UNIT 2 NUMBERS AND SEQUENCES

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EUCLID'S DIVISION LEMMA

Key Points

Theorem 1: Euclid's Division Lemma

Let a and b ($a > b$) be any two positive integers. Then there exist unique integers q and r such that $a = bq + r$, $0 \leq r < b$.

Generalized form of Euclid's Division Lemma.

If a and b are any two integers then there exist unique integers q and r such that $a = bq + r$. where $0 \leq r < |b|$

Theorem 2: (Euclid's Division Algorithm)

If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r and vice-versa.

Algorithm

An Algorithm means a series of mathematical step by step procedure of calculating successively on the results of earlier steps till the desired answer is obtained.

We use Euclid's Division Algorithm to find out H.C.F. of two positive integers easily.

Euclid's Division Algorithm

To find the Highest common factor of two positive integers a and b, where $a > b$

Step 1: Using Euclid's Division Lemma, $a = bq + r$; $0 \leq r < b$. where q $\rightarrow$ quotient, r $\rightarrow$ remainder. If $r = 0$, then H.C.F. $(a, b) = b$.

Step 2 : Otherwise applying Euclid's Division Lemma divide b by r to get $b = rq + r_1$, $0 \leq r_1 < r$.

Step 3 : If $r_1 = 0$ then r is the Highest common factor of a and b.

Step 4 : Otherwise using Euclid's Division Lemma repeat the process until we get $r = 0$. In that case, corresponding divisor is H.C.F. of (a, b).

Theorem 3:

If a, b are two positive integers with $a > b$ then G.C.D. of (a, b) = G.C.D. of $(a - b, b)$.

H.C.F. of three numbers

Let a, b, c be the given positive integers.

(i) First find H.C.F. (a, b) call it as d. i.e., $d = (a, b)$

(ii) Find H.C.F. of d and c.

This will be the H.C.F. of three numbers a, b, and c.

Worked Examples

2.1 We have 34 cakes. Each box can hold 5 cakes only. How many boxes we need to pack and how many cakes are unpacked?

Sol :

We see that 6 boxes are required to pack 30 cakes with 4 cakes left over. This distribution of cakes can be understood as follows.

34	=	5	×	6	+	4
Total number of cakes	=	Number of cakes in each box	×	Number of boxes	+	Number of cakes left over
↓		↓		↓		↓
Dividend a	=	Divisor b	×	Quotient q	+	Remainder r

2.2 Find the quotient and remainder when a is divided by b in the following cases
(i) $a = -12, b = 5$
(ii) $a = 17, b = -3$
(iii) $a = -19, b = -4$.

Sol :

(i) $a = -12, b = 5$

By Euclid's Division Lemma

$a = bq + r$, where $0 \leq r < |b|$

$-12 = 5 \times (-3) + 3 \qquad 0 \leq r < |5|$

Therefore, Quotient $q = -3$, Remainder $r = 3$

(ii) $a = 17, \quad b = -3$

By Euclid's Division Lemma

$a = bq + r$, where $0 \leq r < |b|$

$17 = (-3) \times (-5) + 2, \qquad 0 \leq r < |-3|$

Therefore Quotient $q = -5$

Remainder $r = 2$

(iii) $a = -19, b = -4$

By Euclid's Division Lemma

$a = bq + r$, where $0 \leq r < |b|$

$-19 = (-4) \times (5) + 1 \qquad 0 \leq r < |-4|$

Therefore Quotient $q = 5$, Remainder $r = 1$

2.3 Show that the square of an odd integer is of the form $4q + 1$, for some integer q.

Sol : Let x be any odd integer. Since any odd integer is one more than an even integer, we have $x = 2k + 1$, for some integer k.

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4k(k+1) + 1 \\ &= 4q + 1, \text{ where } q = k(k+1) \\ &\quad \text{is some integer.} \end{aligned}$$

2.4 If the Highest Common Factor of 210 and 55 is expressible in the form $55x - 325$, find x.

Sol : Using Euclid's Division Algorithm, we have

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \end{aligned}$$

The remainder is zero.

So, the last divisor 5 is the Highest Common Factor (H.C.F.) of 210 and 55.

Since,

H.C.F. is expressible in the form $55x - 325 = 5$.

$\Rightarrow \quad 55x = 330$

Hence $x = 6$

2.5 Find the greatest number that will divide 445 and 572 leaving remainders 4 and 5 respectively.

Sol : Since the remainders are 4, 5 respectively the required number is the H.C.F. of the number $445 - 4 = 441$, $572 - 5 = 567$.

Hence, we will determine the H.C.F. of 441 and 567. Using Euclid's Division Algorithm, we have

$$\begin{aligned} 567 &= 441 \times 1 + 126 \\ 441 &= 126 \times 3 + 63 \\ 126 &= 63 \times 2 + 0 \end{aligned}$$

Therefore, H.C.F. of 441, 567 = 63 and so the required number is 63.

2.6 Find the H.C.F. of 396, 504, 636.

Sol : To find H.C.F. of three given numbers, first we have to find the H.C.F. of first two numbers.

To find H.C.F. of 396 and 504

Using Euclid's division algorithm,

we get $504 = 396 \times 1 + 108$

The remainder is $108 \neq 0$

Again applying Euclid's division algorithm

$396 = 108 \times 3 + 72$

The remainder is $72 \neq 0$

Again applying Euclid's division algorithm

$108 = 72 \times 1 + 36$

The remainder is $36 \neq 0$

Again applying Euclid's division algorithm

$72 = 36 \times 2 + 0$

Here the remainder is zero. Therefore, H.C.F. of 396, 504 = 36. To find the H.C.F. of 636 and 36.

Using Euclid's division algorithm,

we get $636 = 36 \times 17 + 24$

The remainder is $24 \neq 0$

Again Applying Euclid's algorithm

$36 = 24 \times 1 + 12$

The remainder is $12 \neq 0$

Again applying Euclid's division algorithm

$24 = 12 \times 2 + 0$

Here the remainder is zero. Therefore,

H.C.F. of 636, 36 = 12

Therefore, Highest Common Factor of 396, 504 and 636 is 12.

Progress Check

1. Find q and r for the following pairs of integers a and b satisfying a = bq + r

(i) a = 13, b = 3 (ii) a = 18, b = 4
(iii) a = 21, b = – 4 (iv) a = – 32, b = – 12
(v) a = – 31, b = 7

Ans :

(i) a = 13, b = 3
By Euclid's Division Lemma
a = bq + r, where $0 \le r < |b|$
13 = 3 (4) + 1, $0 \le r < |3|$
∴ Quotient q = 4
Remainder r = 1

(ii) a = 18, b = 4
By Euclid's Division Lemma
a = bq + r where $0 \le r < |b|$
18 = $4 \times 4 + 2$, $0 \le r < |4|$
∴ Quotient q = 4
Remainder r = 2

(iii) a = 21, b = – 4
By Euclid's Division Lemma
a = bq + r where $0 \le r < |b|$
21 = $(-4) \times (-5) + 1$, $0 \le r < |-4|$
∴ Quotient q = – 5
Remainder r = 1

(iv) a = – 32, b = – 12
By Euclid's Division Lemma
a = bq + r, where $0 \le r < |b|$
– 32 = $(-12) \times 3 + (4)$, $0 \le r < |-12|$
∴ Quotient q = 3
Remainder r = 4

(v) a = – 31, b = 7
By Euclid's Division Lemma
a = bq + r where $0 \le r < |b|$
– 31 = $7 \times (-5) + 4$, $0 \le r < |7|$
∴ Quotient q = – 5
Remainder r = 4

2. Euclid's division algorithm is a repeated application of division Lemma until we get______.

Ans : Zero remainder.

3. The H.C.F. of two equal positive integers k, k is ______

Ans : k

Thinking Corner

1. When a positive integer is divided by 3
(i) What are the possible remainders?
(ii) In which form it can be written?

Ans :

(i) Let the positive integer be a.
By Euclid's Division Lemma
a = bq + r, where $0 \le r < |b|$
So, a = 3q + r, $0 \le r < |3|$
a = 3q + r, r = 0, 1, 2
Possible remainders are 0, 1 and 2.

(ii) The positive integer can be written as
∴ a = 3q; a = 3q + 1, a = 3q + 2

Exercise 2.1

1. Find all positive integers when divided by 3 leaves remainder 2.

Sol : Let the required positive integer be a.
Given a is divided by 3 leaves remainder 2.
By Euclid's Division Lemma a and b are any positive integers, then there exist unique integer q and r such that
a = bq + r where $0 \le r < |b|$
Here a = 3q + r where $0 \le r < |3|$
r = 0, 1, 2.
Considering r = 2
a = 3q + 2, where q ≥ 0
i.e., a = 3q + 2, q = 0, 1, 2, 3, ...
a = 3 (0) + 2, 3 (1) + 2, 3 (2) + 2, 3 (3) +2, ...
a = 2, 5, 8, 11, ...
∴ The required numbers are 2, 5, 8, 11, ...

2. A man has 532 flower pots. He wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over.

Sol :
Given number of flower pots = 532
By Euclid's Division Lemma
a = bq + r where $0 \le r < |b|$
532 = $21 \times 25 + 7$
where $0 \le r < |21|$
∴ Number of completed rows = 25
Number of flower pots left over = 7

3. Prove that the product of two consecutive positive integers is divisible by 2.

Sol : Let n, n+1 be two consecutive positive integers.

Let b = 2.

By Euclid's Division Lemma

n = bq + r where $0 \le r < |b|$

n = 2q + r where $0 \le r < |2|$ i.e., r = 0, 1

$\therefore$ n = 2q or n = 2q + 1

Case 1:

If n = 2q then n + 1 = 2q + 1

Their product n (n + 1) = 2q × (2q + 1)

$= (2q)(2q) + 2q$

$= 4q^2 + 2q$

$= 2q(2q+1)$

= 2 m where m = q (2q + 1) and m is positive.

$\therefore$ n (n + 1) is divisible by 2

Case 2:

If n = 2q + 1 then n + 1 = (2q + 1) + 1 = 2q + 2

$\therefore n(n+1) = (2q+1)(2q+2)$

$= 4q^2 + 2q + 4q + 2$

$= 4q^2 + 6q + 2$

$= 2(2q^2 + 3q + 1)$

= 2 p, where $p = 2q^2 + 3q + 1$ and p is positive.

$\therefore$ n (n + 1) is divisible by 2.

Thus we conclude that,

Product of two consecutive positive integer is divisible by 2.

4. When the positive integer a, b and c are divided by 13, the respective remainders are 9, 7 and 10. Show that a + b + c is divisible by 13.

Sol : Given the positive integers a, b and c divided by 13 leaves remainders 9, 7, 10 respectively.

By Euclid's division Lemma we have

$$a = 13q_1 + 9$$
$$b = 13q_2 + 7$$
$$c = 13q_3 + 10$$
$$a + b + c = 13q_1 + 9 + 13q_2 + 7 + 13q_3 + 10$$
$$= 13q_1 + 13q_2 + 13q_3 + 26$$
$$= 13q_1 + 13q_2 + 13q_3 + 13(2)$$
$$= 13[q_1 + q_2 + q_3 + 2] = 13m$$

where $m = q_1 + q_2 + q_3 + 2$ and m is positive integer.

which is divisible by 13.

$\therefore$ a + b + c is divisible by 13.

5. Prove that square of any integer leaves the remainder either 0 or 1 when divided by 4.

Sol : All the integers 'a' must be either even or odd.

If it is even then a = 2q.

If it is odd then a= 2q + 1

Case 1:

If a = 2q

$a^2 = (2q)^2$

$a^2 = 4q^2$, remainder 0 when divided by 4.

Case 2:

If a = 2q + 1

$$a^2 = (2q+1)^2$$
$$= 4q^2 + 4q + 1$$
$$= 4q(q+1) + 1$$

$a^2 = 4m + 1$ where m = q (q + 1) is an integer.

It is of the form bq + 1 where 1 is the remainder when divided by 4.

$\therefore$ The square of any integer leaves the remainder either 0 or 1 when divided by 4.

6. Use Euclid's Division Algorithm to find the Highest Common Factor (H.C.F) of

(i) 340 and 412 (ii) 867 and 255

(iii) 10224 and 9648 (iv) 84, 90 and 120

Sol :

(i) To find the H.C.F. of (340, 412),

412 > 340 = H.C.F. (412, 340)

Using Euclid's division algorithm we have

$412 = 340 \times 1 + 72$

The remainder $72 \neq 0$

Again applying Euclid's division algorithm

$340 = 72 \times 4 + 52$

The remainder $52 \neq 0$

Again applying Euclid's division algorithm

$72 = 52 \times 1 + 20$

The remainder $20 \neq 0$

Again applying Euclid's division algorithm

$52 = 20 \times 2 + 12$

The remainder $12 \neq 0$

$\therefore$ Again applying Euclid's division algorithm

$20 = 12 \times 1 + 8$

The remainder $8 \neq 0$

Again applying Euclid's Division Algorithm

$12 = 8(1) + 4$

The remainder $4 \neq 0$,

Again applying Euclid's Division Algorithm

$8 = 4(2) + 0$

Now the remainder $= 0$

$\therefore$ H.C.F. (340, 412) $= 4$

(ii) 867 and 255

Here $867 > 255$

Applying repeatedly Euclid's Division Algorithm until getting remainder zero, we have.

$$867 = 255\,(3) + 102$$
$$255 = 102\,(2) + 51$$
$$102 = 51\,(2) + 0$$

The remainder $= 0$

$\therefore$ H.C.F. (867, 255) $= 51$

(iii) 10224 and 9648

Here $10224 > 9648$

Applying repeatedly Euclid's Division Algorithm until getting remainder zero, we have,

$$10224 = 9648\,(1) + 576$$
$$9648 = 576\,(16) + 432$$
$$576 = 432\,(1) + 144$$
$$432 = 144\,(3) + 0$$

The remainder $= 0$

$\therefore$ H.C.F. (10224, 9648) $= 144$

(iv) 84, 90 and 120

First we will find the H.C.F. of 84 and 90.

H.C.F. (90, 84)

Applying Euclid's Division Algorithm until we get remainder zero.

$$90 = 84\,(1) + 6$$
$$84 = 6\,(14) + 0$$

Remainder $= 0$

$\therefore$ H.C.F. (90, 84) $= 6$

Now finding H.C.F. (120, 6) we have

$$120 = 6\,(20) + 0$$

Remainder $= 0$

$\therefore$ H.C.F. is 6.

So H.C.F. of 84, 90 and 120 is 6.

7. Find the largest number which divides 1230 and 1926 leaving remainder 12 in each case.

Sol : Since the remainder is 12 in each case, the required number is H.C.F. of the numbers $1230 - 12 = 1218$ and $1926 - 12 = 1914$. Now to find H.C.F. of 1218 and 1914, we use Euclid's Division Algorithm, we have,

$$1914 = 1218 \times 1 + 696$$
$$1218 = 696 \times 1 + 522$$
$$696 = 522 \times 1 + 174$$
$$522 = 174 \times 3 + 0$$

$\therefore$ H.C.F. of 1218, 1914 $= 174$

$\therefore$ The required number is 174.

8. If d is the Highest Common Factor of 32 and 60, find x and y satisfying $d = 32x + 60y$.

Sol : Using Euclid's Division Algorithm, we have

$$60 = 32 \times 1 + 28 \quad \ldots (1)$$
$$32 = 28 \times 1 + 4 \quad \ldots (2)$$
$$28 = 4 \times 7 + 0$$

The remainder is zero.

$\therefore$ The last divisor 4 is the H.C.F. of 60 and 32.

Given $d = 32x + 60y$

From (2), we have $4 = 32 - 28 \times 1$ ... (3)

Also from (1) we have $28 = 60 - 32 \times 1$ ... (4)

Substituting (4) in (3), we have,

$$4 = 32 - (60 - 32 \times 1) \times 1$$
$$4 = 32 - 60 + 32 \times 1$$

$4 = 32\,(2) + 60\,(-1)$ which is of the form

$d = 32x + 60y$ where $x = 2$ and $y = -1$

$\therefore x = 2;\ y = -1$

9. A positive integer when divided by 88 gives the remainder 61. What will be the remainder when the same number is divided by 11?

Sol : Let the positive integer be 'n'

So $n = 88\,(p) + 61$, where p be an integer

$n = 88\,(p) + (5 \times 11 + 6)$

$n = 8 \times 11 \times p + 5 \times 11 + 6$

$n = 11\,(8p + 5) + 6$

Dividing both the sides by 11, we get

$$\frac{n}{11} = (8p+5) + \frac{6}{11}$$

$\therefore$ When the same number 'n' is divided by 11 the remainder will be 6.

10. Prove that two consecutive positive integers are always coprime.

Sol : Let the two consecutive positive integers be n and n + 1.

Here $n + 1 > n$

By Euclid's Division Algorithm, we have

$n + 1 = n \times q + 1$ for some integer q.

Remainder $= 1$

$\therefore$ Again applying Euclid' s Division Algorithm

$$n = 1 \times (n) + 0$$

Here the remainder $= 0$

$\therefore$ H.C.F. is the last divisor 1.

$\therefore$ H.C.F. $(n, n + 1) = 1$

i.e., H.C.F. of two consecutive positive integers = 1

$\therefore$ Two consecutive positive integers are always coprime.

FUNDAMENTAL THEOREM OF ARITHMETIC

Key Points

Theorem: (Fundamental Theorem of Arithmetic)

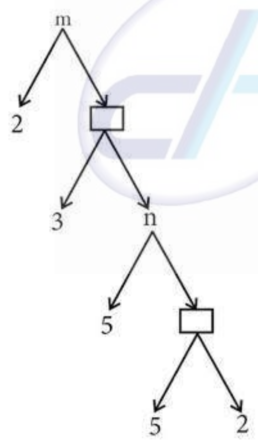
"Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written".

- For a composite number N, we decompose it uniquely in the form

  $N = p_1^{q_1} \times p_2^{q_2} \times p_3^{q_3} \times ... \times p_n^{q_n}$ where $p_1, p_2, p_3, ..., p_n$ are primes and $q_1, q_2, q_3, ..., q_n$ are natural numbers.
- Thus every composite number can be written uniquely as the product of power of primes.
- If a prime number p divides ab then either p divides a or p divides b. That is p divides atleast one of them.

Worked Examples

2.7 In the given factor tree, find the numbers m and n.



Sol :

Value of the first box from bottom

$= 5 \times 2 = 10$

Value of $n = 5 \times 10 = 50$

Value of the second box from bottom

$= 3 \times 50 = 150$

Value of $m = 2 \times 150 = 300$

Thus, the required numbers are

$m = 300, n = 50$

2.8 Can the number 6^n, n being a natural number end with the digit 5? Give reason for your answer.

Sol :

Since $6^n = (2 \times 3)^n = 2^n \times 3^n$

2 is a factor of 6^n

So, 6^n is always even.

But any number whose last digit is 5 is always odd.

Hence, 6^n cannot end with the digit 5.

2.9 Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.

Sol :

Yes, the given number is a composite number.

Because,

$7 \times 5 \times 3 \times 2 + 3 = 3 \times (7 \times 5 \times 2 + 1) = 3 \times 71$

Since the given number can be factorized in terms of two primes, it is a composite number.

2.10 'a' and 'b' are two positive integers such that $a^b \times b^a = 800$. Find 'a' and 'b'.

Sol :

The number 800 can be factorized as

$800 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^5 \times 5^2$

Hence, $a^b \times b^a = 2^5 \times 5^2$

The implies that a = 2 and b = 5 (or)

a = 5 and b = 2.

Progress Check

1. Every natural number except _____ can be expressed as ______.

Ans : 1, product of primes.

2. In how many ways a composite number can be written as product of power of primes?

Ans : In a unique way.

3. The number of divisors of any prime number is ______.

Ans : 2

4. Let m divides n. Then GCD and LCM of m, n are _____ and ______.

Ans : m and n

5. The HCF of numbers of the form 2^m and 3^n is _____.

Ans : 1

Thinking Corner

1. Is 1 a prime number?

Ans : No, 1 is not a prime number.

Because the definition of a prime number is "a positive integer that has exactly two positive divisors (1 and itself) is a prime number".

But 1 has only one positive divisor 1 itself.

$\therefore$ So it is not a prime.

2. Can you think of positive integers a, b such that $a^b = b^a$?

Ans : Take a = 2 and b = 4

We have $2^4 = 4^2$

Exercise 2.2

1. For what values of natural number n, 4^n can end with the digit 6?

Sol : for some natural number n,

$4^n = (2)^{2n}$

So 2 is a factor of 4^n

By fundamental theorem of arithmetic, we know the factorization of 4^n is unique

$\therefore$ Only factor of 4^n is 2, even number of times.

But 4^n always end with 4 or 6.

If n is odd then 4^n end with 4.

If n is even, then 4^n end with the digit 6.

2. If m, n are natural numbers, for what values of m, does $2^n \times 5^m$ ends in 5?

Sol : Consider $2^n \times 5^m$

Since the product has 2 as a factor

$2^n \times 5^m$ is even [$\because$ *n is natural*]

But if a number ends with the digit 5, then the number is an odd number

It is impossible.

For no value of m, $2^n \times 5^m$ ends in 5.

3. Find the HCF of 252525 and 363636.

Sol :

5	252525
5	50505
3	10101
7	3367
13	481
	37

3	363636
3	121212
2	40404
2	20202
3	10101
7	3367
13	481
	37

$252525 = 3^1 \times 5^2 \times 7^1 \times 13^1 \times 37^1$

$363636 = 2^2 \times 3^3 \times 7^1 \times 13^1 \times 37^1$

$\text{H.C.F.} = 3^1 \times 7^1 \times 13^1 \times 37^1$

$= 3 \times 3367$

$= 10101$

4. If $13824 = 2^a \times 3^b$ then find a and b.

Sol :

2	13824
2	6912
2	3456
2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
	3

The number 13824 can be factorized as

$2^a \times 3^b = 13824 = 2^9 \times 3^3$

$\therefore$ a = 9 and b = 3.

5. If $p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 113400$ where p_1, p_2, p_3, p_4 are primes in ascending order and x_1, x_2, x_3, x_4 are integers, find the value of p_1, p_2, p_3, p_4 and x_1, x_2, x_3, x_4.

Sol :

Given $113400 = p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4}$

$$\begin{array}{r|l}
2 & 113400 \\ \hline
2 & 56700 \\ \hline
2 & 28350 \\ \hline
3 & 14175 \\ \hline
3 & 4725 \\ \hline
3 & 1575 \\ \hline
3 & 525 \\ \hline
5 & 175 \\ \hline
5 & 35 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array}$$

The number 113400 can be factorized as

$113400 = 2^3 \times 3^4 \times 5^2 \times 7^1$ where

2, 3, 5, 7 are primes in ascending order.

$\therefore$ Comparing

$p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 2^3 \times 3^4 \times 5^2 \times 7^1$

$p_1 = 2;\ p_2 = 3;\ p_3 = 5;\ p_4 = 7$

$x_1 = 3;\ x_2 = 4;\ x_3 = 2;\ x_4 = 1.$

6. Find the L.C.M. and H.C.F. of 408 and 170 by applying the fundamental theorem of Arithmetic.

Sol : By fundamental theorem, every composite number can be expressed as a product of primes.

$$\begin{array}{r|l}
2 & 170 \\ \hline
5 & 85 \\ \hline
 & 17
\end{array}
\qquad
\begin{array}{r|l}
2 & 408 \\ \hline
2 & 204 \\ \hline
2 & 102 \\ \hline
3 & 51 \\ \hline
 & 17
\end{array}$$

$\therefore$ Factorizing 408 and 170 we get

$408 = 2^3 \times 3^1 \times 17^1$

$170 = 2^1 \times 5^1 \times 17^1$

H.C.F. of 408 and 170 $= 2^1 \times 17^1 = 34$

Also we know that H.C.F. $\times$ L.C.M.

= product of two numbers

$34 \times \text{L.C.M.} = 408 \times 170$

$$\text{L.C.M.} = \frac{408 \times 170}{34} = 2040$$

$\therefore$ H.C.F. (408, 170) = 34; L.C.M. (408, 170) = 2040

7. Find the greatest number consisting 6 digits which is exactly divisible by 24, 15, 36?

Sol :

$$\begin{array}{r|l}
3 & 24,\ 15,\ 36 \\ \hline
2 & 8,\ 5,\ 12 \\ \hline
2 & 4,\ 5,\ 6 \\ \hline
 & 2,\ 5,\ 3
\end{array}$$

L.C.M. $= 3 \times 2 \times 2 \times 2 \times 5 \times 3 = 360$

Greatest number of 6 digit is 999999

L.C.M. of 24, 15 and 36 = 360

$$\begin{array}{r|l}
 & 2777 \\
360 & 999999 \\
 & 720 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 279
\end{array}$$

On dividing 999999 by 360 remainder obtained is 279.

$\therefore$ Greatest number of 6 digit, divisible by 24, 15 and 36 = 999999 – 279 = 999720

Hence the required number is = 999720

8. What is the smallest number that when divided by 35, 56 and 91 leaves remainder 7 in each case?

Sol :

$$\begin{array}{r|l}
7 & 35,\ 56,\ 91 \\ \hline
 & 5,\ 8,\ 13
\end{array}$$

The required number = L.C.M. of (35, 56, 91) + 7

$\therefore$ L.C.M. of 35, 56, 91 $= 7 \times 5 \times 8 \times 13$

= 3640

The required number = 3640 + 7

= 3647

9. Find the least number that is divisible by the first ten natural numbers.

Sol :

The required number is the L.C.M. of first ten natural numbers

i.e., L.C.M. of

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

$$\begin{array}{r|l}
5 & 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ \hline
2 & 1, 2, 3, 4, 1, 6, 7, 8, 9, 2 \\ \hline
3 & 1, 1, 3, 2, 1, 3, 7, 4, 9, 1 \\ \hline
2 & 1, 1, 1, 2, 1, 1, 7, 4, 3, 1 \\ \hline
 & 1, 1, 1, 1, 1, 1, 7, 2, 3, 1
\end{array}$$

L.C.M. is $5 \times 2 \times 3 \times 2 \times 7 \times 2 \times 3 = 2520$

$\therefore$ The least number divisible by first ten natural numbers is 2520.

MODULAR ARITHMETIC

Key Points

Modular Arithmetic

1. Modular arithmetic is a system of arithmetic for integers where numbers wrap around a certain value.
2. Modular arithmetic process cyclically.
3. The German mathematician Carl Friedrich Gauss developed modular arithmetic. He is hailed as Prince of mathematicians.

Congruence modulo

1. If b – a = kn for some integer k. Then it can be written as $a \equiv b \pmod{n}$. Here n is called modulus.
2. If $a \equiv b \pmod{n}$ then a – b is divisible by n. Example: $61 \equiv 5 \pmod{7}$. Here 61 – 5 = 56 is divisible by 7.
3. When a positive integer is divided by n, then the possible remainders are 0, 1, 2......., n – 1.
4. The equation n = mq + r through Euclid's Division lemma can also be written as $n \equiv r \pmod{m}$
5. Two integers a and b are congruent modulo m, written as $a \equiv b \pmod{m}$ if they leave the same remainder when divided by m.
6. While solving congruent equations we get infinitely many solutions.

Theorem:

1. a, b, c and d are integers and m is a positive integer such that if $a \equiv b \pmod{m}$ and then $c \equiv d \pmod{m}$ then
   (i) $(a + c) = (b + d) \pmod{m}$
   (ii) $(a - c) = (b - d) \pmod{m}$
   (iii) $(a \times c) = (b \times d) \pmod{m}$

Theorem:

If $a \equiv b \pmod{m}$ then
(i) $ac \equiv bc \pmod{m}$
(ii) $a \pm c \equiv b \pm c \pmod{m}$ for some integer c.

Worked Examples

2.11 **Find the remainders when 70004 and 778 is divided by 7.**

Sol :

Since 70000 is divisible by 7.

$$70000 \equiv 0 \pmod{7}$$
$$70000 + 4 \equiv 0 + 4 \pmod{7}$$
$$70004 \equiv 4 \pmod{7}$$

Therefore, the remainder when 70004 is divided by 7 is 4.

Since 777 is divisible by 7

$$777 \equiv 0 \pmod{7}$$
$$777 + 1 \equiv 0 + 1 \pmod{7}$$
$$778 \equiv 1 \pmod{7}$$

Therefore, the remainder when 778 is divided by 7 is 1

2.12 **Determine the value of d such that $15 \equiv 3 \pmod{d}$.**

Sol :

$15 \equiv 3 \pmod{d}$ means $15 - 3 = kd$, for some integer k.

$$12 = kd$$

$\Rightarrow$ d divides 12.

The divisors of 12 are 1, 2, 3, 4, 6, 12. But d should be larger than 3 and so the possible values for d are 4, 6, 12.

2.13 Find the least positive value of x such that

(i) $67+x\equiv 1 \pmod 4$ **(ii)** $98\equiv(x+4) \pmod 5$

Sol :

(i) $67 + x \equiv 1 \pmod 4$

$\Rightarrow$ $67 + x - 1 = 4n$, for some integer n

$66 + x = 4n$

$\Rightarrow$ $66 + x$ is a multiple of 4.

Therefore, the least positive value of x must be 2, since 68 is the nearest multiple of 4 more than 66.

(ii) $98 \equiv (x + 4) \pmod 5$

$\Rightarrow$ $98 - (x + 4) = 5n$, for some integer n.

$94 - x = 5n$

$\Rightarrow$ $94 - x$ is a multiple of 5

Therefore, the least positive value of x must be 4 Since $94 - 4 = 90$ is the nearest multiple of 5 less than 94

2.14 Solve $8x \equiv 1 \pmod{11}$

Sol :

$8x\equiv 1 \pmod{11}$ can be written as $8x - 1 = 11k$, for some integer k.

$$x = \frac{11k+1}{8}$$

When we put $k = 5, 13, 21, 29,....$ then $k + 1$ is divisible by 8.

$$x = \frac{11\times 5+1}{8} = 7$$

$$x = \frac{11\times 13+1}{8} = 18$$

Therefore, the solutions are 7, 18, 29, 40, ...

2.15 Compute x, such that $10^4 \equiv x \pmod{19}$

Sol : $10^2 = 100 \equiv 5 \pmod{19}$

$10^4 = (10^2)^2 \equiv 5^2 \pmod{19}$

$10^4 \equiv 25 \pmod{19}$

$10^4 \equiv 6 \pmod{19}$

(since $25 \equiv 6 \pmod{19}$)

Therefore, $x = 6$.

2.16 Find the number of integer solutions of $3x\equiv 1 \pmod{15}$

Sol : $3x \equiv 1 \pmod{15}$ can be written as

$3x - 1 = 15k$ for some integer k

$3x = 15k + 1$

$$x = \frac{15k+1}{3}$$

$$x = 5k+\frac{1}{3}$$

Since 5k is an integer, $5k+\frac{1}{3}$ cannot be an integer.

So there is no integer solution.

2.17 A man starts his journey from Chennai to Delhi by train. He starts at 22.30 hours on Wednesday. If it takes 32 hours of travelling time and assuming that the train is not late, when will he reach Delhi?

Sol :

Starting time 22.30, Travelling time 32 hours.

Here we use modulo 24.

The reaching time is

$22.30 + 32 \pmod{24} \equiv 54.30 \pmod{24}$

$\equiv 6.30 \pmod{24}$

Since $32 = (1 \times 24)$(Thursday) + 8(Friday)

Thus, he will reach Delhi at 6.30 hours on Friday.

2.18 Kala and Vani are friends. Kala says "Today is my birthday" and she asked Vani "When will you celebrate your birthday". Vani replied "Today is Monday and I celebrated my brithday 75 days ago". Find the day when Vani celebrated her birthday?

Sol :

Let us associate the numbers 0, 1, 2, 3, 4, 5, 6 to represent the weekdays from Sunday to Saturday respectively.

Vani says today is Monday. So the number for Monday is 1. Since Vani's birthday was 75 days ago, we have to subtract 75 from 1 and take the modulo 7, since a week contain 7 days.

$$-74 \pmod 7 \equiv -4 \pmod 7 \equiv 7-4 \pmod 7 \equiv 3 \pmod 7$$

(Since, $-74 - 3 = -77$ is divisible by 7)

Thus, $1 - 75 \equiv 3 \pmod 7$

The day for the number 3 is Wednesday.

Therefore, Vani's birthday must be on Wednesday.

Progress Check

1. Two integers a and b are congruent modulo n if ______

Ans : They differ by an integer multiple of n or if $b - a = kn$ for some integer k.

2. The set of all positive integers which leave remainder 5 when divided by 7 are _____

Ans : $n \equiv 5 \pmod 7$

3. The positive values of k such that $(k-3) \equiv 5 \pmod{11}$ are ______

Ans : 19, 30, 41, 52, ...

4. If $59 \equiv 3 \pmod 7$, $46 \equiv 4 \pmod 7$ then $105 \equiv$ _____ (mod 7), $13 \equiv$ _______ (mod 7), $413 \equiv$ _____ (mod 7), $368 \equiv$ _____ (mod 7)

Ans : 7, –1, 0, 4

5. The remainder when $7 \times 13 \times 19 \times 23 \times 29 \times 31$ is divided by 6 is _____

Ans : 1

Thinking Corner

1. How many integers exist which leave a remainder of 2 when divided by 3?

Ans : Infinitely many numbers exist.

Exercise 2.3

1. Find the least positive value of x such that

(i) $71 \equiv x \pmod 8$ (ii) $78 + x \equiv 3 \pmod 5$

(iii) $89 \equiv (x+3) \pmod 4$

(iv) $96 \equiv \frac{x}{7} \pmod 5$ (v) $5x \equiv 4 \pmod 6$

Sol :

(i) $71 \equiv x \pmod 8$

$71 - x = 8n$ for some integer n

$71 - x$ is a multiple of 8

$\therefore$ The least positive value of x must be 7.

(ii) $78 + x = 3 \pmod 5$

$78 + x - 3 = 5n$ for some integer n.

$75 + x = 5n$ for some integer n.

$75 + x$ is a multiple of 5

$\therefore$ least positive value of $x = 5$

(iii) $89 \equiv (x+3) \pmod 4$

$89 - (x+3) = 4n$ for some integer n

$89 - x - 3 = 4n$ for some integer n.

$86 - x - 4n$, for some integer n.

$86 - x$ is a multiple of 4.

$\therefore$ x= 2 is the least positive value.

(iv) $96 \equiv \frac{x}{7} \pmod 5$

$\left(96 - \frac{x}{7}\right) = 5n$ for some integer n.

$\frac{672 - x}{7} = 5n$ for some integer n.

$\therefore$ Least positive value of $x = 7$

(v) $5x \equiv 4 \pmod 6$

$5x - 4 = 6n$ for some integer n

$5x - 4$ is a multiple of 6.

$x = 2$ is the least positive x.

2. If x is congruent to 13 modulo 17 then $7x - 3$ is congruent to which number modulo 17?

Sol :

Given
$$x \equiv 13 \pmod{17}$$
$$7x \equiv 13 \times 7 \pmod{17}$$
$$7x - 3 \equiv (91 - 3) \pmod{17}$$
$$7x - 3 \equiv 88 \pmod{17}$$
$$7x - 3 \equiv 3 \pmod{17}$$

$7x - 3$ is congruent to 3 modulo 17.

3. Solve $5x \equiv 4 \pmod 6$

Sol :

$5x \equiv 4 \pmod 6$

$5x - 4 = 6k$ for some integer k.

$$x = \frac{6k+4}{5}$$

When we put k = 1, 6, 11, 16...

then $6k + 4$ is divisible by 5

$$\therefore \quad x = \frac{6(1)+4}{5} = \frac{10}{5} = 2$$
$$x = \frac{6(6)+4}{5} = \frac{40}{5} = 8$$
$$x = \frac{6(11)+4}{5} = \frac{70}{5} = 14$$
$$x = \frac{6(16)+4}{5} = \frac{100}{5} = 20$$

Therefore, the solutions are 2, 8, 14, 20,...

4. Solve $3x - 2 \equiv 0 \pmod{11}$

Sol :

$3x - 2 \equiv 0 \pmod{11}$

$3x - 2 = 11k$ for some k

$3x = 11k + 2$

$$x = \frac{11k+2}{3}$$

When we put k = 2, 5, 8, 11,..., $11k + 2$ is divisible by 3.

$$\therefore x = \frac{11(2)+2}{3} = \frac{24}{3} = 8$$
$$x = \frac{11(5)+2}{3} = \frac{57}{3} = 19$$

$$x = \frac{11(8)+2}{3} = \frac{90}{3} = 30$$

$$x = \frac{11(11)+2}{3} = \frac{123}{3} = 41$$

$\therefore$ The solutions are 8, 19, 30, 41,.......

5. What is the time 100 hours after 7 a.m.?

Sol : Starting from 7 o' clock 100 hours.

we use modulo 24.

7 o' clock a.m. + 100 (modulo 24) = 7 o' clock + 4 hrs

=11 o' clock a.m.

$\therefore$ 100 hrs after 7 o' clock a.m. is 11 o' clock a.m.

6. What is the time 15 hours before 11 p.m?

Sol : We have to calculate the time from 11 o' clock p.m.

It is 15 hours before 11 o' clock pm.

Subtracting 15 from 11.

$$11 - 15 \equiv -4 \bmod 12$$
$$\equiv 12 - 4 \pmod{12}$$
$$\equiv 8 \pmod{12}$$
$$11 - 15 \equiv 8 \pmod{12}$$

$\therefore$ It is 8 'o' clock a.m.

7. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming?

Sol :

Starting from Tuesday we have to calculate the day after 45 days.

The number for Tuesday is 2.

$$2 + 45 \pmod 7 \equiv 47 \pmod 7$$
$$\equiv 5 \pmod 7$$

Number 5 stands for Friday.

$\therefore$ Uncle will be coming on Friday.

8. Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

Sol :

7 is divisible by 7

$$\therefore 7 \equiv 0 \pmod 7$$
$$7 + 2 \equiv 0 + 2 \pmod 7$$

[$\because$ Adding the constant 2]

$$9 \equiv 2 \pmod 7$$
$$9^n \equiv 2^n \pmod 7$$
$$6 \times 9^n \equiv 6 \times 2^n \pmod 7$$

[$\because$ multiplying by the constant 6]

$$2^n + 6 \times 9^n \equiv 2^n + 6 \times 2^n \pmod 7$$

[$\because$ Adding 2^n]

$$\equiv 2^n (1 + 6) \pmod 7$$
$$\equiv 2^n \times 7 \pmod 7$$
$$\equiv 2^n \times 0 \pmod 7$$
$$\equiv 0 \pmod 7$$

$\therefore$ When $2^n + 6 \times 9^n$ is divided by 7 we get remainder 0.

Hence $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

9. Find the remainder when 2^{81} is divided by 17.

Sol :

First take $2^5 \equiv 15 \pmod{17}$

$$(2^5)^2 \equiv 15^2 \pmod{17}$$
$$\equiv 4 \pmod{17}$$
$$2^{10} \equiv 4 \pmod{17}$$
$$(2^{10})^4 \equiv 4^4 \pmod{17}$$
$$2^{40} \equiv 1 \bmod 17$$
$$(2^{40})^2 \equiv 1^2 \pmod{17}$$
$$2.2^{80} \equiv 2 \times 1 \pmod{17}$$
$$2^{81} \equiv 2 \bmod 17$$

Remainder when 2^{81} is divided by 17 is 2

10. The duration of flight travel from Chennai to London through British Airlines is approximately 11 hours. The airplane begins its journey on Sunday at 23:30 hours. If the time to Chennai is four and half hours ahead to that of London's time, then find the time at London, when the flight lands at London Airport?

Sol :

Starting time from Chennai = 23.30 hrs

Travelling time = 11 hrs

Here we use modulo 24.

$\therefore$ Reaching time = 23.30 + 11 (mod 24)

= 34.30 (mod 24)

= 10.30 (mod 24)

$$\text{Since } 11 = \underbrace{0 \times 24}_{\text{Sunday}} + \underbrace{11}_{\text{Monday}}$$

It reaches London on Monday at 10.30 a.m

Chennai time = 4.30 hrs + London time

London time = Chennai time – 4.30 a.m

= 10.30 – 4.30 = 6 a.m

$\therefore$ The flight will land at London Airport on Monday at 6 a.m

SEQUENCES

Key Points

Definition

1. A real valued sequence is a function defined on the set of natural numbers and taking real values.
2. Each element in the sequence is called 'a term' of the sequence.
3. General form of a sequence is $a_1, a_2, a_3 \ldots\ldots a_n, \ldots\ldots$
4. If the number of elements in the sequence is finite then it is called a finite sequence.
5. If the number of elements in a sequence is infinite then it is called an infinite sequence.
6. A sequence can be considered as a function defined on the set of natural numbers N.
7. A sequence is a function f : $N \to R$, where R is the set of all real numbers.

Worked Examples

2.19 Find the next three terms of the sequences

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \ldots$ **(ii)** 5, 2, –1, –4,.... **(iii)** 1, 0.1, 0.01,....

Sol :

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \ldots$

+4 +4 +4

In the above sequence the numerators are same and the denominator is increased by 4. So the next three terms are

$$a_5 = \frac{1}{14+4} = \frac{1}{18}$$

$$a_6 = \frac{1}{18+4} = \frac{1}{22}$$

$$a_7 = \frac{1}{22+4} = \frac{1}{26}$$

(ii) 5, 2, –1 –4,...

–3 –3 –3

Here each term is decreased by 3. So the next three terms are – 7, – 10, – 13.

(iii) 1, 0.1, 0.01 ,...

÷ 10 ÷ 10

Here each term is divided by 10. Hence, the next three terms are

$$a_4 = \frac{0.01}{10} = 0.001$$

$$a_5 = \frac{0.001}{10} = 0.0001$$

$$a_6 = \frac{0.0001}{10} = 0.00001$$

2.20 Find the general term for the following sequences.

(i) 3, 6, 9, ... **(ii)** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

(iii) 5, – 25, 125,...

Sol :

(i) 3, 6, 9,...

Here the terms are multiples of 3. So the general term is

$a_n = 3n,\ n \in N$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_1 = \frac{1}{2}$; $a_2 = \frac{2}{3}$; $a_3 = \frac{3}{4}$

We see that the numerator of n^{th} term is n, and the denominator is one more than the numerator. Hence, $a_n = \frac{n}{n+1}, n \in N$

(iii) 5, – 25, 125,...

The terms of the sequence have + and – sign alternatively and also they are in powers of 5.

So the general term $a_n = (-1)^{n+1}5^n, n \in N$

2.21 The general term of a sequence is defined as

$$a_n = \begin{cases} n(n+3);\ n \in N \text{ is odd} \\ n^2+1;\ n \in N \text{ is even} \end{cases}$$

Find the eleventh and eighteenth terms.

Sol :

To find a_{11}, since 11 is odd, we put n = 11 in

$a_n = n(n+3) \Rightarrow a_{11} = 11 \times 14$

Thus, the eleventh term $a_{11} = 154$

To find a_{18}, since 18 is even,

we put n = 18 in $a_n = n^2 + 1$

Thus, the eighteenth term $a_{18} = 18^2 + 1 = 325$

2.22 Find the first five terms of the following sequence.

$$a_1 = 1, a_2 = 1, a_n = \frac{a_{n-1}}{a_{n-2}+3}; n \geq 3, n \in N$$

Sol : The first two terms of this sequence are given by $a_1 = 1, a_2 = 1$. The third term a_3 depends on the first and second terms.

$$a_3 = \frac{a_{3-1}}{a_{3-2}+3} = \frac{a_2}{a_1+3} = \frac{1}{1+3} = \frac{1}{4}$$

Similarly the fourth term a_4 depends upon a_2 and a_3.

$$a_4 = \frac{a_{4-1}}{a_{4-2}+3} = \frac{a_3}{a_2+3} = \frac{\frac{1}{4}}{1+3} = \frac{\frac{1}{4}}{4} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

In the same way, the fifth term a_5 can be calculated as

$$a_5 = \frac{a_{5-1}}{a_{5-2}+3} = \frac{a_4}{a_3+3} = \frac{\frac{1}{16}}{\frac{1}{4}+3} = \frac{1}{16} \times \frac{4}{13} = \frac{1}{52}$$

Therefore, the first five terms of the sequence are

$1, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{25}$

Progress Check

1. **Fill in the blanks for the following sequences.**
   (i) 7, 13, 19, _____,…
   (ii) 2, _____, 10, 17, 26, …
   (iii) 1000, 100, 10, 1, _____,…

   Ans : (i) 25, (ii) 5, (iii) 1/10

2. **A sequence is a function defined on the set of _____**

   Ans : Natural Numbers.

3. **The n^{th} term of a sequence 0, 2, 6, 12, 20, … can be expressed as _____**

   Ans : $a_n = n(n-1)$

4. **Say True or False**
   (i) All sequences are functions
   Ans : True.
   (ii) All functions are sequences
   Ans : False.

Exercise 2.4

1. **Find the next three terms of the following sequence.**
   (i) 8, 24, 72, …. **(ii) 5, 1, – 3**
   (iii) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \ldots$

Sol :

(i) 8, 24, 72, ...

$$8 \xrightarrow{\times 3} 24 \xrightarrow{\times 3} 72 \xrightarrow{\times 3} 216 \xrightarrow{\times 3} 648 \xrightarrow{\times 3} 1944$$

Each term is obtained by multiplying the previous term by 3.

∴ Next three terms are 216, 648, 1944.

(ii) 5, 1, –3 $\quad 5 \xrightarrow{-4} 1, \xrightarrow{-4} -3, \ldots$

Here each term is obtained by subtracting 4 from the previous term.

∴ Next three terms are $a_4 = -3 - 4 = -7$

$a_5 = -7 - 4 = -11$

$a_6 = -11 - 4 = -15$

(iii) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \ldots$

$$\frac{1}{4} \xrightarrow{+1} \frac{2}{9} \xrightarrow{+1} \frac{3}{16}$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$2^2 \quad 3^2 \quad 4^2$$

This sequence is generated by $a_n = \frac{n}{(n+1)^2}$

The numerator is increased by 1 in successive terms.

The denominators are $2^2, 3^2, 4^2, \ldots$

$$\therefore a_4 = \frac{4}{5^2} = \frac{4}{25}$$

$$a_5 = \frac{5}{6^2} = \frac{5}{36}$$

$$a_6 = \frac{6}{7^2} = \frac{6}{49}.$$

∴ The next three terms are $\frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

2. Find the first four terms of the sequences whose n^{th} terms are given by

(i) $a_n = n^3 - 2$

(ii) $a_n = (-1)^{n+1} n(n+1)$ (iii) $a_n = 2n^2 - 6$

Sol :

(i) $a_n = n^3 - 2$

$\therefore$ The first term $a_1 = 1^3 - 2 = 1 - 2 = -1$

Second term $a_2 = 2^3 - 2 = 8 - 2 = 6$

Third term $a_3 = 3^3 - 2 = 27 - 2 = 25$

Fourth term $a_4 = 4^3 - 2 = 64 - 2 = 62$

$\therefore$ First four terms are $-1, 6, 25, 62$

(ii) $a_n = (-1)^{n+1} n(n+1)$

n^{th} term is given by $a_n = (-1)^{n+1} n(n+1)$

The first term $a_1 = (-1)^{1+1} 1(1+1) = 2$

Second term $a_2 = (-1)^{2+1} 2(2+1) = -6$

Third term $a_3 = (-1)^{3+1} 3(3+1) = 12$

Fourth term $a_4 = (-1)^{4+1} 4(4+1) = -20$

First four terms are $2, -6, 12, -20$.

(iii) $a_n = 2n^2 - 6$

n^{th} term is given by $a_n = 2n^2 - 6$

$\therefore$ First term $a_1 = 2(1)^2 - 6 = -4$

Second term $a_2 = 2(2)^2 - 6 = 2$

Third term $a_3 = 2(3)^2 - 6 = 12$

Fourth term $a_4 = 2(4)^2 - 6 = 26$

$\therefore$ First four terms are $-4, 2, 12, 26$.

3. Find the n^{th} term of the following sequences.

(i) 2, 5, 10, 17,... (ii) $0, \frac{1}{2}, \frac{2}{3}, \ldots$

(iii) 3, 8, 13, 18,...

Sol :

(i) 2, 5, 10, 17, ...

$\Rightarrow 1^2 + 1, 2^2 + 1, 3^2 + 1, 4^2 + 1, \ldots$

Here the every term is obtained by adding 1 to its square.

$\therefore$ The general term $a_n = n^2 + 1$

(ii) $0, \frac{1}{2}, \frac{2}{3}, \ldots$

We see that the numerators of n^{th} term is $n - 1$ and the denominators of n^{th} term is n.

$$\therefore a_n = \frac{n-1}{n}$$

(iii) 3, 8, 13, 18, ...

$a_1 = 3;\quad a_2 = 8;\quad a_3 = 13;\quad a_4 = 18, \ldots$

$a_1 = 3;\quad a_2 = 3 + 5;\quad a_3 = 8 + 5;\quad a_4 = 13 + 5, \ldots$

We see that n^{th} term is the sum of previous term and 5

$\therefore a_n = a_{n-1} + 5$

Another Method

$a_1 = 3;\quad a_2 = 8;\quad a_3 = 13;\quad a_4 = 18, \ldots$

$a_1 = 5 - 2;\quad a_2 = 2(5) - 2;\quad a_3 = 3(5) - 2;\quad a_4 = 4(5) - 2, \ldots$

$a_n = 5n - 2$

4. Find the indicated terms of the sequences whose n^{th} terms are given by

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

Sol :

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

Given $a_n = \frac{5n}{n+2}$

To find a_6, put $n = 6$

$$\therefore a_6 = \frac{5 \times 6}{6+2} = \frac{30}{8} = \frac{15}{4}$$

To find a_{13}, put $n = 13$

$$a_{13} = \frac{5 \times 13}{13+2} = \frac{65}{15} = \frac{13}{3}$$

$$\therefore a_6 = \frac{15}{4} \text{ and } a_{13} = \frac{13}{3}$$

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

Given $a_n = -(n^2 - 4)$

To find a_4 put $n = 4$

$a_4 = -(4^2 - 4) = -(16 - 4) = -12$

To find a_{11} put $n = 11$

$a_{11} = -(11^2 - 4) = -(121 - 4) = -117$

$\therefore a_4 = -12;\quad a_{11} = -117$

5. Find a_8 and a_{15} whose n^{th} term is

$$a_n = \begin{cases} \frac{n^2-1}{n+3}; & n \text{ is even, } n \in N \\ \frac{n^2}{2n+1}; & n \text{ is odd, } n \in N \end{cases}$$

Sol : To find a_8, since $n = 8$ is even, put $n = 8$ in

$$a_n = \frac{n^2-1}{n+3}$$

$$a_8 = \frac{8^2-1}{8+3} = \frac{64-1}{11} = \frac{63}{11}$$

To find a_{15}, since n = 15 is odd

put n = 15 in $a_n = \frac{n^2}{2n+1}$

$$a_{15} = \frac{15^2}{2(15)+1} = \frac{225}{30+1} = \frac{225}{31}$$

$\therefore a_8 = \frac{63}{11}$ and $a_{15} = \frac{225}{31}$

6. If $a_1 = 1$, $a_2 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$, $n \geq 3$, $n \in N$. Then find the first six terms of the sequence.

Sol : Given first two terms of the sequence.

$a_1 = 1$ and $a_2 = 1$

For $n \geq 3$ $a_n = 2a_{n-1} + a_{n-2}$

When n = 3 $a_3 = 2a_{3-1} + a_{3-2} = 2a_2 + a_1$

$a_3 = 2 + 1 = 3$

When n = 4, $a_4 = 2a_{4-1} + a_{4-2} = 2a_3 + a_2$

$a_4 = 2 \times 3 + 1 = 6 + 1 = 7$

When n = 5 $a_5 = 2a_{5-1} + a_{5-2} = 2a_4 + a_3$

$a_5 = 2 \times 7 + 3 = 14 + 3 = 17$

When n = 6 $a_6 = 2a_{6-1} + a_{6-2} = 2a_5 + a_4$

$a_6 = 2(17) + 7 = 34 + 7 = 41$

$\therefore$ First 6 terms of the sequence are 1, 1, 3, 7, 17, 41

ARITHMETIC PROGRESSION

Key Points

Definition:

Let a and d be real numbers. Then the numbers of the form a, a + d, a + 2d, a + 3d, a + 4d, ... is said to form Arithmetic progression denoted by A. P. The number 'a' is called the first term and d is called the common difference.

1. Arithmetic progression is a sequence whose successive terms differ by a constant number. Example: 2, 4, 6, 8 ...
2. The difference between any two consecutive terms of an A.P. is always constant. The constant value is called the **common difference.**
3. If there are finite number of terms in an A.P. then it is called Finite Arithmetic Progression.
4. If there are infinitely many terms in an A.P. then it is called Infinite Arithmetic Progression.

Terms and common difference of an A.P.

1. n^{th} term of an A.P. is denoted by t_n and $t_n = a + (n-1)d$.
2. To find the common difference we subtract first term from the second, second term from the third and so on.

   $\therefore$ Common difference = $t_2 - t_1 = t_3 - t_2 = ...$
3. The common difference 'd' of an A.P. can be positive, negative or zero.
4. An Arithmetic progression having a common difference of zero is called a **Constant Arithmetic Progression.** Example: –1, –1, –1,.....
5. If finite A.P. whose first term is a and last term l, then the number of terms in the A.P. is given by

$$l = a + (n-1)d$$

$$\Rightarrow n = \frac{l-a}{d} + 1$$

6. All the sequences are functions but all the functions are not sequences.
7. If every term is added or subtracted by a constant then the resulting sequence is also an A.P.
8. If every term is multiplied or divided by a non zero number then the resulting sequence is also an A.P

9. If the sum of three consecutive terms of an A.P. is given then they can be taken as a – d, a, a + d. Here common difference = d.
10. If the sum of four consecutive terms of an A.P. is given they can be taken as a – 3d, a – d, a + d and a + 3d. Here common difference = 2d
11. Three non zero numbers a, b, c are in A.P. if and only if 2b = a + c.

Formulae:

1. Sum up to n terms $S_n = \frac{n}{2}[2a+(n-1)d]$
2. If last term is given then $S_n = \frac{n}{2}(a+l)$
3. Sum of first n odd natural numbers = n^2
4. Sum of first n even natural numbers = n (n + 1)

Worked Examples

2.23 Check whether the following sequences are in A.P. or not?

(i) x + 2, 2x + 3, 3x + 4,…

(ii) 2, 4, 8, 16,…

(iii) $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}$,…

Sol :

To check that the given sequence is in A.P., it is enough to check if the differences between the consecutive terms are equal or not.

(i) $t_2 - t_1 = (2x+3) - (x+2) = x+1$

$t_3 - t_2 = (3x+4) - (2x+3) = x+1$

$t_2 - t_1 = t_3 - t_2$

Thus, the differences between consecutive terms are equal. Hence the sequence x + 2, 2x + 3, 3x + 4,... is in A.P.

(ii) $t_2 - t_1 = 4 - 2 = 2$

$t_3 - t_2 = 8 - 4 = 4$

$t_2 - t_1 \neq t_3 - t_2$

Thus, the differences between consecutive terms are not equal. Hence 2, 4, 8, 16,... are not in A.P.

(iii) $t_2 - t_1 = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

$t_3 - t_2 = 7\sqrt{2} - 5\sqrt{2} = 2\sqrt{2}$

$t_4 - t_3 = 9\sqrt{2} - 7\sqrt{2} = 2\sqrt{2}$

Thus the difference between consecutive terms are equal. Hence the terms of the sequence $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}$,... are in A.P.

2.24 Write an A.P. whose first term is 20 and common difference is 8.

Sol :

First term = a = 20; Common difference = d = 8

Arithmetic progression a, a + d, a + 2d, a + 3d,...

In this case, we get 20, 20 + 8, 20 + 2(8), 20 + 3 (8),...

So, the required A.P. is 20, 28, 36, 44,...

2.25 Find the 15^{th}, 24^{th} and n^{th} term (general term) of an A.P. given by 3, 15, 27, 39,...

Sol :

We have, first term = a = 3 and

common difference = d = 15 – 3 = 12.

We know that n^{th} term (general term) of an A.P. with first term a and common difference d is given by $t_n = a + (n-1)d$

$t_{15} = a + (15-1)d = a + 14d = 3 + 14(12) = 171$

(Here a = 3 and d = 12)

$t_{24} = a + (24-1)d = a + 23d = 3 + 23(12) = 279$

The n^{th} (general term) term is given by

$t_n = a + (n-1)d$

Thus, $t_n = 3 + (n-1)12$

$t_n = 12n - 9$

2.26 Find the number of terms in the A.P. 3, 6, 9, 12,..., 111.

Sol : First term a = 3; Common difference d = 6 – 3 = 3; Last term l = 111.

We know that, $n = \left(\frac{l-a}{d}\right)+1$

$n = \left(\frac{111-3}{3}\right)+1 = 37$

Thus the A.P. contains 37 terms.

2.27 Determine the general term of an A.P. whose 7^{th} term is – 1 and 16^{th} term is 17.

Sol :

Let the A.P. be $t_1, t_2, t_3, t_4,...t_n,...$

It is given that $t_7 = -1$ and $t_{16} = 17$

$a + (7-1)d = -1$ and $a + (16-1)d = 17$

$a + 6d = -1$ ...(1)

$a + 15d = 17$ ...(2)

Subtracting equation (1) from equation (2),

we get $9d = 18 \Rightarrow d = 2$

putting $d = 2$ in equation (1), we get $a + 12 = -1$

so, $a = -13$

Hence, General term $t_n = a + (n-1)d$

$= -13 + (n-1) \times 2 = 2n - 15$

2.28 If l^{th}, m^{th} and n^{th} terms of an A.P. are x, y, z respectively, then show that

(i) $x(m-n) + y(n-l) + z(l-m) = 0$

(ii) $(x-y)n + (y-z)l + (z-x)m = 0$

Sol :

(i) Let a be the first term and d be the common difference.

It is given that $t_l = x$, $t_m = y$, $t_n = z$

Using the general term formula

$a + (l-1)d = x$ ...(1)

$a + (m-1)d = y$ ...(2)

$a + (n-1)d = z$ ...(3)

We have , $x(m-n) + y(n-l) + z(l-m)$

$= a[(m-n) + (n-l) + (l-m)] + d[(m-n)(l-1) + (n-l)(m-1) + (l-m)(n-1)]$

$= a[0] + d[lm - ln - m + n + mn - lm - n + l + ln - mn - l + m]$

$= a(0) + d(0) = 0$

(ii) On subtracting equation (2) from equation (1), equation (3) from equation (2), and equation (1) from equation (3), we get

$x - y = (l-m)d$

$y - z = (m-n)d$

$z - x = (n-l)d$

$(x-y)n + (y-z)l + (z-x)m = [(l-m)n + (m-n)l + (n-l)m]d$

$=[ln - mn + lm - nl + nm - lm]d = 0$

2.29 In an A.P., sum of four consecutive terms is 28 and their sum of their squares is 276. Find the four numbers.

Sol : Let us take the four terms in the form of $(a-3d), (a-d), (a+d)$ and $(a+3d)$.

Since sum of the four terms is 28,

$a - 3d + a - d + a + d + a + 3d = 28$

$4a = 28 \Rightarrow a = 7$

Similarly, since sum of their squares is 276,

$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 276$

$a^2 - 6ad + 9d^2 + a^2 - 2ad + d^2 + a^2 + 2ad + d^2 + a^2 + 6ad + 9d^2 = 276$

$4a^2 + 20d^2 = 276 \Rightarrow 4(7)^2 + 20d^2 = 276.$

$d^2 = 4 \Rightarrow d = \pm 2$

If d = 2 then the four numbers are

$7 - 3(2), 7 - 2, 7 + 2, 7 + 3(2)$

That is the four numbers are 1, 5, 9 and 13.

If $a = 7$, $d = -2$ then the four numbers are

13, 9, 5 and 1.

Therefore, the four consecutive terms of the A.P. are 1, 5, 9 and 13.

2.30 A mother divides ₹ 207 into three parts such that the amounts are in A.P. and gives it to her three children. The product of the two least amounts that the children had ₹ 4623. Find the amount received by each child.

Sol :

Let the amount received by the children be in the form of A.P. given by $a - d, a, a + d$.

Since, sum of the amount is ₹ 207, we have

$(a-d) + a + (a+d) = 207$

$3a = 207 \Rightarrow a = 69$

It is given that product of the two least amounts is 4623

$(a-d)a = 4623$

$(69-d)69 = 4623$

$d = 2$

Therefore, Amount given by the mother to her three children are ₹ (69 – 2), ₹ 69, ₹ (69 + 2). That is, ₹ 67, ₹ 69 and ₹ 71.

Progress Check

1. **The difference between any two consecutive terms of an A.P. is _____**

   Ans : always constant.

2. **If a and d are the first term and common difference of an A.P. then the 8^{th} term is _____**

   Ans : $t_8 = a + 7d$

3. **If t_n is the n^{th} term of an A.P. then $t_{2n} - t_n$ is ___**

   Ans : nd

4. **The common difference of a constant A.P. is ___**

   Ans : Zero.

5. If a and l are first and last terms of an A.P. then the number of terms is _______.

Ans : $n = \frac{l-a}{d} + 1$

6. If every term of an A.P. is multiplied by 3, then the common difference of the new A.P. is _____

Ans : 3 × old common difference.

7. Three numbers a, b and c will be in A.P. if and only if _____

Ans : 2b = a + c

Thinking Corner

1. If t_n is the n^{th} term of an A.P. then the value of $t_{n+1} - t_{n-1}$ is _____

Ans : $t_{n+1} - t_{n-1} = a + ((n+1) - 1)d - [a + ((n-1) - 1)d]$

$= [a + d(n+1) - d] - [a + d(n-1) - d]$

$= [a + nd + d - d] - [a + nd - d - d]$

$= a + nd - a - nd + 2d$

$= 2d.$

Exercise 2.5

1. Check whether the following sequences are in A.P.

(i) a –3, a – 5, a – 7,… **(ii)** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},\ldots$

(iii) 9, 13, 17, 21, 25,…

(iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3},\ldots$

(v) 1, –1, 1, –1, 1, –1,….

Sol :

(i) a – 3, a – 5, a – 7,....

$t_2 - t_1 = a - 5 - [a - 3]$

$= a - 5 - a + 3$

$= -2$

$t_3 - t_2 = a - 7 - (a - 5)$

$= a - 7 - a + 5$

$= -2$

$t_2 - t_1 = t_3 - t_2$

∴ The difference between consecutive terms are equal.

∴ a – 3, a – 5, a – 7,.... are in A.P.

(ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},\ldots.$

$t_2 - t_1 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$

$t_3 - t_2 = \frac{1}{4} - \frac{1}{3} = \frac{3-4}{12} = \frac{-1}{12}$

$t_4 - t_3 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \frac{-1}{20}$

$\therefore t_2 - t_1 \neq t_3 - t_2 \neq t_4 - t_3$

∴ The difference between consecutive terms are not equal.

$\therefore \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},\ldots.$ are not in A.P.

(iii) 9, 13, 17, 21, 25,...

$t_2 - t_1 = 13 - 9 = 4$

$t_3 - t_2 = 17 - 13 = 4$

$t_4 - t_3 = 21 - 17 = 4$

$t_5 - t_4 = 25 - 21 = 4$

$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = t_5 - t_4 = 4.$

The difference between consecutive terms are equal.

∴ 9, 13, 17, 21, 25, ... are in A.P.

(iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3},\ldots$

$t_2 - t_1 = 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}$

$t_3 - t_2 = \frac{1}{3} - 0 = \frac{1}{3}$

$t_4 - t_3 = \frac{2}{3} - \frac{1}{3} = \frac{2-1}{3} = \frac{1}{3}$

$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \frac{1}{3}$

∴ The differences between consecutive terms are equal.

$\therefore \frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3},\ldots$ are in A.P.

(v) 1, –1, 1, –1, 1, –1,...

$t_2 - t_1 = (-1) - 1 = -2$

$t_3 - t_2 = 1 - (-1) = 1 + 1 = 2$

$t_4 - t_3 = -1 - 1 = -2$

$t_5 - t_4 = 1 - (-1) = 1 + 1 = 2$

Here $t_2 - t_1 \neq t_3 - t_2$

∴ The difference between consecutive terms are not equal.

∴ 1, –1, 1, –1, 1, –1,... are not in A.P.

2. First term a and common difference d are given below. Find the corresponding A.P.

(i) a = 5, d = 6, **(ii) a = 7, d = – 5**

(iii) $a = \frac{3}{4}, d = \frac{1}{2}$

Sol :

(i) a = 5, d = 6

First term a = 5; common difference d = 6.

A.P.is given by a, a + d, a + 2d, a+ 3d,...

In this case 5, 5 + 6, 5 + 2(6), 5 + 3 (6),...

$\Rightarrow$ 5, 11, 17, 23,...

$\therefore$ The required A.P. is 5, 11, 17, 23,....

(ii) a = 7; d = – 5

A.P is given by a, a + d, a + 2d, a + 3d,...

In this case 7, 7 + (– 5), 7 + 2(– 5), 7 + 3 (– 5),...

$\Rightarrow$ 7, 2, – 3, – 8,...

$\therefore$ The required A.P. is 7, 2, –3, – 8,...

(iii) $a=\frac{3}{4}, d=\frac{1}{2}$

A.P. is given by a, a + d, a + 2d, a + 3d,...

In this case $\frac{3}{4}, \frac{3}{4}+\frac{1}{2}, \frac{3}{4}+2\left(\frac{1}{2}\right), \frac{3}{4}+3\left(\frac{1}{2}\right), \ldots$

$\Rightarrow \frac{3}{4}, \frac{3+2}{4}, \frac{3+4}{4}, \frac{3+6}{4}, \ldots$

$\Rightarrow \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \ldots.$

$\therefore$ The required A.P. is $\frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \ldots$

3. Find the first term and common difference of the Arithmetic progressions whose nth terms are given below.

(i) $t_n = -3 + 2n$ (ii) $t_n = 4 - 7n$

Sol :

(i) $t_n = -3 + 2n$

Given the nth term of the A.P. is $t_n = -3 + 2n$

Put n = 1 $\Rightarrow$ $t_1 = -3 + 2(1) = -3 + 2$

$a = t_1 = -1$

Put n = 2 $\Rightarrow$ $t_2 = -3 + 2(2) = -3 + 4$

$t_2 = 1$

Common difference $d = t_2 - t_1 = 1 - (-1)$

$= 1 + 1 = 2$

$\therefore$ First term a = – 1; Common difference d = 2

(ii) $t_n = 4 - 7n$

Given the nth term of the A.P. is $t_n = 4 - 7n$

Put n = 1 $\Rightarrow$ $t_1 = 4 - 7(1) = 4 - 7 = -3$

$a = t_1 = -3$

$t_2 = 4 - 7(2) = 4 - 14 = -10$

Common difference $d = t_2 - t_1 = -10 - (-3)$

$= -10 + 3 = -7$

First term a = – 3; Common difference d = – 7

4. Find the 19th term of an A.P. – 11, –15, – 19, ...

Sol : Given the A.P. – 11, – 15, – 19,....

Here First term a = – 11

Common difference $d = t_2 - t_1 = -15 - (-11)$

$= -15 + 11$

$d = -4$

$\therefore$ nth term of an A.P. is $t_n = a + (n - 1)d$

19th term $(t_{19}) = -11 + (19 - 1)(-4)$

$= -11 + 18(-4)$

$= -11 + (-72) = -83$

$\therefore$ 19th term of –11, – 15, – 19,...-is – 83.

5. Which term of an A.P. 16, 11, 6, 1,... is – 54?

Sol : Given the A.P. 16, 11, 6, 1, ...

Here a = 16, d = 11 – 16 = – 5

We have the nth term of an A.P. $t_n = a + (n - 1)d$

Put $t_n = -54$

$-54 = 16 + (n - 1)(-5)$

$-54 - 16 = (n - 1)(-5)$

$\frac{-70}{-5} = n - 1$

$14 = n - 1$

$14 + 1 = n$

$n = 15$

$\therefore$ – 54 is the 15th term of 16, 11, 6, 1,...

6. Find the middle term (s) of an A.P. 9, 15, 21, 27,..., 183.

Sol : Given A.P. is 9, 15, 21, 27,... 183

a = 9, d = 15 – 9 = 6

Let last term $l = 183$

We have $l = a + (n - 1)d$

$183 = 9 + (n - 1)(6)$

$183 - 9 = (n - 1)6$

$174 = (n - 1)6$

$\frac{174}{6} = n - 1$

$29 = n - 1$

$n = 29 + 1 = 30$

$\therefore$ Number of terms in the A.P. are 30.

30 is an even number.

$\therefore \frac{30}{2}^{th}, \left(\frac{30}{2}+1\right)^{th}$ terms are middle term.

15th, 16th terms are middle terms

$t_{15} = 9 + (15 - 1)(6)$

$= 9 + 14(6) = 9 + 84$

$t_{15} = 93$

$t_{16} = 9 + (16 - 1)(6)$

$= 9 + 15(6) = 9 + 90$

$t_{16} = 99$

Middle terms are $t_{15} = 93$ and $t_{16} = 99$

7. If nine times ninth term is equal to the fifteen times fifteenth term, show that six times twenty fourth term is zero.

Sol : We know that n^{th} term of an A.P. is

$t_n = a + (n-1)d$

Given 9 times 9^{th} term = 15 times 15^{th} term

$$\begin{aligned} 9 \times t_9 &= 15 \times t_{15} \\ 9\,[a + (9-1)d] &= 15\,[a + (15-1)d] \\ 9\,(a + 8d) &= 15\,(a + 14\,d) \\ 9a + 72\,d &= 15\,a + 210\,d \\ 15\,a + 210\,d - 9a - 72\,d &= 0 \\ 6a + 138\,d &= 0 \\ 6\,(a + 23\,d) &= 0 \\ 6\,[a + (24-1)d] &= 0 \\ 6 \times t_{24} &= 0 \end{aligned}$$

$\therefore$ 6 times 24^{th} term = 0

8. If 3 + k, 18 – k, 5k + 1 are in A.P. then find k

Sol :

Given 3 + k, 18 – k, 5k + 1 are in A.P.

$t_1 = 3 + k$, $t_2 = 18 - k$, $t_3 = 5k + 1$

$\therefore$ Difference between the consecutive terms must be equal.

i.e., $t_2 - t_1 = t_3 - t_2$

$$\begin{aligned} (18-k) - (3+k) &= (5k+1) - (18-k) \\ 18 - k - 3 - k &= 5\,k + 1 - 18 + k \\ 15 - 2k &= 6k - 17 \\ 15 + 17 &= 6k + 2k \\ 32 &= 8k \\ k &= \frac{32}{8} = 4 \end{aligned}$$

9. Find x, y and z given that the numbers x, 10, y, 24, z are in A.P.

Sol :

Let $t_1 = x$, $t_2 = 10$, $t_3 = y$, $t_4 = 24$, $t_5 = z$.

Given that x, 10, y, 24, z are in A.P.

Then difference between consecutive terms are equal.

i.e., $t_2 - t_1 = t_3 - t_2 = t_4 - t_3$

$10 - x = 24 - y$ = common difference 'd'

Also $t_4 - t_2 = 2d$

$$\begin{aligned} 24 - 10 &= 2d \\ 14 &= 2d \\ d &= \frac{14}{2} = 7 \end{aligned}$$

$$\begin{aligned} 10 - x &= 24 - y = 7 \\ \text{If } 10 - x &= 7; \quad x = 10 - 7 = 3 \\ \text{If } 24 - y &= 7; \quad y = 24 - 7 = 17 \\ \text{Also } z - 24 &= 7; \quad z = 24 + 7 = 31 \end{aligned}$$

$\therefore$ x = 3; y = 17; z = 31.

10. In a theatre, there are 20 seats in the front row and 30 rows were allotted. Each successive row contains two additional seats than its front row. How many seats are there in the last row?

Sol :

Let the number of seats in the front row

$t_1 = 20$

Number of seats in the second row

$t_2 = 20 + 2 = 22$

$\therefore$ Let the numbers of seats in 30^{th} row be t_{30}.

Since the additional seats in each row is 2.

$t_1, t_2, ...t_{30}$ form an A.P.

i.e., 20, 22, 24,... upto 30 terms

Here $a = 20; d = 2$

$$\begin{aligned} t_n &= a + (n-1)\,d \text{ form an A.P.} \\ t_{30} &= 20 + (30-1)(2) = 20 + 29 \times 2 \\ &= 20 + 58 = 78 \end{aligned}$$

$\therefore$ In the last row there are 78 seats.

11. Sum of three consecutive terms in an A.P. is 27 and their product is 288. Find the three terms.

Sol :

Let the three consecutive terms be a – d, a, a +d.

Given their sum is 27

$$\begin{aligned} (a-d) + a + (a+d) &= 27 \\ a - d + a + a + d &= 27 \\ 3a &= 27 \\ a &= \frac{27}{3} = 9 \end{aligned}$$

$$\begin{aligned} \text{Product} &= 288 \\ (a-d)\,a\,(a+d) &= 288 \\ a\,(a^2 - d^2) &= 288 \\ 9\,(9^2 - d^2) &= 288 \\ 81 - d^2 &= \frac{288}{9} \\ 81 - d^2 &= 32 \\ d^2 &= 81 - 32 = 49 \\ d \times d &= 7 \times 7 \\ d &= 7 \end{aligned}$$

a = 9 and d = 7

$\therefore$ The terms are $a - d, a, a + d$

$\Rightarrow 9 - 7, 9, 9 + 7 \Rightarrow 2, 9, 16$

$\therefore$ The required terms are 2, 9, 16

12. The ratio of 6th and 8th term of an A.P. is 7 : 9. Find the ratio of 9th to 13th term.

Sol :

Given in an A.P. 6^{th} term : 8^{th} term = 7 : 9

$a + (6 - 1)\,d : a + (8 - 1)\,d = 7 : 9$

$a + 5d : a + 7d = 7 : 9$

Product of the extremes = product of the means.

$$9\,(a + 5d) = 7\,(a + 7d)$$
$$9a + 45\,d = 7a + 49\,d$$
$$9a - 7a = 49\,d - 45\,d$$
$$2a = 4d$$
$$a = 2d$$

To find the ratio of 9^{th} term : 13^{th} term

$$a + (9 - 1)\,d : a + (13 - 1)\,d = a + 8d : a + 12\,d$$
$$= 2d + 8d : 2d + 12\,d$$
$$= 10\,d : 14\,d$$
$$= 5 : 7.$$

The ratio of 9^{th} term to 13^{th} term is 5 : 7.

13. In a winter season let us take the temperature of Ooty from Monday to Friday to be in A.P. The sum of temperatures from Monday to Wednesday is 0°C and the sum of the temperatures from Wednesday to Friday is 18°C. Find the temperature of each of the five days.

Sol :

Given the temperatures are in A.P.

Let the temperatures of Monday, Tuesday, Wednesday, Thursday and Friday be t_1, t_2, t_3, t_4 and t_5 respectively.

Given $t_1 + t_2 + t_3 = 0\,°C$

$a + a + d + a + 2d = 0\,°C$

$$3a + 3d = 0 \qquad ...(1)$$

And $t_3 + t_4 + t_5 = 18\,°C$

$a + 2d + a + 3d + a + 4d = 18\,°C$

$$3a + 9d = 18\,°C \qquad ...(2)$$

Subtracting (1) from (2)

$$6d = 18\,°C$$
$$d = \frac{18}{6}$$
$$d = 3\,°C$$

Put $d = 3\,°C$ in (1)

$$3a + 3 \times 3 = 0$$
$$3a = -9$$
$$a = \frac{-9}{3} = -3\,°C$$

$\therefore$ $t_1 = -3\,°C$, $t_2 = (-3) + 3 = 0\,°C$,

$t_3 = -3 + 2(3) = 3\,°C$,

$t_4 = -3 + 3\,(3) = 6\,°C$, $t_5 = -3 + 4(3) = 9\,°C$.

$\therefore$ Five days temperatures are $-3\,°C$, $0\,°C$, $3\,°C$, $6\,°C$, $9\,°C$

14. Priya earned ₹ 15,000 in the first month. Thereafter her salary increases by ₹ 1500 per year. Her expenses are ₹ 13,000 during the first year and the expenses increases by ₹ 900 per year. How long will it take her to save ₹ 20,000 per month?

Sol :

Let the first month earning be

t_1 = ₹ 15,000

Increase per year d = ₹ 1,500

Expenses for first year = ₹ 13,000

Expenses increase per year = ₹ 900

$\therefore$ Saving for every year will be

(15000 – 13000), (16500 – 13900), (18000 – 14800),...

ie. 2000, 2600, 3200, ...

Since $t_2 - t_1 = t_3 - t_2$, this sequence form an A.P.

$a = 2000$, $d = 2600 - 2000 = 600$

Let the n^{th} years saving be ₹ 20,000 then

$$a + (n - 1)\,d = t_n$$
$$2000 + (n - 1)\,(600) = 20{,}000$$
$$(n - 1)\,600 = 20{,}000 - 2000$$
$$(n - 1)\,600 = 18000$$
$$n - 1 = \frac{18000}{600} = 30$$
$$n = 30 + 1 = 31$$

$\therefore$ To save ₹ 20000, it takes 31 years.

SERIES

Key Points

1. The sum of the terms of a sequence is called series.
2. If $a_1, a_2, a_3, \ldots, a_n$ be the sequence then the series of real numbers is defined as $a_1 + a_2 + a_3 + \ldots$
3. If a series have finite number of terms then it is called a finite series.
4. If a series have infinite number of terms then it is called an infinite series.

Sum to 'n' terms of an A.P.

1. Sum upto n terms of the series a + (a + d) + (a + 2d) + ... is $S_n = \frac{n}{2}[2a + (n-1)d]$
2. If l is the last term then $S_n = \frac{n}{2}(a + l)$

A series whose terms are in Arithmetic progression is called Arithmetic series.

Let a, a + d, a + 2d, a + 3d,.... be the Arithmetic Progression.

The sum of first n terms of a Arithmetic Progression denoted by S_n is given by

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] \quad \ldots(1)$$

Rewriting the above in reverse order.

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \ldots + (a + d) + a \quad \ldots(2)$$

Adding (1) and (2), We get

$$2S_n = [a + a + (n-1)d] + [a + d + a + (n-2)d] + \ldots + [a + (n-2)d + (a + d)] + [a + (n-1)d + a]$$

$$= [2a + (n-1)d] + [2a + (n-1)d + \ldots + [2a + (n-1)d] \text{ (n terms)}$$

$$2S_n = n \times [2a + (n-1)d] \Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$$

Note

If the first term a, and the last term l (n^{th} term) are given then

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a + (n-1)d]$ Since, $l = a + (n-1)d$ We have

$$S_n = \frac{n}{2}[a + l]$$

Worked Examples

2.31 **Find the sum of first 15 terms of the A.P. $8, 7\frac{1}{4}, 6\frac{1}{2}, 5\frac{3}{4}, \ldots$**

Sol : Here the first term a = 8,

common difference $d = 7\frac{1}{4} - 8 = -\frac{3}{4}$,

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}\left[2 \times 8 + (15-1)\left(-\frac{3}{4}\right)\right]$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right] = \frac{165}{4}$$

2.32 Find the sum of 0.40 + 0.43 + 0.46 + .. + 1.

Sol : Here the value of n is not given. But the last term is given. From this, we can find the value of n.

Given a = 0.40, $l = 1$.

We find d = 0.43 – 0.40 = 0.03

Therefore, $n = \left(\frac{l-a}{d}\right)+1$

$$=\left(\frac{1-0.40}{0.03}\right)+1 = 21$$

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[a+l]$$

Here, n = 21, Therefore,

$$S_{21} = \frac{21}{2}[0.40+1] = 14.7$$

So, the sum of 21 terms of the given series is 14.7.

2.33 How many terms of the series 1 + 5 + 9 + ... must be taken so that their sum is 190?

Sol : Here we have to find the value of n, such that $S_n = 190$.

First term a = 1, common difference d = 5 – 1 = 4.

Sum of first n terms of an A.P

$$S_n = \frac{n}{2}[2a+(n-1)d] = 190$$

$$\frac{n}{2}[2\times 1+(n-1)\times 4] = 190$$

$$n[4n-2] = 380$$

$$2n^2 - n - 190 = 0$$

$$(n-10)(2n+19) = 0$$

But n = 10 as $n = -\frac{19}{2}$ is impossible.

Therefore, n = 10.

2.34 The 13th term of an A.P. is 3 and the sum of first 13 terms is 234. Find the common difference and the sum of first 21 terms.

Sol :

Given the 13th term = 3, so $t_{13} = a + 12d = 3$ ...(1)

Sum of first 13 terms = 234

$$\Rightarrow S_{13} = \frac{13}{2}[2a+12d] = 234$$

$$2a + 12d = 36 \qquad ...(2)$$

Solving (1) and (2) we get,

$$a = 33, d = \frac{-5}{2}$$

Therefore, common difference is $\frac{-5}{2}$

Sum of first 21 terms

$$S_{21} = \frac{21}{2}\left[2\times 33+(21-1)\times\left(-\frac{5}{2}\right)\right]$$

$$= \frac{21}{2}[66-50) = 168.$$

2.35 In an A.P. the sum of first n terms is $\frac{5n^2}{2}+\frac{3n}{2}$. Find the 17th term.

Sol :

The 17th term can be obtained by subtracting the sum of first 16 terms from the sum of first 17 terms.

$$S_{17} = \frac{5\times(17)^2}{2}+\frac{3\times 17}{2} = \frac{1445}{2}+\frac{51}{2} = 748$$

$$S_{16} = \frac{5\times(16)^2}{2}+\frac{3\times 16}{2} = \frac{1280}{2}+\frac{48}{2} = 664$$

Now, $t_{17} = S_{17} - S_{16} = 748 - 664 = 84$

2.36 Find the sum of all natural numbers between 300 and 600 which are divisible by 7.

Sol : The natural numbers between 300 and 600 which are divisible by 7 are 301, 308, 315, ... 595.

The sum of all natural numbers between 300 and 600 is 301 + 308 + 315 + ...+ 595.

The terms of the above series are in A.P.

First term a = 301; common difference d = 7;

Last term $l = 595$.

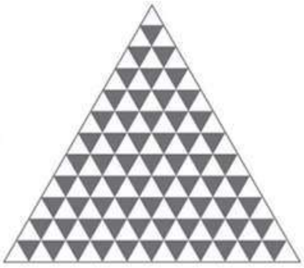
$$n = \left(\frac{l-a}{d}\right)+1 = \left(\frac{595-301}{7}\right)+1 = 43$$

Since, $S_n = \frac{n}{2}[a+l]$, we have

$$S_{43} = \frac{43}{2}[301+595] = 19264.$$

2.37 A mosaic is designed in the shape of an equilateral triangle, 12 ft on each side. Each tile in the mosaic is in the shape of an equilateral triangle of 12 inch side. The tiles are alternate in color as shown in the figure. Find the number of tiles of each colour and total number of tiles in the mosaic.

Sol :



Since the mosaic is in the shape of an equilateral triangle of 12 ft and the tile is in the shape of an equilateral triangle of 12 inch (1 ft), there will be 12 rows in the mosaic.

From the figure, it is clear that number of white tile in each row are 1, 2, 3, 4,...,12 which clearly forms an Arithmetic Progression.

Similarly the number of blue tiles in each row are 0, 1, 2, 3, ..., 11 which is also an Arithmetic Progression.

$$\text{Number of white tiles} = 1 + 2 + 3 + ... + 12 = \frac{12}{2}[1+12] = 78$$

$$\text{Number of blue tiles} = 0 + 1 + 2 + 3 + ... + 11 = \frac{12}{2}[0+11] = 66$$

The total number of tiles in the mosaic $= 78 + 66 = 144$

2.38 The houses of a street are numbered from 1 to 49. Senthil's house is numbered such that the sum of numbers of the houses prior to Senthil's house is equal to the sum of numbers of the houses following Senthil's house. Find Senthil's house number.

Sol :

Let Senthil's house number be x.

It is given that $1 + 2 + 3 + ... + (x - 1) = (x + 1) + (x + 2) + ... + 49$

$$1 + 2 + 3 ... + (x - 1) = [1 + 2 + 3 + ...+ 49] - [1 + 2 + 3 + ... + x]$$

$$\frac{x-1}{2}[1+(x-1)] = \frac{49}{2}[1+49] - \frac{x}{2}[1+x]$$

$$\frac{x(x-1)}{2} = \frac{49 \times 50}{2} - \frac{x(x+1)}{2}$$

$$x^2 - x = 2450 - x^2 - x \Rightarrow 2x^2 = 2450$$

$$x^2 = 1225 \Rightarrow x = 35$$

Therefore Senthil's house number is 35.

2.39 The sum of first n, 2n and 3n terms of an A.P. are S_1, S_2, S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.

Sol :

If S_1, S_2, S_3 are sum of first n, 2n, 3n terms of an A.P. respectively then

$$S_1 = \frac{n}{2}[2a+(n-1)d],$$

$$S_2 = \frac{2n}{2}[2a+(2n-1)d], \quad S_3 = \frac{3n}{2}[2a+(3n-1)d]$$

Consider,

$$S_2 - S_1 = \frac{2n}{2}[2a+(2n-1)d] - \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}\left[[4a+2(2n-1)d] - [2a+(n-1)d]\right]$$

$$S_2 - S_1 = \frac{n}{2} \times [2a+(3n-1)d]$$

$$3(S_2 - S_1) = \frac{3n}{2}[2a+(3n-1)d]$$

$$3(S_2 - S_1) = S_3$$

Progress Check

1. The sum of terms of a sequence is called _____.

Ans : series

2. If a series have finite number of terms then it is called _____.

Ans : finite series.

3. A series whose terms are in _____ is called Arithmetic series.

Ans : Arithmetic progression.

4. If the first and last terms of an A.P. are given then the formula to find the sum is

Ans : $S_n = \frac{n}{2}(a+l)$

5. State True or false.

(i) The n^{th} term of any A.P. is of the form pn + q where p and q are some constants.

Ans : True

(ii) The sum upto n^{th} term of any A.P. is of the form $pn^2 + qn + r$ where p, q, r are some constants.

Ans : False

Thinking Corner

1. The value of n must be positive. Why?

Ans : n denotes the number of terms of a series.

$\therefore$ n must be positive. If it is negative means less.

2. What is the sum of first *n* odd natural numbers?

Ans : First n odd natural numbers are

1, 3, 5, 7,... upto n terms.

$a = 1, \ d = 3 - 1 = 2;$

$S_n = \frac{n}{2}[2a+(n-1)d]$

Now $1 + 3 + 5 + 7 + ... = \frac{n}{2}[2(1)+(n-1)(2)]$

$= \frac{n}{2}[2+2n-2] = \frac{n}{2}[2n] = n^2$

$\therefore$ Sum of first n odd natural numbers = n^2

3. What is the sum of first *n* even natural numbers?

Ans : Sum of n numbers $S_n = \frac{n}{2}[2a+(n-1)d]$

Let the first n even natural numbers be 2, 4, 6, 8, ... upto n terms.

a = 2; d = 4 – 2 = 2

$S_n = \frac{n}{2}[2(2)+(n-1)(2)]$

$= \frac{n}{2}[4+2n-2] = \frac{n}{2}[2n+2]$

$= \frac{n}{2}\times 2(n+1) = n(n+1)$

$\therefore$ Sum of first n even natural numbers = n (n + 1)

Exercise 2. 6

1. Find the sum of the following

(i) 3, 7, 11,… upto 40 terms

(ii) 102, 97, 92, … upto 27 terms.

(iii) 6 + 13 + 20 + …+ 97

Sol :

(i) 3, 7, 11, ... upto 40 terms.

We have sum of n terms of an A.P.

$S_n = \frac{n}{2}\times[2a+(n-1)d]$

Here d = 7 – 3 = 11 – 7 = 4

$\therefore t_2 - t_1 = t_3 - t_2$ and it forms an A.P.

where a = 3, d = 4.

$\therefore S_{40} = \frac{40}{2}[2(3)+(40-1)(4)]$

$= 20[6+(39\times 4)] = 20[6+156]$

$= 20[162] = 3240$

Sum upto 40 terms = 3240

(ii) 102, 97, 92, ...upto 27 terms

Here $t_2 - t_1 = 97 - 102 = -5$

$t_3 - t_2 = 92 - 97 = -5$

$t_2 - t_1 = t_3 - t_2$

$\therefore$ It is an A.P. with a = 102, d = – 5

Sum upto n terms $S_n = \frac{n}{2}[2a+(n-1)d]$

$S_{27} = \frac{27}{2}[2(102)+(27-1)(-5)]$

$= \frac{27}{2}[204+26(-5)]$

$= \frac{27}{2}[204-130]$

$= \frac{27}{2}\times 74$

$= 27\times 37$

Sum upto 27 terms is = 999

(iii) 6 + 13 + 20 + ... + 97

Here $t_2 - t_1 = t_3 - t_2$

$\Rightarrow$ 13 – 6 = 20 – 13 = 7

$\therefore$ It is an Arithmetic series

$\therefore$ Sum $S_n = \frac{n}{2}(a+l)$

a = 6; l = 97

a + (n – 1) d = 97

6 + (n – 1) (7) = 97

(n – 1) 7 = 97 – 6

(n – 1) 7 = 91

$n - 1 = \frac{91}{7} = 13$

n = 13 + 1 = 14

Now $S_n = \frac{14}{2}(6+97) = 7\times 103$

6 + 13 + 20 + ...+ 97 = 721

2. How many consecutive odd integers beginning with 5 will sum to 480?

Sol :

Let the consecutive odd integers beginning with 5 be, 5, 7, 9, 11,....

Here a = 5, d = 2

We know that sum of n consecutive integers

$S_n = \frac{n}{2}[2a+(n-1)d]$

$480 = \frac{n}{2}[2(5)+(n-1)(2)]$

$480 = \frac{2n}{2}[5+n-1]$

480 = n (n + 4)

$n^2 + 4n - 480 = 0$

(n – 20) (n + 24) = 0

n = 20 (or) – 24

Number of terms cannot be negative.

$\therefore \quad n = 20$

20 consecutive odd integers beginning with 5 will sum to 480.

3. Find the sum of first 28 terms of an A.P. whose n^{th} term is $4n - 3$.

Sol : Given $n = 28$

$$t_n = 4n - 3$$
$$t_1 = 4(1) - 3 = 4 - 3 = 1 = a$$
$$t_2 = 4(2) - 3 = 8 - 3 = 5$$
$$d = t_2 - t_1 = 5 - 1 = 4$$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{28}{2}[2(1) + (28-1)(4)]$$
$$= 14[2 + 27(4)] = 14[2 + 108]$$
$$= 14 \times 110 = 1540$$

Sum of first 28 terms of the given A.P. = 1540

4. The sum of first n terms of a certain series is given as $2n^2 - 3n$. Show that the series is an A.P.

Sol :

Given sum of first n terms $S_n = 2n^2 - 3n$

$\therefore$ sum of first term $S_1 = 2(1)^2 - 3(1) = 2 - 3$

$= -1 = t_1$

Sum of first two terms $S_2 = 2(2)^2 - 3(2)$

$$= 8 - 6 = 2$$
$$\therefore t_2 = S_2 - S_1$$
$$= 2 - (-1) = 2 + 1$$
$$t_2 = 3$$

Sum of first three terms $S_3 = 2(3)^2 - 3(3)$

$$= 18 - 9 = 9$$
$$\therefore t_3 = S_3 - S_2 = 9 - 2$$
$$t_3 = 7$$

Here $t_2 - t_1 = 3 - (-1) = 3 + 1 = 4$

$$t_3 - t_2 = 7 - 3 = 4$$
$$\therefore t_2 - t_1 = t_3 - t_2$$

$\therefore$ The series is in A.P.

5. The 104^{th} term and 4^{th} term of an A.P. are 125 and 0. Find the sum of first 35 terms.

Given 104^{th} terms = 125

$$t_{104} = 125$$
$$a + (104 - 1)d = 125$$
$$a + 103d = 125 \quad ...(1)$$

4^{th} term is 0

i,e., $a + (4 - 1)d = 0$

$$a + 3d = 0 \quad ...(2)$$
$$(1) - (2) \Rightarrow 100d = 125$$
$$d = \frac{125}{100} = \frac{5}{4}$$

Put $d = \frac{5}{4}$ in (2)

$$a + 3d = 0$$
$$a + 3\left(\frac{5}{4}\right) = 0$$
$$a + \frac{15}{4} = 0$$
$$a = \frac{-15}{4}$$

$\therefore$ Sum of first n terms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Sum of first 35 terms

$$S_{35} = \frac{35}{2}\left[2\left(\frac{-15}{4}\right) + (35-1)\left(\frac{5}{4}\right)\right]$$
$$= \frac{35}{2}\left[\frac{-15}{2} + 34\left(\frac{5}{4}\right)\right]$$
$$= \frac{35}{2}\left[\frac{-15}{2} + 17\frac{(5)}{2}\right]$$
$$= \frac{35}{2}\left[\frac{-15}{2} + \frac{85}{2}\right]$$
$$= \frac{35}{2}\left[\frac{70}{2}\right] = \frac{2450}{4} = \frac{1225}{2}$$

Sum of first 35 terms of the given series

= 612.5

6. Find the sum of all odd positive integers less than 450.

Sol :

Series of odd positive integers less then 450 is

$1 + 3 + 5 + 7 + ... + 449$

Here $a = 1$; $d = 2$; $l = 449$.

$$\text{Number of terms } n = \left(\frac{l-a}{d}\right) + 1$$
$$n = \frac{449-1}{2} + 1 = \frac{448}{2} + 1 = 224 + 1 = 225$$

Sum of first n odd integers = n^2

$\therefore$ Sum of first 225 odd integers = $225^2 = 225 \times 225$

$1 + 3 + 5 + ... + 449 = 50625$

7. Find the sum of all natural numbers between 602 and 902 which are not divisible by 4.

Sol :

$$\left\{\begin{array}{l}\text{Sum of natural numbers between}\\ \text{602 and 902 not divisible by 4}\end{array}\right\} =$$

$$\left\{\begin{array}{l}\text{Sum of all natural}\\ \text{numbers between}\\ \text{602 and 902}\end{array}\right\} - \left\{\begin{array}{l}\text{Sum of all natural}\\ \text{numbers between}\\ \text{602 and 902 divisible}\\ \text{by 4}\end{array}\right\}$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$\therefore$ Sum of Natural numbers upto n = $\dfrac{n(n+1)}{2}$

$603 + 604 + ... + 901 = (1 + 2 + ... + 901) - (1 + 2 + 3 + ... + 602)$

$$= \frac{901 \times (901+1)}{2} - \frac{602 \times (602+1)}{2}$$

$$= \frac{901 \times 902}{2} - \frac{602 \times 603}{2}$$

$= 4,06,351 - 1,81,503$

$= 2,24,848$

Again numbers divisible by 4 between 602 and 902 are $604 + 608 + ... + 900$

$$n = \left(\frac{l-a}{d}\right) + 1 = \left(\frac{900 - 604}{4}\right) + 1$$

$$= \left(\frac{296}{4}\right) + 1 = 74 + 1 = 75$$

$$S_n = \frac{75}{2}[2(604) + (75-1)(4)]$$

$$= \frac{75}{2} \times 2[604 + (74 \times 2)]$$

$= 75 \times 752 = 56400$

Required sum $= 224848 - 56400$

$= 1,68,448$

8. Raghu wish to buy a Laptop. He can buy it by paying ₹ 40,000 cash or by giving it in 10 installments as ₹ 4800 in the first month, ₹ 4750 in the second month, ₹ 4700 in the third month and so on. If he pays the money in this fashion, Find

(i) Toal amount paid in 10 installments

(ii) How much extra amount that he has to pay than the cost?

Sol :

(i) If paid in cash, cost of laptop = ₹ 40000

The amount he pays in installments are

₹ 4800 + ₹ 4750 + ₹ 4700 + ...10 months.

$\therefore$ This form an Arithmetic series with

a = ₹ 4800

$d = 4750 - 4800 = -50$

$n = 10$

$$S_n = \frac{10}{2}[2(4800) + (10-1)(-50)]$$

$$= \frac{10 \times 2}{2}[4800 + 9 \times (-25)]$$

$= 10\,[4800 - 225]$

$= 10 \times 4575$

= ₹ 45750

$\therefore$ Total amount paid in 10 installments

= ₹ 45750

(ii) Extra amount he pays in installments

= ₹ 45750 – ₹ 40000

= 5750

He pays ₹ 5750 extra by installments.

9. A man repays a loan of ₹ 65000 by paying ₹ 400 in the first month and then increasing the payment by ₹ 300 every month. How long will it take him to clear the loan?

Sol :

Total amount to repay = ₹ 65000

He pays ₹ 400 in the first installment and increasing the payment by ₹ 300 every month.

This form an A.P.

i.e., 400, 700, 1000, ...

$a = 400,\ d = 300$

$\therefore$ Sum upto n terms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$65000 = \frac{n}{2}[2(400) + (n-1)300]$$

$$65000 = \frac{n}{2} \times 2\,[400 + (n-1)150]$$

$= n\,[400 + 150n - 150]$

$= n\,[150n + 250]$

$65000 = 150n^2 + 250n$

Divided by 50

$1300 = 3n^2 + 5n$

$3n^2 + 5n - 1300 = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(3)(-1300)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 15600}}{6} = \frac{-5 \pm \sqrt{15625}}{6}$$

$$= \frac{-5 \pm 125}{6}$$

$$= \frac{-5 - 125}{6}, \frac{-5 + 125}{6}$$

$$n = \frac{-130}{6}, \frac{120}{6}$$

$$n = \frac{-130}{6}, 20$$

n cannot be negative

$\therefore n = 20.$

He will clear the loan by 20 months.

10. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two bricks less than the previous step.

(i) How many bricks are required for the top most step?

(ii) How many bricks are required to build the staircase?

Sol : Total number of steps n = 30.

Bottom step requires a = 100 bricks.

Each successive steps requires 2 less than the previous step.

$\therefore$ Number of bricks in each step form an A.P.

100, 98, 96, ...upto 30 terms.

a = 100, d = – 2, n = 30

(i) Number of bricks required for the topmost step is l

$$\begin{aligned} l &= a + (n-1)d \\ &= 100 + (30-1)(-2) \\ &= 100 + 29(-2) \\ &= 100 - 58 = 42 \end{aligned}$$

$\therefore$ Topmost step requires 42 bricks.

(ii) Number of bricks required to build the staircase, $S_n = \frac{n}{2}(l + a) = \frac{30}{2}(100 + 42)$

$= 15 \times 142 = 2130$

Total number of bricks required = 2130

11. If $S_1, S_2, S_3, \ldots S_m$ are the sums of n terms of 'm' A.P.'s whose first terms are 1, 2, 3,… m and whose common differences are 1, 3, 5,… (2m – 1) respectively, then show that $(S_1 + S_2 + S_3 + \ldots + S_m) = \frac{1}{2}mn(mn+1)$.

Sol :

S_1 is the sum of n terms of an A.P. with a = 1 and d = 1.

S_2 is the sum of n terms of an A.P. with a= 2 and d = 3.

S_m is the sum of n terms of an A.P. with a = m and d = 2m – 1.

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_1 = \frac{n}{2}[2(1) + (n-1)(1)] = \frac{n}{2}[2 + n - 1]$$

$$S_1 = \frac{n}{2}[n+1]$$

$$S_2 = \frac{n}{2}[2(2) + (n-1)(3)] = \frac{n}{2}[4 + 3n - 3]$$

$$S_2 = \frac{n}{2}[3n+1]$$

Similarly, $S_m = \frac{n}{2}[2(m) + (n-1)(2m-1)]$

$$= \frac{n}{2}[2m + 2mn - n - 2m + 1]$$

$$S_m = \frac{n}{2}[2mn - n + 1]$$

Now we find $S_1 + S_2 + \ldots\ldots + S_m$ where S_1 is the first term $a = \frac{n}{2}(n+1)$ and

S_m = last term $l = \frac{n}{2}[2mn - n + 1]$

Here number of terms = m.

$$\therefore S_n = \frac{n}{2}(l + a)$$

$$S_1 + S_2 + S_3 + \ldots S_m = \frac{m}{2}\left[\frac{n}{2}[2mn - n + 1] + \frac{n}{2}(n+1)\right]$$

$$= \frac{m}{2} \times \frac{n}{2}[2mn - n + 1 + n + 1]$$

$$= \frac{1}{4} \times mn(2mn + 2) = \frac{1}{4}mn \times 2(mn+1)$$

$$S_1 + S_2 + S_3 + \ldots S_m = \frac{1}{2}mn(mn+1)$$

12. Find the sum

$$\left[\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+\ldots \text{to 12 terms}\right]$$

Sol :

Here $t_2 - t_1 = \frac{3a-2b}{a+b}-\frac{a-b}{a+b}$

$= \frac{3a-2b-a+b}{a+b} = \frac{2a-b}{a+b}$

$t_3 - t_2 = \frac{5a-3b}{a+b}-\frac{3a-2b}{a+b}$

$= \frac{5a-3b-3a+2b}{a+b} = \frac{2a-b}{a+b}$

$\therefore \quad t_2 - t_1 = t_3 - t_2$

$\therefore$ The sequence form an A.P. with $d = \frac{2a-b}{a+b}$

$\therefore$ Sum of $S_n = \frac{n}{2}[2a+(n-1)d]$

$$S_{12} = \frac{12}{2}\left[2\left(\frac{a-b}{a+b}\right)+(12-1)\left(\frac{2a-b}{a+b}\right)\right]$$

$$= 6\left[2\left(\frac{a-b}{a+b}\right)+11\left(\frac{2a-b}{a+b}\right)\right]$$

$$= 6\times\left[\left(\frac{2a-2b}{a+b}\right)+\frac{22a-11b}{a+b}\right]$$

$$= 6\left[\frac{2a-2b+22a-11b}{a+b}\right]$$

$$= 6\left[\frac{[24a-13b]}{a+b}\right]$$

Sum $S_{12} = \frac{6}{a+b}\left[24a-13b\right]$

GEOMETRIC SEQUENCE OR GEOMETRIC PROGRESSION

Key Points

Definition:

1. A Geometric progression is a sequence in which each term is obtained by multiplying a fixed non-zero number to the proceeding term except the first term.
2. The fixed number is called common ratio. It is denoted by r.
3. The numbers a, ar, ar^2, ..., ar^{n-1} ... is called a geometric progression,where 'a' is the first term $r \neq 0$ is the common ratio.
4. The general term or n^{th} term of a G.P. is $t_n = ar^{n-1}$.
5. The ratio between any two consecutive terms of a G.P. is always constant and that constant is the common ratio. i.e., $r = \frac{t_2}{t_1}$
6 When the product of three consecutive terms of a G.P. are given, we take them as $\frac{a}{r}, a, ar$.
7. When the product of four consecutive terms are given for a G.P, we take them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
8. When a non-zero constant is multiplied or divided in each term of a G.P, the resulting sequence is also a G.P.
9. Three non-zero number a, b, c are in G.P. if and only if $b^2 = ac$.

Formulae:

1. Sum up to n terms of a GP, $S_n = \frac{a(r^n-1)}{r-1}, r \neq 1.$ $S_n = na$, if $r = 1$

2. The sum of infinite G.P. $a + ar + ar^2 + \ldots = \frac{a}{1-r}, -1 < r < 1$.
3. $\sum_{K=1}^{n} K = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
4. $\sum_{K=1}^{n} (2K-1) = 1 + 2 + 3 + \ldots + (2n-1) = n^2$
5. $\sum_{K=1}^{n} K^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{K=1}^{n} K^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[n\frac{(n+1)}{2}\right]^2$

Worked Examples

2.40 Which of the following sequences form a Geometric Progression?

(i) 7, 14, 21, 28,...

(ii) $\frac{1}{2}, 1, 2, 4, \ldots$, (iii) 5, 25, 50, 75,...

Sol : To check if a given sequence form a G.P. we have to see if the ratio between successive terms are equal.

(i) 7, 14, 21, 28,...

$$\frac{t_2}{t_1} = \frac{14}{7} = 2; \quad \frac{t_3}{t_2} = \frac{21}{14} = \frac{3}{2};$$

$$\frac{t_4}{t_3} = \frac{28}{21} = \frac{4}{3}$$

Since the ratios between successive terms are not equal, the sequence 7, 14, 21. 28,... is not a Geometric Progression.

(ii) $\frac{1}{2}, 1, 2, 4, \ldots$

$$\frac{t_2}{t_1} = \frac{1}{\frac{1}{2}} = 2; \quad \frac{t_3}{t_2} = \frac{2}{1} = 2; \quad \frac{t_4}{t_3} = \frac{4}{2} = 2$$

Here the ratios between successive terms are equal. Therefore the sequence $\frac{1}{2}, 1, 2, 4, \ldots$ is a Geometric Progression with common ratio r = 2.

(iii) 5, 25, 50, 75,...

$$\frac{t_2}{t_1} = \frac{25}{5} = 5; \quad \frac{t_3}{t_2} = \frac{50}{25} = 2; \quad \frac{t_4}{t_3} = \frac{75}{50} = \frac{3}{2}$$

Since the ratios between successive terms are not equal, the sequence 5, 25, 50, 75,... is not a Geometric Progression.

2.41 Find the geometric progression whose first term and common ratios are given by (i) a = – 7, r = 6 (ii) a = 256, r = 0.5

Sol :

(i) The general form of Geometric progression is a, ar, ar^2,...

$a = -7$, $ar = -7 \times 6 = -42$, $ar^2 = -7 \times 6^2 = -252$

Therefore the required Geometric Progression is – 7, – 42, – 252,...

(ii) The general form of Geometric progression is a, ar, ar^2,...

$a = 256$, $ar = 256 \times 0.5 = 128$, $ar^2 = 256 \times (0.5)^2 = 64$

Therefore the required Geometric progression is 256, 128, 64,...

2.42 Find the 8th term of the G.P. 9, 3, 1, ...

Sol :

To find the 8th term we have to use the nth term formula $t_n = ar^{n-1}$

First term a = 9, Common ratio $r = \frac{t_2}{t_1} = \frac{3}{9} = \frac{1}{3}$

$$t_8 = 9 \times \left(\frac{1}{3}\right)^{8-1} = 9 \times \left(\frac{1}{3}\right)^7 = \frac{1}{243}$$

Therefore the 8th term of the G.P. is $\frac{1}{243}$.

2.43 In a Geometric progression, the 4th term is $\frac{8}{9}$ and the 7th term is $\frac{64}{243}$. Find the Geometric Progression.

Sol :

4th term, $t_4 = \frac{8}{9} \Rightarrow ar^3 = \frac{8}{9}$ ...(1)

7th term $t_7 = \frac{64}{243} \Rightarrow ar^6 = \frac{64}{243}$ ...(2)

Dividing (2) by (1) we get, $\frac{ar^6}{ar^3} = \frac{\frac{64}{243}}{\frac{8}{9}}$

$\Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$

Substituting the value of r in (1) we get

$a \times \left[\frac{2}{3}\right]^3 = \frac{8}{9} \Rightarrow a = 3$

Therefore the Geometric Progression is

$a, ar, ar^2, \ldots$ That is $3, 2, \frac{4}{3}, \ldots$

2.44 The product of three consecutive terms of a Geometric Progression is 343 and their sum is $\frac{91}{3}$. Find the three terms.

Sol :

Since the product of 3 consecutive terms is given.

We can take them as $\frac{a}{r}, a, ar$.

Product of the terms = 343

$\frac{a}{r} \times a \times ar = 343$

$a^3 = 7^3 \Rightarrow a = 7$

Sum of the terms $= \frac{91}{3}$

Hence $a\left(\frac{1}{r} + 1 + r\right) = \frac{91}{3} \Rightarrow 7\left(\frac{1+r+r^2}{r}\right) = \frac{91}{3}$

$3 + 3r + 3r^2 = 13r \Rightarrow 3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0 \Rightarrow r = 3$ or $r = \frac{1}{3}$

If $a = 7, r = 3$ then the three terms are $\frac{7}{3}, 7, 21$.

If $a = 7, r = \frac{1}{3}$ then the three terms are $21, 7, \frac{7}{3}$.

2.45 The present value of a machine is ₹ 40,000 and its value depreciates each year by 10% . Find the estimated value of the machine in 6th year.

The value of the machine at present is ₹ 40,000.

Since it is depreciated at the rate of 10% after one year the value of the machine is 90% of the initial value.

That is value of the machine at the end of the first year is $40{,}000 \times \frac{90}{100}$

After two years, the value of the machine is 90% of the value in the first year.

Value of the machine at the end of the 2nd year is $40{,}000 \times \left(\frac{90}{100}\right)^2$

Continuing this way, the value of the machine depreciates as

$40000, 40000 \times \frac{90}{100}, 40000 \times \left(\frac{90}{100}\right)^2 \ldots$

This sequence is in the form of G.P. with first term 40,000 and common ratio $\frac{90}{100}$. For finding the value of the machine at the end of 5th year (i.e., in 6th year)

We need to find the sixth term of this G.P.

Thus, $n = 6, a = 40{,}000, r = \frac{90}{100}$.

Using $t_n = ar^{n-1}$, we have $t_6 = 40{,}000 \times \left(\frac{90}{100}\right)^{6-1}$

$= 40000 \times \left(\frac{90}{100}\right)^5$

$t_6 = 40000 \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} = 23619.6$

Therefore the value of the machine in 6th year = ₹ 23619.60

Progress Check

1. A G.P. is obtained by multiplying _____ to the preceding term.

Ans : a constant or a fixed non-zero number.

2. The ratio between any two consecutive terms of the G.P. is ____ and it is called _____.

Ans : constant, common ratio

3. Fill in the blanks if the following are in G.P.

(i) $\frac{1}{8}, \frac{3}{4}, \frac{9}{2}, \_\_$ (ii) $7, \frac{7}{2}, \_\_\_$

(iii) $\_\_\_, 2\sqrt{2}, 4, \ldots$

Ans : (i) 27, (ii) $\frac{7}{4}$, (iii) 2

4. If first term = a, common ratio = r, t_9 = _____, t_{27} = _____

Ans : ar^8, ar^{26}

5. In a G.P. if $t_1 = \frac{1}{5}$ and $t_2 = \frac{1}{25}$ then the common ratio is ____

Ans : $\frac{1}{5}$

6. Three non-zero numbers a, b, c are in G.P. if and only if _____

Ans : $b^2 = ac$

Thinking Corner

1. Is the sequence $2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$ is a G.P.?

Ans : $\frac{t_2}{t_1} = \frac{2^2}{2} = \frac{2\times 2}{2} = 2$

$$\frac{t_3}{t_2} = \frac{2^{2^2}}{2^2} = \frac{2^2 \times 2^2}{2^2} = 2^2 = 4$$

$$\frac{t_4}{t_3} = \frac{2^{2^{2^2}}}{2^{2^2}} = \frac{2^{2^2} \times 2^{2^2}}{2^{2^2}} = 2^{2^2} = 16$$

$$\frac{t_2}{t_1} \neq \frac{t_3}{t_2} \neq \frac{t_4}{t_3}$$

$\therefore$ The sequence is not in G.P.

2. Split 64 into three parts such that the numbers are in G.P.

Ans :

Let the three numbers be ar^{n-1}, a, ar

Let a = 4 and r = 4

The three numbers are 1, 4, 16.

3. If a, b, c,… are in G.P. then 2a, 2b, 2c, … are in ____

Ans : G.P.

4. If 3, x, 6.75 are in G.P. then x is _____

Ans : 4.5

Exercise 2.7

1. Which of the following sequences are in G.P.?

(i) 3, 9, 27, 81,…

(ii) 4, 44, 444, 4444,

(iii) 0.5, 0.05, 0.005,… **(iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$,**

(v) 1, –5, 25, –125,… **(vi) 120, 60, 30, 18…**

(vii) 16, 4, 1 $\frac{1}{4}$,…

Sol :

(i) 3, 9, 27, 81,...

Here $\frac{t_2}{t_1} = \frac{9}{3} = 3$

$$\frac{t_3}{t_2} = \frac{27}{9} = 3$$

$$\frac{t_4}{t_3} = \frac{81}{27} = 3$$

The ratios between successive terms are equal.

$\therefore$ The sequence 3, 9, 27, 81,... are in G.P.

(ii) 4, 44, 444, 4444

$$\frac{t_2}{t_1} = \frac{44}{4} = 11$$

$$\frac{t_3}{t_2} = \frac{444}{44} = \frac{111}{11}$$

$$\frac{t_4}{t_3} = \frac{4444}{444} = \frac{1111}{111}$$

The ratios between the successive terms are not equal. Therefore the sequence 4, 44, 444,... are not in G.P.

(iii) 0.5, 0.05, 0.005,...

$$\frac{t_2}{t_1} = \frac{0.05}{0.5} = \frac{0.5}{5} = \frac{5}{50} = \frac{1}{10}$$

$$\frac{t_3}{t_2} = \frac{0.005}{0.05} = \frac{5}{50} = \frac{1}{10}$$

The ratios between the successive terms are equal. $\therefore$ 0.5, 0.05, 0.005,... are in G.P.

(iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$

$$\frac{t_2}{t_1} = \frac{1/6}{1/3} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{1/12}{1/6} = \frac{1}{12} \times \frac{6}{1} = \frac{1}{2}$$

The ratios between successive terms are equal

$\therefore$ $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$ are in G.P.

(v) 1, – 5, 25, – 125,...

$$\frac{t_2}{t_1} = \frac{-5}{1} = -5$$

$$\frac{t_3}{t_2} = \frac{25}{-5} = -5$$

$$\frac{t_4}{t_3} = \frac{-125}{25} = -5$$

The ratios between successive terms are equal. Therefore, 1, – 5, 25, – 125 ...... are in G.P.

(vi) 120, 60, 30, 18, ...

$$\frac{t_2}{t_1} = \frac{60}{120} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{30}{60} = \frac{1}{2}$$

$$\frac{t_4}{t_3} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$$

The ratios between successive terms are not equal. Therefore, 120, 60, 30, 18,... are not a G.P.

(vii) $16, 4, 1, \frac{1}{4}, \ldots$

$$\frac{t_2}{t_1} = \frac{4}{16} = \frac{1}{4}$$

$$\frac{t_3}{t_2} = \frac{1}{4}$$

$$\frac{t_4}{t_3} = \frac{1/4}{1} = \frac{1}{4}$$

The ratios between successive terms are equal.

$\therefore$ $16, 4, 1, \frac{1}{4}, \ldots$ in G.P.

2. Write the first three terms of the G.P. whose first term and the common ratio are given below.

(i) a = 6, r = 3 **(ii)** $a=\sqrt{2}, r=\sqrt{2}$

(iii) $a=1000, r=\frac{2}{5}$

Sol :

(i) a = 6, r = 3

The three terms of G.P. are a, ar, ar^2

$\Rightarrow$ $6, 6\times 3, 6(3)^2$

$\Rightarrow$ 6, 18, 54

$\therefore$ First three terms are 6, 18, 54.

(ii) $a=\sqrt{2}, r=\sqrt{2}$

Let the three terms of G.P. are a, ar, ar^2

$\sqrt{2}, \sqrt{2}\sqrt{2}, \sqrt{2}(\sqrt{2})^2$

$\sqrt{2}, 2, 2\sqrt{2}$

$\therefore$ First three terms of the G.P. $\sqrt{2}, 2, 2\sqrt{2}$

(iii) $a=1000 \quad r=\frac{2}{5}$

Let the three terms of G.P. are a, ar, ar^2

$$1000, 1000\left(\frac{2}{5}\right), 1000\left(\frac{2}{5}\right)^2 \Rightarrow 1000, 400, 160$$

$\therefore$ First three terms of the G.P. are 1000, 400, 160.

3. In a G.P. 729, 243, 81,... find t_7.

Sol : n^{th} term of G.P. = ar^{n-1}

Here a = 729

$$r = \frac{t_2}{t_1} = \frac{243}{729}$$

$$r = \frac{1}{3}$$

$$t_7 = ar^{7-1} = ar^6 = 729\left(\frac{1}{3}\right)^6$$

$$= 729\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}$$

$$t_7 = 1$$

4. Find x so that x + 6, x + 12 and x + 15 are consecutive terms of a Geometric progression.

Sol :

If the given numbers are consecutive terms of a G.P. then

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$

i.e., $$\frac{x+12}{x+6} = \frac{x+15}{x+12}$$

$$(x+12)^2 = (x+6)(x+15)$$

$$x^2 + 24x + 144 = x^2 + 6x + 15x + 90$$

$$x^2 + 24x + 144 - x^2 - 6x - 15x - 90 = 0$$

$$24x - 21x + 144 - 90 = 0$$

$$3x + 54 = 0$$

$$3x = -54$$

$$x = \frac{-54}{3} = -18$$

5. Find the number of terms in the following G.P.

(i) 4, 8, 16, ..., 8192

(ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{2187}$

Sol :

(i) 4, 8, 16, ...8192

$$a = 4, r = \frac{8}{4} = 2$$

n^{th} term of a G.P. $t_n = ar^{n-1}$

$$8192 = 4r^{n-1}$$

$$r^{n-1} = \frac{8192}{4}$$
$$r^{n-1} = 2048 = 2^{11}$$
$$n - 1 = 11 \quad [\because r = 2]$$
$$n = 11 + 1 = 12$$

2	2048
2	1024
2	512
2	256
2	128
2	64
2	32
2	16
2	8
2	4
	2

$\therefore$ Number of terms in the given G.P. is 12.

(ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{2187}$

Here $a = \frac{1}{3}$, $r = \frac{t_2}{t_1} = \frac{1/9}{1/3} = \frac{1}{9} \times \frac{3}{1}$

$$r = \frac{1}{3}$$

n^{th} term of the G.P. $\quad t_n = ar^{n-1}$

$$\frac{1}{2187} = \frac{1}{3} r^{n-1}$$
$$\frac{3}{2187} = r^{n-1}$$
$$\frac{1}{729} = r^{n-1}$$
$$\left(\frac{1}{3}\right)^6 = r^{n-1}$$

$n - 1 = 6 \quad \left[\because \ r = \frac{1}{3}\right]$

$n = 6 + 1 = 7$

$\therefore$ The number of terms in this G.P. is 7.

6 In a G.P. the 9th term is 32805 and 6th term is 1215. Find the 12th term.

Sol :

9^{th} term $= ar^{9-1} = 32805 \Rightarrow ar^8 = 32805 \quad \ldots(1)$

6^{th} term $= ar^{6-1} = 1215 \Rightarrow ar^5 = 1215 \quad \ldots(2)$

$$\frac{(1)}{(2)} \Rightarrow \frac{ar^8}{ar^5} = \frac{32805}{1215}$$
$$r^3 = 27$$
$$r^3 = 3^3$$
$$r = 3$$

Put $\quad r = 3$ in (2)

$$ar^5 = 1215$$
$$a(3^5) = 1215$$
$$a = \frac{1215}{3^5}$$
$$a = 5$$

$\therefore \quad 12^{th}$ term $t_{12} = ar^{12-1}$

$$= 5(3)^{11} = 5 \times 1{,}53{,}147$$

12th term of the G.P. = 7,65,735

7 Find the 10th term of a G.P. whose 8th term is 768 and the common ratio is 2.

Sol :

n^{th} term of a G.P. $t_n = ar^{n-1}$

Given 8^{th} term $t_8 = 768$ and $r = 2$

$ar^{8-1} = 768$

$a(2)^7 = 768$

$$a = \frac{768}{2^7} = \frac{2^7 \times 2^1 \times 3}{2^7} = 6$$

2	768
2	384
2	192
2	96
2	48
2	24
2	12
2	6
	3

10^{th} term $t_{10} = ar^{10-1} = ar^9$

$$= 6 \times 2^9 = 6 \times 512 = 3072$$

10^{th} term of the G.P. is 3072.

8 If a, b, c are in A.P. then show that 3^a, 3^b, 3^c are in G.P.

Sol : Given a, b, c are in A.P.

$$\therefore t_2 - t_1 = t_3 - t_2$$
$$b - a = c - b$$
$$b + b = c + a$$
$$2b = a + c$$

If we multiply both the sides by same number value will not change.

$$\therefore \quad 3^{2b} = 3^{a+c}$$
$$3^{b+b} = 3^{a+c}$$
$$3^b . 3^b = 3^a . 3^c$$
$$\frac{3^b}{3^a} = \frac{3^c}{3^b}$$

Thus 3^a, 3^b, 3^c are in G.P.

9. In a G.P. the product of three consecutive terms is 27 and the sum of product of terms taken two at a time is $\frac{57}{2}$. Find the three terms.

Sol :

Let the three terms be ar^{-1}, a, ar

$$\text{Given product} = 27$$
$$ar^{-1} \times a \times ar = 27$$
$$a^3 = 27$$
$$a^3 = 3^3$$
$$a = 3$$

Sum of the product taken two at a time $= \frac{57}{2}$

$$(ar^{-1} \times a) + (a \times ar) + (ar^{-1} \times ar) = \frac{57}{2}$$

$$a^2r^{-1} + a^2r + a^2 = \frac{57}{2}$$

$$a^2\left(\frac{1}{r} + r + 1\right) = \frac{57}{2}$$

$$3^2\left(\frac{1+r^2+r}{r}\right) = \frac{57}{2}$$

$$\frac{1+r^2+r}{r} = \frac{57}{2\times3\times3}$$

$$\frac{1+r^2+r}{r} = \frac{19}{6}$$

$$6 + 6r^2 + 6r = 19r$$

$$6r^2 + 6r - 19r + 6 = 0$$

$$6r^2 - 13r + 6 = 0$$

$$6r^2 - 9r - 4r + 6 = 0$$

$$3r(2r-3) - 2(2r-3) = 0$$

$$(2r-3)(3r-2) = 0$$

$$2r - 3 = 0 \text{ (or) } 3r - 2 = 0$$

$$2r = 3 \text{ (or) } 3r = 2$$

$$r = \frac{3}{2} \text{ (or) } = \frac{2}{3}$$

If $r = \frac{3}{2}$, the terms are $\frac{3}{3/2}, 3, 3\left(\frac{3}{2}\right) = 2, 3, \frac{9}{2}$

If $r = \frac{2}{3}$ the terms are $\frac{3}{2/3}, 3, 3\left(\frac{2}{3}\right) = \frac{9}{2}, 3, 2$

$\therefore$ The three terms of G.P. are $\frac{9}{2}, 3, 2$

10. A man joined a company as Assistant Manager. The company gave him a starting salary of ₹ 60,000 and agreed to increase his salary 5% annually. What will be his salary after 5 years?

Sol :

Initial salary for the first year = ₹ 60000

At the end of first year salary = $60000 \times \left(\frac{105}{100}\right)$

At the end of second year salary

$$= (60000)\left(\frac{105}{100}\right)^2$$

Continuing this way the salary increase will be like $60000, 60000\left(\frac{105}{100}\right), 60000\left(\frac{105}{100}\right)^2 \ldots$

This form a G.P. with $a = 60000$ and $r = \frac{105}{100}$

Salary after 5 years (6^{th} year) $= ar^{n-1} = ar^{6-1} = ar^5$

$$= 60000\left(\frac{105}{100}\right)^5$$

$$= 60000 \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100}$$

$\therefore$ After 5 years his salary = ₹ 76577

11. Sivamani is attending an interview for a job and the company gave two offers to him.
Offer A : ₹ 20,000 to start with followed by a guaranteed annual increase of 6% for the first 5 years.
Offer B : ₹ 22,000 to start with followed by a guaranteed annual increase of 3% for the first 5 years. What is his salary in the 4^{th} year with respect to the offers A and B?

Sol : Offer A:

Starting Salary = ₹ 20,000

Increase in salary per year = 6% = $\frac{6}{100}$

$\therefore$ The salary after first year = $20,000 \times \frac{106}{100}$

$\therefore$ Salary after second year = $20000 \times \left(\frac{106}{100}\right)^2$

Continuing like this the salary will be in G.P.

with a = ₹ 20000, $r = \frac{106}{100}$

$\therefore$ In the fourth year salary $= t_n = ar^{n-1}$

$$t_4 = ar^{4-1} = ar^3$$

$$= 20000 \times \left(\frac{106}{100}\right)^3 = 23820$$

$\therefore$ Salary in the fourth year = ₹ 23, 820 by offer A.

Offer B:

Starting salary = ₹ 22000

Annual Increment = 3% = $\frac{3}{100}$

$\therefore$ Salary at the end of first year = $22000 \times \frac{103}{100}$

Salary at second year = $22000\left(\frac{103}{100}\right)^2$

Continuing like this we get a G.P. with

a = ₹ 22000, $r = \frac{103}{100}$

Salary at fourth year $= ar^{n-1}$

$$= 22000\left(\frac{103}{100}\right)^{4-1} = 22000\left(\frac{103}{100}\right)^{3}$$

Salary at 4th year = ₹ 24040 by offer B.

12. If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. then prove that $x^{b-c} \times y^{c-a} \times z^{a-b} = 1$.

Sol : Given a, b, c are in A.P.

$$b - a = c - b$$
$$b + b = c + a$$
$$2b = c + a \quad ...(1)$$

Given x, y, z are three consecutive terms of a G.P.

$$\therefore x = a$$
$$y = ar$$
$$z = ar^2$$

Now $x^{b-c} \times y^{c-a} \times y^{a-b} = a^{b-c} \times (ar)^{c-a} \times (ar^2)^{a-b}$

$$= a^{b-c} \times a^{c-a} \times r^{c-a} \times a^{a-b}\, r^{2(a-b)}$$
$$= a^{b-c+c-a+a-b}\; r^{c-a+2a-2b}$$
$$= a^0\; r^{(a+c)-2b}$$
$$= 1.r^{2b-2b} \text{ [from (1) } a + c = 2b]$$
$$= 1.\, r^0$$
$$= 1 \times 1 = 1$$
$$= \text{RHS}$$

$\therefore x^{b-c} \times y^{c-a} \times z^{a-b} = 1$

Hence proved.

SUM TO n TERMS OF A G.P.

Key Points

1. A series whose terms are in Geometric progression is called Geometric series.
2. Sum upto n terms of a G.P. $S_n = \dfrac{a(r^n - 1)}{r-1}, r \neq 1$. $S_n = na$ if $r = 1$.
3. The sum of infinite G.P. is given by $a + ar + ar^2 + ar^3 + ... = \dfrac{a}{1-r}, -1 < r < 1$.

Worked Examples

2.46 Find the sum of 8 terms of the G.P. 1, –3, 9, – 27...

Sol : Here the first term a = 1,

common ratio $r = \dfrac{-3}{1} = -3 < 1$, Here n = 8.

Sum to n terms of a G.P. is

$$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{if } r \neq 1$$

Hence. $S_8 = \dfrac{1\left((-3)^8 - 1\right)}{(-3)-1} = \dfrac{6561-1}{-4} = -1640$

2.47 Find the first term of a G.P. in which $S_6 = 4095$ and r = 4?

Sol :

Common ratio = 4 >1, Sum of first 6 terms

$$S_6 = 4095$$

Hence, $S_6 = \dfrac{a(r^n - 1)}{r-1} = 4095$

Since, r = 4, $\dfrac{a(4^6 - 1)}{4-1} = 4095$

$$\Rightarrow \frac{a(4096-1)}{3} = 4095 \Rightarrow a = 3$$

2.48 How many terms of the series 1 + 4 + 16 + ... make the sum 1365?

Sol : Let n be the number of terms to be added to get the sum 1365.

$$a = 1, r = \frac{4}{1} = 4 > 1$$

$$S_n = 1365 \Rightarrow \frac{a(r^n - 1)}{r-1} = 1365$$

$$\frac{1(4^n - 1)}{4-1} = 1365 \Rightarrow (4^n - 1) = 4095$$

$$4^n = 4096 \Rightarrow 4^n = 4^6$$
$$n = 6$$

2.49 Find the sum $3 + 1 + \frac{1}{3} + ... \infty$

Sol :

Here $a = 3, r = \dfrac{t_2}{t_1} = \dfrac{1}{3}$

Sum of infinite terms $= \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{3}} = \dfrac{9}{2}$

2.50 Find the rational form of the number 0.6666...

Sol :

We can express the number 0.6666... as follows

0.6666...= 0.6 + 0.06 + 0.006 + 0.0006 + ...

We now see that numbers 0.6, 0.06, 0.006... form an G.P. whose first term a = 0.6

and common ratio $r = \frac{0.06}{0.6} = 0.1$.

Also $-1 < r = 0.1 < 1$.

Using the infinite G.P. formula,

we have $0.6666... = 0.6 + 0.06 + 0.006 + 0.0006 + ...$

$$= \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{2}{3}$$

Thus the rational number equivalent of

0.6666 ... is $\frac{2}{3}$

2.51 Find the sum to n terms of the series 5 + 55 + 555 + ...

Sol : The series is neither Arithmetic nor Geometric series. So it can be split into two series and then find the sum.

5 + 55 + 555 + ... + n terms = 5 [1 + 11 + 111 + ...+ n terms]

$$= \frac{5}{9}\left[9 + 99 + 999 + ... + n\ terms\right]$$

$$= \frac{5}{9}\left[(10-1) + (100-1) + (1000-1) + ... + n\ terms\right]$$

$$= \frac{5}{9}\left[(10 + 100 + 1000 + ... + n\ terms) - n\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{(10-1)} - n\right] = \frac{50(10^n-1)}{81} - \frac{5n}{9}$$

2.52 Find the least positive integer n such that $1 + 6 + 6^2 + ... + 6^n > 5000$.

Sol :

We want to find the least number of terms for which the sum must exceed 5000.

That is to find the least value of n such that $S_n > 5000$

We have $S_n = \frac{a(r^n-1)}{r-1} = \frac{1(6^n-1)}{6-1} = \frac{6^n-1}{5}$

$$S_n > 5000 \Rightarrow \frac{6^n-1}{5} > 5000$$

$6^n - 1 > 25000 \Rightarrow 6^n > 25001$

Since, $6^5 = 7776$ and $6^6 = 46656$

The least positive value of n is 6 such that $1 + 6 + 6^2 + ... + 6^n > 5000$.

2.53 A person saved money every year half as much as he could in the previous year. If he had totally saved ₹ 7875 in 6 years then how much did he save in the first year?

Sol :

Total amount saved in 6 years is $S_6 = 7875$

Since he saved half as much money as every year he saved in the previous year.

We have $r = \frac{1}{2} < 1$

Since, $$\frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 7875.$$

$$\frac{a\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = 7875 \Rightarrow a \times \frac{63}{32} = 7875$$

$$a = \frac{7875 \times 32}{63} \Rightarrow a = 4000$$

The amount saved in the first year is ₹ 4000.

Progress Check

1. **A series whose terms are in Geometric progression is called _____**

   Ans : Geometric series.

2. **When r = 1, the formula for finding sum to n terms of a G.P. is ______**

   Ans : $S_n = na$

3. **When $r \neq 1$, the formula for finding sum to n terms of a G.P. is _____**

   Ans : $S_n = \frac{a(r^n-1)}{r-1}$

4. **Sum to infinite number of terms of a G.P. is ______**

   Ans : $S_\infty = \frac{a}{1-r}, -1 < r < 1$

5. **For what values of r, does the formula for infinite G.P. is valid?**

   Ans : For $-1 < r < 1$, the formula is valid.

6. Is the series 3 + 33 + 333 + … a Geometric series?

Ans : In the series 3 + 33 + 333 + ...

$$\frac{t_2}{t_1} = \frac{33}{3} = 11$$

$$\frac{t_3}{t_2} = \frac{333}{33} = \frac{111}{11}$$

$$\frac{t_2}{t_1} \neq \frac{t_3}{t_2}$$

∴ It is not a Geometric series.

7. The value of r, such that $1 + r + r^2 + r^3 \ldots = \frac{3}{4}$ is

Ans : We have for an infinite series

$$S_\infty = \frac{a}{1-r} \quad -1<r<1$$

Here a = 1

$$\frac{3}{4} = \frac{1}{1-r}$$

$$3(1-r) = 4$$

$$3-3r = 4$$

$$-3r = 4-3$$

$$-3r = 1$$

$$r = \frac{-1}{3}$$

∴ The value of $r = \frac{-1}{3}, -1 < \frac{-1}{3} < 1$.

Exercise 2. 8

1. Find the sum of first n terms of the G.P

(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, \ldots$ (ii) 256, 64, 16,…

Sol :

(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, \ldots$

It is a geometric progression with

$a = 5, r = \frac{-3}{5} \neq 1$

Sum upto n terms $S_n = \frac{a(r^n-1)}{r-1}$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-3}{5}-1}$$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-3-5}{5}}$$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-8}{5}}$$

$$= \frac{-25}{8}\left[\left(\frac{-3}{5}\right)^n - 1\right]$$

$$= \frac{25}{8}\left[1-\left(\frac{-3}{5}\right)^n\right]$$

(ii) 256, 64, 16,....

Here a = 256, $r = \frac{64}{256} = \frac{1}{4} \neq 1$

Sum upto n terms

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$= \frac{256\left[\left(\frac{1}{4}\right)^n - 1\right]}{\frac{1}{4}-1} = \frac{256\left[\left(\frac{1}{4}\right)^n - 1\right]}{-\frac{3}{4}}$$

$$= \frac{-1024}{3}\left[\left(\frac{1}{4}\right)^n - 1\right] = \frac{1024}{3}\left[1-\left(\frac{1}{4}\right)^n\right]$$

2. Find the sum of first six terms of the G.P. 5, 15, 45, …

Sol :

We have to find 5 + 15 + 45 + ...upto 6 terms.

Sum upto n terms of a G.P. $S_n = \frac{a(r^n-1)}{r-1}$

Here a = 5, $r = \frac{15}{5} = 3$

$$\therefore \text{Sum} = \frac{5(3^6-1)}{3-1} = \frac{5(729-1)}{2}$$

$$= \frac{5\times 728}{2} = 5\times 364$$

5 + 15 + 45 + ...upto 6 terms = 1820

3. Find the first term of the G.P. whose common ratio 5 and whose sum to first 6 terms is 46872.

Sol :

Given r = 5 and $S_6 = 46872$

Sum upto n terms of a G.P., $S_n = \dfrac{a(r^n-1)}{r-1}$

$$46872 = \frac{a(5^6-1)}{5-1}$$

$$46872 = a\frac{(15625-1)}{4} = a \times \frac{15624}{4}$$

$$46872 = a \times 3906$$

$$\frac{46872}{3906} = a$$

$$a = 12$$

$\therefore$ First term of the G.P., a = 12

4. Find the sum to infinity of (i) 9 + 3 + 1 + ... (ii) $21+14+\frac{28}{3}+...$

Sol :

(i) $9 + 3 + 1 + ...$

Here $\frac{3}{9} = \frac{1}{3}$

$\therefore$ It is a geometric series with

$a = 9$ and $r = \frac{1}{3}, -1 < r < 1$

$$\therefore \text{Sum} = \frac{a}{1-r} = \frac{9}{1-\frac{1}{3}}$$

$$= \frac{9}{\cancel{2}/3} = 9 \times \frac{3}{2}$$

$$\therefore 9+3+1+.... = \frac{27}{2} = 13.5$$

(ii) $21+14+\frac{28}{3}+...$

Here $\frac{14}{21} = \frac{2}{3} = \frac{t_2}{t_1}$

$$\frac{t_3}{t_2} = \frac{28/3}{14} = \frac{28}{3} \times \frac{1}{14} = \frac{2}{3}$$

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$

It forms a geometric series with

$a = 21, \quad r = \frac{2}{3}, -1 < r < 1.$

$$\text{Sum} = \frac{a}{1-r} = \frac{21}{1-\frac{2}{3}}$$

$$= \frac{21}{1/3} = 21 \times \frac{3}{1}$$

$$21 + 14 + \frac{28}{3} + ... = 63.$$

5. If the first term of an infinite G.P. is 8 and its sum to infinity is $\frac{32}{3}$ then find the common ratio.

Sol : First term a = 8

$$S_\infty = \frac{32}{3}$$

Sum upto infinity of a G.P.,

$$S_\infty = \frac{a}{1-r}$$

$$\frac{32}{3} = \frac{8}{1-r}$$

$$32(1-r) = 24$$

$$1-r = \frac{24}{32}$$

$$1-r = \frac{3}{4}$$

$$1-\frac{3}{4} = r$$

$$\frac{1}{4} = r$$

Common ratio $r = \frac{1}{4}$

6. Find the sum to n terms of the series

(i) 0.4 + 0.44 + 0.444 + … to n terms

(ii) 3 + 33 + 333 + … to n terms

Sol :

(i) 0.4 + 0.44 + 0.444 + ... to n terms

Let S_n = 0.4 + 0.44 + 0.444 + ... to n terms.

= 4 [0.1 + 0.11 + 0.111 + ... to n terms]

(multiply and divide by 9)

$$= \frac{4}{9}[0.9 + 0.99 + 0.999 + ... \text{ to n terms}]$$

$$= \frac{4}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + ...\textit{to n terms}\right]$$

$$= \frac{4}{9}\left[\frac{10-1}{10} + \frac{100-1}{100} + \frac{1000-1}{100} + ...\textit{n terms}\right]$$

$$=\frac{4}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots n\ terms\right]$$

$$=\frac{4}{9}\left[n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots up\ to\ n\ terms\right)\right]$$

$$=\frac{4}{9}\left[n-\left[\frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right]\right]$$

$$=\frac{4}{9}\left[n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{\frac{9}{10}}\right]$$

$$=\frac{4}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]=\left[\frac{4n}{9}-\frac{4}{81}\left(1-\frac{1}{10^n}\right)\right]$$

$$0.4+0.44+0.444\ldots=\left[\frac{4n}{9}-\frac{4}{81}\left(1-\frac{1}{10^n}\right)\right]$$

$$=\frac{4}{9}n-\frac{4\left(1-\left(\frac{1}{10}\right)^n\right)}{81}$$

(ii) 3 + 33 + 333 + ... to n terms.

Let S_n =3 + 33 + 333 + ...upto n terms.

=3 [1 + 11 + 111 + upto n terms]

$=\frac{3}{9}$ [9 + 99 + 999 + ... upto n terms] (multiply and divide by 9)

$=\frac{1}{3}$ [(10 – 1) + (100 – 1) + (1000 – 1) + ... n terms]

$=\frac{1}{3}$ [(10 – 1) + (10^2 – 1) + (10^3 – 1) + ... upto n terms]

$=\frac{1}{3}$ {[10 + 10^2 + 10^3 + ...upto n terms] – n}

10 + 10^2 + ... is a G.P. with a = 10, r = 10.

$$\therefore\ S_n=\frac{a(r^n-1)}{r-1}$$

$$S_n=\frac{1}{3}\left\{\left[\frac{10(10^n-1)}{10-1}\right]-n\right\}$$

$$=\frac{1}{3}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$=\frac{10}{27}(10^n-1)-\frac{n}{3}$$

$$3+33+333+\ldots \text{ to n terms}=\frac{10}{27}(10^n-1)-\frac{n}{3}$$

7. Find the sum of the Geometric series 3 + 6 + 12 + … + 1536.

Sol : a = 3, r = $\frac{6}{3}$ = 2

n^{th} term of the G.P., $t_n = ar^{n-1}$

$1536 = 3(2)^{n-1}$

$\frac{1536}{3} = 2^{n-1}$

$512 = 2^{n-1}$

$2^9 = 2^{n-1}$

n – 1 = 9

n = 9 + 1 = 10

∴ Number of terms in the series n = 10.

∴ Sum upto n terms $S_n = \frac{a(r^n-1)}{r-1}$

$$S_{10}=\frac{3(2^{10}-1)}{2-1}$$

$$=\frac{3\times(1024-1)}{1}=3\times1023$$

3 + 6 + 12 + ..+ 1536 = 3069.

8. Kumar writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they continue the process similarly. Assuming that the process is unaltered and it costs ₹ 2 to mail one letter, find the amount spent on postage when 8^{th} set of letters is mailed.

Sol : Total letters in the first set = 4

Total letters in the second set = 4^2 = 16

Total letters in the third set = 4^3 = 64

So the sequence of letters is 4, 16, 64,....

Here a = 4, r = $\frac{16}{4}$ = 4 and n = 8

$$S_n=\frac{a(r^n-1)}{r-1}=\frac{4(4^8-1)}{4-1}$$

$$=\frac{4}{3}(65536-1)=\frac{4}{3}\times65535$$

= 87380

Since amount of postage per letter is ₹ 2

Total amount spend on postage = 87380 × 2

= ₹ 174760.

9. Find the rational form of the number $0.\overline{123}$.

Sol : We have $0.\overline{123} = 0.123123123....$

$= 0.123 + 0.000123 + 0.000000123 + ...$

$$\frac{t_2}{t_1} = \frac{0.000123}{0.123} = \frac{0.123}{123} = 0.001$$

$\therefore$ It is a G.P. with $a = 0.123$ and $r = 0.001$

$$\text{Sum of infinity} = \frac{a}{1-r}$$

$$= \frac{0.123}{1-0.001} = \frac{0.123}{1-\frac{1}{1000}}$$

$$= \frac{0.123}{\frac{999}{1000}} = \frac{123}{999}$$

$$\therefore 0.\overline{123} = \frac{41}{333}$$

10. If $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$n terms then prove that

$$(x-y)\,S_n = \left[\frac{x^2\,(x^n-1)}{x-1} - \frac{y^2\,(y^n-1)}{y-1}\right]$$

Sol : Given $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...$ n terms)

$$= \frac{x-y}{x-y}\,[(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...\text{n terms})]$$

[multiply and divide by $(x - y)$]

$$= \frac{1}{x-y}\left[(x-y)(x+y) + (x-y)(x^2+xy+y^2) + (x-y)(x^3+x^2y+xy^2+y^3+....n)\right]$$

$$= \frac{1}{x-y}\,[(x^2 - y^2) + (x^3 - y^3) + (x^4 - y^4) + ...\text{n terms}]$$

$$= \frac{1}{x-y}\,[x^2 + x^3 + x^4 + ...\text{n terms}] - [y^2 + y^3 + y^4 + \ldots \text{n terms}]$$

$$S_n = \frac{1}{x-y}\left[\frac{x^2\,(x^n-1)}{x-1} - \frac{y^2\,(y^n-1)}{y-1}\right]$$

$$\left[\because S_n = \frac{a\,(r^n-1)}{r-1}\right]$$

$$(x-y)\,S_n = \left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2\,(y^n-1)}{y-1}\right]$$

SPECIAL SERIES

Key Points

1. Special series are some series whose sum can be expressed by explicit formula.
2. $1^k + 2^k + 3^k + \ldots + n^k = (x+1)^{k+1} - x^{k+1}$
3. $1 + 2 + 3 + \ldots + n = \dfrac{n\,(n+1)}{2} = \displaystyle\sum_{k=1}^{n} k$
4. $1 + 3 + 5 + \ldots + (2n-1) = n^2 = \displaystyle\sum_{k=1}^{n} (2k-1)$
5. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6} = \displaystyle\sum_{k=1}^{n} k^2$
6. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\dfrac{n\,(n+1)}{2}\right]^2 = \displaystyle\sum_{k=1}^{n} k^3$
7. Sum of divisors of one number excluding itself gives the other number. Such numbers are called Amicable Numbers or Friendly Numbers. Example: 220 and 284.
8. The sum of first n natural numbers are called Triangular Numbers because they form triangle shapes.
9. The sum of squares of first n natural numbers are called Square Pyramidal Numbers because they form pyramid shapes with square base.

Worked Examples

2.54 **Find the value of (i) $1 + 2 + 3 + ... + 50$**
(ii) $16 + 17 + 18 + ... + 75$

Sol :

(i) $1+2+3+...+50$

Using $1+2+3+....+n = \frac{n(n+1)}{2}$

$$1+2+3+...+50 = \frac{50\times(50\times 1)}{2} = 1275$$

(ii) $16+17+18+...+75$

$$16+17+18+...+75 = (1+2+3+...+75) - (1+2+3+...+15)$$

$$= \frac{75(75+1)}{2} - \frac{15(15+1)}{2}$$

$$= 2850 - 120 = 2730$$

2.55 **Find the sum of (i) $1+3+5+...+$ to 40 terms**
(ii) $2 + 4 + 6 + ... + 80$
(iii) $1 + 3 + 5 + ... + 55$

Sol :

(i) $1 + 3 + 5 + ...$ 40 terms $= 40^2 = 1600$

$[1 + 3 + 5 + ... + (2n - 1)] = n^2$

(ii) $2 + 4 + 6 + ... + 80 = 2(1 + 2 + 3 + ... + 40)$

$$= 2\times\frac{40\times(40+1)}{2} = 1640$$

(iii) $1 + 3 + 5 + ... + 55$

Here the number of terms is not given. Now we have to find the number of terms using the formula, $n = \frac{(l-a)}{d} + 1 \Rightarrow n = \frac{(55-1)}{2} + 1 = 28$

Therefore, $1 + 3 + 5 + ... + 55 = (28)^2 = 784$

2.56 **Find the sum of (i) $1^2 + 2^2 + ... + 19^2$**
(ii) $5^2 + 10^2 + 15^2 + ... + 105^2$
(iii) $15^2 + 16^2 + 17^2 + ... + 28^2$

Sol :

(i) $1^2 + 2^2 + ... + 19^2 = \frac{19\times(19+1)(2\times 19+1)}{6}$

$$= \frac{19\times 20\times 39}{6} = 2470$$

(ii) $5^2 + 10^2 + 15^2 + ...+ 105^2 = 5^2(1^2 + 2^2 + 3^2 + ... + 21^2)$

$$= 25\times\frac{21\times(21+1)(2\times 21+1)}{6}$$

$$= \frac{25\times 21\times 22\times 43}{6} = 82775$$

(iii) $15^2 + 16^2 + 17^2 + ... + 28^2$

$= (1^2 + 2^2 + 3^2 + ... + 28^2) - (1^2 + 2^2 + 3^2 + ... + 14^2)$

$$= \frac{28\times 29\times 57}{6} - \frac{14\times 15\times 29}{6}$$

$$= 7714 - 1015 = 6699.$$

2.57 **Find the sum of (i) $1^3 + 2^3 + 3^3 + ... + 16^3$**
(ii) $9^3 + 10^3 + ... + 21^3$

Sol :

(i) $1^3 + 2^3 + 3^3 + ... + 16^3$

$$= \left[\frac{16\times(16+1)}{2}\right]^2 = (136)^2 = 18496$$

(ii) $9^3 + 10^3 + ... + 21^3 = (1^3 + 2^3 + 3^3 + ... + 21^3) - (1^3 + 2^3 + 3^3 + ... + 8^3)$

$$= \left[\frac{21\times(21+1)}{2}\right]^2 - \left[\frac{8\times(8+1)}{2}\right]^2$$

$$= (231)^2 - (36)^2 = 53361 - 1296 = 52065.$$

2.58 **If $1 + 2 + 3 + ... + n = 666$ then find n.**

Since, $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$,

we have $\frac{n(n+1)}{2} = 666$

$n^2 + n - 1332 = 0 \Rightarrow (n + 37)(n - 36) = 0$

$\Rightarrow n = -37$ or $n = 36$

But $n \neq -37$ (Since n is a natural number);

Hence $n = 36$.

Progress Check

1. The sum of cubes of first n natural numbers is ______ of the first n natural numbers.

Ans : Square of the sum

2. The average of first 100 natural numbers is ______

Ans :

Sum of first 100 natural numbers $= \frac{n(n+1)}{2}$

$$= \frac{100\times 101}{2} = 5050$$

$$\text{Average} = \frac{5050}{100} = 50.5$$

Say True or False

1. The sum of first n odd natural numbers is always an odd number.

Ans : False

2. The sum of consecutive even numbers is always an even number.

Ans : True

3. The difference between the sum of squares of first n natural numbers and the sum of first n natural numbers is always divisible by 2.

Ans : False

4. The sum of cubes of first n natural numbers is always a square number.

Ans : True

Thinking Corner

1. How many squares are there in a standard chess board?

Ans : 64

2. How many rectangles are there in a standard chess board?

Ans : 1296

Exercise 2. 9

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \dots + 60$

(ii) $3 + 6 + 9 + \dots + 96$

(iii) $51 + 52 + 53 + \dots + 92$

(iv) $1 + 4 + 9 + 16 + \dots + 225$

(v) $6^2 + 7^2 + 8^2 + \dots + 21^2$

(vi) $10^3 + 11^3 + 12^3 + \dots + 20^3$

(vii) $1 + 3 + 5 + \dots + 71.$

Sol :

(i) $1+2+3+ \dots +60$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$1+2+3+\dots+60 = \frac{60\times(60+1)}{2} = \frac{60\times 61}{2}$$

$$1+2+3+\dots+60 = 1830$$

(ii) $3+6+9+\dots+96.$

$$3+6+9+..+96 = 3\,(1+2+3+\dots+32)$$

We know that $1+2+3+\dots+n = \frac{n(n+1)}{2}$

$$\therefore\ 3+6+9+\dots+96 = 3\left[\frac{32\times(32+1)}{2}\right]$$

$$= 3\left[\frac{32\times 33}{2}\right] = 1584$$

$$3+6+9+\dots+96 = 1584$$

(iii) $51+52+53+\dots+92$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$51+52+53+\dots+92 = (1+2+3+\dots+92) - (1+2+\dots+50)$$

$$= \frac{92\times(92+1)}{2} - \frac{50\times(50+1)}{2}$$

$$= \frac{92\times 93}{2} - \frac{50\times 51}{2}$$

$$= 4278 - 1275 = 3003$$

$$51+52+53+\dots+92 = 3003$$

(iv) $1+4+9+16+\dots+225$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1+4+9+16+\dots+225 = 1^2+2^2+3^2+\dots+15^2$$

$$= \frac{15(15+1)[2(15)+1]}{6}$$

$$= \frac{15\times 16\times 31}{6}\ 1240$$

$$1+4+9+16+\dots+225 = 1240$$

(v) $6^2+7^2+8^2+\dots+21^2$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$6^2+7^2+8^2+\dots+21^2 = (1^2+2^2+3^2+\dots+21^2) - (1^2+2^2+\dots+5^2)$$

$$= \frac{21\times(21+1)[2(21)+1]}{6} - \frac{5\times 6\times 11}{6}$$

$$= \frac{21\times 22\times 43}{6} - 55 = 3311 - 55 = 3256$$

$$6^2+7^2+8^2\dots+21^2 = 3256$$

(vi) $10^3+11^3+12^3+\dots+20^3$

$$1^3+2^3+3^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= (1^3+2^3+3^3+\dots+20^3) - (1^3+2^3+\dots+9^3)$$

$$= \left[\frac{20(20+1)}{2}\right]^2 - \left[\frac{9\times(9+1)}{2}\right]^2$$

$= \left[\frac{20\times 21}{2}\right]^2 - \left[\frac{9\times 10}{2}\right]^2 = (210)^2 - (45)^2$

$= 44100 - 2025 = 42075$

$10^3 + 11^3 + 12^3 + ... + 20^3 = 42075$

(vii) $1 + 3 + 5 + .. + 71$

$a = 1, d = 3 - 1 = 2$

Number of terms $n = \frac{l-a}{d} + 1$

$= \frac{71-1}{2} + 1 = \frac{70}{2} + 1 = 36$

Sum of first n odd numbers $= n^2$

$1 + 3 + 5 + ... + 71 = (36)^2 = 1296$

2. If $1 + 2 + 3 + ... + k = 325$, then find $1^3 + 2^3 + 3^3 + ... + k^3$.

Sol :

Sum of first k natural numbers $= \frac{k(k+1)}{2} = 325$

Sum of cube of first k natural numbers

$= \left[\frac{k(k+1)}{2}\right]^2$

$= (325)^2 = 1,05,625$

$1^3 + 2^3 + 3^3 + ... + k^3 = 1,05,625$

3. If $1^3 + 2^3 + 3^3 + ... + k^3 = 44100$, then find $1 + 2 + 3 + ... + k$.

Sol :

$1^3 + 2^3 + 3^3 + ... + k^3 = \left[\frac{k(k+1)}{2}\right]^2 = 44100 = (210)^2$

$\therefore 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} = 210$

$1 + 2 + 3 + ... + k = 210$

4. How many terms of the series $1^3 + 2^3 + 3^3 + ...$ should be taken to get the sum 14400?

Sol :

$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$1^3 + 2^3 + 3^3 + ... + n^3 = (120)^2$

$\therefore \frac{n(n+1)}{2} = 120$

$n(n+1) = 120 \times 2$

$n(n+1) = 240$

$n^2 + n - 240 = 0$

$(n - 15)(n + 16) = 0$

$n = 15 \text{ (or) } (-16)$

Number of terms cannot be negative

$\therefore$ Number of terms to be taken $= 15$

5. The sum of the squares of the first n natural numbers is 285, while the sum of their cubes is 2025. Find the value of n.

Sol :

$1^2 + 2^2 + 3^2 + ... + n^2 = 285$

$\frac{n(n+1)(2n+1)}{6} = 285 \quad ...(1)$

$1^3 + 2^3 + 3^3 + ... + n^3 = 2025$

$\left[\frac{n(n+1)}{2}\right]^2 = 2025 = (45)^2 \Rightarrow \frac{n(n+1)}{2} = 45$

$(1) \Rightarrow \left[\frac{n(n+1)}{2}\right]\left[\frac{(2n+1)}{3}\right] = 285$

$45 \times \left(\frac{2n+1}{3}\right) = 285$

$2n + 1 = 285 / 15$

$2n + 1 = 19$

$2n = 19 - 1 = 18$

$n = \frac{18}{2} = 9$

6. Rekha has 15 square colour papers of sizes 10 cm, 11 cm, 12 cm, ..., 24 cm. How much area can be decorated with these colour papers?

Sol :

$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

With the square colour papers are decorated

$= 10^2 + 11^2 + 12^2 + ... + 24^2$

$10^2 + 11^2 + 12^2 + ... + 24^2 = (1^2 + 2^2 + 3^2 + ... + 24^2) - (1^2 + 2^2 + ... + 9^2)$

$= \frac{24\times(24+1)[2(24)+1]}{6} - \frac{9\times(9+1)[2(9)+1]}{6}$

$= (4 \times 25 \times 49) - \frac{9\times 10\times 19}{6}$

$= 4900 - 285 = 4615$

4615 cm^2 area can be decorated.

7. Find the sum of the series $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + ...$ to (i) n terms (ii) 8 terms

Sol :

(i) $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + \ldots$ n terms

General term of the given series $= (2n)^3 - (2n-1)^3$

$= 8n^3 - [(2n)^3 - 3(2n)^2(1) + 3(2n)(1) - 1^3]$

$= 8n^3 - [8n^3 - 12n^2 + 6n - 1]$

$= 8n^3 - 8n^3 + 12n^2 - 6n + 1$

$= 12n^2 - 6n + 1$

$= 12\left[\frac{n(n+1)(2n+1)}{6}\right] - 6\left[\frac{n(n+1)}{2}\right] + n$

$= 2n(n+1)(2n+1) - 3n(n+1) + n$

$= 2n(2n^2 + n + 2n + 1) - 3n^2 - 3n + n$

$= 4n^3 + 6n^2 + 2n - 3n^2 - 3n + n$

$= 4n^3 + 3n^2$

Hence the sum of n terms $= 4n^3 + 3n^2$

(ii) $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + \ldots$ 8 terms

Sum of n term $= 4n^3 + 3n^2$

Here $n = 8$

Sum $= 4(8)^3 + 3(8)^2$

$= 2048 + 192 = 2240$

Sum $= 2240$

Exercise 2. 10

Multiple choice question.

1. Euclid's division lemma states that for positive integers a and b, there exist unique integers q and r such that a = bq + r, where r must satisfy

(1) $1 < r < b$ (2) $0 < r < b$
(3) $0 \le r < b$ (4) $0 < r \le b$

[Ans : (3)]

2. Using Euclid's division lemma, if the cube of any positive integer is divided by 9 then the possible remainders are

(1) 0, 1, 8 (2) 1, 4, 8
(3) 0, 1, 3 (4) 1, 3, 5 **[Ans : (1)]**

3. If the H.C.F. of 65 and 117 is expressible in the form of 65 m – 117, then the value of m is

(1) 4 (2) 2
(3) 1 (4) 3 **[Ans : (2)]**

Sol :

H.C.F. of 65 and 117 is 13

$65\,m - 117 = 13$

$65\,m = 13 + 117 = 130$

$m = \frac{130}{65} = 2$

4. The sum of the exponents of the prime factors in the prime factorization of 1729 is

(1) 1 (2) 2
(3) 3 (4) 4 **[Ans : (3)]**

Sol : $1729 = 7^1 + 13^1 + 19^1$

Sum of powers $= 1 + 1 + 1 = 3$

5. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(1) 2025 (2) 5220
(3) 5025 (4) 2520 **[Ans : (4)]**

Sol : L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9 is 2520

6. $7^{4k} \equiv$ _____ (mod 100)

(1) 1 (2) 2
(3) 3 (4) 4 **[Ans : (1)]**

7. Given $F_1 = 1$, $F_2 = 3$ and $F_n = F_{n-1} + F_{n-2}$ then F_5 is

(1) 3 (2) 5
(3) 8 (4) 11 **[Ans : (4)]**

Sol :

$F_1 = 1, F_2 = 3, F_3 = 3 + 1 = 4, F_4 = 4 + 3 = 7,$
$F_5 = 7 + 4 = 11$

8. The first term of an arithmetic progression is unity and the common difference is 4. Which of the following will be a term of this A.P

(1) 4551 (2) 10091
(3) 7881 (4) 13531 **[Ans : (3)]**

Sol :

$a = 1, d = 4$

$t_n = a + (n-1)d = 1 + (n-1)4;$

The number should be multiple of 4 + 1 ;

$\therefore$ 7881.

9. If 6 times of 6^{th} term of an A.P. is equal to 7 times the 7^{th} term, then the 13^{th} term of the A.P. is

(1) 0 (2) 6
(3) 7 (4) 13 **[Ans : (1)]**

Sol :

$6t_6 = 7t_7$

$6[a + 5d] = 7[a + 6d]$

$6a + 30d = 7a + 42d$

$7a - 6a + 42d - 30d = 0$

$a + 12d = 0$

$a + (13-1)d = 0$

$t_{13} = 0$

10. An A.P. consists of 31 terms. If its 16^{th} term is m, then the sum of all the terms of this A.P. is

(1) 16 m (2) 62 m

(3) 31 m (4) $\frac{31}{2}m$ **[Ans : (3)]**

Sol : $n = 31, t_{16} = m$

$$a + 15d = m$$

$$s_{31} = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{31}{2}[a+a+30d] = \frac{31}{2}[a+15d+a+15d]$$

$$= \frac{31}{2}[2m] = 31\text{ m}$$

11. In an A.P., the first term is 1 and the common difference is 4. How many terms of the A.P. must be taken for their sum to be equal to 120?

(1) 6 (2) 7

(3) 8 (4) 9 **[Ans : (3)]**

Sol : $a = 1,\ d = 4$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2(1)+(n-1)4]$$

$$120 \times 2 = n[2 + 4(n-1)]$$

$$240 = n[2 + 4n - 4]$$

$$= n[4n - 2]$$

$$240 = 2n[2n - 1]$$

$$120 = n[2n - 1]$$

$$2n^2 - n - 120 = 0$$

$$2n^2 - 16n + 15n - 120 = 0$$

$$2n(n-8) + 15(n-8) = 0$$

$$(n-8)(2n+15) = 0$$

$$n = 8 \text{ or } n = \frac{-15}{2}$$

$$n = \frac{-15}{2} \text{ is not possible}$$

$$\therefore n = 8$$

12. If $A = 2^{65}$ and $B = 2^{64} + 2^{63} + 2^{62} + \dots + 2^0$ which of the following is true?

(1) B is 2^{64} more than A

(2) A and B are equal

(3) B is larger than A by 1

(4) A is larger than B by 1 **[Ans : (4)]**

Sol :

$$A = 2^{65}$$

$$B = 2^{64} + 2^{63} + 2^{62} + \dots + 2^0$$

$$\text{Where } a = 2^{64};\ r = \frac{2^{63}}{2^{64}} = \frac{1}{2};\ n = 65$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$B = \frac{2^{64}\left(\left(\frac{1}{2}\right)^{65} - 1\right)}{-\frac{1}{2}} = 2^{65}\left(\frac{1}{2^{65}} - 1\right)(-1)$$

$$= (1 - 2^{65})(-1)$$

$$B = 2^{65} - 1$$

$$\text{But } A = 2^{65}$$

$$\therefore B = A - 1$$

$$A = B + 1$$

13. The next term of the sequence $\frac{3}{16}, \frac{1}{8}, \frac{1}{12}, \frac{1}{18}, \dots$ is

(1) $\frac{1}{24}$ (2) $\frac{1}{27}$

(3) $\frac{2}{3}$ (4) $\frac{1}{81}$ **[Ans : (2)]**

Sol :

$$a = \frac{3}{16};\ r = \frac{\frac{1}{8}}{\frac{3}{16}} = \frac{1}{8} \times \frac{16}{3} = \frac{2}{3}.$$

It is a G.P.

$$\therefore \text{Next term } \frac{1}{18} \times \frac{2}{3} = \frac{1}{27}$$

14. If the sequence $t_1, t_2, t_3, \dots$ are in A.P. Then the sequence $t_6, t_{12}, t_{18} \dots$ is

(1) a Geometric progression

(2) an Arithmetic progression

(3) neither an Arithmetic progression nor a Geometric progression

(4) a constant sequence. **[Ans : (2)]**

Sol :

Given $t_1, t_2, t_3, \dots$ are in A.P.

then $t_6, t_{12}, t_{18}, \dots$ is an A.P. with common difference 6d.

15. The value of $(1^3 + 2^3 + 3^3 + \ldots + 15^3) - (1 + 2 + 3 + \ldots + 15)$

(1) 14400 (2) 14200

(3) 14280 (4) 14520 **Ans : (3)**

Sol :

$(1^3 + 2^3 + 3^3 + \ldots + 15^3) - (1 + 2 + 3 + \ldots + 15)$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{n(n+1)}{2}\right]$$

$$= \left[\frac{15 \times 16}{2}\right]^2 - \left[\frac{15 \times 16}{2}\right] = (120)^2 - 120$$

$= 14400 - 120 = 14,280$

UNIT EXERCISE - 2

1. Prove that $n^2 - n$ divisible by 2 for every positive integer n.

Sol :

We know that every positive integers is of the form 2q or 2q + 1 for some integer q.

Case 1:

So Let $n = 2q$

$$n^2 - n = (2q)^2 - (2q)$$
$$= 4q^2 - 2q$$
$$= 2q(2q - 1)$$
$$n^2 - n = 2r \quad \text{where} \quad r = q(2q - r)$$

$\therefore n^2 - n$ is even and divisible by 2

Case 2:

Let $n = 2q + 1$

$$n^2 - n = (2q + 1)^2 - (2q + 1)$$
$$= 4q^2 + 4q + 1 - 2q - 1$$
$$= 4q^2 + 2q$$
$$= 2q(2q + 1)$$
$$= 2r \quad \text{where} \quad r = q(2q + 1)$$

$\therefore n^2 - n$ is even and divisible by 2 . Hence, it is proved that $n^2 - n$ is divisible by 2 for every positive integer n.

2. A milk man has 175 litres of cow's milk and 105 litres of buffalo's milk. He wishes to sell the milk by filling the two types of milk in cans of equal capacity. Calculate the following

(i) Capacity of a can

(ii) Number of cans of cow's milk

(iii) Number of cans of buffalo's milk

Sol :

(i) Cow's milk= 175 litres

Buffalo's = 105 litres

The types of cans are of equal capacity.

$\therefore$ Capacity of a can = H.C.F. of 105, 175

By Euclid's division Algorithm

$$175 = 105 \times 1 + 70$$
$$105 = 70 \times 1 + 35$$
$$70 = 35 \times 2 + 0$$

Remainder = 0

$\therefore$ H.C.F. (105, 175) = 35

Capacity of a can = 35 litres.

(ii) Number of cans of Cow's milk

$$= \frac{\textit{Cow's Milk}}{\textit{Capacity of a can}}$$
$$= \frac{175}{35} = 5$$

5 cans of cow's milk.

(iii) Number of cans of Buffalo's milk

$$= \frac{\text{Buffalo's milk}}{\text{Capacity of a can}}$$
$$= \frac{105}{35} = 3$$

3 cans buffalo's milk is there.

3. When the positive integers a, b and c are divided by 13 the respective remainders are 9, 7 and 10. Find the remainder when a + 2b + 3c is divided by 13.

Sol :

When a, b, c are divided by 13 leaves the remainder 9, 7, 10 respectively.

$\therefore \; a = 13q_1 + 9, \; b = 13q_2 + 7, \; c = 13q_3 + 10$

Now $a + 2b + 3c = (13q_1 + 9) + 2(13q_2 + 7) + 3(13q_3 + 10)$

$= 13q_1 + 9 + 26q_2 + 14 + 39q_3 + 30$

$= 13(q_1 + 2q_2 + 3q_3) + 53$

$= 13(q_1 + 2q_2 + 3q_3) + (4 \times 13 + 1)$

$=13(q_1 + 2q_2 + 3q_3 + 4) + 1$

$\therefore$ a + 2b + 3c is divided by 13, the remainder is 1.

4. Show that 107 is of the form 4q + 3 for any integer q.

Sol :

Given the number 107,

It is a positive odd integer.

Let a = 107 and b = 4

Applying division algorithm we have,

$107 = 4q + r$ where $0 \leq r < 4$

$\therefore$ The possible r = 0, 1, 2, 3.

But 107 is odd, the remainders cannot be 0 or 2.

i.e, 4q or 4q + 2 is not possible to express 107.

The other possibilities are 4q + 1 or 4q + 3

Suppose $4q + 1 = 107$

$4q = 107 - 1$

$= 106$

$q = \frac{106}{4}$ not a natural number.

$\therefore$ Only possibility is $107 = 4q + 3$

$\therefore\ 107 = 4q + 3$

$4q = 107 - 3 = 104$

$q = \frac{104}{4} = 26$

$q = 26$

5. If $(m + 1)^{th}$ term of an A.P. is twice the $(n + 1)^{th}$ term, then prove that $(3m + 1)^{th}$ term is twice the $(m + n + 1)^{th}$ term.

Sol :

Let a-first term and d-common difference.

Given $(m + 1)^{th}$ term = $(n + 1)^{th}$ term × 2

We know that $t_n = a + (n - 1)\,d$

$t_{m+1} = a + (m + 1 - 1)\,d = a + md$

$t_{n+1} = a + (n + 1 - 1)\,d = a + nd$

$t_{m+1} = 2t_{n+1}$

$a + md = 2(a + nd)$

$a + md = 2a + 2nd$

$md - 2nd = 2a - a$

$(m - 2n)\,d = a$

$(m + n + 1)^{th}$ term $= t_{m+n+1} = a + (m + n + 1 - 1)\,d$

$= a + (m + n)\,d$

$= (m - 2n)\,d + (m + n)\,d$

$= md - 2nd + md + nd$

$= 2md - nd$

twice $(m + n + 1)^{th}$ term $= 2(2md - nd)$

$= 2(2m - n)\,d$ ... (1)

Taking $(3m + 1)^{th}$ term

$= t_{3m+1} = a + (3m + 1 - 1)\,d$

$= a + 3md$

$= (m - 2n)\,d + 3md$

$= md - 2nd + 3md$

$= 4md - 2nd$

$= 2(2m - n)\,d$ ... (2)

From (1) and (2) we proved that $(3m + 1)^{th}$ term = twice the $(m + n + 1)^{th}$ term.

6. Find the 12^{th} term from the last term of the A.P. – 2, – 4, – 6 … – 100.

Sol :

We have to find out the 12^{th} term from the end.

So we can assume the last term is the first term.

i.e., – 100, ..., – 6, – 4, – 2 be the A.P

first term $a = -100$

$d = -4 - (-6) = -4 + 6 = 2$

$t_n = a + (n - 1)\,d$

$t_{12} = -100 + (12 - 1)(2)$

$= -100 + 22 = -78$

12^{th} term from the last is – 78.

7. Two A.P's have the same common difference. The first term of one A.P. is 2 and that of the other is 7. Show that the difference between their 10^{th} terms is the same as the difference between their 21^{st} terms, which is the same as the difference between any two corresponding terms.

Sol :

Given two A.P.'s have same common difference.

Let it be d.

For first A.P. $a = 2$

10^{th} term $t_{10} = a + 9\,d$

10^{th} term $t_{10} = 2 + 9d$

For second A.P. $a = 7$

10^{th} term $t_{10} = a + 9d$

$= 7 + 9d$

Their Difference $= (7 + 9d) - (2 + 9d)$

$= 7 + 9d - 2 - 9d$

$= 5$ ... (1)

Also 21^{th} term of 1^{st} AP $= a + 20\,d$

$= 2 + 20\,d$

21^{st} term of 2^{nd} AP $= 7 + 20\,d$

Difference $= 7 + 20\,d - (2 + 20\,d)$

$= 7 + 20\,d - 2 - 20\,d = 5$...(2)

From (1) and (2) the difference is same.

Also Difference between n^{th} term of two A.P.'s

$= 7 + (n-1)d - [2 + (n-1)\,d]$

$= 7 + nd - d - 2 - nd + d$

$= 7 - 2 = 5$

$\therefore$ Difference between any two corresponding terms is 5 always.

8. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the preceding year. How much did he save in the first year?

Sol :

Let the amount he saved in the first year be x

Then x + (x + 100) + (x + 200) + ... 10 terms $= 16500$

x + x + 100 + x + 200 + ... 10 terms = 16500

(x + x + ... 10 terms) + (100 + 200 + ... 9 terms) $= 16500$

$$10\,x + \frac{9}{2}\left[2\,(100) + 8\,(100)\right] = 16500$$

$$10\,x = 16500 - 4500$$

$$10x = 12000$$

$$x = \frac{12000}{10} = 1200$$

His 1st year saving = ₹ 1200

9. Find the G.P. in which the 2^{nd} term is $\sqrt{6}$ and the 6^{th} term is $9\sqrt{6}$.

Sol :

n^{th} term of a G.P $= ar^{n-1}$

Given 2^{nd} term $= \sqrt{6}$

$ar^{2-1} = \sqrt{6}$

$ar = \sqrt{6}$

6^{th} term $= 9\sqrt{6}$

$ar^{6-1} = 9\sqrt{6}$

$ar^5 = 9\sqrt{6}$

$$\therefore \frac{ar^5}{ar} = \frac{9\sqrt{6}}{\sqrt{6}}$$

$r^4 = 9$

$r^2 = 3$

$r = \sqrt{3}$

Also $ar = \sqrt{6}$

$$a = \frac{\sqrt{6}}{r} = \frac{\sqrt{2}\,\sqrt{3}}{\sqrt{3}}$$

$a = \sqrt{2}$

$\therefore$ The G.P. is a, ar, ar^2,...

$$= \sqrt{2}, \sqrt{6}, \sqrt{2}\left(\sqrt{3}\right)^2 \ldots\ldots$$

$$= \sqrt{2}, \sqrt{6}, 3\sqrt{2} \ldots..$$

10. The value of a motor cycle depreciates at the rate of 15% per year. What will be the value of the motor cycle 3 year hence, which is now purchased for ₹ 45,000?

Sol :

Value of the motor cycle

= ₹ 45000

The value decreases at the rate of 15%

$$= \frac{15}{100}$$

$\therefore$ Value of the machine after a year

$$= 45000 \times \frac{85}{100}$$

Continuing this way the value of the machine

$45000,\ 45000 \times \frac{85}{100},\ 45000 \times \left(\frac{85}{100}\right)^2$, ...form a G.P.

$$a = 45000,\ r = \frac{85}{100}$$

$\therefore$ We need to find the value of the machine after 3 years i.e., n = 4.

$$t_n = ar^{n-1}$$

$$t_4 = 45000 \times \left(\frac{85}{100}\right)^{4-1}$$

$$= 45000 \times \frac{85}{100} \times \frac{85}{100} \times \frac{85}{100}$$

$$= 27635.625$$

$\therefore$ Value of the machine after 3 years = ₹ 27636

CREATIVE QUESTIONS

I. Multiple Choice Questions

Euclid's Division Lemma and Algorithm

1. Euclid's division lemma can be used to find the _____ of any two positive integers.

(1) HCF (2) Multiples
(3) Both (4) None of these

[Ans: (1)]

2. Euclid's division lemma is not applicable for which values of b?

(1) Positive integer
(2) Zero
(3) Negative integer
(4) All of these **[Ans: (2)]**

3. Using Euclid's division lemma HCF of 455 and 42 can be expressed as _____

(1) $455 = 42 \times 9 + 77$
(2) $455 = 42 \times 10 + 35$
(3) $455 = 42 \times 11 - 7$
(4) $455 = 42 \times 12 - 49$ **[Ans: (2)]**

Fundamental Theorem of Arithmetic

4. The number 132 is to be written as product of its prime factors. Which of the following is correct?

(1) $132 = 2 \times 6 \times 11$
(2) $132 = 2^2 \times 3 \times 11$
(3) $132 = 2^2 \times 3^2 \times 5$
(4) $132 = 3 \times 4 \times 11$ **[Ans: (2)]**

Sol :

$$\begin{array}{r|l} 2 & 132 \\ \hline 2 & 66 \\ \hline 3 & 33 \\ \hline 11 & 11 \\ \hline & 1 \end{array}$$

$$\begin{aligned} 132 &= 2 \times 2 \times 3 \times 11 \\ &= 2^2 \times 3 \times 11 \end{aligned}$$

5. What is the sum of the prime factors of 240?

(1) 16 (2) 14
(3) 12 (4) 10 **[Ans: (1)]**

Sol :

$$\begin{array}{r|l} 2 & 240 \\ \hline 2 & 120 \\ \hline 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

Prime factors of $240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5$
Their sum $= 2 + 2 + 2 + 2 + 3 + 5 = 16$

6. Solve the following $25 + 37 \equiv$ _____ (mod 12)

(1) 2 (2) 3
(3) 1 (4) 62 **[Ans: (1)]**

Sol :

$$\begin{aligned} 25 + 37 &\equiv x \pmod{12} \\ 62 &\equiv x \pmod{12} \\ 62 - x &= 12k \text{ for some } k \\ \frac{62 - x}{12} &= k \text{ for some } k \\ \frac{62 - 2}{12} &= 5 \\ \therefore\ x &= 2. \end{aligned}$$

7. What does 144 reduces to mod 11?

(1) 144 mod 11 (2) 1 mod 11
(3) 2 mod 11 (4) 143 mod 11 **[Ans: (2)]**

Sol : $144 \equiv x \pmod{11}$

$144 - x$ is a multiple of 11

$143 = (144 - 1)$ is a multiple of 11

$\therefore\ x = 1$

$144 \equiv 1 \pmod{11}$.

Sequences

8. First term and common difference in the sequence 7, 10, 13, …

(1) 1, 7 (2) 7, 10
(3) 7, 3 (4) 13, 10 **[Ans: (3)]**

Sol :

The sequence is 7, 10, 13, ...

$t_2 - t_1 = t_3 - t_2 \Rightarrow 10 - 7 = 13 - 10 = 3$

$\therefore$ Common difference $d = 3$

First term is 7.

Arithmetic Progression

9. If the first term of an A.P. is a and n^{th} term is b, then the common difference is _____.

(1) $\frac{b-a}{n+1}$ (2) $\frac{b-a}{n-1}$

(3) $\frac{b-a}{n}$ (4) $\frac{b+a}{n-1}$ **[Ans: (2)]**

Sol :

First term = a

Let common difference be d

$$n^{th} \text{ term} = b$$
$$a + (n-1)d = b$$
$$(n-1)d = b - a$$
$$d = \frac{b-a}{n-1}$$

10. The common differences of the A.P. $\frac{1}{3}, \frac{1-3b}{3}, \frac{1-6b}{3}, \ldots$ is

(1) $\frac{1}{3}$ (2) $\frac{-1}{3}$

(3) $-b$ (4) b **[Ans: (3)]**

Sol :

Common difference = $t_2 - t_1$

$$= \frac{1-3b}{3} - \frac{1}{3} = \frac{1-3b-1}{3} = -\frac{3b}{3} = -b$$

Series

11. The sum of n terms of an A.P. is $3n^2 + 5n$, then which of its term is 164?

(1) 26^{th} (2) 27^{th}

(3) 28^{th} (4) None of these

[Ans: (2)]

Sol :

$$S_n = 3n^2 + 5n$$
$$S_1 = 3(1)^2 + 5 = 8 = a = t_1$$
$$S_2 = 3(2)^2 + 5(2) = 3(4) + 10$$
$$= 12 + 10 = 22$$
$$t_2 = S_2 - S_1 = 22 - 8 = 14$$
$$d = t_2 - t_1 = 14 - 8 = 6$$
$$a + (n-1)d = 164$$
$$8 + (n-1)6 = 164$$
$$n - 1 = \frac{164-8}{6} = \frac{156}{6} = 26$$
$$n = 26 + 1 = 27$$

12. The first, second and last term of an A.P. are a, b and 2a respectively, its sum is

(1) $\frac{ab}{2(b-a)}$ (2) $\frac{ab}{b-a}$

(3) $\frac{3ab}{2(b-a)}$ (4) None of these

[Ans: (3)]

Sol :

$$n = \frac{l-a}{d} + 1$$
$$= \frac{2a-a}{b-a} + 1 = \frac{a}{b-a} + 1$$
$$= \frac{a+b-a}{b-a} = \frac{b}{b-a}$$
$$\text{Sum} = \frac{n}{2}(l+a)$$
$$= \frac{b}{2(b-a)}(2a+a)$$
$$\text{Sum} = \frac{3ab}{2(b-a)}$$

Geometric Sequences

13. 7^{th} term of a G.P. 2, 6, 18, . . . is

(1) 5832 (2) 2919

(3) 1458 (4) 729 **[Ans: (3)]**

Sol : $a = 2,\ r = \frac{t_2}{t_1} = \frac{6}{2} = 3$

$$t_n = ar^{n-1}$$
$$t_7 = 2(3)^{7-1} = 2(3)^6$$
$$= 2 \times 729 = 1458.$$

14. No term of a geometric sequence be

(1) 3 (2) 1

(3) 2 (4) 0 **[Ans: (4)]**

Sum of G.P.

15. Sum of n terms of a G.P. is

(1) $\frac{n}{2}[2a + (n-1)d]$ (2) $\frac{a(1-r^n)}{1-r}$

(3) $\frac{2ab}{(a+b)}$ (4) $\frac{a+b}{2}$ **[Ans: (2)]**

16. Sum of 7 terms of – 2, 6, – 18, . . . is

(1) 1094 (2) – 1094

(3) 9041 (4) – 9041 **[Ans: (2)]**

Sol : $a = -2,\ r = \frac{6}{-2} = -3$

$$\text{Sum} = \frac{a(r^n - 1)}{r-1}$$

$$S_7 = \frac{(-2)[(-3)^7 - 1]}{-3-1}$$

$$= \frac{(-2)(-2187-1)}{-4} = \frac{(-2)(-2188)}{-4}$$

$$= -1094.$$

Special Series

17. $\dfrac{5+9+13+\ldots \text{ to } n \text{ terms}}{7+9+11+\ldots \text{ to } (n+1) \text{ terms}} = \dfrac{17}{16}$ **then n = ?**

(1) 8 (2) 7
(3) 10 (4) 11 **[Ans: (2)]**

Sol :

Sum of 5 + 9 + 13 + ... to n terms

$$= \frac{n}{2}\{2a + (n-1)d\}$$

d = 9 – 5 = 4; a = 5

$$\text{Sum} = \frac{n}{2}[2\times 5 + (n-1)4]$$

$$= \frac{n}{2}[10 + 4n - 4]$$

$$= \frac{n}{2}[4n+6] = n(2n+3)$$

Sum of 7 + 9 + 11 + ... to (n + 1) terms.

$$= \frac{n+1}{2}[2\times 7 + (n+1-1)2]$$

$$= \frac{n+1}{2}[14 + 2n]$$

$$= \frac{(n+1)}{2}2(7+n)$$

$$= (n+1)(7+n)$$

$$\frac{5+9+13+\ldots \text{ to } n \text{ terms}}{7+9+11+\ldots \text{ to } (n+1) \text{ terms}} = \frac{17}{16}$$

$$\frac{n(3+2n)}{(n+1)(7+n)} = \frac{17}{16}$$

$16n(3+2n) = 17(n+1)(7+n)$
$48n + 32n^2 = 17n^2 + 136n + 119$
$48n + 32n^2 - 17n^2 - 136n - 119 = 0$
$15n^2 - 88n - 119 = 0$
$15n^2 - 105n + 17n - 119 = 0$
$15n(n-7) + 17(n-7) = 0$
$(n-7)(15n+17) = 0$
$n = 7$ or $n = -\frac{17}{15}$
$n = -\frac{17}{15}$ is not possible.
$\therefore\ n = 7.$

18. The sum of first n odd natural number is

(1) $2n - 1$ (2) $2n + 1$
(3) n^2 (4) $n^2 - 1$ **[Ans: (3)]**

19. If 1 + 2 + 3 + . . . + 10 = 55, then, $1^3 + 2^3 + 3^3 + \ldots + 10^3 = ?$

(1) 55^2 (2) 10^2
(3) 55^3 (4) 10^3 **[Ans: (1)]**

20. $1^2 + 2^2 + 3^2 \ldots + n^2 = ?$

(1) $\left[\frac{n(n+1)}{2}\right]^2$ (2) $\frac{n(n+1)}{2}$
(3) n^2 (4) $\frac{n(n+1)(2n+1)}{6}$ **[Ans: (4)]**

II. Very Short Answer Questions

1. State the fundamental theorem of Arithmetic.

Sol :

Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written.

2. If 6^n is a number such that n is a natural number. Check whether is any value of $n \in N$ for which 6^n is divisible by 7.

Sol :

We have $6^n = (2\times 3)^n$
$= 2^n \times 3^n$

Prime factorization of 6^n does not contain the prime number 7. Therefore 6^n is not divisible by 7.

3. Find the H.C.F. and L.C.M. of 100 and 190 by fundamental theorem of arithmetic.

Sol :

By fundamental theorem we have every composite number can be expressed as a product of primes.

2	100
2	50
5	25
	5

2	190
5	95
	19

$\therefore$ Factorizing 100 and 190

$$100 = 2^2 \times 5^2$$
$$190 = 2^1 \times 5^1 \times 19^1$$
$$\therefore \text{H.C.F. of 100 and 190} = 2^1 \times 5^1 = 10$$
$$\text{H.C.F.} \times \text{L.C.M.} = \text{Product of two numbers}$$
$$10 \times \text{L.C.M.} = 100 \times 190$$
$$\text{L.C.M.} = \frac{100 \times 190}{10} = 1900$$

4. Write the H.C.F. of smallest composite number and the smallest prime number.

Sol :

$$\text{Smallest composite number} = 4 = 2^2$$
$$\text{Smallest prime number} = 2 = 2^1$$
$$\therefore \text{H.C.F. of 4 and 2} = 2$$

5. The traffic lights at three different road crossings change after every 48 sec, 72 sec and 108 sec respectively. If they all change simultaneously at 8.20 am, then at what time will they again change simultaneously?

Sol :

Interval of change = L.C.M. of (48, 72, 108) sec

$$\begin{array}{r|l} 12 & 48, 72, 108 \\ \hline 3 & 4, 6, 9 \\ \hline 2 & 4, 2, 3 \\ \hline & 2, 1, 3 \end{array}$$

$$\text{LCM} = 12 \times 3 \times 2 \times 2 \times 3 = 432$$

So the lights will again change simultaneously after every 432 sec.

$$= 432/60 = 7 \text{ min } 12 \text{ sec.}$$

Hence next change will be at 8 : 27 : 12 am.

6. Does 7 divides $(2^{29} + 3)$?

Sol :

We have

$$2^3 \equiv 1 \pmod 7$$
$$(2^3)^8 \equiv 1^8 \pmod 7$$
$$2^3 2^{24} \equiv 1 \times 1^8 \pmod 7$$
$$2^{27} \equiv 1 \pmod 7$$
$$2^2\, 2^{27} \equiv 4 \times 1\ (1 \bmod 7)$$
$$2^{29} \equiv 4 \pmod 7$$
$$2^{29} + 3 = (4 + 3) \pmod 7$$
$$2^{29} + 3 \equiv 0 \pmod 7$$

$2^{29} + 3$ is divisible by 7.

7. What is the remainder when $3^{202} + 5^9$ is divided by 8?

Sol :

$$3^2 = 1 \pmod 8$$
$$(3^2)^{101} \equiv 1^{101} \pmod 8$$
$$3^{202} \equiv 1 \pmod 8$$
$$5^2 \equiv 1 \pmod 8$$
$$5 \equiv 5 \pmod 8$$
$$(5^2)^4 \equiv 1^4 \pmod 8$$
$$5^8.5^1 \equiv 5 \pmod 8$$
$$3^{202} + 5^9 = 6 \pmod 8$$

Remainder is 6 when divided by 8.

8. Write the first three terms of the sequence defined by $a_n = (-1)^{n-1}, 2^n$.

Sol :

Given
$$a_n = (-1)^{n-1}\, 2^n$$
$$a_1 = (-1)^{1-1}\, 2^1 = (-1)^0\, 2^1 = 1 \times 2 = 2$$
$$a_2 = (-1)^{2-1} \times 2^2 = (-1)^1 \times 4 = -4$$
$$a_3 = (-1)^{3-1} \times 2^3 = (-1)^2 \times 8 = 8$$

$\therefore$ First three terms are 2, – 4, 8

9. What is the 15th term of the sequence defined by $a_n = \dfrac{n(n-3)}{n+4}$?

Sol :

Given
$$a_n = \frac{n(n-3)}{n+4}$$
$$a_{15} = \frac{15(15-3)}{15+4} = \frac{15 \times 12}{19}$$
$$a_{15} = \frac{180}{19}$$

10. Find the n^{th} term of the sequence 5, 8, 11,...

Sol :

Given the sequence 5, 8, 11,... Every term is 3 more than the previous term but the first term is 5.

$\therefore$ The general term may be $a_n = 3n + 2$

11. Write the general term of the sequence $\dfrac{-1}{2}, 0, \dfrac{3}{2}, \dfrac{8}{2}, \ldots$

Sol :

In each term the numerator increases by n (n – 2) and denominator is 2.

$\therefore$ The general term is $a_n = \dfrac{n(n-2)}{2}$

12. Find the first three terms of $a_n = \dfrac{2n-3}{6}$

Sol :

Given $a_n = \dfrac{2n-3}{6}$

$$\therefore a_1 = \frac{2(1)-3}{6} = \frac{2-3}{6} = \frac{-1}{6}$$

$$a_2 = \frac{2(2)-3}{6} = \frac{4-3}{6} = \frac{1}{6}$$

$$a_3 = \frac{2(3)-3}{6} = \frac{6-3}{6} = \frac{3}{6}$$

$\therefore$ First three terms are $\dfrac{-1}{6}, \dfrac{1}{6}, \dfrac{3}{6}$.

13. If 2x, x + 10, 3x + 2 are in A.P. Find x.

Sol :

Given 2x, x + 10, 3x + 2 are in A.P

$$x + 10 - 2x = 3x + 2 - [x + 10]$$

$$-x + 10 = 3x + 2 - x - 10 = 2x - 8$$

$$10 + 8 = 2x + x$$

$$3x = 18$$

$$x = \frac{18}{3} = 6$$

14. Find four terms of an A.P. whose sum is 20 and the sum of whose squares is 120.

Sol :

Let the four terms be (a – 3d),(a – d) (a + d), (a + 3d)

$$\text{Sum} = 20$$

$$(a-3d) + (a-d) + (a+d) + (a+3d) = 20$$

$$a - 3d + a - d + a + d + a + 3d = 20$$

$$4a = 20$$

$$a = \frac{20}{4} = 5$$

$$\text{Given sum of squares} = 120$$

$$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$$

$$a^2 + 9d^2 - 6ad + a^2 + d^2 - 2ad + a^2 + d^2 + 2ad + a^2 + 9d^2 + 6ad = 120$$

$$4a^2 + 20d^2 = 120$$

$$a^2 + 5d^2 = 30$$

$$25 + 5d^2 = 30$$

$$5d^2 = 5$$

$$d = \pm 1$$

If d = 1 then the numbers are 2, 4, 6, 8.

If d = – 1 then the numbers are 8, 6, 4, 2.

$\therefore$ The four numbers are 2, 4, 6, 8.

15. Which term of the A.P. –1, 3, 7, 11, …. is 95?

Sol :

$a = -1,\quad d = t_2 - t_1 = 3 - (-1) = 4$

Let 95 be the n^{th} term of the A.P.

Then $t_n = 95$

$$a + (n-1)d = 95$$

$$-1 + (n-1)4 = 95$$

$$(n-1)4 = 95 + 1 = 96$$

$$n - 1 = \frac{96}{4} = 24$$

$$n = 24 + 1 = 25$$

$\therefore$ 95 is the 25^{th} term of the A.P

16. How many terms are there in the sequence 3, 6, 9, 12, …., 111?

Sol :

Here $t_2 - t_1 = 6 - 3 = 3$

$t_3 - t_2 = 9 - 6 = 3$ and so on.

$\therefore$ It is an A.P.

$a = 3, d = 3.$

Let the n^{th} term of the A.P. = 111

$$t_n = 111$$

$$a + (n-1)d = 111$$

$$3 + (n-1)3 = 111$$

$$(n-1)3 = 111 - 3 = 108$$

$$(n-1) = \frac{108}{3}$$

$$n - 1 = 36$$

$$n = 36 + 1 = 37$$

$\therefore$ The A.P. has 37 terms.

17. Is 184 a term of the sequence 3, 7, 11, …?

Sol :

Clearly 7 – 3 = 4, 11 – 7 = 4 and it is an A.P.

Suppose 184 is the n^{th} term of the A.P.

then $t_n = 184$

$$a + (n-1)d = 184$$

$$3 + (n-1)4 = 184$$

$$(n-1)4 = 184 - 3$$

$$(n-1)4 = 181$$

$$n - 1 = \frac{181}{4}$$

$$n - 1 = 45.25$$

$$n = 45.25 + 1 = 46.25$$

Since n is not a natural number. 184 is not a term of the given A.P.

18. Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ is the first negative term?

Sol :

Here $t_2 - t_1 = 19\frac{1}{4} - 20 = -\frac{3}{4}$

$$t_3 - t_2 = 18\frac{1}{2} - 19\frac{1}{4} = -\frac{3}{4}$$

$$t_2 - t_1 = t_3 - t_2$$

$\therefore$ The sequence is an A.P. with $a = 20; d = -\frac{3}{4}$

Let the n^{th} term of A.P. be the first negative term.

i.e., $t_n < 0$

$$a + (n-1)d < 0$$

$$20 + (n-1) \times \left(\frac{-3}{4}\right) < 0.$$

$$(n-1)\left(-\frac{3}{4}\right) < -20$$

$$(n-1)\left(\frac{3}{4}\right) > 20$$

$$n - 1 > \frac{20}{\left(\frac{3}{4}\right)}$$

$$n - 1 > 20 \times \frac{4}{3}$$

$$n > \frac{80}{3} + 1$$

$$n > \frac{80+3}{3} = \frac{83}{3}$$

$$n > 27\frac{2}{3}$$

$$n \geq 28 \quad [\because n \text{ is a natural number}]$$

$\therefore$ If $n \geq 28$, the terms t_n becomes negative.

$\therefore$ 28^{th} term is the first negative term.

19. How many numbers of two digits are divisible by 7?

Sol :

We know that first two digit number divisible by 7 is 14.

Last two digit number divisible by 7 is 98.

It is enough to find the number of terms of the A.P. 14, 21, 28, ..., 98.

$$a = 14, d = 7$$

$$n^{th} \text{ term} = 98$$

$$a + (n-1)d = 98$$

$$14 + (n-1)(7) = 98$$

$$(n-1)(7) = 98 - 14 = 84$$

$$n - 1 = \frac{84}{7} = 12$$

$$n = 12 + 1 = 13$$

$\therefore$ There are 13 numbers of two digits which are divisible by 7.

20. Find the sum of first n natural numbers.

Sol :

First n natural numbers are given by 1, 2, 3, 4,.... n

The first term $a = 1; d = 2 - 1 = 1, l = n$.

$$\text{Sum } S_n = \frac{n}{2}(a+l)$$

$$= \frac{n}{2}(1+n) = \frac{n(n+1)}{2}$$

$\therefore$ Sum of first n natural numbers is $\frac{n(n+1)}{2}$

21. How many terms of the A.P. 9, 17, 25,... must be taken to give a sum of 636?

Sol :

Here $a = 9, d = 17 - 9 = 8$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$636 = \frac{n}{2}\{2(9) + (n-1)(8)\}$$

$$= \frac{n}{2} \times 2\{9 + (n-1)4\}$$

$$= n\{9 + 4n - 4) = n\{4n + 5\}$$

$$636 = 4n^2 + 5n$$

$$4n^2 + 5n - 636 = 0$$

$$n = \frac{-5 \pm \sqrt{25 - 4(4)(-636)}}{2(4)}$$

$$= \frac{-5 \pm \sqrt{25 + 10176}}{8}$$

$$= \frac{-5 \pm \sqrt{10201}}{8}$$

$$= \frac{-5 \pm 101}{8}$$

$$= \frac{-5+101}{8}, \frac{-5-101}{8}$$

$$= \frac{96}{8}, \frac{-106}{8} = 12 \text{ or } \frac{-53}{4}$$

$$n = -\frac{53}{4} \text{ is not possible}$$

$\therefore n = 12$

$\therefore$ 12 terms are taken to get the sum 636.

22. If S_n the sum of first n terms of an A.P. is given by $S_n = 5n^2 + 3n$. Then find the n^{th} term.

Sol :

$$S_n = 5n^2 + 3n$$
$$S_{n-1} = 5(n-1)^2 + 3(n-1)$$
$$= 5(n^2 - 2n + 1) + 3(n-1)$$
$$= 5n^2 - 10n + 5 + 3n - 3$$
$$= 5n^2 - 7n + 2$$
$$\text{Now } n^{th} \text{ term} = S_n - S_{n-1}$$
$$\therefore \text{The required } n^{th} \text{ term} = [5n^2 + 3n] - [5n^2 - 7n + 2]$$
$$= 5n^2 + 3n - 5n^2 + 7n - 2$$
$$t_n = 10n - 2$$

23. Find the 9^{th} term and the general term of the G.P. $\frac{1}{4}, \frac{-1}{2}, 1, -2, \ldots$

Sol :

$$\text{Were } a = \frac{1}{4}, \quad r = -2$$
$$a_n = ar^{n-1}$$
$$a_9 = \frac{1}{4}(-2)^8 = 64$$
$$a_n = \frac{1}{4}(-2)^{n-1}$$
$$a_n = (-1)^{n-1} 2^{n-3}$$

24. The fourth, seventh and last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.

Sol :

Let the first term = a, common ratio = r

$$a_4 = 10, a_7 = 80, a_n = 2560$$
$$ar^3 = 10$$
$$ar^6 = 80$$
$$\frac{ar^6}{ar^3} = \frac{80}{10}$$
$$r^3 = 8$$
$$r = 2$$
$$ar^3 = 10$$
$$\therefore a(2)^3 = 10$$
$$a(8) = 10$$
$$a = \frac{10}{8} = \frac{5}{4}$$
$$ar^{n-1} = 2560$$
$$\frac{5}{4}(2)^{n-1} = 2560 \Rightarrow 2^{n-1} = 2560 \times \frac{4}{5}$$
$$= 512 \times 4$$
$$= 2048$$
$$2^{n-1} = 2^{11}$$
$$\therefore n - 1 = 11 \Rightarrow n = 12.$$

25. The 7^{th} term of G.P. is 8 times the 4^{th} term and 5^{th} term is 48. Find the G.P.

Sol :

$$a_7 = 8a_4 \quad \text{and} \quad a_5 = 48$$
$$ar^6 = 8ar^3 \quad \text{and} \quad ar^4 = 48$$
$$r^3 = 8 \qquad \Rightarrow a(2)^4 = 48$$
$$r = 2 \qquad a = \frac{48}{16} = 3$$
$$a = 3$$

$\therefore$ The required G.P. is 3, 6, 12, 24, 48...

26. What term of the G.P. 5, 10, 20, 40,… is 5120?

Sol : $a = 5, \; r = 2$

$$a_n = 5120$$
$$a_n = ar^{n-1}$$
$$5120 = 5(2)^{n-1}$$
$$2^{n-1} = 1024$$
$$2^{n-1} = 2^{10}$$
$$n - 1 = 10$$
$$n = 10 + 1 = 11$$

11^{th} term of the G.P. is 5120.

27. Find the sum upto infinity of the G.P. $1, \frac{1}{3}, \frac{1}{9}, \ldots$

Sol :

$$a = 1, r = \frac{1}{3}$$
$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1.5$$

28. Find the sum of infinity of the G.P. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots$

Sol :

$$\text{Here } a = \frac{-3}{4}, r = \frac{-1}{4}$$
$$\text{Sum} = \frac{a}{1-r} = \frac{-\frac{3}{4}}{1-\left(\frac{-1}{4}\right)}$$

$$\text{Sum} = \frac{\frac{-3}{4}}{\frac{5}{4}} = \frac{-3}{5}$$

29. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times .... = 3$

Sol :

$$\text{L.H.S} = 3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times ....$$

$$= 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....}$$

Now take $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$

It is a G.P. with $a = \frac{1}{2}$, $r = \frac{1/4}{1/2} = \frac{1}{2}$

$$\text{Sum} = \frac{a}{1-r}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\therefore 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...} = 3^1 = 3 = \text{RHS}$$

30. If $y = x + x^2 + x^3 + ...$prove that $x = \frac{y}{1+y}$

Sol :

$$\text{Given } y = x + x^2 + x^3 + ...$$
$$= x(1 + x + x^2 + ...)$$
$$y = x\left[\frac{1}{1-x}\right] \qquad \left[\because \; S_\infty = \frac{a}{1-r}\right]$$
$$y = \frac{x}{1-x}$$
$$y - yx = x$$
$$x + xy = y \quad \Rightarrow x(1+y) = y$$
$$x = \frac{y}{1+y}$$

31. Using Geometric Series rationalise $0.1\overline{42}$

Sol :

$$0.1\overline{42} = 0.1424242...$$
$$= 0.1 + 0.042 + 0.00042 + 0.0000042 + ...$$
$$= 0.1 + \frac{42}{10^3} + \frac{42}{10^5} + \frac{42}{10^7} + ...$$
$$= \frac{1}{10} + \left[\frac{\frac{42}{10^3}}{1-\frac{1}{10^2}}\right] = \frac{1}{10} + \frac{42}{990}$$
$$\therefore 0.1\overline{42} = \frac{141}{990}$$

32. Find the sum: 1 + 2 + 3 + 4 + … + 80.

Sol :

$$\text{We have } 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$
$$\therefore 1 + 2 + 3 + ... + 80 = \frac{80 \times 81}{2} = 3240$$

33. Find the sum of $1^3 + 2^3 + 3^3 + ... + 70^3$.

Sol :

$$\text{We have } 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$
$$= \left[\frac{70 \times 71}{2}\right]^2$$
$$= (35 \times 71)^2 = (2485)^2$$
$$1^3 + 2^3 + 3^3 + ... + 70^3 = 61,75,225$$

34. Find the sum of $1^2 + 2^2 + 3^2 + ... + 23^2$

Sol :

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\text{Sum of } 1^2 + 2^2 + 3^2 + ... + 23^2 = \frac{23 \times 24 \times 47}{6}$$
$$= 4324$$

III. Short Answer Questions:

1. What is the greatest possible length which can be used to measure exactly the lengths 7 m; 3 m 85 cm; 12 m 95 cm?

Sol :

Required Length = H.C.F. of (7 m, 3 m 85 cm, 12 m 95 cm)

= H.C.F. of 700 cm, 385 cm, 1295 cm

5	700
7	140
2	20
2	10
5	5
	1

5	385
7	77
	11

5	1295
7	259
	37

$$700 = 2^2 \times 5^2 \times 7^1$$
$$385 = 5^1 \times 7^1 \times 11^1$$
$$1295 = 5^1 \times 7^1 \times 37^1$$
$$\text{H.C.F. is } 5^1 \times 7^1 = 35$$

$\therefore$ The required length is 35 cm.

2. Find the greatest number that will divide 43, 91 and 183 so as to leave the same remainder in each case.

Sol :

Required number = H.C.F. of (91 – 43), (183 – 91) and (183 – 43)

= H.C.F. of 48, 92, and 140

$$\begin{array}{r|l} 2 & 48 \\ \hline 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array} \qquad \begin{array}{r|l} 2 & 92 \\ \hline 2 & 46 \\ \hline & 23 \end{array} \qquad \begin{array}{r|l} 7 & 140 \\ \hline 2 & 20 \\ \hline 2 & 10 \\ \hline & 5 \end{array}$$

$$48 = 2^4 \times 3^1$$
$$92 = 2^2 \times 23^1$$
$$140 = 2^2 \times 5^1 \times 7^1$$
$$\therefore \text{H.C.F. is } 2^2 = 4$$

$\therefore$ The required number is 4

3. What will be the least number which when doubled will be exactly divisible by 12, 18, 21 and 30?

Sol :

$$\begin{array}{r|l} 2 & 12, 18, 21, 30 \\ \hline 3 & 6, 9, 21, 15 \\ \hline & 2, 3, 7, 5 \end{array}$$

$$\text{L.C.M. of } 12, 18, 21, 30 = 2^2 \times 3^2 \times 5^1 \times 7^1$$
$$= 4 \times 9 \times 35$$
$$= 1260$$
$$\text{Required number} = 1260 \div 2$$
$$= 630$$

4. If the exams are over by Monday. The results will be published 29 days after exams. What day the results will be published.

Sol :

Monday stands for 1.

29 days after Monday is 1 + 29 (mod 7)

$\equiv$ 30 (mod 7)

$\equiv$ 2 (mod 7)

2 stands for Tuesday.

Results will be published on Tuesday.

5. Let the sequence defined by $a_1 = 3$, $a_n = 3a_{n-1} + 1$ for all $n > 1$. Find the first three terms of the sequence.

Sol :

$$\text{Given } a_1 = 3.$$
$$a_n = 3a_{n-1}+1 \quad \text{for all } n > 1$$
$$\text{Put } n = 2;$$
$$a_2 = 3a_{2-1}+1 = 3a_1 + 1$$
$$= 3(3) + 1 = 9 + 1 = 10$$
$$\text{Put } n = 3;$$
$$a_3 = 3a_{3-1} + 1 = 3a_2 + 1$$
$$= 3(10) + 1 = 30 + 1 = 31$$

$\therefore$ First three terms are 3, 10, 31.

6. If $a_n = (-1)^n n$ find a_3, a_5 and a_8

Sol :

Given $a_n = (-1)^n n$

We know that

$$(-1)^{\text{odd number}} = \text{'}-\text{' ve}$$
$$(-1)^{\text{even number}} = +\text{ ve}$$
$$a_3 = (-1)^3\, 3 = -3$$
$$a_5 = (-1)^5\, 5 = -5$$
$$a_8 = (-1)^8\, 8 = 8$$
$$a_3 = -3;\ a_5 = -5;\ a_8 = 8$$

7. If $a_n = (n-1)(2-n)(3+n)$ find a_1, a_2, a_3.

Sol :

Given $a_n = (n-1)(2-n)(3+n)$

Put $n = 1$; $a_1 = (1-1)(2-1)(3+1)$

$a_1 = 0 \times 1 \times 4 = 0$

Put $n = 2$; $a_2 = (2-1)(2-2)(3+2)$

$a_2 = 1 \times (0) \times 5 = 0$

Put $n = 3$; $a_3 = (3-1)(2-3)(3+3)$

$= 2 \times (-1) \times 6 = -2 \times 6$

$a_3 = -12$

$a_1 = 0;\ a_2 = 0;\ a_3 = -12$

8. For what value of n the n^{th} term of the A.P. 69, 68, 67,… and 1, 7, 13, 19 … are the same

Sol :

Consider the A.P. 69, 68, 67,...

$$a = 69;\ d = 68 - 69 = -1$$
$$t_n = a + (n-1)d$$
$$t_n = 69 + (n-1)(-1) \qquad \text{...(1)}$$

For the A.P. 1, 7, 13, 19,...

$$a = 1;\ d = 7 - 1 = 6$$
$$T_n = 1 + (n-1)6 \qquad \text{...(2)}$$

If the two A.P's has an identical term then $t_n = T_n$ for some n.

$$69 + (n-1)(-1) = 1 + (n-1)6$$

$$69 - n + 1 = 1 + 6n - 6$$
$$69 + 1 - 1 + 6 = 6n + n$$
$$75 = 7n$$
$$n = \frac{75}{7}\text{, which is not a natural number.}$$

$\therefore$ The two A.P.'s do not have an identical term for any n.

9. If the 8th term of an A.P. is 31 and the 15th term is 16 more than the 11th term, find the A.P.

Sol :

Let a be the first term and d be the common difference of the A.P.

Given $a_8 = 31$ and $a_{15} = 16 + a_{11}$

$$a + (n-1)d = t_n$$
$$a + 7d = 31 \text{ and}$$
$$a + 14d = 16 + (a + 10d)$$
$$a + 14d = 16 + a + 10d$$
$$a - a + 14d - 10d = 16$$
$$4d = 16$$
$$d = \frac{16}{4} = 4$$

Taking
$$a + 7d = 31$$
$$a + 7(4) = 31$$
$$a + 28 = 31$$
$$a = 31 - 28 = 3.$$

$\therefore$ The A.P. is a, a + d, a + 2d, ...

$\Rightarrow$ 3, 7, 11, 15, 19,...

10. Which term of the A.P. 5, 15, 25,... will be 130 more than its 31st term?

Sol :

We have $a = 5; d = 10$

$$a_{31} = a + 30d = 5 + 30 \times 10 = 305$$

Let the nth term of the Given A.P. is 130 more than the 31st term.

$$a_n = 130 + a_{31}$$
$$a + (n-1)d = 130 + 305$$
$$5 + 10(n-1) = 435$$
$$10(n-1) = 430$$
$$n - 1 = 43$$
$$n = 44$$

$\therefore$ 44th term of the given A.P. is 130 more than its 31st term.

11. If the first term of an A.P. is 17 and last term is 350, common difference is 9, how many terms are there in the A.P.? What is their sum?

Sol :

First term $a = 17$

Last term $l = 350 = t_n$

Common difference $d = 9$

$$t_n = a + (n-1)d$$
$$350 = 17 + (n-1)9$$
$$\frac{350 - 17}{9} = n - 1$$
$$n = \frac{333}{9} + 1$$
$$= 37 + 1 = 38$$

$\therefore$ There are 38 terms in the A.P.

$$S_n = \frac{n}{2}(a + l)$$
$$S_{38} = \frac{38}{2}(17 + 350)$$
$$= 19 \times 367$$

$\therefore$ Required sum $= 6973$

12. Find the sum of all three digit numbers which are divisible by 7.

Sol :

Three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994.

It is in A.P. where $a = 105, d = 7$

$$t_n = 994 = l$$
$$t_n = a + (n-1)d$$
$$994 = 105 + (n-1)7$$
$$n - 1 = \frac{994 - 105}{7} = \frac{889}{7} = 127$$
$$n = 127 + 1 = 128$$
$$\text{Now } S_n = \frac{n}{2}(a + l)$$
$$= \frac{128}{2}(105 + 994)$$
$$= 64(1099)$$

$\therefore$ The required sum $= 70336$

13. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a > b > c$ and $a + b + c = 3/2$ then find the value of a.

Sol :

Let $b = a + d, c = a + 2d$ ...(1)

a^2, b^2, c^2 are in G.P. $(b^2)^2 = a^2c^2$

$\pm b^2 = ac$ ...(2)

$\therefore$ a, b, c are in A.P.

$2b = a + c$

Given $a + b + c = \frac{3}{2}$

$\Rightarrow 3b = \frac{3}{2}$

$\Rightarrow b = \frac{1}{2}$

From (1) $a = b - d$

$\Rightarrow a = \frac{1}{2} - d$

$c = a + 2d$

$= \left(\frac{1}{2} - d\right) + 2d = \frac{1}{2} + d$

From (2) $\pm\frac{1}{4} = \left(\frac{1}{2} - d\right)\left(\frac{1}{2} + d\right)$

$\Rightarrow \pm\frac{1}{4} = \frac{1}{4} - d^2$

Taking negative sign $\therefore d = \pm\frac{1}{\sqrt{2}}$

$\therefore a = \frac{1}{2} - d = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$

$\Rightarrow a = \frac{1}{2} + \frac{1}{\sqrt{2}}$ $[\because a > b > c]$

14. If x, (2x + 2), (3x + 3),.... are in G.P. then find the next term of the G.P.

Sol :

Since x, 2x + 2, 3x + 3,.... are in G.P.

$\frac{2x+2}{x} = \frac{3x+3}{2x+2} = r \ (x \neq -1)$

$r = \frac{3}{2}$

$\therefore \frac{2x+2}{x} = \frac{3}{2}$

$4x + 4 = 3x$

$x = -4$

$\therefore$ The next term $= r(3x + 3)$

$= \frac{3}{2}(-12 + 3)$

$= \frac{-27}{2} = -13.5$

15. The 5^{th}, 8^{th} and 11^{th} term of a G.P. are p, q and s respectively. Show that $q^2 = ps$.

Sol :

Let 'a' be the first term and r be the common ratio of the given G.P.

Here $a_5 = p \Rightarrow ar^4 = p$ ...(1)

$a_8 = q \Rightarrow ar^7 = q$ ...(2)

$a_{11} = s \Rightarrow ar^{10} = s$ ...(3)

Squaring both sides of (2), we get

$q^2 = (ar^7)^2$

$q^2 = a^2r^{14}$

$q^2 = (ar^4)(ar^{10})$

$q^2 = ps$ $[\because p = ar^4 \text{ and } s = ar^{10}]$

16. Find the sum to infinity of the series

$\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + ...$

Sol : This series may be written as

$\left(\frac{1}{7} + \frac{1}{7^3} + \frac{1}{7^5} + ...\right) + \left(\frac{2}{7^2} + \frac{2}{7^4} + \frac{2}{7^6} + ...\right)$

$= \left(\frac{\frac{1}{7}}{1 - \frac{1}{7^2}}\right) - \left(\frac{\frac{2}{7^2}}{1 - \frac{1}{7^2}}\right)$ $\left[\text{Using sum} = \frac{a}{1-r}\right]$

$= \frac{\frac{1}{7}}{\frac{48}{49}} + \frac{\frac{2}{49}}{\frac{48}{49}}$

$= \frac{7}{48} + \frac{2}{48} = \frac{9}{48}$

$\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + ... = \frac{9}{48}$

17. Find the sum to infinity of the G.P.

$\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + ...$

Sol :

The given series may be written as

$\left(\frac{2}{5} + \frac{2}{5^3} + \frac{2}{5^5} + ...\right) + \left(\frac{3}{5^2} + \frac{3}{5^4} + \frac{3}{5^6} + ...\right)$

$S_\infty = \frac{a}{1-r}$

$\text{Sum} = \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2} + \frac{\frac{3}{5^2}}{1 - \left(\frac{1}{5}\right)^2}$

$$= \frac{\frac{2}{5}}{1-\frac{1}{25}} + \frac{\frac{3}{25}}{1-\frac{1}{25}}$$

$$= \frac{5}{12} + \frac{1}{8} = \frac{13}{24}$$

$$\therefore \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots = \frac{13}{24}$$

18. Find $16^2 + 17^2 + 18^2 + \ldots + 30^2$

Sol :

We have

$1^2 + 2^2 + 3^2 + \ldots + 30^2$

$$= \frac{n(n+1)(2n+1)}{6}$$

$16^2 + 17^2 + 18^2 + \ldots + 30^2$

$$= (1^2 + 2^2 + \ldots + 30^2) - (1^2 + 2^2 + \ldots + 15^2)$$

$$= \frac{30 \times 31 \times 61}{6} - \frac{15 \times 16 \times 31}{6}$$

$$= 9455 - 1240 = 8215$$

19. Find $8^3 + 9^3 + 10^3 + \ldots + 23^3$.

Sol :

We have

$1^3 + 2^3 + 3^3 + \ldots + n^3$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

$8^3 + 9^3 + 10^3 + \ldots + 23^3$

$$= (1^3 + 2^3 + \ldots + 23^3) - (1^3 + 2^3 + \ldots + 7^3)$$

$$= \left[\frac{23 \times 24}{2}\right]^2 - \left[\frac{7 \times 8}{2}\right]^2$$

$$= (23 \times 12)^2 - (7 \times 4)^2$$

$$= (276)^2 - (28)^2$$

$$= 76176 - 784$$

$$= 75,392$$

20. 1 + 3 + 5 + … + 20 terms

Sol :

Sum of first n odd number $= n^2$

Sum of first 20 odd numbers $= 20^2$

$= 400$

IV. Long Answer Questions

1. Show that the square of an odd positive integer is of the form 8q + 1, for some integer q.

Sol :

First we will prove "any odd positive integer n is of the form 4q + 1 or 4q + 3, where q is some integer".

By Euclid's division Lemma,

If 'a' and 'b' are two positive integers then

$a = bq + r$ where $0 \le r < |b|$.

Suppose the positive integer be 'a' and b = 4

then $a = 4q + r$ where $0 \le r < |4|$

$a = 4q$, $= 2(2q)$; $=$ even;

$a = 4q + 1$; $a = 4q + 1$ $=$ odd;

$a = 4q + 2$; $= 2(2q + 1)$; $=$ even;

$a = 4q + 3$ $=$ odd

$\therefore$ Any positive odd integer is of the form 4q + 1 or 4q + 3

Case (1):

If $a = 4q + 1$

$a^2 = (4q + 1)^2$

$= 16q^2 + 8q + 1$

$= 8q(2q + 1) + 1$

$= 8m + 1$ where $m = q(2q + 1)$

Case (2):

If $a = 4q + 3$

then $a^2 = (4q + 3)^2 = 16q^2 + 24q + 9$

$= 8[2q^2 + 3q] + 8 + 1$

$= 8[2q^2 + 3q + 1] + 1$

$= 8m + 1$ where $m = 2q^2 + 3q + 1$

$\therefore$ We conclude that the square of an odd positive integer is of the form 8q + 1, for some integer q.

2. Show that if x and y are both odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

Sol :

Let m and n be any integers, then

$x = 2m + 1$ and $y = 2n + 1$ since x and y are odd

$x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4(m^2 + n^2) + 4(m + n) + (1 + 1)$

$= 4(m^2 + n^2) + 4(m + n) + 2$

$= 4q + 2$, where $q = (m^2 + n^2) + (m + n)$

$= 4q + 2$

$x^2 + y^2 = 2[2q + 1]$ which is an even number.

Thus $4q + 2$ is an even number, which is not divisible by 4.
i.e., It leaves the remainder 2.
Hence $x^2 + y^2$ is even but not divisible by 4.

3. If the H.C.F. of 210 and 55 is expressible in the form 210 × 5 + 55y. Find y.

Sol :

Let us find the H.C.F. of 210 and 55, Applying Euclid's Division Algorithm

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \\ \text{Remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 210 and 55 is 5

$$\begin{aligned} \text{Given} \quad 5 &= 210 \times 5 + 55\,y \\ 5 - 210 \times 5 &= 55\,y \\ 55\,y &= 5 - 1050 \\ &= -1045 \\ y &= \frac{-1045}{55} \\ y &= -19 \end{aligned}$$

4. If d is the H.C.F. of 56 and 72, find x and y satisfying d = 56x + 72y.

Sol :

Applying Euclid's division Algorithm to find the H.C.F. of 56 and 72, we have.

$$\begin{aligned} 72 &= 56 \times 1 + 16 & \text{... (1)} \\ 56 &= 16 \times 3 + 8 & \text{... (2)} \\ 16 &= 8 \times 2 + 0 \\ \text{Remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 56 and 72 $= 8$

From (2) we have

$$\begin{aligned} 8 &= 56 - 16 \times 3 \\ 8 &= 56 - (72 - 56 \times 1) \times 3 \\ & \qquad [\because \textit{from } (1)\ 16 = 72 - 56 \times 1] \\ &= 56 - (3 \times 72) + 3 \times 56 \\ &= 56\,(1 + 3) - (3 \times 72) \\ &= 56\,(4) + 72\,(-3) \end{aligned}$$

Comparing with $d = 56x + 72y$, we have

$$x = 4 \text{ and } y = -3$$

5. In an Interview, the number of participants in Mathematics, Physics and Chemistry are 60, 84 and 108 respectively. Find the minimum number of rooms required if in each room the same number of participants to be seated and all of them being in the same subject.

Sol :

The number of participants in each room is the H.C.F. of 60, 84 and 108.

First we find the H.C.F. of 60 and 84, by applying Euclid's Division Algorithm.

$$\begin{aligned} 84 &= 60 \times 1 + 24 \\ 60 &= 24 \times 2 + 12 \\ 24 &= 12 \times 2 + 0 \\ \text{The remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 60 and 84 is 12.

Now applying Euclid's Division Algorithm to 12 and 108

$$\begin{aligned} 108 &= 12 \times 9 + 0 \\ \text{Remainder} &= 0 \\ \therefore \text{H.C.F. of 12 and 108} &= 12 \\ \therefore \text{H.C.F. } (60, 84, 108) &= 12. \end{aligned}$$

$\therefore$ In each room, the minimum 12 number of participants can be seated.

$$\begin{aligned} \text{Total number of participants} &= 60 + 84 + 108 \\ &= 252 \\ \text{Number of rooms required} &= \frac{252}{12} = 21 \end{aligned}$$

H.C.F.

If a composite number n divides ab, then n need not divide neither a nor b. For example 6 divides 4×3, but 6 neither divides 4 nor 3.

6. Find the greatest number which can divide 1356, 1868 and 2764 leaving the same remainder 12 in each case

Sol :

Required number = H.C.F. of (1356 – 12), (1868 – 12), (2764 – 12)
= H.C.F. of 1344, 1856, 2752

$$\begin{array}{c|l} 2 & 2752 \\ \hline 2 & 1376 \\ \hline 2 & 688 \\ \hline 2 & 344 \\ \hline 2 & 172 \\ \hline 2 & 86 \\ \hline & 43 \end{array} \qquad \begin{array}{c|l} 2 & 1344 \\ \hline 2 & 672 \\ \hline 3 & 336 \\ \hline 7 & 112 \\ \hline 2 & 16 \\ \hline 2 & 8 \\ \hline 2 & 4 \\ \hline 2 & 2 \\ \hline & 1 \end{array} \qquad \begin{array}{c|l} 2 & 1856 \\ \hline 2 & 928 \\ \hline 2 & 464 \\ \hline 2 & 232 \\ \hline 2 & 116 \\ \hline 2 & 58 \\ \hline & 29 \end{array}$$

$$2752 = 2^6 \times 43^1$$

$$1344 = 2^6 \times 3^1 \times 7^1$$
$$1856 = 2^6 \times 29^1$$
$$\text{H.C.F. is } 2^6 = 64$$
$\therefore$ The required number = 64

7. Find the smallest number of five digits exactly divisible by 16, 24, 36 and 54.

Sol :

Smallest number of five digits is 10000.

Required number must be divisible by 16, 24, 36 and 54

2	16, 24, 36, 54
2	8, 12, 18, 27
3	4, 6, 9, 27
3	4, 2, 3, 9
2	4, 2, 1, 3
	2, 1, 1, 3

$$\text{L.C.M.} = 2^4 \times 3^3 = 16 \times 27 = 432$$

L.C.M. of 16, 24, 36, 54 = 432

$\therefore$ Required number is divisible by 432.

$$\begin{array}{r|l} & \ \ 231 \\ 432 & 10000 \\ & \ \ 864 \\ \hline & \ \ 1360 \\ & \ \ 1296 \\ \hline & \quad 64 \end{array}$$

On dividing 10000 by 432, we get remainder = 64

$$\text{Required number} = 10000 + (432 - 64) = 10368$$

8. Six bells commence tolling together and toll at intervals of 2, 4, 6, 8, 10 and 12 seconds respectively. In 30 minutes, how many times do they toll together?

Sol :

2	2, 4, 6, 8, 10, 12
3	1, 2, 3, 4, 5, 6
2	1, 2, 1, 4, 5, 2
	1, 1, 1, 2, 5, 1

L.C.M. = 12 × 10 = 120

L.C.M. of 2,4,6,8,10, 12 is 120

$\therefore$ The bells will toll together after every 120 sec. i.e., 2 min.

In 30 minutes they will toll together $\left(\frac{30}{2}+1\right) = 16$ times.

9. Solve $9x \equiv 5 \pmod{13}$

Sol :

$$9x \equiv 5 \pmod{13}$$
$$9x - 5 = 13\text{ k for some integer k.}$$
$$x = \frac{13k+5}{9} \text{ for some integer k.}$$

When we put 1, 10, 19,...... the 13k + 5 is divisible by 13.

$$x = \frac{13(1)+5}{9} = 2$$
$$x = \frac{13(10)+5}{9} = 15$$
$$x = \frac{13(19)+5}{9} = 28$$
$$x = \frac{13(28)+5}{9} = 41$$

$\therefore$ The solutions are 2, 15, 28, 41,...

10. Find the next five terms of the given sequence $a_1 = 1, a_n = a_{n-1} + 2, n \geq 2$

Sol :

Given $a_1 = 1$

$a_n = a_{n-1} + 2$ for $n \geq 2$

Put n = 2, $a_2 = a_{2-1} + 2 = a_1 + 2 = 1 + 2$

$a_2 = 3$

Put n = 3, $a_3 = a_{3-1} + 2 = a_2 + 2 = 3 + 2$

$a_3 = 5$

Put n = 4, $a_4 = a_{4-1} + 2 = a_3 + 2 = 5 + 2$

$a_4 = 7$

Put n = 5, $a_5 = a_{5-1} + 2 = a_4 + 2 = 7 + 2$

$a_5 = 9$

Put n = 6, $a_6 = a_{6-1} + 2 = a_5 + 2 = 9 + 2$

$a_6 = 11$

The next five terms are $a_2 = 3$; $a_3 = 5$; $a_4 = 7$; $a_5 = 9$ and $a_6 = 11$.

11. Find the next five terms of the sequence given by $a_1 = 4; a_n = 4a_{n-1} + 3, n > 1$

Sol :

Given $a_1 = 4$

$a_n = 4a_{n-1} + 3$ for $n > 1$

Put $n = 2$; $a_2 = 4a_{2-1} + 3 = 4a_1 + 3$
$= 4(4) + 3 = 16 + 3$
$a_2 = 19$

Put $n = 3$; $a_3 = 4a_{3-1} + 3 = 4a_2 + 3$
$= 4(19) + 3 = 76 + 3$
$a_3 = 79$

Put $n = 4$; $a_4 = 4a_{4-1} + 3 = 4a_3 + 3$
$= 4(79) + 3 = 316 + 3$
$a_4 = 319$

Put $n = 5$; $a_5 = 4a_{5-1} + 3 = 4a_4 + 3$
$= 4(319) + 3 = 1276 + 3$
$a_5 = 1279$

Put $n = 6$; $a_6 = 4a_{6-1} + 3 = 4a_5 + 3$
$= 4(1279) + 3 = 5116 + 3$
$a_6 = 5119$

$\therefore$ The next five terms are $a_2 = 19, a_3 = 79, a_4 = 319, a_5 = 1279, a_6 = 5119$.

12. A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6$ show that the first three terms of the sequence are zero and all other terms are positive.

Sol :

Given $a_n = n^3 - 6n^2 + 11n - 6$

Put $n = 1$, $a_1 = 1^3 - 6(1)^2 + 11(1) - 6$
$= 1 - 6 + 11 - 6 = 12 - 12$
$a_1 = 0$

Put $n = 2$, $a_2 = 2^3 - 6(2)^2 + 11(2) - 6$
$= 8 - (6 \times 4) + 22 - 6$
$= 8 - 24 + 22 - 6 = 30 - 30$
$a_2 = 0$

Put $n = 3$, $a_3 = 3^3 - 6(3)^2 + 11(3) - 6$
$= 27 - 54 + 33 - 6 = 60 - 60$
$a_3 = 0$

So we have $a_1 = a_2 = a_3 = 0$.

That is the value of the cubic polynomial $n^3 - 6n^2 + 11n - 6$ becomes zero for $n = 1, 2, 3$

$\therefore$ $(n - 1), (n - 2)$ and $(n - 3)$ are factors of a_n.

$\therefore$ $a_n = (n - 1)(n - 2)(n - 3)$ for which $a_n > 0$ for all $n > 3$.

$\therefore$ Other terms are positive.

13. The 10th term an A.P. is 52 and 16th term is 82. Find the 32nd term and the general term.

Sol :

$t_n = a + (n - 1)d$

10th term of an A.P. is 52.

$t_{10} = a + (10 - 1)d = 52$

$a + 9d = 52$ ...(1)

16th term is 82

$a + 15d = 82$ ...(2)

$a + 9d = 52$ ...(1)

Solving (1) and (2)

(2) – (1), $6d = 30$
$d = 5$

$a + 9d = 52$ from (1)
$a + 9(5) = 52$
$a + 45 = 52$
$a = 52 - 45 = 7$

$\therefore$ $t_{32} = a + 31d$
$= 7 + 31(5) = 7 + 155$
$t_{32} = 162$

The general term $a_n = 5n + 2$

14. The sum of 5th and 9th terms of an A.P. is 72 and the sum of 7th and 12th term is 97. Find the A.P.

Sol :

Given $a_5 + a_9 = 72$ and $a_7 + a_{12} = 97$

$a + 4d + a + 8d = 72$ and $a + 6d + a + 11d = 97$

$2a + 12d = 72$ ...(1) and $2a + 17d = 97$ ... (2)

Solving (1) and (2)

$2a + 12d = 72$ ...(1)

$2a + 17d = 97$ ...(2)

(2) – (1) $\Rightarrow$ $5d = 25$
$d = 5$

$2a + 12d = 72$
$2a + 12(5) = 72$
$2a + 60 = 72$
$2a = 72 - 60 = 12$

$$a = \frac{12}{2} = 6$$

$\therefore$ The A.P. is a, a + d, a + 2d. a + 3d ...

6, 6 + 5, 6 + 2(5), 6 + 3(5) ...

6, 11, 16, 21,...

15. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. then prove that a^2, b^2, c^2 are in A.P

Sol :

Given $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\therefore t_2 - t_1 = t_3 - t_2$$

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c+a)-(a+b)}{(a+b)(c+a)}$$

$$\frac{b+c-c-a}{cb+ab+c^2+ac} = \frac{c+a-a-b}{ca+bc+a^2+ab}$$

$$\frac{b-a}{ab+cb+ac+c^2} = \frac{c-b}{ab+bc+ca+a^2}$$

$$(b-a)(ab+bc+ca+a^2) = (c-b)(ab+cb+ac+c^2)$$

$$ab^2+b^2c+cab+a^2b-a^2b-abc-a^2c-a^3 = abc+c^2b+ac^2+c^3-ab^2-cb^2-abc-bc^2$$

$$ab^2+b^2c-a^2c-a^3 = ac^2+c^3-ab^2-cb^2$$

$$b^2c-a^2c+ab^2-a^3 = c^3-cb^2+ac^2-ab^2$$

$$c(b^2-a^2)+a(b^2-a^2) = c(c^2-b^2)+a(c^2-b^2)$$

$$(b^2-a^2)(c+a) = (c^2-b^2)(c+a)$$

$$b^2-a^2 = c^2-b^2$$

$t_2 - t_1 = t_3 - t_2$ of the sequence a^2, b^2, c^2.

$\therefore$ a^2, b^2, c^2 are in A.P.

16. Find the 6th term of the A.P. $\frac{2m+1}{m}, \frac{2m-1}{m}, \frac{2m-3}{m}, \ldots$

Sol :

Here $t_1 = \frac{2m+1}{m}$ $t_2 = \frac{2m-1}{m}$

$$d = t_2 - t_1 = \frac{2m-1}{m} - \frac{2m+1}{m} = \frac{2m-1-2m-1}{m} = \frac{-2}{m}$$

Now $t_n = a + (n-1)d$

$$t_n = \left[\frac{2m+1}{m}\right] + (n-1)\left[\frac{-2}{m}\right] = \left[\frac{2m+1}{m}\right] + \left[\frac{-2n}{m}\right] - 1\left[\frac{-2}{m}\right] = \frac{2m+1}{m} - \frac{2n}{m} + \frac{2}{m} = \frac{2m+1-2n+2}{m}$$

Thus n^{th} term $= \frac{2m-2n+3}{m}$

$\therefore$ 6^{th} term $t_6 = \frac{2m-2(6)+3}{m}$

6^{th} term $= \frac{2m-9}{m}$

17. If S_n the sum of n terms of an A.P. is given by $3n^2 - 4n$. Find the n^{th} term.

Sol :

We have $S_n = 3n^2 - 4n$

$$S_{n-1} = 3(n-1)^2 - 4(n-1) = 3(n^2-2n+1) - 4n + 4 = 3n^2 - 6n + 3 - 4n + 4 = 3n^2 - 10n + 7$$

$\therefore n^{th}$ term $= S_n - S_{n-1}$

$$= [3n^2-4n] - [3n^2-10n+7] = 3n^2 - 4n - 3n^2 + 10n - 7$$

$\therefore$ n^{th} term is $= 6n - 7$

18. The sum of first six terms of an A.P. is 42. The ratio of 10^{th} term to its 30^{th} term is 1 : 3. Calculate the first term and 13^{th} term of the A.P.

Sol :

$$S_6 = \frac{6}{2}\{2a+(6-1)d\} = 42$$

$\therefore$ $6a + 15d = 42$ ...(1)

Given $t_{10} : t_{30} = 1 : 3$

$$\frac{a+9d}{a+29d} = \frac{1}{3}$$

$3(a+9d) = a + 29d$

$3a + 27d = a + 29d$

$3a - a = 29d - 27d$

$2a = 2d$

$a = d$

From (1) $6a + 15d = 42$

$6d + 15d = 42$ $[\because a = d]$

$21d = 42$

$d = \frac{42}{21} = 2$

$a = d = 2$

Now $t_{13} = a + 12d = 2 + 12(2) = 2 + 24 = 26$

13^{th} term $= 26$

19. Which term of the A.P. 3, 15, 27, 39,... will be 120 more than its 21^{st} term?

Sol :

Let the first term a and common difference d.

$$a = 3;\ d = 15 - 3 = 12.$$

$$t_n = a + (n-1)d$$

$$t_{21} = 3 + (21-1) \times 12$$

$$= 3 + 20 \times 12 = 3 + 240$$

$$t_{21} = 243$$

Let the required term be the n^{th} term.

$$n^{th} \text{ term} = 120 + 21^{st} \text{ term}$$

$$= 120 + 243 = 363$$

$$t_n = a + (n-1)d$$

$$363 = 3 + (n-1) \times 12$$

$$363 - 3 = (n-1) \times 12$$

$$n - 1 = \frac{360}{12} = 30$$

$$n = 30 + 1\ n = 31$$

$\therefore$ The required term is 31^{st} term of the A.P.

20. If p^{th} term of an A.P. . is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of the first pq terms is $\frac{1}{2}(pq+1)$

Sol :

n^{th} term of an A.P.

$$t_n = a + (n-1)d$$

Given $\quad p^{th} \text{ term} = \frac{1}{q}$

$$a + (p-1)d = \frac{1}{q} \quad \text{...(1)}$$

$$q^{th} \text{ term} = \frac{1}{p} \Rightarrow a + (q-1)d = \frac{1}{p} \quad \text{...(2)}$$

$$(1) - (2) \Rightarrow [(p-1) - (q-1)]d = \frac{1}{q} - \frac{1}{p}$$

$$(p - 1 - q + 1)d = \frac{p-q}{pq}$$

$$(p-q)d = \frac{p-q}{pq}$$

$$d = \frac{p-q}{pq(p-q)}$$

$$d = \frac{1}{pq}$$

Substituting $d = \frac{1}{pq}$ in (1)

$$a + (p-1)\left(\frac{1}{pq}\right) = \frac{1}{q}$$

$$a = \frac{1}{q} - \frac{p-1}{pq}$$

$$= \frac{p - p + 1}{pq}$$

$$a = \frac{1}{pq}$$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore S_{pq} = \frac{pq}{2}\left[2\left(\frac{1}{pq}\right) + (pq-1)\left(\frac{1}{pq}\right)\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + \frac{pq}{pq} - \frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{1}{pq} + 1\right]$$

$$= \frac{1}{2}\left[\frac{pq}{pq} + pq\right] = \frac{1}{2}(1 + pq)$$

$$\therefore S_{pq} = \frac{1}{2}(pq+1)$$

21. If the sum of first 7 terms of an A.P. is 49 and that of first 17 terms is 289. Find the sum of n terms.

Sol :

Let 'a' be the first term.

'd' be the common difference.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_7 = \frac{7}{2}[2a + 6d] = 49$$

$$\frac{7}{2} \times 2(a + 3d) = 49$$

$$7(a + 3d) = 49$$

$$a + 3d = \frac{49}{7} = 7 \quad \text{...(1)}$$

$$S_{17} = \frac{17}{2}[2a + 16d] = 289$$

$$\Rightarrow \frac{17}{2}\times 2[a+8d] = 289$$

$$17(a+8d) = 289$$

$$a+8d = \frac{289}{17} = 17 \quad ...(2)$$

$$(2)-(1)\ a+8d-a-3d = 17-7$$

$$5d = 10$$

$$d = \frac{10}{5} = 2$$

From (1) we have

$$a+3(2) = 7$$

$$a+6 = 7$$

$$a = 7-6 = 1$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[2(1)+(n-1)(2)]$$

$$= \frac{n}{2}[2+2n-2]$$

$$= \frac{n}{2}[2n] = n^2$$

$\therefore$ Sum of n terms $= n^2$

22. Three numbers, the third which is being 12 form decreasing G.P. If the last term were 9 instead of 12, the three numbers would have formed an A.P. Find the common ratio of the G.P.

Sol :

The numbers a, b, 12 are in GP.

$$b^2 = 12a \quad ...(1)$$

a, b, 9 are in A.P

$$2b = a+9$$

$$a = 2b-9 \quad ...(2)$$

From (1) and (2)

$$b^2 = 12(2b-9)$$

$$b^2-24b+108 = 0$$

$$(b-18)(b-6) = 0$$

$$b = 6, 18$$

From (2) $\quad a = 2b-9 = 2(6)-9 = 3$ and

$$a = 2b-9 = 2(18)-9 = 27$$

$$\therefore \text{Common ratio} = \frac{b}{a} = \frac{6}{3} \text{ and } \frac{18}{27}$$

$$= 2 \text{ and } \frac{2}{3}$$

Since it is a decreasing G.P., Common ratio $= \frac{2}{3}$

23. If each term of a G.P. is positive and each term is the sum of its two succeeding terms find the common ratio of the G.P.

Sol :

If a, ar, ar^2,.... are in GP.

$$a = ar+ar^2 \Rightarrow r^2+r-1 = 0$$

$$r = \frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)} \quad \left[\because x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$r = \frac{-1+\sqrt{5}}{2}$$

$$r = \frac{\sqrt{5}-1}{2}$$

24. Given $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x\neq 0)$ then show that a, b, c, d are in G.P.

Sol :

Given $\quad \frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$

$$\frac{\frac{a}{b}+x}{\frac{a}{b}-x} = \frac{\frac{b}{c}+x}{\frac{b}{c}-x} = \frac{\frac{c}{d}+x}{\frac{c}{d}-x}$$

From first two relations

$$\frac{2\frac{a}{b}}{2x} = \frac{2\frac{b}{c}}{2x}$$

$$\therefore \frac{a}{b} = \frac{b}{c} \quad ...(1)$$

From last two relations

$$\frac{2\frac{b}{c}}{2x} = \frac{2\frac{c}{d}}{2x} \Rightarrow \frac{b}{c} = \frac{c}{d} \quad ...(2)$$

From (1) and (2) $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

$\therefore$ a, b, c, d are in G.P.

25. Find the sum to infinity of the G.P. $(\sqrt{2}+1)+1+(\sqrt{2}-1)+...$

Sol :

$$a = \sqrt{2}+1,$$

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$= \frac{\sqrt{2}-1}{2-1} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1$$

$$\text{Sum} = \frac{a}{1-r}$$

$$= \frac{\sqrt{2}+1}{1-\left(\sqrt{2}-1\right)} = \frac{\sqrt{2}+1}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}\left(\sqrt{2}-1\right)}$$

$$= \frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{\sqrt{2}\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$$

$$= \frac{3+2\sqrt{2}}{\sqrt{2}}$$

$$\text{Sum} = \frac{4+3\sqrt{2}}{2}$$

26. The fourth term of a G.P. is 4. Find the product of its first seven terms.

Sol :

Let a be the first term and r be the common ratio.

$$t_4 = 4 \Rightarrow ar^{4-1} = 4 \Rightarrow ar^3 = 4$$

Product of seven terms $t_1 \times t_2 \times t_3 \times t_4 \times t_5 \times t_6 \times t_7$

$$= (a)\,(ar)\,(ar^2)\,(ar^3)\,(ar^4)\,(ar^5)\,(ar^6)$$

$$= a^7 r^{21} = (ar^3)^7$$

$$= (4)^7 = 16384$$

Product of first seven terms $= 16384$

27. Find the 1025^{th} term in the sequence 1, 22, 4444, 88888888,....

Sol :

Number of digits in the given sequence are 1, 2, 4, 8,...
Which are in G.P. Let 1025^{th} term $= 2^n$

Then $1 + 2 + 4 + 8 + ...+ 2^{n-1} < 1025 \leq 1 + 2 + 4 + ...+ 2^n$

$$1.\frac{2^n-1}{2-1} < 1025 \leq 1.\frac{2^{n+1}-1}{2-1}$$

$2^n - 1 < 1025 \leq 2^{n+1} - 1$

$2^n < 1026 < 2^{n+1} - 1$ ...(1)

$2^{n+1} \geq 1026 > 1024$

$2^{n+1} > 2^{10}$

$n + 1 > 10$

$n > 9$

$\therefore$ 1025^{th} term is 2^{10}

28. Find the sum of 0.6 + 0.66 + 0.666 + ...upto n terms.

Sol :

Let $S_n = 0.6 + 0.66 + 0.666 + ...\text{upto n terms.}$

$= 6\,[0.1 + 0.11 + 0.111 + ...\text{upto n terms}]$

$= \frac{6}{9}\,[0.9 + 0.99 + 0.999 + ...\text{upto n terms}]$

(multiply and divide by 9)

$$= \frac{2}{3}\left[\frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+....\text{upto n terms}\right]$$

$$= \frac{2}{3}\left[\left(1-\frac{1}{10}\right)\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+...\text{ up to n terms}\right]$$

$$= \frac{2}{3}\left[n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+...\text{ upto n terms}\right)\right]$$

$$= \frac{2}{3}\left[n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}\right]$$

$$= \frac{2}{3}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$$

$$0.6 + 0.666 + ...\text{ n terms} = \left[\frac{2n}{3}-\frac{2}{27}\left(1-\frac{1}{10^n}\right)\right]$$

29. Find the sum to n terms $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

Sol :

$(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + ... = 2 + 6 + 12 + 20 + ...+ \text{n terms}$

$= (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + ... + \text{n terms}$

$= (1^2 + 2^2 + 3^2 + ...\text{ n terms}) + (1 + 2 + 3 + ... + \text{n terms})$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)}{2} \times \frac{2n+4}{3}$$

$$= \frac{n(n+1)(n+2)}{3}$$

30. Find the sum to n terms $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ... +$ n terms

Sol :

$(3 \times 1^2) + (5 \times 2^2) + (7 \times 3^2) + ... + \text{n terms} = 3 + 20 + 63 + ... + \text{n terms}$

$= (2\times 1^3 + 1^2) + (2\times 2^3 + 2^2) + (2\times 3^3 + 3^2) + ... + (2\times n^3 + n^2)$

$= 2(1^3 + 2^3 + 3^3 + ... + n^3) + (1^2 + 2^2 + ... + n^2)$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2[n(n+1)]^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{(2n+1)}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3(n^2+n)+(2n+1)}{3}\right]$$

$$= \frac{n(n+1)(3n^2+3n+2n+1)}{6}$$

$$= \frac{n(n+1)(3n^2+5n+1)}{6}$$

31. Find the sum to n terms $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$

Sol :

$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + ...$ n terms

$$= \frac{1}{4}[1^2+2.1+1] + \frac{1}{4}[2^2+2.2+1] + ... + \frac{1}{4}(n^2+2n+1)$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + \frac{n}{1}\right]$$

$$= \frac{n}{4}\left[\frac{(n+1)(2n+1)}{6} + (n+1) + \frac{1}{1}\right]$$

$$= \frac{n}{4}\left[\frac{2n^2+3n+1+6n+6+6}{6}\right]$$

$$= \frac{n}{24}[2n^2+9n+13]$$

32. Find the value of the sum 3 + 5 + 6 + 9 + 10 + 12 + 15 + 18 + 20 + … + 100

Sol :

$3 + 5 + 6 + 9 + 10 + 12 + 15 + 18 + 20 + ... + 100$

$= (3 + 6 + 9 + 12 + 15 + 18 + ... 99) + (5 + 10 + 15 + 20 + ... + 100) - (15 + 30 + ... 90)$

$= 3(1 + 2 + 3 + ... + 33) + 5(1 + 2 + 3 + ... + 20) - 15(1 + 2 + ... + 6)$

$$= 3\left(\frac{33\times 34}{2}\right) + 5\left(\frac{20\times 21}{2}\right) - 15\left(\frac{6\times 7}{2}\right)$$

$= (3\times 33\times 17) + (5\times 10\times 21) - (15\times 3\times 7)$

$= 1683 + 1050 - 315$

$= 2733 - 315$

$= 2418.$