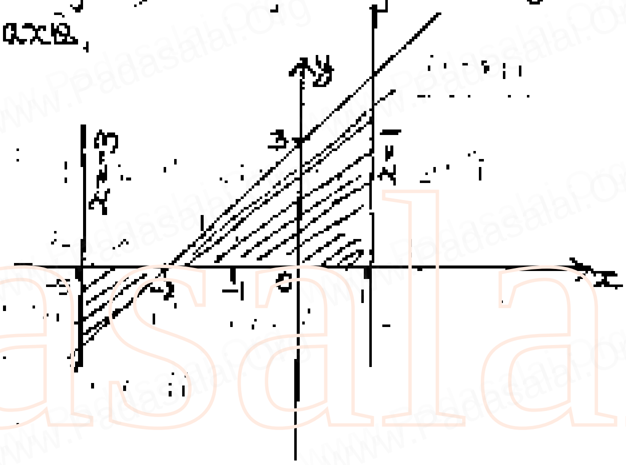
(ii) $x=g(y)$, $x=f(y)$ $\quad f(y)\geq g(y)\ \forall y\in[c,d]$

Area bounded by two curves $\quad A=\int_c^d (x_R-x_L)\,dy$

(using horizontal strips).

<u>Exercise 9.8</u>

1) Find the area of the region bounded by $3x-2y+6=0$, $x=-3$, $x=1$ and $x$ axis.




$3x-2y+6=0$

$-2y=-3x-6$

$y=\frac{3}{2}x+3$

$x=0 \Rightarrow y=3 \quad (0,3)$

$y=0 \Rightarrow 3x=-6 \quad (-2,0)$

$x=-2$

$$\text{Area}=\int_{-3}^{1} y\,dx$$

$$=\int_{-3}^{-2} -y\,dx+\int_{-2}^{1} y\,dx$$

$$=\int_{-3}^{-2}\left(-\frac{3}{2}x-3\right)dx+\int_{-2}^{1}\left(\frac{3}{2}x+3\right)dx$$

$$=\left[-\frac{3x^2}{4}-3x\right]_{-3}^{-2}+\left[\frac{3x^2}{4}+3x\right]_{-2}^{1}$$

$$=\left(-\frac{3(4)}{4}-3(-2)\right)-\left(-\frac{3(9)}{4}-3(-3)\right)+\left[\left(\frac{3}{4}+3\right)-\left(\frac{+3(4)}{4}+3(-2)\right)\right]$$

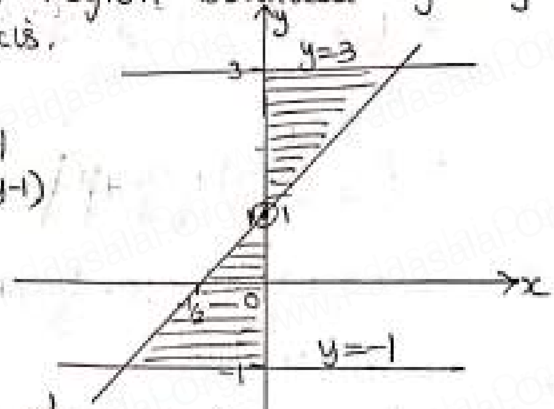
$$=(-3+6)-\left(-\frac{27}{4}+9\right)+\left[\frac{3}{4}+\cancel{3}-\cancel{3}+6\right]$$

$$=3+\frac{27}{4}-9+\frac{3}{4}+6$$

$$=\frac{30}{4}=\frac{15}{2}$$

$$=7.5 \text{ sq units}$$

2) Find the area of the region bounded by $2x-y+1=0$ $y=-1$, $y=3$ and $y$-axis.




$$2x-y+1=0 \qquad 2x = y-1$$
$$-y=-2x-1 \qquad x=\tfrac{1}{2}(y-1)$$
$$y=2x+1$$

$x=0 \Rightarrow y=1 \quad (0,1)$

$y=0 \Rightarrow x=-\tfrac{1}{2} \quad (-\tfrac{1}{2},0)$

$$\text{Area} = \int_{-1}^{3} x\,dy = \int_{-1}^{1} -x\,dy + \int_{1}^{3} x\,dy$$

$$= \int_{-1}^{1} -\tfrac{1}{2}(y-1)\,dy + \int_{1}^{3} \tfrac{1}{2}(y-1)\,dy$$

$$= -\tfrac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{-1}^{1} + \tfrac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{1}^{3}$$

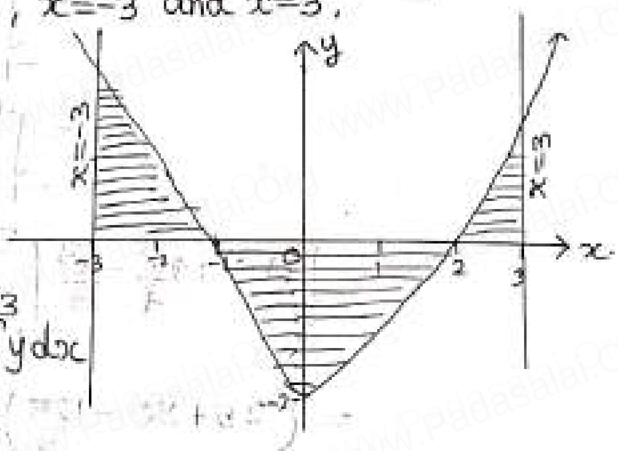
$$= -\tfrac{1}{4}\left[(1-1)^2-(-1-1)^2\right] + \tfrac{1}{4}\left[(3-1)^2-(1-1)^2\right]$$

$$= -\tfrac{1}{4}(0-4) + \tfrac{1}{4}(4-0)$$

$$= \tfrac{4}{4} + \tfrac{4}{4} = 1+1$$

$$= 2 \text{ sq units}$$

3) Find the area of the region bounded by the curve $2+x-x^2+y=0$, $x$ axis, $x=-3$ and $x=3$.




$$y=x^2-x-2$$

put $y=0$ $\quad x^2-x-2=0$

$$(x-2)(x+1)=0$$
$$x=2,-1$$
$$(2,0), (-1,0)$$

$$\text{Area} = \int_{-3}^{-1} +y\,dx + \int_{-1}^{2} -y\,dx + \int_{2}^{3} y\,dx$$

$$= \int_{-3}^{-1} (x^2-x-2)\,dx - \int_{-1}^{2} (x^2-x-2)\,dx + \int_{2}^{3} (x^2-x-2)\,dx$$

$$= \left[\left(\frac{x^3}{3}-\frac{x^2}{2}-2x\right)\right]_{-3}^{-1} - \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} + \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{2}^{3}$$

$$= \left(-\frac{1}{3}-\frac{1}{2}+2\right)-\left(-\frac{27}{3}-\frac{9}{2}+6\right)-\left(\frac{8}{3}-\frac{4}{2}-4\right)+\left(-\frac{1}{3}-\frac{1}{2}+2\right)$$
$$+\left(\frac{27}{3}-\frac{9}{2}-6\right)-\left(\frac{8}{3}-\frac{4}{2}-4\right)$$

$$=\left(-\frac{2}{3}-\frac{2}{2}+4\right)+\left(\frac{54}{3}+\frac{9}{2}-\frac{9}{2}-12\right)-\left(\frac{16}{3}-\frac{8}{2}-8\right)$$

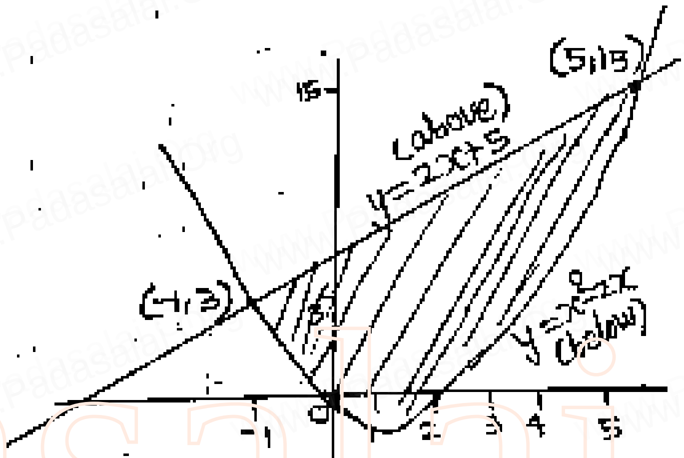
$$=-\frac{2}{3}+3+\frac{54}{3}-12-\frac{16}{3}+12=\frac{54-18}{3}+3=\frac{36}{3}+3$$

$=15$ sq.units

4) Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.

$y=2x+5 \quad \text{---} (1)$

$y=x^2-2x \quad \text{---} (2)$




$x^2-2x=2x+5$

$x^2-4x-5=0$

$(x-5)(x+1)=0$

$x=5,-1$

$x=5 \quad (1) \Rightarrow$

$y=10+5$

$y=15$

$(5,15)$

$x=-1$

$(1) \Rightarrow$

$y=-2+5$

$y=3$

$(-1,3)$

The point of intersections $(-1,3),(5,15)$

$$\text{Area}=\int_{-1}^{5}(y_U-y_L)\,dx$$

$$=\int_{-1}^{5}\left((2x+5)-(x^2-2x)\right)dx$$

$$=\int_{-1}^{5}(5+4x-x^2)\,dx$$

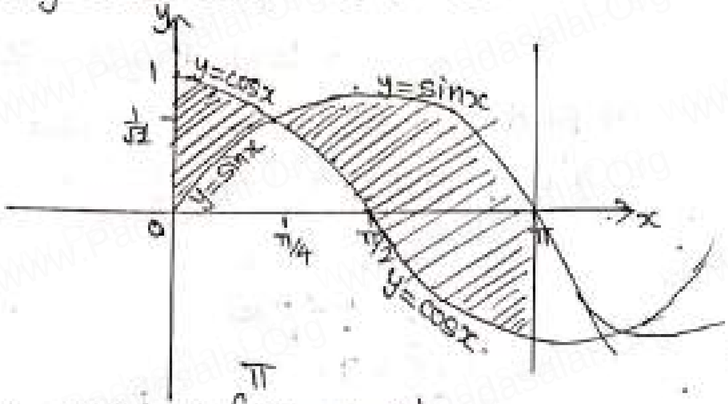
$$=\left[5x+\frac{4x^2}{2}-\frac{x^3}{3}\right]_{-1}^{5}$$

$$=\left(25+50-\frac{125}{3}\right)-\left(-5+2+\frac{1}{3}\right)$$

$$=75-\frac{125}{3}+3-\frac{1}{3}=78-\frac{126}{3}$$

$=36$ sq.units

5) Find the area of the region bounded between the curves $y=\sin x$ and $y=\cos x$ and the lines $x=0$ and $x=\pi$.




$y=\sin x$
$y=\cos x$
$\sin x=\cos x$
$x=\pi/4$

Area of the bounded region

$$=\int_0^{\pi/4}(y_U-y_L)dx+\int_{\pi/4}^{\pi}(y_U-y_L)dx$$

$$=\int_0^{\pi/4}(\cos x-\sin x)dx+\int_{\pi/4}^{\pi}(\sin x-\cos x)dx$$

$$=\left[+\sin x+\cos x\right]_0^{\pi/4}+\left[-\cos x-\sin x\right]_{\pi/4}^{\pi}$$

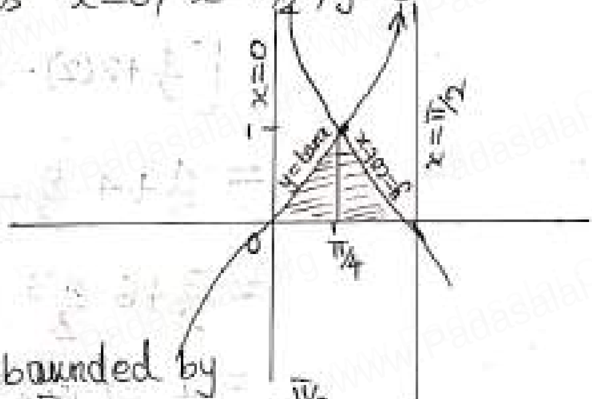
$$=\left[\left(\sin\tfrac{\pi}{4}+\cos\tfrac{\pi}{4}\right)-(\sin 0+\cos 0)\right]-\left[(\cos\pi+\sin\pi)-\left(\cos\tfrac{\pi}{4}+\sin\tfrac{\pi}{4}\right)\right]$$

$$=\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1\right]-\left[-1-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right]$$

$$=\frac{2}{\sqrt{2}}-1+1+\frac{2}{\sqrt{2}}=\frac{4}{\sqrt{2}}=\frac{2\sqrt{2}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$

$=2\sqrt{2}$ sq units.

6) Find the area of the region bounded by $y=\tan x$, $y=\cot x$ and the lines $x=0$, $x=\pi/2$, $y=0$




$y=\tan x$
$y=\cot x$
$\tan x=\cot x$
$x=\pi/4$

Area of the region bounded by the curves $=\int_0^{\pi/4}\tan x\,dx+\int_{\pi/4}^{\pi/2}\cot x\,dx$

$$=\left[\log\sec x\right]_0^{\pi/4}+\left[\log\sin x\right]_{\pi/4}^{\pi/2}$$

$$=\left[\log\sec\pi/4-\log\sec 0\right]+\left[\log\sin\pi/2-\log\sin\pi/4\right]$$

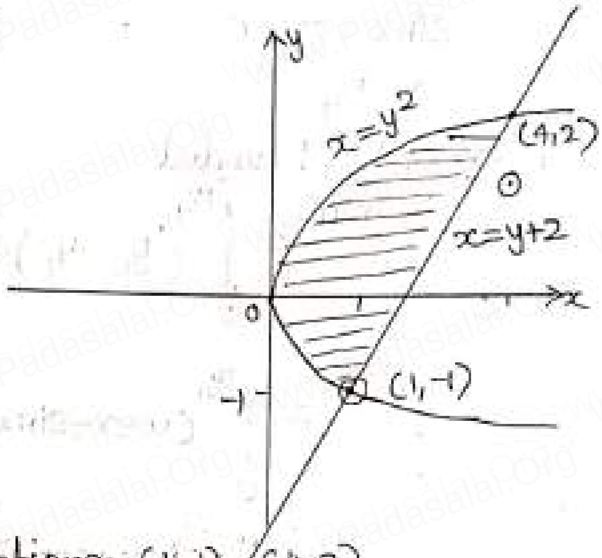
$= \log\sqrt{2} - \log 1 + \log 1 - \log\frac{1}{\sqrt{2}}$

$= \log\sqrt{2} + \log\sqrt{2} = \log 2$ sq units.

7) Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x-2$

$y^2 = x$ —— ①

$y = x-2$ —— ②




$(x-2)^2 = x$.

$x^2 - 4x + 4 - x = 0$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 1, 4$

② ⇒

$x = 1 \Rightarrow y = 1-2 = -1 \quad (1,-1)$

$x = 4 \Rightarrow y = 4-2 = 2 \quad (4,2)$

The point of intersections $(1,-1), (4,2)$

Area of the bounded region

$$= \int_{-1}^{2} (x_R - x_L)\,dy$$

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$$= \int_{-1}^{2} (y+2-y^2)\,dy$$

$$= \left[\frac{y^2}{2} + 2x - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2} + 2(2) - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right]$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{3}{2} + 6 - \frac{\cancel{9}\,3}{\cancel{3}} = \frac{3}{2} + 3 = \frac{3+6}{2}$$

$$= \frac{9}{2} \text{ sq units}$$

8) Father of a family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three parts for his wife, daugher and son. Is it possible to divide? If so find the area to be divided among them.

$x^2=4y$ —① 

$y^2=4x$ —②

$x=\frac{y^2}{4}$

① ⇒ $\frac{y^4}{16}=4y$

$y^4=64y$

$y^4-64y=0$

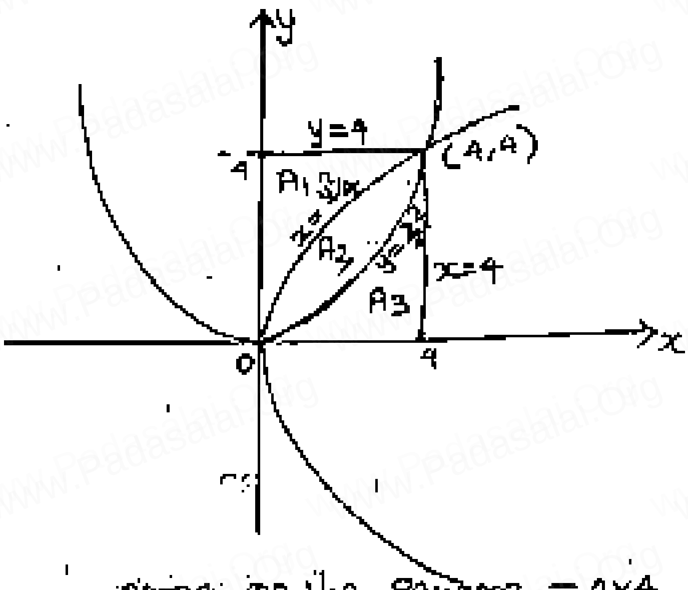
$y(y^3-4^3)=0$

$y=0,\ y=4$

② ⇒ $y=0 \Rightarrow x=0$

$y=4 \Rightarrow 16=4x \quad x=4$

∴ point of intersections $(0,0),(4,4)$




Area of the Square $=4\times4$

$=16$ sq units

Area $A_1=\int_0^4 x\,dy$

$=\int_0^4 \frac{y^2}{4}\,dy$

$=\left[\frac{y^3}{4\times3}\right]_0^4 = \frac{1}{12}\left[4^3-0\right]$

$=\frac{4\times16}{4\times3}$

$A_1=\frac{16}{3}$ sq. units

Area $A_3=\int_0^4 y\,dx = \int_0^4 \frac{x^2}{4}\,dx$

$=\frac{1}{4}\int_0^4 x^2\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$

$=\frac{1}{4\times3}\left[4^3-0\right] = \frac{1}{4\times3}\,4\times16 = \frac{16}{3}$ sq units

∴ Area of $A_2$ = Total Area $-(A_1+A_3)$

$=16-\frac{16}{3}-\frac{16}{3}=\frac{16}{3}$ sq. units

∴ The total Area is divided into 3 equal parts

Each area is $\frac{16}{3}$ sq. units

[ Area of $A_2=\int_0^4 (y_U-y_L)\,dx$

$$= \int_0^4 \left(2x^{1/2} - \frac{x^2}{4}\right) dx$$

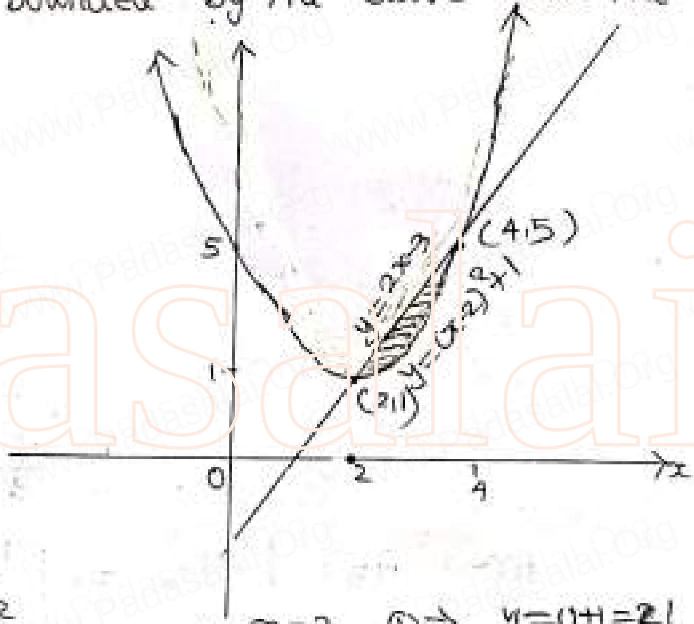
$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^3}{4\times3}\right]_0^4$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{1}{12}x^3\right]_0^4$$

$$= \left(\frac{4}{3}4\sqrt{4} - \frac{1}{\cancel{12}_3}\cancel{4}\times16\right) - (0-0)$$

$$A_2 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units.}$$

9) The curve $y=(x-2)^2+1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.




$y=(x-2)^2+1$ — ①

diff w.r to $x$

$y' = \frac{dy}{dx} = 2(x-2)$

$\frac{dy}{dx} = 0 \Rightarrow 2(x-2)=0$

$x-2=0$

$x=2$

$y''=2$

when $x=2 \Rightarrow y''=2>0$

$y$ has minimum value at $x=2$

① $\Rightarrow$ $y=(2-2)^2+1=1$

minimum point $P(2,1)$

Slope at Q $m=2$

Equation of PQ is $y=2x+c$ — ②

This passing through $P(2,1)$

$1=2(2)+c$

$1-4=c$ $\quad c=-3$

Equation of PQ is $\boxed{y=2x-3}$ — ③

①, ② $\Rightarrow$

$(x-2)^2+1=2x-3$

$x^2-4x+4+1-2x+3=0$

$x^2-6x+8=0$

$(x-2)(x-4)=0$

$x=2,4$

$x=2$ ① $\Rightarrow$ $y=0+1=1$ $(2,1)$

$x=4$ ① $\Rightarrow$ $y=4+1=5$ $(4,5)$

Area bounded by the curves

$$= \int_2^4 (y_U - y_L)\, dx$$

$$= \int_2^4 (2x-3-(x-2)^2-1)\, dx$$

$$= \int_2^4 (2x-4-(x-2)^2)\, dx$$

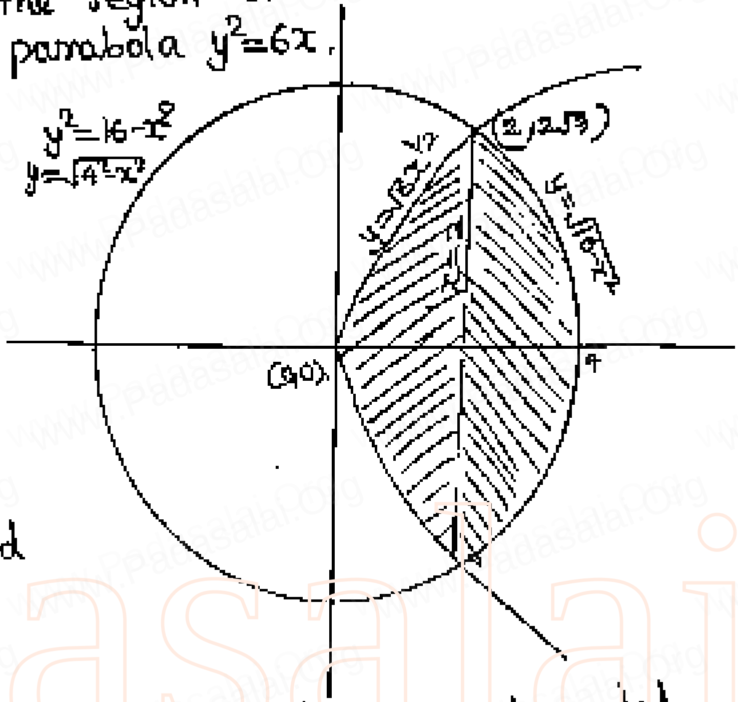
$$= \left[\cancel{4}\frac{x^2}{2} - 4x - \frac{(x-2)^3}{3}\right]_2^4$$

$$= \left(\cancel{16} - \cancel{16} - \frac{2^3}{3}\right) - (4 - 8 - 0)$$

$$= -\frac{8}{3} + 4 = \frac{-8+12}{3}$$

$$= \frac{4}{3} \text{ sq. units.}$$

10) Find the area of the region common to the circle $x^2+y^2=16$ and the parabola $y^2=6x$.




$x^2+y^2=16$ —— ①

$y^2=6x$ —— ②

① ⇒ $x^2+6x=16$

$x^2+6x-16=0$

$(x+8)(x-2)=0$

$x=-8, 2$

$x=-8$ ② ⇒ $y^2=6(-8)$ not valid

$x=2$ ② ⇒ $y^2=6(2)$

$y^2=4\times3$

$y=\pm2\sqrt{3}$

∴ point of intersection $(2,2\sqrt{3}), (2,-2\sqrt{3})$

Area of the region bounded by the curves

$= 2$ [Area lie on the first quadrant]

$$= 2\left[\int_0^2 \sqrt{6}x^{1/2}\,dx + \int_2^4 \sqrt{4^2-x^2}\,dx\right]$$

$$= 2\left(\left[\sqrt{6}\,\frac{x^{3/2}}{3/2}\right]_0^2 + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4\right)$$

$$= 2\left(\left[\sqrt{6}\,\frac{2}{3}\left(2^{3/2}-0\right)\right] + \left[\left(\frac{4}{2}(0) + 8\sin^{-1}\left(\frac{4}{4}\right)\right) - \left(\frac{2}{2}\sqrt{16-4} + 8\sin^{-1}\frac{1}{2}\right)\right]\right)$$

$$= 2\left(\frac{2\sqrt{6}}{3}\,2\sqrt{2} + \frac{\overset{4}{\cancel{8}}\pi}{\cancel{2}} - \sqrt{12} - \overset{4}{\cancel{8}}\left(\frac{\pi}{\cancel{6}_3}\right)\right)$$

$$= 2\left(\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right)$$

$$= 2\left(\frac{8\sqrt{3}-6\sqrt{3}}{3} + \frac{12\pi-4\pi}{3}\right) = \frac{2}{3}\left[2\sqrt{3}+8\pi\right] = \frac{4}{3}(4\pi+\sqrt{3})$$

sq. units

## Volume

① The curve $y=f(x)$ and the line $x=a$ and $x=b$ then the volume of solid of revolution about $x$ axis is $V=\pi\int_a^b y^2 dx$

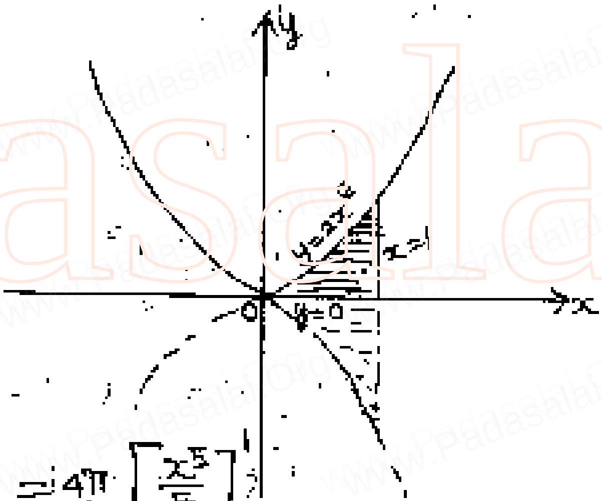
② The curve $x=f(y)$ and the line $y=c$, $y=d$ the volume of solid of revolution about $y$ axis is $V=\pi\int_c^d x^2 dy$.

## Exercise 9.9

1) Find by integration, the volume of the solid generated by revolving about the $x$-axis, the region enclosed by $y=2x^2$, $y=0$ and $x=1$

$y=2x^2$

$y=0 \Rightarrow x=0$
$x=1$




$$\text{volume} = \pi\int_0^1 y^2 dx$$

$$=\pi\int_0^1 4x^4 dx$$

$$=4\pi\int_0^1 x^4 dx = 4\pi\left[\frac{x^5}{5}\right]_0^1$$

$$=4\pi\left[\frac{1}{5}-0\right]$$

$$=\frac{4\pi}{5} \text{ cu. units}$$

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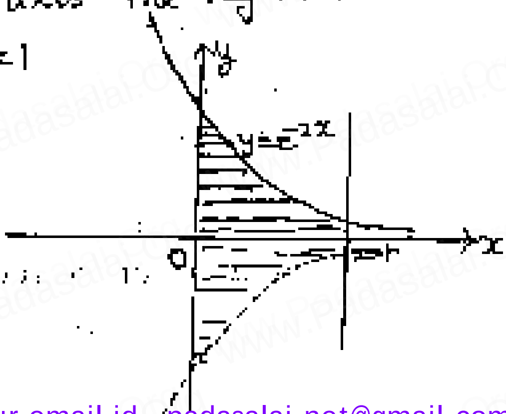
② Find by integration the volume of solid generated by revolving about the $x$-axis the region enclosed by $y=e^{-2x}$ $y=0$, $x=0$ and $x=1$

$y=e^{-2x}$

$y^2=e^{-4x}$ $x=0, x=1$



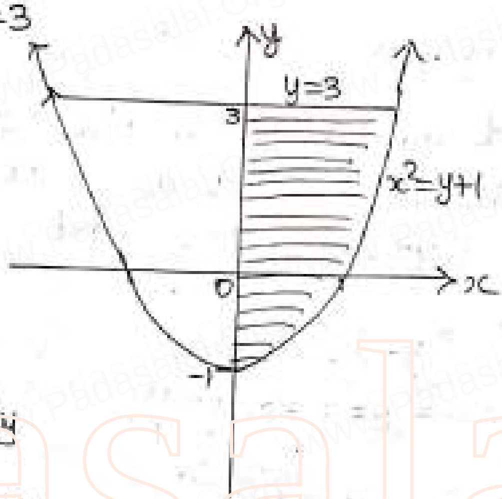
$$\text{volume} = \pi\int_0^1 y^2\,dx$$

$$= \pi\int_0^1 e^{-4x}\,dx$$

$$= \pi\left[\frac{e^{-4x}}{-4}\right]_0^1$$

$$= -\frac{\pi}{4}\left[e^{-4}-e^{0}\right] = \frac{\pi}{4}\left(1-e^{-4}\right) \text{ cu.units}$$

3) Find, by integration the volume of the solid generated by revolving about the y axis, the region enclosed by $x^2=1+y$ and $y=3$




$x^2=1+y$

$x=0 \Rightarrow y=-1$.

$$\text{volume} = \pi\int_{-1}^{3} x^2\,dy$$

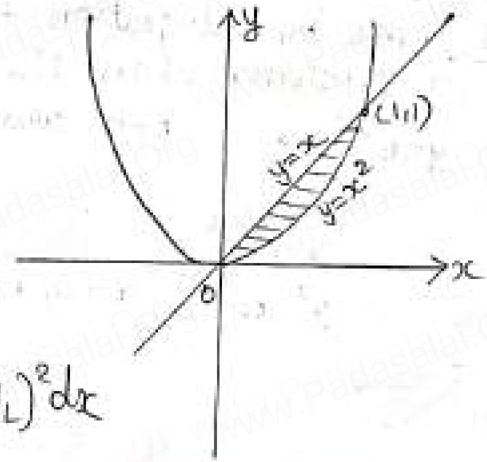
$$= \pi\int_{-1}^{3}(1+y)\,dy$$

$$= \pi\left[y+\frac{y^2}{2}\right]_{-1}^{3}$$

$$= \pi\left[\left(3+\frac{9}{2}\right)-\left(-1+\frac{1}{2}\right)\right]$$

$$= \pi\left[3+\frac{9}{2}+1-\frac{1}{2}\right] = \pi\left[4+\frac{8}{2}\right] = 8\pi \text{ cu units}$$

4) The region enclosed between the graphs of $y=x$ and $y=x^2$ is denoted by R. Find the volume of the generated when R is rotated through 360° about x axis.




$y=x \quad y=x^2$

$x^2=x$

$x^2-x=0$

$x=0,\ x=1$

volume of solid rotated about x axis

$$V = \pi\int_0^1 (y_U-y_L)^2\,dx$$

$$V = \pi \int_0^1 (x - x^2)^2 \, dx$$

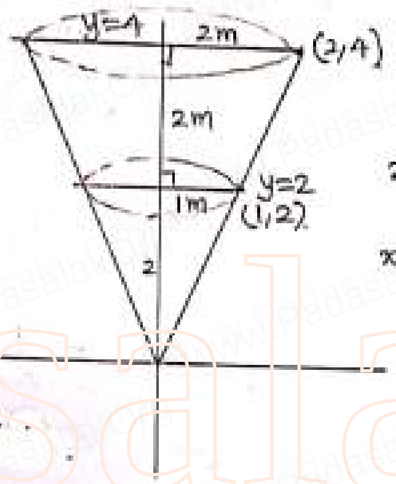
$$= \pi \int_0^1 (x^2 + x^4 - 2x^3) \, dx$$

$$= \pi \left[ \frac{x^3}{3} + \frac{x^5}{5} - \frac{2x^4}{4} \right]_0^1$$

$$= \pi \left( \left[ \frac{1}{3} + \frac{1}{5} - \frac{2^1}{4_2} \right] - 0 \right) = \pi \left[ \frac{10+6-15}{30} \right]$$

$$V = \frac{\pi}{30}$$ (Book answer varies)

5) Find by Integration the volume of the container which is in the shape of a right circular conical frustum as in the figure.



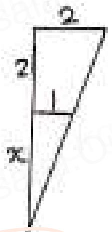

Equation of line joining (1,2), (2,4) is

$$\frac{y-2}{4-2} = \frac{x-1}{2-1}$$

$$y - 2 = 2x - 2$$

$$y = 2x$$

$$y^2 = 4x^2$$




By similar $\Delta$

$$\frac{x}{1} = \frac{x+2}{2}$$

$$2x = x + 2$$

$$x = 2$$

volume of frustum

$$= \pi \int_2^4 x^2 \, dy$$

$$= \pi \int_2^4 \frac{y^2}{4} \, dy$$

$$= \frac{\pi}{4} \left[ \frac{y^3}{3} \right]_2^4$$

$$= \frac{\pi}{3 \times 4} \left[ 4^3 - 2^3 \right] = \frac{\pi}{12} [64 - 8]$$

$$= \frac{\pi}{3_{12}} 56^{14} = \frac{14\pi}{3}$$ cu.units

$R = 2$, $r = 1$, $h = 2$

Volume of frustum $= \frac{1}{3}\pi h (R^2 + r^2 + Rr)$

$$= \frac{1}{3}\pi (2)(4 + 1 + 2)$$

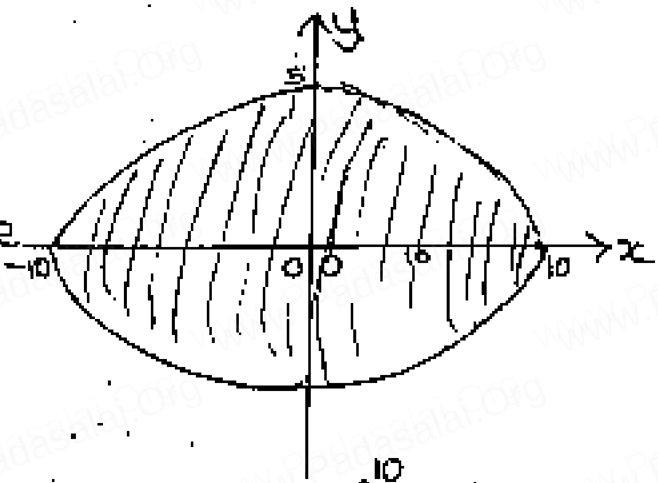
$$= \frac{14\pi}{3}$$

6). A water melon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20 cm and minor axis 10 cm about its major axis. Find its volume using integration.

$2a = 20 \qquad 2b = 10$

$a = 10 \qquad b = 5$

$a^2 = 100 \qquad b^2 = 25$




Equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = \frac{100 - x^2}{100}$$

$$y^2 = \frac{25}{100}(100 - x^2)$$

$$y^2 = \frac{1}{4}(100 - x^2)$$

volume = 2 (volume generated by area bounded in I quadrant).

$$\text{volume} = 2\int_0^{10} \pi y^2 dx$$

$$= 2\pi \int_0^{10} \frac{1}{4}(100 - x^2) dx$$

$$= \frac{\pi}{2}\left[100x - \frac{x^3}{3}\right]_0^{10}$$

$$= \frac{\pi}{2}\left[\left(1000 - \frac{1000}{3}\right) - 0\right]$$

$$= \frac{\pi}{2}\left[\frac{2000}{3}\right]$$

$$= \frac{1000\pi}{3} \text{ cu. units}$$

———x———

Need suggestions
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