8. Differentials and Partial Derivatives

Linear approximation and differentials

* Linear approximation L of f at x_0 is

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \forall x \in (a,b)$$

* Absolute error = Actual value − Approximate value.

* Relative error = $\dfrac{\text{Absolute error}}{\text{Actual value}}$

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* percentage error = Relative error × 100

Exercise 8.1

1) Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$, use the linear approximation to approximate $\sqrt[3]{27.2}$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = \sqrt[3]{x} = x^{1/3} \qquad x_0 = 27 \qquad \Delta x = 0.2$$

$$f(x_0) = f(27) = \sqrt[3]{27} = 3$$

$$f'(x) = \frac{1}{3} x^{1/3 - 1} \qquad f'(27) = \frac{1}{3} \frac{(27)^{1/3}}{27^1} = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{27} = \frac{1}{27}$$

$$\therefore L(x) = 3 + \frac{1}{27}(x - 27)$$

$$= 3 + \frac{x}{27} - \frac{\cancel{27}}{\cancel{27}}$$

$$L(x) = \frac{x}{27} + 2$$ This is the required linear approximation

$x = 27.2 \quad f(27.2) \simeq L(27.2)$

$$f(27.2) = \sqrt[3]{27.2} \approx \frac{27.2}{27} + 2$$

$$= 1.0074 + 2 = 3.0074$$

$$\begin{array}{r} 1.0074 \\ 27 \overline{)27.2} \\ 27 \\ \hline 0.200 \\ 189 \\ \hline 110 \end{array}$$

2) Use the linear approximation to find approximate values of

(i) $(123)^{2/3}$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = x^{2/3} \qquad x_0 = 125 \qquad \Delta x = -2$$

$$f(x_0) = (125)^{2/3} = 5^2 = 25$$

$$f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} \frac{x^{2/3}}{x}$$

$$f'(x_0) = \frac{2}{3} \frac{(125)^{2/3}}{125} = \frac{2}{3} \frac{\cancel{25}}{\cancel{125}\,5} = \frac{2}{15}$$

$$\therefore L(x) = 25 + \frac{2}{15}(x - 125)$$

$$= 25 + \frac{2x}{15} - \frac{\cancel{125}^{25}}{\cancel{15}_3} \times 2$$

$$= \frac{2x}{15} + 25 - \frac{50}{3}$$

$$= \frac{2x}{15} + \frac{75 - 50}{3}$$

$$\boxed{L(x) = \frac{2x}{15} + \frac{25}{3}}$$

$x = 123$

$\boxed{f(123) \approx L(123)}$

$$f(123) = (123)^{2/3} \approx \frac{2(\cancel{123}^{41})}{\cancel{15}\,5} + \frac{25}{3}$$

$$= \frac{82}{5} + \frac{25}{3}$$

$$= \frac{246 + 125}{15} = \frac{371}{15}$$

$$(123)^{2/3} = 24.733$$

$$\begin{array}{r} 24.733 \\ 15\,)\,371 \\ 30 \\ \hline 71 \\ 60 \\ \hline 110 \\ 105 \\ \hline 50 \\ 45 \\ \hline 50 \end{array}$$

(ii) $\sqrt[4]{15}$

$$f(x) = x^{1/4} \qquad x_0 = 16 \qquad \Delta x = -1$$

$$f(x_0) = f(16) = 16^{1/4} = (2^4)^{1/4} = 2$$

$$f'(x) = \frac{1}{4} x^{1/4 - 1}$$

$$= \frac{1}{4} \frac{x^{1/4}}{x}$$

$$f'(16) = \frac{1}{4} \frac{16^{1/4}}{16} = \frac{\cancel{2}}{64} = \frac{1}{32}$$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$= 2 + \frac{1}{32}(x - 16)$

$= 2 + \frac{x}{32} - \frac{16}{32}$

$= \frac{x}{32} + 2 - \frac{1}{2} = \frac{x}{32} + \frac{3}{2}$

$L(x) = \frac{x}{32} + \frac{3}{2}$

$$\begin{array}{r} 1.968 \\ 32\overline{)63} \\ 32 \\ \hline 310 \\ 288 \\ \hline 220 \\ 192 \\ \hline 280 \\ 256 \end{array}$$

$x = 15$ $f(15) = \sqrt[4]{15} \simeq \frac{15}{32} + \frac{3}{2} = \frac{15+48}{32} = \frac{63}{32}$

$\sqrt[4]{15} = 1.968$

(iii) $\sqrt[3]{26}$

$x_0 = 27$

$\Delta x = -1$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $f(x_0) = \sqrt[3]{27} = 3$

$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}\frac{x^{1/3}}{x}$

$f'(x_0) = f'(27) = \frac{1}{3}\frac{(27)^{1/3}}{27} = \frac{1}{3} \times \frac{3}{27} = \frac{1}{27}$

Linear approximation

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$= 3 + \frac{1}{27}(x - 27)$

$= 3 + \frac{x}{27} - 1$

$L(x) = \frac{x}{27} + 2$

$$\begin{array}{r} 0.9629 \\ 27\overline{)260} \\ 243 \\ \hline 170 \\ 162 \\ \hline 80 \\ 54 \\ \hline 260 \end{array}$$

$x = 26$ $f(26) = \sqrt[3]{26} \simeq L(26)$

$\simeq \frac{26}{27} + 2$

$\simeq 0.9629 + 2$

$\sqrt[3]{26} \simeq 2.9629$

3) Find the linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12$, $x_0 = 2$

$f(x_0) = 2^3 - 5(2) + 12$

$= 8 - 10 + 12$

$f'(x) = 3x^2 - 5$

$f'(x_0) = f'(2) = 3(2^2) - 5$
$= 7$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$L(x) = 10 + 7(x-2)$
$= 10 + 7x - 14$
$= 7x - 4$

(ii) $g(x) = \sqrt{x^2+9}$, $x_0 = -4$

$g(x_0) = \sqrt{16+9} = \sqrt{25} = 5$

$g'(x) = \frac{1}{\cancel{2}\sqrt{x^2+9}} \cancel{2}x$

$g'(x_0) = g'(-4) = \frac{-4}{\sqrt{16+9}} = \frac{-4}{5}$

Linear approximation

$L(x) = g(x_0) + g'(x_0)(x - x_0)$
$= 5 + \left(-\frac{4}{5}\right)(x+4)$
$= 5 - \frac{4x}{5} - \frac{16}{5}$
$= \frac{25-16}{25} - \frac{4x}{5}$

$L(x) = \frac{9-4x}{5}$

(iii) $h(x) = \frac{x}{x+1}$, $x_0 = 1$

$h(x_0) = h(1) = \frac{1}{1+1} = \frac{1}{2}$

$h'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \frac{x+1-x}{(x+1)^2}$

$h'(x_0) = h'(1) = \frac{1}{(1+1)^2} = \frac{1}{2^2} = \frac{1}{4}$

Linear approximation

$L(x) = h(x_0) + h'(x_0)(x - x_0)$

$= \frac{1}{2} + \frac{1}{4}(x-1)$

$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4}$ $L(x) = \frac{x+1}{4}$ //

4) The radius of a circular plate is measured as 12.65cm instead of the actual length 12.5 cm find the following in calculating the area of the circular plate.
(i) Absolute error (ii) Relative error (iii) percentage error

$r = 12.65 \quad \Delta r = -0.15$

Area of circle $A = \pi r^2$

$dr = \Delta r = -0.15$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r\, dr$$

$$= 2\pi \times 12.65 \times (-0.15)$$

$$= -3.795\pi \text{ cm}^2$$

approximate error $= -3.795\pi$

Actual Error $= A(12.5) - A(12.65)$

$$= \pi(12.5)^2 - \pi(12.65)^2$$

$$= \pi(156.25 - 160.0225)$$

$$= -3.7725\pi \text{ cm}^2$$

(i) Absolute ~~Relative~~ Error = Actual error − Appr. Error

$$= -3.7725\pi - (-3.795\pi)$$

$$= 0.0225\pi \text{ cm}^2$$

(ii) Relative error $= \dfrac{\text{absolute error}}{\text{actual error}}$

$$= \frac{0.0225\pi}{-3.7725\pi}$$

$$= -0.00596$$

$$= -0.006$$

(iii) ~~Absolut~~ percentage error = Relative error×100

$$= -0.6\%$$

5) A sphere is made of ice having radius 10cm Its radius decreases from 10cm to 9.8cm. Find approximations for the following

(i) change in the volume (ii) change in the surface area

radius $r=10$

$dr = \Delta r = 9.8 - 10$

$dr = -0.2$

volume of sphere $V = \frac{4}{3}\pi r^3$

$$V'(r) = \frac{4}{3} 3\pi r^2$$

(i) change in the volume $= V(9.8) - V(10)$

$$\simeq V'(r)\, dr$$

$$\simeq 4\pi (10)^2 (0.2)$$

$$\simeq -80\pi \text{ cm}^3$$

Change in volume $\simeq$ decreased by $80\pi \text{ cm}^3$

(ii) change in SA

$$S = 4\pi r^2$$

$$S'(r) = 8\pi r$$

change in $SA = S(9.8) - S(10)$

$$\simeq S'(r)\, dr$$

$$\simeq 8\pi(10)(-0.2)$$

$$\simeq -16\pi \text{ cm}^2$$

surface area decreased by $16\pi \text{ cm}^2$

6) The time T, taken for a complete oscillation of a single pendulum with length l is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate % error in calculate T corresponding to an error of a number 2% in the value of l.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

taking log on both sides

$$\log T = \log\left(2\pi \left(\frac{l}{g}\right)^{1/2}\right)$$

$$= \log 2\pi + \frac{1}{2}\log l/g$$

$$\log T = \log 2\pi + \frac{1}{2}\left[\log l - \log g\right]$$

diff w.r to l

$$\frac{1}{T}\frac{dT}{dl} = 0 + \frac{1}{2}\left(\frac{1}{l} - 0\right)$$

$$\frac{dT}{T} = \frac{1}{2}\frac{dl}{l} \quad \text{—(1)}$$

$dl = 2\%$ of $l = \frac{2}{100} l$

$$\frac{dT}{T} = \frac{2l/100}{2l}$$

$$\frac{dT}{T} = \frac{1}{100}$$

relative error $\frac{\Delta T}{T} \simeq \frac{dT}{T} = \frac{1}{100}$

$$\% \text{ error} = \frac{1}{100} \times 100$$

$$= 1\%$$

7) Show that percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the % error in the number.

Let the number be x.

i.e. n th root $x^{1/n} = y$

$$y = x^{1/n}$$

taking log $\quad \log y = \log x^{1/n}$

$$\log y = \frac{1}{n} \log x$$

diff w.r.to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{n} \cdot \frac{1}{x}$$

$$\frac{dy}{y} = \frac{1}{n}\left(\frac{dx}{x}\right)$$

$$\frac{dy}{y} \times 100 = \frac{1}{n}\left(\frac{dx}{x} \times 100\right)$$

$$\frac{\Delta y}{y} \times 100 \simeq \frac{dy}{y} \times 100 \simeq \frac{1}{n}\left(\frac{dx}{x} \times 100\right)$$

$\therefore \%$ error of $y \simeq \frac{1}{n}$ (% error on x)

———x———

DIFFERENTIALS

① $y = f(x)$ $\quad df = f'(x)\,dx$ or $dy = f'(x)\,dx$

② $\quad \Delta f = f(x + \Delta x) - f(x)$

Exercise 8.2

1) Find the differential dy

i) $y = \frac{(1-2x)^3}{3-4x}$

$$\frac{dy}{dx} = \frac{(3-4x)\,3(1-2x)^2(-2) - (1-2x)^3(0-4)}{(3-4x)^2}$$

$$= (1-2x)^2\left[\frac{-6(3-4x)+4(1-2x)}{(3-4x)^2}\right]$$

$$= (1-2x)^2\left[\frac{-18+24x+4-8x}{(3-4x)^2}\right]$$

$$dy = \frac{(1-2x)^2(16x-14)}{(3-4x)^2}\,dx$$

$$dy = \frac{2(1-2x)^2(8x-7)}{(3-4x)^2}\,dx$$

(ii) $y=(3+\sin 2x)^{2/3}$

$$\frac{dy}{dx} = \frac{2}{3}(3+\sin 2x)^{2/3-1}(0+\cos 2x\; 2)$$

$$= \frac{4}{3}(3+\sin 2x)^{-1/3}\cos 2x$$

$$dy = \frac{4\cos 2x}{3(3+\sin 2x)^{1/3}}\,dx$$

(iii) $y=e^{x^2-5x+7}\cos(x^2-1)$

$$\frac{dy}{dx} = e^{x^2-5x+7}(-\sin(x^2-1)(2x)) + \cos(x^2-1)\,e^{x^2-5x+7}(2x-5)$$

$$\frac{dy}{dx} = e^{x^2-5x+7}\left[-2x\sin(x^2-1)+(2x-5)\cos(x^2-1)\right]$$

$$dy = e^{x^2-5x+7}\left[(2x-5)\cos(x^2-1)-2x\sin(x^2-1)\right]$$

2) Find df for $f(x)=x^2+3x$ and evaluate it for (i) $x=2$ and $dx=0.1$ (ii) $x=3$ and $dx=0.02$

$$f(x)=x^2+3x$$

$$\frac{df}{dx}=2x+3$$

$$df=(2x+3)dx$$

$$=(2(2)+3)(0.1)$$

$$=0.7$$

(ii) $x=3$ and $dx=0.02$

$$df=(2x+3)dx$$
$$df=(2(3)+3)\,0.02$$
$$=0.18$$

3) Find Δf and df for the function f for the indicated values of x, Δx and compare

(i) $f(x)=x^3-2x^2$; $x=2$ $\Delta x=dx=0.5$

$$f'(x)=3x^2-2(2x)$$
$$\frac{df}{dx}=3x^2-4x$$
$$df=(3(2)^2-4(2))\,0.5$$
$$=(12-8)\,0.5$$
$$df=2.0$$

$$\Delta f=f(x+\Delta x)-f(x)$$
$$=f(2+0.5)-f(2)$$
$$=f(5/2)-f(2)$$
$$=\frac{125}{8}-2\left(\frac{25}{4}\right)-(8-8)$$
$$=\frac{25}{8}=3.125$$
$$\Delta f=3.125$$

(ii) $f(x)=x^2+2x+3$, $x=-0.5$, $\Delta x=dx=0.1$

$$\frac{df}{dx}=2x+2$$
$$df=(2x+2)dx$$
$$=(2(-0.5)+2)\,0.1$$
$$=1(0.1)$$
$$df=0.1$$

$$\Delta f=f(x+\Delta x)-f(x)$$
$$=f(-0.4)-f(-0.5)$$
$$=(0.16-0.8+3)-(0.25-1+3)$$
$$=3.16-0.8-2.25$$
$$\Delta f=0.11$$

4) Assuming $\log_{10} e = 0.4343$. Find an approximate value of $\log_{10} 1003$

Let $y = f(x) = \log_{10} x$

$x = 1000$

$\Delta x = 3$

$x + \Delta x = 1003$

$\Delta x \simeq dx = 3$

$f(1000) = \log_{10} 1000 = \log_{10} 10^3 = 3$

$y = \log_{10} x$

$\frac{dy}{dx} = \frac{1}{x} \log_{10} e$

$dy = \frac{0.4343}{x} dx$

$= \frac{0.4343}{1000} \times 3$

$dy = 0.001303$

$f(x+\Delta x) \simeq y + dy$

$f(1000+3) \simeq 3 + 0.001303$

$\log_{10} 1003 \simeq 3.001303$

5) The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

(i) Approximately, how much did the tree's diameter grew?

(ii) what is the percentage increase in area of the tree's cross-section?

diameter $2r = 30$

$r = 15$

Circumference $= 30\pi$

Increase in circumference $= 6$

$2\pi r_2 - 2\pi r_1 = 6$

$r_2 - r_1 = \frac{6}{2\pi}$

$\Delta r = dr = \frac{3}{\pi}$

Area $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r dr$

 $dA = 2\pi(15)\frac{3}{\pi}$

$= 90$

$\%$ of Increasing $= \frac{\text{Increasing area}}{\text{actual area}} \times 100$

$= \frac{90}{\pi \times 15 \times 15} \times 100$

$= \frac{40}{\pi}\ \%$

6) An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5mm and radius to the outside of the shell is 5.3 mm, find the volume of shell approximately.

$r = 5$

$\Delta r = dr = 5.3 - 5$

$dr = 0.3$

volume of sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3} 3\pi r^2$

$dV = 4\pi r^2 dr$

volume of shell $= V(5.3) - V(5)$

$\simeq dV$

$= 4\pi(5\times 5)\,0.3$

volume of shell $\simeq 30\pi\ mm^3$

7) Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2mm to 2.1mm, how much is cross-sectional area increased approximately?

$r = 2mm$

$\Delta r \simeq dr = 2.1 - 2$

$= 0.1$

Area of circle $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r\,dr$

Area Increased $= A(2.1) - A(2)$

$\simeq dA$

$= 2\pi(2)(0.1)$

$= 0.4\pi$

Area Increased $\simeq 0.4\pi\ mm^2$

8) In a newly developed city, it is exstimated that the voting population (In thousands) will increased according to $V(t) = 30 + 12t^2 - t^3$, $0 \le t \le 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

$V(t) = 30 + 12t^2 - t^3$

$t = 4$

$\Delta t \simeq dt = 4\frac{1}{6} - 4$

$dt = \frac{1}{6}$

$\frac{dv}{dt} = 24t - 3t^2$

$dv = (24t - 3t^2)dt$

chang in $V = V(4\frac{1}{6}) - V(4)$

$\simeq dv$

$= (24(4) - 3(4)^2)\frac{1}{6}$

$= (96 - 48)\frac{1}{6}$

$= 8$ thousand

Change in voters $\simeq 8000$

9) The relation between the number of words y a person learns in x hours is $y = 52\sqrt{x}$, $0 \le x \le 9$. What is the approximate number of words learned when x changes from

(i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

i) $y = 52\sqrt{x}$

$x = 1$ $\quad dx = \Delta x = 1.1 - 1 = 0.1$

$\frac{dy}{dx} = \frac{\cancel{52}^{26}}{} \frac{1}{\cancel{2}\sqrt{x}}$

$dy = 26 \frac{1}{\sqrt{x}} dx$

change in word learn $\simeq dy$

$= 26 \frac{1}{\sqrt{1}}(0.1)$

$= 2.6$

$\simeq 3$ words.

(ii) $x=4$ $\quad dx=\Delta x=4.1-4$

$=0.1$

$$dy=26\frac{1}{\sqrt{x}}dx$$

Change in words learn $\simeq 26\frac{1}{\sqrt{4}}(0.1)=1.3$

$\simeq 1$ word.

10) A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate % change in the area.

$$r=10.5$$

$$\Delta r=dr=10.75-10.5$$

$$dr=0.25$$

Area of circular plate $A=\pi r^2$

$dA=2\pi r\,dr$

(i) Approximate change in Area $\simeq dA$

$\simeq 2\pi(10.5)(0.25)$

$=5.25\pi\ cm^2$

Approximate change in Area $=5.25\pi\ cm^2$

(ii) approximate % of change $=\frac{dA}{A}\times 100$

$$=\frac{5.25\cancel{\pi}}{\cancel{\pi}\times 10.5\times 10.5}\times 100$$

$$=\frac{525}{10.5\times 10.5}$$

$$=4.76\%$$

11) A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10 cm. Use the differentials to find approximately how many cubic cm of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

edge of cube $a=10$ cm

thickness $\Delta a=da=0.2$ cm

volume of cube $V = a^3$

$$\frac{dV}{da} = 3a^2$$

$$dV = 3a^2 da$$

volume of paint $\simeq dV$

$$\simeq 3(10)^2 \times 0.2$$

$$= 60\,cm^3$$

volume of paint $\simeq 60\,cm^3$

Exact volume of paint $= V(10.2) - V(10)$

$$= (10.2)^3 - 10^3$$

$$= 1061.208 - 1000$$

$$= 61.208\,cm^3.$$

—x—

Limits and continuity of functions of Two variables.

Limit of a function

$F : A \to R$ has a limit L at (u,v)

If for every neighbourhood $(L-\epsilon, L+\epsilon)$, $\epsilon > 0$ of L, there exist a δ-neighbourhood $B_\delta((u,v)) \subset A$ of (u,v) such that

$$(x,y) \in B_\delta((u,v)) - \{(u,v)\}, \ \delta > 0.$$

$$\Rightarrow f(x) \in (L-\epsilon, L+\epsilon)$$

we denote $\lim\limits_{(x,y)\to(u,v)} F(x,y) = L$ if such a limit exists.

continuity

$F : A \to R$ is continuous at (u,v)

If 1) F is defined at (u,v)

2) $\lim\limits_{(x,y)\to(u,v)} F(x,y) = L$ exists

3) $L = F(u,v)$.

Exercise 8.3

1) Evaluate $\lim_{(x,y)\to(1,2)} g(x,y)$, If the limit exists, where

$$g(x,y) = \frac{3x^2-xy}{x^2+y^2+3}$$

$$\lim_{(x,y)\to(1,2)} g(x,y) = \lim_{(x,y)\to(1,2)} \frac{3x^2-xy}{x^2+y^2+3}$$

$$= \frac{3(1)^2-1(2)}{1^2+2^2+3}$$

$$= \frac{3-2}{1+4+3}$$

$$= 1/8$$

2) Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3+y^2}{x+y+2}\right)$, If the limit exists

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^2}{x+y+2}\right) = \cos\left(\frac{0+0}{0+0+2}\right)$$

$$= \cos 0$$

$$= 1$$

3) Let $f(x,y) = \frac{y^2-xy}{\sqrt{x}-\sqrt{y}}$ for $(x,y) \neq (0,0)$, show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^2-xy}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y\overset{-1}{\cancel{(y-x)}}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$$

$$= \lim_{(x,y)\to(0,0)} -y(\sqrt{x}+\sqrt{y})$$

$$= 0(0+0)$$

$$= 0.$$

4). Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$, If the limit exists

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right) = \cos\left(\lim_{(x,y)\to(0,0)} e^x \frac{\sin y}{y}\right)$$

[∵ By composite function Theorem]

$$= \cos\left(\lim_{(x,y)\to(0,0)} e^x \cdot \lim_{(x,y)\to(0,0)} \frac{\sin y}{y}\right)$$

$$= \cos\left(e^0 \cdot 1\right)$$

$$= \cos 1$$

5) Let $g(x,y) = \frac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0)=0$

(i) show that $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ along every line $y=mx$, $m\in R$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 mx}{x^4+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^2} mx}{\cancel{x^2}(x^2+m^2)}$$

$$= \frac{0}{0+m^2}$$

$$= 0$$

(ii) show that $\lim_{(x,y)\to(0,0)} g(x,y) = \frac{k}{1+k^2}$ along every parabola $y=kx^2$, $k\in R-\{0\}$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2kx^2}{x^4+k^2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^4}k}{\cancel{x^4}(1+k^2)}$$

$$= \frac{k}{1+k^2}$$

6) show that $f(x,y) = \frac{x^2-y^2}{y^2+1}$ is continuous at every $(x,y)\in R^2$

Let $(a,b)\in R^2$ be an arbitrary point.

(i) $f(a,b) = \frac{a^2-b^2}{b^2+1}$ is defined. for $\forall (a,b) \in R^2$

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} \frac{x^2-y^2}{y^2+1}$

$$= \frac{a^2-b^2}{b^2+1} = L$$

i.e Limit exist at $(a,b) \in R^2$

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L = f(a,b) = \frac{a^2-b^2}{b^2+1}$

$\therefore f$ is continuous at every point on R^2.

7) Let $g(x,y) = \frac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0,0) = 1$

show that g is continuous at $(0,0)$.

$$|g(x,y) - g(0,0)| = \left| \frac{e^y \sin x}{x} - 1 \right| = \left| \frac{e^y \sin x - x}{x} \right|$$

(i) $g(0,0) = 1$ g is defined at $(0,0)$

(ii) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \lim\limits_{(x,y)\to(0,0)} e^y \frac{\sin x}{x}$

$$= \lim_{(x,y)\to(0,0)} e^y \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$$

$$= 1 \cdot 1$$

$$= 1 \qquad \boxed{L = 1}$$

limit exist at $(0,0)$

(iii) $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 1 = g(0,0)$

$\therefore g$ is continuous at $(0,0)$

Partial Derivatives

Clairaut's Theorem

$F : A \to R$ IF F_{xy} and F_{yx} exist in A. are continuous in A then $F_{xy} = F_{yx}$ in A. where $A \subset R^2$

Laplace's Equation

Let $A \subset R^2$, u is a function $u: A \to R^2$ is said to be harmonic in A. If it satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\forall (x,y) \in A$

This equation is called Laplace's Equation.

Exercise 8.4

1). Find the partial derivatives of the following functions at the indicated points.

(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$

$$\frac{\partial f}{\partial x} = 6x - 2y + 5$$

$$\frac{\partial f}{\partial x}(2,-5) = 6(2) - 2(-5) + 5 = 12 + 10 + 5 = 27$$

$$\frac{\partial f}{\partial y} = 0 - 2x + 2y$$

$$\frac{\partial f}{\partial y}(2,-5) = -2(2) + 2(-5) = -4 - 10 = -14$$

(ii) $g(x,y) = 3x^2 + y^2 + 5x + 2$, $(1,-2)$

$$\frac{\partial g}{\partial x} = 6x + 5$$

$$\frac{\partial g}{\partial x}(1,-2) = 6(1) + 5 = 11$$

$$\frac{\partial g}{\partial y} = 0 + 2y \qquad \frac{\partial g}{\partial y}(1,-2) = 2(-2) = -4$$

(iii) $h(x,y,z) = x \sin(xy) + z^2 x$, $(2, \pi/4, 1)$

$$\frac{\partial h}{\partial x} = x \cos(xy)\, y + \sin xy\, (1) + z^2$$

$$\frac{\partial h}{\partial x}(2, \pi/4, 1) = 2 \cos 2\pi/4_2 \; \pi/4 + \sin 2\pi/4_2 + 1^2$$

$$= 0 + 1 + 1$$

$$= 2$$

$$\frac{\partial h}{\partial y} = x \cos xy\, (x) + 0$$

$$\frac{\partial h}{\partial y}(2, \pi/4, 1) = 2 \cos 2\pi/4_2 (2) = 0$$

$$\frac{\partial h}{\partial z} = 0 + 2zx$$

$$\frac{\partial h}{\partial z}(2, \pi/4, 1) = 2(2)(1) = 4.$$

(iv) $G(x,y) = e^{x+3y}\log(x^2+y^2)$, $(-1,1)$

$$\frac{\partial G}{\partial x} = e^{x+3y}\left(\log x^2+y^2\right)' + \log(x^2+y^2)\left(e^{x+3y}\right)'$$

$$= e^{x+3y}\frac{1}{x^2+y^2}(2x) + \log(x^2+y^2)\, e^{x+3y}(1)$$

$$= e^{x+3y}\left(\frac{2x}{x^2+y^2} + \log(x^2+y^2)\right)$$

$$\frac{\partial G}{\partial x}(-1,1) = e^{-1+3}\left(\frac{-2}{1+1} + \log(1+1)\right)$$

$$= e^2(\log 2 - 1$$

$$\frac{\partial G}{\partial y} = e^{x+3y}\left(\frac{2y}{x^2+y^2}\right) + \log(x^2+y^2)\, e^{x+3y}(3)$$

$$= e^{x+3y}\left(\frac{2y}{x^2+y^2} + 3\log(x^2+y^2)\right)$$

$$\frac{\partial G}{\partial y}(-1,1) = e^{-1+3}\left(\frac{2}{1+1} + 3\log(1+1)\right)$$

$$= e^2\left(\frac{2}{2} + 3\log 2\right)$$

$$= e^2(1+\log 2^3)$$

$$= e^2(1+\log 8)$$

2) For each of the following functions find the f_x, f_y and show that $f_{xy} = f_{yx}$

(i) $f(x,y) = \dfrac{3x}{y+\sin x}$

$$f_x = \frac{(y+\sin x)(3) - 3x(\cos x)}{(y+\sin x)^2}$$

$$f_x = \frac{3y + 3\sin x - 3x\cos x}{(y+\sin x)^2}$$

$$f_y = \frac{(y+\sin x)(0) - 3x(1)}{(y+\sin x)^2}$$

$$f_y = \frac{-3x}{(y+\sin x)^2}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{-3x}{(y+\sin x)^2}\right)$$

$$= \frac{(y+\sin x)^2(-3) + 3x(2(y+\sin x)(\cos x)}{(y+\sin x)^4}$$

$$= \frac{\cancel{(y+\sin x)}\left(-3(y+\sin x) + 6x\cos x\right)}{(y+\sin x)^{\cancel{4}\,3}}$$

$$f_{xy} = \frac{3(2x\cos x - y - \sin x)}{(y+\sin x)^3} \text{ --- } (1)$$

$$f_{yx} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{3y+3\sin x-3x\cos x}{(y+\sin x)^2}\right)$$

$$= \frac{(y+\sin x)^2(3+0-0) - (3y+3\sin x-3x\cos x)(2(y+\sin x))}{(y+\sin x)^4}$$

$$= \frac{\cancel{(y+\sin x)}\left(3(y+\sin x - 2\times 3(y+\sin x - x\cos x)\right)}{(y+\sin x)^{\cancel{4}\,3}}$$

$$= \frac{3(y+\sin x - 2y - 2\sin x + 2x\cos x)}{(y+\sin x)^3}$$

$$f_{yx} = \frac{3(2x\cos x - y - \sin x)}{(y+\sin x)^3} \text{ --- } (2)$$

From (1) & (2)

$$f_{xy} = f_{yx}$$

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(ii) $f(x,y) = \tan^{-1}\frac{x}{y}$

$$f_x = \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y} = \frac{y^2}{x^2+y^2}\cdot\frac{1}{y}$$

$$f_x = \frac{y}{x^2+y^2}$$

$$f_y = \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{-x}{y^2} = \frac{y^2}{x^2+y^2}\left(\frac{-x}{y^2}\right)$$

$$f_y = \frac{-x}{x^2+y^2}$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{(x^2+y^2)(-x)' - (-x)(x^2+y^2)'}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+x(2x)}{(x^2+y^2)^2}$$

$$f_{xy} = \frac{x^2-y^2}{(x^2+y^2)^2} \text{ --- } (1)$$

$$f_{yx} = \frac{\partial}{\partial y}(f_x) = \frac{(x^2+y^2)(y)' - y(x^2+y^2)'}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2}$$

$$f_{yx} = \frac{x^2-y^2}{(x^2+y^2)^2} \text{ —— } (2)$$

From (1) & (2) $f_{xy} = f_{yx}$.

(iii) $f(x,y) = 3\cos(x^2-3xy)$

$$f_x = -\sin(x^2-3xy)\,(2x-3y)$$

$$f_y = -\sin(x^2-3xy)\,(0-3x)$$

$$= 3x\sin(x^2-3xy)$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y)$$

$$= 3x(\cos(x^2-3xy)\,(2x-3y) + \sin(x^2-3xy)\,3$$

$$f_{xy} = 3\left[(2x^2-3xy)\cos(x^2-3xy) + \sin(x^2-3xy)\right] \text{ —— } (1)$$

$$f_{yx} = \frac{\partial}{\partial y}(f_x)$$

$$= -(2x-3y)\cos(x^2-3xy)(0-3x) + \sin(x^2-3xy)\,(-(0-3))$$

$$f_{yx} = 3\left[(2x^2-3xy)\cos(x^2-3xy) + \sin(x^2-3xy)\right] \text{ —— } (2)$$

From (1) & (2)

$$f_{xy} = f_{yx}$$

3) If $U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$, find $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$.

$$U = \frac{x^2}{xy} + \frac{y^2}{xy} + 3z^2y$$

$$U = \frac{x}{y} + \frac{y}{x} + 3z^2y$$

$$\frac{\partial U}{\partial x} = \frac{1}{y} + y\left(\frac{-1}{x^2}\right) + 0 = \frac{1}{y} - \frac{y}{x^2} = \frac{x^2-y^2}{yx^2}$$

$$\frac{\partial U}{\partial y} = -\frac{x}{y^2} + \frac{1}{x} + 3z^2 = \frac{-x^2+y^2}{xy^2} + 3z^2$$

$$\frac{\partial U}{\partial z} = 0 + 0 + 3y(2z) = 6yz.$$

4) If $U(x,y,z) = \log(x^3+y^3+z^3)$, find $\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}$

$$U = \log(x^3+y^3+z^3)$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3+y^3+z^3}(3x^2)$$

$$\frac{\partial U}{\partial y} = \frac{3y^2}{x^3+y^3+z^3}, \quad \frac{\partial U}{\partial z} = \frac{3z^2}{x^3+y^3+z^3}$$

$$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = \frac{3x^2+3y^2+3z^2}{x^3+y^3+z^3}$$

$$= \frac{3(x^2+y^2+z^2)}{x^3+y^3+z^3}$$

(5) For each of the following functions find the g_{xy}, g_{xx}, g_{yy} and g_{yx}

(i) $g(x,y) = xe^y+3x^2y$

$$g_x = e^y+3y(2x) = e^y+6xy$$

$$g_y = xe^y+3x^2$$

$$g_{xy} = \frac{\partial}{\partial x}(g_y) = e^y+6x$$

$$g_{xx} = \frac{\partial}{\partial x}(g_x) = 0+6y = 6y$$

$$g_{yy} = \frac{\partial}{\partial y}(g_y) = xe^y+0 = xe^y$$

$$g_{yx} = \frac{\partial}{\partial y}(g_x) = e^y+6x$$

(ii) $g(x,y) = \log(5x+3y)$

$$g_x = \frac{1}{5x+3y}(5) \qquad g_y = \frac{1}{5x+3y}(3)$$

$$g_{xx} = 5\left(\frac{-1}{(5x+3y)^2}\right)(5) \qquad g_{yy} = 3\left(\frac{-1}{(5x+3y)^2}\right)3$$

$$g_{xx} = \frac{-25}{(5x+3y)^2} \qquad g_{yy} = \frac{-9}{(5x+3y)^2}$$

$$g_{yx} = \frac{\partial}{\partial y}(g_x) \qquad g_{xy} = \frac{\partial}{\partial x}(g_y)$$

$$= 5\left(\frac{-1}{(5x+3y)^2}\right)3 \qquad = 3\left(\frac{-1}{(5x+3y)^2}\right)5$$

$$g_{yx} = \frac{-15}{(5x+3y)^2} \qquad g_{xy} = \frac{-15}{(5x+3y)^2}$$

(iii) $g(x,y) = x^2+3xy-7y+\cos 5x$.

$g_x = 2x+3y-\sin 5x(5)$ $\qquad g_y = 3x-7$

$g_{xy} = \frac{\partial}{\partial x}(g_y) = 3(1) = 3$ $\qquad g_{yy} = 0$

$g_{xx} = \frac{\partial}{\partial x}(g_x) = 2-5\cos 5x(5)$

$= 2-25\cos 5x$

$g_{yx} = \frac{\partial}{\partial y}(g_x) = 3-0 = 3$

6). Let $w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $(x,y,z) \neq (0,0,0)$ show that $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2} = 0$

$w = (x^2+y^2+z^2)^{-1/2}$

$\frac{\partial w}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}x)$

$\frac{\partial w}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial^2 w}{\partial x^2} = -x\left(-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{3}{2}-1}(\cancel{2}x)\right) + (x^2+y^2+z^2)^{-3/2}(-1)$

$= \frac{3x^2}{(x^2+y^2+z^2)^{3/2+1}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$

$= \frac{3x^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$

$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ ——— (1)

similarly $\frac{\partial^2 w}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ ——— (2)

$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$ ——— (3)

adding (1), (2) & (3)

$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2} = \frac{\cancel{2x^2}-\cancel{y^2}-\cancel{z^2}+\cancel{2y^2}-\cancel{x^2}-\cancel{z^2}+\cancel{2z^2}-\cancel{x^2}-\cancel{y^2}}{(x^2+y^2+z^2)^{5/2}}$

$= \frac{0}{(x^2+y^2+z^2)^{5/2}}$

$= 0$

7) If $v(x,y) = e^x(x\cos y - y\sin y)$ then prove that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

$$\frac{\partial v}{\partial x} = e^x[\cos y] + (x\cos y - y\sin y)[e^x]$$

$$\frac{\partial v}{\partial x} = e^x[\cos y + x\cos y - y\sin y]$$

$$\frac{\partial^2 v}{\partial x^2} = e^x[\cos y] + (\cos y + x\cos y - y\sin y)[e^x]$$

$$= e^x[\cos y + \cos y + x\cos y - y\sin y]$$

$$\frac{\partial^2 v}{\partial x^2} = e^x[2\cos y + x\cos y - y\sin y] \quad \text{——} ①$$

$$\frac{\partial v}{\partial y} = e^x[-x\sin y - y\cos y - \sin y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x[-x\cos y + y\sin y - \cos y - \cos y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x[-2\cos y - x\cos y + y\sin y] \quad \text{——} ②$$

$① + ② \Rightarrow$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = [\cancel{2\cos y} + \cancel{x\cos y} - \cancel{y\sin y} - \cancel{2\cos y} - \cancel{x\cos y} + \cancel{y\sin y}]e^x$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$$

8). If $w(x,y) = xy + \sin(xy)$, then prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

$$\frac{\partial w}{\partial x} = y + \cos xy\,(y)$$

$$\frac{\partial w}{\partial x} = y(1+\cos xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = y(0 - \sin xy\,(x) + (1+\cos xy)(1))$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 - xy\sin xy + \cos xy \quad \text{——} ①$$

$$\frac{\partial w}{\partial y} = x + \cos xy\,(x)$$

$$\frac{\partial w}{\partial y} = x(1+\cos xy)$$

$$\frac{\partial^2 w}{\partial x \partial y} = x(0 - \sin xy\,(y) + (1+\cos xy)(1))$$

$$\frac{\partial^2 w}{\partial x \partial y} = 1 - xy\sin xy + \cos xy \quad \text{——} (2)$$

From (1) & (2) $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ proved.

9) If $V(x,y,z) = x^3+y^3+z^3+3xyz$, show that $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$

$$\frac{\partial V}{\partial z} = 0+0+3z^2+3xy$$

$$\frac{\partial V}{\partial z} = 3z^2+3xy$$

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$$\frac{\partial^2 V}{\partial y \partial z} = 3x \quad \text{——} (1)$$

$$\frac{\partial V}{\partial y} = 0+3y^2+0+3xz$$

$$\frac{\partial^2 V}{\partial z \partial y} = 3x \quad \text{——} (2)$$

From (1), (2) $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$

10) A firm produces two types of calculators each week, x number of type A, and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x+90y+0.04xy-0.05y^2$ and $C(x,y) = 8x+6y+2000$ respectively

(i) Find the profit function $P(x,y)$

(ii) Find $\frac{\partial P}{\partial x}(1200,1800)$ and $\frac{\partial P}{\partial y}(1200,1800)$ and interpret these results.

(i) Profit = Revenue − cost

$$P(x,y) = R(x,y) - C(x,y)$$

$$= 80x+90y+0.04xy-0.05y^2$$
$$-8x-6y-2000$$

$$P(x,y) = 72x+84y+0.04xy-0.05x^2-0.05y^2-2000$$

(ii) $\frac{\partial P}{\partial x} = 72+0.04y-0.05(2x$

$$\frac{\partial P}{\partial x}(1200,1800) = 72+(0.04)(1800)-0.1(1200)$$

$$= 72+72-120$$

$$= 24$$

$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.05(2y)$

$\frac{\partial P}{\partial y}(1200,1800) = 84 + 0.04(1200) - 0.1(1800)$

$= 84 + 48 - 180$

$= -48$

at $(1200,1800)$ $\frac{\partial P}{\partial x} = 24$, $\frac{\partial P}{\partial y} = -48$

$\therefore$ keeping y constant and increase x values Then we get profit Increasing.

— x —

Linear approximation and Differential of several variables

① The Linear approximation of F at (x_0,y_0) is

$$L(x,y) = F(x_0,y_0) + \frac{\partial F}{\partial x}_{(x_0,y_0)}(x-x_0) + \frac{\partial F}{\partial y}_{(x_0,y_0)}(y-y_0)$$

② The differential of F is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

where $dx = \Delta x$
$dy = \Delta y$

③ The linear approximation of F at $(x_0,y_0,z_0) \in A$ is defined to be

$$F(x,y,z) = F(x_0,y_0,z_0) + \frac{\partial F}{\partial x}_{(x_0,y_0,z_0)}(x-x_0) + \frac{\partial F}{\partial y}_{(x_0,y_0,z_0)}(y-y_0) + \frac{\partial F}{\partial z}_{(x_0,y_0,z_0)}(z-z_0)$$

④ The differential of F is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

where $dx = \Delta x$
$dy = \Delta y$
$dz = \Delta z$

Exercise 8.5

1) If $w(x,y) = x^3 - 3xy + 2y^2$, $x,y \in R$ find the linear approximation for w at $(1,-1)$

$\frac{\partial w}{\partial x} = 3x^2 - 3y$ $\frac{\partial w}{\partial y} = -3x + 4y$

$\frac{\partial w}{\partial x}(1,-1) = 3+3 = 6$ $\frac{\partial w}{\partial y}(1,-1) = -3-4 = -7$

$w(1,-1) = 1-3(1)(-1)+2(-1)^2$
$= 1+3+2$
$= 6$

Linear approximation $L(x) = f(x_0,y_0) + \frac{\partial F}{\partial x}(x-x_0) + \frac{\partial F}{\partial y}(y-y_0)$

$L(x,y) = 6 + 6(x-1) + (-7)(y+1)$
$= \cancel{6}+6x\cancel{-6}-7y-7$
$= 6x-7y-7$

② Let $z(x,y) = x^2y+3xy^4$, $x,y \in R$ Find the linear approximation for z at $(2,-1)$

$z(2,1) = 2^2(-1)+3(2)(-1)^4$
$= -4+6 = 2$

$\frac{\partial z}{\partial x} = 2xy+3y^4$ $\qquad \frac{\partial z}{\partial x}(2,-1) = 2(2)(-1)+3(-1)^4 = -4+3$
$= -1$

$\frac{\partial z}{\partial y} = x^2+12xy^3$ $\qquad \frac{\partial z}{\partial y}(2,-1) = 2^2+12(2)(-1)^3 = 4-24 = -20$

Linear approximation
$L(x,y) = z(2,-1) + \frac{\partial z}{\partial x}(2,-1)(x-2) + \frac{\partial z}{\partial y}(2,-1)(y+1)$
$= 2 + (-1)(x-2) + (-20)(y+1)$
$= 2-x+2-20y-20$
$= -x-20y+16$
$L(x,y) = -(x+20y-16)$

3) If $v(x,y) = x^2-xy+\frac{1}{4}y^2+7$, $x,y \in R$, find the differential dv

$\frac{\partial v}{\partial x} = 2x-y$ $\qquad \frac{\partial v}{\partial y} = -x+\frac{y}{2}$

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$dv = (2x-y)dx + (-x+\frac{y}{2})dy$

4) Let $w(x,y,z) = x^2-xy+3\sin z$, $x,y,z \in R$, Find the linear approximation at $(2,-1,0)$

$w(2,-1,0) = 2^2-2(-1)+3\sin 0$
$= 4+2+0$
$= 6$

$\frac{\partial w}{\partial x} = 2x - y \qquad \frac{\partial w}{\partial x}(2,-1,0) = 2(2)+1 = 5$

$\frac{\partial w}{\partial y} = 0 - x + 0 \qquad \frac{\partial w}{\partial y}(2,-1,0) = -2$

$\frac{\partial w}{\partial z} = 0 + 0 + 3\cos z \qquad \frac{\partial w}{\partial z}(2,-1,0) = 3\cos 0 = 3$

Linear approximation

$$L(x,y,z) = w(2,-1,0) + \frac{\partial w}{\partial x}(2,-1,0)(x-2) + \frac{\partial w}{\partial y}(2,-1,0)(y+1) + \frac{\partial w}{\partial z}(2,-1,0)(z-0)$$

$$= 6 + 5(x-2) - 2(y+1) + 3(z)$$

$$= 6 + 5x - 10 - 2y - 2 + 3z$$

$$L(x,y,z) = 5x - 2y + 3z - 6$$

5) Let $V(x,y,z) = xy + yz + zx$, $x,y,z \in R$ Find the differential dV.

$\frac{\partial V}{\partial x} = y + 0 + z = y + z$

$\frac{\partial V}{\partial y} = x + z + 0 = x + z$

$\frac{\partial V}{\partial z} = 0 + y + x = y + x$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$dV = (y+z)dx + (x+z)dy + (x+y)dz.$$

Function of Function Rule

① $w = f(x,y)$, x, y are function of t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

② $w = f(x,y)$ $\quad x = x(s,t) \quad y = y(s,t)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

Exercise 8.6

1) If $u(x,y) = x^2y + 3xy^4$, $x = e^t$ and $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$.

$\frac{\partial u}{\partial x} = 2xy + 3y^4 \qquad \frac{\partial u}{\partial y} = x^2 + 12xy^3$

$x = e^t \qquad y = \sin t$

$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = \cos t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (2xy+3y^4)e^t + (x^2+12xy^3)\cos t$$

$$\frac{du}{dt} = (2e^t\sin t+3\sin^4 t)e^t + (e^{2t}+12e^t\sin^3 t)\cos t$$

put $t=0$ $\quad \frac{du}{dt} = (2e^0\sin 0+3\sin^4 0)e^0 + (e^0+12e^0\sin^3 0)\cos 0$

$$= (0+0)1 + (1+0)1$$

$$= 0+1$$

$\frac{du}{dt}$ at $t=0$ is 1

$$\frac{du}{dt} = e^t\left[2e^t\sin t + 3\sin^4 t + e^t\cos t + 12\sin^3 t\cos t\right]$$

$\frac{du}{dt} = 1$ at $t=0$

② If $u(x,y,z) = xy^2z^3$, $x=\sin t$, $y=\cos t$ $z=1+e^{2t}$, find $\frac{du}{dt}$

$$u(x,y,z) = xy^2z^3$$

$$\frac{\partial u}{\partial x} = y^2z^3, \quad \frac{\partial u}{\partial y} = 2xyz^3, \quad \frac{\partial u}{\partial z} = 3xy^2z^2$$

$x=\sin t$ $\qquad y=\cos t$ $\qquad z=1+e^{2t}$

$\frac{dx}{dt}=\cos t$ $\qquad \frac{dy}{dt}=-\sin t$ $\qquad \frac{dz}{dt} = 2e^{2t}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (y^2z^3)\cos t + 2xyz^3(-\sin t) + 3xy^2z^2(2e^{2t})$$

$$\frac{du}{dt} = \cos^2 t(1+e^{2t})^3\cos t + 2\sin t\cos t(1+e^{2t})^3(-\sin t) + 3\sin t\cos^2 t(1+e^{2t})^2 2e^{2t}$$

$$\frac{du}{dt} = (1+e^{2t})^2\left[\cos^3 t(1+e^{2t}) - 2\sin^2 t\cos t(1+e^{2t}) + 6e^{2t}\sin t\cos^2 t\right]$$

$$= (1+e^{2t})^2\left[\cos^3 t(1+e^{2t}) - \sin t\sin 2t(1+e^{2t}) + 6e^{2t}\sin t\cos^2 t\right]$$

3) If $w(x,y,z)=x^2+y^2+z^2$, $x=e^t$, $y=e^t\sin t$ and $z=e^t\cos t$

find $\frac{dw}{dt}$

$$w=x^2+y^2+z^2$$

$$\frac{\partial w}{\partial x}=2x \quad \frac{\partial w}{\partial y}=2y \quad \frac{\partial w}{\partial z}=2z$$

$$x=e^t \quad y=e^t\sin t \quad z=e^t\cos t$$

$$\frac{dx}{dt}=e^t, \quad \frac{dy}{dt}=e^t\cos t+\sin t\, e^t \quad \frac{dz}{dt}=e^t(-\sin t)+\cos t\, e^t$$

$$=e^t(\cos t+\sin t) \qquad =e^t(\cos t-\sin t)$$

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$=2x e^t+2y e^t(\cos t+\sin t)+2z\, e^t(\cos t-\sin t)$$

$$=2e^t\left[x+y(\cos t+\sin t)+z(\cos t-\sin t)\right]$$

$$\frac{dw}{dt}=2e^t\left[e^t+e^t\sin t(\cos t+\sin t)+e^t\cos t(\cos t-\sin t)\right]$$

$$=2e^te^t\left[1+\cancel{\sin t\cos t}+\sin^2 t+\cos^2 t-\cancel{\sin t\cos t}\right]$$

$$=2e^{2t}[1+1]$$

$$\frac{dw}{dt}=4e^{2t}$$

4) Let $U(x,y,z)=xyz$, $x=e^{-t}$, $y=e^{-t}\cos t$, $z=\sin t$, $t\in R$

Find $\frac{dU}{dt}$

$$U=xyz$$

$$\frac{\partial U}{\partial x}=yz, \quad \frac{\partial U}{\partial y}=xz, \quad \frac{\partial U}{\partial z}=xy$$

$$x=e^{-t} \quad y=e^{-t}\cos t \quad z=\sin t$$

$$\frac{dx}{dt}=-e^{-t} \quad \frac{dy}{dt}=e^{-t}(-\sin t)+e^{-t}(-1)\cos t \quad \frac{dz}{dt}=\cos t$$

$$=-e^{-t}(\sin t+\cos t)$$

$$\frac{dU}{dt}=\frac{\partial U}{\partial x}\frac{dx}{dt}+\frac{\partial U}{\partial y}\frac{dy}{dt}+\frac{\partial U}{\partial z}\frac{dz}{dt}$$

$$=yz(-e^{-t})+xz(-e^{-t}(\sin t+\cos t))+xy(\cos t)$$

$$=e^{-t}\cos t\sin t(-e^{-t})+e^{-t}\sin t(-e^{-t}(\sin t+\cos t))$$
$$+e^{-t}e^{-t}\cos t\cos t$$

$$= e^{-2t}\left[-\sin t\cos t - \sin t(\sin t+\cos t) + \cos^2 t\right]$$

$$= -e^{-2t}\left[\sin t\cos t + \sin^2 t + \sin t\cos t - \cos^2 t\right]$$

$$= -e^{-2t}\left[2\sin t\cos t - (\cos^2 t - \sin^2 t)\right]$$

$$\frac{dw}{dt} = -e^{-2t}\left[\sin 2t - \cos 2t\right]$$

5) If $w(x,y) = 6x^3 - 3xy + 2y^2$, $x = e^s$, $y = \cos s$ $s \in R$

find $\frac{dw}{ds}$, and evaluate at $s=0$

$$w = 6x^3 - 3xy + 2y^2$$

$$\frac{\partial w}{\partial x} = 18x^2 - 3y \qquad \frac{\partial w}{\partial y} = -3x + 4y$$

$$x = e^s \qquad y = \cos s$$

$$\frac{dx}{ds} = e^s \qquad \frac{dy}{ds} = -\sin s$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$$

$$= (18x^2 - 3y)e^s + (-3x+4y)(-\sin s)$$

$$= (18e^{2s} - 3\cos s)e^s + (-3e^s + 4\cos s)(-\sin s)$$

$$\frac{dw}{ds} = 18e^{3s} - 3e^s\cos s - 4\sin s\cos s + 3e^s\sin s$$

put $s=0$ $\frac{dw}{ds} = 18e^0 - 3e^0\cos 0 - 4\sin 0\cos 0 + 3e^0\sin 0$

$$= 18 - 3 - 0 + 0$$

$$\frac{dw}{ds} = 15$$

6) If $z(x,y) = x\tan^{-1}(xy)$, $x = t^2$, $y = se^t$, $s,t \in R$

Find $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$ at $s=t=1$

$$z = x\tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = x\frac{1}{1+x^2y^2}y + \tan^{-1}(xy)(1) \qquad \frac{\partial z}{\partial y} = x\frac{1}{1+x^2y^2}x$$

$$x = t^2 \qquad y = se^t$$

$$\frac{\partial x}{\partial s} = 0, \quad \frac{\partial x}{\partial t} = 2t, \qquad \frac{\partial y}{\partial s} = e^t \qquad \frac{\partial y}{\partial t} = se^t$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \left(\frac{xy}{1+x^2y^2} + \tan^{-1}xy\right) 0 + \left(\frac{x^2}{1+x^2y^2}\right) e^t$$

$$= \frac{x^2 e^t}{1+x^2y^2}$$

$$\frac{\partial z}{\partial s} = \frac{t^4 e^t}{1+t^4 e^{2t} s^2}$$

put $s=t=1$ $\quad \frac{\partial z}{\partial s} = \frac{1\, e^1}{1+1e^2(1)} = \frac{e}{1+e^2}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(\frac{xy}{1+x^2y^2} + \tan^{-1}xy\right) 2t + \left(\frac{x^2}{1+x^2y^2}\right) se^t$$

$$= \left(\frac{t^2 se^t}{1+t^4 s^2 e^{2t}} + \tan^{-1} t^2 se^t\right) 2t + \left(\frac{t^4}{1+t^4 s^2 e^{2t}}\right) se^t$$

put $s=t=1$

$$\frac{\partial z}{\partial t} = \left(\frac{1.e}{1+e^2} + \tan^{-1}e\right) 2 + \left(\frac{1}{1+e^2}\right) e$$

$$= \frac{2e}{1+e^2} + 2\tan^{-1}e + \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1+e^2} + 2\tan^{-1}e.$$

7) Let $u(x,y) = e^x \sin y$ where $x = st^2$, $y = s^2 t$, $s,t \in R$

Find $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}$ and evaluate them at $s=t=1$

$$u = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \sin y \qquad \frac{\partial u}{\partial y} = e^x \cos y$$

$$x = st^2 \qquad y = s^2 t$$

$$\frac{\partial x}{\partial s} = t^2 \quad \frac{\partial x}{\partial t} = 2st \qquad \frac{\partial y}{\partial s} = 2st \quad \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= e^x \sin y\,(t^2) + e^x \cos y\,(2st)$$

$$= e^x (t^2 \sin y + 2st \cos y)$$

$$\frac{\partial u}{\partial s} = e^{st^2}(t^2 \sin s^2 t + 2st \cos s^2 t)$$

$$= te^{st^2}(t \sin s^2 t + 2s \cos s^2 t)$$

put $s=t=1$

$$\frac{\partial u}{\partial s} = 1e^{1(1)}\left[\sin 1 + 2(1)\cos 1\right]$$

$$\frac{\partial u}{\partial s} = e(\sin 1 + \cos 1)$$

(ii) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

$$= e^x \sin y\,(2st) + e^x \cos y\,(s^2)$$

$$= se^x(2t\sin y + s\cos y)$$

$$\frac{\partial u}{\partial t} = se^{st^2}(2t\sin s^2t + s\cos s^2t)$$

put $s=t=1$ $\quad \frac{\partial u}{\partial t} = e^1(2\sin 1 + \cos 1)$

8) Let $z(x,y) = x^3 - 3x^2y^3$, where $x = se^t$, $y = se^{-t}$ $s,t\in\mathbb{R}$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$$z = x^3 - 3x^2y^3$$

$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3 \qquad \frac{\partial z}{\partial y} = 0 - 9x^2y^2$

$x = se^t \qquad y = se^{-t}$

$\frac{\partial x}{\partial s} = e^t \quad \frac{\partial x}{\partial t} = se^t \qquad \frac{\partial y}{\partial s} = e^{-t} \quad \frac{\partial y}{\partial t} = -se^{-t}$

i) $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$$= (3x^2 - 6xy^3)e^t + (-9x^2y^2)e^{-t}$$

$$= (3s^2e^{2t} - 6se^t s^3e^{-3t})e^t + (-9s^2e^{2t}s^2e^{-2t})e^{-t}$$

$$= 3s^2e^t\left[e^{2t} - 2e^{-2t}s^2 - 3s^2e^{-2t}\right]$$

$$\frac{\partial z}{\partial s} = 3s^2e^t\left[e^{2t} - 5e^{-2t}s^2\right]$$

(ii) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$$= (3x^2 - 6xy^3)se^t + (-9x^2y^2)(-se^{-t})$$

$$= (3s^2e^{2t} - 6se^ts^3e^{-3t})se^t + 9s^2\cancel{e^{2t}}s^2\cancel{e^{-2t}}\,se^{-t}$$

$$= (3s^2e^{2t} - 6s^4e^te^{-3t})se^t + 9s^5e^{-t}$$

$$= 3s^3e^{3t} - 6s^5e^{-t} + 9s^5s^{-t}$$

$= 3s^3e^{3t} + 3s^5e^{-t}$

9) $w(x,y,z) = xy+yz+zx$ $\quad x=u-v,\ y=uv,\ z=u+v,\ u,v\in R$

Find $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$, and evaluate them at $(\frac{1}{2},1)$

$$w = xy+yz+zx$$

$\frac{\partial w}{\partial x} = y+z \qquad \frac{\partial w}{\partial y} = x+z \qquad \frac{\partial w}{\partial z} = y+x$

$x = u-v \qquad y = uv \qquad z = u+v$

$\frac{\partial x}{\partial u} = 1,\ \frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial u} = v,\ \frac{\partial y}{\partial v} = u \qquad \frac{\partial z}{\partial u} = 1 \quad \frac{\partial z}{\partial v} = 1$

(i) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

$= (y+z)(1) + (x+z)v + (y+x)1$

$= uv+u+\cancel{v} + (u-\cancel{v}+u+\cancel{v})v + (uv+u-\cancel{v})$

$= 2uv+2u+2uv$

$= 4uv+2u$

$\frac{\partial w}{\partial u} = 2u(2v+1)$

$\frac{\partial w}{\partial u}(\frac{1}{2},1) = \cancel{2}\frac{1}{\cancel{2}}(2+1) = 3$

(ii) $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$

$= (y+z)(-1) + (x+z)u + (y+x)(1)$

$= -\cancel{y}-z + (x+z)u + \cancel{y}+x$

$= -\cancel{u}-v + (u-\cancel{v}+u+\cancel{v})u + \cancel{u}-v$

$= -2v+2u^2$

$\frac{\partial w}{\partial v} = 2(u^2-v)$

$\frac{\partial w}{\partial v}(\frac{1}{2},1) = 2(\frac{1}{4}-1)$

$= \frac{1}{2}-2$

$= -\frac{3}{2}$

Homogeneous Functions and Euler's Theorem

① F is a homogeneous function

If (i) $F(\lambda x, \lambda y) = \lambda^p F(x,y)$

or $\forall \lambda \in R$

(ii) $F(\lambda x, \lambda y, \lambda z) = \lambda^p F(x,y,z)$

p is called degree of F

② Euler's Theorem

F is homogeneous on A with degree p then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Exercise 8.7

1). In each of the following cases the function homogeneous or not. If it is so, find the degree.

(i) $f(x,y) = x^2y + 6x^3 + 7$

$f(\lambda x, \lambda y) = (\lambda x)^2 \lambda y + 6(\lambda x)^3 + 7$

$= \lambda^3 x^2 y + \lambda^3 6x^3 + 7$

we cannot take λ^3 out side as common

$\therefore$ F is not a homogeneous function.

(ii) $h(x,y) = \dfrac{6x^2y^3 - \pi y^3 + 9x^4y}{2020x^2 + 2019y^2}$

$$h(\lambda x, \lambda y) = \frac{6\lambda^2 x^2 \lambda^3 y^3 - \pi \lambda^5 y^5 - 9\lambda^4 x^4 \lambda y}{2020\lambda^2 x^2 + 2019\lambda^2 y^2}$$

$$= \frac{\lambda^5(6x^2y^3 - \pi y^5 + 9x^4y)}{\lambda^2(2020x^2 + 2019y^2)}$$

$h(\lambda x, \lambda y) = \lambda^3 h(x,y)$

$h(x,y)$ is a homogeneous function of degree $= 3$

(iii) $g(x,y,z) = \dfrac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$

$$g(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{3\lambda^2 x^2 + 5\lambda^2 y^2 + \lambda^2 z^2}}{4\lambda x + 7\lambda y}$$

$$= \frac{\cancel{\lambda}\sqrt{3x^2 + 5y^2 + z^2}}{\cancel{\lambda}(4x + 7y)}$$

$g(\lambda x,\lambda y,\lambda z)=\lambda^0 g(x,y,z)$

g is a homogeneous function degree=0.

(iv) $u(x,y,z)=xy+\sin\left(\frac{y^2-2z^2}{xy}\right)$

$$u(\lambda x,\lambda y,\lambda z)=\lambda^2 xy+\sin\left(\frac{\lambda^2y^2-2\lambda^2z^2}{\lambda x\lambda y}\right)$$

$$=\lambda^2xy+\sin\frac{\lambda^2(y^2-2z^2)}{\lambda^2xy}$$

$$=\lambda^2xy+\sin\left(\frac{y^2-2z^2}{xy}\right)$$

we cannot take λ^2 outside the function as common.

$\therefore u(x,y,z)$ is not homogeneous.

2) prove that $f(x,y)=x^3-2x^2y+3xy^2+y^3$ is homogeneous what is the degree? verify Euler's Theorem for f.

$$f(\lambda x,\lambda y)=\lambda^3x^3-2\lambda^2x^2\lambda y+3\lambda x\lambda^2y^2+\lambda^3y^3$$

$$=\lambda^3(x^3-2x^2y+3xy^2+y^3)$$

$$f(\lambda x,\lambda y)=\lambda^3 f(x,y)$$

f is a homogeneous function of degree 3

By Euler's Theorem $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f$

<u>verification</u>:

$$\frac{\partial f}{\partial x}=3x^2-4xy+3y^2$$

$$x\frac{\partial f}{\partial x}=3x^3-4x^2y+3xy^2 \quad \text{---} ①$$

$$\frac{\partial f}{\partial y}=0-2x^2+6xy+3y^2$$

$$y\frac{\partial f}{\partial y}=-2x^2y+6xy^2+3y^3 \quad \text{---} ②$$

$$①+② \Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3x^3-4x^2y+3xy^2-2x^2y+6xy^2+3y^3$$

$$=3x^3-6x^2y+9xy^2+3y^3$$

$$=3(x^3-2x^2y+3xy^2+y^3)$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f$$

verified.

3) prove that $g(x,y)=x\log\frac{y}{x}$ is homogeneous, what is the degree? verify Euler's Theorem for g.

$$g(x,y)=x\log\left(\frac{y}{x}\right)$$

$$g(\lambda x,\lambda y)=\lambda x\log\left(\frac{\cancel{\lambda} y}{\cancel{\lambda} x}\right)$$

$$=\lambda\left(x\log\frac{y}{x}\right)$$

$$g(\lambda x,\lambda y)=\lambda^1 g(x,y)$$

g is a homogeneous function of degree $n=1$

∴ By Euler's Theorem

$$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=1g$$

verification

$$\frac{\partial g}{\partial x}=x\left(\log\left(\frac{y}{x}\right)\right)'+\log\left(\frac{y}{x}\right)(x)'$$

$$=x\,\frac{1}{y/x}\left(\frac{-y}{x^2}\right)+\log\frac{y}{x}\,(1)$$

$$=\frac{x^2}{y}\left(\frac{-y}{x^2}\right)+\log\frac{y}{x}$$

$$\frac{\partial g}{\partial x}=-1+\log\frac{y}{x}$$

$$x\frac{\partial g}{\partial x}=-x+x\log\frac{y}{x} \quad \text{--- (1)}$$

$$\frac{\partial g}{\partial y}=x\,\frac{1}{y/x}\,(1/x)$$

$$=\frac{x^2}{y}(1/x)=\frac{x}{y}$$

$$y\frac{\partial g}{\partial y}=x \quad \text{--- (2)}$$

$(1)+(2) \Rightarrow$ $x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=\cancel{-x}+x\log\frac{y}{x}\cancel{+x}$

$$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=1g$$

Euler's Theorem verified.

4) If $u(x,y)=\frac{x^2+y^2}{\sqrt{x+y}}$ prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}u$.

$$u(\lambda x,\lambda y)=\frac{\lambda^2x^2+\lambda^2y^2}{\sqrt{\lambda x+\lambda y}}$$

$$=\frac{\lambda^2(x^2+y^2)}{\lambda^{1/2}\sqrt{x+y}}$$

$$u(\lambda x,\lambda y)=\lambda^{3/2}u(x,y)$$

u is a homogeneous function of degree $n=3/2$

$\therefore$ By Euler's Theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}u$$

5) If $v(x,y)=\log\left(\frac{x^2+y^2}{x+y}\right)$, Prove that $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1$

M-I v is not a homogeneous function

$$v=\log\left(\frac{x^2+y^2}{x+y}\right)$$

taking antilog

$$e^v=\frac{x^2+y^2}{x+y}=u \text{ (say)}$$

$$u(\lambda x,\lambda y)=\frac{\lambda^{\cancel{2}}(x^2+y^2)}{\cancel{\lambda}(x+y)}$$

$$u(\lambda x,\lambda y)=\lambda^1 u(x,y)$$

u is a homogeneous function of degree $n=1$

$\therefore$ By Euler's Theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u$$

$$x\frac{\partial}{\partial x}(e^v)+y\frac{\partial}{\partial y}(e^v)=1e^v$$

$$xe^v\frac{\partial v}{\partial x}+ye^v\frac{\partial v}{\partial y}=e^v$$

divide by e^v $\quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1$

M-II $\quad v=\log(x^2+y^2)-\log(x+y)$

$$\frac{\partial v}{\partial x}=\frac{1}{x^2+y^2}2x-\frac{1}{x+y}\quad (1)$$

$$x\frac{\partial v}{\partial x}=\frac{2x^2}{x^2+y^2}-\frac{x}{x+y}$$

similarly $\quad y\frac{\partial v}{\partial y}=\frac{2y^2}{x^2+y^2}-\frac{y}{x+y}$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{2x^2+2y^2}{x^2+y^2}-\left(\frac{x+y}{x+y}\right)$$

$$=2\left(\frac{x^2+y^2}{x^2+y^2}\right)-1$$

$$=2-1=1$$

Hence proved.

(6) If $w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$

find. $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$

w is not a homogeneous function

$$w = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$$

taking anti log

$$e^w = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2} = u \text{ (say)}$$

$$u(\lambda x, \lambda y, \lambda z) = \frac{\lambda^7(5x^3y^4+7y^2xz^4-75y^3z^4)}{\lambda^2(x^2+y^2)}$$

$$u(\lambda x, \lambda y, \lambda z) = \lambda^5 u(x,y,z)$$

u is a homogeneous function of degree 5

$\therefore$ By Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 5u$$

$$x\frac{\partial}{\partial x}(e^w) + y\frac{\partial}{\partial y}(e^w) + z\frac{\partial}{\partial z}(e^w) = 5e^w$$

$$xe^w\frac{\partial w}{\partial x} + ye^w\frac{\partial w}{\partial y} + ze^w\frac{\partial w}{\partial z} = 5e^w$$

divide by e^w

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

need suggestions

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