Chapter 6: Differential Equations

Exercise 6.1 | Q 1 | Page 193

Determine the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 2\sin x$$

SOLUTION

The given D.E. is

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right) + y = 2\sin x$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

∴ the given D.E. is of order 2 and degree 1.

Exercise 6.1 | Q 2 | Page 193

Determine the order and degree of the following differential equation:

$$\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$$

SOLUTION

The given D.E. is

$$\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$$

On cubing both sides, we get

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^3$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 3.

∴ the given D.E. is of order 2 and degree 3.

Exercise 6.1 | Q 3 | Page 193

Determine the order and degree of the following differential equation:

$$\frac{dy}{dx} = \frac{2\sin x + 3}{\frac{dy}{dx}}$$

SOLUTION

The given D.E. is

$$\frac{dy}{dx} = \frac{2\sin x + 3}{\frac{dy}{dx}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = 2\sin x + 3$$

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 2.

∴ the given D.E. is of order 1 and degree 2.

Exercise 6.1 | Q 4 | Page 193

Determine the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$$

SOLUTION

The given D.E. is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \sqrt{1 + \frac{d^3y}{dx^3}}$$

On squaring both sides, we get

$$\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\right)^2 = 1 + \frac{d^3y}{dx^3}$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 1.

∴ the given D.E. has order 3 and degree 1.

Exercise 6.1 | Q 5 | Page 193

Determine the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 7x + 5 = 0$$

SOLUTION

The given D.E. is

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 7x + 5 = 0$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

∴ the given D.E. has order 2 and degree 1.

Exercise 6.1 | Q 6 | Page 193

Determine the order and degree of the following differential equation:

$$(y''')^2 + 3y'' + 3xy' + 5y = 0$$

SOLUTION

The given D.E. is

$$(y''')^2 + 3y'' + 3xy' + 5y = 0$$

This can be written as:

$$\left(\frac{d^3y}{dx^3}\right)^2 + 3\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right) + 5y = 0$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 2.

∴ the given D.E. has order 3 and degree 2.

Exercise 6.1 | Q 7 | Page 193

Determine the order and degree of the following differential equation:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

SOLUTION

The given D.E. is

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$.

∴ order = 2

Since, this D.E. cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

Exercise 6.1 | Q 8 | Page 193

Determine the order and degree of the following differential equation:

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$$

SOLUTION

The given D.E. is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\frac{d^2y}{dx^2}$$

On squaring both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 8^2\cdot\left(\frac{d^2y}{dx^2}\right)^2$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 2.

$\therefore$ the given D.E. has order 2 and degree 2.

Exercise 6.1 | Q 9 | Page 193

Determine the order and degree of the following differential equation:

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}}\cdot\left(\frac{dy}{dx}\right)^{\frac{1}{3}} = 20$$

SOLUTION

The given D.E. is

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}}\cdot\left(\frac{dy}{dx}\right)^{\frac{1}{3}} = 20$$

$$\therefore \left(\frac{d^3y}{dx^3}\right)^3 \cdot \left(\frac{dy}{dx}\right)^2 = 20^6$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 3.

$\therefore$ the given D.E. has order 3 and degree 3.

Exercise 6.1 | Q 10 | Page 193

Determine the order and degree of the following differential equation:

$$x + \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$$

SOLUTION

The given D.E. is

$$x + \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^2}$$

On squaring both sides, we get

$$\left(x + \frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore x^2 + 2x\frac{d^2y}{dx^2} + \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore x^2 + 2x\frac{d^2y}{dx^2} - 1 = 0$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

$\therefore$ the given D.E. has order 2 and degree 1.

EXERCISE 6.2 [PAGE 196]

Exercise 6.2 | Q 1.01 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:
$x^3 + y^3 = 4ax$

SOLUTION

$x^3 + y^3 = 4ax$ .....(1)

Differentiating both sides w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 4a \times 1$$

$$\therefore 3x^2 + 3y^2\frac{dy}{dx} = 4a$$

Substituting the value of 4a in (1), we get

$$x^3 + y^3 = \left(3x^2 + 3y^2\frac{dy}{dx}\right)x$$

$$\therefore x^3 + y^3 = 3x^3 + 3xy^2\frac{dy}{dx}$$

$$\therefore 2x^3 + 3xy^2\frac{dy}{dx} - y^3 = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.02 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$Ax^2 + By^2 = 1$

SOLUTION

$Ax^2 + By^2 = 1$

Differentiating both sides w.r.t. x, we get

$$A \times 2x + B \times 2y \frac{dy}{dx} = 0$$

$$\therefore Ax + By\frac{dy}{dx} = 0 \quad ....(1)$$

Differentiating again w.r.t. x, we get

$$A \times 1 + B\left[y\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{dy}{dx}\right] = 0$$

$$\therefore A + B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$$

$$\therefore A = -B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]$$

Substituting the value of A in (1), we get

$$-B\,x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + B\,y\frac{dy}{dx} = 0$$

$$\therefore -x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] + y\frac{dy}{dx} = 0$$

$$\therefore -xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$$

$$\therefore xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.03 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

y = A cos (log x) + B sin (log x)

SOLUTION

y = A cos (log x) + B sin (log x) ...(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -A \ \sin(\log x)\cdot\frac{d}{dx}(\log \ x) + B\cos(\log x)\cdot\frac{d}{dx}(\log x)$$

$$= \frac{-A\sin(\log x)}{x} + \frac{B\cos(\log x)}{x}$$

$\therefore x\dfrac{dy}{dx}$ = - A sin (log x) + B cos (log x)

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-A\sin(\log x)}{x} + \frac{B\cos(\log x)}{x}$$

$\therefore x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ = - [A cos (log x) + B sin (log x)] = - y .....[By (1)]

$\therefore x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ is the required D.E.

Exercise 6.2 | Q 1.04 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$y^2 = (x + c)^3$

SOLUTION

$y^2 = (x + c)^3$ ...(1)

Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 3(x+c)^2 \cdot (1) = 3(x+c)^2$$

$$\therefore (x+c)^2 = \frac{2y}{3} \cdot \frac{dy}{dx}$$

$$\therefore (x+c)^6 = \left(\frac{2y}{3} \cdot \frac{dy}{dx}\right)^3$$

$$\therefore \left(y^2\right)^2 = \frac{8y^3}{27} \cdot \left(\frac{dy}{dx}\right)^3 \quad \text{....[By (1)]}$$

$$\therefore 27y^4 = 8y^3\left(\frac{dy}{dx}\right)^3$$

$$\therefore 27y = 8\left(\frac{dy}{dx}\right)^3$$

$$\therefore 8\left(\frac{dy}{dx}\right)^3 - 27y = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.05 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$y = Ae^{5x} + Be^{-5x}$

SOLUTION

$y = Ae^{5x} + Be^{-5x}$ ...(1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{5x} \times 5 + Be^{-5x} \times (-5)$$

$$\therefore \frac{dy}{dx} = 5Ae^{5x} - 5Be^{-5x}$$

and $\frac{d^2y}{dx^2} = Ae^{5x} \times 5 + Be^{-5x} \times (-5)$

$$= 25Ae^{5x} + 25Be^{-5x}$$

$$= 25\left(Ae^{5x} + Be^{-5x}\right) = 25y \quad \text{....[By(1)]}$$

$$\therefore \frac{d^2y}{dx^2} - 25y = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.06 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:
$(y - a)^2 = 4(x - b)$

SOLUTION

$(y - a)^2 = 4(x - b)$

Differentiating twice w.r.t. x, we get

$$2(y - a) \cdot \frac{d}{dx}(y - a) = 4\frac{d}{dx}(x - b)$$

$$\therefore 2(y - a) \cdot \left(\frac{dy}{dx} - 0\right) = 4(1 - 0)$$

$$\therefore 2(y - a)\frac{dy}{dx} = 4$$

$$\therefore (y - a)\frac{dy}{dx} = 2 \quad \text{....(1)}$$

Differentiating again w.r.t. x, we get

$$(y - a)\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(y - a) = 0$$

$$\therefore (y - a)\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0\right) = 0$$

$$\therefore (y - a)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore \frac{2}{\frac{dy}{dx}} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{.....[By (1)]}$$

$$\therefore 2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.07 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

y = a + a/x

SOLUTION

$$y = a + \frac{a}{x} \qquad \text{....(1)}$$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(a + \frac{a}{x}\right) = 0 + a\left(-\frac{1}{x^2}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{a}{x^2}$$

$$\therefore a = -x^2\frac{dy}{dx}$$

Substituting the value of a in (1), we get

$$y = - x^2\frac{dy}{dx} + \frac{1}{x}\left(-x^2\frac{dy}{dx}\right)$$

$\therefore$ y = $-x^2 \dfrac{dy}{dx} - x\dfrac{dy}{dx}$

$\therefore \left(x^2 + x\right)\dfrac{dy}{dx} + y = 0$

$\therefore$ x(x + 1) $\dfrac{dy}{dx} + y = 0$

This is the required D.E.

Exercise 6.2 | Q 1.08 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$y = c_1e^{2x} + c_2e^{5x}$

SOLUTION

$y = c_1e^{2x} + c_2e^{5x}$ ....(1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = c_1e^{2x} \times 2 + c_2e^{5x} \times 5$$

$$\therefore \frac{dy}{dx} = 2c_1e^{2x} + 5c_2e^{5x} \quad \text{....(2)}$$

and $\dfrac{d^2y}{dx^2} = 2c_1e^{2x} \times 2 + 5c_2e^{5x} \times 5$

$$\therefore \frac{d^2y}{dx^2} = 4c_1e^{2x} + 25c_2e^{5x} \quad \text{.....(3)}$$

The equations (1), (2) and (3) are consistent in c_1e^{2x} and c_2e^{5x}

$\therefore$ determinant of their consistency is zero.

$$\therefore \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 2 & 5 \\ \frac{d^2y}{dx^2} & 4 & 25 \end{vmatrix} = 0$$

$$\therefore y(50 - 20) - 1\left(25\frac{dy}{dx} - 5\frac{d^2y}{dx^2}\right) + 1\left(4\frac{dy}{dx} - 2\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore 30y - 25\frac{dy}{dx} + 5\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2\frac{d^2y}{dx^2} = 0$$

$$\therefore 3\frac{d^2y}{dx^2} - 21\frac{dy}{dx} + 30y = 0$$

$$\therefore \frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.09 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$c_1x^3 + c_2y^2 = 5$

SOLUTION

$c_1x^3 + c_2y^2 = 5$ .....(1)

Differentiating twice w.r.t. x, we get

$$c_1 \times 3x^2 + c_2 \times 2y\frac{dy}{dx} = 0$$

$$\therefore 3c_1x^2 + 2c_2y\frac{dy}{dx} = 0 \text{ ....(2)}$$

Differentiating again w.r.t. x, we get

$$3c_1 \times 2x + 2c_2\left[y.\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{dy}{dx}\right] = 0$$

$$\therefore 6c_1x + 2c_2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$$

The equations (1), (2) and (3) in c_1, c_2 are consistent.

∴ determinant of their consistency is zero.

$$\therefore \begin{vmatrix} x^3 & y^2 & 5 \\ 3x^2 & 2y\frac{dy}{dx} & 0 \\ 6x & 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 & 0 \end{vmatrix}$$

$$\therefore x^3(0-0) - y^2(0-0) + 5\left[6x^2y\frac{d^2y}{dx^2} + 6x^2\left(\frac{dy}{dx}\right)^2 - 12xy\frac{dy}{dx}\right] = 0$$

$$\therefore 6x^2y\frac{d^2y}{dx^2} + 6x^2\left(\frac{dy}{dx}\right)^2 - 12xy\frac{dy}{dx} = 0$$

$$\therefore xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} = 0$$

This is the required D.E.

Exercise 6.2 | Q 1.1 | Page 196

Obtain the differential equation by eliminating the arbitrary constants from the following equation:
$y = e^{-2x}$ (A cos x + B sin x)

SOLUTION

$y = e^{-2x}$ (A cos x + B sin x)

∴ e^{-2x} y = A cos x + B sin x ....(1)

Differentiating twice w.r.t. x, we get

$$e^{2x}\cdot\frac{dy}{dx} + y\cdot e^{2x}\times 2 = A(-\sin x) + B\cos x$$

$$\therefore e^{2x}\left(\frac{dy}{dx}+2y\right) = -A \sin x + B \cos x$$

Differentiating again w.r.t. x, we get

$$e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}+2y\right)\cdot e^{2x}\times 2 = -A \cos x + B(-\sin x)$$

$$\therefore e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2\frac{dy}{dx}+4y\right) = -(A \cos x + B \sin x)$$

$$\therefore e^{2x}\left(\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y\right) = -e^{2x}.y \quad \text{....[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y = -y$$

$$\therefore \frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y = 0$$

This is the required D.E.

Exercise 6.2 | Q 2 | Page 196

Form the differential equation of the family of lines having intercepts a and b on the coordinate axes respectively.

SOLUTION

The equation of the line having intercepts a and b on the coordinate axes respectively, is

$$\frac{x}{a}+\frac{y}{b} = 1 \quad \text{....(1)}$$

where a and b are arbitrary constants.

Differentiating (1) w.r.t. x, we get

$$\frac{1}{a}(1)+\left(\frac{1}{b}\right)\cdot\frac{dy}{dx} = 0$$

$$\therefore \left(\frac{1}{b}\right)\frac{dy}{dx} = -\frac{1}{a}$$

$$\therefore \frac{dy}{dx} = -\frac{b}{a}$$

Differentiating again w.r.t. x, we get

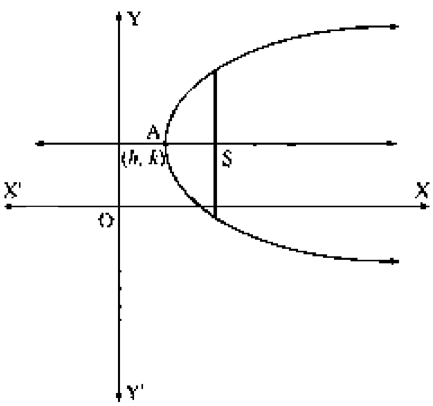
$$\frac{d^2y}{dx^2} = 0$$

This is the required D.E.

Exercise 6.2 | Q 3 | Page 196

Find the differential equation all parabolas having a length of latus rectum 4a and axis is parallel to the axis.

SOLUTION



Let A(h, k) be the vertex of the parabola whose length of the latus rectum is 4a.

Then the equation of the parabola is

$(y - k)^2 = 4a(x - h)$, where h and k are arbitrary constants. Differentiating w.r.t. x, we get

$$2(y - k) \cdot \frac{d}{dx}(y - k) = 4a\frac{d}{dx}(x - h)$$

$$\therefore 2(y - k)\left(\frac{dy}{dx} - 0\right) = 4a(1 - 0)$$

$$\therefore 2(y - k)\frac{dy}{dx} = 4a$$

$$\therefore (y - k)\frac{dy}{dx} = 2a \qquad \text{...(1)}$$

Differentiating again w.r.t. x, we get

$$(y - k) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(y - k) = 0$$

$$\therefore (y - k)\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0\right) = 0$$

$$\therefore (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore \frac{2a}{\left(\frac{dy}{dx}\right)} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \text{....[By (1)]}$$

$$\therefore 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

This is the required D.E.

Exercise 6.2 | Q 4 | Page 196

Find the differential equation of the ellipse whose major axis is twice its minor axis.

SOLUTION

Let 2a and 2b be lengths of major axis and minor axis of the ellipse.

Then 2a = 2(2b)

∴ a = 2b

∴ equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e. $\dfrac{x^2}{(2b)^2} + \dfrac{y^2}{b^2} = 1$

$$\therefore \frac{x^2}{4b^2} + \frac{y^2}{b^2} = 1$$

$\therefore x^2 + 4y^2 = 4b^2$

Differentiating w.r.t. x, we get

$$2x + 4 \times 2y\frac{dy}{dx} = 0$$

$$\therefore x + 4y\frac{dy}{dx} = 0$$

This is the required D.E.

Exercise 6.2 | Q 5 | Page 196

Form the differential equation of family of lines parallel to the line 2x + 3y + 4 = 0.

SOLUTION

The equation of the line parallel to the line 2x + 3y + 4 = 0 is

2x + 3y + 4 = 0, where c is an arbitrary constant.

Differentiating w.r.t. x, we get

$$2 \times 1 + 3\frac{dy}{dx}\ 0 = 0$$

$$\therefore\ 3\frac{dy}{dx} + 2 = 0$$

This is the required D.E.

Exercise 6.2 | Q 6 | Page 196

Find the differential equation of all circles having radius 9 and centre at point (h, k).

SOLUTION

Equation of the circle having radius 9 and centre at point (h, k) is

$(x - h)^2 + (y - k)^2 = 81,$ .....(1)

where h and k are arbitrary constant.

Differentiating (1) w.r.t. x, we get

$$2(x - h) \cdot \frac{d}{dx}(x - h) + 2(y - k) \cdot \frac{d}{dx}(y - k) = 0$$

$$\therefore (x - h)(1 - 0) + (y - k)\left(\frac{dy}{dx} - 0\right) = 0$$

$$\therefore (x - h) + (y - k)\frac{dy}{dx} = 0 \quad .....(2)$$

Differentiating again w.r.t. x, we get

$$\frac{d}{dx}(x - h) + (y - k) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(y - k) = 0$$

$$\therefore (1 - 0) + (y - k)\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0\right) = 0$$

$$\therefore (y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

$$\therefore (y - k)\frac{d^2y}{dx^2} = -\left[\left(\frac{dy}{dx}\right)^2 + 1\right]$$

$$\therefore y - k = \frac{-\left(\frac{dy}{dx}\right)^2 + 1}{\frac{d^2y}{dx^2}} \quad ....(3)$$

From (2), $x - h = -(y - k)\frac{dy}{dx}$

Substituting the value of (x - h) in (1), we get

$$(y - k)^2\left(\frac{dy}{dx}\right)^2 + (y - k)^2 = 81$$

$$\therefore \left(\frac{dy}{dx}\right)^2 + 1 = \frac{81}{(y - k)^2}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 + 1 = \frac{81 \cdot \frac{d^2y}{dx^2}}{\left[\left(\frac{dy}{dx}\right)^2 + 1\right]^2}$$

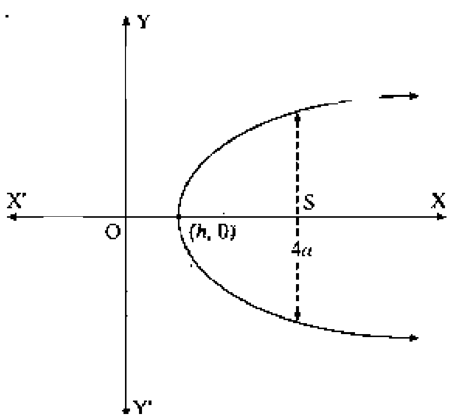
$$\therefore 81\left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3$$

This is the required D.E.

Exercise 6.2 | Q 7 | Page 196

Form the differential equation of all parabolas whose axis is the X-axis.

SOLUTION



The equation of the parbola whose axis is the X-axis is $y^2 = 4a(x - h)$, ...(1)

where a and h are arbitrary constants.

Differentiating (1) w.r.t. x, we get

$$2y\left(\frac{dy}{dx}\right) = 4a(1 - 0)$$

$$\therefore y\frac{dy}{dx} = 2a$$

Differentiating again w.r.t. x, we get

$$y \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{dy}{dx} = 0$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

This is the required D.E.

EXERCISE 6.3 [PAGES 200 - 201]

Exercise 6.3 | Q 1.1 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

xy = log y +c; $\frac{dy}{dx} = \frac{y^2}{1 - xy}$

SOLUTION

xy = log y +c

Differentiating w.r.t. x, we get

$$x \cdot \frac{dy}{dx} + y \times 1 = \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore x\frac{dy}{dx} + y = \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\left(x - \frac{1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore \left(\frac{xy - 1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore \frac{dy}{dx} = \frac{-y^2}{xy - 1} = \frac{y^2}{1 - xy}, \text{ if } xy \neq 1$$

Hence, xy = log y + c is a solution of the D.E.

$\frac{dy}{dx} = \frac{y^2}{1 - xy}$, if xy ≠ 1.

Exercise 6.3 | Q 1.2 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

$$y = \left(\sin^{-1} x\right)^2 + c; \left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

SOLUTION

$$y = \left(\sin^{-1} x\right)^2 + c \qquad .....(1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x\right)^2 + 0$$

$$\therefore \frac{dy}{dx} = 2\left(\sin^{-1} x\right) \cdot \frac{d}{dx}\left(\sin^{-1} x\right)$$

$$= 2\sin^{-1} x \times \frac{1}{\sqrt{1 - x^2}}$$

$$\therefore \sqrt{1 - x^2}\,\frac{dy}{dx} = 2\sin^{-1} x$$

$$\therefore \left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 = 4\left(\sin^{-1} x\right)^2$$

$$\therefore \left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 = 4(y - c) \qquad ....[\text{By } (1)]$$

Differentiating again w.r.t. x, we get

$$\left(1 - x^2\right) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \cdot \frac{d}{dx}\left(1 - x^2\right) = 4\frac{d}{dx}(y - c)$$

$$\therefore \left(1 - x^2\right) \cdot 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{dy}{dx} - 0\right)$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$\left(1-x^{2}\right)\frac{d^{2}y}{dx^{2}}-x\frac{dy}{dx}=2$$

Hence, $y = (\sin^{-1} x)^2 + c$ is a solution of the D.E.

$$\left(1-x^{2}\right)\frac{d^{2}y}{dx^{2}}-x\frac{dy}{dx}=2$$

Exercise 6.3 | Q 1.3 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

$y = e^{-x} + Ax + B$; $e^{x}\frac{d^{2}y}{dx^{2}}=1$

SOLUTION

$y = e^{-x} + Ax + B$;

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{-x}\times(-1)+A\times 1+0$$

$$\therefore \frac{dy}{dx}=e^{-x}+A$$

Differentiating again w.r.t. x, we get

$$\frac{d^{2}y}{dx^{2}}=-e^{-x}\times(-1)+0$$

$$\therefore \frac{d^{2}y}{dx^{2}}=\frac{1}{e^{x}}$$

$$\therefore e^{x}\frac{d^{2}y}{dx^{2}}=1$$

Hence, $y = e^{-x} + Ax + B$ is a solution of the D.E.

$$e^{x}\frac{d^{2}y}{dx^{2}}=1$$

Exercise 6.3 | Q 1.4 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

$y = x^m; x^2\frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my = 0$

SOLUTION

$y = x^m$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^m) = mx^{m-1}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(mx^{m-1}\right) = m\frac{d}{dx}\left(x^{m-1}\right) = m(m-1)x^{m-2}$

$$\therefore x^2\frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my$$

$$= x^2 \cdot m(m-1)x^{m-2} - mx \cdot mx^{m-1} + m \cdot x^m$$

$$= m(m - 1)x^m - m^2x^m + mx^m$$

$$= \left(m^2 - m - m^2 + m\right)x^m = 0$$

This shows that $y = x^m$ is a solution of the D.E.

$$x^2\frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my = 0.$$

Exercise 6.3 | Q 1.5 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

$y = a + \frac{b}{x}; x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

SOLUTION

$y = a + \frac{b}{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 + b\left(-\frac{1}{x^2}\right) = -\frac{b}{x^2}$$

$$\therefore x^2\frac{dy}{dx} = -b$$

Differentiating again w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}\left(x^2\right) = 0$$

$$\therefore x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 2x = 0$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Hence, $y = a + \frac{b}{x}$ is a solution of the D.E.

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Exercise 6.3 | Q 1.6 | Page 200

In the following example verify that the given expression is a solution of the corresponding differential equation:

$y = e^{ax}; x\frac{dy}{dx} = y\log y$

y = e^{ax}

∴ log y = log e^{ax} = ax log e

∴ log y = ax ......(1) .....[∵ log e = 1]

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = a \times 1$$

$$\therefore \frac{dy}{dx} = ay$$

$$\therefore x\frac{dy}{dx} = (ax)y$$

$$\therefore x\frac{dy}{dx} = y\log y \quad \text{....[By (1)]}$$

Hence, y = e^{ax} is a solution of the D.E. $x\frac{dy}{dx} = y\log y$.

Exercise 6.3 | Q 2.01 | Page 201

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{(1+y)^2}{(1+x)^2}$$

SOLUTION

$$\frac{dy}{dx} = \frac{(1+y)^2}{(1+x)^2}$$

$$\therefore \frac{1}{1+y^2}dy = \frac{1}{1+x^2}dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2}dx = \int \frac{1}{1+x^2}dx$$

$\therefore$ $\tan^{-1} y = \tan^{-1} x + c$

This is the general solution.

Exercise 6.3 | Q 2.02 | Page 201

Solve the following differential equation:

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

SOLUTION

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

$$\therefore \frac{dy}{dx} = e^{2x+3y} = e^{2x} . e^{3y}$$

$$\therefore \frac{1}{e^{3y}}dy = e^{2x}dx$$

Integrating both sides, we get

$$\int e^{-3y}dy = \int e^{2x'}\ dx$$

$$\therefore \int e^{-3}ydy = \int e^{2x}dx$$

$$\therefore \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c_1$$

$$\therefore 2e^{-3y} = -3e^{2x} + 6c_1$$

$\therefore$ $2e^{-3y} + 3e^{2x} = c$, where $c = 6c_1$

This is the general solution.

Exercise 6.3 | Q 2.03 | Page 201

Solve the following differential equation:

$$y - x\frac{dy}{dx} = 0$$

SOLUTION

$$y - x\frac{dy}{dx} = 0$$

$$\therefore x\frac{dy}{dx} = y$$

$$\therefore \frac{1}{x}dx = \frac{1}{y}dy$$

Integrating both sides, we get

$$\int \frac{1}{x}dx = \int \frac{1}{y}dy$$

$\therefore$ log |x| = log |y| + log c

$\therefore$ log |x| = log |cy|

$\therefore$ x = cy

This is the general solution.

Exercise 6.3 | Q 2.04 | Page 201

Solve the following differential equation:

$$\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$$

SOLUTION

$$\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$$

$$\therefore \frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating both sides, we get

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f\prime(x)}{f(x)}\ dx\ = \log|f(x)| + c$$

$\therefore$ the general solution is

log |tan x| + log |tan y| = log c, where c_1 = log c,

$$\therefore \log|\tan x \cdot \tan y| = \log c$$

$$\therefore \tan x \cdot \tan y = c$$

This is the general solution.

Exercise 6.3 | Q 2.05 | Page 201

Solve the following differential equation:

$$\cos x \cdot \cos y\ dy - \sin x \cdot \sin y dx = 0$$

SOLUTION

$$\cos x \cdot \cos y\ dy - \sin x \cdot \sin y dx = 0$$

$$\frac{\cos y}{\sin y} dy - \frac{\sin x}{\cos x} dx = 0$$

Integrating both sides, we get

$$\int \cot y\ dy - \int \tan x\ dx = c_1$$

$\therefore \log|\sin y| - [-\log|\cos x|] = \log c$,where c_1 = log c

$$\therefore \log|\sin y| + \log|\cos x| = \log c$$

$\therefore \log|\sin y| + \log|\cos x| = \log c$

$\therefore \log|\sin y \cdot \cos x| = \log c$

$\therefore \sin y \cdot \cos x = c$

This is the general solution.

Exercise 6.3 | Q 2.06 | Page 201

Solve the following differential equation:

$\frac{dy}{dx} = -k$, where k is a constant.

SOLUTION

$$\frac{dy}{dx} = -k,$$

$\therefore$ dy = - k dx

Integrating both sides, we get

$$\int dy = -k \int dx$$

$\therefore$ y = - kx + c

This is the general solution.

Exercise 6.3 | Q 2.07 | Page 201

Solve the following differential equation:

$$\frac{\cos^2 y}{x} dy + \frac{\cos^2 x}{y} dx = 0$$

SOLUTION

$$\frac{\cos^2 y}{x}dy + \frac{\cos^2 x}{y}dx = 0$$

$\therefore$ y $\cos^2 y$ dy + x $\cos^2$ x dx = 0

$$\therefore x\left(\frac{1-\cos 2x}{2}\right)dx + y\left(1+\frac{\cos 2y}{2}\right)dy = 0$$

$\therefore$ x(1 + cos 2x) dx + y(1 + cos 2y)dy = 0

$\therefore$ x dx + x cos 2x dx + y dy + y cos 2y dy = 0

Integrating both sides, we get

$$\int x\,dx + \int y\,dy + \int x\cos 2x\,dx + \int y\cos 2y\,dy = c_1 \qquad .....(1)$$

Using integration by parts

$$\int x\cos 2x\,dx = x\int \cos 2x\,dx - \int\left[\frac{d}{dx}(x)\int \cos 2x\,dx\right]dx$$

$$= x\left(\frac{\sin 2x}{2}\right) - \int 1\cdot\frac{\sin 2x}{2}dx$$

$$= \frac{x\sin 2x}{2} + \frac{1}{2}\cdot\frac{\cos 2x}{2} = \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

Similarly,

$$\int y\cos 2y\,dy = \frac{y\sin 2y}{2} + \frac{\cos 2y}{4}$$

$\therefore$ from (1), we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + \frac{y\sin 2y}{2} + \frac{\cos 2y}{4} = c_1$$

Multiplying throughout by 4, this becomes

$2x^2 + 2y^2 + 2x\sin 2x + \cos 2x + 2y\sin 2y + \cos 2y = 4c_1$

$\therefore 2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + \cos 2y + \cos 2x + c = 0$

where $c = -4c_1$

This is the general solution.

Exercise 6.3 | Q 2.08 | Page 201

Solve the following differential equation:

$$y^3 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$$

SOLUTION

$$y^3 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$$

$$\therefore y^3 = \frac{dy}{dx} + x^2 \frac{dy}{dx}$$

$$\therefore y^3 = \left(1 + x^2\right) \frac{dy}{dx}$$

$$\therefore \frac{1}{1 + x^2} dx = \frac{1}{y^3} dy$$

Integrating both sides, we get

$$\int \frac{1}{1 + x^2} dx = \int y^{-3} dy$$

$$\therefore \tan^{-1} x = \frac{y^{-2}}{-2} + c_1$$

$$\therefore \tan^{-1} x = -\frac{1}{2y^2} + c_1$$

$\therefore 2y^2 \tan^{-1} x = -1 + 2c_1 y^2$

$\therefore 2y^2 \tan^{-1} x + 1 = cy^2$, where $c = 2c_1$

This is the general solution.

Exercise 6.3 | Q 2.09 | Page 201

Solve the following differential equation:

$$2e^{x+2y}dx - 3dy = 0$$

SOLUTION

$$2e^{x+2y}dx - 3dy = 0$$

$$\therefore 2e^{x} \cdot e^{2y}dx - 3dy = 0$$

$$\therefore 2e^{x}dx - \frac{3}{e^{2y}}dy = 0$$

Integrating both sides, we get

$$2\int e^{x}dx - 3\int e^{-2y}dy = c_1$$

$$\therefore 2e^{x} - 3 \cdot \frac{e^{-2y}}{-2} = c_1$$

$$\therefore 4e^{x} + 3e^{-2y} = 2c_1$$

$\therefore 4e^{x} + 3e^{-2y} = c$, where $c = 2c_1$.

This is the general solution.

Exercise 6.3 | Q 2.1 | Page 201

Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2e^{y}$$

SOLUTION

$$\therefore \frac{dy}{dx} = e^{x} \cdot e^{y} + x^2 \cdot e^{y} = e^{y}\left(e^{x} + x^2\right)$$

$$\therefore \frac{1}{e^{y}}dy = \left(e^{x} + x^2\right)dx$$

Integrating both sides, we get

$$\int e^{-y}dy = \int (e^x + x^2)\,dx$$

$$\therefore \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c_1$$

$$\therefore e^x + e^{-y} + \frac{x^3}{3} = -c_1$$

$\therefore 3e^x + 3e^{-y} + x^3 = -3c_1$

$\therefore 3e^x + 3e^{-y} + x^3 = c$, where $c = -3c_1$

This is the general solution.

Exercise 6.3 | Q 3.1 | Page 201

For the following differential equation find the particular solution satisfying the given condition:
$3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$, when $x = 0$, $y = \pi$.

SOLUTION

$3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$, when $x = 0$, $y = \pi$.

$$\therefore \frac{3e^x}{1+e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating both sides, we get

$$3\int \frac{e^x}{1+e^x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$$

$\therefore$ the general solution is

3 log |1 + e^x| + log |tan y| = log c, where c_1 =log c

∴ log |(1 + e^x)^3 * tan y| = log c

∴ (1 + e^x)^3 tan y = c

When x = 0, y = π, we have

(1 + e^0)^3 tan π = c

∴ c = 0

∴ the particular solution is (1 + e^x)^3 tan y = 0

Exercise 6.3 | Q 3.2 | Page 201

For the following differential equation find the particular solution satisfying the given condition:

$\left(\mathbf{x} - \mathbf{y}^2\mathbf{x}\right)\mathbf{dx} - \left(\mathbf{y} + \mathbf{x}^2\mathbf{y}\right)\mathbf{dy} = \mathbf{0},$ when x = 2, y = 0

SOLUTION

$\left(\mathbf{x} - \mathbf{y}^2\mathbf{x}\right)\mathbf{dx} - \left(\mathbf{y} + \mathbf{x}^2\mathbf{y}\right)\mathbf{dy} = \mathbf{0},$ when x = 2, y = 0

∴ x (1 - y^2) dx - y (1 + x^2)dy = 0

$$\therefore \frac{x}{1+x^2}dx - \frac{y}{1-y^2}dy = 0$$

$$\therefore \frac{2x}{1+x^2} - \frac{2y}{1+y^2}dy = 0$$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f\prime(x)}{f(x)} dx = \log|f(x)| + c$$

$\therefore$ the general solution is

$\log|1 + x^2| + \log|1 - y^2| = \log c$, where $c_1 = \log c$

$\therefore \log|(1 + x^2)(1 - y^2)| = \log c$

$\therefore (1 + x^2)(1 - y^2) = c$

When $x = 2$, $y = 0$, we have

$(1 + 4)(1 - 0) = c$

$\therefore c = 5$

$\therefore$ the particular solution is $(1 + x^2)(1 - y^2) = 5$.

Exercise 6.3 | Q 3.3 | Page 201

For the following differential equation find the particular solution satisfying the given condition:

$y(1 + \log x)\frac{dx}{dy} - x \log x = 0, y = e^2$, when $x = e$

SOLUTION

$$y(1 + \log x)\frac{dx}{dy} - x \log x = 0$$

$$\therefore \frac{1 + \log x}{x \log x} dx - \frac{dy}{y} = 0$$

Integrating both sides, we get

$$\therefore \int \frac{1 + \log x}{x \log x} dx - \frac{dy}{y} = c_1 \qquad .....(1)$$

Put $x \log x = t$

Then $\left[x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)\right] dx = dt$

$\therefore \left[\frac{x}{x} + (\log x)(1)\right] dx = dt$

$\therefore \int \frac{1 + \log x}{x \log x} dx = \int \frac{dt}{t} = \log|t| = \log|x \log x|$

$\therefore$ from (1), the general solution is

log |x log x| - log |y| = log c, where c_1 = log c

$\therefore \log\left|\frac{x \log x}{y}\right| = \log c$

$\therefore \frac{x \log x}{y} = c$

$\therefore$ x log x = cy

This is the general solution.

Now, y = e^2, when x = e

$\therefore$ e log e = c.e^2

$\therefore$ 1 = c.e

$\therefore$ c = $\frac{1}{e}$

$\therefore$ the particular solution is x log x = $\left(\frac{1}{e}\right)y$

$\therefore$ y = ex log x.

Exercise 6.3 | Q 3.4 | Page 201

For the following differential equation find the particular solution satisfying the given condition:

$(e^y + 1)\cos x + e^y \sin x \frac{dy}{dx} = 0, \text{ when} x = \frac{\pi}{6}$, y = 0

SOLUTION

$$(e^y+1)\cos x + e^y \sin x \frac{dy}{dx} = 0$$

$$\therefore \frac{\cos x}{\sin x}dx + \frac{e^y}{e^y+1}dy = 0$$

Integrating both sides, we get

$$\int \frac{\cos x}{\sin x}dx + \int \frac{e^y}{e^y+1}dy = c_1$$

Now, $\frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(e^y+1) = e^y$ and

$\int \frac{f\prime(x)}{f(x)}$ dx = log |f(x)| + c

∴ from (1), the general solution is

∴ log |sin x| + log $|e^y$ + 1| = log c, where c_1 = log c

$$\therefore \sin x \cdot (e^y+1) = c$$

When x = $\frac{\pi}{4}$, y = 0, we get

$$\left(\frac{\sin \pi}{4}\right)(e^0+1) = c$$

∴ c = $\frac{1}{\sqrt{2}}(1+1) = \sqrt{2}$

∴ the particular solution is sin x$\cdot(e^y+1) = \sqrt{2}$

Exercise 6.3 | Q 3.5 | Page 201

For the following differential equation find the particular solution satisfying the given condition:

$(x+1)\frac{dy}{dx} - 1 = 2e^{-y}, y = 0$, when x = 1

SOLUTION

$$(x+1)\frac{dy}{dx} - 1 = 2e^{-y}$$

$$\therefore (x+1)\frac{dy}{dx} = \frac{2}{e^{y}} + 1 = \left(2 + e^{-y}\right)$$

$$\therefore \frac{e^{y}}{2+e^{-y}}dy = \frac{1}{x+1}dx$$

Integrating both sides, we get

$$\int \frac{e^{y}}{2+e^{-y}}dy = \int \frac{1}{x+1}dx$$

$\therefore$ log $|2 + e^{y}|$ = log $|x + 1|$ + log c ...... $\left[\therefore \frac{d}{dy}(2+e^{y}) = e^{y} \text{ and } \int \frac{f\prime(y)}{f(y)}dy = \log|f(y)| + c\right]$

$\therefore$ log $|2 + e^{y}|$ = log $|c\,(x + 1)|$

$\therefore 2 + e^{y} = c(x + 1)$

This is the general solution.

Now, y = 0, when x = 1

$\therefore 2 + e^{0} = c\,(1 + 1)$

$\therefore 3 = 2c$

$\therefore c = \frac{3}{2}$

$\therefore$ the particular solution is $2 + e^{y} = \frac{3}{2}(x+1)$

$\therefore 2(2 + e^{y}) = 3(x + 1)$

Exercise 6.3 | Q 3.6 | Page 201

For the following differential equation find the particular solution satisfying the given condition:

$$\cos\left(\frac{dy}{dx}\right) = a, a \in R, y(0) = 2$$

$$\cos\left(\frac{dy}{dx}\right) = a$$

$$\therefore \frac{dy}{dx} = \cos^{-1} a$$

$\therefore$ dy = $(\cos^{-1}$ a) dx

Integrating both sides, we get

$$\int dy = \left(\cos^{-1} a\right) \int dx$$

$\therefore$ y = $(\cos^{-1}$ a) x + c

$\therefore$ y = x $\cos^{-1}$ a + c

This is a general solution.

Now, y(0) = 2, i.e. y = 2, when x = 0

$\therefore$ 2 = 0 + c

$\therefore$ c = 2

$\therefore$ the particular solution is

y = x $\cos^{-1}$ a + 2

$\therefore$ y - 2 = x $\cos^{-1}$ a

$$\therefore \frac{y-2}{x} = \cos^{-1} a$$

$$\therefore \cos\left(\frac{y - 2}{x}\right) = a.$$

Exercise 6.3 | Q 4.1 | Page 201

Reduce the following differential equation to the variable separable form and hence solve:

$$\frac{dy}{dx} = \cos(x + y)$$

SOLUTION

Put x + y = u. Then $1 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{dy}{dx} = \frac{du}{dx} - 1$

$\therefore$ (1) becomes, $\frac{du}{dx} - 1$ = cos u

$\therefore \frac{du}{dx}$ = 1 + cos u

$\therefore \frac{1}{1 + \cos u} du$ = dx

Integrating both sides, we get

$$\int \frac{1}{1 + \cos u} du = \int dx$$

$$\therefore \int \frac{1}{2 \cos^2\left(\frac{u}{2}\right)} du = \int dx$$

$$\therefore \frac{1}{2} \int \sec^2\left(\frac{u}{2}\right) du = \int dx$$

$$\therefore \frac{1}{2} \frac{\tan\left(\frac{u}{2}\right)}{\frac{1}{2}} = x + c$$

∴ tan (("x + y")/2) = x + c

This is the general solution.

Exercise 6.3 | Q 4.2 | Page 201

Reduce the following differential equation to the variable separable form and hence solve:

$$(x - y)^2 \frac{dy}{dx} = a^2$$

SOLUTION

$(x - y)^2 \frac{dy}{dx} = a^2$ .....(1)

Put x - y = u

∴ x - u = y

∴ 1 - $\frac{du}{dx} = \frac{dy}{dx}$

∴ (1) becomes, $u^2\left(1 - \frac{du}{dx}\right) = a^2$

∴ $u^2 - u^2\frac{du}{dx} = a^2$

∴ $u^2 - a^2 = u^2\frac{du}{dx}$

∴ dx = $\frac{u^2}{u^2 - a^2}$du

Integrating both sides, we get

$\int dx = \int \frac{(u^2 - a^2) + a^2}{u^2 - a^2}$du

∴ x = $\int 1\ du + a^2 \int \frac{du}{u^2 - a^2} + c_1$

$\frac{1}{2a}\log\left|\frac{u - a}{u + a}\right| + c_1$

∴ x = x - y + $\frac{a}{2}\log\left|\frac{x - y - a}{x - y + a}\right| + c_1$

∴ - c_1 + y = $\frac{a}{2}\log\left|\frac{x - y - a}{x - y + a}\right|$

$$\therefore -2c_1 + 2y = a \log \left|\frac{x - y - a}{x - y + a}\right|$$

$$\therefore c + 2y = a \log \left|\frac{x - y - a}{x - y + a}\right|, \text{ where } c = -2c_1$$

This is the general solution.

Exercise 6.3 | Q 4.3 | Page 201

Reduce the following differential equation to the variable separable form and hence solve:

$$x + y\frac{dy}{dx} = \sec\left(x^2 + y^2\right)$$

SOLUTION

$$x + y\frac{dy}{dx} = \sec\left(x^2 + y^2\right) \quad \text{....(1)}$$

Put $x^2 + y^2 = u$

$$\therefore 2x + 2y\frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore x + y\frac{dy}{dx} = \frac{1}{2} \cdot \frac{du}{dx}$$

$$\therefore \text{(1) becomes, } \frac{1}{2} \cdot \frac{du}{dx} = \sec u$$

$$\therefore \frac{1}{\sec u} = 2 \cdot dx$$

Integrating both sides, we get

$$\int \cos u \, du = 2 \int dx$$

$$\therefore \sin u = 2x + c$$

$$\therefore \sin\left(x^2 + y^2\right) = 2x + c$$

This is the general solution.

Exercise 6.3 | Q 4.4 | Page 201

Reduce the following differential equation to the variable separable form and hence solve:

$$\cos^2(x - 2y) = 1 - 2\frac{dy}{dx}$$

SOLUTION

$$\cos^2(x - 2y) = 1 - 2\frac{dy}{dx} \quad .....(1)$$

Put x - 2y = u. Then $1 - 2\frac{dy}{dx} = \frac{du}{dx}$

∴ (1) becomes, $\cos^2 u = \frac{du}{dx}$

∴ $dx = \frac{1}{\cos^2 u} du$

Integrating both sides, we get

$\int dx = \int \sec^2 u \, du$

∴ x = tan u + c

∴ x = tan (x - 2y) + c

This is the general solution.

Exercise 6.3 | Q 4.5 | Page 201

Reduce the following differential equation to the variable separable form and hence solve:

(2x - 2y + 3)dx - (x - y + 1)dy = 0, when x = 0, y = 1.

SOLUTION

(2x - 2y + 3)dx - (x - y + 1)dy = 0

∴ (x - y + 1)dy = (2x - 2y + 3) dx

∴ $\frac{dy}{dx} = \frac{2(x - y + 3)}{(x - y) + 1}$ ....(1)

Put x - y = u. Then $1 - \frac{dy}{dx} = \frac{du}{dx}$

∴ $\frac{dy}{dx} = 1 - \frac{du}{dx}$

∴ (1) becomes, $1 - \frac{du}{dx} = \frac{2u + 3}{u + 1}$

∴ $\frac{du}{dx} = 1 - \frac{2u + 3}{u + 1} = \frac{u + 1 - 2u - 3}{u + 1}$

∴ $\frac{du}{dx} = \frac{-u - 2}{u + 1} = -\left(\frac{u + 2}{u + 1}\right)$

∴ $\frac{u + 2}{u + 1}$du = - dx

Integrating both sides, we get

$$\int \frac{u + 2}{u + 1} du = -\int 1 dx$$

∴ $\int \frac{(u + 2) - 1}{u + 2} du = -\int 1 dx$

∴ $\int \left(1 - \frac{1}{u + 2}\right) du = -\int 1 dx$

∴ u = log |u + 2| = - x + c

∴ x - y - log |x - y + 2| = - x + c

∴ (2x - y) - log |x - y + 2| = c

This is the general solution.

Now, y = 1, when x = 0

∴ (0 - 1) - log |0 - 1 + 2| = c

∴ - 1 - 0 = c

∴ c = - 1

∴ the particular solution is

(2x - y) - log |x - y + 2| = - 1

∴ (2x - y) - log |x - y + 2| + 1 = 0

EXERCISE 6.4 [PAGE 203]

Exercise 6.4 | Q 1 | Page 203

Solve the following differential equation:

$$\mathrm{x}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right)\mathrm{dy} = \left[\mathrm{y}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right) - \mathrm{x}\right]\mathrm{dx}$$

SOLUTION

$$\mathrm{x}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right)\mathrm{dy} = \left[\mathrm{y}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right) - \mathrm{x}\right]\mathrm{dx}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{y}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right) - \mathrm{x}}{\mathrm{x}\sin\left(\frac{\mathrm{y}}{\mathrm{x}}\right)} \quad .....(1)$$

Put y = vx

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{v} + \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}} \text{ and } \frac{\mathrm{y}}{\mathrm{x}} = \mathrm{v}$$

∴ (1) becomes, v + x$\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}\ \sin \mathrm{v} - \mathrm{x}}{\mathrm{x}\sin \mathrm{v}}$

$$\therefore \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{v}\sin(\mathrm{v} - 1)}{\sin \mathrm{v}} - \mathrm{v}$$

$\therefore \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{v \sin v - 1 - v \sin v}}{\sin \mathrm{v}} = \frac{-1}{\sin \mathrm{v}}$

$\therefore$ sin v dv = $-\frac{1}{\mathrm{x}}$dx

Integrating both sides, we get

$$\int \sin \mathrm{vdv} = -\int \frac{1}{\mathrm{x}}\mathrm{dx}$$

$\therefore$ - cos v = - log x - c

$\therefore$ cos $\left(\frac{\mathrm{y}}{\mathrm{x}}\right) = \log \mathrm{x} + \mathrm{c}$

This is the general solution.

Exercise 6.4 | Q 2 | Page 203

Solve the following differential equation:

$(x^2 + y^2)dx - 2xy\ dy = 0$

SOLUTION

$(x^2 + y^2)dx - 2xy\ dy = 0$

$\therefore 2xy\ dy = (x^2 + y^2)dx$

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{x}^2 + \mathrm{y}^2}{2\mathrm{xy}}$ ....(1)

Put y = vx

$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{v} + \mathrm{x}du/dx$

$\therefore$ (1) becomes, v + x$\frac{\mathrm{du}}{\mathrm{dx}} = \frac{\mathrm{x}^2 + \mathrm{v}^2\mathrm{x}^2}{2\mathrm{x}(\mathrm{vx})}$

$\therefore \mathrm{v} + \mathrm{x}\frac{\mathrm{du}}{\mathrm{dx}} = \frac{1 + \mathrm{v}^2}{2\mathrm{v}}$

$$\therefore x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2}dv = \frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$$-\int \frac{-2v}{1-v^2}dv = \int \frac{1}{x}dx$$

$\therefore$ - log |1 - v^2| = log x + log c_1 .... $\left[\because \frac{d}{dv}\left(1-v^2\right) = -2v \text{ and } \int \frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right]$

$$\therefore \log\left|\frac{1}{1-v^2}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{1}{1-\left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{x^2}{x^2-y^2}\right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1}x$$

$\therefore x^2 - y^2 = cx$, where c = $\frac{1}{c_1}$

This is the general solution.

Exercise 6.4 | Q 3 | Page 203

Solve the following differential equation:

$$\left(1+2e^{\frac{x}{y}}\right)+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\frac{dy}{dx}=0$$

SOLUTION

$$\left(1+2e^{\frac{x}{y}}\right)+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\frac{dy}{dx}=0$$

$$\therefore \left(1+2e^{\frac{x}{y}}\right)+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\cdot\frac{1}{\frac{dy}{dx}}=0$$

$$\therefore \left(1+2e^{\frac{x}{y}}\right)\frac{dx}{dy}+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)=0 \quad \text{....(1)}$$

Put $\frac{x}{y}=u$

$\therefore$ x = uy

$$\therefore \frac{dx}{dy}=u+y\frac{du}{dy}$$

$\therefore$ (1) becomes, $(1+2e^{u})\left(u+y\frac{du}{dy}\right)+2e^{u}(1-u)=0$

$$u+2ue^{u}+y(1+2e^{u})\frac{du}{dy}+2e^{u}-2ue^{u}=0$$

$$\therefore (u+2e^{u})+y(1+2e^{u})\frac{du}{dy}=0$$

Integrating both sides, we get

$$\int \frac{1}{y} dy + \int \frac{1 + 2e^u}{u + 2e^u} du = c_1$$

$\therefore$ log |y| + log |u + $2e^u$| = log c, where c_1 = log c ......

$$\left[\because \frac{d}{du}(u + 2e^u) = 1 + 2e^u \text{ and } \int \frac{f\prime(u)}{f(u)} du = \log|f(u) + c|\right]$$

$\therefore$ log |y (u + $2e^u$)| = log c

$\therefore$ y(u + $2e^u$) = c

$\therefore y\left(\frac{x}{y} + 2e^{\frac{x}{y}}\right)$ = c

$\therefore$ x + $2ye^{\frac{x}{y}} = c$

This is the general solution.

Exercise 6.4 | Q 5 | Page 203

Solve the following differential equation:

$(x^2 - y^2)dx + 2xy\, dy = 0$

SOLUTION

$(x^2 - y^2)dx + 2xy\, dy = 0$

$\therefore - 2xy\, dy = (x^2 - y^2)dx$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{-2xy} \quad ....(1)$$

put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) becomes, v + x $\frac{dv}{dx} = \frac{x^2 - v^2x^2}{-2x(vx)}$

$$\therefore v + x\frac{dv}{dx} = \frac{1 - v^2}{-2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 - v^2}{-2v} - v = \frac{1 - v^2 + 2v^2}{\text{-}2v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{\text{-}2v}$$

$$\therefore \frac{-2v}{1 + v^2}dv = \frac{1}{x}dx$$

Integrating both sides, we get

$$\therefore \int \frac{-2v}{1 + v^2}dv = \int \frac{1}{x}dx$$

$$\therefore \log\left|1 + v^2\right| = \log x + \log c_1$$

$$\text{....}\left[\because \frac{d}{dx}(1 + v^2) = 2v \text{ and } \int\left[\frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right]\right.$$

$$\therefore \log\left|\frac{1}{1 + v^2}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{1}{1 + \left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{x^2}{x^2 + y^2}\right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2 + y^2} = c_1 x$$

$$\therefore x^2 + y^2 = \frac{1}{c_1}x$$

$\therefore x^2 + y^2 = cx$ where $c = \dfrac{1}{c_1}$

This is the general solution.

Exercise 6.4 | Q 6 | Page 203

Solve the following differential equation:

$$\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0$$

SOLUTION

$$\frac{dy}{dx} + \frac{x - 2y}{2x - y} = 0 \quad ....(1)$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x - 2y}{2x - y}\right)$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{du}{dx}$$

$\therefore$ (1) becomes, $v + x\frac{du}{dx} = -\left(\frac{x - 2vx}{2x - vx}\right)$

$$\therefore v + x\frac{du}{dx} = -\left(\frac{1 - 2v}{2 - v}\right)$$

$$\therefore x\frac{dv}{dx} = -\left(\frac{1 - 2v}{2 - v}\right) - v$$

$$\therefore x\frac{dv}{dx} = \frac{-1 + 2v - 2v + v^2}{2 - v}$$

$$\therefore x\frac{dv}{dx} = \frac{v^2 - 1}{2 - v}$$

$$\therefore \frac{2 - v}{v^2 - 1}dv = \frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{2-v}{v^2-1}dv = \int \frac{1}{x}dx$$

$$\therefore 2\int \frac{1}{v^2-1}dv - \frac{1}{2}\int \frac{2v}{v^2-1}dv = \int \frac{1}{x}dx$$

$$\therefore 2 \times \frac{1}{2}\log\left|\frac{v - 1}{v+1}\right| - \frac{1}{2}\log\left|v^2-1\right| = \log|x| + \log c_1 \quad .....$$

$$\left[\because \frac{d}{dx}\left(v^2-1\right) = 2v \text{ and } \int \frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right.$$

$$\therefore \log\left|\frac{v - 1}{v+1}\right| - \log\left|\left(v^2-1\right)^{\frac{1}{2}}\right| = \log|c_1 x|$$

$$\therefore \log\left|\frac{v - 1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}}\right| = \log|c_1 x|$$

$$\therefore \frac{v - 1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}} = c_1 x$$

$$\therefore \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \cdot \frac{1}{\sqrt{\frac{y^2}{x^2}-1}} = c_1 x$$

$$\therefore \frac{y - x}{y+x} \cdot \frac{x}{\sqrt{y^2-x^2}} = c_1 x$$

$$\therefore \frac{y - x}{y+x} = c_1\sqrt{y^2-x^2}$$

$$\therefore \frac{y - x}{y+x} = c_1\sqrt{y - x} \cdot (y+x)$$

$$\therefore \sqrt{y - x} = c_1(y+x)^{\frac{3}{2}}$$

$\therefore$ y - x = $c_1^2(x+y)^3$

$\therefore$ y - x = c(x + y)^3, wherec = c_1^2

$\therefore$ y = c(x + y)^3 + x

This is the general solution.

Exercise 6.4 | Q 7 | Page 203

Solve the following differential equation:

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

SOLUTION

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0 \quad ...(1)$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \text{ and } \frac{y}{x} = v$$

$\therefore$ (1) becomes, x $\left(v + x\frac{dv}{dx}\right) - vx + x\sin v = 0$

$$\therefore vx + x^2\frac{dv}{dx} - vx + x\sin v = 0$$

$$\therefore x^2\frac{dv}{dx} + x\sin v = 0$$

$$\therefore \frac{1}{\sin v}dv + \frac{1}{x}dx = 0$$

Integrating, we get

$$\therefore \int \operatorname{cosec} v\, dv + \int \frac{1}{x}dx = c_1$$

$\therefore \log|\operatorname{cosec} v \text{ - } \cot v| + \log|x| = \log c$, where c_1 = log c

$$\therefore \log|x(\operatorname{cosec} v - \cot v)| = \log c$$

$$\therefore \log|x(\operatorname{cosec} v - \cot v)| = \log c$$

$$\therefore x\left(\frac{1}{\sin v} - \frac{\cos v}{\sin v}\right) = c$$

$\therefore$ x(1 - cos v) = c sin v

$$\therefore x\left[1 - \cos\left(\frac{y}{x}\right)\right] = c\sin\left(\frac{y}{x}\right)$$

This is the general solution.

Exercise 6.4 | Q 8 | Page 203

Solve the following differential equation:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

SOLUTION

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\therefore \left(1 + e^{\frac{x}{y}}\right)\frac{dx}{dy} + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0 \quad .....(1)$$

Put $\frac{x}{y} = u$

$\therefore$ x = uy

$$\therefore \frac{dx}{dy} = u + y\frac{du}{dy}$$

$\therefore$ (1) becomes, $(1 + e^u)\left(u + y\frac{du}{dy}\right) + e^u(1 - u) = 0$

$$\therefore u + ue^u + y(1 + e^u)\frac{du}{dy} + e^u - ue^u = 0$$

$$\therefore (u + e^u) + y(1 + e^u)\frac{du}{dy} = 0$$

$$\therefore \frac{dy}{y} + \frac{1+e^u}{u+e^u} du = 0$$

$$\therefore \int \frac{dy}{y} + \int \frac{1+e^u}{u+e^u} du = c_1 \quad ....(2)$$

$$\therefore \frac{d}{du}(u+e^u) = 1+e^u \text{ and } \int \frac{f'(u)}{f(u)} du = \log|f(u)| + c$$

$\therefore$ from (2), the general solution is

$\log|y| + \log|u + e^u| = \log c$, where $c_1 = \log c$

$\therefore \log|y(u + e^u)| = \log c$

$\therefore y(u + e^u) = c$

$$\therefore y\left(\frac{x}{y} + e^{\frac{x}{y}}\right) = c$$

$\therefore x + ye^{\frac{x}{y}} = c$

This is the general solution.

Exercise 6.4 | Q 9 | Page 203

Solve the following differential equation:

$$y^2 - x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

SOLUTION

$$y^2 - x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$\therefore x^2 \frac{dy}{dx} + xy \frac{dy}{dx} = y^2$$

$$\therefore (x^2 + xy) \frac{dy}{dx} = y^2$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x^2 + xy} \qquad ......(1)$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) becomes, $v + x\dfrac{dv}{dx} = \dfrac{v^2x^2}{x^2 + x \cdot vx} = \dfrac{v^2}{1+v}$

$$\therefore x\frac{dv}{dx} = \frac{v^2}{1+v} - v = \frac{v^2 - v - v^2}{1+v}$$

$$\therefore x\frac{dv}{dx} = \frac{-v}{1+v}$$

$$\therefore \frac{1+v}{v}dv = -\frac{1}{x}dx$$

Integrating, we get

Integrating, we get

$$\int \frac{1+v}{v}dv = -\int \frac{1}{x}dx$$

$$\int \left(\frac{1}{v} + 1\right)dv = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{1}{v}dv + \int 1dv = -\int \frac{1}{x}dx$$

$\therefore$ log |v| + v = - log |x| + c

$\therefore$ log $\left|\dfrac{y}{x}\right| + \dfrac{y}{x} = -\log|x| + c$

$\therefore$ log |y| - log |x| + $\dfrac{y}{x}$ = - log |x| + c

$$\therefore \frac{y}{x} + \log|y| = c$$

This is the general solution.

Exercise 6.4 | Q 10 | Page 203

Solve the following differential equation:

$$xy\frac{dy}{dx} = x^2 + 2y^2, y(1) = 0$$

SOLUTION

$$xy\frac{dy}{dx} = x^2 + 2y^2, y(1) = 0$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \quad \text{....(1)}$$

Put y = vx. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 2v^2x^2}{x \cdot vx} = \frac{1 + 2v^2}{v}$

$$\therefore x\frac{dv}{dx} = \frac{1 + 2v^2}{v} - v = \frac{1 + 2v^2 - v^2}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore \frac{v}{1 + v^2}dv = \frac{1}{x}dx$$

Integrating, we get

$$\therefore \int \frac{v}{1 + v^2}dv = \int \frac{1}{x}dx$$

$$\therefore \frac{1}{2}\int \frac{2v}{1 + v^2}dv = \int \frac{1}{x}dx + \log c_1$$

$$\therefore \frac{1}{2}\log\left|1 + v^2\right| = \log|x| + \log c_1$$

$$\therefore \log\left|1 + v^2\right| = 2\log\left|x^2\right| + 2\log c_1^2$$

$$\therefore \log\left|1 + v^2\right| = \log\left|cx^2\right|, \text{ where } c = c_1^2$$

$\therefore$ 1 + v^2 = cx^2

$$\therefore 1 + \frac{y^2}{x^2} = cx^2$$

$$\therefore \frac{x^2 + y^2}{x^2} = cx^2$$

$$\therefore x^2 + y^2 = cx^4$$

This is the general solution.

Now, y(1) = 0, i.e. when x = 1, y = 0, we get

1 + 0 = c(1)

$\therefore$ c = 1

$\therefore$ the particular solution is $x^2 + y^2 = x^4$.

Exercise 6.4 | Q 11 | Page 203

Solve the following differential equation:

x dx + 2y dx = 0, when x = 2, y = 1

SOLUTION

x dx + 2y dx = 0

$\therefore$ x dy = - 2y dx

$$\therefore \frac{1}{y} dy = \frac{-2}{x} dx$$

Integrating, we get

$$\int \frac{1}{y} dy = -2 \int \frac{1}{x} dx$$

$\therefore$ log |y| = - 2 log |x| + log c

$\therefore$ log |y| = - log $|x^2|$ + log c

$\therefore \log|y| = \log\left|\frac{c}{x^2}\right|$

$\therefore y = \frac{c}{x^2}$

$\therefore x^2y = c$

This is the general solution.

When x = 2, y = 1, we get

4(1) = c

$\therefore$ c = 4

$\therefore$ the particular solution is

$x^2y = 4$.

Exercise 6.4 | Q 12 | Page 203

Solve the following differential equation:

$$x^2\frac{dy}{dx} = x^2 + xy + y^2$$

SOLUTION

$$x^2\frac{dy}{dx} = x^2 + xy + y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \quad ...(1)$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + x \cdot vx + v^2x^2}{x^2}$

$$\therefore \mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=1+\mathrm{v}+\mathrm{v}^2$$

$$\therefore \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=1+\mathrm{v}^2$$

$$\therefore \frac{1}{1+\mathrm{v}^2}\mathrm{dv}=\frac{1}{\mathrm{x}}\mathrm{dx}$$

Integrating, we get

$$\int\frac{1}{1+\mathrm{v}^2}\mathrm{dv}=\int\frac{1}{\mathrm{x}}\mathrm{dx}$$

∴ tan-1 v = log |x| + c

$$\therefore \tan^{-1}\left(\frac{\mathrm{y}}{\mathrm{x}}\right)=\log|\mathrm{x}|+\mathrm{c}$$

This is the general solution.

Exercise 6.4 | Q 13 | Page 203

Solve the following differential equation:

(9x + 5y) dy + (15x + 11y)dx = 0

SOLUTION

(9x + 5y) dy + (15x + 11y)dx = 0

∴ (9x + 5y) dy = - (15x + 11y) dx

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{-(15\mathrm{x}+11\mathrm{y})}{9\mathrm{x}+5\mathrm{y}} \quad ...(1)$$

Put y = vx

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}}=\mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}$$

∴ (1) becomes, $\mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{-(15\mathrm{x}+11\mathrm{y})}{9\mathrm{x}+5\mathrm{y}}$

$$\therefore \mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{-(15+11\mathrm{v})}{9+5\mathrm{v}}$$

$$\therefore \mathrm{v}+\mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{-(15+11\mathrm{v})}{9+5\mathrm{v}}-\mathrm{v}=\frac{-15-11\mathrm{v}-9\mathrm{v}-5\mathrm{v}^2}{9+5\mathrm{v}}$$

$$\therefore \mathrm{x}\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{-5\mathrm{v}^2-20\mathrm{v}-15}{9+5\mathrm{v}}=-\left(\frac{5\mathrm{v}^2+20\mathrm{v}+15}{5\mathrm{v}+9}\right)$$

$$\therefore \frac{5\mathrm{v}+9}{5\mathrm{v}^2+20\mathrm{v}+15}\mathrm{dv}=-\frac{1}{\mathrm{x}}\mathrm{dx} \quad \text{....(2)}$$

Integrating, we get

$$\frac{1}{5}\int\frac{5\mathrm{v}+9}{\mathrm{v}^2+4\mathrm{v}+3}\mathrm{dv}=-\int\frac{1}{\mathrm{x}}\mathrm{dx}$$

Let $\frac{5\mathrm{v}+9}{\mathrm{v}^2+4\mathrm{v}+3}=\frac{5\mathrm{v}+9}{(\mathrm{v}+3)(\mathrm{v}+1)}=\frac{\mathrm{A}}{\mathrm{v}+3}+\frac{\mathrm{B}}{\mathrm{v}+1}$

∴ 5v + 9 = A(v + 1) + B(v + 3)

Put v + 3 = 0, i.e. v = - 3, we gwr

- 15 + 9 = A(- 2) + B(0)

∴ - 6 = - 2A

∴ A = 3

Put v + 1 = 0, i.e. v = - 1, we get

- 5 + 9 = A(0) + B(2)

∴ 4 = 2B

∴ B = 2

$$\therefore \frac{5\mathrm{v}+9}{\mathrm{v}^2+4\mathrm{v}+3}=\frac{3}{\mathrm{v}+3}+\frac{2}{\mathrm{v}+1}$$

∴ (2) becomes,

$$\frac{1}{5}\int\left(\frac{3}{v+3}+\frac{2}{v+1}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \frac{3}{5}\int\frac{1}{v+3}dv+\frac{2}{5}\int\frac{1}{v+1}dv=-\int\frac{1}{x}dx$$

$$\therefore \frac{3}{5}\log|v+3|+\frac{2}{5}\log|v+1|=-\log|x|+c$$

∴ 3 log |v + 3| + 2 log |v + 1| = - 5 log x + $5c_1$

$\therefore \log\left|(v+3)^3\right|+\log\left|(v+1)^2\right|=-\log\left|x^5\right|+\log c$, where $5c_1$ = log c

$$\therefore \log\left|(v+3)^3(v+1)^2\right|=\log\left|\frac{c}{x^5}\right|$$

$$\therefore (v+3)^3(v+1)^2=\frac{c}{x^5}$$

$$\therefore \left(\frac{y}{x}+3\right)^3\left(\frac{y}{x}+1\right)^2=\frac{c}{x^5}$$

$$\therefore (x+y)^2(3x+y)^3=c$$

This is the general solution.

Exercise 6.4 | Q 14 | Page 203

Solve the following differential equation:

(x^2 + 3xy + y2)dx - x^2 dy = 0

SOLUTION

(x^2 + 3xy + y2)dx - x^2 dy = 0

∴ x^2 dy = (x^2 + 3xy + y2)dx

$$\therefore \frac{dy}{dx}=\frac{x^2+3xy+y^2}{x^2} \quad ....(1)$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 3x \cdot vx + v^2x^2}{x^2}$

$$\therefore v + x\frac{dv}{dx} = 1 + 3v + v^2$$

$$x\frac{dv}{dx} = v^2 + 2v + 1 = (v+1)^2$$

$$\therefore \frac{1}{(v+1)^2}dv = \frac{1}{x}dx$$

Integrating, we get

$$\int (v+1)^{-2}dv = \int \frac{1}{x}dx$$

$$\therefore \frac{(v+1)^{-1}}{-1} = \log|x| + c_1$$

$$\therefore -\frac{1}{v+1} = \log|x| + c_1$$

$$\therefore -\frac{1}{\frac{y}{x}+1} = \log|x| + c_1$$

$$\therefore -\frac{x}{y+x} = \log|x| + c_1$$

$$\therefore \log|x| + \frac{x}{x+y} = -c_1$$

$\therefore \log|x| + \frac{x}{x+y} = c$, where c = - c_1

This is the general solution.

Exercise 6.4 | Q 15 | Page 203

Solve the following differential equation:

$(x^2 + y^2)dx - 2xy\ dy = 0$

SOLUTION

$(x^2 + y^2)dx - 2xy\ dy = 0$

$\therefore 2xy\ dy = (x^2 + y^2)dx$

$\therefore \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ ....(1)

Put y = vx

$\therefore \frac{dy}{dx} = v + x du/dx$

$\therefore$ (1) becomes, $v + x\frac{du}{dx} = \frac{x^2 + v^2x^2}{2x(vx)}$

$\therefore v + x\frac{du}{dx} = \frac{1 + v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v}$

$\therefore x\frac{dv}{dx} = \frac{1 - v^2}{2v}$

$\therefore \frac{2v}{1 - v^2}dv = \frac{1}{x}dx$

Integrating both sides, we get

$\int \frac{2v}{1 - v^2}dv = \int \frac{1}{x}dx$

$-\int \frac{-2v}{1 - v^2}dv = \int \frac{1}{x}dx$

$\therefore -\log|1 - v^2| = \log x + \log c_1$ .... $\left[\because \frac{d}{dv}(1 - v^2) = -2v \text{ and } \int \frac{f\prime(x)}{f(x)}dx = \log|f(x)| + c\right]$

$$\therefore \log\left|\frac{1}{1-v^2}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{1}{1-\left(\frac{y^2}{x^2}\right)}\right| = \log c_1 x$$

$$\therefore \log\left|\frac{x^2}{x^2-y^2}\right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1} x$$

$\therefore x^2 - y^2 = cx$, where c = $\frac{1}{c_1}$

This is the general solution.

EXERCISE 6.5 [PAGES 206 - 207]

Exercise 6.5 | Q 1.01 | Page 206

Solve the following differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$$

SOLUTION

$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$ ...(1)

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where P = $\frac{1}{x}$ and Q = $x^3 - 3$

$\therefore$ I.F. = $e^{\int P dx} = e^{\int \frac{1}{x} dx}$

$= e^{\log x}$ = x

$\therefore$ the solution of (1) is given by

y(I.F.) = ∫ Q. (I.F.)dx + c_1

$$\therefore \mathrm{y} \cdot \mathrm{x} = \int \left(\mathrm{x}^3 - 3\right)\mathrm{x}\mathrm{dx} + \mathrm{c}_1$$

$$\therefore \mathrm{xy} = \int \left(\mathrm{x}^4 - 3\mathrm{x}\right)\mathrm{dx} + \mathrm{c}_1$$

$$\therefore \mathrm{xy} = \frac{\mathrm{x}^5}{5} - 3 \cdot \frac{\mathrm{x}^2}{2} + \mathrm{c}_1$$

$\therefore \frac{\mathrm{x}^2}{5} - \frac{3\mathrm{x}^2}{2} - \mathrm{xy} = \mathrm{c}$, where c = - c_1

$\therefore$ This is the general solution.

Exercise 6.5 | Q 1.02 | Page 206

Solve the following differential equation:

$$\cos^2 \mathrm{x} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \tan \mathrm{x}$$

SOLUTION

$$\cos^2 \mathrm{x} \cdot \frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y} = \tan \mathrm{x}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{1}{\cos^2 \mathrm{x}} \cdot \mathrm{y} = \frac{\tan \mathrm{x}}{\cos^2 \mathrm{x}}$$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} + \sec^2 \mathrm{x} \cdot \mathrm{y} = \tan \mathrm{x} \cdot \sec^2 \mathrm{x} \qquad ....(1)$$

This is the linear differential equation of the form

$\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{P} \cdot \mathrm{y} = \mathrm{Q}$, where P = $\sec^2 x$ and Q = $\tan \mathrm{x} \cdot \sec^2 \mathrm{x}$

$\therefore$ I.F. = $\mathrm{e}^{\int \mathrm{Pdx}} = \mathrm{e}^{\int \sec^2 \mathrm{x}\, \mathrm{dx}} = \mathrm{e}^{\tan \mathrm{x}}$

$\therefore$ the solution of (1) is given by

$$\mathrm{y}\cdot(\mathrm{I.F.}) = \int \mathrm{Q}(I.F.)\mathrm{dx} + \mathrm{c}$$

$$\therefore \mathrm{y}\cdot \mathrm{e}^{\tan x} = \int \tan x \cdot \sec^2 x \cdot \mathrm{e}^{\tan x}\mathrm{dx} + \mathrm{c}$$

Put tan x = t

$$\therefore \sec^2 x\, \mathrm{dx} = \mathrm{dt}$$

$$\therefore \mathrm{y}\cdot \mathrm{e}^{\tan x} = \int \mathrm{t}\cdot \mathrm{e}^{\mathrm{t}}\mathrm{dt} + \mathrm{c}$$

$$\therefore \mathrm{y}\cdot \mathrm{e}^{\tan x} = \mathrm{t}\int \mathrm{e}^{\mathrm{t}}\mathrm{dt} - \int\left[\frac{\mathrm{d}}{\mathrm{dt}}(\mathrm{t})\int \mathrm{e}^{\mathrm{t}}\mathrm{dt}\right]\mathrm{dt} + \mathrm{c}$$

$$= \mathrm{t}\cdot \mathrm{e}^{\mathrm{t}} - \int 1\cdot \mathrm{e}^{\mathrm{t}}\mathrm{dt} + \mathrm{c}$$

$$= \mathrm{t}\cdot \mathrm{e}^{\mathrm{t}} - \mathrm{e}^{\mathrm{t}} + \mathrm{c}$$

$$= \mathrm{e}^{\mathrm{t}}(\mathrm{t} - 1) + \mathrm{c}$$

$$\therefore \mathrm{y}\cdot \mathrm{e}^{\tan x} = \mathrm{e}^{\tan x}(\tan x - 1) + \mathrm{c}$$

This is the general solution.

Exercise 6.5 | Q 1.03 | Page 206

Solve the following differential equation:

$$\left(\mathrm{x} + 2\mathrm{y}^3\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}$$

SOLUTION

$$\left(\mathrm{x} + 2\mathrm{y}^3\right)\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}$$

$$\therefore \frac{\mathrm{x} + 2\mathrm{y}^3}{\mathrm{y}} = \frac{1}{\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}$$

$$\therefore \frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\therefore \frac{dy}{dx} - \frac{1}{y} \cdot x = 2y^2 \quad .....(1)$$

This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where P = $-\frac{1}{y}$ and Q = $2y^2$

$\therefore$ I.F. = $e^{\int P dy} = e^{\int -\frac{1}{y} dy}$

$\therefore$ = $e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$

$\therefore$ the solution of (1) is given by

$$\therefore x \cdot (\text{I.F.}) = \int Q(\text{I.F.}) dy + c$$

$$\therefore x\left(\frac{1}{y}\right) = \int 2y^2 \times \frac{1}{y} dy + c$$

$$\therefore \frac{x}{y} = 2\int y dx + c$$

$$\therefore \frac{x}{y} = 2 \cdot \frac{y^2}{2} + c$$

$\therefore$ x = $y(c + y^2)$

This is the general solution.

Exercise 6.5 | Q 1.04 | Page 206

Solve the following differential equation:

$$\frac{dy}{dx} + y \cdot \sec x = \tan x$$

SOLUTION

$$\frac{dy}{dx} + y \cdot \sec x = \tan x$$

$$\therefore \frac{dy}{dx} + (\sec x) \cdot y = \tan x \quad .....(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where P = sec x and Q = tan x

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)}$

= sec x + tan x

$\therefore$ the solution of (1) is given by

$\therefore$ y(I.F.) = $\int Q \cdot (I.F.)dx + c$

$\therefore$ y (sec x + tan x) = ∫ tan x (sec x + tan x) dx + c

$\therefore$ (sec x + tan x) y = ∫ (sec x tan x + $\tan^2 x$) dx + c

$\therefore$ (sec x + tan x) y = ∫ (sec x tan x + $\sec^2 x$ - 1) dx + c

$\therefore$ (sec x + tan x) y = sec x + tan x - x + c

$\therefore$ y (sec x + tan x) = sec x + tan x - x + c

This is the general solution.

Exercise 6.5 | Q 1.05 | Page 206

Solve the following differential equation:

$$x\frac{dy}{dx} + 2y = x^2 \cdot \log x$$

SOLUTION

$$x\frac{dy}{dx} + 2y = x^2 \cdot \log x$$

$$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x \cdot \log x \quad ...(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where P = $\frac{2}{x}$ and $Q = x \cdot \log x$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int \frac{2}{x} dx} = e^{2\int \frac{1}{x} dx}$

$$= e^{2\log x} = e^{\log x^2} = x^2$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$$

$$\therefore y \cdot x^2 = \int (x \log x) \cdot x^2 dx + c$$

$$\therefore x^2 \cdot y = \int x^3 \cdot \log x \; dx + c$$

$$= (\log x)\int x^3 dx - \int \left[\frac{d}{dx}(\log x)\int x^3 dx\right] dx + c$$

$$= (\log x) \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4} dx + c$$

$$= \frac{1}{4}x^4 \log x - \frac{1}{4}\int x^3 dx + c$$

$$\therefore x^2 \cdot y = \frac{1}{4}x^4 \log x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$

$$\therefore x^2 \cdot y = \frac{x^4 \log x}{4} - \frac{x^2}{16} + c$$

This is the general solution.

Exercise 6.5 | Q 1.06 | Page 206

Solve the following differential equation:

$$(x + y)\frac{dy}{dx} = 1$$

SOLUTION

$$(x + y)\frac{dy}{dx} = 1$$

$$\therefore \frac{dx}{dy} = x + y$$

$$\therefore \frac{dx}{dy} - x = y$$

$$\therefore \frac{dx}{dy} + (-1)x = y \quad \text{....(1)}$$

This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where P = - 1 and Q = y

$\therefore$ I.F. = $e^{\int P\,dy} = e^{\int -1 dy} = e^{-y}$

$\therefore$ the solution of (1) is given by

x.(I.F.) = $\int Q \cdot (I.F.)dy + c$

$$\therefore x \cdot e^{-y} = \int y \cdot e^{-y}dy + c$$

$$\therefore e^{-y} \cdot x = y\int e^{-y}dy - \int\left[\frac{d}{dx}(y)\int e^{-y}dy\right]dy + c$$

$$= y \cdot \frac{e^{-y}}{-1} - \int 1 \cdot \frac{e^{-y}}{-1}dy + c$$

$$= -ye^{-y} + \int e^{-y}dy + c$$

$$\therefore e^{-y} \cdot x = -ye^{-y} + \frac{e^{-y}}{-1} + c$$

$$\therefore e^{-y} \cdot x + ye^{-y} + e^{-y} = c$$

$$\therefore e^{-y}(x + y + 1) = c$$

$\therefore$ x + y + 1 = ce^y

This is the general solution.

Exercise 6.5 | Q 1.07 | Page 206

Solve the following differential equation:

$$(x + a)\frac{dy}{dx} - 3y = (x + a)^5$$

SOLUTION

$$(x + a)\frac{dy}{dx} - 3y = (x + a)^5$$

$$\therefore \frac{dy}{dx} - \frac{3y}{x + a} = (x + a)^4$$

$$\therefore \frac{dy}{dx} + \left(\frac{-3}{x + a}\right)y = (x + a)^4 \quad ...(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where P = $\frac{-3}{x + a}$ and Q = $(x + a)^4$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int \frac{-3}{x+a}dx} = e^{-3\int \frac{1}{x+a}dx}$

$$= e^{-3\log|x + a|} = e^{\log(x + a)^{-3}}$$

$$= (x + a)^{-3} = \frac{1}{(x + a)^3}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$$

$$\therefore y \cdot \frac{1}{(x+a)^3} = \int (x+a)^4 \cdot \frac{1}{(x+a)^3} dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \int (x+a)dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \frac{x+a^2}{2} + c$$

$\therefore$ 2y = $(x + a)^5$ + 2c $(x + a)^3$

This is the general solution.

Exercise 6.5 | Q 1.08 | Page 206

Solve the following differential equation:

dr + (2r cot θ + sin 2θ) dθ = 0

SOLUTION

dr + (2r cot θ + sin 2θ) dθ = 0

$$\therefore \frac{dr}{d\theta} + (2r \cot\theta + \sin 2\theta) = 0$$

$$\therefore \frac{dr}{d\theta} + (2\cot\theta)r = -\sin 2\theta \qquad .....(1)$$

This is the linear differential equation of the form

$\frac{dr}{d\theta} + P \cdot r = Q$, where P = 2 cot θ and Q = - sin 2θ

$\therefore$ I.F. = $e^{\int P \, d\theta} = e^{\int e \cot\theta} d\theta$

$$= e^{2\int \cot\theta d\theta} = e^{2\log\sin\theta}$$

$$= e^{\log(\sin^2\theta)} = \sin^2\theta$$

$\therefore$ the solution of (1) is given by

$$\mathrm{r} \cdot (\mathrm{I.F.}) = \int \cdot (\mathrm{I.F.}) \mathrm{d}\theta + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta = \int - \sin 2\theta \cdot \sin^2 \theta \mathrm{d}\theta + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta = \int -2 \sin \theta \cos \theta \cdot \sin^2 \theta \mathrm{d}\theta + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta = -2 \int \sin^3 \theta \cos \theta \, \mathrm{d}\theta + \mathrm{c}$$

Put sin θ = t

$\therefore$ cos θ dθ = dt

$$\therefore \mathrm{r} \cdot \sin^2 \theta = -2 \int \mathrm{t}^3 \mathrm{dt} + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta = -2 \cdot \frac{\mathrm{t}^4}{4} + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta = -\frac{1}{2} \sin^4 \theta + \mathrm{c}$$

$$\therefore \mathrm{r} \cdot \sin^2 \theta + \frac{\sin^4 \theta}{2} = \mathrm{c}$$

This is the general solution.

Exercise 6.5 | Q 1.09 | Page 206

Solve the following differential equation:

$y \, dx + (x - y^2) \, dy = 0$

SOLUTION

y dx + (x - y^2) dy = 0

∴ y dx = - (x - y^2) dy

$$\therefore \frac{dx}{dy} = -\frac{(x - y^2)}{y} = -\frac{x}{y} + y$$

$$\therefore \frac{dx}{dy} + \left(\frac{1}{y}\right) \cdot x = y \quad ....(1)$$

This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where P = $\frac{1}{y}$ and Q = y

∴ I.F. = $e^{\int P\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

∴ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (I.F.)dy + c_1$$

$$\therefore xy = \int y \cdot y\, dy + c_1$$

$$\therefore xy = \int y^2 dy + c_1$$

$$\therefore xy = \frac{y^3}{3} + c_1$$

$\therefore \frac{y^3}{3} = xy + c$, where c = $-c_1$

This is the general solution.

Exercise 6.5 | Q 1.1 | Page 207

Solve the following differential equation:

$$(1 - x^2)\frac{dy}{dx} + 2xy = x(1 - x^2)^{\frac{1}{2}}$$

SOLUTION

$$\left(1-x^2\right)\frac{dy}{dx}+2xy=x\left(1-x^2\right)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx}+\left(\frac{2x}{1-x^2}\right)y=\frac{x}{\left(1-x^2\right)^{\frac{1}{2}}}$$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where P = $\frac{2x}{1-x^2}$ and Q = $\frac{x}{\left(1-x^2\right)^{\frac{1}{2}}}$

$\therefore$ I.F. = $e^{\int P\,dx}=e^{\int \frac{2x}{1-x^2}dx}$

$$=e^{-\int \frac{-2x}{1-x^2}}=e^{-\log\left|1-x^2\right|}$$

$$=e^{\log\left|\frac{1}{1-x^2}\right|}=\frac{1}{1-x^2}$$

$\therefore$ the solution of (1) is given by

$$y\cdot(\mathrm{I.F.})=\int Q\cdot(\mathrm{I.F.})dx+c$$

$\therefore y\cdot\frac{1}{1-x^2}=\int\frac{x}{(1-x)^{\frac{1}{2}}}\cdot\frac{1}{1-x^2}$ dx + c

$$\therefore \frac{y}{\left(1-x^2\right)}=\int\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}dx+c$$

Put 1 - x^2 = t

$\therefore$ x dx = - $\frac{dt}{2}$

$$\therefore \frac{y}{1-x^2}=\int\frac{1}{t^{\frac{3}{2}}}\cdot\frac{-dt}{2}+c$$

$$\therefore \frac{y}{1-x^2}=-\frac{1}{2}\int t^{-\frac{3}{2}}dt+c$$

$$\therefore \frac{y}{1-x^2} = -\frac{1}{2} \cdot \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$\therefore \frac{y}{1-x^2} = \frac{1}{(1-x^2)^{\frac{1}{2}}} + c$$

$\therefore$ y = $\sqrt{1-x^2} + c\left(1-x^2\right)$

This is the general solution.

Exercise 6.5 | Q 1.11 | Page 207

Solve the following differential equation:

$$\left(1+x^2\right)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

SOLUTION

$$\left(1+x^2\right)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\therefore \frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{e^{\tan^{-1}x}}{1+x^2} \qquad \text{....(1)}$$

This is the linear differential equation of the form

$\dfrac{dy}{dx} + P \cdot y = Q$, where P = $\dfrac{1}{1+x^2}$ and Q = $\dfrac{e^{\tan^{-1}x}}{1+x^2}$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx}$

$= e^{\tan^{-1}x}$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$$

$$\therefore y \cdot e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot e^{\tan^{-1}x}dx + c$$

$$\therefore y \cdot e^{\tan^{-1} x} = \int \left(e^{\tan^{-1} x}\right) \cdot \left(\frac{e^{\tan^{-1} x}}{1 + x^2}\right) dx + c$$

Put $e^{\tan^{-1} x} = t$

$$\therefore \frac{e^{\tan^{-1} x}}{1 + x^2} dx = dt$$

$$\therefore y \cdot e^{\tan^{-1} x} = \int t \, dt + c$$

$$\therefore y \cdot e^{\tan^{-1} x} = \frac{t^2}{2} + c$$

$$\therefore y \cdot e^{\tan^{-1} x} = \frac{1}{2} \left(e^{\tan^{-1} x}\right)^2 + c$$

$$\therefore y = \frac{1}{2} e^{\tan^{-1} x} + c e^{-\tan^{-1} x}$$

This is the general solution.

Exercise 6.5 | Q 2 | Page 207

Find the equation of the curve which passes through the origin and has the slope x + 3y - 1 at any point (x, y) on it.

SOLUTION

Let A (x, y) be the point on the curve y = f(x).

Then slope of the tangent to the curve at point A is $\frac{dy}{dx}$.

According to the given condition,

$$\frac{dy}{dx} = x + 3y - 1$$

$$\therefore \frac{dy}{dx} - 3y = x - 1 \qquad ....(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where P = - 3 and Q = x - 1

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int -3dx} = e^{-3x}$

$\therefore$ the solution of (1) is given by

$$y \cdot (I.F.) = \int Q \cdot (I.F.)dx + c$$

$$\therefore y \cdot e^{-3x} = \int (x - 1) \cdot e^{-3x}dx + c$$

$$\therefore e^{-3x} \cdot y = (x - 1)\int e^{-3x} - \int \left[\frac{d}{dx}(x - 1) \cdot \int e^{-3x}dx\right]dx + c_1$$

$$\therefore e^{-3x} \cdot y = (x - 1) \cdot \frac{e^{-3x}}{-3} - \int 1 \cdot e^{-3x} \cdot \frac{y}{-3}dx + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1) \cdot e^{-3x} + \frac{1}{3}\int e^{-3x}dx + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1)e^{-3x} + \frac{1}{3} \cdot \frac{e^{-3x}}{-3} + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1)e^{-3x} - \frac{1}{9}e^{-3x} + c_1$$

$\therefore$ 9y = - 3(x - 1) - 1 + $9c_1 \cdot e^{3x}$

$\therefore$ 9y + 3(x - 1) + 1 = $9c_1 \cdot e^{3x}$

$\therefore$ 9y + 3x - 3 + 1 = $9c_1 \cdot e^{3x}$

$\therefore$ 3(x + 3y) = 2 + $9c_1 \cdot e^{3x}$

$\therefore$ 3(x + 3y) = 2 + $c \cdot e^{3x}$ where c = $9c_1$ ....(2)

This is the general equation of the curve.

But the required curve is passing through the origin (0, 0).

∴ by putting x = 0 and y= 0 in (2), we get

0 = 2 + c

∴ c = - 2

∴ from (2), the equation of the required curve is

$3(x + 3y) = 2 - 2e^{3x}$

i.e. $3(x + 3y) = 2(1 - e^{3x})$.

Exercise 6.5 | Q 3 | Page 207

Find the equation of the curve passing through the point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ having a slope of the tangent to the curve at any point (x, y) is $-\frac{4x}{9y}$.

SOLUTION

Let A(x, y) be the point on the curve y = f(x).

Then the slope of the tangent to the curve at point A is $\frac{dy}{dx}$.

According to the given condition

$$\frac{dy}{dx} = -\frac{4x}{9y}$$

$$\therefore y\,dy = -\frac{4}{9}x\,\,dx$$

Integrating both sides, we get

$$\int y\,dy = -\frac{4}{9}\int x\,dx$$

$$\therefore \frac{y^2}{2} = -\frac{4}{9}\cdot\frac{x^2}{2} + c_1$$

$$\therefore 9y^2 = -4x^2 + 18c_1$$

$\therefore 4x^2 + 9y^2 = c_1$ where $c = 18c_1$ .....(1)

This is the general equation of the curve.

But the required curve is passing through the point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$.

$\therefore$ by putting $x = \frac{3}{\sqrt{2}}$ and $y = \sqrt{2}$ in (1), we get

$$4\left(\frac{3}{\sqrt{2}}\right)^2 + 9\left(\sqrt{2}\right)^2 = c$$

$\therefore 18 + 18 = c$

$\therefore c = 36$

$\therefore$ from (1), the equation of the required curve is $4x^2 + 9y^2 = 36$.

Exercise 6.5 | Q 4 | Page 207

The curve passes through the point (0, 2). The sum of the coordinates of any point on the curve exceeds the slope of the tangent to the curve at any point by 5. Find the equation of the curve.

SOLUTION

Let A(x, y) be any point on the curve.

Then slope of the tangent to the curve at point A is $\frac{dy}{dx}$.

According to the given condition

$$x + y = \frac{dy}{dx} + 5$$

$$\therefore \frac{dy}{dx} - y = x - 5 \quad ...(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where P = - 1 and Q = x - 5

∴ I.F. = $e^{\int P\,dx} = e^{\int -1 dx} = e^{-x}$

∴ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot e^{-x} = \int (x - 5) e^{-x} dx + c$$

$$\therefore e^{-x} \cdot y = (x - 5) \int e^{-x} dx - \int \left[\frac{d}{dx}(x - 5) \int e^{-x} dx \right] dx + c$$

$$\therefore e^{-x} \cdot y = (x - 5) \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} dx + c$$

$$\therefore e^{-x} \cdot y = -(x - 5) \cdot e^{-x} + \int e^{-x} dx + c$$

$$\therefore e^{-x} \cdot y = -(x - 5) e^{-x} + \frac{e^{-x}}{-1} + c$$

∴ y = - (x - 5) - 1 + ce^x

∴ y = - x + 5 - 1 + ce^x

∴ y = 4 - x + ce^x ....(2)

This is the general equation of the curve.

But the required curve is passing through the point (0, 2).

∴ by putting x = 0, y = 2 in (2), we get

2 = 4 - 0 + c

∴ c = - 2

∴ from (2), the equation of the required curve is

$y = 4 - x - 2e^x$

Exercise 6.5 | Q 5 | Page 207

If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also, the curve passes through the point (0, 1). Find the equation of the curve.

SOLUTION

Let A (x, y) be any point on the curve y = f(x).

Then the slope of the tangent to the curve at point A is $\frac{dy}{dx}$.

According to the given condition

$$\frac{dy}{dx} = x + xy$$

$$\therefore \frac{dy}{dx} - xy = x \qquad .....(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where P = - x and Q = x

$$\therefore \text{I.F.} = e^{\int P\,dx} = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$$

∴ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$$

$$\therefore y \cdot e^{-\frac{x^2}{2}} = \ = \int x \cdot e^{-\frac{x^2}{2}}dx + c$$

$$\therefore e^{-\frac{x^2}{2}} \cdot y = \int x \cdot e^{-\frac{x^2}{2}}dx + c$$

$\therefore$ Put $-\frac{x^2}{2} = t$

$\therefore$ - x dx = dt

$\therefore$ x dx = - dt

$\therefore e^{-\frac{x^2}{2}} \cdot y = \int e^t \cdot (-dt) + c$

$\therefore e^{-\frac{x^2}{2}} \cdot y = -\int e^t \cdot dt + c$

$\therefore e^{-\frac{x^2}{2}} \cdot y = -e^t + c$

$\therefore e^{-\frac{x^2}{2}} \cdot y = -e^{-\frac{x^2}{2}} + c$

$\therefore$ y = - 1 + $ce^{\frac{x^2}{2}}$

$\therefore$ 1 + y = $ce^{\frac{x^2}{2}}$ .....(2)

This is the general equation of the curve.

But the required curve is passing through the point (0, 1).

$\therefore$ by putting x = 0 and y = 1 in (2), we get

1 + 1 = c

$\therefore$ c = 2

$\therefore$ from (2), the equation of the required curve is

1 + y = $2e^{\frac{x^2}{2}}$.

EXERCISE 6.6 [PAGE 213]

Exercise 6.6 | Q 1 | Page 213

In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

SOLUTION

Let x be the number of bacteria in the culture at time t.

Then the rate of increase is $\frac{dy}{dx}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt}$ = kx, where k is a constant

$\therefore \frac{dx}{x}$ = k dt

On integrating, we get

$$\int \frac{dx}{x} = k \int 1 dt + c$$

$\therefore$ log x = kt + c

Initially, i.e. when t = 0, let x = x_0

$\therefore$ log x_0 = k × 0 + c

$\therefore$ c = log x_0

$\therefore$ log x = kt + log x_0

$\therefore$ log x - log x_0 = kt

$\therefore \log\left(\frac{x}{x_0}\right)$ = kt ...(1)

Since the number doubles in 4 hours, i.e. when t = 4,

x = $2x_0$

$\therefore \log\left(\frac{2x_0}{x_0}\right)$ = 4k

$\therefore$ k = $\frac{1}{4}$ log 2

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = \frac{t}{4}$ log 2

When t = 12, we get

$\log\left(\frac{x}{x_0}\right) = \frac{12}{4}$ log 2 = 3 log 2

$\therefore \log\left(\frac{x}{x_0}\right)$ = log 8

$\therefore \frac{x}{x_0} = 8$

$\therefore$ x = $8x_0$

$\therefore$ number of bacteria will be 8 times the original number in 12 hours.

Exercise 6.6 | Q 2 | Page 213

If the population of a country doubles in 60 years; in how many years will it be triple (treble) under the assumption that the rate of increase is proportional to the number of inhabitants?
(Given log = 20.6912, log 3 = 1.0986)

SOLUTION

Let P be the population at time t years. Then dP/dt, the rate of increase of population is proportional to P.

$\therefore \frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt}$ = kP, where k is a constant

$\therefore \frac{dP}{P}$ = k dt

On integrating, we get

$$\int \frac{dP}{P} = k \int dt + c$$

$\therefore$ log P = kt + c

Initially, i.e. when t = 0, let P = P_0

$\therefore$ log P_0 = k × 0 + c

$\therefore$ c = log P_0

$\therefore$ log P = kt + log P_0

$\therefore$ log P - log P_0 = kt

$\therefore \log\left(\frac{P}{P_0}\right)$ = kt ...(1)

Since the population doubles in 60 hours, i.e. when t = 60, P = $2P_0$

$\therefore \log\left(\frac{2P_0}{P_0}\right)$ = 60k

$\therefore$ k = $\frac{1}{60}$ log 2

$\therefore$ (1) becomes, $\log\left(\frac{P}{P_0}\right) = \frac{t}{60}$ log 2

When population becomes triple, i.e. when P = $3P_0$, we get

$$\log\left(\frac{3P_0}{P_0}\right) = \frac{t}{60}\log 2$$

$$\therefore \log 3 = \left(\frac{t}{60}\right)\log 2$$

$$\therefore t = 60\left(\frac{\log 3}{\log 2}\right) = 60\left(\frac{1.0986}{0.6912}\right)$$

$= 60 \times 1.5894 = 95.364 \approx 95.4$ years

∴ the population becomes triple in 95.4 years (approximately).

Exercise 6.6 | Q 3 | Page 213

If a body cools from 80°C to 50°C at room temperature of 25°C in 30 minutes, find the temperature of the body after 1 hour.

SOLUTION

Let θ° C be the temperature of the body at time t minutes. Room temperature is given to be 25°C.

Then by Newton's law of cooling, dΘ/dt the rate of change of temperature, is proportional to (θ - 25).

$$\text{i.e. } \frac{d\theta}{dt} \propto (\theta - 25)$$

$$\therefore \frac{d\theta}{dt} = -k(\theta - 25), \text{ where } k > 0$$

$$\therefore \frac{d\theta}{\theta - 25} = -k\,dt$$

On integrating, we get

$$\int \frac{1}{\theta - 25}d\theta = -k\int dt + c$$

$$\therefore \log(\theta - 25) = -kt + c$$

Initially, i.e. when $t = 0$, $\theta = 80$

$$\therefore \log(80 - 25) = -k \times 0 + c \quad \therefore c = \log 55$$

$\therefore$ log (θ - 25) = - kt + log 55

$\therefore$ log (θ - 25) - log 55 = - kt

$$\log\left(\frac{\theta-25}{55}\right)=-\mathrm{kt} \quad ....(1)$$

Now, when t = 30, θ = 50

$$\therefore \log\left(\frac{50-25}{55}\right)=-30\mathrm{k}$$

$$\therefore \mathrm{k} = -\frac{1}{30}\log\left(\frac{5}{11}\right)$$

$\therefore$ (1) becomes, $\log\left(\frac{\theta-25}{55}\right)=\frac{\mathrm{t}}{30}\log\left(\frac{5}{11}\right)$

When t = 1 hour = 60 minutes, then

$$\log\left(\frac{\theta-25}{55}\right)=\frac{60}{30}\log\left(\frac{5}{11}\right)=2\log\left(\frac{5}{11}\right)$$

$$\therefore \log\left(\frac{\theta-25}{55}\right)=\log\left(\frac{5}{11}\right)^2$$

$$\therefore \frac{\theta-25}{55}=\left(\frac{5}{11}\right)^2=\frac{25}{121}$$

$$\therefore \theta-25=55\times\frac{25}{121}=\frac{125}{11}$$

$$\therefore \theta=25+\frac{125}{11}=\frac{400}{11}=36.36$$

$\therefore$ θ the temperature of the body will be 36.36° C after 1 hour.

Exercise 6.6 | Q 4 | Page 213

The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hours.

[Take $\sqrt{2} = 1.414$]

SOLUTION

Let x be the number of bacteria at time t.

Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt}$ = kx, where k is a constant

$\therefore \frac{dx}{x} = k \ \ dt$

On integrating, we get

$$\int \frac{dx}{x} = k \int dt + c$$

$\therefore$ log x = kt + c

Initially, i.e. when t = 0, x = 1000

$\therefore$ log 1000 = k × 0 + c $\quad \therefore$ c = log 1000

$\therefore$ log x = ly + log 1000

$\therefore$ log x - log 1000 = kt

$\therefore$ log$\left(\frac{x}{1000}\right) = kt$ ...(1)

Now, when t = 1, x = 2 × 1000 = 2000

$\therefore \log\left(\frac{2000}{1000}\right)$ = k $\quad \therefore$ k = log 2

$\therefore$ (1) becomes, log`("x"/1000) = t log 2

If t = $2\frac{1}{2} = \frac{5}{2}$, then

$$\log\left(\frac{x}{1000}\right) = \frac{5}{2}\log 2 = \log(2)^{\frac{5}{2}}$$

$$\therefore \left(\frac{x}{1000}\right) = (2)^{\frac{5}{2}} = 4\sqrt{2} = 4 \times 1.414 = 5.656$$

$\therefore$ x = 5.656 × 1000 = 5656

$\therefore$ number of bacteria after $2\frac{1}{2}$hours = 5656.

Exercise 6.6 | Q 5 | Page 213

The rate of disintegration of a radioactive element at any time t is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.

SOLUTION

Let m be the mass of the radioactive element at time t.

Then the rate of disintegration is $\frac{dm}{dt}$ which is proportional to m.

$$\therefore \frac{dm}{dt} \propto m$$

$\therefore \frac{dm}{dt}$ = - km, where k > 0

$\therefore \frac{dm}{m}$ = - k dt

On integrating, we get

$$\int \frac{1}{m} dm = -k \int dt + c$$

$\therefore$ log m = - kt + c

Initially, i.e. when t = 0, m = 1.5

$$\int \frac{1}{m} dm = -k \int dt + c$$

$\therefore \log m = - kt + c$

Initially, i.e. when t = 0, m = 1.5

$\therefore \log(1.5) = - k \times 0 + c \qquad \therefore c = \log\left(\frac{3}{2}\right)$

$\therefore \log m = - kt + \log\left(\frac{3}{2}\right)$

$\therefore \log m - \log\frac{3}{2} = - kt$

$\therefore \log\left(\frac{2m}{3}\right) = - kt$

When $m = 0.5 = \frac{1}{2}$, then

$$\log\left(\frac{2 \times \frac{1}{2}}{3}\right) = -kt$$

$\therefore \log\left(\frac{1}{3}\right) = - kt$

$\therefore \log(3)^{-1} = - kt$

$\therefore - \log 3 = - kt$

$\therefore t = \frac{1}{k}\log 3$

$\therefore$ the original mass will disintegrate to 0.5 gm when $t = \frac{1}{k}\log 3$

Exercise 6.6 | Q 6 | Page 213

The rate of decay of certain substances is directly proportional to the amount present at that instant. Initially, there is 25 gm of certain substance and two hours later it is found that 9 gm are left. Find the amount left after one more hour.

SOLUTION

Let x gm be the amount of the substance left at time t.

Then the rate of decay is $\frac{dx}{dt}$, which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = -kx$, where $k > 0$

$\therefore \frac{1}{x} dx = -k\, dt$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt + c$$

$\therefore \log x = -kt + c$

Initially i.e. when $t = 0$, $x = 25$

$\therefore \log 25 = -k \times 0 + c \quad \therefore c = \log 25$

$\therefore \log x = -kt + \log 25$

$\therefore \log x - \log 25 = -kt$

$\therefore \log\left(\frac{x}{25}\right) = -kt \quad$ ....(1)

Now, when $t = 2$, $x = 9$

$\therefore \log\left(\frac{9}{25}\right) = -2k$

$\therefore -2k = \log\left(\frac{3}{5}\right)^2 = 2\log\left(\frac{3}{5}\right)$

$\therefore k = -\log\left(\frac{3}{5}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{x}{25}\right) = t \log\left(\frac{3}{5}\right)$

When t = 3, then

$$\log\left(\frac{x}{25}\right) = 3\log\left(\frac{3}{5}\right) = \log\left(\frac{3}{5}\right)^3$$

$\therefore \frac{x}{25} = \frac{27}{125}$ $\therefore x = \frac{27}{5}$

$\therefore$ the amount left after 3 hours = $\frac{27}{5}$ gm.

Exercise 6.6 | Q 7 | Page 213

Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.

SOLUTION

Let P be the population of the city at time t.

Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to P.

$\therefore \frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt}$ = kP, where k is a constant.

$\therefore \frac{dP}{P}$ = k dt

On integrating, we get

$$\int \frac{1}{P} dP = k\int dt + c$$

$\therefore$ log P = kt + c

Initially, i.e. when t = 0, P = 30000

$\therefore$ log 30000 = k × 0 + c $\therefore$ c = log 30000

$\therefore \log P = kt + \log 30000$

$\therefore \log P - \log 30000 = kt$

$\therefore \log\left(\frac{P}{30000}\right) = kt \qquad .....(1)$

Now, when t = 40, P = 40000

$\therefore \log\left(\frac{40000}{30000}\right) = k \times 40$

$\therefore k = \frac{1}{40}\log\left(\frac{4}{3}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{P}{30000}\right) = \frac{t}{40}\log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore P = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore$ the population of the city at time t = $30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$

Exercise 6.6 | Q 8 | Page 213

A body cools according to Newton's law from 100° C to 60° C in 20 minutes. The temperature of the surrounding being 20° C. How long will it take to cool down to 30° C?

SOLUTION

Let θ°C be the temperature of the body at time t. The temperature of the surrounding is given to be 20° C.

According to Newton's law of cooling

$$\frac{d\theta}{dt} \propto \theta - 20$$

$\therefore \frac{d\theta}{dt} = -k(\theta - 20)$, where k > 0

$$\therefore \frac{d\theta}{\theta - 20} = -k\,dt$$

On integrating, we get

$$\int \frac{1}{\theta - 20} d\theta = -k \int dt + c$$

$\therefore \log(\theta - 20) = -kt + c$

Initially, i.e. when $t = 0$, $\theta = 100$

$\therefore \log(100 - 20) = -k \times 0 + c \qquad \therefore c = \log 80$

$\therefore \log(\theta - 20) = -kt + \log 80$

$\therefore \log(\theta - 20) - \log 80 = -kt$

$\therefore \log\left(\frac{\theta - 20}{80}\right) = -kt \quad$ ....(1)

Now, when $t = 20$, $\theta = 60$

$$\therefore \log\left(\frac{60 - 20}{80}\right) = -k \times 20$$

$$\therefore \log\left(\frac{40}{80}\right) = -20k$$

$$\therefore k = -\frac{1}{20}\log\left(\frac{1}{2}\right)$$

$\therefore$ (1) becomes, $\log\left(\frac{\theta - 20}{80}\right) = \frac{t}{20}\log\left(\frac{1}{2}\right)$

When $\theta = 30$, then

$$\log\left(\frac{30-20}{80}\right) = \frac{t}{20}\log\left(\frac{1}{2}\right)$$

$$\therefore \log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{20}} = \frac{1}{8} = \left(\frac{1}{2}\right)^{3}$$

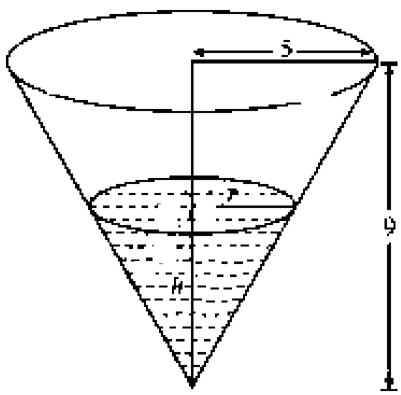
$$\therefore \frac{t}{20} = 3 \quad \therefore t = 60$$

∴ the body will cool down to 30° C in 60 minutes, i.e. in 1 hour.

Exercise 6.6 | Q 9 | Page 213

A right circular cone has height 9 cm and radius of the base 5 cm. It is inverted and water is poured into it. If at any instant the water level rises at the rate of (π/A) cm/sec, where A is the area of the water surface A at that instant, show that the vessel will be full in 75 seconds.

SOLUTION



Let r be the radius of the water surface and h be the height of the water at time t.

∴ area of the water surface A = πr^2 sq cm.

Since the height of the right circular cone is 9 cm and radius of the base is 5 cm.

$\frac{r}{h} = \frac{5}{9}$ $\therefore$ r = $\frac{5}{9}h$

$\therefore$ area of water surface, i.e. A = $\pi\left(\frac{5}{9}h\right)^2$

$\therefore$ A = $\frac{25\pi h^2}{81}$ .....(1)

The water level, i.e. the rate of change of h is $\frac{dh}{dt}$ rises at the rate of $\left(\frac{\pi}{A}\right)$ cm/sec.

$\therefore \frac{dh}{dt} = \frac{\pi}{A} = \frac{\pi \times 81}{25\pi h^2}$ ....[By (1)]

$\therefore \frac{dh}{dt} = \frac{81}{25h^2}$

$\therefore h^2 dh = \frac{81}{25} dt$

On integrating, we get

$\int h^2 dh = \frac{81}{25} \int dt + c$

$\therefore \frac{h^3}{3} = \frac{81}{25} \cdot t + c$

Initially, i.e. when t = 0, h = 0

$\therefore$ 0 = 0 + c $\therefore$ c = 0

$\therefore \frac{h^3}{3} = \frac{81}{25} t$

When the vessel will be full, h = 9

$\therefore \frac{(9)^3}{3} = \frac{81}{25} \times t$

$\therefore$ t = $\frac{81 \times 9 \times 25}{3 \times 81} = 75$

Hence, the vessel will be full in 75 seconds.

Exercise 6.6 | Q 10 | Page 213

Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the raindrop at any time t.

SOLUTION

Let r be the radius, V be the volume and S be the surface area of the spherical raindrop at time t.

Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

The rate at which the raindrop evaporates is dV/dt which is proportional to the surface area.

$$\therefore \frac{dV}{dt} \propto S$$

$$\therefore \frac{dV}{dt} = -kS, \text{ where } k > 0 \quad ...(1)$$

Now, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$$\therefore \frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore \text{(1) becomes, } 4\pi r^2 \frac{dr}{dt} = -k\left(4\pi r^2\right)$$

$$\therefore \frac{dr}{dt} = -k$$

$$\therefore dr = -k\,dt$$

On integrating, we get

$$\int dr = -k \int dt + c$$

$$\therefore r = -kt + c$$

Initially, i.e. when $t = 0$, $r = 3$

$\therefore 3 = - k \times 0 + c \quad \therefore c = 3$

$\therefore r = - kt + 3$

When $t = 1, r = 2$

$\therefore 2 = - k \times 1 + 3$

$\therefore k = 1$

$\therefore r = - t + 3$

$\therefore r = 3 - t$, where $0 \leq t \leq 3$.

This is the required expression for the radius of the raindrop at any time t.

Exercise 6.6 | Q 11 | Page 213

The rate of growth of the population of a city at any time t is proportional to the size of the population. For a certain city, it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years, if the initial population is 10,000. [Take e = 2.7182]

SOLUTION

Let P be the population of the city at time t.

Then the rate of growth of population is $\frac{dP}{dt}$ which is proportional to P.

$$\therefore \frac{dP}{dt} \propto P$$

$$\therefore \frac{dP}{dt} = kP, \text{ where } k = 0.04$$

$$\therefore \frac{dP}{dt} = (0.04)P$$

$$\therefore \frac{1}{P}dP = (0.04)dt$$

On integrating, we get

$$\int \frac{1}{P}dP = (0.04)\int dt + c$$

$\therefore \log P = (0.04)t + c$

Initially, i.e., when t = 0, P = 10000

$\therefore \log 10000 = (0.04) \times 0 + c$

$\therefore c = \log 10000$

$\therefore \log P = (0.04)\, t + \log 10000$

$\therefore \log P - \log 10000 = (0.04)\, t$

$\therefore \log\left(\frac{P}{10000}\right) = (0.04)t$

When t = 25, then

$\log\left(\frac{P}{10000}\right) = (0.04) \times 25 = 1$

$\therefore \log\left(\frac{P}{10000}\right) = \log e$ ....[$\because \log e = 1$]

$\therefore \frac{P}{10000} = e = 2.7182$

$\therefore P = 2.7182 \times 10000 = 27182$

$\therefore$ the population of the city after 25 years will be 27,182.

Exercise 6.6 | Q 12 | Page 213

Radium decomposes at the rate proportional to the amount present at any time. If p percent of the amount disappears in one year, what percent of the amount of radium will be left after 2 years?

SOLUTION

Let x be the amount of the radium at time t.

Then the rate of decomposition is $\frac{dx}{dt}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt}$ = - kx, where k > 0

$\therefore \frac{1}{x}dx$ = - k dt

On integrating, we get

$$\int \frac{1}{x}dx = -k \int dt$$

$\therefore$ log x = - kt + c

Let the original amount be x_0, i.e. x = x_0, when t = 0.

$\therefore$ log x_0 = - k × 0 + c $\quad \therefore$ c = log x_0

$\therefore$ log x = - kt + log x_0

$\therefore$ log x - log x_0 = - kt

$\therefore \log\left(\frac{x}{x_0}\right)$ = - kt .....(1)

But p% of the amount disappears in one year,

$\therefore$ when t = 1, x= x_0 - p % of x_0 , i.e. x = x_0 - $\frac{px_0}{100}$

$\therefore \log\left(\frac{x_0 - \frac{px_0}{100}}{x_0}\right) = -k \times 1$

$\therefore$ k = $-\log\left(1 - \frac{p}{100}\right) = -\log\left(\frac{100 - p}{100}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = t\log\left(\frac{100 - p}{100}\right)$

When t = 2, then

$$\log\left(\frac{x}{x_0}\right) = 2\log\left(\frac{100-p}{100}\right) = \log\left(\frac{100-p}{100}\right)^2$$

$$\therefore \frac{x}{x_0} = \left(\frac{100-p}{100}\right)^2$$

$$\therefore x = \left(\frac{100-p}{100}\right)^2 x_0 = \left(1-\frac{p}{100}\right)^2 x_0$$

$$\therefore \text{\% left after 2 years} = \frac{100 \times \left(1-\frac{p}{100}\right)^3 x_0}{x_0}$$

$$\therefore = 100\left(1-\frac{p}{100}\right)^2 = \left[10\left(1-\frac{p}{100}\right)\right]^2$$

$$= \left(10-\frac{p}{10}\right)^2$$

Hence, $= \left(10-\frac{p}{10}\right)^2$ % of the amount will be left after 2 years.

MISCELLANEOUS EXERCISE 6 [PAGES 214 - 216]

Miscellaneous exercise 6 | Q 1.01 | Page 214

Choose the correct option from the given alternatives:

The order and degree of the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ are respectively.

- 2, 1
- 1, 2
- 3, 2
- **2, 3**

SOLUTION

2, 3

Miscellaneous exercise 6 | Q 1.02 | Page 214

Choose the correct option from the given alternatives:

The differential equation of y = $c^2 + \frac{c}{x}$ is

- $x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$
- $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
- $x^3\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y$
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

SOLUTION

$$x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$$

Miscellaneous exercise 6 | Q 1.03 | Page 215

Choose the correct option from the given alternatives:

$x^2 + y^2 = a^2$ is a solution of

- $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$
- $y = x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} + a^2y$
- **$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$**
- $\frac{d^2y}{dx^2} = (x + 1)\frac{dy}{dx}$

SOLUTION

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Hint:

$$x^2 + y^2 = a^2 \quad \therefore 2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= x\left(-\frac{x}{y}\right) + a\sqrt{1 + \frac{x^2}{y^2}} = -\frac{x^2}{y} + a \times \frac{a}{y}$$

$$= \frac{a^2 - x^2}{y} = \frac{y^2}{y} = y$$

Miscellaneous exercise 6 | Q 1.04 | Page 215

Choose the correct option from the given alternatives:
The differential equation of all circles having their centres on the line y = 5 and touching the X-axis is

$$y^2\left(1 + \frac{dy}{dx}\right) = 25$$

$$(y - 5)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 25$$

$$(y - 5)^2 + \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 25$$

$$(y - 5)^2\left[1 - \left(\frac{dy}{dx}\right)^2\right] = 25$$

SOLUTION

$$(y - 5)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 25$$

Hint: Equation of the circle is

$(x - h)^2 + (y - 5)^2 = 5^2$ ....(1)

$\therefore 2(x - h) + 2(y - 5)\frac{dy}{dx} = 0$

$\therefore (x - h)^2 = (y - 5)^2 \left(\frac{dy}{dx}\right)^2$

$\therefore 25 - (y - 5)^2 = (y - 5)^2 \left(\frac{dy}{dx}\right)^2$ ...[By (1)]

$\therefore (y - 5)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 25$

Miscellaneous exercise 6 | Q 1.05 | Page 215

Choose the correct option from the given alternatives:

The differential equation $y\frac{dy}{dx} + x = 0$ represents family of

- circles
- parabolas
- ellipses
- hyperbolas

SOLUTION

circles

Hint:

$y\frac{dy}{dx} + x = 0 \therefore \int y\, dy + \int x\, dx = c$

$\therefore \dfrac{y^2}{2} + \dfrac{x^2}{2} = c$

$\therefore x^2 + y^2 = 2c$ which is a circle.

Miscellaneous exercise 6 | Q 1.06 | Page 215

Choose the correct option from the given alternatives:

The solution of $\dfrac{1}{x} \cdot \dfrac{dy}{dx} = \tan^{-1} x$ is

$\dfrac{x^2 \tan^{-1} x}{2} + c = 0$

$x \tan^{-1} x + c = 0$

$x - \tan^{-1} x = c$

$y = \dfrac{x^2 \tan^{-1} x}{2} - \dfrac{1}{2}\left(x - \tan^{-1} x\right) + c$

SOLUTION

$y = \dfrac{x^2 \tan^{-1} x}{2} - \dfrac{1}{2}\left(x - \tan^{-1} x\right) + c$

Miscellaneous exercise 6 | Q 1.07 | Page 215

Choose the correct option from the given alternatives:

The solution of $(x + y)^2 \dfrac{dy}{dx} = 1$ is

$x = \tan^{-1} (x + y) + c$

$y \tan^{-1} \left(\dfrac{x}{y}\right) = c$

$y = \tan^{-1} (x + y) + c$

$y + \tan^{-1} (x + y) + c$

SOLUTION

y = $\tan^{-1}$ (x + y) + c

Hint:

$$(x+y)^2 \frac{dy}{dx} = 1$$

Put x + y = u $\quad \therefore 1 + \frac{dy}{dx} = \frac{du}{dx}$

$$\therefore u^2\left(\frac{du}{dx} - 1\right) = 1$$

$$\therefore u^2 \frac{du}{dx} = u^2 + 1$$

$$\therefore \int \frac{u^2}{u^2+1} du = \int dx$$

$$\therefore \int \frac{(u^2+1) - 1}{u^2+1} du = \int dx$$

$$\therefore \int \left(1 - \frac{1}{u}\right) du = \int dx$$

∴ u - $\tan^{-1}$ u = x + c

∴ x + y - $\tan^{-1}$ (x + y) = x + c

∴ y = $\tan^{-1}$ (x + y) + c.

Miscellaneous exercise 6 | Q 1.08 | Page 215

Choose the correct option from the given alternatives:

The solution of $\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$ is

$$\sin^{-1}\left(\frac{y}{x}\right) = 2\log|x| + c$$

$$\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + c$$

$$\sin\left(\frac{y}{x}\right) = \log|x| + c$$

$$\sin\left(\frac{y}{x}\right) = 2\log|x| + c$$

SOLUTION

$$\sin^{-1}\left(\frac{y}{x}\right) = \log|x| + c$$

Hint:

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$$

Put y = vx $\quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 - v^2x^2}}{x} = v + \sqrt{1 - v^2}$$

$$\therefore x\frac{dv}{dx} = \sqrt{1 - v^2}$$

$$\therefore \int \frac{1}{\sqrt{1 - v^2}} dv = \int \frac{1}{x} dx$$

$\therefore \sin^{-1} v = \log|x| + c$

$\therefore \sin^{-1} v \left(\dfrac{y}{x}\right) = \log|x| + c.$

Miscellaneous exercise 6 | Q 1.09 | Page 215

Choose the correct option from the given alternatives:

The solution of $\dfrac{dy}{dx} + y = \cos x - \sin x$

$ye^x = \cos x + c$

$ye^x + e^x \cos x = c$

$ye^x = e^x \cos x + c$

$y^2e^x = e^x \cos x + c$

SOLUTION

$ye^x = e^x \cos x + c$

Hint:

$$\frac{dy}{dx} + y = \cos x - \sin x$$

I.F. = $e^{\int 1 dx} = e^x$

∴ the solution is $y \cdot e^x = \int (\cos x - \sin x) e^x + c$

∴ $ye^x = e^x \cos x + c$

Miscellaneous exercise 6 | Q 1.1 | Page 216

Choose the correct option from the given alternatives:

The integrating factor of linear differential equation $x\frac{dy}{dx} + 2y = x^2 \log x$ is

$\frac{1}{x}$

k

$\frac{1}{n^2}$

x^2

SOLUTION

x^2

Hint: I.F. = $e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$

Miscellaneous exercise 6 | Q 1.11 | Page 216

Choose the correct option from the given alternatives:

The solution of the differential equation $\frac{dy}{dx} = \sec x - y \tan x$

y sec x + tan x = c

y sec x = tan x + c

sec x + y tan x = c

sec x = y tan x + c

SOLUTION

y sec x = tan x + c

Hint:

$$\frac{dy}{dx} = \sec x - y \tan x$$

$$\therefore \frac{dy}{dx} + y \tan x = \sec x$$

I.F. = $e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

∴ the solution is

$$y \cdot \sec x = \int \sec x \cdot \sec x dx + c$$

∴ y sec x = tan x + c

Miscellaneous exercise 6 | Q 1.12 | Page 216

Choose the correct option from the given alternatives:

The particular solution of $\frac{dy}{dx} = xe^{y-x}$, when x = y = 0 is

$e^{x-y} = x + 1$

e^{x+y} = x + 1

$e^{x} + e^{y}$ = x + 1

e^{y-x} = x - 1

SOLUTION

$e^{x - y} = x + 1$

Hint:

$$\frac{dy}{dx} = xe^{y - x}$$

$$\therefore \int e^{-y} dy = \int xe^{-x} dx$$

$$\therefore e^{-y} = x \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} dx + c$$

$$\therefore -e^{-y} = -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$\therefore e^{-y} = \frac{x}{e^x} + \frac{1}{e^x} - c$$

$$\therefore e^{x - y} = x + 1 - ce^x$$

When x = y = 0, we get

1 = 1 - c $\quad\quad$ ∴ c = 0

∴ particular solution is

$e^{x - y} = x + 1$

Miscellaneous exercise 6 | Q 1.13 | Page 216

Choose the correct option from the given alternatives:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a solution of

$$\frac{d^2y}{dx^2} + yx + \left(\frac{dy}{dx}\right)^2 = 0$$

$$xy \cdot \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

$$y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+y=0$$

$$xy\frac{dy}{dx}+y\frac{d^2y}{dx^2}=0$$

SOLUTION

$$xy\cdot\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

Hint:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad .....(1)$$

$$\therefore \frac{1}{a^2}\times 2x-\frac{1}{b^2}\times 2y\frac{dy}{dx}=0$$

$$\therefore \frac{x}{a^2}-\frac{y}{b^2}\frac{dy}{dx}=0 \quad ....(2)$$

and $\frac{1}{a^2}\times 1-\frac{1}{b^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0 \quad ....(3)$

Equations (1), (2) and (3) are consistent

$$\therefore \begin{vmatrix} x^2 & -y^2 & 1 \\ x & -y\frac{dy}{dx} & 0 \\ 1 & -\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] & 0 \end{vmatrix}=0$$

$$\therefore xy\cdot\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

Miscellaneous exercise 6 | Q 1.14 | Page 216

Choose the correct option from the given alternatives:

The decay rate of certain substances is directly proportional to the amount present at that instant. Initially there are 27 grams of substance and 3 hours later it is found that 8 grams left. The amount left after one more hour is

$5\frac{2}{3}$ grams

$5\frac{1}{3}$ grams

5.1 grams

5 grams

SOLUTION

$5\frac{1}{3}$ grams

Miscellaneous exercise 6 | Q 1.15 | Page 216

Choose the correct option from the given alternatives:
If the surrounding air is kept at 20° C and a body cools from 80° C to 70° C in 5 minutes, the temperature of the body after 15 minutes will be

1. 51.7° C
2. **54.7° C**
3. 52.7° C
4. 50.7° C

SOLUTION

54.7° C

MISCELLANEOUS EXERCISE 6 [PAGES 216 - 218]

Miscellaneous exercise 6 | Q 1.1 | Page 216

Determine the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = x^3$$

SOLUTION

The given D.E. is

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = x^3$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

$\therefore$ the given D.E. is of order 2 and degree 1.

Miscellaneous exercise 6 | Q 1.2 | Page 216

Determine the order and degree of the following differential equation:

$$\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt[5]{1+\frac{dy}{dx}}$$

SOLUTION

The given D.E. is

$$\left(\frac{d^3y}{dx^3}\right)^2 = \sqrt[5]{1+\frac{dy}{dx}}$$

$$\left(\frac{d^3y}{dx^3}\right)^{2\times 5} = 1 + \frac{dy}{dx}$$

$$\left(\frac{d^3y}{dx^3}\right)^{10} = 1 + \frac{dy}{dx}$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 10.

$\therefore$ the given D.E. is of order 3 and degree 10.

Miscellaneous exercise 6 | Q 1.3 | Page 216

Determine the order and degree of the following differential equation:

$$\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}=\frac{d^2y}{dx^2}$$

SOLUTION

The given D.E. is

$$\sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}=\frac{d^2y}{dx^2}$$

On cubing both sides, we get

$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{d^2y}{dx^2}\right)^3$$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 3.

$\therefore$ the given D.E. is of order 2 and degree 3.

Miscellaneous exercise 6 | Q 1.4 | Page 216

Determine the order and degree of the following differential equation:

$$\frac{dy}{dx}=3y+\sqrt[4]{1+5\left(\frac{dy}{dx}\right)^2}$$

SOLUTION

The given D.E. is

$$\frac{dy}{dx}=3y+\sqrt[4]{1+5\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \frac{dy}{dx}-3y=\sqrt[4]{1+5\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \left(\frac{dy}{dx} - 3y\right)^4 = 1 + 5\left(\frac{dy}{dx}\right)^2$$

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 4.

$\therefore$ the given D.E. is of order 1 and degree 4.

Miscellaneous exercise 6 | Q 1.5 | Page 216

Determine the order and degree of the following differential equation:

$$\frac{d^4y}{dx^4} + \sin\left(\frac{dy}{dx}\right) = 0$$

SOLUTION

The given D.E. is

$$\frac{d^4y}{dx^4} + \sin\left(\frac{dy}{dx}\right) = 0$$

This D.E. has highest order derivative $\frac{d^4y}{dx^4}$.

$\therefore$ order = 4

Since this D.E. cannot be expressed as a polynomial in differential coefficient, the degree is not defined.

Miscellaneous exercise 6 | Q 2.1 | Page 217

In the following example verify that the given function is a solution of the differential equation.

$$x^2 + y^2 = r^2; x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y$$

SOLUTION

$x^2 + y^2 = r^2$ .....(1)

Differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore 2y\frac{dy}{dx} = -2x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= x\left(-\frac{x}{y}\right) + r\sqrt{1 + \left(-\frac{x}{y}\right)^2}$$

$$= -\frac{x^2}{y} + r\sqrt{1 + \frac{x^2}{y^2}}$$

$$= -\frac{x^2}{y} + r\sqrt{\frac{y^2 + x^2}{y^2}}$$

$$= -\frac{x^2}{y} + r\sqrt{\frac{r^2}{y^2}} \quad \text{....[By (1)]}$$

$$= -\frac{x^2}{y} + \frac{r^2}{y} = \frac{r^2 - x^2}{y}$$

$$\therefore \frac{y^2}{y} = y$$

Hence, $x^2 + y^2 = r^2$ is a solution of the D.E.

$$x\frac{dy}{dx} + r\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y$$

Miscellaneous exercise 6 | Q 2.2 | Page 217

In the following example verify that the given function is a solution of the differential equation.

$$y = e^{ax}\sin bx; \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$$

SOLUTION

$$y = e^{ax}\sin bx$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(e^{ax}\sin bx\right)$$

$$= e^{ax}\cdot\frac{d}{dx}(\sin bx) + \sin bx\cdot\frac{d}{dx}\left(e^{ax}\right)$$

$$= e^{ax}\times\cos bx\cdot\frac{d}{dx}(bx) + \sin bx\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

$$= e^{ax}\cos bx\times b + e^{ax}\sin bx\times a$$

$$= e^{ax}(b\cos bx + a\sin bx)$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[e^{ax}(b\cos bx + a\sin bx)\right]$$

$$= e^{ax}\cdot\frac{d}{dx}(b\cos bx + a\sin bx) + (b\cos bx + a\sin bx)\cdot\frac{d}{dx}\left(e^{ax}\right)$$

$$= e^{ax}\left[b(-\sin bx)\cdot\frac{d}{dx}(bx) + a\cos bx\cdot\frac{d}{dx}(bx)\right] + (b\cos bx + a\sin bx)\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

$$= e^{ax}[-b\sin bx\times b + a\cos bx\times b] + (b\cos bx + a\sin bx)\cdot e^{ax}\times a$$

$$= e^{ax}\left(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx\right)$$

$$= e^{ax}\left[\left(a^2 - b^2\right)\sin bx + 2ab\cos bx\right]$$

$$\therefore \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y$$

$$= e^{ax}\left[\left(a^2 - b^2\right)\right] \sin bx + 2ab\cos bx - 2a \cdot e^{ax}(b\cos bx + a\sin bx) + \left(a^2 + b^2\right) \cdot e^{ax} \sin bx$$

$$= e^{ax}\left[\left(a^2 - b^2\right) \sin bx + 2ab\cos bx - 2ab\cos bx - 2a^2 \sin bx + \left(a^2 + b^2\right) \sin bx\right]$$

$$= e^{ax}\left[\left(a^2 - b^2\right) \sin bx - \left(a^2 - b^2\right) \sin bx\right]$$

$$= e^{ax} \times 0 = 0$$

Hence. y = e^{ax} sin bx is a solution of the D.E.

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$$

Miscellaneous exercise 6 | Q 2.3 | Page 217

In the following example verify that the given function is a solution of the differential equation.

$$y = 3\cos(\log x) + 4\sin(\log x); x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

SOLUTION

$$y = 3\cos(\log x) + 4\sin(\log x) \quad ...(1)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3\frac{d}{dx}[\cos(\log x)] + 4\frac{d}{dx}[\sin(\log x)]$$

$$= 3[-\sin(\log x)]\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$= -3\sin(\log x) \times \frac{1}{x} + 4\cos(\log x) \times \frac{1}{x}$$

$$\therefore x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Differentiating again w.r.t. x, we get,

$$x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) = -3\frac{d}{dx}[\sin(\log x)] + 4\frac{d}{dx}[\cos(\log x)]$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 = -3\cos(\log x) \cdot \frac{d}{dx}(\log x) + 4[-\sin(\log x)] \cdot \frac{d}{dx}(\log x)$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -3\cos(\log x) \times \frac{1}{x} - 4\sin(\log x) \times \frac{1}{x}$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[3\cos(\log x) + 4\sin(\log x)] = -y \quad \text{...[By (1)]}$$

$$\therefore x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Hence, y = 3 cos (log x) + 4 sin (log x) is a solution of the D.E. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Miscellaneous exercise 6 | Q 2.4 | Page 217

In the following example verify that the given function is a solution of the differential equation.

$xy = ae^x + be^{-x} + x^2; x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x^2 = xy + 2$

SOLUTION

$$xy = ae^x + be^{-x} + x^2$$

$$\therefore xy - x^2 = ae^x + be^{-x} \quad \text{....(1)}$$

Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x) - 2x = ae^x + be^{-x} \times (-1)$$

$$\therefore x\frac{dy}{dx} + y - 2x = ae^x - be^{-x}$$

Differentiating again w.r.t. x, we get

$$x \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) + \frac{dy}{dx} - 2 \times 1 = ae^x - be^{-x}(-1)$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 + \frac{dy}{dx} - 2 = ae^x + be^{-x}$$

$$\therefore x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-2=xy-x^2$$ ....[By (1)]

$$\therefore x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+x^2=xy+2$$

Hence, xy = $ae^x - be^{-x} + x^2$ is a solution of the D.E.

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+x^2=xy+2$$

Miscellaneous exercise 6 | Q 2.5 | Page 217

In the following example verify that the given function is a solution of the differential equation.

$x^2 = 2y^2 \log y,\ x^2+y^2 = xy\frac{dx}{dy}$

SOLUTION

$x^2 = 2y^2 \log y$ .....(1)

Differentiating both sides w.r.t. y, we get

$$2x\frac{dx}{dy} = 2\frac{d}{dy}\left(y^2 \log y\right)$$

$$= 2\left[y^2\frac{d}{dy}(\log y)+(\log y)\cdot\frac{d}{dy}\left(y^2\right)\right]$$

$$= 2\left[y^2\times\frac{1}{y}+(\log y)\times 2y\right]$$

$$\therefore x\frac{dx}{dy} = y+2y\log y$$

$$\therefore xy\frac{dx}{dy} = y^2+2y^2\log y$$

= $y^2 + x^2$ ....[By (1)]

$$\therefore x^2+y^2 = xy\frac{dx}{dy}$$

$$x^2 + y^2 = xy\frac{dx}{dy}$$

Miscellaneous exercise 6 | Q 3.1 | Page 217

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$$y^2 = a(b - x)(b + x)$$

SOLUTION

$$y^2 = a(b - x)(b + x) = a\left(b^2 - x^2\right)$$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = a(0 - 2x) = -2ax$$

$$\therefore y\frac{dy}{dx} = -ax \quad \text{....(1)}$$

Differentiating again w.r.t. x, we get

$$y \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{dy}{dx} = -a \times 1$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a$$

$$\therefore xy\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -ax$$

$$\therefore xy\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y\frac{dy}{dx} \quad \text{....[By (1)]}$$

$$\therefore xy\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y\frac{dy}{dx} = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 3.2 | Page 217

Obtain the differential equation by eliminating the arbitrary constants from the following equation:
y = a sin (x + b)

SOLUTION

y = a sin (x + b)

$$\therefore \frac{dy}{dx} = a\frac{d}{dx}[\sin(x + b)]$$

$$= a\cos(x + b) - \frac{d}{dx}(x + b)$$

= a cos (x + b) × (1 + 0)

= a cos (x + b)

and $\frac{d^2y}{dx^2} = a\frac{d}{dx}[\cos(x + b)]$

$$= a[-\sin(x + b)] \cdot \frac{d}{dx}(x + b)$$

$$= -a\sin(x + b) \times (1 + 0)$$

$$\therefore \frac{d^2y}{dx^2} = -y \qquad \text{.....[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 3.3 | Page 217

Obtain the differential equation by eliminating the arbitrary constants from the following equation:
$(y - a)^2 = b(x + 4)$

SOLUTION

$(y - a)^2 = b(x + 4)$ ....(1)

Differentiating both sides w.r.t. x, we get

$$2(y - a) \cdot \frac{d}{dx}(y - a) = b\frac{d}{dx}(x + 4)$$

$$\therefore 2(y - a) \cdot \left(\frac{dy}{dx} - 0\right) = b(1 + 0)$$

$$\therefore 2(y - a)\frac{dy}{dx} = b$$

$$\therefore 2(y - a)\frac{dy}{dx} = \frac{(y - a)^2}{x + 4} \quad \text{....[By (1)]}$$

$$2(x + 4)\frac{dy}{dx} = y - a$$

Differentiating again w.r.t. x, we get

$$2\left[(x + 4) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x + 4)\right] = \frac{dy}{dx} - 0$$

$$\therefore 2\left[(x + 4)\frac{d^2y}{dx^2} + \frac{dy}{dx} \times (1 + 0)\right] = \frac{dy}{dx}$$

$$\therefore 2(x + 4)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\therefore 2(x + 4)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 3.4 | Page 217

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

$$y = \sqrt{a\cos(\log x) + b\sin(\log x)}$$

SOLUTION

y = $\sqrt{\mathrm{a}\cos(\log \mathrm{x})+\mathrm{b}\sin(\log \mathrm{x})}$

∴ y^2 = a cos (log x) + b sin (log x) ....(1)

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx}=a\frac{d}{dx}[\cos(\log x)]+b\frac{d}{dx}[\sin(\log x)]$$

$$=a[-\sin(\log x)]\cdot\frac{d}{dx}(\log x)+b\cos(\log x)\cdot\frac{d}{dx}(\log x)$$

$$=-a\sin(\log x)\times\frac{1}{x}+b\cos(\log x)\times\frac{1}{x}$$

$$\therefore\ 2xy\frac{dy}{dx}=-a\sin(\log x)+b\cos(\log x)$$

Differentiating again w.r.t. x, we get

$$2\left[xy\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(xy)\right]$$

$$=-a\frac{d}{dx}[\sin(\log x)]+b\frac{d}{dx}[\cos(\log x)]$$

$$\therefore\ 2\left[xy\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(x\frac{dy}{dx}+y\times 1\right)\right]$$

$$=-a\cos(\log x)\cdot\frac{d}{dx}(\log x)+b[-\sin(\log x)]\cdot\frac{d}{dx}(\log x)$$

∴ `2"xy" ("d"^2"y")/"dx"^2 + 2"x" ("dy"/"dx")^2 + "2y" "dy"/"dx"

$$=-a\cos(\log x)\times\frac{1}{x}-b\sin(\log x)\times\frac{1}{x}$$

$$\therefore\ 2x^2y\frac{d^2y}{dx^2}+2x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

$=-[a\cos(\log x)+b\sin(\log x)]=-y^2$ ......[By (1)]

$$\therefore 2x^2y\frac{d^2y}{dx^2} + 2x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y^2 = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 3.5 | Page 217

Obtain the differential equation by eliminating the arbitrary constants from the following equation:

y = $Ae^{3x+1} + Be^{-3x+1}$

SOLUTION

y = $Ae^{3x+1} + Be^{-3x+1}$ ....(1)

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{3x+1} \cdot \frac{d}{dx}(3x+1) + Be^{-3x+1} \cdot \frac{d}{dx}(-3x+1)$$

$$= Ae^{3x+1} \times (3+1+0) + Be^{-3x+1} \times (-3 \times 1 + 0)$$

$$= 3Ae^{3x+1} - 3Be^{-3x+1}$$

and $\frac{d^2y}{dx^2} = 3Ae^{3x+1} \cdot \frac{d}{dx}(3x+1) - 3Be^{-3x+1} \cdot \frac{d}{dx}(-3x+1)$

$$= 3Ae^{3x+1} \times (3 \times 1 + 0) - 3Be^{-3x+1} \times (-3 \times 1 + 0)$$

$$= 9Ae^{3x+1} - 9Be^{-3x+1}$$

$$= 9\left(Ae^{3x+1} + Be^{-3x+1}\right)$$

= 9y ....[By (1)]

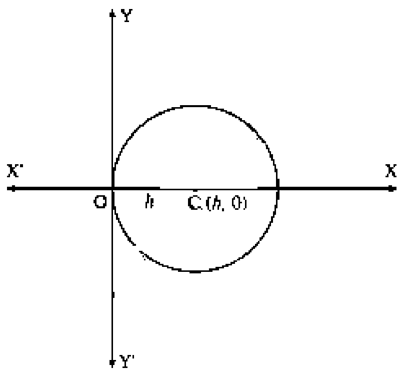
$$\therefore \frac{d^2y}{dx^2} - 9y = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 4.1 | Page 217

Form the differential equation of all circles which pass through the origin and whose centers lie on X-axis.

SOLUTION



Let C(h, 0) be the center of the circle which passes through the origin. Then the radius of the circle is h.

∴ equation of the circle is $(x - h)^2 + (y - 0)^2 = h^2$

∴ $x^2 - 2hx + h + y^2 = h^2$

∴ $x^2 + h^2 = 2hx$ ....(1)

Differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 2h$$

Substituting the value of 2h in equation (1), we get

$$x^2 + y^2 = \left(2x + 2y\frac{dy}{dx}\right)x$$

$$\therefore x^2 + y^2 = 2x^2 + 2xy\frac{dy}{dx}$$

$$\therefore 2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 4.2 | Page 217

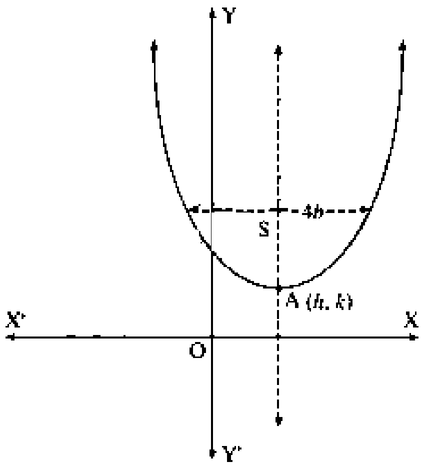
Form the differential equation of all parabolas which have 4b as latus rectum and whose axis is parallel to the Y-axis.

SOLUTION

Let A(h, k) be the vertex of the parabola which has 4b as a latus rectum and whose axis is parallel to Y-axis. Then the equation of the parabola is

$(x - h)^2 = 4b(y - k)$ .....(1)

where h and k are arbitrary constants.



Differentiating both sides of (1) w.r.t. x, we get

$$2(x - h) \cdot \frac{d}{dx}(x - h) = 4b\frac{d}{dx}(y - k)$$

$$\therefore 2(x - h) \times (1 - 0) = 4b\left(\frac{dy}{dx} - 0\right)$$

$$\therefore (x - h) = 2b\frac{dy}{dx}$$

Differentiating again w.r.t. x, we get

$$1 - 0 = 2b\frac{d^2y}{dx^2}$$

$$\therefore 2b\frac{d^2y}{dx^2} - 1 = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 4.3 | Page 217

Find the differential equation of the ellipse whose major axis is twice its minor axis.

SOLUTION

Let 2a and 2b be lengths of major axis and minor axis of the ellipse.

Then 2a = 2(2b)

∴ a = 2b

∴ equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

i.e. $$\frac{x^2}{(2b)^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{x^2}{4b^2} + \frac{y^2}{b^2} = 1$$

$$\therefore x^2 + 4y^2 = 4b^2$$

Differentiating w.r.t. x, we get

$$2x + 4 \times 2y\frac{dy}{dx} = 0$$

$$\therefore x + 4y\frac{dy}{dx} = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 4.4 | Page 217

Form the differential equation of all the lines which are normal to the line 3x + 2y + 7 = 0.

SOLUTION

Slope of the line 3x - 2y + 7 = 0 is $\frac{-3}{-2} = \frac{3}{2}$

$\therefore$ slope of normal to this line is $-\frac{2}{3}$

Then the equation of the normal is

y = $-\frac{2}{3}x + k$, where k is an arbitrary constant.

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{2}{3} \times 1 + 0$$

$$\therefore 3\frac{dy}{dx} + 2 = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 4.5 | Page 217

Form the differential equation of the hyperbola whose length of transverse and conjugate axes are half of that of the given hyperbola $\frac{x^2}{16} - \frac{y^2}{36} = k$.

SOLUTION

The equation of the hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{36} = k \text{ i.e. } \frac{x^2}{16\,k} - \frac{y^2}{36k} = 1$$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

a^2 = 16k, b^2 = 36k

$\therefore$ a = $4\sqrt{k}, b = 6\sqrt{k}$

$\therefore$ l(transverse axis) = 2a = $8\sqrt{k}$

and l(conjugate axis) = 2b = $12\sqrt{k}$

Let 2A and 2B be the lengths of the transverse and conjugate axes of the required hyperbola.

Then according to the given condition

2A = a = $4\sqrt{k}$ and $2B = b = 6\sqrt{k}$

∴ A = $2\sqrt{k}$ and B = $3\sqrt{k}$

∴ equation of the required hyperbola is

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

i.e. $\frac{x^2}{4k} - \frac{y^2}{9k} = 1$

∴ $9x^2 - 4y^2 = 36k$, where k is an arbitrary constant.

Differentiating w.r.t. x, we get

$$9 \times 2x - 4 \times 2y\frac{dy}{dx} = 0$$

$$\therefore 9x - 4y\frac{dy}{dx} = 0$$

This is the required D.E.

Miscellaneous exercise 6 | Q 5.1 | Page 217

Solve the following differential equation:

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

SOLUTION

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

$$\therefore \frac{dy}{dx} = e^{2x+3y} = e^{2x}.e^{3y}$$

$$\therefore \frac{1}{e^{3y}}dy = e^{2x}dx$$

Integrating both sides, we get

$$\int e^{-3y}dy = \int e^{2x'} \ dx$$

$$\therefore \int e^{-3}ydy = \int e^{2x}dx$$

$$\therefore \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c_1$$

$$\therefore 2e^{-3y} = -3e^{2x} + 6c_1$$

$\therefore 2e^{-3y} + 3e^{2x} = c$, where $c = 6c_1$

This is the general solution.

Miscellaneous exercise 6 | Q 5.2 | Page 217

Solve the following differential equation:

$$\frac{dy}{dx} = x^2y + y$$

SOLUTION

$$\frac{dy}{dx} = x^2y + y$$

$$\therefore \frac{dy}{dx} = y(x^2 + 1)$$

$$\therefore \frac{1}{y}dy = (x^2+1)dx$$

Integrating both sides, we get

$$\int \frac{1}{y}dy = \int (x^2+1)dx$$

$$\therefore \log|y| = \frac{x^3}{3} + x + c$$

This is the general solution.

Miscellaneous exercise 6 | Q 5.4 | Page 217

Solve the following differential equation:

x dy = (x + y + 1) dx

SOLUTION

x dy = (x + y + 1) dx

$$\therefore \frac{dy}{dx} = \frac{x+y+1}{x} = \frac{x+1}{x} + \frac{y}{x}$$

$$\therefore \frac{dy}{dx} - \frac{1}{x} \cdot y = \frac{x+1}{x} \quad ....(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where P = $-\frac{1}{x}$ and $Q = \frac{x+1}{x}$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int -\frac{1}{x}dx}$

$$= e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dx + c$$

$$\therefore y \cdot \frac{1}{x} = \int \frac{x+1}{x} \times \frac{1}{x} dx + c$$

$$\therefore \frac{y}{x} = \int \frac{x+1}{x^2} dx + c$$

$$\therefore \frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx + c$$

$$\therefore \frac{y}{x} = \int \frac{1}{x} dx + \int x^{-2} dx + c$$

$$\therefore \frac{y}{x} = \log|x| + \frac{x^{-1}}{-1} + c$$

$\therefore$ y = x log x - 1 + cx

This is the general solution.

Miscellaneous exercise 6 | Q 5.5 | Page 217

Solve the following differential equation:

$$\frac{dy}{dx} + y \cot x = x^2 \cot x + 2x$$

SOLUTION

$$\frac{dy}{dx} + y \cot x = x^2 \cot x + 2x \quad ....(1)$$

This is the linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where P = cot x and Q = x^2 cot x + 2x

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int \cot x\,dx}$

$= e^{\log(\sin x)} = \sin x.$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$\therefore$ y sin x = $\int (x^2 \cot x + 2x) \sin x\, dx + c$

$\therefore$ y sin x = $\int (x^2 \cot x \cdot \sin x + 2x \sin x) dx + c$

$\therefore$ y sin x = $\int x^2 \cos x \, dx + 2 \int x \sin x \, dx + c$

$\therefore$ y sin x = $x^2 \int \cos x \;\; dx - \int \left[\frac{d}{dx} (x^2) \int \cos x \;\; dx \right] dx + 2 \int x \sin x \, dx + c$

$= x^2 (\sin x) - \int 2x (\sin x) dx + 2 \int x \sin x \, dx + c$

$= x^2 \sin x - 2 \int x \sin x \, dx + 2 \int x \sin x \, dx + c$

$\therefore$ y sin x = x^2 sin x + c

$\therefore$ y = x^2 + c cosec x

This is the general solution.

Miscellaneous exercise 6 | Q 5.6 | Page 217

Solve the following differential equation:

y log y = (log y^2 - x) $\frac{dy}{dx}$

SOLUTION

y log y = (log y^2 - x) $\frac{dy}{dx}$

$$\therefore \frac{1}{\frac{dy}{dx}} = \frac{\log y^2 - x}{y \log y}$$

$$\therefore \frac{dx}{dy} = \frac{2 \log y \text{ - } x}{y \log y}$$

$$\therefore \frac{dx}{dy} = \frac{2}{y} - \frac{x}{y \log y}$$

$$\therefore \frac{dx}{dy} + \frac{x}{y \log y} = \frac{2}{y} \quad .....(1)$$

This is the linear differential equation of the form

$\frac{dx}{dy} + Px = Q$ where P = $\frac{1}{y\log y}$ and $Q = \frac{2}{y}$

∴ I.F. = $e^{\int P\,dy} = e^{\int \frac{1}{y\log y}dy}$

$= e^{\int \frac{1/y}{\log y}dy} = e^{\log|\log y|}$ = log y

∴ the solution of (1) is given by

$$x \cdot (I.F.) = \int Q \cdot (I.F.)dy + c$$

$$\therefore x \cdot \log y = \int \frac{2}{y} \cdot \log y\, dy + c$$

$$\therefore (\log y) \cdot x = 2\int \frac{\log y}{y}dy + c$$

Put log y = t

$$\therefore \frac{1}{y}dy = dt$$

$$\therefore (\log y) \cdot x = 2\int t\, dt + c$$

∴ x log y = $2 \cdot \frac{t^2}{2} + c$

∴ x log y = $(\log y)^2$ + c

This is the general solution.

Miscellaneous exercise 6 | Q 5.7 | Page 217

Solve the following differential equation:

$$\frac{dx}{dy} + 8x = 5e^{-3y}$$

SOLUTION

$$\frac{dx}{dy} + 8x = 5e^{-3y}$$

$$\therefore \frac{dx}{dy} + 2x = \frac{5}{4}e^{-3y} \quad .....(1)$$

This is the linear differential equation of the form

$\frac{dx}{dy} + Px = Q$ where P = 2 and $Q = \frac{5}{4}e^{-3y}$

$\therefore$ I.F. = $e^{\int P\,dy} = e^{2dy} = e^{2y}$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})dy + c_1$$

$$\therefore x \cdot e^{2y} = \int \frac{5}{4}e^{-3y} \cdot e^{2y}dy + c_1$$

$$\therefore x \cdot e^{2y} = \frac{5}{4}\int e^{-y}dy + c_1$$

$$\therefore x\, e^{2y} = \frac{5}{4} \cdot \frac{e^{-y}}{-1} + c_1$$

$\therefore$ $4xe^{2y} = -5e^{-y} + 4c_1$

$\therefore$ $4xe^{2y} + -5e^{-y} = c$, where $c = 4c_1$

This is the general solution.

Miscellaneous exercise 6 | Q 6.1 | Page 218

Find the particular solution of the following differential equation:

$y(1 + \log x) = (\log x^x)\frac{dy}{dx}$, when $y(e) = e^2$

SOLUTION

$$y(1+\log x)=(\log x^x)\frac{dy}{dx}$$

$$y(1+\log x)-(\log x^x)\frac{dy}{dx}=0$$

$$y(1+\log x)\frac{dx}{dy}-x\log x=0$$

$$\therefore \frac{1+\log \mathrm{x}}{\mathrm{x}\log \mathrm{x}}\mathrm{dx}-\frac{\mathrm{dy}}{\mathrm{y}}=0$$

Integrating both sides, we get

$$\therefore \int \frac{1+\log \mathrm{x}}{\mathrm{x}\log \mathrm{x}}\mathrm{dx}-\frac{\mathrm{dy}}{\mathrm{y}}=\mathrm{c}_1 \qquad .....(1)$$

Put x log x = t

Then $\left[\mathrm{x}\cdot\frac{\mathrm{d}}{\mathrm{dx}}(\log \mathrm{x})+(\log \mathrm{x})\cdot\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{x})\right]\mathrm{dx}=\mathrm{dt}$

$$\therefore \left[\frac{\mathrm{x}}{\mathrm{x}}+(\log \mathrm{x})(1)\right]\mathrm{dx}=\mathrm{dt}$$

$$\therefore \int \frac{1+\log \mathrm{x}}{\mathrm{x}\log \mathrm{x}}\mathrm{dx}=\int \frac{\mathrm{dt}}{\mathrm{t}}=\log|\mathrm{t}|=\log|\mathrm{x}\log \mathrm{x}|$$

∴ from (1), the general solution is

log |x log x| - log |y| = log c, where c_1 = log c

$$\therefore \log\left|\frac{\mathrm{x}\log \mathrm{x}}{\mathrm{y}}\right|=\log \mathrm{c}$$

$$\therefore \frac{\mathrm{x}\log \mathrm{x}}{\mathrm{y}}=\mathrm{c}$$

∴ x log x = cy

This is the general solution.

Now, y = e^2, when x = e

$\therefore$ e log e = c.e^2

$\therefore$ 1 = c.e

$\therefore$ c = $\dfrac{1}{e}$

$\therefore$ the particular solution is x log x = $\left(\dfrac{1}{e}\right)y$

$\therefore$ y = ex log x.

Miscellaneous exercise 6 | Q 6.2 | Page 218

Find the particular solution of the following differential equation:

$\left(x + 2y^2\right)\dfrac{dy}{dx} = y$, when x = 2, y = 1

SOLUTION

$$\left(x + 2y^2\right)\frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$$

$$\therefore \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y \qquad \text{....(1)}$$

This is the linear differential equation of the form

$\dfrac{dx}{dy} + Px = Q$ where P = $-\dfrac{1}{y}$ and Q = 2y.

$\therefore$ I.F. = $e^{\int P\,dy} = e^{\int -\frac{1}{y}dy}$

$$= e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore x \times \frac{1}{y} = \int 2y \times \frac{1}{y} dy + c$$

$$\therefore \frac{x}{y} = 2\int 1 dy + c$$

$$\therefore \frac{x}{y} = 2y + c$$

$\therefore x = 2y^2 + cy$

This is the general solution.

When x = 2, y = 1, we have

$2 = 2(1)^2 + c(1)$

$\therefore c = 0$

$\therefore$the particular solution is $x = 2y^2$.

Miscellaneous exercise 6 | Q 6.3 | Page 218

Find the particular solution of the following differential equation:

$\frac{dy}{dx} - 3y \cot x = \sin 2x$, when $y\left(\frac{\pi}{2}\right) = 2$

SOLUTION

$$\frac{dy}{dx} - 3y \cot x = \sin 2x$$

$$\therefore \frac{dy}{dx} - (3 \cot x)y = \sin 2x \quad ....(1)$$

This is the linear differential equation of the form

$\frac{dx}{dy} + Px = Q$ where P = $-3 \cot x$ and Q = sin 2x.

$\therefore$ I.F. = $e^{\int P\, dy} = e^{\int -3 \cot x dx}$

$$= e^{-3\log\sin x} = e^{\log(\sin x)^{-3}}$$

$$= (\sin x)^{-3} = \frac{1}{\sin^3 x}$$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore y \times \frac{1}{\sin^3 x} = \int \sin 2x \times \frac{1}{\sin 3x} dx + c$$

$$\therefore y \operatorname{cosec}^3 x = \int 2\sin x \cos x \times \frac{1}{\sin^3 x} dx + c$$

$$\therefore y \operatorname{cosec}^3 x = 2\int \frac{\cos x}{\sin^2 x} dx + c$$

Put $\sin x = t$ $\quad\therefore \cos x\, dx = dt$

$$\therefore y \operatorname{cosec}^3 x = 2\int \frac{1}{t^2} dt + c$$

$$\therefore y \operatorname{cosec}^3 x = 2\int t^{-2} dt + c$$

$$\therefore y \operatorname{cosec}^3 x = 2\left[\frac{t^{-1}}{-1}\right] + c$$

$$\therefore y \operatorname{cosec}^3 x = \frac{-2}{\sin x} + c$$

$\therefore y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = c$

This is the general solution.

Now, $y\left(\frac{\pi}{2}\right) = 2$, i.e. $y = 2$, when $x = \frac{\pi}{2}$

$$\therefore 2\operatorname{cosec}^3 \frac{\pi}{2} + 2\operatorname{cosec}\frac{\pi}{2} = c$$

$\therefore 2(1)^3 + 2(1) = c$

$\therefore$ c = 4

$\therefore$ the particular solution is

$y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = 4$

$\therefore y \operatorname{cosec}^2 x + 2 = 4 \sin x$

Miscellaneous exercise 6 | Q 6.4 | Page 218

Find the particular solution of the following differential equation:
(x + y)dy + (x - y)dx = 0; when x = 1 = y

SOLUTION

$(x + y)dy + (x - y)dx = 0$

$\therefore (x + y)dy = -(x - y)dx$

$$\therefore \frac{dy}{dx} = \frac{y - x}{y + x} \quad ...(1)$$

$$\text{Put } y = vx \qquad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v - 1}{v + 1}$$

$$\therefore x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = \frac{v - 1 - v^2 - v}{v + 1}$$

$$\therefore x\frac{dv}{dx} = -\left(\frac{1 + v^2}{1 + v}\right)$$

$$\therefore \frac{1 + v}{1 + v^2}dv = -\frac{1}{x}dx$$

Integrating both sides, we get

$$\int \frac{1 + v}{1 + v^2}dv = -\int \frac{1}{x}dx$$

$$\therefore \int\left(\frac{1}{1+v^2}+\frac{v}{1+v^2}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \int\frac{1}{1+v^2}dv+\frac{1}{2}\int\frac{2v}{1+v^2}dv=-\int\frac{1}{x}dx$$

$$\therefore \tan^{-1}v+\frac{1}{2}\log\left|1+v^2\right|=-\log x+c \quad .....\left[\because \frac{d}{dv}\left(1+v^2\right)=2v \text{ and } \int\frac{f\prime(x)}{f(x)}dv=\log|f(v)|+c\right]$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\log\left|1+\frac{y^2}{x^2}\right|=\log|x|+c$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\log\left|\frac{x^2+y^2}{x^2}\right|=-\log|x|+c$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}\log\ x^2+y^2-\frac{1}{2}\log\left|x^2\right|=-\log|x|+c$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\log\sqrt{x^2+y^2}-\log|x|=-\log|x|+c$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\log\sqrt{x^2+y^2}=c$$

This is the general solution.

When x = 1 = y, we have

$$\tan^{-1}(1)+\log\sqrt{1^2+1^2}=c$$

$$\therefore \tan^{-1}\left(\frac{\tan\pi}{4}\right)+\log\sqrt{2}=c$$

$$\therefore c=\frac{\pi}{4}+\log\sqrt{2}$$

∴ the particular solution is

$$\therefore \tan^{-1}\left(\frac{y}{x}\right)+\log\sqrt{x^2+y^2}=\frac{\pi}{4}+\log\sqrt{2}$$

Miscellaneous exercise 6 | Q 6.5 | Page 218

Find the particular solution of the following differential equation:

$\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$ when y(0) = 1

SOLUTION

$$\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$$

$$\therefore\left(1+2e^{x/y}\right)dx=-2e^{x/y}\left(1-\frac{x}{y}\right)dy$$

$$\therefore\left(1+2e^{x/y}\right)dx=2e^{x/y}\left(\frac{x}{y}-1\right)dy$$

$$\therefore\frac{dy}{dx}=\frac{2e^{x/y}\left(\frac{x}{y}-1\right)}{1+2e^{x/y}}\quad .....(1)$$

Put x = vy

$$\therefore\frac{dx}{dy}=v+y\frac{dv}{dy}$$

$\therefore$ (1) becomes, $v+y\dfrac{dv}{dy}=\dfrac{2e^{v}(v\text{ - }1)}{1+2e^{v}}$

$$\therefore y\frac{dv}{dy}=\frac{2e^{v}(v\text{ - }1)}{1+2e^{v}}-v$$

$$=\frac{2ve^{v}-2e^{v}-v-2ve^{v}}{1+2e^{v}}$$

$$=-\left(\frac{v+2e^{v}}{1+2e^{v}}\right)$$

$\therefore \left(\frac{1+2e^v}{v+2e^v}\right)dv \equiv -\frac{1}{y}dy$

Integrating both sides, we get

$\int\left(\frac{1+2e^v}{v+2e^v}\right)dv \equiv -\int\frac{1}{y}dy$

$\therefore \log|v + 2e^v| = -\log y + \log c$ .... $\left[\because \frac{d}{dx}(v+2e^v) = 1+2e^v \text{ and } \int\frac{f\prime(v)}{f(v)}dv = \log|f(v)|+c\right]$

$\therefore \log|v + 2e^v| + \log y = \log c$

$\therefore \log|y(v + 2e^v)| = \log c$

$\therefore y(v + 2e^v) = c$

$\therefore y\left(\frac{x}{y} + 2e^{x/y}\right) = c$

$\therefore x + 2ye^{x/y} = c$

This is the general solution.

Now, y(0) = 1, i.e. when x = 0, y = 1

$\therefore 0 + 2(1)e^0 = c$

$\therefore c = 2$

$\therefore$ the particular solution is $x + 2ye^{x/y} = 2$

Miscellaneous exercise 6 | Q 7 | Page 218

Show that the general solution of differential equation $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$ is given by (x + y + 1) = (1 - x - y - 2xy).

SOLUTION

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$

$$\therefore \frac{dy}{dx}=-\left(\frac{y^2+y+1}{x^2+x+1}\right)$$

$$\therefore \frac{1}{y^2+y+1}dy=-\frac{1}{x^2+x+1}dx$$

Integrating both sides, we get

$$\int\frac{1}{y^2+y+1}dy=-\int\frac{1}{x^2+x+1}dx$$

$$\therefore \int\frac{1}{\left(y^2+y+\frac{1}{4}\right)+\frac{3}{4}}dy=-\int\frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}dx$$

$$\therefore \int\frac{1}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dy=-\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$\therefore \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]=-\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+c_1$$

$$\therefore \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)=c_1$$

$$\therefore \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\left(\frac{2y+1}{\sqrt{3}}\right)+\left(\frac{2x+1}{\sqrt{3}}\right)}{1-\left(\frac{2y+3}{\sqrt{3}}\right)\left(\frac{2x+1}{\sqrt{3}}\right)}\right]=c_1$$

$$\therefore \tan^{-1}\frac{\left(\frac{2y+1+2x+1}{\sqrt{3}}\right)}{\left(\frac{3-4xy-2y-2x-1}{3}\right)}=\frac{\sqrt{3}}{2}c_1$$

$$\therefore \frac{(2y+2x+2)\sqrt{3}}{2-2x-2y-4xy}=\tan\left(\frac{\sqrt{3}}{2}c_1\right)$$

$$\therefore \frac{x+y+z}{1-x-y-2xy} = \frac{1}{\sqrt{3}} \tan\left(\frac{\sqrt{3}}{2}\ c_1\right) = c, \text{ where } c = \frac{1}{\sqrt{3}} \tan\left(\frac{\sqrt{3}}{2}\ c_1\right)$$

$\therefore$ (x + y + 1) = c(1 - x - y - 2xy)

This is the general solution.

Miscellaneous exercise 6 | Q 8 | Page 218

The normal lines to a given curve at each point (x, y) on the curve pass through (2, 0). The curve passes through (2, 3). Find the equation of the curve.

SOLUTION

Let P(x, y) be a point on the curve y = f(x).

Then slope of the normal to the curve is $-\frac{1}{\frac{dy}{dx}}$

$\therefore$ equation of the normal is

$$(Y - y) = -\frac{1}{\frac{dy}{dx}}(X - x)$$

$$\therefore (Y - y)\frac{dy}{dx} = -(X - x)$$

$$\therefore (Y - y)\frac{dy}{dx} + (X - x) = 0$$

Since, this normal passes through (2, 0), we get

$$(0 - y)\frac{dy}{dx} + (2 - x) = 0$$

$$\therefore -y\frac{dy}{dx} = x - 2$$

$$\therefore -y\ dy = (x - 2)dx$$

Integrating both sides, we get

$$\int -y\,dy = \int (x - 2)dx$$

$$\therefore -\frac{y^2}{2} = \frac{x^2}{2} - 2x + c_1$$

$$\therefore \frac{x^2}{2} + \frac{y^2}{2} - 2x + c_1 = 0$$

$\therefore x^2 + y^2 - 4x + 2c_1 = 0$

$\therefore x^2 + y^2 = 4x - 2c_1$

$\therefore x^2 + y^2 = 4x + c$, where $c = -2c_1$

This is the general equation of the curve.

Since, the required curve passed through the point (2, 3), we get

$2^2 + 3^2 = 4(2) + c$

$\therefore c = 5$

$\therefore$ equation of the required curve is

$x^2 + y^2 = 4x + 5.$

Miscellaneous exercise 6 | Q 9 | Page 218

The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

SOLUTION

Let r be the radius and V be the volume of the spherical balloon at any time t. Then rate of change in volume of spherical balloon is dV/dt

$\therefore \dfrac{\mathrm{dV}}{\mathrm{dt}} = \mathrm{k}'$, where k' is a constant.

But V = $\dfrac{4}{3}\pi \mathrm{r}^3$

$\therefore \dfrac{\mathrm{dV}}{\mathrm{dt}} = \dfrac{4\pi}{3} \times 3\mathrm{r}^2 \dfrac{\mathrm{dr}}{\mathrm{dt}} = 4\pi\mathrm{r}^2 \dfrac{\mathrm{dr}}{\mathrm{dt}}$

$\therefore 4\pi\mathrm{r}^2 \dfrac{\mathrm{dr}}{\mathrm{dt}}$ = k'

$\therefore \mathrm{r}^2 \dfrac{\mathrm{dr}}{\mathrm{dt}} = \dfrac{k\prime}{4\pi}$ = k, where k = $\dfrac{k\prime}{4\pi}$

$\therefore$ r^2 dr = k dt

Integrating both sides, we get

$$\int \mathrm{r}^2 \mathrm{dr} = \mathrm{k} \int \mathrm{dt}$$

$$\therefore \frac{\mathrm{r}^3}{3} = \mathrm{kt} + c$$

Initially the radius is 3 units

i.e. r = 3, when t = 0

Miscellaneous exercise 6 | Q 10 | Page 218

A person’s assets start reducing in such a way that the rate of reduction of assets is proportional to the square root of the assets existing at that moment. If the assets at the beginning ax ‘ 10 lakhs and they dwindle down to ‘ 10,000 after 2 years, show that the person will be bankrupt in $2\dfrac{2}{9}$ years from the start.

SOLUTION

Let x be the assets of the person at time t years. Then the rate of reduction is dx/dt which is proportional to √x.

$\therefore \frac{dx}{dt} \propto \sqrt{x}$

$\therefore \frac{dx}{dt} = -k\sqrt{x}$, where k > 0

$\therefore \frac{dx}{\sqrt{x}}$ = - k dt

Integrating both sides, we get

$\int x^{-\frac{1}{2}} dx = -k \int dt$

$\therefore \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = -kt + c$

$\therefore 2\sqrt{x}$ = - kt + c

At the beginning, i.e. at t = 0, x = 10,00,000

$\therefore 2\sqrt{1000000}$ = - k (0) + c

∴ c = 2000

$\therefore 2\sqrt{x} = -kt$ + 2000 ...(1)

Also, when t = 2, x = 10,000

$\therefore 2\sqrt{10000} = -k \times 2 + 2000$

∴ 200 = - 2k + 2000

∴ 2k = 1800

∴ k = 900

∴ (1) becomes,

$\therefore 2\sqrt{x} = -900t + 2000$

When the person will be bankrupt, x = 0

$\therefore$ 0 = - 900 t + 2000

$\therefore$ 900 t = 2000

$\therefore$ t = $\frac{20}{9} = 2\frac{2}{9}$

Hence, the person will be bankrupt in $2\frac{2}{9}$ years.