Harsh Pathak
2016041

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Q3 $\beta_n = \dfrac{\alpha}{\theta_n}$ where $\theta_n = \theta_{n-1} + \alpha(1-\theta_{n-1})$

$\theta_0 = 0$

we know that

$$\theta_0 = 0$$
$$\theta_1 = \theta_0(1-\alpha) + \alpha$$
$$\theta_2 = \theta_1(1-\alpha) + \alpha$$
$$\theta_3 = \theta_2(1-\alpha) + \alpha$$
$$\theta_3 = \{\theta_1(1-\alpha) + \alpha\}(1-\alpha) + \alpha$$
$$= (1-\alpha)^2\theta_1 + \alpha(1-\alpha) + \alpha$$
$$= (1-\alpha)^2\{\theta_0(1-\alpha)+\alpha\} + \alpha(1-\alpha) + \alpha$$
$$= (1-\alpha)^3\theta_0 + \alpha(1-\alpha)^2 + \alpha(1-\alpha) + \alpha$$

in general

$$\theta_n = (1-\alpha)^n\theta_0 + \{\alpha + \alpha(1-\alpha) + \dots + \alpha(1-\alpha)^{n-1}\}$$
$$= \alpha\left\{\frac{1-(1-\alpha)^n}{\alpha}\right\} = 1-(1-\alpha)^n$$

hence $\boxed{\theta_n = 1-(1-\alpha)^n}$ and $\theta_0 = 0$

Now $Q_{n+1} = \beta_n R_n + (1-\beta_n)Q_n$

$$= \beta_n R_n + (1-\beta_n)[\beta_{n-1}R_{n-1} + (1-\beta_{n-1})Q_{n-1}]$$
$$= \beta_n R_n + (1-\beta_n)(\beta_{n-1})R_{n-1} + (1-\beta_n)(1-\beta_{n-1})Q_{n-1}$$

It turns out that coefficient of $\boxed{Q_i = \prod_{k=i}^{k=n}(1-\beta_k)}$

coefficient of $Q_1 = \boxed{(1-\beta_n)(1-\beta_{n-1})\dots(1-\beta_1)}$

Further $1-\beta_j = 1-\dfrac{\alpha}{\theta_j} = \dfrac{(\theta_{j-1})(1-\alpha)}{\theta_j}$

Coeff of $Q_1 = \dfrac{(1-\alpha)\theta_{n-1}}{\theta_n} \cdot \dfrac{(1-\alpha)\theta_{n-2}}{\theta_{n-1}} \cdot \dfrac{(1-\alpha)\theta_{n-3}}{\theta_{n-2}} \cdot \dots \cdot \dfrac{(1-\alpha)\theta_0}{\theta_1} = 0$

Since $\theta_0 = 0$

Since coeff. of $\theta_i = \prod_{k=i}^{k=n}(1-\beta_k)$ this is an exponential recency weighted average.

We also need to prove that sum of weights of $R_i = 1$

$$\text{coeff of } R_i = \beta_i \prod_{k=i+1}^{n}(1-\beta_k)$$

$$= \beta_i \cdot \frac{(1-\alpha)\theta_i}{\theta_{i+1}} \cdot \frac{(1-\alpha)\theta_{i+1}}{\theta_{i+2}} \cdot \frac{(1-\alpha)\theta_{i+2}}{\theta_{i+3}} \cdot \ldots \cdot \frac{(1-\alpha)\theta_{n-1}}{\theta_n}$$

$$= \beta_i(1-\alpha)^{n-i}\frac{\theta_i}{\theta_n} = \frac{\alpha}{\theta_i}(1-\alpha)^{n-i}\frac{\theta_i}{\theta_n} = \frac{\alpha(1-\alpha)^{n-i}}{\theta_n}$$

$$\sum_{i=1}^{n} \text{coeff}(R_i) = \sum_{i=1}^{n} \frac{\alpha(1-\alpha)^{n-i}}{\theta_n}$$

$$= \frac{\alpha}{\theta_n}\sum_{i=1}^{n}(1-\alpha)^{n-i}$$

$$= \frac{\alpha}{\theta_n}\left\{1+(1-\alpha)+(1-\alpha)^2+\ldots+(1-\alpha)^{n-1}\right\}$$

$$= \frac{\alpha}{\theta_n}\left\{\frac{1-(1-\alpha)^n}{\alpha}\right\} = \frac{1-(1-\alpha)^n}{\theta_n} = 1$$

Since $\theta_n = 1-(1-\alpha)^n$, sum $= 1$