Chapter - 5.

Numerical Method

Aim to find approximate values.

x - variable.

$f(x) = y$, next term add h with x.

$\Rightarrow x+h$.

Δ = Del.

∇ = Tel (nelpha).

E = Shifting operator.

Forward Diff. operator.

(FDO) $\Rightarrow \Delta$

$\Delta y_n = y_{n+1} - y_n$ first Diff $= \Delta'$

eg: $\Delta y_0 = y_1 - y_0$

* $f(x) = f(x+h) - f(x)$

Hints: * $\Delta c = 0$, * $\Delta (c \cdot f(x)) = c \cdot \Delta f(x)$.

* $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

* $\Delta (f(x) \cdot g(x)) = f(x) \Delta g(x) + g(x) \Delta f(x)$.

* $\Delta \dfrac{f(x)}{g(x)} = \dfrac{g(x) \Delta f(x) - f(x) \Delta g(x)}{g(x)\, g(x+h)}$.

$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$.

$\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n$.

$\vdots$

$\Delta^{k+1} y_n = \Delta^k y_{n+1} - \Delta^k y_n$

x	$y\,f(x)$	Δy	$\Delta^2 y_n$	$\Delta^3 y_0$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	$\Delta^5 y_0$
x_3	y_3	Δy_2		$\Delta^3 y_1$		
		Δy_3	$\Delta^3 y_1$		$\Delta^4 y_1$	
x_4	y_4			$\Delta^3 y_2$		
		Δy_4	$\Delta^4 y_1$			
x_5	y_5					

Backward diff. operation.

BDO $\Rightarrow \nabla$.

$\nabla y_{n+1} = y_{n+1} - y_n$.

$\nabla y_1 = y_1 - y_0$

$\nabla^2 y_1 = \nabla y_1 - \nabla y_0$

$\nabla^3 y_1 = \nabla^2 y_1 - \nabla^2 y_0$

$\nabla f(x+nh) = \nabla^n f(x)$.

Shifting operator : 'E'

(or)

Displacement operator.

$E(f(x_0)) = f(x_0+h)$

$E\,f(x) = f(x+h)$.

$E^n f(x) = f(x+nh)$.

$E^{-n} f(x) = f(x-nh)$.

$E y_0 = y_1$

$E y_{n-1} = y_n$

$E^2 y_0 = E(E y_0) = E y_1 = y_2$.

$\therefore E^n y_0 = y_n$.

i) $\Delta = E-1$

ii) $E\Delta = \Delta E$.

iii) $\nabla \equiv \frac{E-1}{E}$.

Eg : 5.1.

FDO Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

Eg : 5.2 :

$y = f(x) = x^3 + 2x + 1$. $x = 1, 2, 3, 4, 5$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	4				
		9			
2	13		12		
				6	
		21			
3	34		18		0
		39		6	
4	73		24		
		63			
5	136				

Eg : 5.3

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	8					
		4				
2	12		3			
		7		0		
3	19		3			
		10		0		
4	29		3			
		13				
5	42		$K-55$			
		$K-42$				
6	K					

Gn 2^{nd} is Constant.

$K - 55 = 3$

$K = 58$

$\therefore 6^{th}$ term is 58.

1. $\Delta(\log ax)$.

$\Delta f(x) = f(x+h) - f(x)$.

$\Delta(\log ax) = \log(ax+h) - \log(ax)$.

$= \log \frac{ax+h}{ax}$

$= \log\left(\frac{ax}{ax} + \frac{h}{ax}\right)$

$= \log\left(1 + \frac{h}{ax}\right)$.

4. $f(x) = x^2+3x$ ST $\Delta f(x) = 2x+4$

$f(x) = x^2+3x$

$\Delta f(x) = \Delta(x^2+3x)$.

$\because \left[\Delta f(x) = f(x+h) - f(x)\right]$.

$= (x+h)^2 + 3(x+h) - (x^2+3x)$

$= x^2+h^2+2xh+3x+3h-x^2-3x$.

$= h^2+2xh+3h$.

$\boxed{h=1}$

$= 1 + 2x + 3(1)$

$= 2x+4$.

$\therefore$ Hence Proved.

3. $h=1$; Pt $= (E^{-1}\Delta)x^3 = 3x^2-3x+1$.

$E^{-h}\Delta f(x) = \Delta f(x+nh)$.

$E^{-3}\Delta f(x^3) = \Delta(x-1)^3$.

$\Rightarrow f(x) = f(x+h) - f(x)$.

$= (x-1+1) - (x-1)^3$.

$= x^3 - (1 \times x^3 - 3x^2 \times 1 + 3x \times 1^2 - 1 \times 1^3)$.

$= x^3 - (x^3 - 3x^2 + 3x - 1)$

$= x^3 - x^3 + 3x^2 - 3x + 1$.

$= 3x^2 - 3x + 1$.

$\therefore$ L.H.S = R.H.S

Proved.

Eg: 5.4

(i) Δe^{ax} $\quad \because f(x) = f(x+h) - f(x)$

$= e^{a(x+h)} - e^{ax}$

$= e^{ax} \cdot e^{h} - e^{ax}$

$= e^{ax}(e^{h}-1)$

(ii) $\Delta^2 e^x = \Delta[\Delta e^x]$ $\quad f(x) = f(x+h) - f(x)$

$= \Delta[e^{x+h} - e^x]$

$= \Delta[e^x e^h - e^x]$

$= \Delta e^x [e^h - 1]$ $\quad \Delta e^x = [e^{x+h} - e^x]$

$f(x) = f(x+h) - f(x)$ $= [e^h-1][e^h-1] e^x$ $\quad = e^x e^h - e^x$

$= e^x[e^h - 1]$

$= (e^h-1)^2 e^x$

iii) $\Delta \log x = \log(x+h) - \log x$

$= \log \frac{x+h}{x}$ $\quad f(x) = f(x+h) - f(x)$

$= \log \frac{\cancel{x}}{\cancel{x}} + \frac{h}{x}$

$= \log\left[1 + \frac{h}{x}\right]$

Eg: 5.5

$U_0 = 1,\ U_1 = 11,\ U_2 = 21,\ U_3 = 28,\ U_4 = 29$. find $\Delta^4 U_0$

$\Delta^4 U_0$:

$(E-1)^4 U_0 = 0$

$= (1 \times E^4 - 4E^3 \times 1 + 6E^2 \times 1^2 - 4 \times E^1 \times 1^3 + 1 \times 1^4)$

$= E^4 U_0 - 4E^3 U_0 + 6E^2 U_0 - 4E U_0 + U_0)$. $\quad +1 \quad -4 \quad +6 \quad -4$

$= U_4 - 4U_3 + 6U_2 - 4U_1 + U_0$

$= 29 - 112 + 126 - 44 + 1$

$\Delta^4 U_0 = 0$.

6)

x	0	1	2	3	4
y_x	1	3	9	-	81

4^{th} entry $= 0$.

$\Delta^4 y_0 = 0$. $\quad +1 \quad -4 \quad +6 \quad -4 \quad +1$.

$(E-1)^4 y_0 = 0$.

$(E^4 - 4E^3 + 6E^2 - 4E^2 + E) y_0 = 0$.

$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E^2 y_0 + y_0 = 0$

$(E^n y_0 = y_n)$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$81 - 4y_3 + (6 \times 9) - (4 \times 3) + 1 = 0$

$-4y_3 = -124$

$y_3 = \frac{124}{4}$ $\quad y_3 = 31$.

8.

x	0	1	2	3	4	5
$y=f(x)$	0	-	8	15	-	35

$\Delta^4 y_K = 0.$

$(E-1)^4 y_K = 0.$

$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_K = 0.$

K=0

$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$

$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$

$(E^n y_0 = y_n)$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$

$(15\times4) \quad (6\times8).$

$y_4 - 60 + 48 - 4y_1 + 0 = 0$

$y_4 - 4y_1 = 12 \ldots (1).$

K=1

$(E^4 - 4E^3 + 6E^2 - 4E + 1) y_1 = 0.$

$E^4 y_1 - 4E^3 y_1 + 6E^2 y_1 - 4E y_1 + y_1 = 0$

$(E^n y_1 = y_{n+1})$

$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0.$

$(6\times15). \quad (4\times8)$

$35 - 4y_4 + 90 - 32 + y_1 = 0$

$-4y_4 + y_1 = -93 \ldots (2)$

$(2) \to \cancel{-4y_4} + y_1 = 93$

$4\times(1) \to \cancel{4y_4} - 16y_1 = 48$

$-15y_1 = -45$

$\boxed{y_1 = 3.}$

$(1) \Rightarrow y_4 - 4y_1 = 12.$

$y_4 - 4(3) = 12$

$y_4 - 12 = 12$

$\boxed{y_4 = 24.}$

Eg: 5.11.

Year (x)	1961	1962	1963	1964	1965	1966	1967.
Production (y)	200	220	260	-	350	-	430.

b entries = 0.

$\Delta^5 y_K = 0$

$(E-1)^5 y_K = 0.$

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E^1 - 1) y_K = 0.$

K=0

$(E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E^1 y_0 - y_0) = 0$

$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0.$

$y_5 - 5(350) + 10y_3 - 10(260) + 5(220) - 200 = 0$

$y_5 + 10y_3 = 3450 \ldots (1).$

K=1

$(E^5 y_1 - 5E^4 y_1 + 10E^3 y_1 - 10E^2 y_1 + 5E^1 y_1 - y_1) = 0$

$y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$430 - 5y_5 + 10(350) - 10y_3 + 5(260) - 220 = 0$

$-5y_5 - 10y_3 = -5010 \ldots (2).$

$(2) \to \quad -5y_5 - 10y_3 = -5010$

$(1)\times 5 \to \quad 5y_5 + 50y_3 = 17250$

$40y_3 = 12240$

$\boxed{y_3 = 306.}$

$(1) \to y_5 + 10y_3 = 3450$

$y_5 + 10(306) = 3450$

$y_5 + 3060 = 3450$

$\boxed{y_5 = 390.}$

Eg : 5.10

	0	1	2	3	4.
x	2	3	4	5	6
$f(x)$	45.0	49.2	54.1	-	67.4.

4 entries = 0

$\Delta^4 y_0 = 0$

$(E-1)^4 y_0 = 0$

$(E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = 0.$

$E^4y_0 - 4E^3y_0 + 6E^2y_0 - 4Ey_0 + y_0 = 0$

$E^n y_0 = y_n.$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$67.4 - 4y_3 + 6(54.1) - 4(49.2) + 45.0 = 0.$

$67.4 - 4y_3 + 324.6 - 196.8 + 45.0 = 0$

$-4y_3 = -240.2$

$\boxed{y_3 = 60.05.}$

Eg : 5.9

$y_3 = 2, y_4 = -6, y_5 = 8, y_6 = 9, y_7 = 17$, Calculate $\Delta^4 y_3$.

$\Delta^4 y_3 = 0$

$(E-1)^4 y_3 = 0.$

$(E^4 - 4E^3 + 6E^2 - 4E + I)y_3 = 0$

$(E^n y_3 = y_{3+n}).$

$E^4y_3 - 4E^3y_3 + 6E^2y_3 - 4Ey_3 + y_3 = 0$

$y_7 - 4y_6 + 6y_5 - 4y_4 + y_3 = 0.$

$= 17 - 4(9) + 6(8) - 4(-6) + 2$

$= 17 - 36 + 48 + 24 + 2$

$= 55.$

5. $\Delta\left(\frac{1}{(x+1)(x+2)}\right)$

$\frac{1}{(x+1)(x+2)} = \frac{1}{(x+1)} - \frac{1}{(x+2)}.$

$\Delta\left(\frac{1}{(x+1)(x+2)}\right) = \Delta\left(\frac{1}{x+1}\right) - \Delta\left(\frac{1}{x+2}\right)$

$= \left(\frac{1}{x+1+1} - \frac{1}{x+1}\right) - \left(\frac{1}{x+1+2} - \frac{1}{x+2}\right)$

$= \frac{1}{x+2} - \frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+2}$

$= \frac{2}{x+2} - \frac{1}{x+1} - \frac{1}{x+3}.$

$= \frac{2(x+1)(x+3) - 1(x+2)(x+3) - 1(x+1)(x+2)}{(x+1)(x+2)(x+3)}$

$$= \frac{2(x^2+4x+3) - 1(x^2+5x+6) - 1(x^2+3x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{2x^2+8x+6-x^2-5x-6-x^2-3x-2}{(x+1)(x+2)(x+3)}$$

$$= \frac{-2}{(x+1)(x+2)(x+3)}$$

Eg: 5.6

$$\Delta^2\left(\frac{1}{x}\right) = \Delta\left(\Delta\left(\frac{1}{x}\right)\right)$$

$$= \Delta\left(\frac{1}{x+1} - \frac{1}{x}\right)$$

$$= \Delta\frac{1}{x+1} - \Delta\frac{1}{x}$$

$$= \frac{1}{x+1+1} - \frac{1}{x+1} - \left(\frac{1}{x+1} - \frac{1}{x}\right)$$

$$= \frac{1}{x+2} - \frac{1}{x+1} - \frac{1}{x+1} + \frac{1}{x}$$

$$= \frac{-2}{x+1} + \frac{1}{x+2} - \frac{1}{x}$$

$$= \frac{-2(x)(x+2) + 1(x)(x+1) - 1(x+1)(x+2)}{(x)(x+1)(x+2)}$$

$$= \frac{-\cancel{2x^2} - \cancel{4x} + \cancel{x^2} + \cancel{x} + \cancel{x^2} + \cancel{3x} + 2}{(x)(x+1)(x+2)}$$

$$= \frac{2}{(x)(x+1)(x+2)}$$

Eg: 5.7

$$f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$f(x+h) = f(x) + \Delta f(x).$$

L.H.S

$$f(4) = f(3) + \Delta f(3)$$

$(x=3,\ h=1)$ $= f(3) + \Delta(f(2) + \Delta f(2))$

$$= f(3) + \Delta f(2) + \Delta^2 f(2).$$

$$= f(3) + \Delta f(2) + \Delta^2(f(1) + \Delta f(1))$$

$$= f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1).$$

L.H.S = R.H.S.

Eg: 5.5

$$\Delta\left[\frac{5x+12}{x^2+5x+6}\right] \qquad (x+3)(x+2)$$

6
3 2
5

$$\frac{5x+12}{x^2+5x+6} = \frac{A}{(x+3)} + \frac{B}{(x+2)}$$

$$= \Delta\left[\frac{3}{(x+3)} + \frac{2}{(x+2)}\right]$$

$$= \Delta\frac{3}{(x+3)} + \Delta\frac{2}{(x+2)} \qquad f(x) = f(x+h) - f(x)$$

$$= \frac{3}{x+1+3} - \frac{3}{x+3} + \frac{2}{x+1+2} - \frac{2}{x+2}$$

$$= \frac{3}{x+4} - \frac{3}{x+3} + \frac{2}{x+3} - \frac{2}{x+2}.$$

$$= \frac{-2}{x+2} - \frac{1}{x+3} + \frac{3}{x+4}$$

$$= \frac{-2(x+3)(x+4) - 1(x+2)(x+4) + 3(x+2)(x+3)}{(x+2)(x+3)(x+4)}$$

$$= \frac{-2(x^2+12+7x) - (x^2+8+6x) + 3(x^2+6+5x)}{(x+2)(x+3)(x+4)}$$

$$= \frac{-2x^2-24-14x-x^2-8-6x+3x^2+18+15x}{(x+2)(x+3)(x+4)}$$

$$= \frac{-5x-14}{(x+2)(x+3)(x+4)}$$

Exercise 5.2

Interpolation
- Graphical
- Algebraic
  - Newton's forward
  - Newton's Backward

Forward:

$$f(x_0+nh) = f(0) + \frac{n}{1!}\Delta f(x_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0) + \dots$$

(or)

$$y(x_0+nh) = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \dots$$

$$n = \frac{x-x_0}{h}.$$

Backward:

$$f(x_n+nh) = f(x_n) + \frac{n}{1!}\Delta f(x_n) + \frac{n(n+1)}{2!}\Delta^2 f(x_n) + \dots$$

(or)

$$y(x_n+h) = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \dots$$

$$n = \frac{x-x_n}{h}.$$

Lagrange's Interpolation formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$$

$$+ \dots \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}\dots y_n$$

12.

x	3	7	11	19
$f(x)$	42	43	47	60

$x_0=3,\ x_1=7,\ x_2=11,\ x_3=19$ Gn. $x=15$.

$y_0=42,\ y_1=43,\ y_2=47,\ y_3=60$

Lagrange's :

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3.$$

$$= \frac{(8)(4)(-4)}{(-4)(-8)(-16)}(42) + \frac{(12)(4)(-4)}{(4)(-4)(-12)}(43) + \frac{(12)(8)(-4)}{(8)(4)(-8)}(47) + \frac{(12)(8)(4)}{(16)(12)(8)}(60).$$

$= +10.5 - 43 + 70.5 + 15$

$= 53$

when $x=15$, $y=53$.

11.

x	1982	1983	1984	1986
$f(x)$	150	235	365	525

$x_0=1982,\ x_1=1983,\ x_2=1984,\ x_3=1986$. Gn $x=1985$.

$y_0=150,\ y_1=235,\ y_3=365,\ y_4=525$.

Lagrange

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3).$$

$$= \frac{(2)(1)(-1)}{(-1)(-2)(-4)}(150) + \frac{(3)(1)(-1)}{(1)(-1)(-3)}(235) + \frac{(3)(2)(-1)}{(2)(1)(-2)}(365)$$

$$+ \frac{(3)(2)(1)}{(4)(3)(2)}(525).$$

$= 593.75$

10. $x=26,\ x_0=15,\ x_1=25,\ x_2=30,\ x_3=35$.

$y_0=36,\ y_1=40,\ y_2=45,\ y_3=48$.

Lagrange :

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1).$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3).$$

$$= \frac{(1)(-4)(-9)}{(-10)(-15)(-20)}(36) + \frac{(11)(-4)(-9)}{(10)(-5)(-10)}(40) + \frac{(11)(1)(-9)}{(15)(5)(-5)}(45)$$

$$+ \frac{(11)(1)(-4)}{(20)(10)(5)}(48).$$

$= 53.336$

9.

$x = 1986$, $x_0 = 1974$, $x_1 = 1978$, $x_2 = 1982$, $x_3 = 1990$

$y_0 = 25$, $y_1 = 60$, $y_2 = 80$, $y_3 = 170$.

Lagrange:

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$= \frac{(8)(4)(-4)}{(-4)(-4)(-16)}(25) + \frac{(12)(4)(-4)}{(4)(-4)(-12)}(60) + \frac{(12)(8)(-4)}{(8)(4)(-8)}(80) + \frac{(12)(8)(4)}{(16)(12)(8)}(170)$$

$= 25 - 60 + 120 + 42.5$

$= 127.5$

Eg: 5.22

$x = 10$, $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$.

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

Lagrange:

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}(y_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}(y_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}(y_3)$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$= 2 - 4.33 + 11.67 + 5.33$

$= 14.67$.

3.

x	0	1	2	3
$f(x)$	1	2	1	10

$x = x$, $x_0 = 0$, $h = 1$, $n = \frac{x - x_0}{h}$

$= \frac{x - 0}{1}$ $\quad n = x$

$$y(x) = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$(x^2 - x)(x-2)$

$\frac{(x^3 - 2x^2 - x + 2x)(2}{3}$

$$y(x) = 1 + \frac{x}{1}(1) + \frac{\overbrace{x(x-1)}^{(x^2-x)}}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= x + \frac{(-2x^2 + 2x)}{2} + \frac{(x^3 - 2x^2 - x + 2x)12}{6}$$

$$= \frac{x - 2x^2 + 2x + x^3 - 2x^2 - x + 2x\,(12)}{12}$$

$= 1 + x - x^2 + x$

$+ 2x^3 - 4x^2 - 2x^2 + 4x$

$= 2x^3 - 7x^2 + 6x$

6.

x	30	35	40	45	50
$f(x)$	15.9	14.9	14.1	13.3	12.5

forward.

$x = 32,\ x_0 = 30,\ h = 5.\quad n = \frac{x - x_0}{h}$

$= \frac{32-30}{5}$

$n = 0.4$

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	15.9				
		-1			
35	14.9		+0.2		
		-0.8		-0.2	
40	14.1		+0		0.2
		-0.8		0	
45	13.3		+0		
		-0.8			
50	12.5				

$y_{(32)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots$

$= 15.9 + \frac{0.4}{1}(-1) + \frac{(0.4)(0.4-1)}{2!}(0.2) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(-0.2)$

$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(0.2)$

$= 15.9 - 0.4 - 0.024 - 0.0128 - 0.00832$

$= 15.45$

7.

x	40	50	60	70	80	90
T	180	204	226	250	276	304

backward.

$x = 84,\ x_n = 90,\ h = 10,\ n = \frac{x - x_n}{h}$

$= \frac{84-90}{10} = -0.6.$

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	180					
		24				
50	204		-2			
		22		4		
60	226		2		-4	
		24		0		4
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

$y_{(84)} = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \frac{n(n+1)(n+2)(n+3)}{4!}\nabla^4 y_n + \dots$

$= 304 + \frac{(-0.6)}{1}(28) + \frac{(-0.6)(-0.6+1)}{2!}(2) + 0 + 0 + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!}(4)$

$= 304 - 16.8 - 0.24 - 0.0914$

$= 286.86$

8.

x	0	1	2	3
$f(x)$	1	2	11	34

backward.
$x_n = 2.8$

$x = 2.8,\ x_n = 3,\ h = 1,\ n = \frac{x - x_n}{1}$

$= \frac{2.8-3}{1}$

$= -0.2$

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		8	
		9		6
2	11		14	
		23		
3	34			

$$y(2.8) = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \frac{n(n+1)(n+2)}{3!}\Delta^3 y_n.$$

$$= 34 + \frac{(-0.2)}{1!}(23) + \frac{(-0.2)(-0.2+1)}{2!}(14) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{3!}(6)$$

$$= 34 - 4.6 - 1.12 - 0.288$$

$$= 27.992.$$

Eg: 5.14.

x	0	1	2	3	4	5	6
y	2	4	10	16	20	24	38

$x = 3.2,\ x_0 = 3,\ h = 1,\ n = \frac{x - x_0}{h} = \frac{3.2-3}{1} = 0.2.$

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$
3	16			
		4		
4	20		0	
		4		10
5	24		10	
		14		
6	38			

$$y_{(3.2)} = y_0 + \frac{n}{1!}\Delta y_n + \frac{n(n-1)}{2!}\Delta^2 y_n + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_n + \cdots$$

$$= 16 + \frac{0.2}{1}(4) + 0 + \frac{(0.2)(0.2-1)(0.2-2)}{3!}(10)$$

$$= 16 + 0.8 + 0.48$$

$$= 17.28$$

Eg: 5.15

Mark	30-40	40-50	50-60	60-70	70-80
No. of Student	31	42	51	35	31

forward

$x = 45,\ x_0 = 40,\ h = 10,\ n = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5.$

Less than 40

x	$f(x)$	Δ	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$y_{(45)} = y_0 + \frac{n}{1!}(\Delta y_0) + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 31 + \frac{0.5}{1}(42) + \frac{0.5(0.5-1)}{2}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(37).$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453$

$= 47.867$

Eg: 5.16

Weight	20-40	40-60	60-80	80-100	100-120
No. of Student	250	120	100	70	50

Forward

$x = 70, \quad x_0 = 40, \quad h = 20, \quad n = \frac{x - x_0}{h}$

$= \frac{70-40}{20} = 1.5$

x below	f(x)	Δ	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		10	
100	540		-20		
		50			
120	590				

$$y_{(70)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 250 + \frac{1.5}{1}(120) + \frac{1.5(1.5-1)}{2!}(-20) + \frac{1.5(1.5-1)(1.5-2)}{3!}(-10) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!}(20)$$

$= 250 + 180 - 7.5 + 0.625 + 0.46875$

$= 423.59$

Eg: 5.13

Year	1891	1901	1911	1921	1931
Population	98752	1,32,285	1,68,076	1,95,690	2,46,050

Forward

$x = 1905, \quad x_0 = 1891, \quad h = 10, \quad n = \frac{x - x_0}{h}$

$= \frac{1905 - 1891}{10} = 1.4$

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	98752				
		33533			
1901	1,32,285		2258		
		35791		-10455	
1911	1,68,076		-8197		41438
		27614		30983	
1921	1,95,690		22786		
		50360			
1931	2,46,050				

$$y_{(1905)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 98752 + \frac{1.4}{1}(33533) + \frac{1.4(1.4-1)}{2!}(2258) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-10455) + \frac{1.4(1.4-1)(1.4-2)(1.4-3)}{4!}(41438)$$

$= 98752 + 46946.2 + 632.24 + 585.48 + 928.212$

$= 147844$

Forward

Eg : 5.17.

x	1941	1951	1961	1971	1981	1991
y	20	24	29	36	46	51

$x = 1946$. $x_0 = 1941$. $h = 10$, $n = \frac{x - x_0}{h}$

$= \frac{1946 - 1941}{10} = 0.5$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1941	20					
		4				
1951	24		1			
		5		1		
1961	29		2		0	
		7		1		-9
1971	36		3		-9	
		10		-8		
1981	46		-5			
		5				
1991	51					

$y_{(1946)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots$

$= 20 + \frac{0.5}{1}(4) + \frac{0.5(0.5-1)}{2!}(1) + \frac{0.5(0.5-1)(0.5-2)}{3!}(1) + 0$

$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)(0.5-4)}{5!}(-9)$

$= 20 + 2 - 0.125 + 0.0625 + 0 - 0.246093$

$= 21.691407$

Eg : 5.18

Backward

x	140	150	160	170	180
f(x)	3.685	4.854	6.302	8.076	10.225

$x_n = 180$, $x = 175$, $h = 10$, $n = \frac{x - x_n}{h}$

$= \frac{175 - 180}{10} = -0.5$

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$y_{(175)} = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \frac{n(n+1)(n+2)}{3!}\Delta^3 y_n + \dots$ (0.049)

$= 10.225 + \frac{(-0.5)}{1}(2.149) + \frac{(-0.5)(-0.5+1)}{2!}(0.375) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}$

$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.002)$

$= 10.225 - 1.0745 - 0.0468 - 0.0030625 - 0.000078125$

$= 9.1$

Eg : 5.19

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512.

$x = 8$, $x_n = 7.5$, $h = 1$, $n = \frac{x - x_n}{h}$

$= \frac{-8+7.5}{1} = -0.5$

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

$$y(7.5) = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \frac{n(n+1)(n+2)}{3!}\Delta^3 y_n + \cdots$$

$$= 512 + \frac{(-0.5)}{1}(169) + \frac{(-0.5)(-0.5+1)}{2!}42 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(6)$$

$= 512 - 84.5 - 5.25 - 0.375$

$= 421.875$

Eg : 5.20

Age	45	50	55	60	65	backward.
Premium	114.84	96.16	83.32	74.48	68.48	

$x = 63$, $x_n = 65$, $h = 5$, $n = \frac{x - x_n}{h}$

$= \frac{63-65}{5} = -0.4$.

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6.			
65	68.48				

$$y(63) = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \frac{n(n+1)(n+2)}{3!}\Delta^3 y_n + \cdots$$

$$= 68.48 + \frac{-0.4}{1}(-6) + \frac{(-0.4)(-0.4+1)}{2!}(2.84) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(-1.16)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}(0.68)$$

$= 68.48 + 2.4 - 0.3408 + 0.07424 - 0.02828$

$= 70.585.$

Eg : 5.21

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29.

Polynomial.

$x = x$, $x_n = 7$, $h = 1$, $n = \frac{x - x_n}{h}$

$= \frac{x-7}{1} = x-7$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	0
		2		
2	4		1	0
		3		
3	7		1	0
		4		
4	11		1	0
		5		
5	16		1	0
		6		
6	22		1	0
		7		
7	29			

$y_{(x)} = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n+1)}{2!}\Delta^2 y_n + \frac{n(n+1)(n+2)}{3!}\Delta^3 y_n + \dots$

$= 29 + \frac{x-7}{1}(7) + \frac{(x-7)(x-7+1)}{2!}(1)$.

$= 29 + 7x - 49 + \frac{1}{2}\left[(x-7)(x-6)\right]$ × 9 ÷ by 2

$= \frac{1}{2}\left[58 + 14x - 98 + x^2 + 42 - 13x\right]$

$= \frac{1}{2}\left[x^2 + x + 2\right]$