निश्चित समाकल

Ex 10.1

योगफल की सीमा के रूप में (प्रथम सिद्धान्त सै) निम्न निश्चित समाकलों के मान ज्ञात कीजिए :

प्रश्न 1.

$\int_3^5 (x-2)dx$

हल :

$\int_3^5 (x-2)dx$

परिभाषानुसार,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)]$$

यहाँ a = 3, b = 5,f (x) = x − 2, nh = 5 − 3 = 2

$\therefore \int_3^5 (x-2)dx$

$$= \lim_{h\to 0} h\,[f(3+h) + f(3+2h) + f(3+3h) + \ldots + f(3+nh)]$$

$$= \lim_{h\to 0} h\,[(3+h-2) + (3+2h-2) + (3+3h-2) + \ldots + (3+nh-2)]$$

$$= \lim_{h\to 0} h\,[1+h+1+2h+1+3h+\ldots+1+nh]$$

$$= \lim_{h\to 0} h\,[h(1+2+3+\ldots+n) + (1+1+1+\ldots+n\text{ बार})]$$

$$= \lim_{h\to 0} \lambda\left(\frac{n(n+1)}{2}\right)+n$$

$$= \lim_{h\to 0} \frac{h\left(hn^2+h+2n\right)}{2}$$

$$= \lim_{h\to 0} \frac{h^2n^2+h^2+2nh}{2}$$

$$= \lim_{h\to 0} \frac{(nh)^2 + 2(nh) + h^2 n}{2}$$

$$= \frac{(2)^2 + 2\times 2 + 0}{2}$$

$$= \frac{4+4}{2} = \frac{8}{2} = 4$$

प्रश्न 2.

$$\int_a^b x^2 dx$$

हल :

$$\int_a^b x^2 dx$$

परिभाषानुसार,

$$\int_a^b f(x) dx = \lim_{h\to 0} h\, [f(a+h) + f(a+2h) + \ldots + f(a+nh)]$$

यहाँ a = a, b = b, nh = b − a

$$= \lim_{h\to 0} h\, [f(a+h) + f(a+2h) + \ldots + f(a+nh)]$$

$$= \lim_{h\to 0} h\, [(a+h)^2 + (a+2h)^2 + \ldots + (a+nh)^2]$$

$$= \lim_{h\to 0} h\, [(a^2 + h^2 + 2ah) + (a^2 + 2^2h^2 + 2.a^2h) + \ldots + (a^2 + n^2h^2 + 2anh)]$$

$$= \lim_{h\to 0} h\, [(a^2 + a^2 + \ldots + a^2) + h^2(1 + 2^2 + \ldots + n^2) + \ldots + 2ah(1 + 2 + \ldots + n)]$$

$$= \lim_{h\to 0} h \left[a^2 \times n + h^2 \times \frac{n(2n+1)(n+1)}{6} + 2ah \times \frac{n(n+1)}{2}\right]$$

$$= \lim_{h\to 0} \left[a^2 nh + h^3 \frac{(n)(n+1)(2n+1)}{6} + ah^2 n(n+1)\right]$$

$$= \lim_{h\to 0} \left[a^2 (b-a) + \frac{(nh)(nh+h)(2nh+h)}{6} + anh(nh+h)\right]$$

$$= \lim_{h \to 0} \left[a^2 (b-a) + \frac{(b-a)(b-a+h)\{2(b-a)+h\}}{6} + a(b-a)(\overline{b-a}+h) \right]$$

$$= a^2(b-a) + a(b-a)(b-a) + \frac{1}{3}(b-a)(b-a)(b-a)$$

$$= (b-a)a^2 + a(b-a)^2 + \frac{1}{3}(b-a)^3$$

$$= \frac{b-a}{3}\left[3a^2 + 3a(b-a) + (b-a)^2\right]$$

$$= \frac{b-a}{3}\left[3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right]$$

$$= \frac{b-a}{3}[b^2 + ab + a^2]$$

$$= \frac{b^3 - a^3}{3}$$

प्रश्न 3.

$$\int_1^3 (x^2 + 5x)dx$$

हल :

$$\int_1^3 (x^2 + 5x)dx$$

यहाँ $f(x) = x^2 + 5x$, $a = 1, b = 3$ $nh = 3 - 1 = 2$

$$\therefore \int_1^3 (x^2 + 5x)dx$$

$$= \lim_{h \to 0} h\,[f(1) + f(1+h) + \ldots + f(1 + (n-1)h)]$$

$$= \lim_{h \to 0} h\,[(1^2 + 5 \times 1) + \{(1+h)^2 + 5(1+h)\} + \ldots + \{1 + (n-1)h^2 + 5\{1 + (n-1)h]$$

$$= \int_1^3 x^2 dx + \int_1^3 5x dx$$

$$= I_1 + I_2 \qquad \ldots(i)$$

$$I_1 = \int_1^3 x^2\, dx$$

$$= \lim_{h \to 0} h\,[f(1+h) + f(1+2h) + ... + f(1+nh)]$$

$$= \lim_{h \to 0} h\,[(1+h)^2 + (1+2h)^2 + ... + (1+nh)^2]$$

$$= \lim_{h \to 0} h\,[(1+h^2+2h) + (1+2^2h^2+2\cdot 2h) + ... + (1+n^2h^2+2\cdot nh)]$$

$$= \lim_{h \to 0} h\,[(1+1+...+1) + h^2(1+2^2+...+n^2) + 2h(1+2+...+n)]$$

$$= \lim_{h \to 0} h\left[n + h^2\frac{n(n+1)(2n+1)}{6} + 2h\frac{n(n+1)}{2}\right]$$

$$= \lim_{h \to 0}\left[n\lambda + \frac{nh(nh+h)(2nh+h)}{6} + nh\,(nh+h)\right]$$

$$= \lim_{h \to 0}\left[2 + \frac{2(2+h)(4+h)}{6} + 2(2+h)\right]$$

$$= 2 + \frac{(2+0)(4+0)}{3} + 2\,(2+0)$$

$$= 2 + \frac{8}{3} + 4$$

$$= 6 + \frac{8}{3} = \frac{26}{3}$$

$$I_2 = 5\int_1^3 x\,dx$$

$$= \lim_{h \to 0} 5h[f(1+h) + f(1+2h) + ... + f(1+nh)]$$

$$= \lim_{h \to 0} 5h[1+h+1+2h+...+1+nh]$$

$$= \lim_{h \to 0} 5h[(1+1+...+1) + h\,(1+2+...+n)]$$

$$= \lim_{h \to 0} 5h\left[n + h\times\frac{n(n+1)}{2}\right]$$

$$= \lim_{h \to 0} 5nh + \frac{5nh(nh+h)}{2}$$

$$= \lim_{h \to 0} 5\times 2 + \frac{5\times 2(2+h)}{2}$$

$$= 10 + 5\,(2+0) = 20$$

समी. (i) में I1 व I2 का मान रखने पर,

$$\int_0^3 \left(x^2 + 5x\right) dx = \frac{26}{3} + 20$$

$$= \frac{26+60}{3} = \frac{86}{3}$$

प्रश्न 4.

$$\int_a^b e^{-x} dx$$

हल :

यहाँ $f(x) = e^{-x}$, 'a' = a, 'b' = b तथा nh = b − a

$$\therefore \int_a^b e^{-x} dx = \lim_{h \to 0} h[f(a+h) + f(a+2h) + ... + f(a+nh)]$$

$$= \lim_{h \to 0} h[e^{-(a+h)} + e^{-(a+2h)} + ... + e^{-(a+nh)}]$$

$$= \lim_{h \to 0} he^{-a} [e^{-h} + e^{-2h} + ... + e^{-nh}]$$

$$= \lim_{h \to 0} he^{-a} \left[\frac{e^{-h}\{(e^{-h})^n - 1\}}{e^{-h} - 1}\right]$$

$$= he^{-a} \lim_{h \to 0} e^{-1} \left[\frac{e^{-nh} - 1}{e^{-h} - 1}\right]$$

$$= e^{-a} e^{-1} \frac{h[e^{-(b-a)} - 1]}{e^{-h} - 1} \qquad (\because \quad nh = b - a)$$

$$= e^{-a} e^{0} (e^{a-b} - 1) \cdot \frac{1}{\lim_{h \to 0} \frac{e^h - 1}{e^h}}$$

$$= e^{-a} (e^{a-b} - 1) \cdot \frac{1}{-1} \qquad \left(\because \lim_{h \to 0} \frac{e^h - 1}{e^h} = -1\right)$$

$= -(e^{-b} - e^{-a})$

$= e^{-a} - e^{-b}$

प्रश्न 5.

$$\int_0^2 (x+4) dx$$

हल :

$$\int_0^2 (x+4)dx$$

यहाँ $f(x) = x + 4$, $a = 0$, $b = 2$

तथा $nh = b - a = 2 - 0 = 2$

$$\therefore \int_0^2 (x+4)dx$$

$$= \lim_{h \to 0} h\, [f(0+h) + f(0+2h) + ... + f(0+nh)]$$

$$= \lim_{h \to 0} h\, [f(h) + f(2h) + ... + f(nh)]$$

$$= \lim_{h \to 0} h\, [f(h+4) + f(2h+4) + ... + f(nh+4)]$$

$$= \lim_{h \to 0} h\, [h(1 + 2 + ... + n) + (4 + 4 + ... + 4)]$$

$$= \lim_{h \to 0} h \left[h\left(\frac{n(n+1)}{2}\right) + 4n\right]$$

$$= \lim_{h \to 0} \left[\frac{nh(nh+h)}{2} + 4nh\right]$$

$$= \lim_{h \to 0} \left[\frac{2(2+h)}{2} + 4 \times 2\right]$$

$= (2+0)+8 = 10$

प्रश्न 6.

$$\int_1^3 (2x^2+5)dx$$

हल :

$$\int_1^3 (2x^2+5)dx$$

यहाँ $f(x) = 2x^2 + 5$, $a = 1$, $b = 3$

तथा $nh = b - a = 3 - 1 = 2$

$\therefore \int_1^3 \left(2x^2+5\right) dx$

$= \lim_{h \to 0} h\, [f(1+h) + f(1+2h) + ... + f(1+nh)]$

$= \lim_{h \to 0} h\, [2(1+h)^2 + 5 + 2(1+2h)^2 + 5 + ... +$

$2(1+nh)^2 + 5]$

$= \lim_{h \to 0} h\, [2((1+h)^2 + (1+2h)^2 + ... + (1+nh)^2) +$

$(5 + 5 + ... + 5)]$

$= \lim_{h \to 0} h\, [2\{1 + h^2 + 2h + 1 + 2^2h^2 + 2.2h + ... + 1 + n^2h^2$

$+ 2.nh\} + 5n]$

$= \lim_{h \to 0} 2h\, [(1 + 1 + ... + 1) + h^2(1 + 2^2 + ... + n^2)$

$+ 2h\,(1 + 2 + ... + n)] + \lim_{h \to 0} 5nh$

$= \lim_{h \to 0} 2h \left[n + h^2 \frac{n(n+1)(2n+1)}{6} + \frac{2hn(n+1)}{2}\right]$

$+ \lim_{h \to 0} 5 \times 2$

$= \lim_{h \to 0} 2\left[nh + \frac{nh(nh+h)(2nh+h)}{6} + nh(nh+h)\right] + 10$

$= \lim_{h \to 0} 2\left[2 + \frac{2(2+h)(2\times 2+h)}{6} + 2(2+h)\right] + 10$

$= 2\left[2 + \frac{2\times 2\times 4}{6} + 2\times 2\right] + 10$

$= 2\left[2 + \frac{8}{3} + 4\right] + 10$

$= 2\left[6 + \frac{8}{3}\right] + 10$

$= 2 \times \frac{26}{3} + 10$

$= \frac{52+30}{3} = \frac{82}{3}$

Ex 10.2

निम्न समाकलों के मान ज्ञात कीजिए

प्रश्न 1.

$$\int_1^3 (2x+1)^3 \, dx$$

हल :

$$\int_1^3 (2x+1)^3 \, dx$$

माना 2x + 1 = t

2dx = dt

जय x = 1 तो t = 3

2dx = dt

जब x = 3 तौ t = 7

$$\therefore \quad \int_1^3 (2x+1)^3 \, dx = \int_3^7 t^3 \frac{dt}{2}$$

$$= \frac{1}{2}\left(\frac{t^4}{4}\right)_3^7 = \frac{1}{8}(7^4 - 3^4)$$

$$= \frac{2320}{8} = 290$$

प्रश्न 2.

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$$

हल :

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$$

माना cosx = t

तौ -sinx dx = dt

या sinx dx = -dt

जब x = 0

तो t = cos 0 = 1

जब $x = \frac{\pi}{2}$

तो $t = \cos\frac{\pi}{2} = 0$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx = \int_1^0 \frac{-dt}{1+t^2}$$

$$= -\left(\tan^{-1} t\right)_1^0$$

$$= -(\tan^{-1}0 - \tan^{-1}1)$$

$$= -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}$$

प्रश्न 3.

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

हल :

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

माना logx = t

तौ $\frac{1}{x}$ dx = dt

जब x = 1, तौ t = log 1 = 0

जब x = 3, तौ t = log 3

$$\therefore \int_1^3 \frac{\cos(\log 3)}{x} dx = \int_0^{\log 3} \cos t\, dt$$

$$= (\sin t)_0^{\log 3}$$

= sin(log 3)

प्रश्न 4.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

हल :

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

माना √x = t

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2dt$$

जब x = 0, तो t = √0 = 0

जब x = 1, तो t = √1 = 1

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int_0^1 e^t \, 2dt$$

$$= 2\left[e^t\right]_0^1$$

$$= 2[e^1 - e^0] = 2(e - 1)$$

प्रश्न 5.

$$\int_0^{\pi/2} \sqrt{1+\sin x}$$

हल :

$$\int_0^{\pi/2} \sqrt{1+\sin x} dx$$

$$= \int_0^{\pi/2} \sqrt{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\right)} dx$$

$$= \int_0^{\pi/2} \sqrt{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2} dx$$

$$= \int_0^{\pi/2} \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right) dx$$

$$= \int_0^{\pi/2} \sin\frac{x}{2} dx + \int_0^{\pi/2} \cos\frac{x}{2} dx \qquad ...(1)$$

माना $\frac{x}{2} = t$

तो $dx = 2dt$

जब $x = 0$ तो $t = \frac{0}{2} = 0$

जब $x = \frac{\pi}{2}$ तो $t = \frac{\pi}{2\times 2} = \frac{\pi}{4}$

$$\therefore \int_0^{\pi/2} \sqrt{1+\sin x}\,dx = \int_0^{\pi/2} \sin\frac{x}{2}dx + \int_0^{\pi/2} \cos\frac{x}{2}dx$$

समी. (1) से,

$$= \int_0^{\pi/4} \sin t.2dt + \int_0^{\pi/4} \cos t\ 2dt$$

$$= -2(\cos t)_0^{\pi/4} + 2(\sin t)_0^{\pi/4}$$

$$= -2\left[\cos\frac{\pi}{4} - \cos 0\right] + 2\left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= -2\left[\frac{1}{\sqrt{2}} - 1\right] + 2\left[\frac{1}{\sqrt{2}} - 0\right]$$

$$= -\frac{2}{\sqrt{2}} + 2 + \frac{2}{\sqrt{2}}$$

$$= 2$$

प्रश्न 6.

$$\int_o^c \frac{y}{\sqrt{y+c}}dy$$

हल :

$$\int_o^c \frac{y}{\sqrt{y+c}}dy$$

माना y + c = t

तो y + c = t

dy = dt

जय y = 0 तो t = 0 + c = c

जब y = c तौ t = c + c = 2c

$$\therefore \int_0^c \frac{y}{y+c}dy = \int_c^{2c} \frac{t-c}{\sqrt{t}}dt$$

$$= \int_c^{2c}\left(\sqrt{t} - \frac{c}{\sqrt{t}}\right)dt$$

$$= \int_c^{2c} t^{\frac{1}{2}}dt - c\int_c^{2c} t^{-\frac{1}{2}}dt$$

$$= \left[\frac{2}{3}t^{\frac{3}{2}}\right]_c^{2c} - c\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_c^{2c}$$

$$= \frac{2}{3}\left[(2c)^{\frac{3}{2}} - c^{\frac{3}{2}}\right] - 2c\left[(2c)^{\frac{1}{2}} - c^{\frac{1}{2}}\right]$$

$$= \frac{2}{3}\left[c^{\frac{3}{2}}\left(2\sqrt{2}-1\right)\right] - 2c\left[c^{\frac{1}{2}}\left(\sqrt{2}-1\right)\right]$$

$$= \frac{2}{3}c^{\frac{3}{2}}\left(2\sqrt{2}-1\right) - 2c^{\frac{3}{2}}\left(\sqrt{2}-1\right)$$

$$= \frac{2}{3}c^{\frac{3}{2}}\left[2\sqrt{2}-1-3\sqrt{2}+3\right]$$

$$= \frac{2}{3}\left[2-\sqrt{2}\right]c^{\frac{3}{2}}$$

प्रश्न 7.

$$\int_0^{\infty} \frac{e^{\tan^{-1}x}}{1+x^2}dx$$

हल :

$$\int_0^{\infty} \frac{e^{\tan^{-1}x}}{1+x^2}dx$$

माना $\tan^{-1}x = t$

तो $\frac{1}{1+x^2}dx = dt$

जब $x = 0$, तो $t = \tan^{-1} 0 = 0$

जब $x = \infty$, तो $t = \tan^{-1}\infty = \frac{\pi}{2}$

$$\therefore \int_0^\infty \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int_0^{\frac{\pi}{2}} e^t dt$$

$$= \left[e^t\right]_0^{\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}} - e^0 = e^{\frac{\pi}{2}} - 1$$

प्रश्न 8.

$$\int_1^2 \frac{(1+\log x)^2}{x}dx$$

हल :

$$\int_1^2 \frac{(1+\log x)^2}{x}dx$$

माना $1 + \log x = t$

तौ $\frac{1}{x}dx = dt$

जब $x = 1$, तौ $t = 1 + \log 1 = 1$

जब $x = 2$, तौ $t = 1 + \log 2$

$$\therefore \int_1^2 \frac{(1+\log x)^2}{x}dx = \int_1^{1+\log 2} t^2 dt$$

$$= \left[\frac{t^3}{3}\right]_1^{1+\log 2}$$

$$= \frac{1}{3}\left[(1+\log 2)^3 - 1^3\right]$$

$$= \frac{(1+\log 2)^3}{3} - \frac{1}{3}$$

प्रश्न 9.

$$\int_\alpha^\beta \frac{dx}{(x-\alpha)(\beta-x)}, \beta > \alpha$$

हल :

$$\int_{\alpha}^{\beta} \frac{dx}{(x-\alpha)(\beta-x)}$$

माना $$I = \int_{\alpha}^{\beta} \frac{dx}{(x-\alpha)(\beta-x)}$$

$$I = \int_{\alpha}^{\beta} \left(\frac{A}{x-\alpha} + \frac{B}{\beta-x}\right) dx$$

हल करने पर, $A = B = \dfrac{1}{\beta-\alpha}$

$\therefore$ $$I = \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \left(\frac{1}{x-\alpha} + \frac{1}{\beta-x}\right) dx$$

$$= \frac{1}{\beta-\alpha} \left[\log(x-\alpha) + \log(\beta-x)\right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha}$$

$$\left[\log(\beta-\alpha) + \log(\beta-\beta) - \log(\alpha-\alpha) + \log(\beta-\alpha)\right]$$

$$= \frac{1}{\beta-\alpha} \left[\log(\beta-\alpha) + 0 - 0 + \log(\beta-\alpha)\right]$$

$$= \frac{2\log(\beta-\alpha)}{\beta-\alpha}$$

प्रश्न 10.

$$\int_0^{\pi/4} \frac{(\sin x + \cos x)}{9 + 16\sin 2x}$$

हल :

$$\int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)}{9 + 16\sin 2x}$$

माना sinx – cosx = t

⇒ (cosx + sinx)dx = dt

⇒ 1-2 sinx cosx = t^2

⇒ sin 2x = 1 – t^2

जब $x = \frac{\pi}{4}$, तो $t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

जब $x = 0$, तो $t = \sin 0 - \cos 0 = -1$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)}{9 + 16\sin 2x} = \int_{-1}^{0} \frac{dt}{9 + 16\left(1 - t^2\right)}$$

$$= \int_{-1}^{0} \frac{1}{25 - 16t^2}$$

$$= \frac{1}{16} \int_{-1}^{0} \frac{dt}{(5/4)^2 - t^2}$$

$$= \frac{1}{16} \cdot \frac{1}{2 \times \frac{5}{4}} \left[\log \frac{\frac{5}{4} + t}{\frac{5}{4} - t}\right]_{-1}^{0}$$

$$= \frac{1}{40} [\log 1 - (\log 1 - \log 9)]$$

$$= \frac{1}{40} [\log 9]$$

$$= \frac{1}{40} \times 2 \times \log 3$$

$$= \frac{1}{20} \log 3$$

प्रश्न 11.

$$\int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}}$$

हल :

$$\int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}}$$

माना $\quad \log x = t$

तो $\quad \frac{1}{x} dx = dt$

जब $x = \frac{1}{e}$, तो $t = \log \frac{1}{e}$

जब $x = e$, तो $t = \log e$

$$\therefore \quad \int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}} = \int_{\log 1/e}^{\log e} \frac{dt}{t^{1/3}}$$

$$= \left[\frac{t^{-\frac{1}{3}+1}}{\frac{2}{3}}\right]_{\log\frac{1}{e}}^{\log e} = \frac{3}{2}\left[t^{\frac{2}{3}}\right]_{\log\frac{1}{e}}^{\log e}$$

$$= \frac{3}{2}\left[(\log e)^{3/2} - \left(\log\frac{1}{e}\right)^{3/2}\right]$$

$$= \frac{3}{2}\left[1^{2/3} - 1^{2/3}\right] = 0$$

प्रश्न 12.

$$\int_0^{\pi/4} \sin 2x \cos 3x \, dx$$

हल :

$$\int_0^{\pi/4} \sin 2x \cos 3x dx$$

$$= \frac{1}{2} \int_0^{\pi/4} 2 \sin 2x \cos 3x \, dx$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\sin(2x+3x) + \sin(2x-3x)\right] dx$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\sin 5x + \sin(-x)\right] dx$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\sin 5x - \sin x\right] dx$$

$$= \frac{1}{2}\left[\left(-\frac{\cos 5x}{5}\right)_0^{\pi/4} - (-\cos x)_0^{\pi/4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}(\cos 5x)_0^{\pi/4} + (\cos x)_0^{\pi/4}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\left(\cos\frac{5\pi}{4} - \cos 0\right) + \left(\cos\frac{\pi}{4} - \cos 0\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\left(-\frac{1}{\sqrt{2}} - 1\right) + \left(\frac{1}{\sqrt{2}} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{5\sqrt{2}} + \frac{1}{5} + \frac{1}{\sqrt{2}} - 1\right]$$

$$= \frac{1}{2}\left(\frac{3\sqrt{2}}{5} - \frac{4}{5}\right)$$

$$= \frac{1}{10}\left(3\sqrt{2} - 4\right)$$

प्रश्न 13.

$$\int_e^{e^2}\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx$$

हल :

$$\int_e^{e^2}\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx$$

माना $\log x = t$

$$x = e^t$$

$$dx = e^t\, dt$$

$$\int_e^{e^2}\left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx = \int_e^{e^2}\left[\frac{1}{t} - \frac{1}{t^2}\right]e^t\, dt$$

$$= \int_e^{e^2}\frac{d}{dt}\left(e^t \cdot \frac{1}{t}\right)dt$$

$$= \left(\frac{e^t}{t}\right)_e^{e^2} = \left(\frac{x}{\log x}\right)_e^{e^2}$$

$$= \frac{e^2}{\log e^2} - \frac{e}{\log e}$$

$$= \frac{e^2}{2} - \frac{e}{1} = \frac{e^2}{2} - e$$

प्रश्न 14.

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx$$

हल :

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx$$

माना $1 - x^2 = t$

$\Rightarrow - 2x\, dx = dt$

$\Rightarrow x dx = -\frac{1}{2} dt$

$\Rightarrow \sin 2x = 1 - t^2$

जब $x = 0$ तो $t = 1 - 0 = 1$

जब $x = 1$ तो $t = 1 - 1 = 0$

$$\therefore \quad \int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx = \int_0^1 \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx$$

$$= \int_1^0 \frac{1-t}{\sqrt{t}} \frac{dt}{-2}$$

$$= -\frac{1}{2} \int_1^0 (t^{1/2} + t^{1/2})\, dt$$

$$= -\frac{1}{2} \left[\left(2t^{1/2}\right)_1^0 - \left(\frac{2t^{3/2}}{3}\right)_1^0 \right]$$

$$= -\frac{1}{3} (t^{3/2})_1^0 - (t^{1/2})_1^0$$

$$= \frac{1}{3}(0 - 1) - (0 - 1)$$

$$= \frac{1}{3} \times (-1) + 1 = 1 - \frac{1}{3} = \frac{2}{3}$$

प्रश्न 15.

$$\int_{\pi/2}^{\pi} \frac{1-\sin x}{1-\cos x}dx$$

हल :

$$\int_{\pi/2}^{\pi} \frac{1-\sin x}{1-\cos x}dx$$

$$= \int_{\pi/2}^{\pi} \frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$$

$$= \int_{\pi/2}^{\pi} \left[\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right]dx$$

$$= -\frac{1}{2}\left[\cot x\right]_{\pi/2}^{\pi} - 2\left[\log\left(\sin\frac{x}{2}\right)\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{2}\left[\cot x - \cot\frac{\pi}{2}\right] - 2\left[\log\left(\sin\frac{\pi}{2}\right) - \log\left(\sin\frac{\pi}{4}\right)\right]$$

$$= -\frac{1}{2}(\infty - 0) - 2\left(0 - \log\frac{1}{\sqrt{2}}\right)$$

$$= 2\log\frac{1}{\sqrt{2}} = \log\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \log\frac{1}{2} = \log\frac{e}{2}$$

प्रश्न 16.

$$\int_0^{\pi/4} \frac{dx}{4\sin^2 x + 5\cos^2 x}$$

हल :

$$\int_0^{\frac{\pi}{4}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$$

अंश व हर में $\cos^2 x$ से भाग देने पर

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{5 + 4\tan^2 x}dx$$

माना 2 tanx = t

⇒ $2 \sec^2 x\, dx = dt$

⇒ $\sec^2 x\, dx = -\frac{dt}{2} dt$

जब x = 0 तो t = 0

जब $x = \frac{\pi}{4}$, तो $t = 2 \tan\frac{\pi}{4} = 2 \times 1 = 2$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$$

$$= \frac{1}{2} \int_0^2 \frac{1}{\left(\sqrt{5}\right)^2 + t^2} dt$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{5}} \tan^{-1} \frac{t}{\sqrt{5}}\right]_0^2$$

$$= \frac{1}{2\sqrt{5}} \left[\tan^{-1} \frac{2}{\sqrt{5}} - \tan^{-1} 0\right]$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

प्रश्न 17.

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

हल :

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

माना $$I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} dx \qquad \text{...(i)}$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]} dx$$

या $$I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)} \qquad \text{...(ii)}$$

समी (i) तथा (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} + \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)} dx$$

$$= \int_0^{\pi/2} \left[\frac{\sin x + \cos x}{\sin x + \cos x}\right] dx$$

$$= \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

प्रश्न 18.

$$\int_{-1}^{1} x \tan^{-1} x dx$$

हल :

$$\int_{-1}^{1} x \tan^{-1} x \, dx$$

$$= 2\int_0^1 x \tan^{-1} x \, dx \qquad (\because x \tan^{-1} x \text{ सम फलन है।})$$

$$= \left[2 \cdot \frac{x^2}{2} \cdot \tan^{-1} x\right]_0^1 - 2\int_0^1 \frac{1}{2} \frac{x^2}{1+x^2} dx$$

$$= \left[x^2 \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x^2 + 1 - 1}{1+x^2} dx$$

$$= \left[x^2 \tan^{-1} x\right]_0^1 - [x]_0^1 + \left[\tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

प्रश्न 19.

$$\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

हल :

$$\int_0^1 \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

माना $\sin^{-1}x = t,$

तो $x = \sin t$

$$\frac{1}{\sqrt{1-x^2}}dx = dt$$

जब x = 0 तौ t = 0

जब x = 1 तौ t = $\frac{\pi}{2}$

$$\therefore \int_0^1 \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

$$= \int_0^{\pi/2} \underset{\text{II}}{\sin t}.\underset{\text{I}}{t}\, dt$$

$$= t\int_0^{\pi/2}\sin t\, dt - \int_0^{\pi/2}\left[\frac{d}{dt}(t)\int \sin t\, dt\right]dt$$

$$= t[-\cos t]_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos t)dt$$

$$= -t[\cos t]_0^{\pi/2} + \int_0^{\pi/2}\cos t\, dt$$

$$= -t[\cos t]_0^{\pi/2} + [\sin t]_0^{\pi/2}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\frac{\pi}{2}\cos\frac{\pi}{2} - 0\times\cos 0\right)$$

= (1-0)-(0-0)

= 1

प्रश्न 20.

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$$

हल :

$$\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$$

माना $\qquad x^2 = y$

तो $\dfrac{y}{\left(y+a^2\right)\left(y+b^2\right)} = \dfrac{A}{\left(y+a^2\right)} + \dfrac{B}{\left(y+b^2\right)}$

तुलना करके हल करने पर,

$$A = \frac{a^2}{a^2-b^2}$$

तथा $\qquad B = \dfrac{b^2}{b^2-a^2}$

$$\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$$

$$= \frac{a^2}{a^2-b^2} \int_0^\infty \frac{1}{a^2+x^2}dx + \frac{b^2}{b^2-a^2} \int_0^\infty \frac{1}{b^2+x^2}dx$$

$$= \frac{a^2}{a^2-b^2} \cdot \frac{1}{a} \cdot \left(\tan^{-1}\frac{x}{a}\right)_0^\infty + \frac{b^2}{b^2-a^2} \cdot \frac{1}{b} \cdot \left(\tan^{-1}\frac{x}{b}\right)_0^\infty$$

$$= \frac{a}{a^2-b^2} \cdot (\tan^{-1}\infty - \tan^{-1}0) + \frac{b^2}{b^2-a^2}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{a^2}{a^2-b^2}\left(\frac{\pi}{2}-0\right) + \frac{b^2}{b^2-a^2}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{a^2}{a^2-b^2} \times \frac{\pi}{2} + \frac{b^2}{b^2-a^2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{2\left(a^2-b^2\right)}(a-b)$$

$$= \frac{\pi}{2(a+b)}$$

प्रश्न 21.

$$\int_1^2 \log x\, dx$$

हल :

$$\int_1^2 \log x\, dx = \int_1^2 \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1}\, dx$$

$$= \left[\log x \int 1\, dx\right]_1^2 - \int\left[\frac{d}{dx}(\log x) \cdot \int 1\, dx\right] dx$$

$$= [x \log x]_1^2 - \int_1^2 \frac{1}{x} \cdot x\, dx$$

$$= (x \log x)_1^2 - \int_1^2 1\, dx$$

$$= (x \log x)_1^2 - (x)_1^2$$

= (2 log 2 – 1 log 1) – (2 – 1)
= log 2^2 – 0 – 1
= log 4 – 1
= log 4 – log e
= log $\frac{4}{e}$

प्रश्न 22.

$$\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right) \cos\left(\frac{1}{x}\right) dx$$

हल :

$$\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right) \cos\left(\frac{1}{x}\right) dx$$

माना $\frac{1}{x} = t$

$$-\frac{1}{x^2}\, dx = dt$$

जब $x = \frac{4}{\pi}$, तो $t = \frac{\pi}{4}$

जब $x = \frac{2}{\pi}$, तो $t = \frac{\pi}{2}$

$$\therefore \int_{4/\pi}^{2/\pi} \frac{1}{x} . \cos\frac{1}{x} . \left(-\frac{1}{x^2}\right) dx$$

$$= \int_{4/\pi}^{2/\pi} t\cos t\,dt$$

$$= [t\sin t]_{\pi/2}^{\pi/2} - \int_{\pi/4}^{\pi/2} \sin t\,dt$$

$$= \left[\frac{\pi}{2}\sin\frac{\pi}{2} - \frac{\pi}{4}\sin\frac{\pi}{4}\right] + [\cos t]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{\pi}{2}\times 1 - \frac{\pi}{4}\times\frac{1}{\sqrt{2}}\right] + \left[\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} + \left(0 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}$$

प्रश्न 23.

$$\int_0^{\pi/2} \frac{\sin x\cos x dx}{\cos^2 x + 3\cos x + 2}$$

हल :

$$\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + 3\cos x + 2}\,dx$$

माना cosx = t

तो -sin x dx = dt

या sinx dx = -dt

जब x = 0, तो t = 1

जब $x = \frac{\pi}{2}$, तो $t = 0$

$$\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^2 x + 3\cos x + 2}\,dx$$

$$= -\int_1^0 \frac{t}{t^2 + 3t + 2}\,dt$$

$$= -\int_1^0 \frac{t}{(t+1)(t+2)}\,dt$$

अब $\frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

t = A(t + 2) + B (t + 1)

= At + 2A + Bt + B

= (A + B)t + 2A + B

तुलना करने पर,

A + B = 1, 2A + B = 0

⇒ A + (A + B) = 0

⇒ A + 1 = 0

⇒ A = − 1

⇒ − 1 + B = 1

⇒ B = 2

$\therefore \quad \frac{t}{(t+1)(t+2)} = \frac{1}{t+1} + \frac{2}{t+2}$

$$= -\int_1^0 \left[-\frac{1}{(t+1)}\right] dt - \int_1^0 \left[\frac{2}{(t+2)}\right] dt$$

$$= \int_1^0 \frac{1}{t+1} dt - 2 \int_1^0 \frac{1}{t+2} dt$$

$$= \left[\log(t+1)\right]_1^0 - 2\left[\log(t+2)\right]_1^0$$

= [log (0 + 1) − log (1 + 1)] − 2 [log (0 + 2) − log (1 + 2)]

= log 1 − log 2 − 2 log 2 + 2 log 3

= 0 − log 2 − log 2^2 + log 3^2

= log 9 − log 2 x 4

= log $\frac{9}{8}$

प्रश्न 24.

$$\int_0^3 \sqrt{\frac{x}{3-x}} dx$$

हल :

$$\int_0^3 \sqrt{\frac{x}{3-x}} dx$$

माना x = 3 $\sin^2 \theta$

∴ dx = 3 $\sin^2$ dθ

जब x = 0, तौ θ = $\frac{\pi}{2}$

जब x = 0, तौ θ = 0

$$\therefore \quad \int_0^3 \sqrt{\frac{x}{3-x}}dx = \int_0^{\pi/2} \sqrt{\frac{3\sin^2\theta}{3\left(1-\sin^2\theta\right)}} \cdot 3\sin 2\theta \, d\theta$$

$$= 3\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta d\theta$$

$$= 3\int_0^{\pi/2} 2\sin^2\theta \, d\theta$$

$$= 3\int_0^{\pi/2} \left(1-\cos 2\theta\right) d\theta$$

$$= 3\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2}$$

$$= 3\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - (0-0)$$

$$= 3\left(\frac{\pi}{2} - 0\right) = \frac{3\pi}{2}$$

प्रश्न 25.

$$\int_0^1 \frac{x^2}{1+x^2} dx$$

हल :

$$\int_0^1 \frac{x^2}{1+x^2} dx = \int_0^1 \frac{\left(1+x^2\right)-1}{\left(1+x^2\right)} dx$$

$$= \int_0^1 dx - \int_0^1 \frac{1}{1+x^2} dx$$

$$= [x]_0^1 - \left[\tan^{-1} x\right]_0^1$$

$$= (1-0) - (\tan^{-1}1 - \tan^{-1}0)$$

$$= 1 - \left(\frac{\pi}{4} - 0\right)$$

$$= 1 - \frac{\pi}{4}$$

प्रश्न 26.

$$\int_1^2 \frac{dx}{(x+1)(x+2)}$$

हल :

$$\int_1^2 \frac{dx}{(x+1)(x+2)} = \int_1^2 \frac{(x+2-x-1)}{(x+1)(x+2)} dx$$

$$= \int_1^2 \frac{(x+2)}{(x+1)(x+2)} dx - \int_1^2 \frac{(x+1)}{(x+1)(x+2)} dx$$

$$= \int_1^2 \frac{1}{x+1} dx - \int_1^2 \frac{1}{x+2} dx$$

$$= \left[\log(x+1)\right]_1^2 - \left[\log(x+2)\right]_1^2$$

= [log (x +1)] − [log (x + 2)]

= [log (2 + 1) − log (1 + 1]] − [log (2 + 2) − log (1 + 2)]

= log 3 − log 2 − log 4 + log 3

= 2 log 3 − (log 2 + log 4)

= 2 log 3 − log 8

$$= \log \frac{3^2}{8} = \log \frac{9}{8}$$

Ex 10.3

निम्नलिखित समाकलों के मान ज्ञात कीजिए

प्रश्न 1.

$$\int_{-2}^{2} |2x+3|\, dx$$

हल :

माना

$$I = \int_{-2}^{2} |2x+3|\, dx$$

$$= \int_{-2}^{-3/2} -(2x+3)\, dx + \int_{-3/2}^{2} (2x+3)\, dx$$

[∵ अन्तराल $\left(-2, -\frac{3}{2}\right)$ में x का मान ऋणात्मक है

∴ $|2x+3| = -(2x+3)$]

$$= -\left[\frac{2x^2}{2} + 3x\right]_{-2}^{-3/2} + \left[\frac{2x^2}{2} + 3x\right]_{-3/2}^{2}$$

$$= (x^2+3x)_{-3/2}^{2} - (x^2+3x)_{-2}^{-3/2}$$

$$= \left[2^2 + 3\times 2 - \left(\frac{-3}{2}\right)^2 - 3\left(\frac{-3}{2}\right)\right] - \left[\left(\frac{-3}{2}\right)^2 + 3\left(\frac{-3}{2}\right) - (-2)^2 - 3(-2)\right]$$

$$= \left[4+6-\frac{9}{4}+\frac{9}{2}\right] - \left[\frac{9}{4} - \frac{9}{2} - 4 + 6\right]$$

$$= 4+6-\frac{9}{4}+\frac{9}{2}-\frac{9}{4}+\frac{9}{2}+4-6$$

$$= 8+9-\frac{9}{2} = 8+\frac{9}{2} = \frac{16+9}{2} = \frac{25}{2}$$

प्रश्न 2.

$$\int_{-2}^{2} |1-x^2|\, dx$$

हल :

माना

$$I = \int_{-2}^{2} |1 - x^2|\, dx$$

$$\Rightarrow I = \int_{-2}^{-1} |1 - x^2|\, dx + \int_{-1}^{1} |1 - x^2|\, dx + \int_{1}^{2} |1 - x^2|\, dx$$

$$= -\int_{-2}^{-1} (1 - x^2)\, dx + \int_{-1}^{1} (1 - x^2)\, dx - \int_{1}^{2} (1 - x^2)\, dx$$

$$= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} - \left[x - \frac{x^3}{3}\right]_{1}^{2}$$

$$= -\left[1 + \frac{1}{3} - \left(-2 + \frac{8}{3}\right)\right] + \left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right]$$

$$\left[-2 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right]$$

$$= -\left(\frac{-4}{3}\right) + \frac{4}{3} - \left(\frac{-4}{3}\right)$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$$

प्रश्न 3.

$$\int_{1}^{4} f(x)\, dx, \text{ जहाँ } f(x) = \begin{bmatrix} 7x + 3, 1 \le x \le 3 \\ 8x, \quad 3 \le x \le 4 \end{bmatrix}$$

हल :

माना

$$I = \int_{1}^{4} f(x)\, dx$$

$$I = \int_{1}^{3} (7x + 3)\, dx + \int_{3}^{4} 8x\, dx$$

$$= \left[7\left(\frac{x^2}{2}\right) + 3x\right]_{1}^{3} + 8\left[\frac{x^2}{2}\right]_{3}^{4}$$

$$= \frac{7}{2}(3^2 - 1^2) + 3(3 - 1) + 4(4^2 - 3^2)$$

$$= \frac{7}{2} \times 8 + 6 + 4 \times 7$$

= 28 + 6 + 28
= 62

प्रश्न 4.

$$\int_0^3 [x]\, dx,$$

हल : माना

$$I = \int_0^3 [x]\, dx$$

$$= \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \int_2^3 [x]\, dx$$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx$$

$$= 0 + [x]_1^2 + (2x)_2^3$$

= 0 + (2 – 1) + (6 – 4)
= 0 + 1 + 2
= 3

प्रश्न 5.

$$\int_{-\pi/4}^{\pi/4} x^5 \cos^2 x\, dx$$

हल : माना

$$I = \int_{-\pi/4}^{\pi/4} x^5 \cos^2 x\, dx$$

यहाँ $f(x) = x^5 \cos^2 x$

अब $f(x) = (-x)^5 \cos^2 (-x)$
$= -x^5 \cos^2 x$
$= -f(x)$

अतः यह एक विषम फलन है।

$$\therefore \quad \int_{-\pi/4}^{\pi/4} x^5 \cos^2 x\, dx = 0$$

प्रश्न 6.

$$\int_{-\pi}^{\pi} \frac{\sin x \cos x}{1+\cos^2 x}\, dx$$

हल : माना

$$I = \int_{-\pi}^{\pi} \frac{\sin x \cos x}{1+\cos^2 x} dx$$

यहाँ $$f(x) = \frac{\sin x \cos x}{1+\cos^2 x}$$

$$\Rightarrow \quad f(-x) = \frac{\sin(-x).\cos(-x)}{1+[\cos(-x)]^2}$$

$$= \frac{-\sin x \cos x}{1+\cos^2 x}$$

[∵ sin x एक विषम फलन है तथा cos x एक सम फलन हैं
∴ sin (-x) = -sin x
cos(-x) = cos x]
= -f(x)
अत: यह एक विषम फलन है।

$$\therefore \int_{-\pi}^{\pi} \frac{\sin x \cos x}{1+\cos^2 x} dx = 0$$

प्रश्न 7.

$$\int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

हल : माना

$$I = \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} \quad \ldots(i)$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}-x\right)}}{\sqrt{\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}-x\right)}+\sqrt{\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}-x\right)}} dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \quad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx + \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_{\pi/4}^{3\pi/4} 1\,dx$$

$$= [x]_{\pi/4}^{3\pi/4} = \frac{3\pi}{4}+\frac{\pi}{4} = \pi$$

$$\Rightarrow \qquad I = \frac{\pi}{2}$$

$$\Rightarrow \qquad \int_{\pi/4}^{3\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \frac{\pi}{2}$$

प्रश्न 8.

$$\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$$

हल : माना

$$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx \qquad \ldots(i)$$

$$I = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}\,dx$$

$$I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos - x}+e^{\cos x}}\,dx \qquad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx + \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi} \frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi} 1\, dx$$

$$\Rightarrow \qquad 2I = [x]_0^{\pi} = (\pi - 0) = \pi$$

$$\therefore \qquad I = \frac{\pi}{2}$$

$$\Rightarrow \qquad \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx = \frac{\pi}{2}$$

प्रश्न 9.

$$\int_0^{\pi/2} \sin 2x \cdot \log \tan x\, dx$$

हल : माना

$$I = \int_0^{\pi/2} \sin 2x \log \tan x\, dx \qquad \ldots(i)$$

$$I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\pi/2} \sin 2x \log \cot x\, dx \qquad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \sin 2x \log \tan dx + \int_0^{\pi/2} \sin 2x \log \cot x\, dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi/2} \sin 2x (\log \tan x + \log \cot x)\, dx$$

$$\Rightarrow \qquad 2I = \int_0^{\pi/2} \sin 2x \log(\tan x \times \cot x)\, dx$$

$$\Rightarrow \qquad I = \frac{1}{2}\int_0^{\pi/2} \sin 2x \times \log 1\, dx$$

$$\Rightarrow \qquad I = \frac{1}{2}\int_0^{\pi/2} \sin 2x \times 0$$

$$\Rightarrow \qquad I = 0$$

$$\therefore \qquad \int_0^{\pi/2} \sin 2x \log \tan x\, dx = 0$$

प्रश्न 10.

$$\int_{-1}^{1} \log\left[\frac{2-x}{2+x}\right] dx$$

हल : माना

माना $f(x) = \log\left(\frac{2-x}{2+x}\right)$

तब $f(-x) = \log\left(\frac{2-(-x)}{2+(-x)}\right)$

$$= \log\left(\frac{2+x}{2-x}\right) = \log\left(\frac{2-x}{2+x}\right)^{-1}$$

$$= -\log\left(\frac{2-x}{2+x}\right)$$

$$= -f(x)$$

$\therefore f(x)$ विषम है।

$$\therefore \quad \int_{-a}^{a} f(x)\, dx = 0$$

जब $f(x)$ विषम है।

$$\therefore \quad \int_{-1}^{0} \log\left(\frac{2-x}{2+x}\right) dx = 0$$

प्रश्न 11.

$$\int_0^1 \log\left(\frac{1}{x}-1\right) dx$$

हल : माना

$$I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx$$

$$= \int_0^1 \log\frac{x}{(1-x)}\, dx = -I$$

$2I = 0 \Rightarrow I = 0$

$$\int_0^1 \log\left(\frac{1}{x}-1\right) dx = 0$$

प्रश्न 12.

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

हल :

माना

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\dfrac{\sin x}{\cos x}}}\, dx$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad \ldots(i)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\dfrac{\pi}{3}+\dfrac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\dfrac{\pi}{3}+\dfrac{\pi}{6}-x\right)}+\sqrt{\sin\left(\dfrac{\pi}{3}+\dfrac{\pi}{6}-x\right)}}\, dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\dfrac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\dfrac{\pi}{2}-x\right)}+\sqrt{\sin\left(\dfrac{\pi}{2}-x\right)}}\, dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \qquad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$I = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1)\, dx = \frac{1}{2}[x]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

प्रश्न 13.

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$$

हल : माना

$$I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\, dx \qquad \ldots(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} \qquad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$2I = \int_0^{\pi/2} 1\, dx$$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$$

प्रश्न 14.

$$\int_0^{\pi/2} \log \sin 2x\, dx$$

हल : माना

$$\int_0^{\pi/2} \log \sin 2x\, dx$$

$$\int_0^{\pi/2} \log 2\, dt + \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$\int_0^{\pi/2} \log 2\, dx + 2\int_0^{\pi/2} \log \sin x\, dx$$

$$\frac{\pi}{2} \log 2 + 2\int_0^{\pi/2} \log (\sin x)\, dx$$

$$2\int_0^{\pi/2} \log(\sin x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt + \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \log(\sin t)\,dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$= \int_0^{\pi/2} \log(\sin t)\,dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$= \int_0^{\pi/2} \log(\sin x)\,dx + \frac{\pi}{2}\log\frac{1}{2}$$

$$\Rightarrow \quad \int_0^{\pi/2} \log(\sin x)\,dx = \frac{\pi}{2}\log\frac{1}{2}.$$

प्रश्न 15.

$$\int_{-\pi/4}^{\pi/4} \frac{\left(x+\frac{\pi}{4}\right)}{2-\cos 2x}\,dx$$

हल : माना

$$I = \int_{-\pi/4}^{\pi/4} \frac{x+\frac{\pi}{4}}{2-\cos 2x} \qquad \ldots(i)$$

$$= \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx$$

$$= 0 + \frac{\pi}{4}\times 2\int_0^{\pi/4} \frac{1}{2-\cos 2x}\,dx$$

($\because$ $\frac{x}{2-\cos 2x}$, x का विषम फलन तथा $\frac{1}{2-\cos 2x}$, x का सम फलन है)

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{1}{1+2\sin^2 x}\,dx \qquad (\because \cos 2x = 1-2\sin^2 x)$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{1+3\tan^2 x}\,dx$$

($\cos^2 x$ से भाग देने पर तथा $\sec^2 x = 1 + \tan^2 x$ दर में रखने पर)

$$= \frac{\pi}{2}\int_0^{1} \frac{dt}{1+(\sqrt{3}t)^2}$$

जहाँ t = tan x तथा dt = sec² dx

जब x = 0,तो t = tanθ = 0

जब $x = \frac{\pi}{4}$, तो $t = \tan\frac{\pi}{4} = 1$

$$= \frac{\pi}{2}\left[\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}t)\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{3} - 0\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{\sqrt{3}}\times\frac{\pi}{3}\right] = \frac{\pi^2}{6\sqrt{3}}$$

प्रश्न 16.

$$\int_0^{\pi} \log(1-\cos x)\,dx$$

हल : माना

$$I = \int_0^{\pi} \log(1+\cos x)\,dx \quad \ldots(i)$$

$$= \int_0^{\pi} \log[1+\cos(\pi-x)]\,dx$$

$$I = \int_0^{\pi} \log(1+\cos x)\,dx \quad \ldots(ii)$$

(i) तथा (ii) को जोड़ने

$$2I = \int_0^{\pi} \log (1+\cos x)\, dx + \int_0^{\pi} \log (1+\cos x)\, dx$$

$$= \int_0^{\pi} \log (1+\cos x)(1+\cos x)\, dx$$

$$= \int_0^{\pi} \log (1-\cos^2 x)\, dx$$

$$= \int_0^{\pi} \log \sin^2 x\, dx$$

$$= 2\int_0^{\pi} \log \sin x\, dx$$

या $\quad I = \int_0^{\pi} \log \sin x\, dx$

$$= 2\int_0^{\pi/2} \log \sin x\, dx = 2I_1 \qquad \text{...(iii)}$$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx\right]$$

जब $f(2a-x) = f(x)$

अब $\quad I_1 = \int_0^{\pi/2} \log \sin x\, dx \qquad$ ...(iv)

या $\quad I_1 = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-x\right) dx$

या $\quad I_1 = \int_0^{\pi/2} \log \cos x\, dx \qquad$ ...(v)

$$\left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx\right]$$

(iv) तथा (v) को जोड़ने

$$2I_1 = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$= \int_0^{\pi/2} [\log \sin x + \log \cos x]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$= \int_0^{\pi/2} [\log (2\sin x \cos x) - \log 2]\, dx$$

या $2I_1 = \int_0^{\pi/2} \log \sin 2x\, dx - \log 2 \int_0^{\pi/2} dx$

$$2I_1 = \int_0^{\pi/2} \log \sin 2x - \log 2 [x]_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \log 2 \left[\frac{\pi}{2} - 0\right]$$

$$2I_1 = \int_0^{\pi/2} \log \sin 2x - \frac{\pi}{2} \log 2 \quad \ldots(\text{vi})$$

या $2I_1 = I_2 - \frac{\pi}{2} \log 2$

अब $I_2 = \int_0^{\pi/2} \log \sin 2x\, dx$

$2x = t$ रखने पर,

$$2\, dx = dt \text{ या } dx = \frac{1}{2} dt$$

जब x = 0 तब t = 0, जब x = $\frac{\pi}{2}$ तब t = π

$$\therefore \quad I_2 = \int_0^{\pi} \log \sin t \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t\, dt$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log (\sin t)\, dt$$

$$= \int_0^{\pi/2} \log \sin x\, dx = I_1$$

$\therefore \quad I_2 = I_1$

समीकरण (vi) से,

$$2I_1 = I_1 - \frac{\pi}{2} \log 2 \qquad (\because I_2 = I_1)$$

$$\therefore \quad I_1 = -\frac{\pi}{2} \log 2$$

I1 का मान समीकरण (iii) में रखने पर

$$I = 2I_1 = 2\left(-\frac{\pi}{2}\log 2\right) = -\pi \log 2$$

$\Rightarrow \qquad I = -\pi \log 2$

$\Rightarrow \qquad I = \pi \log (2-1)$

$\Rightarrow \qquad I = \pi \log \frac{1}{2}$

$$\int_0^{\pi} \log (1-\cos x)\, dx = \pi \log \frac{1}{2}$$

प्रश्न 17.

$\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx$

हल : माना

$$\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx = 2\int_0^{\pi/4} \sin^2 x\, dx$$

($\because \sin^2 x$ एक सम फलन है)

$$= 2\int_0^{\pi/4} \left[\frac{1-\cos 2x}{2}\right] dx = \int_0^{\pi/4} (1-\cos 2x)\, dx$$

समी करण (i) व (ii) को जोड़ने पर

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4} = \left[\frac{\pi}{4} - \frac{\sin \pi/2}{2}\right] - (0-0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

प्रश्न 18.

$\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

हल : माना

$$I = \int_0^{\pi} \frac{x\, dx}{1+\sin x} \qquad \ldots(1)$$

$$I = \int_0^{\pi} \frac{(\pi - x)\, dx}{1+\sin (\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\, dx}{1+\sin x} \qquad \ldots(2)$$

(1) तथा (2) को जोड़ने पर

$$2I = \int_0^\pi \frac{x\,dx}{1+\sin x} + \int_0^\pi \frac{(\pi - x)\,dx}{1+\sin x}$$

$$= \int_0^\pi \frac{(x+\pi-x)\,dx}{1+\sin x} = \int_0^\pi \frac{\pi}{1+\sin x}$$

$$= \pi\int_0^\pi \frac{1}{1+\sin x}\times\left(\frac{1-\sin x}{1-\sin x}\right)dx$$

$$= \pi\int_0^\pi \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \pi\int_0^\pi \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \pi\int_0^\pi \frac{1}{\cos^2 x}\,dx - \pi\int_0^\pi \frac{\sin x}{\cos^2 x}\,dx$$

$$= \pi\int_0^\pi \sec^2 x\,dx - \pi\int_0^\pi \sec x\tan x\,dx$$

$$= \pi[\tan x]_0^\pi - \pi[\sec x]_0^\pi$$

$= \pi[\tan \pi - \tan 0] - \pi[\sec \pi - \sec 0]$
$= \pi \times 0 - \pi(-1-1) = 2\pi$
$2I = 2\pi$
$\therefore I = \pi$

प्रश्न 19.

$$\int_0^\pi x\sin^3 x\,dx$$

हल : माना

$$\int_0^\pi x\sin^3 x\,dx$$

$$= \int_0^\pi x\left(\frac{3\sin x - \sin 3x}{4}\right)dx$$

$$= \frac{3}{4}\int_0^\pi x\sin x\,dx - \frac{1}{4}\int_0^\pi x\sin 3x\,dx$$

$$= \frac{3}{4}I_1 - \frac{1}{4}I_2 \qquad \ldots(i)$$

$$I_1 = \int_0^\pi \underset{I}{x}\,\underset{II}{\sin x}\,dx$$

$$\int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin x}\, dx = x\int \sin x\, dx - \int\left[\frac{d}{dx}\cdot x \int \sin x\, dx\right] dx$$

$$= x\cdot(-\cos x) - \int 1\cdot(-\cos x)\, dx$$

$$= -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x$$

$$\Rightarrow \int_0^{\pi} x \sin x\, dx = [-x\cos x + \sin x]_0^{\pi}$$

$$= (-\pi\cos\pi + \sin\pi) - (0+0)$$

$$= -\pi(-1) + 0 = \pi \qquad \ldots(\text{ii})$$

अब $\qquad I_2 = \int_0^{\pi} \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin 3x}\, dx$

$$\int \underset{\mathrm{I}}{x} \underset{\mathrm{II}}{\sin 3x}\, dx = x\int \sin 3x\, dx - \int\left[\frac{d(x)}{dx}\int \sin 3x\, dx\right] dx$$

$$= x\left(\frac{-\cos 3x}{3}\right) - \int\left(\frac{-\cos 3x}{3}\right) dx$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{3}\frac{\sin 3x}{3}$$

$$= \frac{-1}{3}x\cos 3x + \frac{1}{9}\sin 3x$$

$$\Rightarrow \int_0^{\pi} x\sin 3x\, dx = \left[\frac{1}{9}\sin 3x - \frac{1}{3}x\cos 3x\right]_0^{\pi}$$

$$= \left(\frac{1}{9}\sin 3\pi - \frac{1}{3}\pi\cos 3\pi\right)$$

$$-\left(\frac{1}{9}\sin 0 - \frac{1}{3}\times 0\times\cos 0\right)$$

$$= 0 - \frac{\pi}{3}\times(-1) - 0$$

$$= \frac{\pi}{3} \qquad \ldots(\text{iii})$$

समी (ii) व (iii) से मान समी (i) में रखने पर

$$\int_0^{\pi} x\sin^3 x\, dx = \frac{3}{4}\pi - \frac{1}{4}\frac{\pi}{3}$$

$$= \frac{9\pi - \pi}{3\times 4} = \frac{2\pi}{3}$$

प्रश्न 20.

$$\int_0^{\frac{\pi}{2}} \log(\tan x + \cot x)\,dx$$

हल : माना

$$I = \int_0^{\pi/2} \log(\tan x + \cot x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) dx$$

$$= \int_0^{\pi/2} \log 1\, dx - \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$= 0 - \int_0^{\pi/2} \log \frac{\sin 2x}{2}\,dx$$

$$= \int_0^{\pi/2} \log 2\, dx - \int_0^{\pi/2} \log(\sin 2x)\,dx$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\pi/2} \log(\sin 2x)\,dx \qquad \ldots(i)$$

$$\int_0^{\pi/2} \log \sin 2x\, dx$$

$$= \int_0^{\pi/2} \log 2dx + \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$= \int_0^{\pi/2} \log 2\, dx + 2\int_0^{\pi/2} \log \sin x\, dx$$

$$= \frac{\pi}{2}\log 2 + 2\int_0^{\pi/2} \log(\sin x)\,dx$$

$$2\int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\int_0^{\pi/2} \log(\sin t)\,dt + \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \log(\sin t)\,dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$= \int_0^{\pi/2} \log(\sin t)\,dt + \frac{\pi}{2}\log\frac{1}{2}$$

$$= \int_0^{\pi/2} \log(\sin x)\,dx + \frac{\pi}{2}\log\frac{1}{2}$$

$$\Rightarrow \therefore \int_0^{\pi/2} \log(\sin x)\, dx = \frac{\pi}{2} \log \frac{1}{2} \qquad \ldots\text{(ii)}$$

समी. (i) व (ii) से

$$\int_0^{\pi/2} \log(\tan x + \cot x)\, dx = \frac{\pi}{2} \log 2 - \frac{\pi}{2} \log \frac{1}{2}$$

$$= \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2$$

$$= 2 \times \frac{\pi}{2} \log 2 = \pi \log 2$$

प्रश्न 21.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\, dx$$

हल : माना

$$I = \int_{\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\, dx$$

$$= \int_{\pi/2}^{0} \frac{\cos x}{1+e^x}\, dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x}\, dx$$

$\int_{\pi/2}^{0} \frac{\cos x}{1+e^x}\, dx$ में x के स्थान पर x रखने पर,

$$\int_{\pi/2}^{0} \frac{\cos(-x)}{1+e^x}\, dx + \int_0^{\pi/2} \frac{e^x \cos x}{1+e^x}\, dx$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{e^x \cos x}{1+e^{-x}}\, dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x}\, dx$$

$$= \int_0^{\pi/2} \frac{(1+e^x) \cos x}{(1+e^x)}\, dx$$

$$= \int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2}$$

$$= \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

$$\int_{\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\, dx = 1$$

प्रश्न 22.

$$\int_a^b \frac{f(x)}{f(x)-f(a+b-x)}\,dx$$

हल : माना

$$I = \int_a^b \frac{f(x)\,dx}{f(x)+f(a+b-x)} \qquad \ldots(i)$$

$$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f[a+b-(a+b-x)]}\,dx$$

$$\Rightarrow \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)} \qquad \ldots(ii)$$

समी करण (i) व (ii) को जोड़ने पर

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(a+b+x)+f(x)}\,dx$$

$$\Rightarrow \quad 2I = \int_a^b 1\,dx = [x]_a^b$$

$$\Rightarrow \quad 2I = b-a$$

$$\Rightarrow \quad I = \frac{b-a}{2}$$

$$\therefore \quad \int_a^b \frac{f(x)}{f(x)-f(a+b-x)}\,dx = \frac{b-a}{2}$$

Miscellaneous Exercise

निम्नलिखित का समाकल कीजिए

प्रश्न 1.

$$\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx$$

का मान है

(a) $2\int_0^{\pi} \sin 3x.x\, dx$ (b) 0

(c) a^2 (d) 1

हल : माना

$$\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}\, dx$$

$$= \int_0^{\pi/4} (\sqrt{\sin x+\cos x})^2\, dx$$

$$= \int_0^{\pi/4} (\sin x+\cos x)\, dx$$

$$= [-\cos x+\sin x]_0^{\pi/4}$$

$$= (\sin x-\cos x)_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)-(\sin 0-\cos 0)$$

$$= \left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-(0-1) = 1$$

अत: विक (d) सही है।

प्रश्न 2.

$$\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}$$

का मान है

(a) 3 (b) 2

(d) $\frac{3}{2}$ (c) $\frac{1}{2}$

हल : माना

$$I = \int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\, dx \quad \ldots(1)$$

$$I = \int_2^5 \frac{\sqrt{2+5-x}}{\sqrt{2+5-x}+\sqrt{7-2-5+x}}\, dx$$

$$I = \int_2^5 \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\, dx \quad \ldots(2)$$

समीकरण (1) व (2) को जोड़ने पर।

$$2I = \int_2^5 \frac{\sqrt{x}+\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\, dx$$

$$= \int_2^5 1\, dx = [x]_2^5 = 5 - 2$$

$$= 3$$

$$I = \frac{3}{2}$$

$$\Rightarrow \quad \int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\, dx = \frac{3}{2}$$

अत: विक (c) सही है।

प्रश्न 3.

$$\int_{a-c}^{b-c} f(x+c)$$

का मान है

(a) $\int_a^b f(x+c)\, dx$ (b) $\int_a^b f(x)\, dx$

(c) $\int_{a-2c}^{b-2c} f(x)\, dx$ (d) $\int_a^b f(x+2c)\, dx$

हल : अत: विक (b) सही है।

प्रश्न 4. यदि

$$A(x) = \int_0^x \theta^2 \, d\theta$$

हो, तो A(3) का मान होगा
(a) 9
(b) 27
(c) 3
(d) 81

हल :

$$A(x) = \int_0^x \theta^2 \, d\theta$$

$$= \left[\frac{\theta^3}{3}\right]_0^x$$

$$= \frac{1}{3}x^3$$

$$A(3) = \frac{1}{3} \times (3)^3 = \frac{1}{3} \times 3 \times 3 \times 3$$

= 3 x 3 = 9

अत: विकल्प (a) सही है।

प्रश्न 5.

$$\int_1^2 \frac{(x+3)}{x(x+2)} \, dx$$

हल : माना

$$\frac{x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$\frac{x+3}{x(x+2)} = \frac{A(x+2)+Bx}{x(x+2)}$$

x + 3 = (A + B)x + 2A
तुलना से, A + B = 1, 2A = 3
⇒ A + B = 1, A = 3/2
B = 1 − 3/2 = − 1/2

$$\therefore \quad \int_1^2 \frac{x+3}{x(x+2)} \, dx = \int_1^2 \frac{3}{2x} \, dx - \int_1^2 \frac{1}{2(x+2)} \, dx$$

$$\Rightarrow \quad \int_1^2 \frac{x+3}{x(x+2)} \, dx = \frac{3}{2}(\log x)_1^2 - \frac{1}{2}[\log (x+2)]_1^2$$

$$\Rightarrow \quad \int_1^2 \frac{x+3}{x(x+2)} \, dx = \frac{3}{2}\log 2 - \frac{1}{2}\log \frac{4}{3}$$

$$\Rightarrow \quad \int_1^2 \frac{x+3}{x(x+2)} \, dx = \frac{1}{2}\left[\log 2^3 - \log \frac{2^2}{3}\right]$$

$$\Rightarrow \quad \int_1^2 \frac{x+3}{x(x+2)}\,dx = \frac{1}{2}\log\left(\frac{2^3 \times 3}{2^2}\right) = \frac{1}{2}\log 6$$

अतः $\int_1^2 \frac{x+3}{x(x+2)}\,dx = \frac{1}{2}\log 6$

प्रश्न 6.

$$\int_1^2 \frac{xe^x}{(1+x)^2}\,dx$$

हल : माना

$$I = \int_1^2 \frac{xe^x}{(1+x)^2}\,dx$$

$$= \int_1^2 \frac{(x+1)-1}{(1+x)^2}\,e^x\,dx$$

$$= \int_1^2 \left(\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right) e^x\,dx$$

$$= \left[\frac{e^x}{1+x}\right]_1^2$$

$$= \frac{e^2}{1+2} - \frac{e^1}{1+1}$$

$$= \frac{e^2}{3} - \frac{e}{2} = \frac{e}{6}(2e-3)$$

प्रश्न 7.

$$\int_0^{\frac{\pi}{2}} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$$

हल : माना

$$I = \int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$$

तब $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$= \int e^x\left(\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$

$$= \frac{1}{2}\int e^x\left(\sec^2\frac{x}{2}\right)dx + \int e^x\left(\tan\frac{x}{2}\right)dx$$

$$= \int \underset{I}{\tan\frac{x}{2}}.\underset{II}{e^x}\,dx + \frac{1}{2}\int e^x \sec^2\frac{x}{2}\,dx$$

$$= \tan\frac{x}{2}.e^x - \int \sec^2\frac{x}{2}.\frac{1}{2}e^x\,dx + \frac{1}{2}\int e^x \sec^2\frac{x}{2}\,dx$$

$$= e^x \tan\frac{x}{2} - \frac{1}{2}\int e^x \sec^2 x\,dx + \frac{1}{e}\int e^x \sec^2\frac{x}{2}\,dx$$

$$= e^x \tan\frac{x}{2}$$

$$\therefore \int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \left[e^x \tan\frac{x}{2}\right]_0^{\pi/2}$$

$$= \left[e^{\pi/2}\tan\frac{\pi}{4} - e^0 \tan 0\right]$$

$= e^{\pi/2} \times 1 = 0$
$= e^{\pi/2}$

प्रश्न 8.

$$\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4}\,dx$$

हल :

$$\int_{1/3}^{1} \frac{(x-x^3)^{1/3}\,dx}{x^4} = \int_{1/3}^{1} \frac{x\left(\frac{1}{x^2}-1\right)^{1/3} dx}{x^4}$$

$$= \int_{1/3}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3}\,dx$$

अब माना $\frac{1}{x^2}-1 = t$ जब $x = \frac{1}{3}$, तो $t = 8$

$$\frac{dx}{x^3} = \frac{-dt}{2} \text{ तथा } x = 1 \text{ तो } t = 0$$

अतः $\frac{-1}{2}\int_8^0 (t)^{1/3}\, dt = \frac{1}{2}\int_0^8 t^{1/3}\, dt$

$$= \frac{1}{2}\times\frac{3}{4}\times[t^{4/3}]_0^8$$

$$= \frac{3}{8}\times(8)^{4/3} = 6$$

प्रश्न 9.

$\int_0^{\frac{\pi}{2}} x^2 \cos^2 x\, dx$

हल :

$$\int_0^{\pi/2} x^2 \cos^2 x\, dx$$

$$= \int_0^{\pi/2} x^2\left[\frac{1}{2}(1+\cos 2x)\right]dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x^2\, dx + \frac{1}{2}\int_0^{\pi/2} x^2 \cos 2x\, dx \quad \ldots\text{(i)}$$

अब $\int x^2 \cos 2x\, dx$

$$= x^2 \int \cos 2x\, dx - \int\left[\frac{d}{dx}(x^2)\int \cos 2x\, dx\right]dx$$

$$= x^2\frac{\sin 2x}{2} - \int 2x.\frac{\sin 2x}{2}\, dx$$

$$= \frac{1}{2}x^2 \sin 2x - \int x \sin 2x\, dx \quad \ldots\text{(ii)}$$

$\int x \sin 2x\, dx$

$$= x\int \sin 2x\, dx - \int\left[\frac{d}{dx}(x)\int \sin 2x\, dx\right]dx$$

$$= x\left(\frac{-\cos 2x}{2}\right) - \int 1\times\left(\frac{-\cos 2x}{2}\right)dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}\cos 2x + \frac{1}{4}\sin 2x \qquad \ldots\text{(iii)}$$

समीकरण (i) (ii) व (iii) को जोड़ने पर।

$$\int_0^{\pi/2} x^2 \cos^2 x\, dx = \frac{1}{6}(x^3)_0^{\pi/2} + \left[\frac{1}{4}x^2 \sin 2x + \frac{1}{8}x\cos 2x - \frac{1}{8}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{1}{6}\times\frac{\pi^3}{8} + \frac{\pi^2}{16}\sin \pi + \frac{\pi}{8}\cos \pi - \frac{1}{8}\sin \pi$$

$$= \frac{\pi^3}{6\times 8} + 0 - \frac{\pi}{8} + 0 = \frac{\pi^2}{48}(\pi - 6)$$

प्रश्न 10.

$$\int_0^1 \tan^{-1} x\, dx$$

हल : माना

$$I = \int_0^1 \tan^{-1} x\, dx$$

$$= \int_0^1 \underset{\text{I}}{\tan^{-1} x} \cdot \underset{\text{II}}{1}\, dx \qquad \ldots\text{(i)}$$

$$\int \tan^{-1} x \cdot 1\, dx = (\tan^{-1} x)x - \int \frac{1}{1+x^2}\, x\, dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx$$

(माना $1 + x^2 = t$)

$$= x\tan^{-1} x - \frac{1}{2}\frac{dt}{t}$$

$$= x\tan^{-1} x - \frac{1}{2}\log t$$

$$\int_0^1 \tan^{-1} x\, dx = \left[x\tan^{-1} x - \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \left[1 \tan^{-1} x - \frac{1}{2} \log (1+1^2)\right] - 0$$

$$= \frac{\pi}{4} - \frac{1}{2} \log 2$$

प्रश्न 11.

$$\int_0^{\frac{\pi}{4}} \sin 3x \sin 2x \, dx$$

हल : माना

$$\int_0^{\pi/4} \sin 3x \sin 2x \, dx$$

$$= \frac{1}{2} \int_0^{\pi/4} 2 \sin 3x \sin 2x \, dx$$

$$= \frac{1}{2} \int_0^{\pi/4} [\cos (3x - 2x) - \cos (3x + 2x)] \, dx$$

$$= \frac{1}{2} \int_0^{\pi/4} (\cos x - \cos 5x) \, dx$$

$$= \frac{1}{2} \left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$$

$$= \frac{1}{2} \left[\left(\sin \frac{\pi}{4} - \frac{1}{5} \sin \frac{5\pi}{4}\right) - \left(\sin 0 - \frac{1}{5} \sin 0\right)\right]$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{5} \sin \frac{5\pi}{4} - 0\right]$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{5} \sin \frac{5\pi}{4}\right]$$

$$= \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{5} \times \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{5\sqrt{2}}\right)$$

$$= \frac{6}{2 \times 5\sqrt{2}} = \frac{3}{5\sqrt{2}} = \frac{3\sqrt{2}}{10}$$

प्रश्न 12.

$$\int_{2}^{2} |1-x^2|\,dx$$

हल : माना

$$I = \int_{2}^{2} |1-x^2|\,dx$$

$$= \int_{2}^{1} |1-x^2|\,dx + \int_{1}^{1} |1-x^2|\,dx + \int_{1}^{2} |1-x^2|\,dx$$

$$= -\int_{-2}^{-1} (1-x^2)\,dx + \int_{-1}^{1} (1-x^2)\,dx - \int_{1}^{2} (1-x^2)\,dx$$

$$= -\left[x+\frac{x^3}{3}\right]_{-2}^{-1} + \left[x-\frac{x^3}{3}\right]_{-1}^{1} - \left[x-\frac{x^3}{3}\right]_{1}^{2}$$

$$= -\left[-1+\frac{1}{3}-\left(-2+\frac{8}{3}\right)\right]+\left[1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right]$$
$$-\left[2-\frac{8}{3}-\left(1-\frac{1}{3}\right)\right]$$

$$= -\left(-\frac{4}{3}\right)+\frac{4}{3}-\left(-\frac{4}{3}\right)=\frac{4}{3}+\frac{4}{3}+\frac{4}{3}$$

प्रश्न 13.

$$\int_{\pi}^{\pi} \frac{2x(1+\sin x)}{(1+\cos^2 x)}\,dx$$

हल :

$$I = \int_{\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$$

$$I = \int_{\pi}^{\pi} \frac{2x+2x\sin x}{1+\cos^2 x}\,dx$$

$$I = \int_{\pi}^{\pi} \frac{2x\,dx}{1+\cos^2 x} + \int_{\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

$$I = I_1 + I_2$$

यहाँ $I_1 = \int_{\pi}^{\pi} \frac{2x\,dx}{1+\cos^2 x}$ और $I_2 = \int_{\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$

$\therefore \frac{2x}{1+\cos^2 x}$ एक विषम फलन है $\frac{2x\sin x}{1+\cos^2 x}$ और एक सम फलन है।

$$\therefore \quad I_1 = 0 \text{ और } I_2 = 2\int_0^{\pi} \frac{2x \sin x}{1+\cos^2 x}$$

$$\text{अत: } \quad I_2 = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \qquad \ldots(2)$$

$$\text{या } \quad I_2 = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$I_2 = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$$

$$I_2 = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx - 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx \qquad \ldots(3)$$

$$I_2 = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx - I_2$$

$$\therefore \quad 2I_2 = 4\pi \int_0^{\pi} \frac{\sin x\, dx}{1+\cos^2 x}$$

माना cosx ⇒ − sinx dx = dt

sinx dx = -dt

जब x = 0,तो t = 1 तथा जब x = π तो t = -1

$$\therefore \quad 2I_2 = 4\pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$2I_2 = 4\pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$I_2 = 2\pi[\tan^{-1} t]_{-1}^{1}$$

$$= 2\pi[\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$= 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = 2\pi \times \frac{2\pi}{4} = \frac{4\pi^2}{4}$$

$$I_2 = \pi^2$$

I1 और I2 का मान समीकरण (1) में रखन पर

$$I = 0 + \pi^2 = \pi^2$$

अर्थात $$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} = \pi^2$$

प्रश्न 14.

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$$

हल : माना

$$I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$I = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$$

माना

$$\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$$

$$I = \int \frac{t\, dt}{(1-\sin^2 t)} = \int \frac{t\, dt}{\cos^2 t} = \int t \sec^2 t\, dt$$

I = ∫t sec² t dt
I= t tan t – ∫1.tan t dt
= t tan t – log sec t + c
I = sin x tan (sin) – log sec(sin x) + c

$$I = \sin^{-1} x \tan\left(\tan^{-1} \frac{x}{\sqrt{1-x^2}}\right)$$

$$- \log \sec\left(\sec^{-1} \frac{x}{\sqrt{1-x^2}}\right) + C$$

$$= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log \frac{1}{\sqrt{1-x^2}} + C$$

$$= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log (1-x^2)^{-\frac{1}{2}} + C$$

$$= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log (1-x^2) + C$$

अब $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

$$= \left[\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log (1-x^2)\right]_0^{1/\sqrt{2}}$$

$$= \left[\frac{\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{1-\frac{1}{2}}} + \frac{1}{2} \log\left(1-\frac{1}{2}\right)\right] - (0+0)$$

$$= \frac{\frac{1}{2} \times \frac{\pi}{4}}{1-\frac{1}{2}} + \frac{1}{2} \log\left(1-\frac{1}{2}\right)$$

$$= \frac{\pi}{8} \times \frac{1}{\left(\frac{1}{2}\right)} + \frac{1}{2} \log\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2} \log \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2} \log 2$$

प्रश्न 15.

$$\int_0^{\infty} (\cot^{-1} x)\, dx$$

हल : माना

$$\int \cot^{-1} x\, dx = \int \cot^{-1} x . 1\, dx$$

$$= \cot^{-1} x . \int 1\, dx - \int \left\{\frac{d}{dx}(\cot^{-1} x) . \int 1\, dx\right\} dx$$

$$= x \cot^{-1} x + \int \frac{x}{1+x^2}\, dx \quad \left[\begin{array}{l} \text{माना } 1+x^2 = t \\ \qquad x\, dx = \frac{dt}{2} \end{array}\right]$$

$$= x \cot^{-1} x + \frac{1}{2} \log |t|$$

$$= x \cot^{-1} x + \frac{1}{2} \log (1+x^2)$$

प्रश्न में limit 0 से 1 रखते है, तब

$$\Rightarrow \int_0^1 \cot^{-1} x\, dx = \left[x \cot^{-1} x + \frac{1}{2} \log (1+x^2)\right]_0^1$$

$$= \left[1 \times \frac{\pi}{4} + \frac{1}{2} \log 2\right] - (0+0)$$

$$= \frac{\pi}{4} + \frac{1}{2} \log 2$$

प्रश्न 16.

$$\int_0^{\pi} \frac{dx}{1-2a \cos x + a^2},\ a > 1$$

हल :

माना

$$I = \int_0^{\pi} \frac{dx}{1-2a \cos x + a^2}$$

$$= \int_0^{\pi} \frac{dx}{(1+a^2)\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) - 2a\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)}$$

$$= \int_0^{\pi} \frac{dx}{(a-1)^2 \cos^2 \frac{x}{2} + (a+1)^2 \sin^2 \frac{x}{2}}$$

$$= \frac{2}{(1+a)^2} \int_0^{\infty} \frac{dt}{\left(\frac{a-1}{a+1}\right)^2 + t^2} \qquad \text{जहाँ } t = \tan \frac{x}{2}, \quad 2dt = \sec^2 \frac{x}{2}\, dx$$

$$= \frac{2}{(1+a)^2} \cdot \frac{a+1}{a-1} \left[\tan^{-1}\left(\frac{a+1}{a-1} \cdot t\right)\right]_0^{\infty}$$

$$= \frac{2}{(a+1)(a-1)} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{2}{a^2-1} \times \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{a^2-1}$$

प्रश्न 17.

$$\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi^2}{2ab}$$

हल : माना

$$I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$$

यहाँ $\qquad f(x) = \dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$

$\because \qquad f(\pi - x) = \dfrac{1}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$

$$= \frac{1}{a^2\cos^2 x + b^2\sin^2 x} = f(x)$$

अतः के निष्कासन नियम से

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

$$= \frac{\pi}{2} \times 2\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$$

(गुणधर्म VII से)

$$= \pi\int_0^{\pi} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

($\cos^2 x$ का अंश-हर में भाग देने पर)

जहाँ माना $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$x = 0$ तो $t = 0$, $x = \frac{\pi}{2}$ तो $t = \infty$

$$= \pi\int_0^{\infty} \frac{dt}{a^2 + b^2t^2}$$

$$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{\pi}{b^2}\frac{1}{\left(\frac{a}{b}\right)}\left[\tan^{-1}\left(\frac{t}{\frac{a}{b}}\right)\right]_0^{\infty}$$

$$= \frac{\pi}{b^2} \times \frac{b}{a}(\tan^{-1}\infty - \tan^{-1}0)$$

$$= \frac{\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{2ab}$$

सिद्ध हुआ।