Chapter6 Applications of Vector Algebra

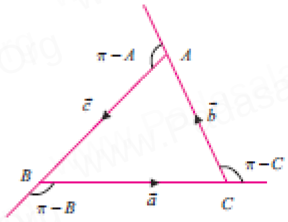
Example 6.1 (Cosine formulae)

With usual notations, in any triangle ABC,

prove the following by vector method.

(i) $a^2 = b^2 + c^2 - 2bc \cos A$

Solution:



In any triangle ABC,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\vec{a} = -\vec{b} - \vec{c}$$

Applying dot product,

$$\vec{a}.\vec{a} = (\vec{b} + \vec{c}).(\vec{b} + \vec{c})$$

$$\vec{a}^2 = \vec{b}^2 + \vec{c}^2 + 2\vec{b}.\vec{c}$$

$$a^2 = b^2 + c^2 + 2bc\cos(\pi - A)$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

(ii) $b^2 = c^2 + a^2 - 2ca \cos B$

(iii) $c^2 = a^2 + b^2 - 2ab \cos C$

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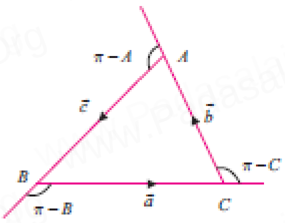
Example 6.2

With usual notations, in any triangle ABC,

prove the following by vector method.

(i) $a = b\cos C + c\cos B$

Solution:



In any triangle ABC,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\vec{a} = -\vec{b} - \vec{c}$$

Applying dot product of $\vec{a}$ on both sides,

$$\vec{a}.\vec{a} = -\vec{b}.\vec{a} - \vec{c}.\vec{a}$$

$$\vec{a}^2 = -|\vec{b}||\vec{a}|\cos(\pi - C) - |\vec{c}||\vec{a}|\cos(\pi - B)$$

$$a^2 = -ba[-\cos(C)] - -ca[-\cos(B)]$$

$$a^2 = ba\cos(C) + ca\cos(B)$$

Dividing by a,

$$a = b\cos C + c\cos B \qquad \text{Hence Proved.}$$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

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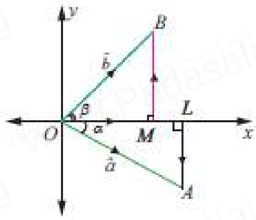
Example 6.3 By vector method, prove that

cos(α +β) = cos α cos β − sin α sin β

Solution:



Take two points A and B on the unit circle with centre as origin O.

So, $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

$\angle AOX = \alpha$ and $\angle BOX = \beta$

$\therefore \angle AOB = \alpha + \beta$

Let $\vec{\imath}$ and $\vec{\jmath}$ be the unit vectors along x , y axis.

Draw AL and BM perpendicular to x axis.

Then $A(cos\alpha, -sin\alpha)$ and $B\ (cos\beta, sin\beta)$ are the co-ordinates.

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$$

$$= \cos\alpha\vec{\imath} - \sin\alpha\vec{\jmath}$$

$$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$$

$$= \cos\beta\vec{\imath} + \sin\beta\vec{\jmath}$$

By definition

$$\overrightarrow{OA}.\overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos(\alpha + \beta)$$

$$= (1)(1)\cos(\alpha+\beta)$$

$$\overrightarrow{OA}\,.\,\overrightarrow{OB} = \cos(\alpha+\beta) \quad ....(1)$$

By value

$$\overrightarrow{OA}\,.\,\overrightarrow{OB} = (\cos\alpha\vec{\imath} - \sin\alpha\vec{\jmath}).(\cos\beta\vec{\imath} + \sin\beta\vec{\jmath})$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad .....(2)$$

From (1) and (2)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

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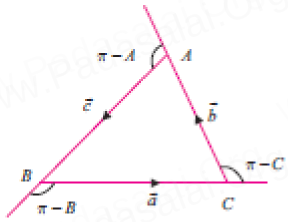
Example 6.4 With usual notations, in any triangle ABC, prove by vector method that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Solution:



Let $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$

Area of triangle ABC =

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{BC}\times\overrightarrow{BA}\right| = \frac{1}{2}\left|\overrightarrow{CB}\times\overrightarrow{CA}\right|$$

So, $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \left|\overrightarrow{BA}\times\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\times\overrightarrow{CA}\right|$

$$cb\sin(\pi - A) = ca\sin(\pi - B) = ab\sin(\pi - C)$$

$$cb\sin A = ca\sin B = ab\sin C$$

Dividing by abc

$$\frac{cb\sin A}{abc} = \frac{ca\sin B}{abc} = \frac{ab\sin C}{abc}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Inverting, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Hence proved.

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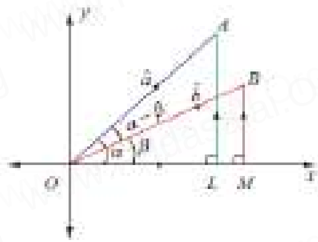
Example 6.5 Prove by vector method that

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Solution:



Take two points A and B on the unit circle with centre as origin O.

So, $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 1$

$\angle AOX = \alpha$ and $\angle BOX = \beta$

$\therefore \angle AOB = \alpha - \beta$

Let $\vec{\imath}$ and $\vec{\jmath}$ be the unit vectors along x , y axis.

Draw AL and BM perpendicular to x axis.

Then $A(cos\alpha, sin\alpha)$ and $B\ (cos\beta, sin\beta)$ are the co-ordinates.

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$$

$$= \cos\alpha\vec{\imath} + \sin\alpha\vec{\jmath}$$

$$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$$

$$= \cos\beta\vec{\imath} + \sin\beta\vec{\jmath}$$

By definition

$$\overrightarrow{OB}\times\overrightarrow{OA} = \left|\overrightarrow{OB}\right|\left|\overrightarrow{OA}\right|\sin(\alpha-\beta)\,\hat{k}$$

$$= (1)(1)\sin(\alpha-\beta)\,\hat{k}$$

$$\overrightarrow{OB}\times\overrightarrow{OA} = \sin(\alpha-\beta)\,\hat{k} \quad ......(1)$$

By value $\overrightarrow{OB}\times\overrightarrow{OA} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$

$$= \pm\hat{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta) \quad ......(2)$$

From (1) and (2)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

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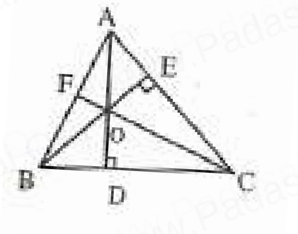
Example 6.7 Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

Solution:



Let ABC be the given triangle.

Let the altitudes AD and BE intersecting at O and take it as the origin.

To prove that CO is perpendicular to AB.

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$

AD $\perp r$ BC

$\Rightarrow \overrightarrow{OA} \perp r\, \overrightarrow{BC}$

$\therefore \overrightarrow{OA} . \overrightarrow{BC} = 0$

$\Rightarrow \vec{a}.(\vec{c} - \vec{b}) = 0$............ (1)

BE $\perp r$ CA

$\Rightarrow \overrightarrow{OB} \perp r\, \overrightarrow{CA}$

$\therefore \overrightarrow{OB} . \overrightarrow{CA} = 0$

$\Rightarrow \vec{b}.(\vec{a} - \vec{c}) = 0$.........(2)

(1) + (2) $\Rightarrow$

$\vec{a}.(\vec{c} - \vec{b}) + \vec{b}.(\vec{a} - \vec{c}) = 0$

$\vec{a}.\vec{c} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{c} = 0$

$\vec{a}.\vec{c} - \vec{b}.\vec{c} = 0$

$(\vec{a} - \vec{b}).\vec{c} = 0$

$(\overrightarrow{OA} - \overrightarrow{OB}).\vec{c} = 0$

$\Rightarrow \overrightarrow{BA}.\overrightarrow{OC} = 0$

$\Rightarrow$ OC $\perp r$ AB

Hence the altitudes of a triangle are concurrent.

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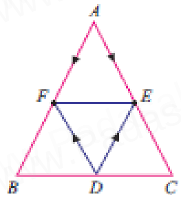
Example 6.8 In triangle ABC , the points D, E, F are the midpoints of the sides BC, CA, and AB respectively. Using vector method, show that the area of Δ DEF is equal to $\frac{1}{4}$ (area of ΔABC).

Solution:



In Triangle ABC, consider A as the origin.

D,E,F are the midpoints of BC,CA and AB.

Then the position vectors of D,E,F are

$\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}, \frac{\overrightarrow{AC}}{2}$, and $\frac{\overrightarrow{AC}}{2}$.

Considering AB and Ac as the adjacent sides ,

area of triangle ABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$\text{Area } \Delta \text{ DEF } = \frac{1}{2}\left|\overrightarrow{DE} \times \overrightarrow{DF}\right|$$

$$= \frac{1}{2}\left|\frac{\overrightarrow{AB}}{2} \times \frac{\overrightarrow{AC}}{2}\right|$$

$$= \frac{1}{2}\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right)$$

$$= \frac{1}{4}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{4}(\text{Area } \Delta \text{ ABC})$$

....

Example 6.9 A particle acted upon by constant forces $2\vec{i} + 5\vec{j} + 6\vec{k}$ and $-\vec{i} - 2\vec{j} - \vec{k}$ is displaced from the point $(4, -3, -2)$ to the point $(6,1,-3)$. Find the total work done by the forces.

Solution: Given Force $\vec{F_1} = 2\vec{i} + 5\vec{j} + 6\vec{k}$

Force $\vec{F_2} = -\vec{i} - 2\vec{j} - \vec{k}$

Hence resultant force $\vec{F} = \vec{F_1} + \vec{F_2}$

$$= 2\vec{i} + 5\vec{j} + 6\vec{k} - \vec{i} - 2\vec{j} - \vec{k}$$

$$\vec{F} = \vec{\imath} + 3\vec{\jmath} + 5\vec{k}$$

Let A and B be the points $(4, -3, -2)$ and $(6, 1, -3)$ respectively.

Then the displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (6\vec{\imath} + \vec{\jmath} - 3\vec{k}) - (4\vec{\imath} - 3\vec{\jmath} - 2\vec{k})$$

$$= 6\vec{\imath} + \vec{\jmath} - 3\vec{k} - 4\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$$

$$\vec{d} = 2\vec{\imath} + 4\vec{\jmath} - \vec{k}$$

Work done $W = \vec{F}.\vec{d}$

$$= (\vec{\imath} + 3\vec{\jmath} + 5\vec{k}).(2\vec{\imath} + 4\vec{\jmath} - \vec{k})$$

$$= 2 + 12 - 5$$

$$= 14 - 5$$

Work done W = 9 units.

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Example 6.10 A particle is acted upon by the forces $3\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$ and $2\vec{\imath} + \vec{\jmath} - \vec{k}$ is displaced from the point $(1,3,-1)$ to the point $(4,-1,\lambda)$ If the work done by the forces is 16units, find the value of λ.

Solution: Given Force $\vec{F}_1 = 3\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$

Force $\vec{F}_2 = 2\vec{\imath} + \vec{\jmath} - \vec{k}$

Hence resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2$

$$= 3\vec{\imath} - 2\vec{\jmath} + 2\vec{k} + 2\vec{\imath} + \vec{\jmath} - \vec{k}$$

$$\vec{F} = 5\vec{\imath} - \vec{\jmath} + \vec{k}$$

Let A and B be the points $(1,3,-1)$ and $(4,-1,\lambda)$respectively.

Then the displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (4\vec{\imath} - \vec{\jmath} + \lambda\vec{k}) - (\vec{\imath} + 3\vec{\jmath} - \vec{k})$$

$$= 4\vec{\imath} - \vec{\jmath} + \lambda\vec{k} - \vec{\imath} - 3\vec{\jmath} + \vec{k}$$

$$= 3\vec{\imath} - 4\vec{\jmath} + \lambda\vec{k} + \vec{k}$$

$$\vec{d} = 3\vec{\imath} - 4\vec{\jmath} + (\lambda + 1)\vec{k}$$

Work done $W = \vec{F}.\vec{d}$

$$= (5\vec{\imath} - \vec{\jmath} + \vec{k}).[3\vec{\imath} - 4\vec{\jmath} + (\lambda + 1)\vec{k}]$$

$$= 15 + 4 + \lambda + 1$$

$$= \lambda + 20$$

Given Work done W = 16 units.

$$\therefore \lambda + 20 = 16$$

$$\lambda = 16 - 20$$

$$\lambda = 4$$

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Example 6.11

Find the magnitude and the direction cosines of the torque about the point $(2,0,-1)$ of a force $2\vec{\imath} + \vec{\jmath} - \vec{k}$, whose line of action passes through the origin.

Solution: Let A be the point $(2,0,-1)$

Then its position vector $\overrightarrow{OA} = -2\vec{\imath} + 0\vec{\jmath} + \vec{k}$

Given Force $\vec{F} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$

So torque $\vec{t} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{\imath}(0-1) - \hat{\jmath}(2-2) + \hat{k}(-2-0)$$

$$= -\hat{\imath} - 2\hat{k}$$

Magnitude $|\hat{t}| = \sqrt{1+4} = \sqrt{5}$

the direction cosines are $-\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}$

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EXERCISE 6.1

1. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.

Solution: AB is the chord.

C is the midpoint of AB.

O is the centre of the circle.

Then OA= OB [Radius]

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Then $\overrightarrow{OC} = \frac{\vec{a}+\vec{b}}{2}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \vec{b} - \vec{a}$

Now $\overrightarrow{OC}.\overrightarrow{AB} = \left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{b} - \vec{a})$

$= \frac{1}{2}(\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

$= \frac{1}{2}\left[\left(\vec{b}\right)^2 - (\vec{a})^2\right]$

$= \frac{1}{2}\left[\left(\overrightarrow{OB}\right)^2 - \left(\overrightarrow{OA}\right)^2\right]$

$= \frac{1}{2}[(OB)^2 - (OA)^2]$

$= 0$

$\therefore$ $\overrightarrow{OC} is$ $\perp$ to $\overrightarrow{AB}$. Hence proved.

....

2. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.

Solution: Let OAB be an isosceles triangle with OA = OB

OC be the median to AB

C is the midpoint of AB.

Take O as the origin.

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Then $\overrightarrow{OC} = \frac{\vec{a}+\vec{b}}{2}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \vec{b} - \vec{a}$

Now $\overrightarrow{OC}.\overrightarrow{AB} = \left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{b} - \vec{a})$

$= \frac{1}{2}(\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

$= \frac{1}{2}\left[\left(\vec{b}\right)^2 - (\vec{a})^2\right]$

$= \frac{1}{2}\left[\left(\overrightarrow{OB}\right)^2 - \left(\overrightarrow{OA}\right)^2\right]$

$= \frac{1}{2}[(OB)^2 - (OA)^2]$

$= 0$

$\therefore$ $\overrightarrow{OC} is$ $\perp$ to $\overrightarrow{AB}$. Hence proved.

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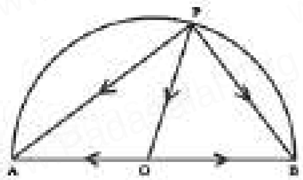
3. Prove by vector method that an angle in a semi-circle is a right angle.

Solution:



Let AB be the diameter of the circle with centre at O.

Let P be any point on the circle.

To prove $\angle APB = 90^0$

We know that OA = OB = OP $(radii)$

Now $\overrightarrow{PA} = \overrightarrow{PO} + \overrightarrow{OA}$ and

$\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB}$

$= \overrightarrow{PO} - \overrightarrow{OA}$ $(since\ \overrightarrow{OB} = -\overrightarrow{OA})$

$\overrightarrow{PA}.\overrightarrow{PB} = (\overrightarrow{PO} + \overrightarrow{OA}).(\overrightarrow{PO} - \overrightarrow{OA})$

$= \left(\overrightarrow{PO}\right)^2 - \left(\overrightarrow{OA}\right)^2$

$= (PO)^2 - (OA)^2$

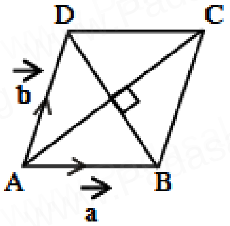
$\overrightarrow{PA}.\overrightarrow{PB} = 0$

That is $\overrightarrow{PA}\ is\ \perp r\ to\ \overrightarrow{PB}$

This gives $\angle APB = 90^0$

4. Prove by vector method that the diagonals of a rhombus bisect each other at right angles.

Solution:



Let ABCD be the Rhombus.

Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

We have AB = BC = CD = DA

That is $|\vec{a}| = |\vec{b}|$ .... (1)

Now $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \vec{a} + \vec{b}$ and

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

$= \vec{b} - \vec{a}$

$\therefore\ \overrightarrow{AC}.\overrightarrow{BD} = (\vec{a} + \vec{b}).(\vec{b} - \vec{a})$

$= (\vec{b} + \vec{a}).(\vec{b} - \vec{a})$

$= (\vec{b})^2 - (\vec{a})^2$

$= 0$ $(since\ |\vec{a}| = |\vec{b}|)$

Thus $\overrightarrow{AC}.\overrightarrow{BD} = 0$

That is $\overrightarrow{AC}$ *is* $\perp r$ *to* $\overrightarrow{BD}$

Hence the diagonals of a rhombus are at right angles.

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5. Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.

Solution: Let ABCD be a parallelogram.

To prove ABCD is a rectangle if the diagonals are equal.

Now $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{BC} - \overrightarrow{AB}$

But $(\overrightarrow{AC})^2 = (\overrightarrow{BD})^2$

$(\overrightarrow{AB} + \overrightarrow{BC})^2 = (\overrightarrow{BC} - \overrightarrow{AB})^2$

$(\overrightarrow{AB})^2 + (\overrightarrow{BC})^2 + 2(\overrightarrow{AB}).(\overrightarrow{BC})$

$= (\overrightarrow{BC})^2 + (\overrightarrow{AB})^2 - 2(\overrightarrow{BC}).(\overrightarrow{AB})$

$2(\overrightarrow{AB}).(\overrightarrow{BC}) = -2(\overrightarrow{BC}).(\overrightarrow{AB})$

$4(\overrightarrow{AB}).(\overrightarrow{BC}) = 0$

$(\overrightarrow{AB}).(\overrightarrow{BC}) = 0$

That is $\overrightarrow{AB}$ *is* $\perp r$ *to* $\overrightarrow{BC}$

That is ABCD is a rectangle.

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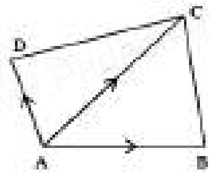
6. Prove by vector method that the area of the quadrilateral *ABCD* having diagonals *AC* and *BD* is $\frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$.

Solution:



Vector area of quadrilateral ABCD

= Vector area of $\Delta\ ABC$ + Vector area of $\Delta\ ACD$

$= \frac{1}{2}(\overrightarrow{AB} \times \overrightarrow{AC}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$

$= -\frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AB}) + \frac{1}{2}(\overrightarrow{AC} \times \overrightarrow{AD})$

$= \frac{1}{2}(\overrightarrow{AC}) \times (-\overrightarrow{AB} + \overrightarrow{AD})$

$= \frac{1}{2}(\overrightarrow{AC}) \times (\overrightarrow{BA} + \overrightarrow{AD})$

$= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$

Vector area of quadrilateral ABCD $= \frac{1}{2}\overrightarrow{AC} \times \overrightarrow{BD}$

....

7. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.

Solution: Let ABCD and $ABC^/D^/$ be two parallelograms on the same base AB and between the same parallel lines.

To prove Area of ABCD = Area of $ABC^/D^/$

Area of $ABC^/D^/ = \left|\overrightarrow{AB} \times \overrightarrow{AD^/}\right|$

$= \left|\overrightarrow{AB} \times \left(\overrightarrow{AD} + \overrightarrow{DD^/}\right)\right|$

$= \left|\left(\overrightarrow{AB} \times \overrightarrow{AD}\right) + \left(\overrightarrow{AB} + \overrightarrow{DD^/}\right)\right|$

$= \left|\left(\overrightarrow{AB} \times \overrightarrow{AD}\right)\right| + 0$

= Area of ABCD

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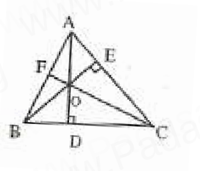
8. If G is the centroid of a $ABC\Delta$, prove that (area of ΔGAB) = (area of ΔGBC) = (area of ΔGCA) = $\frac{1}{3}$(area of Δ ABC)

Solution:



Area of $\Delta GAB = \frac{1}{2}\left(\overrightarrow{AB} \times \overrightarrow{AG}\right)$

$= \frac{1}{2}\left[\left(\overrightarrow{OB} - \overrightarrow{OA}\right) \times \left(\overrightarrow{OG} - \overrightarrow{OA}\right)\right]$

$= \frac{1}{2}\left[\left(\vec{b} - \vec{a}\right) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3} - \vec{a}\right)\right]$

$= \frac{1}{2}\left[\left(\vec{b} - \vec{a}\right) \times \left(\frac{\vec{a}+\vec{b}+\vec{c}-3\vec{a}}{3}\right)\right]$

$= \frac{1}{2}\left[\left(\vec{b} - \vec{a}\right) \times \left(\frac{\vec{b}+\vec{c}-2\vec{a}}{3}\right)\right]$

$= \frac{1}{6}\left[\vec{b} \times \vec{b} + \vec{b} \times \vec{c} - \vec{b} \times 2\vec{a} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} + \vec{a} \times 2\vec{a}\right]$

$= \frac{1}{6}\left[\vec{b} \times \vec{c} - 2\vec{b} \times \vec{a} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}\right]$

$= \frac{1}{6}\left[\vec{b} \times \vec{c} + 2\vec{a} \times \vec{b} - \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right]$

$= \frac{1}{6}\left[\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right]$

$= \frac{1}{3} \times \frac{1}{2}\left[\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right]$

$= \frac{1}{3}$(area of Δ ABC)

Similarly we can prove other results.

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9. Using vector method, prove that $cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Solution:

Take two points A and B on the unit circle with centre as origin O.

So, $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = 1$

$\angle AOX = \alpha$ and $\angle BOX = \beta$

$\therefore \angle AOB = \alpha - \beta$

Let $\vec{\imath}$ and $\vec{\jmath}$ be the unit vectors along x , y axis.

Draw AL and BM perpendicular to x axis.

Then $A(cos\alpha, sin\alpha)$ and $B\ (cos\beta, sin\beta)$ are the co-ordinates.

$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$

$= \cos\alpha\vec{\imath} + \sin\alpha\vec{\jmath}$

$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$

$= \cos\beta\vec{\imath} + \sin\beta\vec{\jmath}$

By definition

$\overrightarrow{OA}.\overrightarrow{OB} = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos(\alpha - \beta)$

$= (1)(1)\cos(\alpha - \beta)$

$\overrightarrow{OA}.\overrightarrow{OB} = \cos(\alpha - \beta)$ .... (1)

By value

$\overrightarrow{OA}.\overrightarrow{OB} = (\cos\alpha\vec{\imath} + \sin\alpha\vec{\jmath}).(\cos\beta\vec{\imath} + \sin\beta\vec{\jmath})$

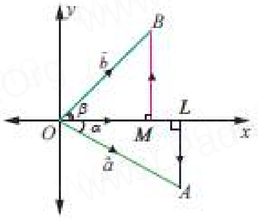
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ..... (2)

From (1) and (2)

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

10. Prove by vector method that
$\sin(\alpha+\beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

Solution:



Take two points A and B on the unit circle with centre as origin O.

So, $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$

$\angle AOX = \alpha$ and $\angle BOX = \beta$

$\therefore \angle AOB = \alpha + \beta$

Let $\vec{i}$ and $\vec{j}$ be the unit vectors along x , y axis.

Draw AL and BM perpendicular to x axis.

Then $A(\cos\alpha, -\sin\alpha)$ and $B\,(\cos\beta, \sin\beta)$ are the co-ordinates.

$$\overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$$

$$= \cos\alpha\vec{i} - \sin\alpha\vec{j}$$

$$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB}$$

$$= \cos\beta\vec{i} + \sin\beta\vec{j}$$

By definition

$$\overrightarrow{OB} \times \overrightarrow{OA} = |\overrightarrow{OB}||\overrightarrow{OA}|\sin(\alpha+\beta)\,\hat{k}$$

$$= (1)(1)\sin(\alpha+\beta)\,\hat{k}$$

$$\overrightarrow{OB} \times \overrightarrow{OA} = \sin(\alpha+\beta)\,\hat{k} \quad \ldots\ldots(1)$$

By value $\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & -\sin\alpha & 0 \end{vmatrix}$

$$= \pm\,\hat{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \quad \ldots\ldots(2)$$

From (1) and (2)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

....

11. A particle acted on by constant forces $8\vec{i}+2\vec{j}-6\vec{k}$ and $6\vec{i}+2\vec{j}-2\vec{k}$ is displaced from the point $(1,2,3)$ to the point $(5,4,1)$. Find the total work done by the forces.

Solution: Given Force $\vec{F}_1 = 8\vec{i}+2\vec{j}-6\vec{k}$

Force $\vec{F}_2 = 6\vec{i}+2\vec{j}-2\vec{k}$

Hence resultant force $\vec{F} = \vec{F}_1 + \vec{F}_2$

$$= 8\vec{i}+2\vec{j}-6\vec{k}+6\vec{i}+2\vec{j}-2\vec{k}$$

$$\vec{F} = 14\vec{i}+4\vec{j}-8\vec{k}$$

Let A and B be the points (1,2, 3) and (5, 4, 1)respectively.

Then the displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (5\vec{i}+4\vec{j}+\vec{k}) - (\vec{i}+2\vec{j}+3\vec{k})$$

$$= 5\vec{i}+4\vec{j}+\vec{k}-\vec{i}-2\vec{j}-3\vec{k}$$

$$\vec{d} = 4\vec{i}+2\vec{j}-2\vec{k}$$

Work done $W = \vec{F}.\vec{d}$

$$= (14\vec{i}+4\vec{j}-8\vec{k}).(4\vec{i}+2\vec{j}-2\vec{k})$$

$$= 56 + 8 + 16$$

$$= 80$$

Work done W = 80 units.

....

12. Forces of magnitudes $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $(3\vec{i}+4\vec{j}+5\vec{k})$ and $(10\vec{i}+6\vec{j}-8\vec{k})$, respectively, act on a particle which is displaced from the point with position vector $(4\vec{i}-3\vec{j}-2\vec{k})$ to the point with position vector $(6\vec{i}+\vec{j}-3\vec{k})$. Find the work done by the forces.

Solution:

Given Force $\vec{F_1} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$

$$|\vec{F_1}| = \sqrt{3^2 + 4^2 + 5^2}$$

$$= \sqrt{9 + 16 + 25}$$

$$= \sqrt{50}$$

$$= \sqrt{25 \times 2}$$

$$= 5\sqrt{2}$$

$\therefore$ Unit Force $\widehat{F_1} = \frac{\vec{F_1}}{|\vec{F_1}|}$

$$5\sqrt{2}\,\widehat{F_1} = 5\sqrt{2}\left(\frac{3\vec{\imath}+4\vec{\jmath}+5\vec{k}}{5\sqrt{2}}\right)$$

First Force $= 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$

Given Force $\vec{F_2} = 10\vec{\imath} + 6\vec{\jmath} - 8\vec{k}$

$$|\vec{F_2}| = \sqrt{10^2 + 6^2 + 8^2}$$

$$= \sqrt{100 + 36 + 64}$$

$$= \sqrt{200}$$

$$= \sqrt{100 \times 2}$$

$$= 10\sqrt{2}$$

$\therefore$ Unit Force $\widehat{F_2} = \frac{\vec{F_2}}{|\vec{F_2}|}$

$$10\sqrt{2}\,\widehat{F_2} = 10\sqrt{2}\left(\frac{10\vec{\imath}+6\vec{\jmath}-8\vec{k}}{10\sqrt{2}}\right)$$

Second Force $= 10\vec{\imath} + 6\vec{\jmath} - 8\vec{k}$

Hence resultant force

$$\vec{F} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k} + 10\vec{\imath} + 6\vec{\jmath} - 8\vec{k}$$

$$\vec{F} = 13\vec{\imath} + 10\vec{\jmath} - 3\vec{k}$$

Then the displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (6\vec{\imath} + \vec{\jmath} - 3\vec{k}) - (4\vec{\imath} - 3\vec{\jmath} - 2\vec{k})$$

$$= 6\vec{\imath} + \vec{\jmath} - 3\vec{k} - 4\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$$

$$\vec{d} = 2\vec{\imath} + 4\vec{\jmath} - \vec{k}$$

Work done W $= \vec{F}.\vec{d}$

$$= (13\vec{\imath} + 10\vec{\jmath} - 3\vec{k}).(2\vec{\imath} + 4\vec{\jmath} - \vec{k})$$

$$= 26 + 40 + 3$$

$$= 69$$

Work done W = 69 units.

....

13. Find the magnitude and direction cosines of the torque of a force represented by $3\vec{\imath} + 4\vec{\jmath} - 5\vec{k}$ about the point with position vector $2\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$ acting through a point whose position vector is $4\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$.

Solution: Given Force $\vec{F} = 3\vec{\imath} + 4\vec{\jmath} - 5\vec{k}$

$$\vec{r} = (\textit{through the point} - \textit{about the point})$$

$$\vec{r} = (4\vec{\imath} + 2\vec{\jmath} - 3\vec{k}) - (2\vec{\imath} - 3\vec{\jmath} + 4\vec{k})$$

$$\vec{r} = 4\vec{\imath} + 2\vec{\jmath} - 3\vec{k} - 2\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$$

$$\vec{r} = 2\vec{\imath} + 5\vec{\jmath} - 7\vec{k}$$

So torque $\vec{t} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix}$$

$$= \hat{\imath}(-25 + 28) - \hat{\jmath}(-10 + 21) + \hat{k}(8 - 15)$$

$$= \hat{\imath}(3) - \hat{\jmath}(11) + \hat{k}(-7)$$

$$= 3\hat{\imath} - 11\hat{\jmath} - 7\hat{k}$$

Magnitude $|\hat{t}| = \sqrt{9 + 121 + 49} = \sqrt{179}$

the direction cosines are $\frac{3}{\sqrt{179}}, -\frac{11}{\sqrt{179}}, -\frac{7}{\sqrt{179}}$

....

14. Find the torque of the resultant of the three forces represented by $-3\vec{\imath} + 6\vec{\jmath} - 3\vec{k}$, $4\vec{\imath} - 10\vec{\jmath} + 12\vec{k}$ and $4\vec{\imath} + 7\vec{\jmath}$ acting at the point with position vector $8\vec{\imath} - 6\vec{\jmath} - 4\vec{k}$, about the point with position vector $18\vec{\imath} + 3\vec{\jmath} - 9\vec{k}$.

Solution: Given Force $\vec{F_1} = -3\vec{\imath} + 6\vec{\jmath} - 3\vec{k}$

Force $\vec{F_2} = 4\vec{\imath} - 10\vec{\jmath} + 12\vec{k}$

Force $\vec{F_3} = 4\vec{\imath} + 7\vec{\jmath}$

Resultant Force $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$

$\vec{F} = 5\vec{\imath} + 3\vec{\jmath} + 9\vec{k}$

$\vec{r} = (at\ the\ point - about\ the\ point)$

$\vec{r} = (8\vec{\imath} - 6\vec{\jmath} - 4\vec{k}) - (18\vec{\imath} + 3\vec{\jmath} - 9\vec{k})$

$\vec{r} = 8\vec{\imath} - 6\vec{\jmath} - 4\vec{k} - 18\vec{\imath} - 3\vec{\jmath} + 9\vec{k}$

$\vec{r} = -10\vec{\imath} - 9\vec{\jmath} + 5\vec{k}$

So torque $\vec{t} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix}$$

$= \hat{\imath}(-81 - 15) - \hat{\jmath}(-90 - 25) + \hat{k}(-30 + 45)$

$= \hat{\imath}(-96) - \hat{\jmath}(-115) + \hat{k}(15)$

$= -96\hat{\imath} + 115\hat{\jmath} + 15\hat{k}$

....

Example 6.12

If $\vec{a} = -3\vec{\imath} - \vec{\jmath} + 5\vec{k}, \vec{b} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$,
$\vec{c} = 4\vec{\jmath} - 5\vec{k}$, find $\vec{a}.(\vec{b} \times \vec{c})$.

Solution: $\vec{a} = -3\vec{\imath} - \vec{\jmath} + 5\vec{k}$,
$\vec{b} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$,
$\vec{c} = 4\vec{\jmath} - 5\vec{k}$

$\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}]$

$$= \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix}$$

$= -3(10 - 4) + 1(-5 - 0) + 5(4 + 0)$

$= -3(6) + 1(-5) + 5(4)$

$= -18 - 5 + 20$

$= -23 + 20$

$= -3$

....

Example 6.13

Find the volume of the parallelepiped whose coterminous edges are given by the vectors
$2\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$, $\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $3\vec{\imath} - \vec{\jmath} + 2\vec{k}$.

Solution: Volume of the parallelepiped with $\vec{a}, \vec{b}$ *and* $\vec{c}$ as coterminous edges is $|[\vec{a}, \vec{b}, \vec{c}]|$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$= 2(4 - 1) + 3(2 + 3) + 4(-1 - 6)$

$= 2(3) + 3(5) + 4(-7)$

$= 6 + 15 - 28$

$= 21 - 28$

$= -7$

Volume of the parallelepiped $|[\vec{a}, \vec{b}, \vec{c}]| = |-7|$

$= 7$ cubic units.

....

Example 6.14

Show that the vectors $\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$, $2\vec{\imath} - \vec{\jmath} + 2\vec{k}$ and $3\vec{\imath} + \vec{\jmath} - \vec{k}$ are coplanar.

Solution: If $[\vec{a}, \vec{b}, \vec{c}] = 0$, then the vectors are coplanar.

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$= 1(1 - 2) - 2(-2 - 6) - 3(2 + 3)$

$= 1(-1) - 2(-8) - 3(5)$

$= -1 + 16 - 15$

$= -16 + 16$

$= 0$

So, the given vectors are coplanar.

....

Example 6.15

If $2\vec{\imath} - \vec{\jmath} + 3\vec{k}$, $3\vec{\imath} + 2\vec{\jmath} + \vec{k}$, $\vec{\imath} + m\vec{\jmath} + 4\vec{k}$ are coplanar, find the value of m .

Solution: Given the vectors are coplanar.

$\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$$

$$2(8-m)+1(12-1)+3(3m-2)=0$$

$$16-2m+11+9m-6=0$$

$$27-6+7m=0$$

$$21+7m=0$$

$$7m=-21$$

$$m=-3$$

....

Example 6.16

Show that the four points $(6,-7,0), (16,-19,-4), (0,3,-6), (2,-5,10)$ lie on a same plane.

Solution: Let A $(6,-7,0)$, B $(16,-19,-4)$

C $(0,3,-6)$ and D $(2,-5,10)$ be the points.

To show the points lie on the same plane, we have to prove that the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (16\vec{i} - 19\vec{j} - 4\vec{k}) - (6\vec{i} - 7\vec{j})$$

$$= 16\vec{i} - 19\vec{j} - 4\vec{k} - 6\vec{i} + 7\vec{j}$$

$$= 10\vec{i} - 12\vec{j} - 4\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= (3\vec{j} - 6\vec{k}) - (6\vec{i} - 7\vec{j})$$

$$= 3\vec{j} - 6\vec{k} - 6\vec{i} + 7\vec{j}$$

$$= -6\vec{i} + 10\vec{j} - 6\vec{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$= (2\vec{i} - 5\vec{j} + 10\vec{k}) - (6\vec{i} - 7\vec{j})$$

$$= 2\vec{i} - 5\vec{j} + 10\vec{k} - 6\vec{i} + 7\vec{j}$$

$$= -4\vec{i} + 2\vec{j} + 10\vec{k}$$

Now, $$[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}$$

$$= 0 \qquad (Since\ R_1 = R_2 + R_3)$$

$\therefore$ The vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

Hence the given points are lie on a plane.

....

Example 6.17

If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also coplanar.

Solution: Given the vectors are coplanar.

$$\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$$

Now, $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$$= [\vec{a}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] + [\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}+\vec{a}] + [\vec{a}, \vec{c}, \vec{c}+\vec{a}]$$

$$+ [\vec{b}, \vec{b}, \vec{c}+\vec{a}] + [\vec{b}, \vec{c}, \vec{c}+\vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{c}] + [\vec{a}, \vec{c}, \vec{a}]$$

$$+ [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}]$$

$$= 2[\vec{a}, \vec{b}, \vec{c}]$$

$$= 0$$

Hence, $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

....

Example 6.18

If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, prove that

$$[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}] = [\vec{a}, \vec{b}, \vec{c}].$$

Solution:

$$[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}] = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} [\vec{a}, \vec{b}, \vec{c}]$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

....

EXERCISE 6.2

1. If $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}, \vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c} = 3\vec{i} + 2\vec{j} + \vec{k}$, find $\vec{a}.(\vec{b} \times \vec{c})$.

Solution: $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$,

$\vec{b} = 2\vec{\imath} + \vec{\jmath} - 2\vec{k}$,

$\vec{c} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$

$$\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}]$$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1(1+4) + 2(2+6) + 3(4-3)$$

$$= 1(5) + 2(8) + 3(1)$$

$$= 5 + 16 + 3$$

$$= 24$$

---

2. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\vec{\imath} + 14\vec{\jmath} + 10\vec{k}$, $14\vec{\imath} - 10\vec{\jmath} - 6\vec{k}$ and $2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$.

Solution: Volume of the parallelepiped with

$\vec{a}, \vec{b}$ *and* $\vec{c}$ as coterminous edges is $|[\vec{a}, \vec{b}, \vec{c}]|$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$$

$$= -6(20+24) - 14(-28+12) + 10(56+20)$$

$$= -6(44) - 14(-16) + 10(76)$$

$$= -264 + 224 + 760$$

$$= -264 + 984$$

$= 720$ cubic units.

---

3. The volume of the parallelepiped whose coterminous edges are $7\vec{\imath} + \lambda\vec{\jmath} - 3\vec{k}$, $\vec{\imath} + 2\vec{\jmath} - \vec{k}$, $-3\vec{\imath} + 7\vec{\jmath} + 5\vec{k}$ is 90 cubic units. Find the value of λ.

Solution: Volume of the parallelepiped with

$\vec{a}, \vec{b}$ *and* $\vec{c}$ as coterminous edges is $|[\vec{a}, \vec{b}, \vec{c}]|$

Given $|[\vec{a}, \vec{b}, \vec{c}]|$=90 cubic units.

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$$

$$7(10+7) - \lambda(5-3) - 3(7+6) = 90$$

$$7(17) - \lambda(2) - 3(13) = 90$$

$$119 - 2\lambda - 39 = 90$$

$$80 - 2\lambda = 90$$

$$-2\lambda = 90 - 80$$

$$-2\lambda = 10$$

$$\lambda = -5$$

---

4. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of

$$(\vec{a} + \vec{b}).(\vec{b} \times \vec{c}) + (\vec{b} + \vec{c}).(\vec{c} \times \vec{a}) + (\vec{c} + \vec{a}).(\vec{a} \times \vec{b}).$$

Solution:

Given Volume of the parallelepiped with

$\vec{a}, \vec{b}$ *and* $\vec{c}$ as edges is $|[\vec{a}, \vec{b}, \vec{c}]| = 4$

Now

$$(\vec{a} + \vec{b}).(\vec{b} \times \vec{c}) = \vec{a}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{c})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{c}]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + 0$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

$$(\vec{b} + \vec{c}).(\vec{c} \times \vec{a}) = \vec{b}.(\vec{c} \times \vec{a}) + \vec{c}.(\vec{c} \times \vec{a})$$

$$= [\vec{b}, \vec{c}, \vec{a}] + [\vec{c}, \vec{c}, \vec{a}]$$

$$= [\vec{b}, \vec{c}, \vec{a}] + 0$$

$$= [\vec{b}, \vec{c}, \vec{a}]$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

$$(\vec{c} + \vec{a}).(\vec{a} \times \vec{b}) = \vec{c}.(\vec{a} \times \vec{b}) + \vec{a}.(\vec{a} \times \vec{b})$$

$$= [\vec{c}, \vec{a}, \vec{b}] + [\vec{a}, \vec{a}, \vec{b}]$$

$$= [\vec{c}, \vec{a}, \vec{b}] + 0$$

$$= [\vec{c}, \vec{a}, \vec{b}]$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

$(\vec{a}+\vec{b}).(\vec{b}\times\vec{c})+(\vec{b}+\vec{c}).(\vec{c}\times\vec{a})$

$+(\vec{c}+\vec{a}).(\vec{a}\times\vec{b})=3\,[\vec{a},\vec{b},\vec{c}]$

$=3\times 4$

$=12$

……………………………………………………………………

5. Find the altitude of a parallelepiped determined by the vectors $\vec{a}=-2\vec{\imath}+5\vec{\jmath}+3\vec{k}$, $\vec{b}=\vec{\imath}+3\vec{\jmath}-2\vec{k}$ and $\vec{c}=-3\vec{\imath}+\vec{\jmath}+4\vec{k}$ if the base is taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.

Solution: Volume = Base Area × Height.

Here Volume $=[\vec{a},\vec{b},\vec{c}]$ and

Area $=|\vec{b}\times\vec{c}|$

To find height 'H'

$$[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix}-2 & 5 & 3\\ 1 & 3 & -2\\ -3 & 1 & 4\end{vmatrix}$$

$=-2(12+2)-5(4-6)+3(1+9)$

$=-2(14)-5(-2)+3(10)$

$=-28+10+30$

$=-28+40$

$=12$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 3 & -2\\ -3 & 1 & 4\end{vmatrix}$$

$=\hat{\imath}(12+2)-\hat{\jmath}(4-6)+\hat{k}(1+9)$

$=\hat{\imath}(14)-\hat{\jmath}(-2)+\hat{k}(10)$

$=14\,\hat{\imath}+2\hat{\jmath}+10\,\hat{k}$

$\therefore |\vec{b}\times\vec{c}|=\sqrt{14^2+2^2+10^2}$

$=\sqrt{196+4+100}$

$=\sqrt{300}$

$=\sqrt{100\times 3}$

$=10\sqrt{3}$

Volume = Base Area × Height

$12=10\sqrt{3}\times$ Height

Height $=\dfrac{12}{10\sqrt{3}}$

$=\dfrac{6}{5\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}$

$=\dfrac{6\sqrt{3}}{5\times 3}$

$=\dfrac{2\sqrt{3}}{5}$

……………………………………………………………………

6. Determine whether the three vectors $2\vec{\imath}+3\vec{\jmath}+\vec{k}$, $\vec{\imath}-2\vec{\jmath}+2\vec{k}$ and $3\vec{\imath}+\vec{\jmath}+3\vec{k}$ are coplanar.

Solution: If $[\vec{a},\vec{b},\vec{c}]=0$, then the vectors are coplanar.

$$[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix}2 & 3 & 1\\ 1 & -2 & 2\\ 3 & 1 & 3\end{vmatrix}$$

$=2(-6-2)-3(3-6)+1(1+6)$

$=2(-8)-3(-3)+1(7)$

$=-16+9+7$

$=-16+16$

$=0$

Hence the given vectors are coplanar.

……………………………………………………………………

7. Let $\vec{a}=\vec{\imath}+\vec{\jmath}+\vec{k}$, $\vec{b}=\vec{\imath}$, and $\vec{c}=c_1\vec{\imath}+c_2\vec{\jmath}+c_3\vec{k}$. If $c_1=1$ and $c_2=2$, find c_3 such that $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar

Solution: Given the vectors are coplanar.

$\therefore [\vec{a},\vec{b},\vec{c}]=0$

Substituting $c_1=1$ and $c_2=2$ in

$\vec{c}=c_1\vec{\imath}+c_2\vec{\jmath}+c_3\vec{k}$ we get,

$\vec{c}=\vec{\imath}+2\vec{\jmath}+c_3\vec{k}$

$$[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 1 & 2 & c_3\end{vmatrix}=0$$

$1(0)-1(c_3-0)+1(2-0)=0$

$$0 - c_3 + 2 = 0$$
$$c_3 = 2$$

8. If $\vec{a} = \vec{i} - \vec{k}, \vec{b} = x\vec{i} + \vec{j} + (1-x)\vec{k}$, $\vec{c} = y\vec{i} + x\vec{j} + (1+x-y)\vec{k}$, show that $[\vec{a}, \vec{b}, \vec{c}]$ depends on neither x *nor* y .

Solution:

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$
$$= 1(1+x-y) - x(1-x) - 1(x^2 - y)$$
$$= 1 + x - y - x + x^2 - x^2 + y$$
$$= 1$$

$\therefore [\vec{a}, \vec{b}, \vec{c}] = 1$, that depends on neither x *nor* y .

9. If the vectors $a\vec{i} + a\vec{j} + c\vec{k}, \vec{i} + \vec{k}$ and $c\vec{i} + c\vec{j} + b\vec{k}$ are coplanar, prove that c is the geometric mean of a and b.

Solution: Given the vectors are coplanar.

$$\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$$
$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$
$$a(o - c) - a(b - c) + c(c - 0) = 0$$
$$ac - ab + ac + c^2 = 0$$
$$-ab + c^2 = 0$$
$$c^2 = ab$$
$$c = \sqrt{ab}$$

That is c is the geometric mean of a and b.

10. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4} |\vec{a}|^2 |\vec{b}|^2$

Solution: Given $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

So, $\vec{c}$ is parallel to $\vec{a} \times \vec{b}$

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a}.(\vec{b} \times \vec{c})$$
$$|[\vec{a}, \vec{b}, \vec{c}]| = |\vec{a}||\vec{b} \times \vec{c}|$$
$$= |\vec{a}||\vec{b}||\vec{c}| \sin\frac{\pi}{6}$$
$$= |\vec{a}||\vec{b}|(1)\left(\frac{1}{2}\right)$$
$$= |\vec{a}||\vec{b}|\left(\frac{1}{2}\right)$$
$$\therefore [\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4} |\vec{a}|^2 |\vec{b}|^2 \text{ Proved.}$$

Example 6.19

Prove $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = [\vec{a}, \vec{b}, \vec{c}]^2$.

Solution:

$$[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = (\vec{a} \times \vec{b}).\{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\}$$
$$= (\vec{a} \times \vec{b}).\{[\vec{b}, \vec{c}, \vec{a}]\vec{c} - [\vec{b}, \vec{c}, \vec{c}]\vec{a}\}$$
$$= (\vec{a} \times \vec{b}).\{[\vec{b}, \vec{c}, \vec{a}]\vec{c} - 0\}$$
$$= (\vec{a} \times \vec{b}).[\vec{b}, \vec{c}, \vec{a}]\vec{c}$$
$$= \{(\vec{a} \times \vec{b}).\vec{c}\}[\vec{a}, \vec{b}, \vec{c}]$$
$$= [\vec{a}, \vec{b}, \vec{c}][\vec{a}, \vec{b}, \vec{c}]$$
$$= [\vec{a}, \vec{b}, \vec{c}]^2 \quad \text{Proved.}$$

Example 6.20

Prove that $\left(\vec{a}.(\vec{b} \times \vec{c})\right)\vec{a} = (\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

Solution: Treating $(\vec{a} \times \vec{b})$ as the first vector on the right hand side of the given equation and using the vector triple product expansion, we get $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

$$= \left((\vec{a} \times \vec{b}).\vec{c}\right)\vec{a} - \left((\vec{a} \times \vec{b}).\vec{a}\right)\vec{c}$$
$$= \left((\vec{a} \times \vec{b}).\vec{c}\right)\vec{a}$$

Example 6.21

For any four vectors $\vec{a},\vec{b},\ \vec{c},\vec{d}$, we have

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

$$= [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$

Solution: Taking $\vec{p} = (\vec{a}\times\vec{b})$ we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{p}\times(\vec{c}\times\vec{d})$$

$$= (\vec{p}.\vec{d})\vec{c} - (\vec{p}.\vec{c})\vec{d}$$

$$= \left((\vec{a}\times\vec{b}).\vec{d}\right)\vec{c} - \left((\vec{a}\times\vec{b}).\vec{c}\right)\vec{d}$$

$$= [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

Similarly taking $\vec{q} = (\vec{c}\times\vec{d})$ we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{q}$$

$$= (\vec{a}.\vec{q})\vec{b} - (\vec{b}.\vec{q})\vec{a}$$

$$= \left(\vec{a}.(\vec{c}\times\vec{d})\right)\vec{b} - \left(\vec{b}.(\vec{c}\times\vec{d})\right)\vec{a}$$

$$= [\vec{a},\vec{c},\vec{d}]\vec{a} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$

....

Example 6.22 If $\vec{a} = -2\vec{\imath} + 3\vec{\jmath} - 2\vec{k}, \vec{b} = 3\vec{\imath} - \vec{\jmath} + 3\vec{k}, \vec{c} = 2\vec{i} - 5\vec{j} + \vec{k}$

find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$.

State whether they are equal.

Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -2 \\ 3 & -1 & 3 \end{vmatrix}$$

$$= \hat{\imath}(9-2) - \hat{\jmath}(-6+6) + \hat{k}(2-9)$$

$$= \hat{\imath}(7) - \hat{\jmath}(0) + \hat{k}(-7)$$

$$= 7\,\hat{\imath} - 7\hat{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 0 & -7 \\ 2 & -5 & 1 \end{vmatrix}$$

$$= \hat{\imath}(0-35) - \hat{\jmath}(7+14) + \hat{k}(-35-0)$$

$$= \hat{\imath}(-35) - \hat{\jmath}(21) + \hat{k}(-35)$$

$$= -35\hat{\imath} - 21\hat{\jmath} - 35\hat{k}$$

LHS: $(\vec{a}\times\vec{b})\times\vec{c} = -35\hat{\imath} - 21\hat{\jmath} - 35\hat{k}$ ....(1)

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 3 \\ 2 & -5 & 1 \end{vmatrix}$$

$$= \hat{\imath}(-1+15) - \hat{\jmath}(3-6) + \hat{k}(-15+2)$$

$$= \hat{\imath}(14) - \hat{\jmath}(-3) + \hat{k}(-13)$$

$$= 14\hat{\imath} + 3\hat{\jmath} - 13\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & -2 \\ 14 & 3 & -13 \end{vmatrix}$$

$$= \hat{\imath}(-39+6) - \hat{\jmath}(26+28) + \hat{k}(-6-42)$$

$$= \hat{\imath}(-33) - \hat{\jmath}(54) + \hat{k}(-48)$$

$$= -33\hat{\imath} - 54\hat{\jmath} - 48\hat{k}$$

RHS: $\vec{a}\times(\vec{b}\times\vec{c}) = -33\hat{\imath} - 54\hat{\jmath} - 48\hat{k}$ ....(2)

Therefore, equations (1) and (2) showed that

$$(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$$

....

Example 6.23 If $\vec{a} = \vec{\imath} - \vec{\jmath}, \vec{b} = \vec{\imath} - \vec{\jmath} - 4\vec{k}, \vec{c} = 3\vec{\jmath} - \vec{k}$ and $\vec{d} = 2\vec{\imath} + 5\vec{\jmath} + \vec{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$

Solution: Given $\vec{a} = \vec{\imath} - \vec{\jmath}, \vec{b} = \vec{\imath} - \vec{\jmath} - 4\vec{k}, \vec{c} = 3\vec{\jmath} - \vec{k}$ and $\vec{d} = 2\vec{\imath} + 5\vec{\jmath} + \vec{k}$

LHS $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix}$$

$$= \hat{\imath}(4-0) - \hat{\jmath}(-4-0) + \hat{k}(-1+1)$$

$$= \hat{\imath}(4) - \hat{\jmath}(-4) + \hat{k}(0)$$

$$= 4\hat{\imath} + 4\hat{\jmath}$$

$$\vec{c}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= \hat{\imath}(3+5) - \hat{\jmath}(0+2) + \hat{k}(0-6)$$

$$= \hat{\imath}(8) - \hat{\jmath}(2) + \hat{k}(-6)$$

$$= 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{\imath}(-24-0) - \hat{\jmath}(-24-0) + \hat{k}(-8-32)$$

$$= \hat{\imath}(-24) - \hat{\jmath}(-24) + \hat{k}(-40)$$

$$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \quad ....(1)$$

RHS $= [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

$$[\vec{a},\vec{b},\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 1(-1+20) + 1(1+8) + 0$$

$$= 1(19) + 1(9)$$

$$= 19 + 9$$

$$= 28$$

$$[\vec{a},\vec{b},\vec{d}]\vec{c} = 28(3\vec{\jmath} - \vec{k})$$

$$= 84\vec{\jmath} - 28\vec{k}$$

$$[\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 1(1+12) + 1(-1-0) + 0$$

$$= 1(13) + 1(-1)$$

$$= 13 - 1$$

$$= 12$$

$$[\vec{a},\vec{b},\vec{c}]\vec{d} = 12(2\vec{\imath} + 5\vec{\jmath} + \vec{k})$$

$$= 24\vec{\imath} + 60\vec{\jmath} + 12\vec{k}$$

$$[\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

$$= (84\vec{\jmath} - 28\vec{k}) - (24\vec{\imath} + 60\vec{\jmath} + 12\vec{k})$$

$$= 84\vec{\jmath} - 28\vec{k} - 24\vec{\imath} - 60\vec{\jmath} - 12\vec{k}$$

$$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \quad ....(2)$$

From (1) and (2)

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

is verified.

$$[\vec{a},\vec{c},\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 1(3+5) + 1(0+2) + 0$$

$$= 1(8) + 1(2)$$

$$= 8 + 2$$

$$= 10$$

$$[\vec{a},\vec{c},\vec{d}]\vec{b} = 10(\vec{\imath} - \vec{\jmath} - 4\vec{k})$$

$$= 10\vec{\imath} - 10\vec{\jmath} - 40\vec{k}$$

$$[\vec{b},\vec{c},\vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$

$$= 1(3+5) + 1(0+2) - 4(0-6)$$

$$= 1(8) + 1(2) - 4(-6)$$

$$= 8 + 2 + 24$$

$$= 34$$

$$[\vec{b},\vec{c},\vec{d}]\vec{a} = 34(\vec{\imath} - \vec{\jmath})$$

$$= 34\vec{\imath} - 34\vec{\jmath}$$

$$[\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$

$$= (10\vec{\imath} - 10\vec{\jmath} - 40\vec{k}) - (34\vec{\imath} - 34\vec{\jmath})$$

$$= 10\vec{\imath} - 10\vec{\jmath} - 40\vec{k} - 34\vec{\imath} + 34\vec{\jmath}$$

$$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k} \quad ....(3)$$

From (1) and (3)

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$

is also verified.

....

EXERCISE 6.3

1. If $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}, \vec{b} = 2\vec{\imath} + \vec{\jmath} - 2\vec{k}$, $\vec{c} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$ Find (i) $(\vec{a}\times\vec{b})\times\vec{c}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})$.

Solution: $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}, \vec{b} = 2\vec{\imath} + \vec{\jmath} - 2\vec{k},$

$\vec{c} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$

To find (i) $(\vec{a} \times \vec{b}) \times \vec{c}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$$

$= \hat{\imath}(4-3) - \hat{\jmath}(-2-6) + \hat{k}(1+4)$

$= \hat{\imath}(1) - \hat{\jmath}(-8) + \hat{k}(5)$

$= \hat{\imath} + 8\hat{\jmath} + 5\hat{k}$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix}$$

$= \hat{\imath}(8-10) - \hat{\jmath}(1-15) + \hat{k}(2-24)$

$= \hat{\imath}(-2) - \hat{\jmath}(-14) + \hat{k}(-22)$

$= -2\hat{\imath} + 14\hat{\jmath} - 22\hat{k}$

To find (ii) $\vec{a} \times (\vec{b} \times \vec{c})$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$\vec{a}.\vec{c} = (\vec{\imath} - 2\vec{\jmath} + 3\vec{k}).(3\vec{\imath} + 2\vec{\jmath} + \vec{k})$

$= 3 - 4 + 3$

$= 2$

$(\vec{a}.\vec{c})\vec{b} = 2(2\vec{\imath} + \vec{\jmath} - 2\vec{k})$

$= 4\vec{\imath} + 2\vec{\jmath} - 4\vec{k}$

$\vec{a}.\vec{b} = (\vec{\imath} - 2\vec{\jmath} + 3\vec{k}).(2\vec{\imath} + \vec{\jmath} - 2\vec{k})$

$= 2 - 2 - 6 = -6$

$(\vec{a}.\vec{b})\vec{c} = -6(3\vec{\imath} + 2\vec{\jmath} + \vec{k})$

$= -18\vec{\imath} - 12\vec{\jmath} - 6\vec{k}$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (4\vec{\imath} + 2\vec{\jmath} - 4\vec{k}) - (-18\vec{\imath} - 12\vec{\jmath} - 6\vec{k})$

$= 4\vec{\imath} + 2\vec{\jmath} - 4\vec{k} + 18\vec{\imath} + 12\vec{\jmath} + 6\vec{k}$

$= 22\vec{\imath} + 14\vec{\jmath} + 2\vec{k}$

2. For any vector $\vec{a}$, prove that

$\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}.$

Solution: $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$

We know $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$\therefore \vec{\imath} \times (\vec{a} \times \vec{\imath}) = (\vec{\imath}.\vec{\imath})\vec{a} - (\vec{\imath}.\vec{a})\vec{\imath}$

$\vec{\imath} \times (\vec{a} \times \vec{\imath}) = \vec{a} - a_1\vec{\imath}$

Similarly $\vec{\jmath} \times (\vec{a} \times \vec{\jmath}) = \vec{a} - a_2\vec{\jmath}$

$\vec{k} \times (\vec{a} \times \vec{k}) = \vec{a} - a_3\vec{k}$

Hence $\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k})$

$= \vec{a} - a_1\vec{\imath} + \vec{a} - a_2\vec{\jmath} + \vec{a} - a_3\vec{k}$

$= 3\vec{a} - (a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k})$

$= 3\vec{a} - \vec{a}$

$= 2\vec{a}$ Proved.

....

3. Prove that $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0.$

Solution:

Now, $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}]$

$= [\vec{a}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] - [\vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}]$

$= [\vec{a}, \vec{b}, \vec{c} - \vec{a}] - [\vec{a}, \vec{c}, \vec{c} - \vec{a}]$

$-[\vec{b}, \vec{b}, \vec{c} - \vec{a}] + [\vec{b}, \vec{c}, \vec{c} - \vec{a}]$

$= [\vec{a}, \vec{b}, \vec{c}] - [\vec{a}, \vec{b}, \vec{a}] - [\vec{a}, \vec{c}, \vec{c}] + [\vec{a}, \vec{c}, \vec{a}]$

$-[\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{c}] - [\vec{b}, \vec{c}, \vec{a}]$

$= [\vec{a}, \vec{b}, \vec{c}] - [\vec{b}, \vec{c}, \vec{a}]$

$= [\vec{a}, \vec{b}, \vec{c}] - [\vec{a}, \vec{b}, \vec{c}]$

$= 0$ Proved.

....

4. If $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}, \vec{b} = 3\vec{\imath} + 5\vec{\jmath} + 2\vec{k},$

$\vec{c} = -\vec{\imath} - 2\vec{\jmath} + 3\vec{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Solution:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & -1\\ 3 & 5 & 2\end{vmatrix}$$

$$=\hat{\imath}(6+5)-\hat{\jmath}(4+3)+\hat{k}(10-9)$$

$$=\hat{\imath}(11)-\hat{\jmath}(7)+\hat{k}(1)$$

$$=11\hat{\imath}-7\hat{\jmath}+\hat{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 11 & -7 & 1\\ -1 & -2 & 3\end{vmatrix}$$

$$=\hat{\imath}(-21+2)-\hat{\jmath}(33+1)+\hat{k}(-22-7)$$

$$=\hat{\imath}(-19)-\hat{\jmath}(34)+\hat{k}(-29)$$

$$=-19\hat{\imath}-34\hat{\jmath}-29\hat{k} \quad \ldots\ldots(1)$$

$$\vec{a}.\vec{c}=(2\vec{\imath}+3\vec{\jmath}-\vec{k}).(-\vec{\imath}-2\vec{\jmath}+3\vec{k})$$

$$=-2-6-3$$

$$=-11$$

$$(\vec{a}.\vec{c})\vec{b}=-11(3\vec{\imath}+5\vec{\jmath}+2\vec{k})$$

$$=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}$$

$$\vec{b}.\vec{c}=(3\vec{\imath}+5\vec{\jmath}+2\vec{k}).(-\vec{\imath}-2\vec{\jmath}+3\vec{k})$$

$$=-3-10+6$$

$$=-7$$

$$(\vec{b}.\vec{c})\vec{a}=-7(2\vec{\imath}+3\vec{\jmath}-\vec{k})$$

$$=-14\vec{\imath}-21\vec{\jmath}+7\vec{k}$$

$$(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$$

$$=(-33\vec{\imath}-55\vec{\jmath}-22\vec{k})-(-14\vec{\imath}-21\vec{\jmath}+7\vec{k})$$

$$=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}+14\vec{\imath}+21\vec{\jmath}-7\vec{k}$$

$$=-19\hat{\imath}-34\hat{\jmath}-29\hat{k} \quad \ldots\ldots(2)$$

From (1) and (2)

$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$ is verified.

(ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 5 & 2\\ -1 & -2 & 3\end{vmatrix}$$

$$=\hat{\imath}(15+4)-\hat{\jmath}(9+2)+\hat{k}(-6+5)$$

$$=\hat{\imath}(19)-\hat{\jmath}(11)+\hat{k}(-1)$$

$$=19\hat{\imath}-11\hat{\jmath}-\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & -1\\ 19 & -11 & -1\end{vmatrix}$$

$$=\hat{\imath}(-3-11)-\hat{\jmath}(-2+19)+\hat{k}(-22-57)$$

$$=\hat{\imath}(-14)-\hat{\jmath}(17)+\hat{k}(-79)$$

$$=-14\hat{\imath}-17\hat{\jmath}-79\hat{k} \quad \ldots\ldots(1)$$

$$\vec{a}.\vec{c}=(2\vec{\imath}+3\vec{\jmath}-\vec{k}).(-\vec{\imath}-2\vec{\jmath}+3\vec{k})$$

$$=-2-6-3$$

$$=-11$$

$$(\vec{a}.\vec{c})\vec{b}=-11(3\vec{\imath}+5\vec{\jmath}+2\vec{k})$$

$$=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}$$

$$\vec{a}.\vec{b}=(2\vec{\imath}+3\vec{\jmath}-\vec{k}).(3\vec{\imath}+5\vec{\jmath}+2\vec{k})$$

$$=6+15-2$$

$$=21-2$$

$$=19$$

$$(\vec{a}.\vec{b})\vec{c}=19(-\vec{\imath}-2\vec{\jmath}+3\vec{k})$$

$$=-19\vec{\imath}-38\vec{\jmath}+57\vec{k}$$

$$(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$$

$$=(-33\vec{\imath}-55\vec{\jmath}-22\vec{k})-(-19\vec{\imath}-38\vec{\jmath}+57\vec{k})$$

$$=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}+19\vec{\imath}+38\vec{\jmath}-57\vec{k}$$

$$=-14\hat{\imath}-17\hat{\jmath}-79\hat{k} \quad \ldots\ldots(2)$$

From (1) and (2)

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ is verified.

5. $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}, \vec{b} = -\vec{i} + 2\vec{j} - 4\vec{k},$

$\vec{c} = \vec{i} + \vec{j} + \vec{k}$, then find the value of

$(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$

Solution:

$$(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c}) = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{c} \end{vmatrix}$$

$\vec{a}.\vec{a} = (2\vec{i} + 3\vec{j} - \vec{k}).(2\vec{i} + 3\vec{j} - \vec{k})$

$= 4 + 9 + 1$

$= 14$

$\vec{a}.\vec{c} = (2\vec{i} + 3\vec{j} - \vec{k}).(\vec{i} + \vec{j} + \vec{k})$

$= 2 + 3 - 1$

$= 5 - 1$

$= 4$

$\vec{b}.\vec{a} = (-\vec{i} + 2\vec{j} - 4\vec{k}).(2\vec{i} + 3\vec{j} - \vec{k})$

$= -2 + 6 + 4$

$= -2 + 10$

$= 8$

$\vec{b}.\vec{c} = (-\vec{i} + 2\vec{j} - 4\vec{k}).(\vec{i} + \vec{j} + \vec{k})$

$= -1 + 2 - 4$

$= -5 + 2$

$= -3$

$$\therefore (\vec{a} \times \vec{b}).(\vec{a} \times \vec{c}) = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 14 & 4 \\ 8 & -3 \end{vmatrix}$$

$= -42 - 32$

$= -74$

....

6. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors, then show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$.

Solution: $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors

$\therefore [\vec{a}, \vec{b}, \vec{d}] = 0 = [\vec{a}, \vec{b}, \vec{c}]$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

$= (0)\vec{c} - (0)\vec{d}$

$= \vec{0}$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$ is proved.

....

7. If $\vec{a} = \vec{\imath} + 2\vec{\jmath} + 3\vec{k}, \vec{b} = 2\vec{\imath} - \vec{\jmath} + \vec{k},$
$\vec{c} = 3\vec{\imath} + 2\vec{\jmath} + \vec{k}$ and
$\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$,
find the values of l, m, n.

Solution:

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$= \hat{\imath}(-1 - 2) - \hat{\jmath}(2 - 3) + \hat{k}(4 + 3)$

$= \hat{\imath}(-3) - \hat{\jmath}(-1) + \hat{k}(7)$

$= \hat{\imath}(-3) - \hat{\jmath}(-1) + \hat{k}(7)$

$= -3\hat{\imath} + \hat{\jmath} + 7\hat{k}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 1 & 7 \end{vmatrix}$$

$= \hat{\imath}(14 - 3) - \hat{\jmath}(7 + 9) + \hat{k}(1 + 6)$

$= \hat{\imath}(11) - \hat{\jmath}(16) + \hat{k}(7)$

$= 11\hat{\imath} - 16\hat{\jmath} + 7\hat{k}$

$l\vec{a} + m\vec{b} + n\vec{c}$

$= l(\vec{\imath} + 2\vec{\jmath} + 3\vec{k}) + m(2\vec{\imath} - \vec{\jmath} + \vec{k}) + n(3\vec{\imath} + 2\vec{\jmath} + \vec{k})$

$= \vec{\imath}(l + 2m + 3n) + \vec{\jmath}(2l - m + 2n) + \vec{k}(3l + m + n)$

Given $\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$

$l + 2m + 3n = 11 \quad \ldots\ldots\ldots(1)$

$2l - m + 2n = -16 \quad \ldots\ldots\ldots(2)$

$3l + m + n = 7 \quad \ldots\ldots\ldots(3)$

Solving (1) and (2)

$$l + 2m + 3n = 11$$

$$(2) \times 2 \quad 4l - 2m + 4n = -32$$

$$5l + 7n = -21 \quad \ldots\ldots\ldots (4)$$

Solving (2) and (3)

$$2l - m + 2n = -16$$

$$3l + m + n = 7$$

$$5l + 3n = -9 \quad \ldots\ldots\ldots (5)$$

Solving (4) and (5)

$$(4) \times 3 \quad 15l + 21n = -63$$

$$(5) \times 7 \quad 35l + 21n = -63$$

$$-10l = 0$$

$$\therefore l = 0$$

Substituting $l = 0$ in

$$5l + 3n = -9$$

$$3n = -9$$

$$n = -3$$

Substituting $l = 0$ and $n = -3$ in

$$l + 2m + 3n = 11$$

$$0 + 2m + 3(-3) = 11$$

$$2m - 9 = 11$$

$$2m = 11 + 9$$

$$2m = 20$$

$$m = 10$$

$$\therefore l = 0, m = 10 \text{ and } n = -3$$

....

8. If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$ find the angle between $\hat{a}$ and $\hat{c}$.

Solution: Given $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$

$$[(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}] = \frac{1}{2}\hat{b}$$

Equating coeff of $\hat{b}$ on either sides,

$$\vec{a}.\vec{c} = \frac{1}{2}$$

$$|\vec{a}||\vec{c}|\cos\theta = \frac{1}{2}$$

$\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = 1 = |\vec{c}|$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

The angle between $\hat{a}$ and $\hat{c}$ is $\frac{\pi}{3}$

....

<u>Straight line</u>

The equation of a straight line passing through **a fixed point** with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$

(i) Parametric form is

$$\vec{r} = \vec{a} + t\,\vec{b}, \text{ where } t \in \mathrm{R}$$

(ii) Non-parametric form of vector equation is

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

(iii) Cartesian form is

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$$

where the point $\mathrm{A}(x_1, x_2, x_3)$and

the vector $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

<u>Straight Line passing through **two given points**</u>

The equation of a straight line passing through two given points with position vector $\vec{a}$ and $\vec{b}$

(i) Parametric form is

$$\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a}), \text{ where } t \in \mathrm{R}$$

(ii) Non-parametric form of vector equation is

$$(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = \vec{0}$$

(iii) Cartesian form is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where the points $\mathrm{A}\ (x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$

Angle between two straight lines

(a) Vector form

The acute angle between two given straight lines $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$.

$$\text{So, } \cos\theta = \frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$$

(b) Cartesian form

If two lines are given in Cartesian form as

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and

$\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$, then the acute angle

θ between the two given lines is given by

$$\cos\theta = \left(\frac{|b_1d_1+b_2d_2+b_3d_3|}{\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}\sqrt{d_1{}^2+d_2{}^2+d_3{}^2}}\right)$$

....

Example 6.24

A straight line passes through the point $(1,2,-3)$ and parallel to $4\vec{i} + 5\vec{j} - 7\vec{k}$.
Find (i) vector equation in parametric form
(ii) vector equation in non-parametric form
(iii) Cartesian equations of the straight line.

Solution:

The line passes through the point $(1,2,-3)$

So, the position vector of the point is

$\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = 4\vec{i} + 5\vec{j} - 7\vec{k}$.

(i) Vector eqn in parametric form $\vec{r} = \vec{a} + t\,\vec{b}$

So, $\vec{r} = (\vec{i} + 2\vec{j} - 3\vec{k}) + t\,(4\vec{i} + 5\vec{j} - 7\vec{k})$

(ii) vector equation in non-parametric form

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

So, $\left(\vec{r} - (\vec{i} + 2\vec{j} - 3\vec{k})\right) \times (4\vec{i} + 5\vec{j} - 7\vec{k}) = \vec{0}$

(iii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$$

Example 6.25

The vector equation in parametric form of a line is $\vec{r} = (3\vec{i} - 2\vec{j} + 6\vec{k}) + t(2\vec{i} - \vec{j} + 3\vec{k})$.Find
(i) the direction cosines of the straight line
(ii) Vector equation in non-parametric form
(iii) Cartesian equations of the line.

Solution: $\vec{r} = (3\vec{i} - 2\vec{j} + 6\vec{k}) + t(2\vec{i} - \vec{j} + 3\vec{k})$

Comparing with $\vec{r} = \vec{a} + t\,\vec{b}$, the line passes through the point $\vec{a} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ and parallel to $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$. Hece the direction ratios of the line are $(2, -1, 3)$

$|\vec{b}| = \sqrt{4+1+9} = \sqrt{14}$

(i) The direction cosines of the straight line are $\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(ii) Vector equation in non-parametric form

$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$

So, $\left(\vec{r} - (3\vec{i} - 2\vec{j} + 6\vec{k})\right) \times (2\vec{i} - \vec{j} + 3\vec{k}) = \vec{0}$

(iii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-6}{3}$$

....

Example 6.26

Find the vector equation in parametric form and Cartesian equations of the line passing through $(-4,2,-3)$ and is parallel to the line $\frac{-x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}$.

Solution:

The line passes through the point $(-4,2,-3)$

So, the position vector of the point is

$\vec{a} = -4\vec{i} + 2\vec{j} - 3\vec{k}$ and rewriting the equation

$\frac{-x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}$ as

$$\frac{x+2}{-4} = \frac{y+3}{-2} = \frac{z-3}{\frac{3}{2}}$$

we have $\vec{b} = -4\vec{i} - 2\vec{j} + \frac{3}{2}\vec{k}$

$= -\frac{1}{2}(8\vec{i} + 4\vec{j} - 3\vec{k})$

So, the parallel vector is $8\vec{i} + 4\vec{j} - 3\vec{k}$

(i) Vector eqn in parametric form $\vec{r} = \vec{a} + t\,\vec{b}$

So, $\vec{r} = (-4\vec{i} + 2\vec{j} - 3\vec{k}) + t\,(8\vec{i} + 4\vec{j} - 3\vec{k})$

(ii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x+4}{8} = \frac{y-2}{4} = \frac{z+3}{-3}$$

....

Example 6.27 Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the straight line crosses the xy - plane.

Solution:

The equation of a straight line passing through two given points with position vector

$\vec{a} = -5\vec{i} + 7\vec{j} - 4\vec{k}$ and $\vec{b} = 13\vec{i} - 5\vec{j} + 2\vec{k}$

$\vec{b} - \vec{a} = (13\vec{i} - 5\vec{j} + 2\vec{k}) - (-5\vec{i} + 7\vec{j} - 4\vec{k})$

$= 13\vec{i} - 5\vec{j} + 2\vec{k} + 5\vec{i} - 7\vec{j} + 4\vec{k}$

$= 18\vec{i} - 12\vec{j} + 6\vec{k}$

(i) Parametric form is

$\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a})$, where $t \in$ R

So, $\vec{r} = (-5\vec{i} + 7\vec{j} - 4\vec{k}) + t\,(18\vec{i} - 12\vec{j} + 6\vec{k})$

$\vec{r} = (-5\vec{i} + 7\vec{j} - 4\vec{k}) + 6t\,(3\vec{i} - 2\vec{j} + \vec{k})$

(ii) Non-parametric form of vector equation is

$(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = \vec{0}$

So, $\left(\vec{r} - (-5\vec{i} + 7\vec{j} - 4\vec{k})\right) \times (3\vec{i} - 2\vec{j} + \vec{k}) = \vec{0}$

(iii) Cartesian form is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1}$$

(iv)To find the point where the straight line crosses the xy - plane.

Let $\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1} = \lambda$

$\frac{x+5}{3} = \lambda$ gives

$x + 5 = 3\lambda$

$x = 3\lambda - 5$

$\frac{y-7}{-2} = \lambda$ gives

$y - 7 = -2\lambda$

$y = -2\lambda + 7$

$\frac{z+4}{1} = \lambda$ gives

$z + 4 = \lambda$

$z = \lambda - 4$

Hence $(x, y, z) = (3\lambda - 5, -2\lambda + 7, \lambda - 4)$

On xy – plane $z = 0$

$\therefore \lambda - 4 = 0$ gives $\lambda = 4$

When $\lambda = 4$,

$x = 3\lambda - 5 = 12 - 5 = 7$

$y = -2\lambda + 7 = -8 + 7 = -1$

$z = 0$

The point where the straight line crosses the xy – plane is $(7, -1, 0)$

....

Example 6.28

Find the angle between the straight line $\frac{x+3}{2} = \frac{y-1}{2} = -z$ with coordinate axes.

Solution: $\frac{x+3}{2} = \frac{y-1}{2} = -z$

Rewriting $\frac{x+3}{2} = \frac{y-1}{2} = \frac{z}{-1}$ and comparing with $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

we have $\vec{b} = 2\vec{i} + 2\vec{j} - \vec{k}$

$|\vec{b}| = \sqrt{4+4+1} = \sqrt{9} = 3$

So, the direction cosines of the straight line are $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$

$$\cos\alpha = \frac{2}{3}, \cos\beta = \frac{2}{3} \text{ and } \cos\gamma = \frac{-1}{3}$$

where α, β, γ are the angles made by the $\vec{b}$ vector with x, y, z axis.

Hence $\alpha = cos^{-1}\left(\frac{2}{3}\right), \beta = cos^{-1}\left(\frac{2}{3}\right)$ and

$\gamma = cos^{-1}\left(\frac{-1}{3}\right)$

....

Example 6.29 Find the angle between the lines $\vec{r} = (\vec{i} + 2\vec{j} + 4\vec{k}) + t(2\vec{i} + 2\vec{j} + \vec{k})$ and the straight line passing through the points $(5,1,4)$ and $(9,2,12)$.

Solution: $\vec{r} = (\vec{i} + 2\vec{j} + 4\vec{k}) + t(2\vec{i} + 2\vec{j} + \vec{k})$

is parallel to the vector $\vec{b} = 2\vec{i} + 2\vec{j} + \vec{k}$

The direction ratios of the straight line passing through the points $(5,1,4)$ and $(9,2,12)$ is $(9-5, 2-1, 12-4) = (4,1,8)$

$$\text{So, } \vec{d} = 4\vec{i} + \vec{j} + 8\vec{k}$$

The acute angle between two given straight lines $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$.

$$\text{So, } \cos\theta = \frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$$

$$\vec{b}.\vec{d} = (2\vec{i} + 2\vec{j} + \vec{k}).(4\vec{i} + \vec{j} + 8\vec{k})$$

$$= 8 + 2 + 8 = 18$$

$$|\vec{b}| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$|\vec{d}| = \sqrt{16+1+64} = \sqrt{81} = 9$$

$$\cos\theta = \frac{18}{(3)(9)}$$

$$= \frac{2}{3}$$

Hence $\theta = cos^{-1}\left(\frac{2}{3}\right)$

....

Example 6.30

Find the angle between the straight lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ and state whether they are parallel or perpendicular

Solution: Comparing the equation $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ with $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$

we have $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{d} = 4\vec{i} - 4\vec{j} + 2\vec{k}$

Angle between the lines

$$\cos\theta = \left(\frac{|b_1d_1 + b_2d_2 + b_3d_3|}{\sqrt{{b_1}^2 + {b_2}^2 + {b_3}^2}\sqrt{{d_1}^2 + {d_2}^2 + {d_3}^2}}\right)$$

$$\vec{b}.\vec{d} = (2\vec{i} + \vec{j} - 2\vec{k}).(4\vec{i} - 4\vec{j} + 2\vec{k})$$

$$= 8 - 4 - 4 = 0$$

$$|\vec{b}| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$|\vec{d}| = \sqrt{16+16+4} = \sqrt{36} = 6$$

$$\cos\theta = (0)$$

Hence $\theta = cos^{-1}(0) = \frac{\pi}{2}$

The lines are perpendicular

....

Example 6.31

Show that the straight line passing through the points $A(6,7,5)$ and $B(8,10,6)$is perpendicular to the straight line passing through the points $C(10,2,-5)$ and $D(8,3,-4)$.

Solution: The direction ratios of the straight line passing through the points $A(6,7,5)$ and $B(8,10,6)$ is $(8-6, 10-7, 6-5) = (2,3,1)$

$$\text{So, } \vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$$

The direction ratios of the straight line passing through the points $C(10,2,-5)$and $D(8,3,-4)$ is $(8-10, 3-2, -4+5) = (-2,1,1)$

$$\text{So, } \vec{d} = -2\vec{i} + \vec{j} + \vec{k}$$

The acute angle between two given straight lines $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$.

So, $\cos\theta = \frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$

$$\vec{b}.\vec{d} = (2\vec{i} + 3\vec{j} + \vec{k}).(-2\vec{i} + \vec{j} + \vec{k})$$
$$= -4 + 3 + 1$$
$$= 0$$

$\cos\theta = (0)$

Hence $\theta = cos^{-1}(0) = \frac{\pi}{2}$

The lines are perpendicular

Example 6.32

Show that the lines $\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ are parallel.

Solution: Comparing the equation

$\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ with $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$

we have $\vec{b} = 4\vec{i} + 6\vec{j} + 12\vec{k}$ and

$$\vec{d} = -2\vec{i} + 3\vec{j} + 6\vec{k}$$

Here $\vec{b} = -2(-2\vec{i} + 3\vec{j} + 6\vec{k})$

$$= -2\vec{d}$$

Hence the given lines are parallel.

EXERCISE 6.4

1. Find the non-parametric form of vector equation and Cartesian equations of the straight line passing through the point with position vector $4\vec{i} + 3\vec{j} - 7\vec{k}$ and parallel to the vector $2\vec{i} - 6\vec{j} + 7\vec{k}$.

Solution: Given $\vec{a} = 4\vec{i} + 3\vec{j} - 7\vec{k}$

and $\vec{b} = 2\vec{i} - 6\vec{j} + 7\vec{k}$

(ii) vector equation in non-parametric form

$$(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$$

So, $\left(\vec{r} - (4\vec{i} + 3\vec{j} - 7\vec{k})\right) \times (2\vec{i} - 6\vec{j} + 7\vec{k}) = \vec{0}$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 4 & 3 & -7 \end{pmatrix}$ and $\begin{pmatrix} b_1 & b_2 & b_3 \\ 2 & -6 & 7 \end{pmatrix}$

(ii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x-4}{2} = \frac{y-3}{-6} = \frac{z+7}{7}$$

2. Find the parametric form of vector equation and Cartesian equations of the straight line passing through the point $(-2,3,4)$ and parallel to the straight line $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{8-z}{6}$.

Solution: Given $\vec{a} = -2\vec{i} + 3\vec{j} + 4\vec{k}$

and $\vec{b} = -4\vec{i} + 5\vec{j} + 6\vec{k}$

(i) Vector eqn in parametric form $\vec{r} = \vec{a} + t\,\vec{b}$

So, $\vec{r} = (-2\vec{i} + 3\vec{j} + 4\vec{k}) + t\,(-4\vec{i} + 5\vec{j} + 6\vec{k})$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ -2 & 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} b_1 & b_2 & b_3 \\ -4 & 5 & 6 \end{pmatrix}$

(ii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x+2}{-4} = \frac{y-3}{5} = \frac{z-4}{6}$$

3. Find the points where the straight line passes through $(6,7,4)$ and $(8,4,9)$ cuts the xz and yz planes.

Solution: Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 6 & 7 & 4 \end{pmatrix}$ and $\begin{pmatrix} x_2 & y_2 & z_2 \\ 8 & 4 & 9 \end{pmatrix}$

Cartesian form is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$
$$\frac{x - 6}{8 - 6} = \frac{y - 7}{4 - 7} = \frac{z - 4}{9 - 4}$$
$$\frac{x - 6}{2} = \frac{y - 7}{-3} = \frac{z - 4}{5}$$

(i) To find the point where the straight line crosses the xz - plane.

Let $\frac{x - 6}{2} = \frac{y - 7}{-3} = \frac{z - 4}{5} = \lambda$

$\frac{x-6}{2} = \lambda$ gives

$x - 6 = 2\lambda$

$x = 2\lambda + 6$

$\frac{y-7}{-3} = \lambda$ gives

$y - 7 = -3\lambda$

$y = -3\lambda + 7$

$\frac{z-4}{5} = \lambda$ gives

$z - 4 = 5\lambda$

$z = 5\lambda + 4$

Hence $(x, y, z) = (2\lambda + 6, -3\lambda + 7, 5\lambda + 4)$

On xz – plane $y = 0$

$\therefore -3\lambda + 7 = 0$

$-3\lambda = -7$

$3\lambda = 7$

$\lambda = \frac{7}{3}$

When $\lambda = \frac{7}{3}$,

$x = 2\lambda + 6$

$x = 2\left(\frac{7}{3}\right) + 6$

$= \frac{14}{3} + 6$

$= \frac{14+18}{3}$

$x = \frac{32}{3}$

When $\lambda = \frac{7}{3}$,

$z = 5\lambda + 4$

$z = 5\left(\frac{7}{3}\right) + 4$

$= \frac{35}{3} + 4$

$= \frac{35+12}{3}$

$z = \frac{47}{3}$ and $y = 0$

The point where the straight line crosses the xz – plane is $\left(\frac{32}{3}, 0, \frac{47}{3}\right)$

(ii)Hence $(x, y, z) = (2\lambda + 6, -3\lambda + 7, 5\lambda + 4)$

On yz – plane $x = 0$

$\therefore 2\lambda + 6 = 0$

$2\lambda = -6$

$\lambda = -3$

When $\lambda = -3$,

$y = -3\lambda + 7$

$= -3(-3) + 7$

$= 9 + 7$

$= 16$

When $\lambda = -3$,

$z = 5\lambda + 4$

$= 5(-3) + 4$

$= -15 + 4$

$= -11$

The point where the straight line crosses the yz – plane is $(0, 16, -11)$

....

4. Find the direction cosines of the straight line passing through the points (5,6,7) and (7,9,13) . Also, find the parametric form of vector equation and Cartesian equations of the straight line passing through two given points.

Solution: The straight line passing through the points (5,6,7) and (7,9,13). The direction ratios of the straight line passing through the points (5,6,7) and (7,9,13) is

$(7-5, 9-6, 13-7) = (2,3,6)$

Hence $\vec{b} - \vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

(i) Parametric form is

$\vec{r} = \vec{a} + t\left(\vec{b} - \vec{a}\right)$, where $t \in$ R

So, $\vec{r} = \left(5\vec{i} + 6\vec{j} + 7\vec{k}\right) + t\left(2\vec{i} + 3\vec{j} + 6\vec{k}\right)$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 5 & 6 & 7 \end{pmatrix}$ and $\begin{pmatrix} x_2 & y_2 & z_2 \\ 7 & 9 & 13 \end{pmatrix}$

Cartesian form is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-5}{7-5} = \frac{y-6}{9-6} = \frac{z-7}{13-7}$$

$$\frac{x-5}{2} = \frac{y-6}{3} = \frac{z-7}{6}$$

we have $\vec{b} = 2\vec{\imath} + 3\vec{\jmath} + 6\vec{k}$

$$|\vec{b}| = \sqrt{4+9+36} = \sqrt{49} = 7$$

So, the direction cosines of the straight line are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

....

5. Find the angle between the following lines.

(i) $\vec{r} = (4\vec{\imath} - \vec{\jmath}) + t(\vec{\imath} + 2\vec{\jmath} - 2\vec{k})$,

$\vec{r} = (\vec{\imath} - 2\vec{\jmath} + 4\vec{k}) + s(-\vec{\imath} - 2\vec{\jmath} + 2\vec{k})$

Solution:

The acute angle between two given straight lines $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + s\,\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$.

So, $\cos\theta = \frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$

we have $\vec{b} = \vec{\imath} + 2\vec{\jmath} - 2\vec{k}$ and

$$\vec{d} = -\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$$

$$\vec{b}.\vec{d} = (\vec{\imath} + 2\vec{\jmath} - 2\vec{k}).(-\vec{\imath} - 2\vec{\jmath} + 2\vec{k})$$

$$= -1-4-4$$

$$= -9$$

$$|\vec{b}.\vec{d}| = 9$$

$$|\vec{b}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$|\vec{d}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\cos\theta = \frac{9}{(3)(3)}$$

$$= \frac{9}{9}$$

$$= 1$$

Hence $\theta = cos^{-1}(1) = 0^0$

(ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}$,

$\vec{r} = 4\vec{k} + t(2\vec{\imath} + \vec{\jmath} + \vec{k})$.

Solution:

The acute angle between two given straight lines $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\vec{r} = \vec{c} + s\,\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$

So, $\cos\theta = \frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$

we have $\vec{b} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$ and

$$\vec{d} = 2\vec{\imath} + \vec{\jmath} + \vec{k}$$

$$\vec{b}.\vec{d} = (3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}).(2\vec{\imath} + \vec{\jmath} + \vec{k})$$

$$= 6+4+5$$

$$= 15$$

$$|\vec{b}.\vec{d}| = 15$$

$$|\vec{b}| = \sqrt{9+16+25}$$

$$= \sqrt{50} = \sqrt{25 \times 2}$$

$$= 5\sqrt{2}$$

$$|\vec{d}| = \sqrt{4+1+1} = \sqrt{6}$$

$$\cos\theta = \frac{15}{(5\sqrt{2})(\sqrt{6})}$$

$$= \frac{3}{\sqrt{12}} = \frac{3}{\sqrt{4\times3}}$$

$$= \frac{\sqrt{3}\times\sqrt{3}}{2\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2}$$

Hence $\theta = cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^0$

(iii) $2x = 3y = -z$ and $6x = -y = -4z$.

Solution: $2x = 3y = -z$ can be written

$$\frac{2x}{1} = \frac{3y}{1} = \frac{-z}{1}$$

$$\frac{x}{\frac{1}{2}} = \frac{y}{\frac{1}{3}} = \frac{z}{-1} \text{ and}$$

$$6x = -y = -4z$$

$$\frac{6x}{1} = \frac{-y}{1} = \frac{-4z}{1}$$

$$\frac{x}{\frac{1}{6}} = \frac{y}{-1} = \frac{z}{-\frac{1}{4}}$$

Comparing the equation with

$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$

we have $\vec{b}=\frac{1}{2}\vec{\imath}+\frac{1}{3}\vec{\jmath}-\vec{k}$ and

$$\vec{d}=\frac{1}{6}\vec{\imath}-1\vec{\jmath}-\frac{1}{4}\vec{k}$$

The acute angle between two given straight lines is $\cos\theta=\frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}$

$$\vec{b}.\vec{d}=\left(\frac{1}{2}\vec{\imath}+\frac{1}{3}\vec{\jmath}-\vec{k}\right).\left(\frac{1}{6}\vec{\imath}-1\vec{\jmath}-\frac{1}{4}\vec{k}\right)$$

$$=\frac{1}{12}-\frac{1}{3}+\frac{1}{4}$$

$$=\frac{1-4+3}{12}$$

$$=0$$

Hence $\theta=cos^{-1}(0)=\frac{\pi}{2}$

....

6. The vertices of ΔABC are $A(7,2,1)$, $B(6,0,3)$ and $C(4,2,4)$. Find $\angle ABC$.

Solution: Given the vertices of ΔABC are

$A(7,2,1)$, $B(6,0,3)$ and $C(4,2,4)$.

To find: $\angle ABC$

That is to find angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$

Now $\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}$

$$=(7\vec{\imath}+2\vec{\jmath}+\vec{k})-(6\vec{\imath}+0\vec{\jmath}+3\vec{k})$$

$$=7\vec{\imath}+2\vec{\jmath}+\vec{k}-6\vec{\imath}-0\vec{\jmath}-2\vec{k}$$

$$=\vec{\imath}+2\vec{\jmath}-2\vec{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$$

$$=(4\vec{\imath}+2\vec{\jmath}+4\vec{k})-(6\vec{\imath}+0\vec{\jmath}+3\vec{k})$$

$$=4\vec{\imath}+2\vec{\jmath}+4\vec{k}-6\vec{\imath}-0\vec{\jmath}-3\vec{k}$$

$$=-2\vec{\imath}+2\vec{\jmath}+\vec{k}$$

The acute angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$

is $\cos\theta=\frac{(\overrightarrow{BA}).(\overrightarrow{BC})}{|\overrightarrow{BA}||\overrightarrow{BC}|}$

$$(\overrightarrow{BA}).(\overrightarrow{BC})=(\vec{\imath}+2\vec{\jmath}-2\vec{k}).(-2\vec{\imath}+2\vec{\jmath}+\vec{k})$$

$$=-2+4-2$$

$$=0$$

Hence $\angle ABC=cos^{-1}(0)=\frac{\pi}{2}$

....

7. If the straight line joining the points $(2,1,4)$ and $(a-1,4,-1)$ is parallel to the line joining the points $(0,2,b-1)$ and $(5,3,-2)$, find the values of a and b.

Solution: Let the straight line L_1 joining the points $A(2,1,4)$ and $B(a-1,4,-1)$

Here $\begin{pmatrix}x_1 & y_1 & z_1\\ 2 & 1 & 4\end{pmatrix}$ and $\begin{pmatrix}x_2 & y_2 & z_2\\ a-1 & 4 & -1\end{pmatrix}$

Cartesian form of AB is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-2}{a-1-2}=\frac{y-1}{4-1}=\frac{z-4}{-1-4}$$

$$\frac{x-6}{a-3}=\frac{y-7}{3}=\frac{z-4}{-5}$$

we have $\vec{b}=(a-3)\vec{\imath}+3\vec{\jmath}-5\vec{k}$

Let the straight line L_2 joining the points $C(0,2,b-1)$ and $D(5,3,-2)$,

Here $\begin{pmatrix}x_1 & y_1 & z_1\\ 0 & 2 & b-1\end{pmatrix}$ and $\begin{pmatrix}x_2 & y_2 & z_2\\ 5 & 3 & -2\end{pmatrix}$

Cartesian form of CD is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-0}{5-0}=\frac{y-2}{3-2}=\frac{z-b+1}{-2-b+1}$$

$$\frac{x}{5}=\frac{y-7}{1}=\frac{z-4}{-1-b}$$

we have $\vec{d}=5\vec{\imath}+\vec{\jmath}+(-1-b)\vec{k}$

Given the lines are parallel.

So, $\vec{a}=m\vec{d}$

$$(a-3)\vec{\imath}+3\vec{\jmath}-5\vec{k}=m[5\vec{\imath}+\vec{\jmath}+(-1-b)\vec{k}]$$

$$(a-3)\vec{\imath}+3\vec{\jmath}-5\vec{k}=5m\vec{\imath}+m\vec{\jmath}+m(-1-b)\vec{k}$$

Equating $\vec{\imath}$

$$a-3=5m$$
$$a=5m+3$$

Equating $\vec{j}$

$$3=m$$

Equating $\vec{k}$

$$-5=m(-1-b)$$

Substituting $m=3$

$$a=5(3)+3$$
$$=15+3$$
$$a=18$$

Substituting $m=3$

$$-5=(3)(-1-b)$$
$$-5=-3-3b$$
$$3b=-3+5$$
$$3b=2$$
$$b=\frac{2}{3}$$

....

8. If the straight lines $\frac{x-5}{5m+2}=\frac{2-y}{5}=\frac{1-z}{-1}$ and $x=\frac{2y+1}{4m}=\frac{1-z}{-3}$ are perpendicular to each other, find the value of m .

Solution:

$$\frac{x-5}{5m+2}=\frac{2-y}{5}=\frac{1-z}{-1}=\frac{x-5}{5m+2}=\frac{y-2}{-5}=\frac{z-1}{1}$$

$$x=\frac{2y+1}{4m}=\frac{1-z}{-3}=\frac{x}{1}=\frac{y+\frac{1}{2}}{2m}=\frac{z-1}{3}$$

Comparing the equation

$\frac{x-5}{5m+2}=\frac{y-2}{-5}=\frac{z-1}{1}$ and $\frac{x}{1}=\frac{y+\frac{1}{2}}{2m}=\frac{z-1}{3}$ with

$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$

we have $\vec{b}=(5m+2)\vec{\imath}-5\vec{j}+\vec{k}$ and

$$\vec{d}=\vec{\imath}+2m\vec{j}+3\vec{k}$$

Given the vectors are perpendicular

$$\therefore \vec{b}.\vec{d}=0$$

$$\left((5m+2)\vec{\imath}-5\vec{j}+\vec{k}\right).\left(\vec{\imath}+2m\vec{j}+3\vec{k}\right)=0$$
$$5m+2-10m+3=0$$
$$-5m+5=0$$
$$-5m=-5$$
$$m=1$$

....

9. Show that the points $(2,3,4)$, $(-1,4,5)$ and $(8,1,2)$ are collinear.

Solution: Let the straight line L_1 joining the points $(2,3,4)$and $(-1,4,5)$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 2 & 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} x_2 & y_2 & z_2 \\ -1 & 4 & 5 \end{pmatrix}$

Cartesian form is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$
$$\frac{x-2}{-1-2}=\frac{y-3}{4-3}=\frac{z-4}{5-4}$$
$$\frac{x-2}{-3}=\frac{y-3}{1}=\frac{z-4}{1}=m$$
$$(x\,,y,z)=(-3m+2,m+3,m+4)$$

Substituting $\mathrm{m}=-2$, we get the third point $(8,1,2)$. Hence the given points are collinear.

....

The shortest distance between the two parallel lines $\vec{r}=\vec{a}+s\,\vec{b}$ and $\vec{r}=\vec{c}+t\,\vec{b}$ is given by

$$d=\frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}, where\ |\vec{b}|\neq 0$$

The shortest distance between the two skew lines $\vec{r}=\vec{a}+s\,\vec{b}$ and $\vec{r}=\vec{c}+t\,\vec{d}$ is given by

$$\delta=\frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|},\quad where\ |\vec{b}\times\vec{d}|\neq 0$$

(1) It follows from theorem (6.14) that two straight lines $\vec{r}=\vec{a}+s\,\vec{b}$ and $\vec{r}=\vec{c}+t\,\vec{b}$ intersect each other (that is, coplanar) if

$$(\vec{c}-\vec{a}).\left(\vec{b}\times\vec{d}\right)=0$$

(2) If two lines

$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$

intersect (that is, coplanar), then we have

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$

....

Example 6.33

Find the point of intersection of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$.

Solution: Every point on the line

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s$

Then $\frac{x-1}{2}=s$

$x-1=2s$

$x=2s+1$

$\frac{y-2}{3}=s$

$y-2=3s$

$y=3s+2$

$\frac{z-3}{4}=s$

$z-3=4s$

$z=4s+3$

Similarly if $\frac{x-4}{5}=\frac{y-1}{2}=z=t$

Then $\frac{x-4}{5}=t$

$x-4=5t$

$x=5t+4$

$\frac{y-1}{2}=t$

$y-1=2t$

$y=2t+1$ and $z=t$

So, at the point of intersection, for some values of s and t, we have

$(2s+1,3s+2,4s+3)=(5t+4,2t+1,t)$

There fore $2s+1=5t+4$

$2s-5t=4-1$

$2s-5t=3$ ....(1)

and $3s+2=2t+1$

$3s-2t=1-2$

$3s-2t=-1$ ....(2)

Solving (1) and (2)

(1) × 2 $\quad 4s-10t=6$

(2) × 5 $\quad 15s-10t=-5$

We have $\quad -11s=11$

gives $s=-1$

Substituting $s=-1$ in $(2s+1,3s+2,4s+3)$

we get the point of intersection is

$(-2+1,-3+2,-4+3)=(-1,-1,-1)$

....

Example 6.34

Find the equation of a straight line passing through the point of intersection of the straight lines $\vec{r}=(\vec{i}+3\vec{j}-\vec{k})+t(2\vec{i}+3\vec{j}+2\vec{k})$ and $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$, and perpendicular to both straight lines.

Solution: Given $\vec{r}=(\vec{i}+3\vec{j}-\vec{k})+t(2\vec{i}+3\vec{j}+2\vec{k})$

we have $\vec{a}=\vec{i}+3\vec{j}-\vec{k}$ and $\vec{b}=2\vec{i}+3\vec{j}+2\vec{k}$

Its Cartesian equation is $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}$

Another equation of the given line is

$\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$. Since the lines are intersecting every point on the line

Let $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}=s$

Then $\frac{x-1}{2}=s$

$x-1=2s$

$x=2s+1$

$\frac{y-3}{3}=s$

$y-3=3s$

$y=3s+3$

$\frac{z+1}{2}=s$

$z+1=2s$

$$z = 2s - 1$$

Similarly if $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4} = t$

Then $\frac{x-2}{1} = t$

$$x - 2 = t$$
$$x = t + 2$$
$$\frac{y-4}{2} = t$$
$$y - 4 = 2t$$
$$y = 2t + 4$$
$$\frac{z+3}{4} = t$$
$$z + 3 = 4t$$
$$z = 4t - 3$$

So, at the point of intersection, for some values of s and t, we have

$$(2s + 1, 3s + 3, 2s - 1) = (t + 2, 2t + 4, 4t - 3)$$

There fore $2s + 1 = t + 2$

$$2s - t = 2 - 1$$
$$2s - t = 1 \quad ....(1)$$

and $3s + 3 = 2t + 4$

$$3s - 2t = 4 - 3$$
$$3s - 2t = 1 \quad ....(2)$$

Solving (1) and (2)

$(1) \times 2 \qquad 4s - 2t = 2$

$(2) \qquad 3s - 2t = 1$

We have $\quad s = 1$

Substituting $s = 1$ in $(2s + 1, 3s + 3, 2s - 1)$

we get the point of intersection is

$$(2 + 1, 3 + 3, 2 - 1) = (3, 6, 1)$$

From the given equation we have

$\vec{b} = 2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$ and $\vec{d} = \vec{\imath} + 2\vec{\jmath} + 4\vec{k}$, then

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \hat{\imath}(12 - 4) - \hat{\jmath}(8 - 2) + \hat{k}(4 - 3)$$
$$= \hat{\imath}(8) - \hat{\jmath}(6) + \hat{k}(1)$$

$= 8\hat{\imath} - 6\hat{\jmath} + \hat{k}$ is a vector perpendicular to both the given straight lines. Therefore, the required straight line passing through (3, 6, 1) and perpendicular to both the given straight lines is the same as the straight line passing through (3, 6, 1) and parallel to $(8\hat{\imath} - 6\hat{\jmath} + \hat{k})$. Thus, the equation of the required straight line is $\vec{r} = \vec{a} + m\,\vec{b}$ that is

$$\vec{r} = (3\vec{\imath} + 6\vec{\jmath} + \vec{k}) + m(8\hat{\imath} - 6\hat{\jmath} + \hat{k})$$

....

Example 6.35

Determine whether the pair of straight lines $\vec{r} = (2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}) + t(2\vec{\imath} + 3\vec{\jmath} + 4\vec{k})$, $\vec{r} = (2\vec{\jmath} - 3\vec{k}) + s(\vec{\imath} + 2\vec{\jmath} + 3\vec{k})$ are parallel. Find the shortest distance between them.

Solution: Comparing the given two equations

$$\vec{r} = (2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}) + t(2\vec{\imath} + 3\vec{\jmath} + 4\vec{k}),$$
$$\vec{r} = (2\vec{\jmath} - 3\vec{k}) + s(\vec{\imath} + 2\vec{\jmath} + 3\vec{k})$$

with $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + s\,\vec{d}$

We have $\vec{a} = 2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$

$$\vec{b} = 2\vec{\imath} + 3\vec{\jmath} + 4\vec{k}$$
$$\vec{c} = 2\vec{\jmath} - 3\vec{k}$$
$$\vec{d} = \vec{\imath} + 2\vec{\jmath} + 3\vec{k}$$

Clearly, $\vec{b}$ is not a scalar multiple of $\vec{d}$.

So, the two vectors are not parallel and hence the two lines are not parallel.

The shortest distance between the two straight lines is given by $\delta = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$

Now, $\vec{c} - \vec{a} = (2\vec{\jmath} - 3\vec{k}) - (2\vec{\imath} + 6\vec{\jmath} + 3\vec{k})$

$$= 2\vec{\jmath} - 3\vec{k} - 2\vec{\imath} - 6\vec{\jmath} - 3\vec{k}$$

$$= -2\vec{i} - 4\vec{j} - 6\vec{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \hat{\imath}(9-8) - \hat{\jmath}(6-4) + \hat{k}(4-3)$$

$$= \hat{\imath}(1) - \hat{\jmath}(2) + \hat{k}(1)$$

$$= \hat{\imath} - 2\hat{\jmath} + \hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = (-2\vec{i} - 4\vec{j} - 6\vec{k}).(\hat{\imath} - 2\hat{\jmath} + \hat{k})$$

$$= -2 + 8 - 6$$

$$= -8 + 8$$

$$= 0$$

Therefore, the distance between the two given straight lines is zero. Thus, the given lines intersect each other.

....

Example 6.36

Find the shortest distance between the two given straight lines

$\vec{r} = (2\vec{i} + 3\vec{j} + 4\vec{k}) + t(-2\vec{i} + \vec{j} - 2\vec{k})$ and

$\frac{x-3}{2} = \frac{y}{-1} = \frac{z+2}{2}$.

Solution:

The parametric form of vector equations of the given straight lines are

$$\vec{r} = (2\vec{i} + 3\vec{j} + 4\vec{k}) + t(-2\vec{i} + \vec{j} - 2\vec{k})$$

$$\vec{r} = (3\vec{i} + 0\vec{j} - 2\vec{k}) + t(2\vec{i} - \vec{j} + 2\vec{k})$$

Comparing with $\vec{r} = \vec{a} + s\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$

We have $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$$\vec{b} = -2\vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{c} = 3\vec{i} + 0\vec{j} - 2\vec{k}$$

$$\vec{d} = 2\vec{i} - \vec{j} + 2\vec{k} = -1(-2\vec{i} + \vec{j} - 2\vec{k})$$

$$= -1(\vec{b})$$

Clearly, $\vec{b}$ is a scalar multiple of $\vec{d}$. Hence the given two lines are parallel.

The shortest distance between the two parallel lines $\vec{r} = \vec{a} + s\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{b}$ is given by

$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}, where\ |\vec{b}| \neq 0$$

Now, $\vec{c} - \vec{a} = (3\vec{i} + 0\vec{j} - 2\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$

$$= 3\vec{i} + 0\vec{j} - 2\vec{k} - 2\vec{i} - 3\vec{j} - 4\vec{k}$$

$$= \vec{i} - 3\vec{j} - 6\vec{k}$$

$$(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= \hat{\imath}(6+6) - \hat{\jmath}(-2-12) + \hat{k}(1-6)$$

$$= \hat{\imath}(12) - \hat{\jmath}(-14) + \hat{k}(-5)$$

$$= 12\hat{\imath} + 14\hat{\jmath} - 5\hat{k}$$

$$|(\vec{c}-\vec{a})\times\vec{b}| = \sqrt{144+196+25} = \sqrt{365}$$

$$|\vec{b}| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\therefore d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$$

$$= \frac{\sqrt{365}}{3}$$

....

Example 6.37

Find the coordinates of the foot of the perpendicular drawn from the point $(-1,2,3)$ to the straight line $\vec{r} = (\vec{i} - 4\vec{j} + 3\vec{k}) + t(2\vec{i} + 3\vec{j} + \vec{k})$. Also, find the shortest distance from the point to the straight line

Solution: Comparing the given equation $\vec{r} = (\vec{i} - 4\vec{j} + 3\vec{k}) + t(2\vec{i} + 3\vec{j} + \vec{k})$ with $\vec{r} = \vec{a} + t\vec{b}$, we get $\vec{a} = (\vec{i} - 4\vec{j} + 3\vec{k})$, and $\vec{b} = (2\vec{i} + 3\vec{j} + \vec{k})$. We denote the given point $(-1,2,3)$ by D, and the point $(1,-4,3)$ on the straight line by A.

If F is the foot of the perpendicular from D to the straight line, then F is of the form $(2t+1, 3t-4, t+3)$ and

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$$
$$= (2t+1, 3t-4, t+3) - (-1,2,3)$$
$$= (2t+2, 3t-6, t)$$
$$\overrightarrow{DF} = (2t+2)\vec{i} + (3t-6)\vec{j} + t\vec{k}$$

Since $\vec{b}$ *is perpendicular to* $(\overrightarrow{DF})$

$$\vec{b}.(\overrightarrow{DF}) = 0$$
$$\vec{b}.(\overrightarrow{DF}) = (2\vec{i}+3\vec{j}+\vec{k}).[(2t+2)\vec{i}+(3t-6)\vec{j}+t\vec{k}]$$
$$2(2t+2)+3(3t-6)+t=0$$
$$4t+4+9t-18+t=0$$
$$14t-14=0$$
$$14t=14$$
$$t=1$$

Substituting $t=1$ in F $(2t+1, 3t-4, t+3)$

We get F $(2+1, 3-4, 1+3) = F(3,-1,4)$

Hence the distance

$$\text{DF} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$= \sqrt{(3+1)^2+(-1-2)^2+(4-3)^2}$$
$$= \sqrt{(4)^2+(-3)^2+(1)^2}$$
$$= \sqrt{16+9+1}$$
$$= \sqrt{26}$$

....

EXERCISE 6.5

1. Find the parametric form of vector equation and Cartesian equations of a straight line passing through $(5, 2, 8)$ and is perpendicular to the straight lines

$\vec{r} = (\vec{i}+\vec{j}-\vec{k}) + t(2\vec{i}-2\vec{j}+\vec{k})$ and

$\vec{r} = (2\vec{i}-\vec{j}+3\vec{k}) + t(\vec{i}+2\vec{j}+2\vec{k})$.

Solution: Given $\vec{b} = (2\vec{i}-2\vec{j}+\vec{k})$ and

$$\vec{d} = (\vec{i}+2\vec{j}+2\vec{k})$$

To find the equation of the line passing through the point $(5,2,8)$, hence $\vec{a} = (5\vec{i}+2\vec{j}+8\vec{k})$

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= \hat{\imath}(-4-2) - \hat{\jmath}(4-1) + \hat{k}(4+2)$$
$$= \hat{\imath}(-6) - \hat{\jmath}(3) + \hat{k}(6)$$
$$= -6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$$

This vector is perpendicular to both the given two vectors.

(i) Thus, the equation of the required straight line in parametric form is

$\vec{r} = \vec{a} + t(\vec{b}\times\vec{d})$ that is

$$\vec{r} = (5\vec{i}+2\vec{j}+8\vec{k}) + t(-6\hat{\imath}-3\hat{\jmath}+6\hat{k})$$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 5 & 2 & 8 \end{pmatrix}$ and $\begin{pmatrix} b_1 & b_2 & b_3 \\ -6 & -3 & 6 \end{pmatrix}$

(ii) Cartesian form $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$$\frac{x-5}{-6} = \frac{y-2}{-3} = \frac{z-8}{6}$$

....

2. Show that the lines

$\vec{r} = (6\vec{i}+\vec{j}+2\vec{k}) + s(\vec{i}+2\vec{j}-3\vec{k})$ and

$\vec{r} = (3\vec{i}+2\vec{j}-2\vec{k}) + t(2\vec{i}+4\vec{j}-5\vec{k})$ are skew lines and hence find the shortest distance between them.

Solution: Comparing the given two equations

$$\vec{r} = (6\vec{i}+\vec{j}+2\vec{k}) + s(\vec{i}+2\vec{j}-3\vec{k}) \text{ and}$$
$$\vec{r} = (3\vec{i}+2\vec{j}-2\vec{k}) + t(2\vec{i}+4\vec{j}-5\vec{k})$$

with $\vec{r} = \vec{a} + s\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$

We have $\vec{a} = 6\vec{i}+\vec{j}+2\vec{k}$

$$\vec{b} = \vec{i}+2\vec{j}-3\vec{k}$$
$$\vec{c} = 3\vec{i}+2\vec{j}-2\vec{k}$$
$$\vec{d} = 2\vec{i}+4\vec{j}-5\vec{k}$$

The shortest distance between the two skew lines $\vec{r} = \vec{a} + s\,\vec{b}$ and $\vec{r} = \vec{c} + t\,\vec{d}$ is given by

$$\delta = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}, \quad where\ |\vec{b}\times\vec{d}| \neq 0$$

Now, $\vec{c}-\vec{a} = (3\vec{i}+2\vec{j}-2\vec{k}) - (6\vec{i}+\vec{j}+2\vec{k})$

$$= 3\vec{\imath} + 2\vec{\jmath} - 2\vec{k} - 6\vec{\imath} - \vec{\jmath} - 2\vec{k}$$

$$= -3\vec{\imath} + \vec{\jmath} - 4\vec{k}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$= \hat{\imath}(-10+12) - \hat{\jmath}(-5+6) + \hat{k}(4-4)$$

$$= \hat{\imath}(2) - \hat{\jmath}(1) + \hat{k}(0)$$

$$= 2\hat{\imath} - \hat{\jmath}$$

$$|\vec{b} \times \vec{d}| = \sqrt{4+1} = \sqrt{5} \neq 0$$

Hence the given lines are skew lines.

$$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (-3\vec{\imath} + \vec{\jmath} - 4\vec{k}).(2\hat{\imath} - \hat{\jmath})$$

$$= -6 - 1$$

$$= -7$$

$$|(\vec{c} - \vec{a}).(\vec{b} \times \vec{d})| = 7$$

The distance between the skew lines

$$\delta = \frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|} = \frac{7}{\sqrt{5}}$$

....

3. If the two lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-m}{2} = z$ intersect at a point, find the value of m .

Solution: Given $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and

$$\frac{x-3}{1} = \frac{y-m}{2} = \frac{z}{1}.$$

Since the lines are intersecting every point on the line.

Let $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = s$

Then $\frac{x-1}{2} = s$

$$x - 1 = 2s$$

$$x = 2s + 1$$

$$\frac{y+1}{3} = s$$

$$y + 1 = 3s$$

$$y = 3s - 1$$

$$\frac{z-1}{4} = s$$

$$z - 1 = 4s$$

$$z = 4s + 1$$

Similarly if $\frac{x-3}{1} = \frac{y-m}{2} = \frac{z}{1} = t$

Then $\frac{x-3}{1} = t$

$$x - 3 = t$$

$$x = t + 3$$

$$\frac{y-m}{2} = t$$

$$y - m = 2t$$

$$y = 2t + m$$

$$\frac{z}{1} = t$$

$$z = t$$

So, at the point of intersection, for some values of s and t, we have

$$(2s+1, 3s-1, 4s+1) = (t+3, 2t+m, t)$$

There fore $2s + 1 = t + 3$

$$2s - t = 3 - 1$$

$$2s - t = 2 \quad ....(1)$$

and $4s + 1 = t$

$$4s - t = -1 \quad ....(2)$$

Solving (1) and (2)

$$2s - t = 2$$

$$4s - t = -1$$

We have $-2s = 3$

$$s = -\frac{3}{2}$$

Substituting $s = -\frac{3}{2}$ in $2s - t = 2$

$$2\left(-\frac{3}{2}\right) - t = 2$$

$$-3 - t = 2$$

$$t = -5$$

Substituting $s = -\frac{3}{2}$ and $t = -5$

in $3s - 1 = 2t + m$

$$3\left(-\frac{3}{2}\right)-1=2(-5)+m$$

$$-\frac{9}{2}-1=-10+m$$

$$-\frac{9}{2}-1+10=m$$

$$-\frac{9}{2}+9=m$$

$$m=\frac{-9+18}{2}=\frac{9}{2}$$

....

4. Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0,$ and $\frac{x-6}{2}=\frac{z-1}{3}, y-2=0$, intersect. Also find the point of intersection.

Solution: $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0$

$$\frac{x-3}{3}=0 \Rightarrow x=3$$

$$\frac{y-3}{-1}=0 \Rightarrow y=3$$

$$z-1=0 \Rightarrow z=1$$

Hence $(x_1,y_1,z_1)=(3,3,1)$ and

$(b_1,b_2,b_3)=(3,-1,0)$

$$\frac{x-6}{2}=\frac{z-1}{3}, y-2=0$$

$$\frac{x-6}{2}=0 \Rightarrow x=6$$

$$y-2=0 \Rightarrow y=2$$

$$\frac{z-1}{3}=0 \Rightarrow z=1$$

Hence $(x_2,y_2,z_2)=(6,2,1)$

$(d_1,d_2,d_3)=(2,0,3)$

If two lines $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$

and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ intersect

(that is, coplanar), then we have

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$=\begin{vmatrix} 6-3 & 2-3 & 1-1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$=\begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$=0 \quad since\ R_1 \equiv R_2$$

So, the lines are intersecting lines.

Any point on the line

$\frac{x-3}{3}=\frac{y-3}{-1}=\alpha$, and $z-1=0 \Rightarrow z=1$

$$\frac{x-3}{3}=\alpha$$

$$x-3=3\alpha \Rightarrow x=3\alpha+3$$

$$\frac{y-3}{-1}=\alpha$$

$y-3=-\alpha \Rightarrow y=-\alpha+3$ and

Any point on the line

$\frac{x-6}{2}=\frac{z-1}{3}=\beta$, and $y-2=0 \Rightarrow y=2$

$$\frac{x-6}{2}=\beta$$

$$x-6=2\beta \Rightarrow x=2\beta+6$$

$$\frac{z-1}{3}=\beta$$

$$z-1=3\beta \Rightarrow z=3\beta+1$$

Equating y, $\quad -\alpha+3=2$

$$-\alpha=2-3$$

$$-\alpha=-1$$

$$\alpha=1$$

Substituting $\alpha=1$, in $x=3\alpha+3$

$$x=3+3=6$$

The point of intersection is $(6,2,1)$

5. Show that the straight lines $x+1=2y=-12z$ and $x=y+2=6z-6$ are skew and hence find the shortest distance between them.

Solution: Given $\frac{x+1}{1}=\frac{2y}{1}=\frac{-12z}{1}$

$$\frac{x+1}{1}=\frac{y}{\frac{1}{2}}=\frac{z}{-\frac{1}{12}}$$

Here $(x_1,y_1,z_1)=(-1,0,0)$ and

$$(b_1,b_2,b_3)=\left(1,\tfrac{1}{2},-\tfrac{1}{12}\right)$$

Another line $x=y+2=6z-6$

$$x=y+2=6(z-1)$$

$$\frac{x}{1}=\frac{y+2}{1}=\frac{6(z-1)}{1}$$

$$\frac{x}{1}=\frac{y+2}{1}=\frac{(z-1)}{\frac{1}{6}}$$

Here $(x_2,y_2,z_2)=(0,-2,1)$ and

$$(d_1,d_2,d_3)=\left(1,1,\tfrac{1}{6}\right)$$

$$\therefore\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}$$

$$=\begin{vmatrix} 0+1 & -2-0 & 1-0\\ 1 & \frac{1}{2} & -\frac{1}{12}\\ 1 & 1 & \frac{1}{6}\end{vmatrix}$$

$$=\begin{vmatrix} 1 & -2 & 1\\ 1 & \frac{1}{2} & -\frac{1}{12}\\ 1 & 1 & \frac{1}{6}\end{vmatrix}$$

$$=1\left(\frac{1}{12}+\frac{1}{12}\right)+2\left(\frac{1}{6}+\frac{1}{12}\right)+1\left(1-\frac{1}{2}\right)$$

$$=1\left(\frac{2}{12}\right)+2\left(\frac{3}{12}\right)+1\left(\frac{1}{2}\right)$$

$$=\frac{1}{6}+\frac{1}{2}+\frac{1}{2}$$

$$=\frac{1}{6}+1=\frac{7}{6}\neq 0$$

So, the lines are skew lines.

$$|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|=\frac{7}{6}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & \frac{1}{2} & -\frac{1}{12}\\ 1 & 1 & \frac{1}{6}\end{vmatrix}$$

$$=\hat{\imath}\left(\frac{1}{12}+\frac{1}{12}\right)-\hat{\jmath}\left(\frac{1}{6}+\frac{1}{12}\right)+\hat{k}\left(1-\frac{1}{2}\right)$$

$$=\hat{\imath}\left(\frac{1}{6}\right)-\hat{\jmath}\left(\frac{1}{4}\right)+\hat{k}\left(\frac{1}{2}\right)$$

$$=\frac{1}{6}\hat{\imath}-\frac{1}{4}\hat{\jmath}+\frac{1}{2}\hat{k}$$

$$|\vec{b}\times\vec{d}|=\sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}}$$

$$=\sqrt{\frac{16+36+144}{576}}$$

$$=\sqrt{\frac{196}{576}}$$

$$=\frac{14}{24}=\frac{7}{12}$$

The distance between the skew lines

$$\delta=\frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}=\frac{\frac{7}{6}}{\frac{7}{12}}=\frac{7}{6}\times\frac{12}{7}=2$$

....

6. Find the parametric form of vector equation of the straight line passing through $(-1,2,1)$ and parallel to the straight line $\vec{r}=(2\vec{\imath}+3\vec{\jmath}-\vec{k})+t(\vec{\imath}-2\vec{\jmath}+\vec{k})$ and hence find the shortest distance between the lines.

Solution:

The parametric form of vector equations of the given straight line

$$\vec{r}=(2\vec{\imath}+3\vec{\jmath}-\vec{k})+t(\vec{\imath}-2\vec{\jmath}+\vec{k})$$

Comparing with $\vec{r}=\vec{a}+t\,\vec{b}$

We have $\vec{a}=2\vec{\imath}+3\vec{\jmath}-\vec{k}$

$$\vec{b}=\vec{\imath}-2\vec{\jmath}+\vec{k}$$

The parametric form of vector equations of the parallel line $\vec{r}=\vec{c}+s\vec{b}$

We have $\vec{c}=-\vec{\imath}+2\vec{\jmath}+\vec{k}$

The shortest distance between the two parallel lines is given by

$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}, where\ |\vec{b}| \neq 0$

Now, $\vec{c}-\vec{a} = (-\vec{i}+2\vec{j}+\vec{k}) - (2\vec{i}+3\vec{j}-\vec{k})$

$= -\vec{i}+2\vec{j}+\vec{k}-2\vec{i}-3\vec{j}+\vec{k}$

$= -3\vec{i}-\vec{j}+2\vec{k}$

$$(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & -1 & 2 \\ 1 & -2 & 1 \end{vmatrix}$$

$= \hat{\imath}(-1+4) - \hat{\jmath}(-3-2) + \hat{k}(6+1)$

$= \hat{\imath}(3) - \hat{\jmath}(-5) + \hat{k}(5)$

$= 3\hat{\imath}+5\hat{\jmath}+7\hat{k}$

$|(\vec{c}-\vec{a})\times\vec{b}| = \sqrt{9+25+49} = \sqrt{83}$

$\vec{b} = \vec{\imath}-2\vec{\jmath}+\vec{k}$

$|\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$

$\therefore d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$

$= \frac{\sqrt{86}}{\sqrt{6}}$

....

7. Find the foot of the perpendicular drawn from the point $(5,4,2)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Also, find the equation of the perpendicular

Solution: Comparing the given equation

$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$ with

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

we get $\vec{a} = (-\vec{\imath}+3\vec{\jmath}+\vec{k})$, and

$\vec{b} = (2\vec{\imath}+3\vec{\jmath}-\vec{k})$.

Any point lie on the line

Let $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = t$

$\frac{x+1}{2} = t$

$x+1 = 2t$

$x = 2t-1$

$\frac{y-3}{3} = t$

$y-3 = 3t$

$y = 3t+3$

$\frac{z-1}{-1} = t$

$z-1 = -t$

$z = -t+1$

If F is the foot of the perpendicular from D$(5,4,2)$ to the straight line, then F is of the form $(2t-1, 3t+3, -t+1)$ and

$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD}$

$= (2t-1, 3t+3, -t+1) - (5,4,2)$

$= (2t-6, 3t-1, -t-1)$

$\overrightarrow{DF} = (2t-6)\vec{\imath} + (3t-1)\vec{\jmath} + (-t-1)\vec{k}$

Since $\vec{b}$ *is perpendicular to* $(\overrightarrow{DF})$

$\vec{b}.(\overrightarrow{DF}) = 0$

$(2\vec{\imath}+3\vec{\jmath}-\vec{k}).[(2t-6)\vec{\imath}+(3t-1)\vec{\jmath}+(-t-1)\vec{k}]$

$2(2t-6)+3(3t-1)+t+1 = 0$

$4t-12+9t-3+t+1 = 0$

$14t-15+1 = 0$

$14t = 14$

$t = 1$

Substituting $t = 1$ in F $(2t-1, 3t+3, -t+1)$

We get F $(2-1, 3+3, -1+1) = F(1,6,0)$

and D $(5,4,2)$

Here $\begin{pmatrix} x_1 & y_1 & z_1 \\ 5 & 4 & 2 \end{pmatrix}$ and $\begin{pmatrix} x_2 & y_2 & z_2 \\ 1 & 6 & 0 \end{pmatrix}$

Cartesian form is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-5}{1-5} = \frac{y-4}{6-4} = \frac{z-2}{0-2}$$

$$\frac{x-2}{-4} = \frac{y-3}{2} = \frac{z-4}{-2}$$

....

(a) The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ *is* $\vec{r}.\hat{d} = p$.

(b) Cartesian equation of a plane in normal form : Let l, m, n be the direction cosines of $\hat{d}$.

Then we have $\hat{d} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$.

Thus, equation (1) becomes

$\vec{r}.(l\hat{\imath} + m\hat{\jmath} + n\hat{k}) = p$ which implies

$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(l\hat{\imath} + m\hat{\jmath} + n\hat{k}) = p$ or

$$lx + my + nz = p \ldots (2)$$

Equation (2) is called the Cartesian equation of the plane in normal form

Equation of a plane perpendicular to a vector and passing through a given point

(a) Vector form of equation

Consider a plane passing through a point A with position vector $\vec{a}$ and $\vec{n}$ is a normal vector to the given plane.

Let $\vec{r}$ be the position vector of an arbitrary point P. Then $\overrightarrow{AP}$ is perpendicular to $\vec{n}$.

So, $\overrightarrow{AP}.\vec{n} = 0$ which gives $(\vec{r} - \vec{a}).\vec{n}=0$ . . . . .. (1)

which is the vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$.

Note: $(\vec{r} - \vec{a}).\vec{n}=0 \Rightarrow \vec{r}.\vec{n} = \vec{a}.\vec{n} \Rightarrow \vec{r}.\vec{n} = q$ where $q = \vec{a}.\vec{n}$.

(b) Cartesian form of equation :If a, b, c are the direction ratios of $\vec{n}$, then we have

$\vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$. Suppose, A is (x_1, y_1, z_1) then equation (1) becomes

$[(x - x_1)\hat{\imath} + (y - y_1)\hat{\jmath} + (z - z_1)\hat{k}].(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) = 0$.

That is, $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

which is the Cartesian equation of a plane, normal to a vector with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1) .

Intercept form: The general equation ax+by+cz+d=0 of first degree in x, y, z represents a plane.

....

Example 6.38

Find the vector and Cartesian form of the equations of a plane which is at a distance of 12 units from the origin and perpendicular to $6\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$.

Solution: Given $\vec{d} = 6\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$ and $p = 12$

If $\hat{d}$ is the unit normal in the direction of the vector then $\hat{d} = \frac{\vec{d}}{|\vec{d}|}$

$$\vec{d} = 6\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$$

$$|\vec{d}| = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$$

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{1}{7}(6\vec{\imath} + 2\vec{\jmath} - 3\vec{k})$$

If $\vec{r}$ is the position vector of a point (x, y, z) on the plane , then using $\vec{r}.\hat{d} = p$, vector equation of the plane in normal form is $\vec{r}.\frac{1}{7}(6\vec{\imath} + 2\vec{\jmath} - 3\vec{k}) = 12$

Applying $\vec{r} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$ in the above equation, we get

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).\frac{1}{7}(6\vec{\imath} + 2\vec{\jmath} - 3\vec{k}) = 12$$

$$\frac{1}{7}(6x + 2y - 3z) = 12$$

$$6x + 2y - 3z = 84$$

is the Cartesian equation of the plane.

....

Example 6.39

If the Cartesian equation of a plane is $3x - 4y + 3z = -8$, find the vector equation of the plane in the standard form.

Solution: If $\vec{r} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$ is the position vector of a point (x, y, z) then the given equation can be written as

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(3\vec{\imath} - 4\vec{\jmath} + 3\vec{k}) = -8 \quad \text{or}$$

$$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(-3\vec{\imath} + 4\vec{\jmath} - 3\vec{k}) = 8$$

That is $\vec{r}.(-3\vec{i}+4\vec{j}-3\vec{k})=8$ is the vector equation of the required plane.

Example 6.40
Find the direction cosines and length of the perpendicular from the origin to the plane $\vec{r}.(3\vec{i}-4\vec{j}+12\vec{k})=5$.

Solution: Given $\vec{r}.(3\vec{i}-4\vec{j}+12\vec{k})=5$

It is of the form $\vec{r}.(\vec{d})=q$

$$\vec{d}=3\vec{i}-4\vec{j}+12\vec{k}$$

$$|\vec{d}|=\sqrt{9+16+144}=\sqrt{169}=13$$

$$\hat{d}=\frac{\vec{d}}{|\vec{d}|}=\frac{1}{13}(3\vec{i}-4\vec{j}+12\vec{k})$$

Dividing $\vec{r}.(3\vec{i}-4\vec{j}+12\vec{k})=5$ by 13

we get $\vec{r}.\left(\frac{3}{13}\vec{i}-\frac{4}{13}\vec{j}+\frac{12}{13}\vec{k}\right)=\frac{5}{13}$

is of the form $\vec{r}.\hat{d}=p$

Hence we get $\hat{d}=\frac{3}{13}\vec{i}-\frac{4}{13}\vec{j}+\frac{12}{13}\vec{k}$ is a unit normal vector to the plane from the origin. The direction cosines are $\frac{3}{13},-\frac{4}{13},\frac{12}{13}$ and the length of the perpendicular to the origin is $p=\frac{5}{13}$

Example 6.41
Find the vector and Cartesian equations of the plane passing through the point with position vector $4\vec{i}+2\vec{j}-3\vec{k}$ and normal to vector $2\vec{i}-\vec{j}+\vec{k}$.

Solution: Given Position vector of the point

$\vec{a}=4\vec{i}+2\vec{j}-3\vec{k}$ and Normal to the vector $\vec{n}=2\vec{i}-\vec{j}+\vec{k}$

(i) The vector equation of the plane passing through the point with position vector $\vec{a}$ and normal to vector $\vec{n}$ is $\vec{r}.\vec{n}=\vec{a}.\vec{n}$

Hence, $\vec{r}.(2\vec{i}-\vec{j}+\vec{k})=(4\vec{i}+2\vec{j}-3\vec{k}).(2\vec{i}-\vec{j}+\vec{k})$

$$\vec{r}.(2\vec{i}-\vec{j}+\vec{k})=8-2-3$$

$$\vec{r}.(2\vec{i}-\vec{j}+\vec{k})=3 \quad ....(A)$$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(ii) The Cartesian equation of the plane

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(2\vec{i}-\vec{j}+\vec{k})=3$$

$$2x-y+z=3 \quad .....(B)$$

Example 6.42
A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point

Solution: The equation of the plane having intercepts a,b,c, on the x,y,z axes respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Since the sum of the reciprocals of the intercepts on the coordinate axes is a constant, we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=k$, where k is a constant, and which can be written as $\frac{1}{a}\left(\frac{1}{k}\right)+\frac{1}{b}\left(\frac{1}{k}\right)+\frac{1}{c}\left(\frac{1}{k}\right)=1$. This shows that the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ passes through the fixed point $\left(\frac{1}{k},\frac{1}{k},\frac{1}{k}\right)$

EXERCISE 6.6

1. Find a parametric form of vector equation of a plane which is at a distance of 7 units from the origin having $3,-4,5$ as direction ratios of a normal to it.

   Solution: Perpendicular from origin $p=7$
   Direction ratios of a normal $=3,-4,5$

$$\vec{d}=3\vec{i}-4\vec{j}+5\vec{k}$$

$$|\vec{d}|=\sqrt{9+16+25}=\sqrt{50}=5\sqrt{2}$$

$$\hat{d}=\frac{\vec{d}}{|\vec{d}|}=\frac{1}{5\sqrt{2}}(3\vec{i}-4\vec{j}+5\vec{k})$$

The parametric form of vector equation of the plane is $\vec{r}.\hat{d} = p$

$\vec{r}.\frac{1}{5\sqrt{2}}(3\vec{\imath} - 4\vec{\jmath} + 5\vec{k}) = 7$

....

2. Find the direction cosines of the normal to the plane $12x + 3y - 4z = 65$. Also, find the non-parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin.

Solution: $12x + 3y - 4z = 65$

It is of the form $\vec{r}.(\vec{d}) = q$

$\vec{r}.(12\vec{\imath} + 3\vec{\jmath} - 4\vec{k}) = 65$

$\vec{d} = 12\vec{\imath} + 3\vec{\jmath} - 4\vec{k}$

$|\vec{d}| = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$

$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{1}{13}(12\vec{\imath} + 3\vec{\jmath} - 4\vec{k})$

Dividing $\vec{r}.(12\vec{\imath} + 3\vec{\jmath} - 4\vec{k}) = 65$ by 13

we get $\vec{r}.\left(\frac{12}{13}\vec{\imath} + \frac{3}{13}\vec{\jmath} - \frac{4}{13}\vec{k}\right) = \frac{65}{13} = 5$

is of the form $\vec{r}.\hat{d} = p$, which is non-parametric form of vector equation of a plane

Hence we get $\hat{d} = \frac{12}{13}\vec{\imath} + \frac{3}{13}\vec{\jmath} - \frac{4}{13}\vec{k}$ is a unit normal vector to the plane from the origin. The direction cosines are $\frac{12}{13}, \frac{3}{13}, -\frac{4}{13}$ and the length of the perpendicular to the origin is $p = 5$

....

3. Find the vector and Cartesian equations of the plane passing through the point with position vector $2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$ and normal to the vector $\vec{\imath} + 3\vec{\jmath} + 5\vec{k}$.

Solution: Given Position vector of the point

$\vec{a} = 2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}$ and Normal to the vector

$\vec{n} = \vec{\imath} + 3\vec{\jmath} + 5\vec{k}$

(i) The vector equation of the plane passing through the point with position vector $\vec{a}$ and normal to vector $\vec{n}$ is $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

Hence ,

$\vec{r}.(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}) = (2\vec{\imath} + 6\vec{\jmath} + 3\vec{k}).(\vec{\imath} + 3\vec{\jmath} + 5\vec{k})$

$\vec{r}.(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}) = 2 + 18 + 15$

$\vec{r}.(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}) = 35 \quad ....(A)$

substituting $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, we get

(ii) The Cartesian equation of the plane

$(x\hat{\imath} + y\hat{\jmath} + z\hat{k}).(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}) = 35$

$x + 3y + 5z = 35 \quad ....(B)$

....

4. A plane passes through the point $(-1,1,2)$ and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.

Solution: Given magnitude $= 3\sqrt{3}$

and passes through point $\vec{a} = -\vec{\imath} + \vec{\jmath} + 2\vec{k}$

The normal makes an equal angle with the coordinate axes. If α, β, γ are the angles then

$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

But $\alpha = \beta = \gamma$. Hence $3cos^2\alpha = 1$

$\therefore cos^2\alpha = \frac{1}{3} = \cos\alpha = \frac{1}{\sqrt{3}}$

So, $\vec{n} = 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\vec{\imath} + \frac{1}{\sqrt{3}}\vec{\jmath} + \frac{1}{\sqrt{3}}\vec{k}\right)$

$= 3\frac{\sqrt{3}}{\sqrt{3}}(\vec{\imath} + \vec{\jmath} + \vec{k})$

$= 3(\vec{\imath} + \vec{\jmath} + \vec{k})$

$\vec{n} = (3\vec{\imath} + 3\vec{\jmath} + 3\vec{k})$

(i) The vector equation of the plane passing through the point with position vector $\vec{a}$ and normal to vector $\vec{n}$ is $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

Hence ,

$\vec{r}.(3\vec{i}+3\vec{j}+3\vec{k}) = (-\vec{i}+\vec{j}+2\vec{k}).(3\vec{i}+3\vec{j}+3\vec{k})$

$\vec{r}.(3\vec{i}+3\vec{j}+3\vec{k}) = -3+3+6$

$\vec{r}.(3\vec{i}+3\vec{j}+3\vec{k}) = 6 \quad ....(A)$

substituting $\vec{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(ii) The Cartesian equation of the plane

$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(3\vec{i}+3\vec{j}+3\vec{k}) = 6$

$3x+3y+3z = 6$

$x+y+z = 2 \quad ....(B)$

5. Find the intercepts cut off by the plane $\vec{r}.(6\vec{i}+4\vec{j}-3\vec{k}) = 12$ on the coordinate axes.

Solution: Given $\vec{r}.(6\vec{i}+4\vec{j}-3\vec{k}) = 12$

Comparing the equation with $\vec{r}.\vec{n} = q$

We have $\vec{n} = 6\vec{i}+4\vec{j}-3\vec{k}$ and $q = 12$

The equation of the plane having intercepts a, b, c, on the x, y, z axes respectively is

$\frac{q}{a} = 6,\ \frac{q}{b} = 4$ *and* $\frac{q}{c} = -3$

Substituting $q = 12$ in $\frac{q}{a} = 6$

we get $\frac{12}{a} = 6 \Rightarrow \frac{12}{6} = a \Rightarrow a = 2$

similarly $\frac{12}{b} = 4 \Rightarrow \frac{12}{4} = b \Rightarrow b = 3$ and

$\frac{12}{c} = -3 \Rightarrow \frac{12}{-3} = c \Rightarrow c = -4$

Hence $x-intercept = 2, y-intercept = 3$ and $z-intercept = -4$

6. If a plane meets the coordinate axes at A, B, C such that the centriod of the triangle ABC is the point (u, v, w), find the equation of the plane.

Solution: Let A$(a,0,0)$, B$(a,0,0)$ and C$(a,0,0)$ be the vertices of triangle ABC.

Then centroid of $\Delta = \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

Given centroid (u, v, w)

$\therefore \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right) = (u, v, w)$

Equating we get, $\frac{a}{3} = u \Rightarrow a = 3u$

$\frac{b}{3} = v \Rightarrow b = 3v$

$\frac{c}{3} = w \Rightarrow c = 3w$

We know the equation of the plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

Substituting the values of a, b, c

$\frac{x}{3u}+\frac{y}{3v}+\frac{z}{3w} = 1$ gives

$\frac{x}{u}+\frac{y}{v}+\frac{z}{w} = 3$

Equation of a plane passing through **a given point** and parallel to two given non-parallel vectors.

(a) Parametric form of vector equation

$\vec{r} = \vec{a}+s\vec{b}+t\vec{c}$

(b) Non-parametric form of vector equation

$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c}) = 0$

(c) Cartesian form of equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Equation of a plane passing through **2 given distinct points** and parallel to a given non-parallel vectors.

(a) Parametric form of vector equation

$\vec{r} = \vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

(b) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times\vec{c}\right)=0$$

(c) Cartesian form of equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

Equation of a plane passing through **three** given non-collinear points

(a) Parametric form of vector equation

$$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t(\vec{c}-\vec{a})$$

(b) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right)=0$$

(c) Cartesian form of equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

....

Example 6.43 Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(0,1,-5)$ and parallel to the straight lines

$\vec{r}=(\vec{i}+2\vec{j}-4\vec{k})+s(2\vec{i}+3\vec{j}+6\vec{k})$ and $\vec{r}=(\vec{i}-3\vec{j}+5\vec{k})+t(\vec{i}+\vec{j}-\vec{k})$.

Solution:

The plane passes through the point $(0,1,-5)$, hence $\vec{a}=0\vec{i}+\vec{j}-5\vec{k}$ and parallel to 2 given straight lines.

So, $\vec{b}=2\vec{i}+3\vec{j}+6\vec{k}$ and $\vec{c}=\vec{i}+\vec{j}-\vec{k}$

Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

(i) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=\hat{\imath}(-3-6)-\hat{\jmath}(-2-6)+\hat{k}(2-3)$$

$$=\hat{\imath}(-9)-\hat{\jmath}(-8)+\hat{k}(-1)$$

$$=-9\hat{\imath}+8\hat{\jmath}-\hat{k}$$

$[\vec{r}-(0\vec{i}+\vec{j}-5\vec{k})].(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=0$ gives

$$\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})-(\vec{j}-5\vec{k}).(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=0$$

$$\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})-(0+8+5)=0$$

$$\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})-13=0$$

$$\vec{r}.(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=13$$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(ii) The Cartesian equation of the plane

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=13$$

$$-9x+8y-z=13$$

$$9x-8y+z=-13$$

$$9x-8y+z+13=0$$

....

Example 6.44 Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1,2,0)$, $(2,2-1)$ and parallel to the straight line $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$.

Solution: Plane is parallel to the vector $\vec{c}=\vec{i}+2\vec{j}-\vec{k}$ and passing through the points $\vec{a}=-\vec{i}+2\vec{j}$, $\vec{b}=2\vec{i}+2\vec{j}-\vec{k}$

So, $\vec{b}-\vec{a}=2\vec{i}+2\vec{j}-\vec{k}+\vec{i}-2\vec{j}$

$$=3\vec{i}-\vec{k}$$

(a) Parametric form of vector equation of plane

$$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$$

$$\vec{r}=(-\vec{i}+2\vec{j})+s(3\vec{i}-\vec{k})+t(\vec{i}+2\vec{j}-\vec{k})$$

(b) Non-parametric form of vector equation

$(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times\vec{c}\right)=0$

$$(\vec{b}-\vec{a})\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 0 & -1\\ 1 & 2 & -1\end{vmatrix}$$

$= \hat{\imath}(0+2)-\hat{\jmath}(-3+1)+\hat{k}(6-0)$

$= \hat{\imath}(2)-\hat{\jmath}(-2)+\hat{k}(6)$

$= 2\hat{\imath}+2\hat{\jmath}+6\hat{k}$

$(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times\vec{c}\right)=0$ gives

$(\vec{r}-(-\vec{\imath}+2\vec{\jmath})).(2\hat{\imath}+2\hat{\jmath}+6\hat{k})=0$

$\vec{r}.(2\hat{\imath}+2\hat{\jmath}+6\hat{k})-(-\vec{\imath}+2\vec{\jmath}).(2\hat{\imath}+2\hat{\jmath}+6\hat{k})=0$

$\vec{r}.(\hat{\imath}+2\hat{\jmath}+6\hat{k})-(-2+4)=0$

$\vec{r}.(\hat{\imath}+2\hat{\jmath}+6\hat{k})-2=0$

$\vec{r}.(\hat{\imath}+2\hat{\jmath}+6\hat{k})=2$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(iii) The Cartesian equation of the plane

$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(\hat{\imath}+2\hat{\jmath}+6\hat{k})=2$

$x+2y+6z=2$

$x+2y+6z-2=0$

....

EXERCISE 6.7

1. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

Solution:

The plane passes through the point (2, 3, 6), hence $\vec{a}=2\vec{\imath}+3\vec{\jmath}+6\vec{k}$ and parallel to 2 given straight lines.

So, $\vec{b}=2\vec{\imath}+3\vec{\jmath}+\vec{k}$ and $\vec{c}=2\vec{\imath}-5\vec{\jmath}-3\vec{k}$

Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

(i) Non-parametric form of vector equation

$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 1\\ 2 & -5 & -3\end{vmatrix}$$

$= \hat{\imath}(-9+5)-\hat{\jmath}(-6-2)+\hat{k}(-10-6)$

$= \hat{\imath}(-4)-\hat{\jmath}(-8)+\hat{k}(-16)$

$= -4\hat{\imath}+8\hat{\jmath}-16\hat{k}$

$[\vec{r}-(2\vec{\imath}+3\vec{\jmath}+6\vec{k})].(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})=0$

gives

$\vec{r}.(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})-(2\vec{\imath}+3\vec{\jmath}+6\vec{k}).(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})=0$

$\vec{r}.(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})—(-8+24-96)=0$

$\vec{r}.(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})—(-104+24)=0$

$\vec{r}.(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})—(-80)=0$

$\vec{r}.(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})+80=0$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(ii) The Cartesian equation of the plane

$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(-4\hat{\imath}+8\hat{\jmath}-16\hat{k})+80=0$

$-4x+8y-16z+80=0$

$4x-8y+16z-80=0$

Dividing by 4, $x-2y+4z-20=0$

....

2. Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points (2,2,1), (9,3,6) and perpendicular to the plane $2x+6y+6z=9$.

Solution: Plane is parallel to the vector $\vec{c}=2\vec{\imath}+6\vec{\jmath}+6\vec{k}$ and passing through the points $\vec{a}=2\vec{\imath}+2\vec{\jmath}+\vec{k}$, $\vec{b}=9\vec{\imath}+3\vec{\jmath}+6\vec{k}$

So, $\vec{b}-\vec{a}=9\vec{i}+3\vec{j}+6\vec{k}-2\vec{i}-2\vec{j}-\vec{k}$

$=7\vec{i}+\vec{j}+5\vec{k}$

(a) Parametric form of vector equation of plane

$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

$\vec{r}=(2\vec{i}+2\vec{j}+\vec{k})+s(7\vec{i}+\vec{j}+5\vec{k})$

$+t\,(2\vec{i}+6\vec{j}+6\vec{k})$

(b) Cartesian form of equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}=0$$

RW $\begin{pmatrix} 1 & 5 & 7 & 1 \\ 6 & 6 & 2 & 6 \end{pmatrix}$

$-24x-32y+40z+48+64-40=0$

$-24x-32y+40z+72=0$

Dividing by -8, $\quad 3x+4y-5z-9=0$

....

3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points $(2,2,1)$, $(1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$.

Solution: Given

$\vec{c}=(-\vec{i}+5\vec{j}-8\vec{k})-(2\vec{i}+\vec{j}-3\vec{k})$

$=-\vec{i}+5\vec{j}-8\vec{k}-2\vec{i}-\vec{j}+3\vec{k}$

$=-3\vec{i}+4\vec{j}-5\vec{k}.$

Plane is parallel to the vector

$\vec{c}=-3\vec{i}+4\vec{j}-5\vec{k}$and passing through the

points $\vec{a}=2\vec{i}+2\vec{j}+\vec{k}$, $\;\vec{b}=\vec{i}-2\vec{j}+3\vec{k}$

So, $\vec{b}-\vec{a}=\vec{i}-2\vec{j}+3\vec{k}-2\vec{i}-2\vec{j}-\vec{k}$

$=-\vec{i}-4\vec{j}+2\vec{k}$

(a) Parametric form of vector equation of plane

$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

$\vec{r}=(2\vec{i}+2\vec{j}+\vec{k})+s(-\vec{i}-4\vec{j}+2\vec{k})$

$+t\,(-3\vec{i}+4\vec{j}-5\vec{k})$

(b) Cartesian form of equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix}=0$$

RW $\begin{pmatrix} -4 & 2 & -1 & -4 \\ 4 & -5 & -3 & 4 \end{pmatrix}$

$12x-11y-16z-24+22+16=0$

$12x-11y-16z+14=0$

....

4. Find the non-parametric form of vector equation of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

Solution:

The plane passes through the point $(1,-2,4)$,

hence $\vec{a}=\vec{i}-2\vec{j}+4\vec{k}$ and parallel to 2 given straight lines.

So, $\vec{b}=\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{c}=3\vec{i}-\vec{j}+\vec{k}$

Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

(i) Non-parametric form of vector equation

$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$

$=\hat{\imath}(2-3)-\hat{\jmath}(1+9)+\hat{k}(-1-6)$

$$= \hat{\imath}(-1) - \hat{\jmath}(10) + \hat{k}(-7)$$

$$= -\hat{\imath} - 10\hat{\jmath} - 7\hat{k}$$

$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$ gives

$$[\vec{r} - (\vec{i} - 2\vec{j} + 4\vec{k})].(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) - (\vec{i} - 2\vec{j} + 4\vec{k}).(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) — (-1 + 20 - 28) = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) — (-29 + 20) = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) — (-9) = 0$$

$$\vec{r}.(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}) + 9 = 0$$

....

5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + t(2\vec{i} - \vec{j} + 4\vec{k})$ and perpendicular to plane $\vec{r}.(\vec{i} + 2\vec{j} + \vec{k}) = 8$.

Solution: The plane passes through the point $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$ and parallel to 2 given straight lines.

So, $\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$

Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

(a) Parametric form of vector equation

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$$

$$\vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + s(2\vec{i} - \vec{j} + 4\vec{k}) + t(\vec{i} + 2\vec{j} + \vec{k})$$

(b) Cartesian form of equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

RW $\begin{pmatrix} -1 & 4 & 2 & -1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$

$$-9x + 2y + 5z + 9 + 2 - 15 = 0$$

$$-9x + 2y + 5z - 4 = 0$$

Dividing by -1, $\quad 9x - 2y - 5z + 4 = 0$

6. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the points $(3,6,-2)$, $(-1,-2,6)$, and $(6,4,-2)$.

Solution: The plane passes through 3 points.

So, $\vec{a} = 3\vec{i} + 6\vec{j} - 2\vec{k}$, $\vec{b} = -\vec{i} - 2\vec{j} + 6\vec{k}$ and $\vec{c} = 6\vec{i} + 4\vec{j} - 2\vec{k}$

Hence, $\vec{b} - \vec{a} = -\vec{i} - 2\vec{j} + 6\vec{k} - 3\vec{i} - 6\vec{j} + 2\vec{k}$

$$= -4\vec{i} - 8\vec{j} + 8\vec{k}$$

and $\quad \vec{c} - \vec{a} = 6\vec{i} + 4\vec{j} - 2\vec{k} - 3\vec{i} - 6\vec{j} + 2\vec{k}$

$$= 3\vec{i} - 2\vec{j}$$

Equation of a plane passing through three given non-collinear points

(a) Parametric form of vector equation

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$$

$$\vec{r} = (3\vec{i} + 6\vec{j} - 2\vec{k}) + s(-4\vec{i} - 8\vec{j} + 8\vec{k}) + t(3\vec{i} - 2\vec{j})$$

(b) Non-parametric form of vector equation

$$(\vec{r} - \vec{a}).\left((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right) = 0$$

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$

$$= \hat{\imath}(0 + 16) - \hat{\jmath}(0 - 24) + \hat{k}(8 + 24)$$

$$= \hat{\imath}(16) - \hat{\jmath}(-24) + \hat{k}(32)$$

$$= 16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}$$

$\therefore (\vec{r} - \vec{a}).\left((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right) = 0$ Gives,

$$[\vec{r} - (3\vec{i} + 6\vec{j} - 2\vec{k})].(16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}) = 0$$

$$\vec{r}.(16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}) - (3\vec{i} + 6\vec{j} - 2\vec{k}).(16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}) = 0$$

$$\vec{r}.(16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}) — (48 + 144 - 64) = 0$$

$$\vec{r}.(16\hat{\imath} + 24\hat{\jmath} + 32\hat{k}) — (192 - 64) = 0$$

$$\vec{r}.(16\hat{\imath}+24\hat{\jmath}+32\hat{k})—(128)=0$$

$$\vec{r}.(16\hat{\imath}+24\hat{\jmath}+32\hat{k})-128=0$$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(c) Cartesian form of equation

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(16\hat{\imath}+24\hat{\jmath}+32\hat{k})-128=0$$

$$16x+24y+32z-128=0$$

Dividing by 8, $\quad 2x+3y+4z-16=0$

....

7. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r}=(6\vec{i}-\vec{j}+\vec{k})+s(-\vec{i}+2\vec{j}+\vec{k})$ $+t(-5\vec{i}-4\vec{j}-5\vec{k})$.

Solution: Comparing with $\vec{r}=\vec{a}+s\vec{b}+t\vec{c}$

given $\vec{a}=6\vec{i}-\vec{j}+\vec{k}, \vec{b}=-\vec{i}+2\vec{j}+\vec{k}$ and

$\vec{c}=-5\vec{i}-4\vec{j}-5\vec{k}$

(i) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -1 & 2 & 1\\ -5 & -4 & -5\end{vmatrix}$$

$$=\hat{\imath}(-10+4)-\hat{\jmath}(5+5)+\hat{k}(4+10)$$

$$=\hat{\imath}(-6)-\hat{\jmath}(10)+\hat{k}(14)$$

$$=-6\hat{\imath}-10\hat{\jmath}+14\hat{k}$$

$$[\vec{r}-(6\vec{i}-\vec{j}+\vec{k})].(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})=0$$

gives

$$\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})-(6\vec{i}-\vec{j}+\vec{k}).(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})=0$$

$$\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})—(-36+10+14)=0$$

$$\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})—(-36+24)=0$$

$$\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})—(-12)=0$$

$$\vec{r}.(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})+12=0$$

substituting $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, we get

(*ii*) The Cartesian equation of the plane

$$(x\hat{\imath}+y\hat{\jmath}+z\hat{k}).(-6\hat{\imath}-10\hat{\jmath}+14\hat{k})+12=0$$

$$-6x-10y+14z+12=0$$

Dividing by -2, $\quad 3x+5y-7z-6=0$

....

Condition for a line to lie in a plane

We observe that a straight line will lie in a plane if every point on the line, lie on the plane and the normal to the plane is perpendicular to the line.

(i) If the line $\vec{r}=\vec{a}+t\vec{b}$ lies in the plane $\vec{r}.\vec{n}=d$, then $\vec{a}.\vec{n}=d$ and $\vec{b}.\vec{n}=0$.

(ii) If the line $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ lies in the plane $Ax+By+Cz+d=0$, then

$Ax_1+By_1+Cz_1+d=0$ and

$aA+bB+cC=0$.

Condition for coplanarity of two lines

(a) Condition in vector form

The two given non-parallel lines

$\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are coplanar if

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

(b) Condition in Cartesian form

$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and

$\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ are coplanar if

$$\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=0$$

....

Example 6.45

Verify whether the line $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$ lies in the plane $5x-y+z=8$.

Solution: Given $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$

comparing with $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

we have, $(x_1, y_1, z_1)=(3,4,-3)$ and

$$(a,b,c)=(-4,-7,12)$$

Direction ratios of the normal to the given plane are $(A,B,C)=(5,-1,1)$

The given point $(3,4,-3)$satisfies the given plane $5x-y+z=8$. Next,

$$aA+bB+cC=(-4)(5)+(-7)(-1)+(12)(1)$$
$$=-20+7+12$$
$$=-20+19$$
$$=-1\neq 0$$

So, the normal to the plane is not perpendicular to the line. Hence, the given line does not lie in the plane.

....

Example 6.46 Show that the lines $\vec{r}=(-\vec{i}-3\vec{j}-5\vec{k})+s(3\vec{i}+5\vec{j}+7\vec{k})$ and $\vec{r}=(2\vec{i}+4\vec{j}+6\vec{k})+t(\vec{i}+4\vec{j}+7\vec{k})$ are coplanar. Also, find the non-parametric form of vector equation of the plane containing these lines.

Solution: Comparing the given line

$\vec{r}=(-\vec{i}-3\vec{j}-5\vec{k})+s(3\vec{i}+5\vec{j}+7\vec{k})$ and

$\vec{r}=(2\vec{i}+4\vec{j}+6\vec{k})+t(\vec{i}+4\vec{j}+7\vec{k})$ with

$$\vec{r}=\vec{a}+s\vec{b} \text{ and } \vec{r}=\vec{c}+t\vec{d}$$

we have, $\vec{a}=-\vec{i}-3\vec{j}-5\vec{k}$, $\vec{b}=3\vec{i}+5\vec{j}+7\vec{k}$ and

$$\vec{c}=2\vec{i}+4\vec{j}+6\vec{k},\ \vec{d}=\vec{i}+4\vec{j}+7\vec{k}$$

The two given non-parallel lines are coplanar if

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

Now, $\vec{c}-\vec{a}=2\vec{i}+4\vec{j}+6\vec{k}+\vec{i}+3\vec{j}+5\vec{k}$

$$=3\vec{i}+7\vec{j}+11\vec{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 3 & 5 & 7\\ 1 & 4 & 7\end{vmatrix}$$

$$=\hat{\imath}(35-28)-\hat{\jmath}(21-7)+\hat{k}(12-5)$$
$$=\hat{\imath}(7)-\hat{\jmath}(14)+\hat{k}(7)$$
$$=7\hat{\imath}-14\hat{\jmath}+7\hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(3\vec{i}+7\vec{j}+11\vec{k}).(7\hat{\imath}-14\hat{\jmath}+7\hat{k})$$
$$=21-98+77$$
$$=98-98=0$$

Therefore the two lines are coplanar.

(i) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$[\vec{r}-(-\vec{i}-3\vec{j}-5\vec{k})].(7\hat{\imath}-14\hat{\jmath}+7\hat{k})=0$ gives

$$\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})-(-\vec{i}-3\vec{j}-5\vec{k}).(7\hat{\imath}-14\hat{\jmath}+7\hat{k})=0$$
$$\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})—(-7+42-35)=0$$
$$\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})—(-42+42)=0$$
$$\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})—(0)=0$$
$$\vec{r}.(7\hat{\imath}-14\hat{\jmath}+7\hat{k})=0$$

....

EXERCISE 6.8

1. Show that the straight lines $\vec{r}=(5\vec{i}+7\vec{j}-3\vec{k})+s(4\vec{i}+4\vec{j}-5\vec{k})$ and $\vec{r}=(8\vec{i}+4\vec{j}+5\vec{k})+t(7\vec{i}+\vec{j}+3\vec{k})$ are coplanar. Find the vector equation of the plane in which they lie.

Solution: Comparing the given line

$\vec{r}=(5\vec{i}+7\vec{j}-3\vec{k})+s(4\vec{i}+4\vec{j}-5\vec{k})$ and

$\vec{r}=(8\vec{i}+4\vec{j}+5\vec{k})+t(7\vec{i}+\vec{j}+3\vec{k})$ with

$$\vec{r}=\vec{a}+s\vec{b} \text{ and } \vec{r}=\vec{c}+t\vec{d}$$

we have, $\vec{a}=5\vec{i}+7\vec{j}-3\vec{k}$, $\vec{b}=4\vec{i}+4\vec{j}-5\vec{k}$ and

$$\vec{c}=8\vec{i}+4\vec{j}+5\vec{k},\ \vec{d}=7\vec{i}+\vec{j}+3\vec{k}$$

The two given non-parallel lines are coplanar if

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0$$

Now, $\vec{c}-\vec{a}=8\vec{i}+4\vec{j}+5\vec{k}-5\vec{i}-7\vec{j}+3\vec{k}$

$$=3\vec{i}-3\vec{j}+8\vec{k}$$

$$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 4 & 4 & -5\\ 7 & 1 & 3\end{vmatrix}$$

$$=\hat{\imath}(12+5)-\hat{\jmath}(12+35)+\hat{k}(4-28)$$

$$=\hat{\imath}(17)-\hat{\jmath}(47)+\hat{k}(-24)$$

$$=17\hat{\imath}-47\hat{\jmath}-24\hat{k}$$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(3\vec{i}-3\vec{j}+8\vec{k}).(17\hat{\imath}-47\hat{\jmath}-24\hat{k})$$

$$=51+141-192$$

$$=192-192$$

$$=0$$

Therefore the two lines are coplanar.

(i) Non-parametric form of vector equation

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{d})=0$$

$$[\vec{r}-(5\vec{i}+7\vec{j}-3\vec{k})].(17\hat{\imath}-47\hat{\jmath}-24\hat{k})=0$$

gives

$$\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})-(5\vec{i}+7\vec{j}-3\vec{k}).(17\hat{\imath}-47\hat{\jmath}-24\hat{k})=0$$

$$\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})—(85-329+72)=0$$

$$\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})—(157-329)=0$$

$$\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})—(-172)=0$$

$$\vec{r}.(17\hat{\imath}-47\hat{\jmath}-24\hat{k})+172=0$$

....

2. Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar.

Also, find the plane containing these lines.

Solution: Condition in Cartesian form

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}\text{ and}$$

$$\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}\text{ are coplanar if}$$

$$\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=0$$

Comparing with the given equation we have,

$(x_1,y_1,z_1)=(2,3,4)$, $(x_2,y_2,z_2)=(1,4,5)$,

$(b_1,b_2,b_3)=(1,1,3)$ and $(d_1,d_2,d_3)=(-3,2,1)$

$$\text{So, }\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=\begin{vmatrix}-1 & 1 & 1\\ 1 & 1 & 3\\ -3 & 2 & 1\end{vmatrix}$$

$$=-1(1-6)-1(1+9)+1(2+3)$$

$$=-1(-5)-1(10)+1(5)$$

$$=5-10+5$$

$$=10-10$$

$$=0$$

Therefore the two lines are coplanar.

(i) Cartesian equation of the plane

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0$$

$$\begin{vmatrix}x-2 & y-3 & z-4\\ 1 & 1 & 3\\ -3 & 2 & 1\end{vmatrix}=0$$

$$\text{RW}\begin{pmatrix}1 & 3 & 1 & 1\\ 2 & 1 & -3 & 2\end{pmatrix}$$

$$-5x-10y+5z+10+30-20=0$$

$$-5x-10y+5z+40-20=0$$

$$-5x-10y+5z+20=0$$

$$\div by-5,\quad x+2y-z-4=0$$

....

3. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

Solution: Condition in Cartesian form

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}\text{ and}$$

$$\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}\text{ are coplanar if}$$

$$\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix}=0$$

Comparing with the given equation we have,

$(x_1, y_1, z_1) = (1,2,3), (x_2, y_2, z_2) = (3,2,1),$

$(b_1, b_2, b_3) = (1,2,m^2),\ (d_1, d_2, d_3) = (1,m^2,2)$

$$\text{So, } \begin{vmatrix} 2 & 0 & -2 \\ 1 & 2 & m^2 \\ 1 & m^2 & 2 \end{vmatrix} = 0$$

$$2(4-m^4) - 2(m^2-2) = 0$$

$$8 - 2m^4 - 2m^2 + 4 = 0$$

$$-2m^4 - 2m^2 + 12 = 0$$

$$\div\ by -2, \quad m^4 + m^2 - 6 = 0$$

$$(m^2+3)(m^2-2) = 0$$

$$m^2 + 3 = 0 \Rightarrow m^2 = -3 \text{ not possible}$$

$$m^2 - 2 = 0 \Rightarrow m^2 = 2$$

$$m = \pm\sqrt{2}$$

....

4. If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Solution: Condition in Cartesian form

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and

$\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Comparing with the given equation we have,

$(x_1, y_1, z_1) = (1,-1,0), (x_2, y_2, z_2) = (-1,-1,0),$

$(b_1, b_2, b_3) = (2,\lambda,2),\ (d_1, d_2, d_3) = (5,2,\lambda)$

$$\text{So, } \begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$$-2(\lambda^2 - 4) = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

(i) When $\lambda = +2$

Cartesian equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$$

$$\text{RW} \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 \end{pmatrix}$$

$$6y - 6z + 6 = 0$$

$$\div\ by\ 6, \quad y - z + 1 = 0$$

(ii) When $\lambda = -2$

Cartesian equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$$

$$\text{RW} \begin{pmatrix} -2 & 2 & 2 & -2 \\ 2 & -2 & 5 & 2 \end{pmatrix}$$

$$14y + 14z + 14 = 0$$

$$\div\ by\ 14, \quad y + z + 1 = 0$$

....

Angle between 2 planes

The acute angle θ between the two planes

$\vec{r}.\vec{n}_1 = p_1$ and $\vec{r}.\vec{n}_2 = p_2$ is $\cos\theta = \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

The acute angle θ between the two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\left|\sqrt{a_1^2 + b_1^2 + c_1^2}\right|\left|\sqrt{a_2^2 + b_2^2 + c_2^2}\right|}$$

Angle between a line and a plane

The acute angle θ between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$ is $\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$

The acute angle θ between the line

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and the plane

$ax + by + cz = p$, then $\vec{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$

and $\vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then

$$\sin\theta = \frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}}$$

<u>Distance of a point from a plane.</u>

(a)Vector form of equation:

The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}.\vec{n} = p$ is given by $\delta = \frac{|\vec{u}.\vec{n}-p|}{|\vec{n}|}$

(b)Cartesian form of equation:

If A(x_1, y_1, z_1) is the given point with position vector $\vec{u}$ and $ax + by + cz = p$ is the equation of the plane, then the perpendicular distance is given by $\delta = \frac{|ax_1+by_1+cz_1-p\,|}{\sqrt{a^2+b^2+c^2}}$

<u>Distance between two parallel lines</u>

$ax + by + cz + d_1 = 0$ and

$ax + by + cz + d_2 = 0$ is given by $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

Equation of a plane passing through the line of intersection of two given planes

(i)The vector equation of a plane which passes through the line of intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$is given by

$(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$, where $\lambda \in \mathbb{R}$

The Cartesian equation of a plane which passes through the line of intersection of the planes

$a_1x + b_1y + c_1z = d_1$ and

$a_2x + b_2y + c_2z = d_2$ is given by

$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$

<u>The coordinates of the image of a point in a plane</u>

Let (a_1, a_2, a_3) be the point $\vec{u}$ whose image in the plane is required.

Then $\vec{u} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$.

Let $ax + by + cz = d$ be the equation of the given plane. Writing the equation in the vector form we get $\vec{r}.\vec{n} = p$ where $\vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

Then the position vector of the image is

$$\vec{v} = \vec{u} + \frac{2[p-(\vec{u}.\vec{n})]}{|\vec{n}|^2}\vec{n}$$

Meeting point of a line and a plane

The position vector of the point of intersection of the straight line line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$ is $\vec{a} + \frac{[p-(\vec{a}.\vec{n})]}{\vec{b}.\vec{n}}\vec{b}$

....

Example 6.47

Find the acute angle between the planes

$\vec{r}.(2\vec{i} + 2\vec{j} + 2\vec{k}) = 11$ and $4x - 2y + 2z = 15$

Solution: Given two planes are

$\vec{r}.(2\vec{i} + 2\vec{j} + 2\vec{k}) = 11$ and $4x - 2y + 2z = 15$

Hence the normal vectors of the planes are

$\vec{n}_1 = 2\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{n}_2 = 4\vec{i} - 2\vec{j} + 2\vec{k}$

The acute angle θ between the two planes

is $\cos\theta = \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

$$= \frac{|8-4+4|}{\sqrt{4+4+4}\sqrt{16+4+4}}$$

$$= \frac{|8|}{\sqrt{12}\sqrt{24}}$$

$$= \frac{8}{12\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}\sqrt{2}}{3\sqrt{2}}$$

$$\cos\theta = \frac{\sqrt{2}}{3}$$

Hence $\theta = cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$

Example 6.48

Find the angle between the straight line $\vec{r} = (2\vec{i} + 3\vec{j} + \vec{k}) + t(\vec{i} - \vec{j} + \vec{k})$ and the plane $2x - y + z = 5$

Solution: Comparing the given equation with the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$, we have $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{n} = 2\vec{i} - \vec{j} + \vec{k}$

The acute angle θ between the line and the plane is $\sin\theta = \frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$

$$= \frac{|2+1+1|}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$= \frac{4}{\sqrt{3}\sqrt{6}} = \frac{4}{\sqrt{18}} = \frac{4}{3\sqrt{2}}$$

Hence $\theta = sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$

Example 6.49

Find the distance of a point $(2,5,-3)$ from the plane $\vec{r}.(6\vec{i} - 3\vec{j} + 2\vec{k}) = 5$.

Solution: Comparing plane $\vec{r}.(6\vec{i} - 3\vec{j} + 2\vec{k}) = 5$ with $\vec{r}.\vec{n} = p$, we get $\vec{n} = 6\vec{i} - 3\vec{j} + 2\vec{k}$ and $p = 5$

the given point $(2,5,-3)$ is $\vec{u} = 2\vec{i} + 5\vec{j} - 3\vec{k}$

The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}.\vec{n} = p$ is given by $\delta = \frac{|\vec{u}.\vec{n} - p|}{|\vec{n}|}$

Now $\vec{u}.\vec{n} = (2\vec{i} + 5\vec{j} - 3\vec{k}).(6\vec{i} - 3\vec{j} + 2\vec{k})$

$$= 12 - 15 - 6$$

$$= 12 - 21$$

$$= -9$$

$$|\vec{u}.\vec{n} - p| = |-9 - 5| = 14$$

$$|\vec{n}| = \sqrt{36 + 9 + 4} = \sqrt{49} = 7$$

$$\therefore \delta = \frac{|\vec{u}.\vec{n} - p|}{|\vec{n}|} = \frac{14}{7} = 2$$

Example 6.50

Find the distance of the point $(5,-5,-10)$ from the point of intersection of a straight line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x - y + z = 5$

Solution: Given the line passes through the points $A(4,1,2) = (x_1, y_1, z_1)$ and

$$B(7,5,4) = (x_2, y_2, z_2)$$

So, equation the line passing through 2 points is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$$\frac{x-4}{7-4} = \frac{y-1}{5-1} = \frac{z-2}{4-2}$$

$$\frac{x-4}{3} = \frac{y-1}{4} = \frac{z-2}{2}$$

Let $\frac{x-4}{3} = \frac{y-1}{4} = \frac{z-2}{2} = \lambda$

$\frac{x-4}{3} = \lambda$ gives

$$x - 4 = 3\lambda \Rightarrow x = 3\lambda + 4$$

$\frac{y-1}{4} = \lambda$ gives

$y - 1 = 4\lambda \Rightarrow y = 4\lambda + 1$ and

$\frac{z-2}{2} = \lambda$ gives

$$z - 2 = 2\lambda \Rightarrow z = 2\lambda + 2$$

Hence the point $(3\lambda + 4, 4\lambda + 1, 2\lambda + 2)$ lies on the plane $x - y + z = 5$

So, $3\lambda + 4 - 4\lambda - 1 + 2\lambda + 2 = 5$

$$\lambda + 5 = 5$$

$$\lambda = 5 - 5 = 0$$

Substituting $\lambda = 0$, we get the point of intersection is $(4,1,2)$

Hence the distance between the points $(4,1,2)$ *and* $(5,-5,-10)$ is

$$d=\sqrt{(4-5)^2+(1+5)^2+(2+10)^2}$$

$$=\sqrt{(-1)^2+(6)^2+(12)^2}$$

$$=\sqrt{1+36+144} \quad =\sqrt{181}$$

....

Example 6.51

Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Solution: $x+2y-2z+1=0$ .... (1)

$$2x+4y-4z+5=0$$

$\div 2,\quad x+2y-2z+\frac{5}{2}=0$ ...... (2)

Comparing with $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$,

we have $d_1=1$ and $d_2=\frac{5}{2}$

Distance between two parallel lines is given by

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|\frac{5}{2}-1\right|}{\sqrt{1+4+4}}$$

$$=\frac{\left|\frac{5-2}{2}\right|}{\sqrt{9}}$$

$$=\frac{\frac{3}{2}}{3}=\frac{3}{2}\times\frac{1}{3}=\frac{1}{2}$$

....

Example 6.52 Find the distance between the planes $\vec{r}.(2\vec{i}-\vec{j}-2\vec{k})=6$ and $\vec{r}.(6\vec{i}-3\vec{j}-6\vec{k})=27$.

Solution: Let $\vec{u}$ be the position vector of a point on the plane $\vec{r}.(2\vec{i}-\vec{j}-2\vec{k})=6$

Then we have $\vec{u}.(2\vec{i}-\vec{j}-2\vec{k})=6$

The perpendicular distance from a point with position vector $\vec{u}$ to the plane

$\vec{r}.(6\vec{i}-3\vec{j}-6\vec{k})=27$ is given by $\delta=\frac{|\vec{u}.\vec{n}-p|}{|\vec{n}|}$

$$\therefore \delta=\frac{|\vec{u}.(6\vec{i}-3\vec{j}-6\vec{k})-p|}{|\vec{n}|}$$

$$=\frac{|3\vec{u}.(2\vec{i}-\vec{j}-2\vec{k})-27|}{|\sqrt{36+9+36}|}$$

$$=\frac{|3(6)-27|}{|9|}$$

$$=\frac{|18-27|}{9}=\frac{9}{9}=1$$

....

Example 6.53Find the equation of the plane passing through the intersection of the planes $\vec{r}.(\vec{i}+\vec{j}+\vec{k})+1=0$ and $\vec{r}.(2\vec{i}-3\vec{j}+5\vec{k})=2$ and the point $(-1,2,1)$.

Solution: The vector equation of a plane which passes through the line of intersection of the planes $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$is given by

$$(\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$$

We have $\vec{r}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, $\vec{n}_1=\vec{i}+\vec{j}+\vec{k}$,

$\vec{n}_2=2\vec{i}-3\vec{j}+5\vec{k}$, $d_1=-1$, $d_2=2$

$$\therefore (\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$$

$$(x+y+z+1)+\lambda(2x-3y+5z-2)=0$$

It passes through the point $(-1,2,1)$.

$$(-1+2+1+1)+\lambda(-2-6+5-2)=0$$

$$(4-1)+\lambda(5-10)=0$$

$$3-5\lambda=0$$

$$5\lambda=3$$

$$\lambda=\frac{3}{5}$$

Substituting $\lambda=\frac{3}{5}$, we get

$$(x+y+z+1)+\frac{3}{5}(2x-3y+5z-2)=0$$

$$5(x+y+z+1)+3(2x-3y+5z-2)=0$$

$$5x+5y+5z+5+6x-9y+15z-6=0$$

$$11x-4y+20z-1=0$$

....

Example 6.54

Find the equation of the plane passing through the intersection of the planes

$2x+3y-z+7=0$ and $x+y-2z+5=0$ and is perpendicular to the plane

$x+y-3z-5=0$

Solution: The equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ is

$$(2x+3y-z+7)+\lambda(x+y-2z+5)=0$$

$$(2+\lambda)x+(3+\lambda)y+(-1-2\lambda)z+(7+5\lambda)=0$$

This is perpendicular to $x+y-3z-5=0$

That is normal are perpendicular. So,

$$(2+\lambda)(1)+(3+\lambda)(1)+(-1-2\lambda)(-3)=0$$

$$2+\lambda+3+\lambda+3+6\lambda=0$$

$$8\lambda+8=0$$

$$\lambda=-1$$

$$(2x+3y-z+7)+\lambda(x+y-2z+5)=0$$

becomes,

$$(2x+3y-z+7)+(-1)(x+y-2z+5)=0$$

$$2x+3y-z+7-x-y+2z-5=0$$

$$x+2y+z+2=0$$

....

Example 6.55

Find the image of the point whose position vector is $\vec{i}+2\vec{j}+3\vec{k}$ in the plane $\vec{r}.(\vec{i}+2\vec{j}+4\vec{k})=38$.

Solution:

$$\vec{u}=\vec{i}+2\vec{j}+3\vec{k} \quad \vec{n}=\vec{i}+2\vec{j}+4\vec{k}, p=38$$

$$(\vec{u}.\vec{n})=(\vec{i}+2\vec{j}+3\vec{k}).(\vec{i}+2\vec{j}+4\vec{k})$$

$$=1+4+12$$

$$=17$$

$$2[p-(\vec{u}.\vec{n})]=2[38-17]$$

$$=2[21]$$

$$=42$$

$$|\vec{n}|^2=1+4+16$$

$$=21$$

The image of the point whose position vector is given by $\vec{v}=\vec{u}+\frac{2[p-(\vec{u}.\vec{n})]}{|\vec{n}|^2}\vec{n}$

$$=(\vec{i}+2\vec{j}+3\vec{k})+\frac{42}{21}(\vec{i}+2\vec{j}+4\vec{k})$$

$$=(\vec{i}+2\vec{j}+3\vec{k})+2(\vec{i}+2\vec{j}+4\vec{k})$$

$$=\vec{i}+2\vec{j}+3\vec{k}+2\vec{i}+4\vec{j}+8\vec{k}$$

$$=3\vec{i}+6\vec{j}+11\vec{k}$$

....

Example 6.56 Find the coordinates of the point where the straight line

$\vec{r}=(2\vec{i}-\vec{j}+2\vec{k})+t(3\vec{i}+4\vec{j}+2\vec{k})$

□intersects the plane $x-y+z-5=0$.

Solution:

Given $\vec{r}=(2\vec{i}-\vec{j}+2\vec{k})+t(3\vec{i}+4\vec{j}+2\vec{k})$

can be written as

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$$

$$\text{Let } \frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=\lambda$$

$\frac{x-2}{3}=\lambda$ gives

$$x-2=3\lambda \Rightarrow x=3\lambda+2$$

$\frac{y+1}{4}=\lambda$ gives

$y+1=4\lambda \Rightarrow y=4\lambda-1$ and

$\frac{z-2}{2}=\lambda$ gives

$$z-2=2\lambda \Rightarrow z=2\lambda+2$$

Hence the point $(3\lambda+2, 4\lambda-1, 2\lambda+2)$ lies on the plane $x-y+z=5$

So, $3\lambda+2-4\lambda+1+2\lambda+2=5$

$$\lambda+5=5$$

$$\lambda=5-5$$

$$\lambda=0$$

Substituting $\lambda=0$ in the point $(3\lambda+2, 4\lambda-1, 2\lambda+2)$ we get $(2,-1,2)$

....

EXERCISE 6.9

1. Find the equation of the plane passing

through the line of intersection of the planes $\vec{r}.(2\vec{i}-7\vec{j}+4\vec{k})=3$ and $3x-5y+4z+11=0$ and the point $(-2,1,3)$.

Solution: Given planes are

$$\vec{r}.(2\vec{i}-7\vec{j}+4\vec{k})=3 \Rightarrow 2x-7y+4z-3=0$$

and $3x-5y+4z+11=0$

Equation of the plane passes through the point of intersection of the planes

$$(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0$$

It passes through $(-2,1,3)$

So,

$$(-4-7+12-3)+\lambda(-6-5+12+11)=0$$

$$(-14+12)+\lambda(-11+23)=0$$

$$-2+12\lambda=0$$

$$12\lambda=2$$

$$\lambda=\frac{2}{12}$$

$$\lambda=\frac{1}{6}$$

Hence the required equation of the plane is

$$(2x-7y+4z-3)+\frac{1}{6}(3x-5y+4z+11)=0$$

$$6(2x-7y+4z-3)+(3x-5y+4z+11)=0$$

$$12x-42y+24z-18+3x-5y+4z+11=0$$

$$15x-47y+28z-7=0$$

....

2. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.

Solution: Equation of the plane passes through the point of intersection of the planes

$$(x+2y+3z-2)+\lambda(x-y+z-3)=0$$

$$(1+\lambda)x+(2-\lambda)y+(3+\lambda)z+(-2-3\lambda)=0$$

The perpendicular distance from the point $(3,1,-1)$ the above plane is

$$\delta=\frac{|ax_1+by_1+cz_1-p|}{\sqrt{a^2+b^2+c^2}}$$

$$\frac{2}{\sqrt{3}}=\frac{|3(1+\lambda)+1(2-\lambda)-1(3+\lambda)-2-3\lambda|}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}$$

$$=\frac{|3+3\lambda+2-\lambda-3-\lambda-2-3\lambda|}{\sqrt{\lambda^2+1+2\lambda+4+\lambda^2-4\lambda+9+\lambda^2+6\lambda}}$$

$$=\frac{|-2\lambda|}{\sqrt{3\lambda^2+4\lambda+14}}$$

Squaring on either sides,

$$\frac{4}{3}=\frac{4\lambda^2}{3\lambda^2+4\lambda+14}$$

$$4(3\lambda^2+4\lambda+14)=4\lambda^2(3)$$

$\div 4,$ $$3\lambda^2+4\lambda+14=3\lambda^2$$

$$4\lambda+14=0$$

$$4\lambda=-14$$

$$\lambda=-\frac{14}{4}$$

$$\lambda=-\frac{7}{2}$$

Substituting $\lambda=-\frac{7}{2}$ in

$$(x+2y+3z-2)+\lambda(x-y+z-3)=0$$

we get,

$$(x+2y+3z-2)-\frac{7}{2}(x-y+z-3)=0$$

$$2(x+2y+3z-2)-7(x-y+z-3)=0$$

$$2x+4y+6z-4-7x+7y-7z+21=0$$

$$-5x+11y-z+17=0$$

$\div -1,$ $$5x-11y+z-17=0$$

....

3. Find the angle between the line $\vec{r}=(2\vec{i}-\vec{j}+\vec{k})+t(\vec{i}+2\vec{j}-2\vec{k})$ and the plane $\vec{r}.(6\vec{i}+3\vec{j}+2\vec{k})=8$.

Solution: Comparing the given equation with the line $\vec{r}=\vec{a}+t\vec{b}$ and the plane $\vec{r}.\vec{n}=p$, we have $\vec{b}=\vec{i}+2\vec{j}-2\vec{k}$ and $\vec{n}=6\vec{i}+3\vec{j}+2\vec{k}$

The acute angle θ between the line and the plane is $\sin\theta=\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}$

$$= \frac{|6+6-4|}{\sqrt{1+4+4}\sqrt{36+9+4}}$$

$$= \frac{8}{\sqrt{9}\sqrt{49}} = \frac{8}{3\times 7} = \frac{8}{21}$$

Hence $\theta = sin^{-1}\left(\frac{8}{21}\right)$

....

4. Find the angle between the planes $\vec{r}.(\vec{\imath}+\vec{\jmath}-2\vec{k}) = 3$ and $2x-2y+z=2$.

Solution: Given two planes are

$\vec{r}.(\vec{\imath}+\vec{\jmath}-2\vec{k}) = 3$ and $2x-2y+z=2$

Hence the normal vectors of the planes are

$\vec{n}_1 = \vec{\imath}+\vec{\jmath}-2\vec{k}$ and $\vec{n}_2 = 2\vec{\imath}-2\vec{\jmath}+\vec{k}$

The acute angle θ between the two planes

is $\cos\theta = \frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$

$$= \frac{|2-2-2|}{\sqrt{1+1+4}\sqrt{4+4+1}}$$

$$= \frac{|-2|}{\sqrt{6}\sqrt{9}}$$

$$= \frac{2}{3\sqrt{6}}$$

$$\cos\theta = \frac{2}{3\sqrt{6}}$$

Hence $\theta = cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$

....

5. Find the equation of the plane which passes through the point $(3,4,-1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the two planes.

Solution: Equation of the given plane is

$2x-3y+5z+7=0$

Any plane parallel to the given plane is

$2x-3y+5z+K=0$

The parallel plane passes through $(3,4,-1)$

Hence $\quad 2(3)-3(4)+5(-1)+K=0$

$$6-12-5+K=0$$

$$6-17+K=0$$

$$-11+K=0$$

$$K=11$$

Required equation of the plane is

$$2x-3y+5z+11=0$$

To find the distance between the planes

$$2x-3y+5z+7=0$$

$$2x-3y+5z+11=0$$

$\div 2 \qquad x-\frac{3}{2}y+\frac{5}{2}z+\frac{7}{2}=0 \quad ......(1)$

$$x-\frac{3}{2}y+\frac{5}{2}z+\frac{11}{2}=0 \quad ......(2)$$

Comparing with $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$,

we have $d_1=\frac{7}{2}$ and $d_2=\frac{11}{2}$

Distance between two parallel lines is given by

$$\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}} = \frac{\left|\frac{11}{2}-\frac{7}{2}\right|}{\sqrt{1+\frac{9}{4}+\frac{25}{4}}}$$

$$= \frac{\left|\frac{11-7}{2}\right|}{\sqrt{\frac{4+9+25}{4}}}$$

$$= \frac{\left|\frac{4}{2}\right|}{\sqrt{\frac{38}{4}}}$$

$$= \frac{|2|}{\frac{\sqrt{38}}{2}}$$

$$= \frac{4}{\sqrt{38}}$$

....

6. Find the length of the perpendicular from the point $(1,-2,3)$ to the plane $x-y+z=5$

Solution: Given $A(1,-2,3)$ and the position vector $\vec{u} = x-y+z=5$

If $A(x_1,y_1,z_1)$ is the given point with position vector $\vec{u}$ and $ax+by+cz=p$ is the equation of the plane, then the perpendicular distance is given by $\delta = \frac{|ax_1+by_1+cz_1-p\,|}{\sqrt{a^2+b^2+c^2}}$

$$= \frac{|1(1)-1(-2)+1(3)-5\,|}{\sqrt{1^2+(-2)^2+1^2}}$$

$$= \frac{|1+2+3-5\,|}{\sqrt{1+4+1}}$$

$$= \frac{|6-5|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

7. Find the point of intersection of the line $x-1=\frac{y}{2}=z+1$ with the plane $2x-y+2z=2$. Also, find the angle between the line and the plane.

Solution:

Equation of the line $x-1=\frac{y}{2}=z+1$ is of the form $\frac{x-1}{1}=\frac{y-0}{2}=\frac{z+1}{1}$

So, $(a_1, b_1, c_1) = (1, 2, 1)$ then

$\vec{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ and the plane $2x - y + 2z = 2$ hence

$\vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

The acute angle θ between the line

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and the plane

$ax + by + cz = p$, then $\vec{b} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$

and $\vec{n} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then

$$\sin\theta = \frac{|aa_1+bb_1+cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$$

$$= \frac{|2(1)-1(2)+2(1)|}{\sqrt{(2)^2+(-1)^2+(2)^2}\sqrt{(1)^2+(2)^2+(1)^2}}$$

$$= \frac{|2-2+2|}{\sqrt{4+1+4}\sqrt{1+4+1}} = \frac{|2|}{\sqrt{9}\sqrt{6}}$$

$$\sin\theta = \frac{2}{3\sqrt{6}} \Rightarrow \theta = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

8. Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point (4,3,2) to the plane $x + 2y + 3z = 2$.

Solution:

Equation of the plane is $x + 2y + 3z = 2$.

So, normal $\vec{n} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

Equation of the plane through the point (4,3,2) with direction of the normal is

$$\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3}$$

Let $\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3} = \lambda$

$\frac{x-4}{1} = \lambda$ gives

$x - 4 = \lambda \Rightarrow x = \lambda + 4$

$\frac{y-3}{2} = \lambda$ gives

$y - 3 = 2\lambda \Rightarrow y = 2\lambda + 3$ and

$\frac{z-2}{3} = \lambda$ gives

$z - 2 = 3\lambda \Rightarrow z = 3\lambda + 2$

Hence the point $(\lambda + 4, 2\lambda + 3, 3\lambda + 2)$ lies on the plane $x + 2y + 3z = 2$

So, $\lambda + 4 + 4\lambda + 6 + 9\lambda + 6 = 2$

$14\lambda + 16 = 2$

$14\lambda = 2 - 16$

$14\lambda = -14$

$\lambda = -1$

Substituting $\lambda = -1$ in the point $(\lambda + 4, 2\lambda + 3, 3\lambda + 2)$ we get $(3, 1, -1)$ which is the coordinates of the foot of the perpendicular

(ii) Length of the perpendicular from the point (4,3,2) to the plane $x + 2y + 3z - 2 = 0$ is

$$\delta = \frac{|ax_1+by_1+cz_1-p|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|1(4)+2(3)+3(2)-2|}{\sqrt{1^2+2^2+3^2}}$$

$$= \frac{|4+6+6-2|}{\sqrt{1+4+9}}$$

$$= \frac{|16-2|}{\sqrt{14}}$$

$$= \frac{14}{\sqrt{14}}$$

$$\delta = \sqrt{14}$$

EXERCISE 6.10

Choose the correct or the most suitable answer from the given four alternatives:

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then is equal to $[\vec{a}, \vec{b}, \vec{c}]$

(1) 2 (2) -1 (3) 1 **(4) 0**

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then

(1) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$ (2) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$

(3) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (4) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$

3. If $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ then the value of $[\vec{a}, \vec{b}, \vec{c}]$ is

(1) $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (2) $\frac{1}{3}|\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(3) 1 (4) -1

4. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a} \times (\vec{b} \times \vec{c})$ is equal to

(1) $\vec{a}$ **(2) $\vec{b}$** (3) $\vec{c}$ (4) $\vec{0}$

5. If $[\vec{a}, \vec{b}, \vec{c}] = 1$ then the value of

$\frac{\vec{a}.(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b}).\vec{a}}$ is

(1) 1 (2) -1 **(3) 2** (4) 3

6. The volume of the parallelepiped with its edges represented by the vectors $\vec{\imath}+\vec{\jmath}, \vec{\imath}+2\vec{\jmath}, \vec{\imath}+\vec{\jmath}+\pi\vec{k}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ **(3) π** (4) $\frac{\pi}{4}$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a}\times\vec{b}] = \frac{\pi}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

8. If $\vec{a} = \vec{\imath}+\vec{\jmath}+\vec{k}$, $\vec{b} = \vec{\imath}+\vec{\jmath}$ and $\vec{c} = \vec{\imath}$

$(\vec{a}\times\vec{b})\times\vec{c} = \lambda\vec{a} + \mu\vec{b}$ then

the value of $\lambda + \mu$ is

(1) 0 (2) 1 (3) 6 (4) 3

9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 3$, then

$\{[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]\}^2$ is equal to

(1) 81 (2) 9 (3) 27 (4) 18

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{2}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ **(2) $\frac{3\pi}{4}$** (3) $\frac{\pi}{4}$ (4) π

11. If the volume of the parallelepiped with

$\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})$, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ and $(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})$ as coterminous edges is

(1) 8cubic units (2) 512cubic units

(3) 64cubic units (4)24 cubic units

12. Consider the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$. Let $P_1$and P_2 be the planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between and $P_1$and $P_2$is

(1) 0° (2) 45° (3) 60° (4) 90°

13. If $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b}.\vec{c}\neq 0$ and $\vec{a}.\vec{b}\neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular **(2) parallel**

(3) inclined at an angle $\frac{\pi}{3}$

(4) inclined at an angle $\frac{\pi}{2}$

14. If $\vec{a}=2\vec{\imath}+3\vec{\jmath}-\vec{k}, \vec{b}=\vec{\imath}+\vec{\jmath}-5\vec{k}$, $\vec{c}=3\vec{\imath}+5\vec{\jmath}-\vec{k}$ then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(1) $-17\vec{\imath}+21\vec{\jmath}-97\vec{k}$ (2) $17\vec{\imath}+21\vec{\jmath}-123\vec{k}$

(3) $-17\vec{\imath}-21\vec{\jmath}+97\vec{k}$ **(4) $-17\vec{\imath}-21\vec{\jmath}-97\vec{k}$**

15. The angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$,and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ **(4) $\frac{\pi}{2}$**

16. If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, then(α,β) is

(1) $(-5,5)$ **(2) $(-6,7)$**

(3) $(5,-5)$ (4) $(6,-7)$

17. The angle between the line $\vec{r}=(\vec{\imath}+\vec{\jmath}+\vec{k})+t(2\vec{\imath}+\vec{\jmath}-2\vec{k})$ and the plane $\vec{r}.(\vec{\imath}+\vec{\jmath})+4$ is

(1) 0° (2) 30° **(3) 45°** (4) 90°

18. The coordinates of the point where the line$\vec{r}=(6\vec{\imath}-\vec{\jmath}-3\vec{k})+t(-\vec{\imath}+4\vec{k})$ meets the plane $\vec{r}.(\vec{\imath}+\vec{\jmath}+\vec{k})=3$ are

(1)(2,1,0) (2) (7,−1,7)

(3) (1,2,−6) **(4) (5,−1,1)**

19. Distance from the origin to the plane $3x-6y+2z+7=0$ is

(1) 0 **(2) 1** (3) 2 (4) 3

20. The distance between the planes

$x+2y+3z+7=0$ and $2x+4y+6z+7=0$ is

(1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$ (3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$

21. If the direction cosines of a line are $\frac{1}{c},\frac{1}{c},\frac{1}{c}$ then

(1) $c=\pm 3$ **(2) $c=\pm\sqrt{3}$**

(3) $c>0$ (4) $0<c<1$

22. The vector equation $\vec{r}=(\vec{i}-2\vec{j}-\vec{k})+t(6\vec{i}-\vec{k})$ represents a straight line passing through the points

(1) $(0,6,-1)$ and $(1,-2,-1)$

(2) $(0,6,-1)$ and $(-1,-4,-2)$

(3) $(1,-2,-1)$ and $(1,4,-2)$

(4) $(1,-2,-1)$ and $(0,6,-1)$

23. If the distance of the point $(1,1,1)$ from the origin is half of its distance from the plane $x+y+z+k=0$, then the values of k are

(1) ± 3 (2) ± 6 (3) $-3, 9$ **(4) $3, -9$**

24. If the planes $\vec{r}.(2\vec{i}-\lambda\vec{j}+\vec{k})=3$ and $\vec{r}.(4\vec{i}+\vec{j}-\mu\vec{k})=5$ are parallel, then the value of λ and μ are

(1) $\frac{1}{2},-2$ (2) $-\frac{1}{2},2$

(3) $-\frac{1}{2},-2$ (4) $\frac{1}{2},2$

25. If the length of the perpendicular from the origin to the plane $2x+3y+\lambda z=1$, $\lambda>0$ is $\frac{1}{5}$, then the value of λ is

(1) $2\sqrt{3}$ (2) $3\sqrt{2}$ (3) 0 (4) 1

....

DEPARTMENT OF MATHEMATICS

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VELLORE DT -632503

UNIT – 6

APPLICATIONS OF VECTOR ALGEBRA