EXERCISE 8.3

Question 1:

Express the trigonometric ratios sin A, sec A and tan A in terms of cot A:

Solution:

(i) We know that $\operatorname{cosec}^2 A - \cot^2 A = 1$

$$\Rightarrow \frac{1}{\sin^2 A} = 1 + \cot^2 A$$

$$\Rightarrow \sin^2 A = \frac{1}{1+\cot^2 A}$$

$$\Rightarrow \mathbf{\sin A} = \sqrt{\frac{1}{1+\cot^2 A}} = \frac{\mathbf{1}}{\sqrt{\mathbf{1+\cot^2 A}}}$$

(ii) $\sec^2 A = 1 + \tan^2 A$

$$\Rightarrow \sec^2 A = 1 + \frac{1}{\cot^2 A} = \frac{\cot^2 A + 1}{\cot^2 A}$$

$$\Rightarrow \sec A = \sqrt{\frac{\cot^2 A + 1}{\cot^2 A}} = \frac{\sqrt{\mathbf{\cot^2 A + 1}}}{\mathbf{\cot A}}.$$

(iii) $\tan A = \dfrac{1}{\cot A}$

Question 2:

Write all the other trigonometric ratios of ∠A in terms of sec A.

Solution:

Since $\sin^2 A + \cos^2 A = 1$, therefore

$$\sin^2 A = 1 - \cos^2 A = \frac{1}{1} - \frac{1}{\sec^2 A} = \frac{\sec^2 A - 1}{\sec^2 A} \Rightarrow \sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$$

$$\cos A = \frac{1}{\sec A}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{\sqrt{\sec^2 A - 1}}{\sec A}}{\frac{1}{\sec A}} = \frac{\sqrt{\sec^2 A - 1}}{1} = \sqrt{\sec^2 A - 1}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{1}{\frac{\sqrt{\sec^2 A - 1}}{\sec A}} = \frac{\sec A}{\sqrt{\sec^2 A - 1}}$$

$$\cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{\sec^2 A - 1}}$$

Question 3:

Evaluate:

(i) $\dfrac{\sin^2 63^\circ + \sin^2 27^\circ}{\cos^2 17^\circ + \cos^2 73^\circ}$ (ii) $\sin 25^\circ \cos 65^\circ + \cos 25^\circ \sin 65^\circ$

Solution:

(i) $\dfrac{\sin^2 63^\circ + \sin^2 27^\circ}{\cos^2 17^\circ + \cos^2 73^\circ}$

$$= \frac{\sin^2 63^\circ + \sin^2 (90^\circ - 63^\circ)}{\cos^2 (90^\circ - 73^\circ) + \cos^2 73^\circ}$$

$$= \frac{\sin^2 63^\circ + \cos^2 63^\circ}{\sin^2 73^\circ + \cos^2 73^\circ} = \mathbf{1}.$$

(ii) $\sin 25^\circ \cos 65^\circ + \cos 25^\circ \sin 65^\circ$

$$= \sin 25^\circ \cos(90 - 25^\circ) + \cos 25^\circ \sin(90 - 25^\circ)$$

$$= \sin^2 25^\circ + \cos^2 25^\circ = \mathbf{1}.$$

Question 4:

Choose the correct option. Justify your choice.

(i) $9 \sec^2 A - 9 \tan^2 A$ = ……
(A) 1
(B) 9
(C) 8
(D) 0

(ii) $(1 + \tan \theta + \sec \theta)(1 + \cot \theta - \operatorname{cosec} \theta)$ = ………..
(A) 0
(B) 1
(C) 2
(D) -1

(iii) $(\sec A + \tan A)(1 - \sin A)$ = ………….
(A) sec A
(B) sin A
(C) cosec A
(D) cos A

(iv) $1+\tan^2 A/1+\cot^2 A$ **= ……….**
(A) $\sec^2 A$
(B) -1
(C) $\cot^2 A$
(D) $\tan^2 A$

Solution:

(i) $9\sec^2 A - 9\tan^2 A = 9(\sec^2 A - \tan^2 A) = 9 \times 1 = 9$

Correct option is (B)

(ii) $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \operatorname{cosec}\theta) = \left(\frac{1}{1} + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\left(\frac{1}{1} + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)$

$$= \left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)\left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right) = \frac{(\cos\theta + \sin\theta)^2 - (1)^2}{\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta - 1}{\cos\theta\sin\theta} = \frac{1 + 2\cos\theta\sin\theta - 1}{\cos\theta\sin\theta} = \frac{2\cos\theta\sin\theta}{\cos\theta\sin\theta} = 2$$

Correct option is (C)

(iii) $(\sec A + \tan A)(1 - \sin A) = \left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)\left(\frac{1 - \sin A}{1}\right) = \left(\frac{1 + \sin A}{\cos A}\right)\left(\frac{1 - \sin A}{1}\right)$

$$= \frac{(1)^2 - (\sin A)^2}{\cos A} = \frac{1 - \sin^2 A}{\cos A} = \frac{\cos^2 A}{\cos A} = \cos A$$

Correct option is (D)

(iv) $$\frac{1 + \tan^2 A}{1 + \cot^2 A} = \frac{\sec^2 A}{\operatorname{cosec}^2 A} = \frac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}} = \frac{1}{\cos^2 A} \times \frac{\sin^2 A}{1} = \frac{\sin^2 A}{\cos^2 A} = \tan^2 A$$

Correct option is (D)

Question 5:

Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

(ii) $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 + \sin A}{\cos A} = 2\sec A$

(iii) $\dfrac{\tan\theta}{1 - \cot\theta} + \dfrac{\cot\theta}{1 - \tan\theta} = 1 + \sec\theta\operatorname{cosec}\theta$

(iv) $\dfrac{1 + \sec A}{\sec A} = \dfrac{\sin^2 A}{1 - \cos A}$

(v) $\dfrac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \operatorname{cosec} A + \cot A$, using the identity $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

(vi) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$

(vii) $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

(viii) $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

(ix) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

(x) $\left(\dfrac{1 + \tan^2 A}{1 + \cot^2 A}\right) = \left(\dfrac{1 - \tan A}{1 - \cot A}\right)^2 = \tan^2 A$

Solution:

(i) We have, $(\text{cosec}\ \theta - \cot\ \theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

$$\begin{aligned}
\text{LHS} &= (\text{cosec}\ \theta - \cot\ \theta)^2 \\
&= \left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2 \\
&= \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta} \\
&= \frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta} \\
&= \text{RHS}.
\end{aligned}$$

Hence, **proved**.

(ii)
$$\begin{aligned}
\text{LHS} &= \frac{\cos A}{1+\sin A} + \frac{1+\sin A}{\cos A} \\
&= \frac{\cos^2 A + \sin^2 A + 1 + 2\sin A}{(1+\sin A)\cos A} \\
&= \frac{2+2\sin A}{\cos A(1+\sin A)} = \frac{2(1+\sin A)}{\cos A(1+\sin A)} \\
&= 2\sec A = \text{RHS}.
\end{aligned}$$

Hence, **proved**.

(iii) $$\text{LHS} = \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta}$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta}}{1-\dfrac{\cos\theta}{\sin\theta}} + \frac{\dfrac{\cos\theta}{\sin\theta}}{1-\dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\sin\theta\times\sin\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos\theta\times\cos\theta}{\sin\theta(\cos\theta-\sin\theta)}$$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} - \frac{\cos^2\theta}{\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{\sin^3\theta-\cos^3\theta}{\cos\theta\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{(\sin\theta-\cos\theta)\left(\begin{array}{r}\sin^2\theta+\cos^2\theta\\ +\sin\theta\cos\theta\end{array}\right)}{\cos\theta\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta}{\cos\theta\sin\theta}$$

$$= \frac{\sin\theta\cos\theta+1}{\cos\theta\sin\theta}$$

$$= \frac{\sin\theta\cos\theta}{\cos\theta\sin\theta} + \frac{1}{\cos\theta}\times\frac{1}{\sin\theta}$$

$$= 1 + \frac{1}{\cos\theta}\frac{1}{\sin\theta}$$

$$= 1 + \sec\theta\,\text{cosec}\,\theta = \text{RHS}.$$

Hence, **proved**.

(iv) $$\text{LHS} = \frac{1+\sec A}{\sec A} = \frac{1+\dfrac{1}{\cos A}}{\dfrac{1}{\cos A}} = 1+\cos A$$

$$= \frac{(1+\cos A)\times(1-\cos A)}{(1-\cos A)}$$

$$= \frac{1-\cos^2 A}{1-\cos A} = \frac{\sin^2 A}{1-\cos A} = \text{RHS}.$$

Hence, **proved**.

(v) LHS $= \dfrac{\cos A - \sin A + 1}{\cos A + \sin A - 1}$

$$= \dfrac{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\sin A} + \dfrac{1}{\sin A}}{\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\sin A} - \dfrac{1}{\sin A}}$$

$$= \frac{\cot A - 1 + \operatorname{cosec} A}{\cot A + 1 - \operatorname{cosec} A}$$

$$= \frac{\operatorname{cosec} A + \cot A - \left(\operatorname{cosec}^2 A - \cot^2 A\right)}{\left(\cot A - \operatorname{cosec} A + 1\right)}$$

$[\because \operatorname{cosec}^2 A = 1 + \cot^2 A \Rightarrow \operatorname{cosec}^2 A - \cot^2 A = 1]$

$$= \frac{\operatorname{cosec} A + \cot A - (\operatorname{cosec} A + \cot A)(\operatorname{cosec} A - \cot A)}{(\cot A - \operatorname{cosec} A + 1)}$$

$$= \frac{(\operatorname{cosec} A + \cot A)(1 - \operatorname{cosec} A + \cot A)}{(\cot A - \operatorname{cosec} A + 1)}$$

$= \operatorname{cosec} A + \cot A =$ RHS.

Hence, **proved**.

(vi) LHS $= \sqrt{\dfrac{1 + \sin A}{1 - \sin A}}$

$$= \sqrt{\frac{(1 + \sin A)(1 + \sin A)}{(1 - \sin A)(1 + \sin A)}}$$

$$= \sqrt{\frac{(1 + \sin A)^2}{1 - \sin^2 A}} = \sqrt{\frac{(1 + \sin A)^2}{\cos^2 A}}$$

$$= \frac{1 + \sin A}{\cos A} = \frac{1}{\cos A} + \frac{\sin A}{\cos A}$$

$= \sec A + \tan A =$ RHS.

Hence, **proved**.

(vii) LHS $= \dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta}$

$$= \frac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$$

$$= \frac{\sin\theta(\sin^2\theta + \cos^2\theta - 2\sin^2\theta)}{\cos\theta\ (2\cos^2\theta - \sin^2\theta - \cos^2\theta)}$$

$$= \frac{\sin\theta(\cos^2\theta - \sin^2\theta)}{\cos\theta(\cos^2\theta - \sin^2\theta)}$$

$$= \frac{\sin\theta}{\cos\theta} = \tan\theta = \text{RHS.}$$

(viii) LHS $= (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$

$= \sin^2 A + \operatorname{cosec}^2 A + 2\sin A \operatorname{cosec} A$

$+ \cos^2 A + \sec^2 A + 2\cos A \sec A$

$= (\sin^2 A + \cos^2 A) + 2 + \operatorname{cosec}^2 A$

$+ \sec^2 A + 2$

$= 1 + 4 + (1 + \cot^2 A) + (1 + \tan^2 A)$

$= 7 + \tan^2 A + \cot^2 A =$ RHS.

Hence, **proved**.

(ix) LHS $= (\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

$$= \left(\operatorname{cosec} A - \frac{1}{\operatorname{cosec} A}\right)\left(\sec A - \frac{1}{\sec A}\right)$$

$$= \left(\frac{\operatorname{cosec}^2 A - 1}{\operatorname{cosec} A}\right)\left(\frac{\sec^2 A - 1}{\sec A}\right)$$

$$= \frac{\cot^2 A}{\operatorname{cosec} A} \times \frac{\tan^2 A}{\sec A}$$

$$= \frac{\sin A}{\tan^2 A} \times \cos A \tan^2 A = \sin A \cos A$$

$$\text{RHS} = \frac{1}{\tan A + \cot A} = \frac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}$$

$$= \frac{\sin A \cos A}{\sin^2 A + \cos^2 A} = \frac{\sin A \cos A}{1}$$

$\because$ LHS = RHS Hence, **proved.**

(x) $\text{LHS} = \dfrac{1+\tan^2 A}{1+\cot^2 A} = \dfrac{\sec^2 A}{\operatorname{cosec}^2 A} = \dfrac{\frac{1}{\cos^2 A}}{\frac{1}{\sin^2 A}}$

$$= \frac{\sin^2 A}{\cos^2 A} = \tan^2 A.$$

$$\text{RHS} = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \left(\frac{1-\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}}\right)^2$$

$$= \left(\frac{\frac{\cos A - \sin A}{\cos A}}{\frac{\sin A - \cos A}{\sin A}}\right)^2$$

$$= \frac{(\cos A - \sin A)^2 \times \sin^2 A}{(\sin A - \cos A)^2 \times \cos^2 A}$$

$$= \frac{(\sin A - \cos A)^2 \times \sin^2 A}{(\sin A - \cos A)^2 \times \cos^2 A}$$

$$= \frac{\sin^2 A}{\cos^2 A} = \tan^2 A.$$

$\because$ LHS = RHS Hence, **proved**.