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CONTROLS AND SYSTEMS

Control System Design for Mechanical Systems Using Contraction Theory

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Abstract—Mechanical systems are highly nonlinear which makes control of these systems difficult to translate into a much understandable systems. Even the control of relatively simple linear mechanical systems is often complicated by the fact, that an undesirable nonlinear effect as friction is present. So, in order to simplify these stale tasks, the contraction theory works with developing some properties for such non linear systems. Therefore there are some controllers which are to be developed and discussed in this paper.

Keywords— First word or key phrase, second word or key phrase, third word or key phrase. (Place between three and six key words or phrases separated by a comma, which represent the theme of your work)

I. INTRODUCTION

In this paper we will be have a scope from continuum physics and also differential geometry. So, in order to consider the contraction analysis we will be working with some of the tool from continuum physics and differential geometry. Continuum Physics: It works on the physics of continuous body in space which have physical quantities based on mathematical continuous functions. In this paper the mail goal is to use the Hyperstability(V.M.Popov), Lyapunov and Barbalat's lemma functions to solve the nonlinear adaptive aerodynamic problem which will be handled in the section IV.

II. CONTRACTION ANALYSIS

The basis of all the linear control systems is through differential approximation which leads to stability in the system. Lyapunov theory is one of the most widely-used approaches to stability analysis of a nonlinear system, which provides a condition for stability with respect to an equilibrium point, a target trajectory, or an invariant set. Contraction theory rewrites suitable Lyapunov stability conditions using a quadratic Lyapunov function of the differential states. In order to figure out the exact generalized results to differential stabilize the non linear system for which we will be using the contraction theory. Contraction theory is the control system tool based on an exact differential analysis of convergence.We will have a general system of the form.

$$\dot{x} = f(x,t)$$

where f is an $n \times 1$ nonlinear vector function and x is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $u(x,t)$. All quantities are assumed to be real and smooth, by which it is meant that any required derivative or partial derivative exists and is continuous. The plant equation (1) can be thought of as an n-dimensional fluid flow, where $\dot{x}$ is the n-dimensional "velocity" vector at the n-dimensional position x and time t. Assuming as we do that $f(x,t)$ is continuously differentiable, (1) yields the exact differential relation

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x$$

where δx is a virtual displacement. Real Displacement: The real displacement is the distance final distance moved from the initial point to the final point in a given time interval(time interval is very important).

Whereas, the virtual displacement is the change in position for a infinitesimal time. Which can be considered in a way that the displacement takes place instantaneously. Consider a system of n-particles, with the position coordinate of ith particle.

$$fi = fi(x1, x2, x3...............xn;t)$$

from a generalized displacement, we have

$$\delta fi = \frac{\partial fi}{\partial x_j}\delta x_j + \frac{\partial fi}{\partial t}\delta t - - - - - (1)$$

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However, if we consider the case of virtual displacement where the displacement is for a infinitesimal fixed time in that case, Ri being a virtual displacement becomes as follow

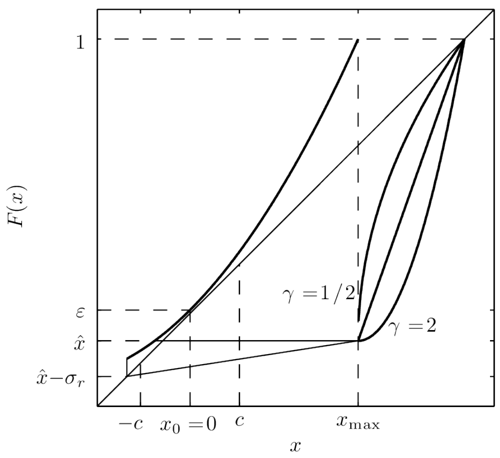
$$\delta fi = \Sigma_j \frac{\partial fi}{\partial x_j} \delta x_j - - - - - (2)$$

The line vector δx can also be expressed using the differential coordinate transformation

$$\delta z = \Theta \delta x - (3)$$

The δz represents the linear differential tangent form and the $\delta z^T \delta z$ represents the quadratic differential tangent form, both of which are differential with respect to time.[11]

Consider now two neighboring trajectories in the flow field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, and the virtual displacement $\boldsymbol{\delta}\mathbf{x}$ between them (Figure 1). The squared distance between these two trajectories can be defined as $\boldsymbol{\delta}\mathbf{x}^T \boldsymbol{\delta}\mathbf{x}$, leading from (2) to the rate of change



$$A = \boldsymbol{\delta}\mathbf{x}^T$$

$$B = \boldsymbol{\delta}\dot{\mathbf{x}} = \frac{d}{dt}(\boldsymbol{\delta}\mathbf{x})$$

$$\frac{d}{dt}\left(\boldsymbol{\delta}\mathbf{x}^T \boldsymbol{\delta}\mathbf{x}\right) = \frac{d}{dt}(\boldsymbol{\delta}\mathbf{x})\, x \boldsymbol{\delta}\mathbf{x}^T$$

$$\frac{d}{dt}\left(\boldsymbol{\delta}\mathbf{x}^T \boldsymbol{\delta}\mathbf{x}\right) = 2\boldsymbol{\delta}\mathbf{x}^T \boldsymbol{\delta}\dot{\mathbf{x}}$$

from equation (2), we know

$$\delta \dot{x} = \frac{\partial f}{\partial x} \delta x$$

A d-dimensional manifold can be informally defined as a set M covered with a “suitable” collection of coordinate patches, or charts, that identify certain subsets of M. Such a collection of coordinate charts can be thought of as the basic structure required to do differential calculus on M. In the literature, a manifold is sometimes simply defined as a set endowed with a differentiable structure. Basically a manifold is a set or structure which can be differentiable.

where $\Theta(x,t)$ is a square matrix. This leads to

$$\delta z^T \delta z = \delta x^T M \delta x - - - - - - - - - (4)$$

where $M(x,t) = \Theta^T \Theta$ represents a symmetric and continuously differentiable metric-formally. The equation (4) defines a Riemann space while (3) is in general not integrable. Always M is to be uniformly positive definite so that exponential convergence of δz to $\mathbf{0}$ implies exponential convergence of δx to 0 . We cannot expect to find explicit new coordinates $z(x,t)$, but δz and $\delta z^T \delta z$ can always be defined. The shortest path length between two points P_1 and P_2 with respect to the metric M.

Accordingly, a ball of center c and radius R is defined as the set of all points whose distance to c with respect to M is strictly less than R. Computing the dynamics of δz (or "virtual” dynamics)

$$\frac{d}{dt}\delta z = F \delta z \quad \text{ where } F = \left(\dot{\Theta} + \Theta \frac{\partial f}{\partial x}\right)\Theta^{-1} - - - - - (5)$$

Definition 1: Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region of the state space is called a contraction region if the Jacobian $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is uniformly negative definite in that region.

By $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ uniformly negative definite we mean that

$$\exists \beta > 0, \forall \mathbf{x}, \forall t \geq 0, \frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}^T}{\partial \mathbf{x}}\right) \leq -\beta \mathbf{I} < 0$$

More generally, by convention all matrix inequalities will refer to the symmetric parts of the square matrices involved - for instance, we shall write the above as $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \leq -\beta \mathbf{I} < 0$. By a region we mean an open connected set. Extending the above definition, a semi-contraction region corresponds to $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ being negative semi-definite, and an indifferent region to $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ being skew-symmetric.

Consider now a ball of constant radius centered about a given trajectory, such that given this trajectory the ball remains within a contraction region at all times (i.e., $\forall t \geq 0$). Because any length within the ball decreases exponentially, any trajectory starting in the ball remains in the ball (since by definition the center of the ball is a particular system trajectory) and converges exponentially to the given trajectory (Figure 2). Thus, as in stable linear time-invariant (LTI) systems, the initial conditions are exponentially "forgotten." This leads to the following theorem:

Theorem 1: Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory.

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region.

This sufficient exponential convergence result may be viewed as a strengthened, an analogy we shall generalize . Note that its proof is very straightforward, even in the non-autonomous case, and even in the non-global case, where it guarantees explicit regions of convergence. Also, note that the ball in the above theorem may not be replaced by an arbitrary convex region - while radial distances would still decrease, tangential velocities could let trajectories.

Theorem 1 could also be derived from the error dynamics

$$\dot{\tilde{x}} = f(\tilde{x},t) = \int_1^0 \frac{\partial f}{\partial x}(x_d + \lambda \tilde{x})d\lambda$$

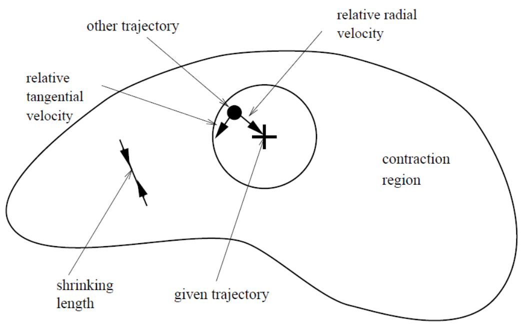
with $\tilde{x} = x - x_d$, where x represents a general trajectory and x_d a given trajectory.

The following Lyapunov analysis

$$\dot{V} = \frac{d}{dt}\tilde{x}^T\tilde{x} = 2\tilde{x}^T \int_1^0 \frac{\partial f}{\partial x}(x_d + \lambda \tilde{x})d\lambda \tilde{x}$$

This shows uniformly negative definite

exponential convergence to zero.



III. COMBINATION OF CONTRACTING SYSTEMS

In multivariable nonlinear autonomous system has a wide scope in stability approach using Lyapunov's method and Popov's hyperstability are been used in such systems as a principal tool.

To study asymptotic stability in a system with a set of error differential equations ,Lyapunov method is used for Lyapunov function. Whereas, in nonautonomous equations this theory is less decisive as compared to the autonomous systems which leads to the difficulty in yeilding a negative semidefinite time derivative V(x,t). So far, the problem for Lyapunov method requires to recast the feedback loop consisting the linear time-invariant operator in the feedback path which can be resolved by using an alternative path of hyperstability.

Firstly, we present the Lyapunov Stability method followed by hyperstability method by V.M.Pupov.

a. Lyapunov's Method

: When we consider the general statement which come with conditions for uniform asymptotic stability of differential equation:

$$\dot{x} = f(x,t) \qquad , \qquad f(0,t) = 0$$

where x and f are n—vectors.For, f is solutions exist for all $t \geq t_0$ assumed to be sufficiently regular.

Definition : A real valued function $\phi(\rho)$ belongs to class K if it is defined, continuous and strictly increasing for all $\rho, 0 \geq \rho \geq p_1$ where ρ_1 is arbitrary and $\phi(0) = 0$.

Theorem 1: If a function V(x,t) that is defined for all x and t satisfies:

i: V(x,t) is continuous with respect to x and t for all t.

ii: V(x,t) is positive definite. i.e. $\alpha\varepsilon$ K exists such that

$$0 \geq \alpha(||x||) \geq V(x,t)$$

iii : V(x,t) is radially unbounded. i.e. α in (ii) is such that

$$\lim_{\rho\to\infty} \alpha(\rho) = \infty$$

iv : V(x,t) is descresent. i.e. there exists $\beta\varepsilon$ K such that

$$V(x,t) \geq \beta(||x||)$$

for all x ε R ^{n}

Positive definiteness of V and negative semi-definiteness of result in stability. Uniform asymptotic stability is guaranteed by the additional requirement that V(x,t) is negative definite.The above theorem has been applied extensively in systems theory. Quite often it is found that V(x,t) is only negative semi-definite and in such cases only uniform stability rather than uniform asymptotic stability can be concluded . For autonomous systems defined by

$$\dot{x} = f(x)$$

the proof of asymptotic stability using Lyapunov 's method as well as asymptotic hyperstability using Popov's theory.

b. Lemma

: If 'g' is a real function of the real variable 't' defined and uniformly continuous for t > 0 and if the limit of the integral

$$\int_t^0 g(\tau)d(\tau)$$

as t tends to infinity exists and is a finite number, then

$$\lim_{t\to\infty} g(t) = 0$$

If $\dot{V}(x,t)$is identified with g(t) in the above lemma, then if $\dot{V}(x,t)$ is uniformly continuous, every solution of the differential equation (1) would be such that

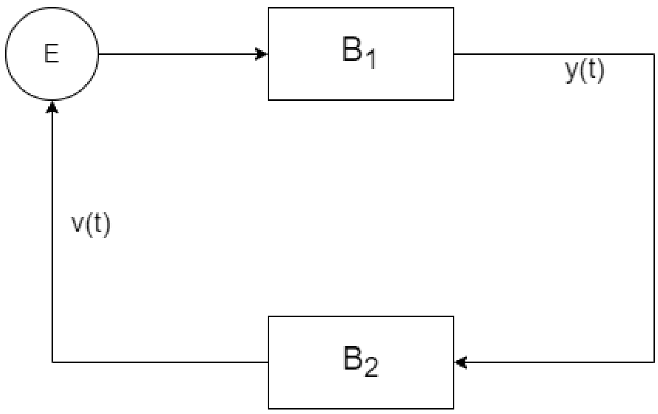
$$\dot{V}(x(t),t) = 0.$$

c. Hyperstability Theory

In 1963 Popov introduced the concept of Hyperstabihity as a natural extension of absolute stability. Consider a feedback system with two blocks B1 and B2 as shown in Figure 3. Let B2 have input y(t) and output v(t) and let

$$\int_t^0 y(\tau)v(\tau)d(\tau) \geq 0$$

for all t ≥ 0 ——————————(4)
The block B1 is said to be hyperstable if every system of the type shown In Figure 3



Consider a completely controllable and completely observable system B_1 with m—inputs and m—outputs described by The below equations belong to B_1 system

$$\dot{x}(t) = Ax(t) + Bw(t)$$

$$y(t) = Cx(t) + Dw(t)$$

where w(t) and y(t) are (mxl) vectors , x(t) is an (nxl) state vector and A,B ,C ,D are constant matrices of appropriate dimensions.
Hyperstability of B_1 is then defined by the property which requires that the state x(t) be bounded for a certain class of inputs v(t). This class is defined by those v which satisfy for all T.

$$\int_T^0 w^T(t)y(t)dt \geq \delta[||x(0)||]$$

single input — single output systems and hence w(t) and y (t) are to be scalars.

IV. The Passivity Formalism

In a dynamic system Lyapunov functions are basically the generalizations of notion of energy. Likewise, the passivity theory formalizes that the Lyapunov functions are combinations of systems which are derived simply by addition of the Lyapunov functions of subsystems.
Every linear or non-linear,physical systems have to satisfy energy-conservation equations of the form

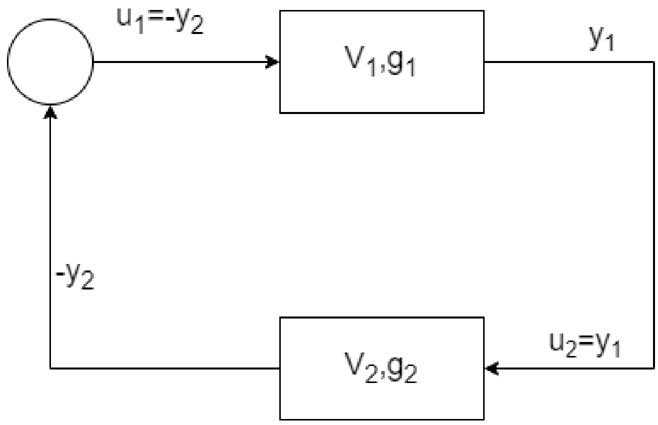
$$\frac{d}{dt}(StoredEnergy) = [ExternalPowerInput] + [InternalPowerGeneration]$$

The external power input term can be represented as the scalar product $y^T u$ of an input ("effort" or "flow") u, and a output ("flow" or "effort") y.
In the following the equation considers the general systems which verifies the form

$$\dot{(V_1)}(t) = y_1^T u_i - g_1(t) - - - - - - - - - - - - - - - - (5)$$

where $V_1(t)$ and $g_1(t)$ are scalar functions of time,y_t is its output and U_1 is the system input. Note that, from a mathematical point of view, the above form is quite general given an arbitrary system, of input $U_1(t)$ and output $y_1(t)$, we can let, for instance,$g_1(t) = 0$ and $V_1(t) = \int_t^0 y_1^T(r)u_1(r)dr$. It is the physical or "Lyapunov-like" properties that $V_1(t)$ and $g_1(t)$ may have. To verify the above equation

$$\dot{V_2}(t) = y_2^T u_2 - g_2(t) - - - - - - - - - - - - -(6)$$



in a feedback configuration, namely $u_2 = y_1$ and $u_1 = -y2$ (Figure 4). Considering u_i and v_j:

$$\frac{d}{dt}[V_1(t) + V_2(t)] = -[g_1(t) + g_2(t)]$$

$$\frac{d}{dt}\sum_i V_i = -\sum_i g_i$$

This equation indirectly helps us to extract the total length by summation of all the lengths V_i individually.

a. Basic Combination

Parallel Combinations:The two systems of the same dimension x_1 and x_2

$$x_1 = f_1(x_1, t)$$

$$x_2 = f_2(x_2, t)$$

while we know the virtual dynamics equation from (2)

$$\delta z_1 = F_1 \delta z$$

$$\delta z_2 = F_2 \delta z$$

$$i = 1, 2, 3.....$$

The above systems are connected in a parallel combination. In any superposition which is uniformly positive considering both the systems are contracting with same metric.

$$\alpha_1(t)\delta \dot{z_1} + \alpha_2(t)\delta \dot{z_2}$$

where $\alpha > 0, \forall t \geq 0, \alpha_i(t) \geq \alpha$
When the contracting system is not time variant then the superposition would be in the following ways:

$$\dot{x} = \alpha_1(t) f_1(x,t) + \alpha_2(t) f_2(x,t)$$

Feedback Combination: Consider two systems, for possibly different dimensions for example $f_i(x_i, x_i, t)$

$$\dot{x}_1 = f_1(x_1, x_2, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, t)$$

and connect them in the feedback combination

$$\frac{d}{dt}\begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix} = \begin{pmatrix} F_1 & G \\ -G^T & F_2 \end{pmatrix}\begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix}$$

Then the augmented system is contracting if the individual plants are contracting.
Hierarchical Combination: Consider a smooth virtual dynamics of the form

$$\frac{d}{dt}\begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix} = \begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}\begin{pmatrix} \delta z_1 \\ \delta z_2 \end{pmatrix}.$$

The result for the two equations can be partitioned similarly for more than two. If some subsystems are indifferent rather than contracting and that will result in variation in the derivations. For instance, if F_{11} is skew-symmetric rather than negative definite, then the dynamics in δz_1 is indifferent.
Hence, for bounded $F_{21}, F_{21}\delta z_1$ represents a bounded disturbance in the second equation. A skew-symmetric F_{22} then leads to a bound on $\|\delta z_2\|$ which increases linearly with time. With linear analysis, these systems are marginally contracting.

b. Barbalat's lemma

Now, given that a function tends towards a finite limit. However, we do not know will the derivative actually converge to zero. Therefore, Barbalat's Lemma proposed that the derivative should itself be smooth.
Lemma: $\underline{Definition}$ - If the differentiable function f(t) has a finite limit as t $\rightarrow \infty$, and if f is uniformly continuous, then f(t)$\rightarrow$ 0 as t$\rightarrow \infty$.
Let's say a function g(t) is continuous on [0, ∞) if

$$\forall t_1 \geq 0$$

$$\forall R > 0$$

,

$$\exists \eta T(R, t_l) > 0$$
$$\forall t \geq 0$$

,

$$|t - t_1| < \eta => |g(t) - g(t_1)| < R$$

While a function g is said to be uniformly continuous on [0,∞) if

$$\forall R > 0$$
$$\exists \eta T(R) > 0$$
$$\forall t_1 \geq 0$$
$$\forall t \geq 0$$
$$|t - t_1| < \eta => |g(t) - g(t_1)| < R$$

If one finds an η which does not depends on a specific point t_1 only then g is uniformly continuous. Also, considering the condition in which η does not shrink as $t_1 \rightarrow \infty$. It should be noted that t and t_1 play a symmetric role in defining uniform contiunity. The function's derivative is determined by the sufficient condition for differential function to be uniformly continuous is derivative be bounded.

c. Adaption

If a part of a system consists of any uncertainity which are unknown constant parameters a, in that case we would use adaptive technique. For instance, a closed-loop plant error dynamics for an error model in n differential equation is

$$\dot{e} = Ae + b\phi^T(t)u(t)$$

$$e_1 = c^T e$$

$$\dot{\tilde{z}} = f(z,t) - f(z_d,t) + W(z,t)a - W(z,t)\hat{a}$$

with parameter estimate vector $\hat{a}$, state vector z, desired state vector z_d,and $\tilde{z} = z - z_d$. Letting $\tilde{a} = \hat{a} - a$, and choosing the parameter adaptation

$$z_d = model\ estimated\ output$$

$$\tilde{z} = tracking\ error\ of\ the\ system$$

$$\hat{a} = adjustable\ parameter$$

Since, according to Barbalat's lemma the asymptotic properties of function and their derivatives. Given the differential function f of time t, the following factors are important
$\dot{f} \rightarrow 0$ f converges
The fact that $\dot{f} \rightarrow 0$ does not imply that f(t) has a limit as t $\rightarrow \infty$
Geometrically, a derivative is diminishing which means the curve is flatter and flatter
Consider, for instance the function f(t)= sin(log t), so
$f(\dot{t}) = \frac{cos(logt)}{t}$ as t $\rightarrow \infty$
Therefore, $f(z,t) = f(z_d,t) = 0$

$$\dot{\tilde{a}} = \dot{\hat{a}} = -W^T(z,t)\tilde{z}$$

Barbalat's lemma and the Lyapunov-like analysis

$$\dot{V} = \frac{d}{dt}\left(\tilde{z}^T\tilde{z} + \tilde{a}^T\tilde{a}\right) = 2\tilde{z}^T\int_0^1 \frac{\partial f}{\partial z}(z_d + \lambda\tilde{z})\,d\lambda\tilde{z}$$

show asymptotic convergence of $\tilde{z}$ to zero for uniformly negative definite $\partial f/\partial z$ and bounded $\ddot{V}$.

d. Time delay

In systems which have transmission delays or computation delays there comes a time delay feedback connections when multiple systems are involved.
Likewise, in the system for underwater vehicle control through acoustic transmission and hand-eye coordination is a case of force reflecting teleoperation.
For example, Let's consider there are two systems which are contracting, probably different dimensions.

$$\dot{z}_1 = f_1(z_1,t) + G_1\tau_2$$

$$\dot{z}_2 = f_2(z_i, t) + G_2\tau_2$$

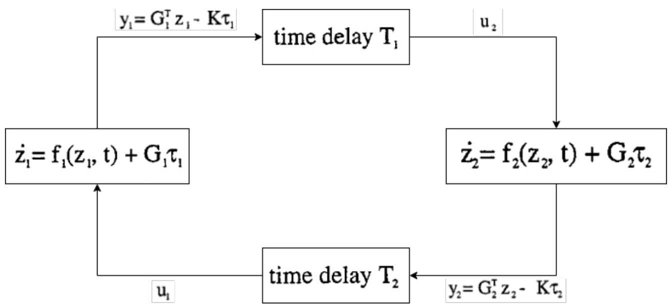
$$u_i = G_i^T z_i + K\tau_i \quad i = 1,2$$
$$y_i = G_i^T z_i - K\tau_i \quad i = 1,2$$

to choose which variables are actually transmitted. Directly inspired by the use of wave variables in force-reflecting teleoperation, define intermediate variables

$$\mathbf{u}_1 = \mathbf{G}_1^T \mathbf{z}_1 + \tau_1$$
$$\mathbf{v}_1 = \mathbf{G}_1^T \mathbf{z}_1 - \tau_1$$
$$\mathbf{u}_2 = \mathbf{G}_2^T \mathbf{z}_2 + \tau_2$$
$$\mathbf{v}_2 = \mathbf{G}_2^T \mathbf{z}_2 - \tau_2$$

and transmit these in place of the more obvious $\mathbf{z}_i$,

$$\mathbf{u}_2(t) = \mathbf{u}_1(t - T)$$
$$\mathbf{v}_1(t) = \mathbf{v}_2(t - T)$$



The rate of change of differential length can then be computed similarly to (Niemeyer and Slotine, 1991)

$$\delta\dot{V} = \frac{1}{2}\frac{d}{dt}\left(\delta\mathbf{z}1^T\delta\mathbf{z}_1 + \delta\mathbf{z}_2^T\delta\mathbf{z}_2 + \frac{1}{2}\int t - T^t\left(\delta\mathbf{u}_1^T\delta\mathbf{u}_1 + \delta\mathbf{v}_2^T\delta\mathbf{v}_2\right)d\tau\right) = \delta\mathbf{z}_1\frac{\partial\mathbf{f}_1}{\partial\mathbf{z}_1}\delta\mathbf{z}_1 + \delta\mathbf{z}_2\frac{\partial\mathbf{f}_2}{\partial\mathbf{z}_2}\delta\mathbf{z}_2$$

We have used

$$\int_{t-T}^{t}\left(\delta\mathbf{u}1^T\delta\mathbf{u}_1(\tau) + \delta\mathbf{v}_2^T\delta\mathbf{v}_2(\tau)\right)d\tau - \int -T^0\left(\delta\mathbf{u}_1^T\delta\mathbf{u}_1(\tau) + \delta\mathbf{v}_2^T\delta\mathbf{v}_2(\tau)\right)d\tau$$
$$= \int_o^t\left(\delta\mathbf{u}_1^T\delta\mathbf{u}_1(\tau) - \delta\mathbf{u}_1^T\delta\mathbf{u}_1(\tau - T) + \delta\mathbf{v}_2^T\delta\mathbf{v}_2(\tau) - \delta\mathbf{v}_2^T\delta\mathbf{v}_2(\tau - T)\right)d\tau$$
$$= \int_o^t\left(\left(\delta\mathbf{u}_1^T\delta\mathbf{u}_1(\tau) - \delta\mathbf{v}_1^T\delta\mathbf{v}_1(\tau)\right) - \left(\delta\mathbf{u}_2^T\delta\mathbf{u}_2(\tau) - \delta\mathbf{v}_2^T\delta\mathbf{v}_2(\tau)\right)\right)d\tau$$
$$=4\int_o^t\left(\delta\mathbf{z}_1^T\mathbf{G}_1\delta\tau_1(\tau) - \delta\mathbf{z}_2^T\mathbf{G}_2\delta\tau_2(\tau)\right)d\tau$$

where the time arguments apply to the whole dot-products and $\mathbf{u}_1 = \mathbf{v}_2 = 0, \forall t \leq 0$. Applying Barbalat's lemma to $\delta\dot{V}$ shows asymptotic convergence of $\delta\mathbf{z}_1$ and $\delta\mathbf{z}_1$ to zero for contracting separate dynamics $\mathbf{f}_1$ and $\mathbf{f}_2$ and bounded $\delta\ddot{V}$. This derivation can be extended straightforwardly to different transmission time delays T_i, and to feedback loops composed of more than two systems. Note that in the special case $T = 0$, the above reduces to $\mathbf{G}_1^T\mathbf{z}_1 - \mathbf{G}_2^T\mathbf{z}_2 = \mathbf{0}$.

V. Mechanical Systems

a. Strap Down Algorithm

Consider the Euler dynamics of a rigid body, with Euler angles $x = (\psi, \theta, \phi)^T$ and rotation vector $\omega = (\omega_1, \omega_2, \omega_3)^T$ in body coordinates A.

$$\dot{x} = J^{-1}\omega - - - - - - - - - - - - - - - - - - - (6)$$

with

$$J = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 0 & -\sin\psi & \cos\theta\cos\psi \end{pmatrix}.$$

δ z is a differential angular displacement

δ x is a differential linear displacement

$\theta = AJ$

Analyzing the differential angular displacement $\delta z = \Theta\delta x$ in inertial coordinates.
where A is the orthonormal transformation matrix from body coordinates to inertial coordinates. The generalized matrix is a Jacobian matrix. Since, we know
ω= $v(linearvelocity)r(length)$
Therefore, the $\delta z = AJ\delta x$ This shows that $\partial A/\partial x$ is bounded which indirectly means it is a contracting.
After solving the matrix of orthogonal and Jacobian we come to a conclusion where

$$F = 0.$$

In inertial navigation, the classical strap down algorithm measures the body turn rate ω and the inertial acceleration in body coordinates and then combines (6) with

$$\dot{v} = A - - - (7)$$

$$\dot{r} = v - - - (7)$$

Since A is the angular velocity orthogonal matrix and r is the inertial coordinate which is differentiated into linear velocity of the system. The equations (6),(7),(7) represent a hierarchy of three indifferent systems which extends the well-known analysis in the linearized case of nonlinear dynamics.

$$A=\begin{pmatrix} \cos\theta\cos\phi & \sin\psi\sin\theta\cos\phi-\cos\psi\sin\phi & \cos\psi\sin\theta\cos\phi+\sin\psi\sin\phi \\ \cos\theta\sin\phi & \sin\psi\sin\theta\sin\phi+\cos\psi\cos\phi & \cos\psi\sin\theta\sin\phi-\sin\psi\cos\phi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}$$

b. An Aircraft Controller

So, for aircraft the definition of every term is provided below,

$$u = Axial\ Velocity/Body\ Velocity$$

$$v = Vertical\ Velocity$$

$$x = Range$$

$$z = Altitude$$

$$\theta = Pitch\ Angle$$

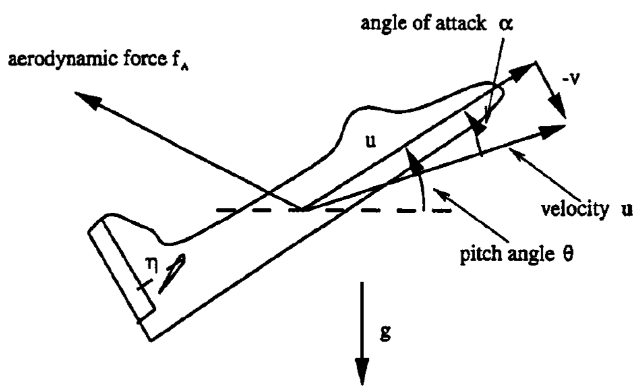
$$q = Pitch\ rate$$

$$I = Inertia$$

$$\zeta = Constant\ Thrust$$

$$M(u,\theta,\dot{\theta}) = Inertia$$

$$f_A(u,\eta) = Aerodynamic\ Force$$



a simplified model of the longitudinal motion of a high-performance aircraft, possibly at high angle of attack As the aircraft has 6Dof for it to maneuver which could be 3 translational(x,y,z) and 3 rotational(θ,\$)

$$\ddot{\theta} = \frac{1}{I}(M+\eta)$$

$$=\begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix} u + \tfrac{1}{m} f_A + \tfrac{1}{m}\begin{pmatrix} \xi \\ 0 \end{pmatrix} + g\begin{pmatrix} -\sin\theta \\ -\cos\theta \end{pmatrix}$$

A simple rotational controller
So, by separation it is imposed that

$$\dot{\hat{z}} = f(\hat{z},t) - (e(\hat{z} - e(z)) + G(z,t)u(\hat{z},t)$$

combined with plant dynamics

$$\dot{z} = f(z,t) + G(z,t)u(\hat{z},t)$$

with control input $u(\hat{z},t)$, state vector z, and the state estimate vector $\hat{z}$. Subtracting the plant dynamics from the observer dynamics, leads with

$\tilde{z} = \hat{z} - z$ to

$$\dot{\tilde{z}} = f(\hat{z},t) - e(\hat{z}) - (f(z,t) - e(z))$$

As per the section for adaption we learned earlier in (IVc) which

$$\eta = -M + I\left(\dot{\theta}_d - \dot{\theta} + \theta_d - \theta + \ddot{\theta}_d\right)$$

figure of aircraft with body-fixed velocity u, pitch angle θ, mass m, inertia I, gravity constant g, constant thrust ξ, elevator torque η and external torque $M(u,\theta,\dot{\theta})$ around the center of mass, and the aerodynamic force $f_A(u,\eta)$. The dynamic structure is hierarchical, reflecting the physics of the system. The above equation guarantees exponential convergence of θ to θ_d. In turn, the virtual velocity dynamics which is derived from the rate of change of differential length can then be computed to the virtual velocity from

$$\delta\dot{z} = \delta z\left(\frac{\partial z}{\partial x}f + \frac{\partial z}{\partial t}\right)$$

$$\delta\dot{u} = \left[\begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix} + \frac{1}{m}\frac{\partial f_A}{\partial u}\right]\delta u$$

the mild physical assumption that $\partial f_A/\partial u$ is quite contracting under this uniformly negative definite-this control-motivated requirement may actually guide aerodynamic design for high angle of attack aircraft. Exponential convergence to a desired trajectory in $\alpha_d(t) = -\arctan(v_d(t)/u_d(t))$ or $u_d(t)$ will be guarantees by using the time-varying $\theta_d(t)$. In the below mentioned example we will describe some of the aerodynamic forces using adaptive stabilization controller.
Example: The aerodynamic forces for a fixed coordinate in a body can be computed from the lift and drag forces:

$$f_a = \begin{pmatrix} sin(\alpha) & -cos\alpha \\ cos(\alpha) & sin(\alpha) \end{pmatrix}(LD)^T$$

As previously mentioned the effective angle of attack $\alpha = -arctan(\frac{v}{u})$. The lift force L($\eta$, u), while the drag force D(η, u). Accordingly, in this example a an assumption is being made about for the periodic lift force which is $\frac{\alpha}{\pi}$ and the periodic drag force is $\frac{2\alpha}{\pi}$. The forces for the lift and drag are as follows:

$$L = \frac{\rho S}{2}\left(u^2+v^2\right)c_{L\max}sin(\alpha)cos(\alpha) = -\frac{\rho S}{2}c_(L)uv$$

$$D = \frac{\rho S}{2}\left(u^2+v^2\right)\left(c_o + c_{d\max}sin^2(\alpha)\right) = -\frac{\rho S}{2}\left(c_o u^2 + (c_{d\max}+c_o)v^2\right)$$

The aerodynamic force is

$$\mathrm{f}_A = -\frac{\rho S}{2}\frac{1}{\sqrt{u^2+v^2}}\begin{pmatrix} c_o u^3 + (c_{\mathrm{imax}} + c_o - c_{L\max})uv^2 \\ (c_{L\max}+c_o)vu^2 + (c_{\max} + c_o)v^3 \end{pmatrix}$$

whose variation is with
$c_1 = c_{\max} + c_o - c_{\mathrm{Lmax}}$, $c_2 = c_{\mathrm{Lmax}} + c_o$, and
$c_3 = c_{\mathrm{imax}} + c_o$

$$\frac{\partial \mathbf{f}_A}{\partial \mathbf{u}} = -\frac{\rho S}{2\sqrt{u^2+v^2}}\left((u^2+v^2)\begin{pmatrix} 3c_o u^2 + c_1 v^2 & 2c_1 uv \\ 2c_2 vu & c_2 u^2 + 3c_3 v^2 \end{pmatrix} - \begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} c_o u^3 + c_1 uv^2 \\ c_2 vu^2 + c_3 v^3 \end{pmatrix}\right)$$
$$= -\frac{\rho S}{2}\sqrt{u^2+v^2}\begin{pmatrix} 2c_o u^4 + 3c_o u^2 v^2 + c_1 v^4 & (-c_o + 2c_1)\, u^3 v + c_1 uv^3 \\ c_2 vu^3 + (-c_3 + 2c_2)\, v^3 u & c_2 u^4 + 3c_3 v^2 u^2 + 2c_3 v^4 \end{pmatrix}$$
$$= -\frac{\rho S}{2}\sqrt{u^2+v^2}\begin{pmatrix} 2c_o \cos^4\alpha + 3c_o \cos^2\alpha \sin^2\alpha + c_1 \sin^4\alpha & (c_o - 2c_1)\cos^3\alpha \sin\alpha + c_1 \cos\alpha \sin^3\alpha \\ -c_2 \sin\alpha \cos^3\alpha + (c_3 - 2c_2)\sin^3\alpha \cos\alpha & c_2 \cos^4\alpha + 3c_3 \sin^2\alpha \cos^2\alpha + 2c_3 \sin^4\alpha \end{pmatrix}$$

maximal lift coefficient $c_{Lmax} > 0$
maximal drag coefficient $c_{Dmax} > 0$
air density ρ
wing area S

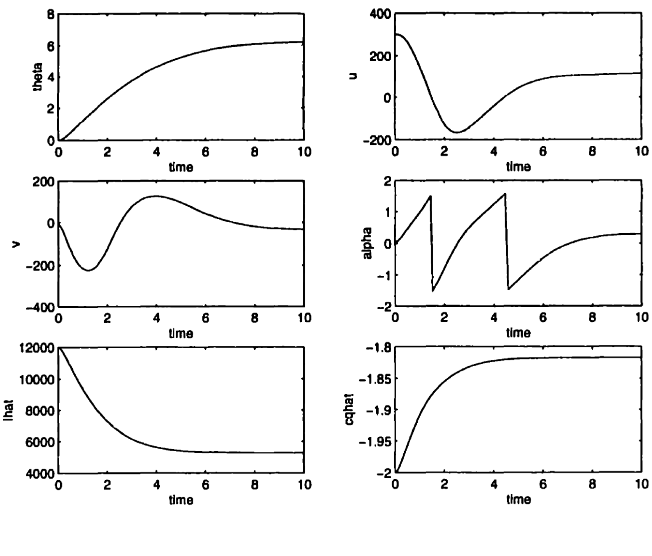
The contraction behavior of the translational motion is only a function of the angle of attack α. Consider now a typical high performance aircraft. The eigen values of the symmetric part of $\frac{\partial f_A}{\partial u}$ divided by $\frac{1}{2}\rho S(\sqrt{u^2+v^2})$ are illustrated in the figure as a function of α. The system is contracting as both the eigenvalues are negative. Note: With higher angle of attack the contraction increases with increase in energy dissipation. Consider the rotational dynamics

$$I\ddot{\theta} = -\frac{1}{2}(\rho S\sqrt{u^2+v^2}c_q \bar{c}^2\dot{\theta}) + \eta$$

The

$$\eta = -\lambda \hat{I}\dot{\theta} + \frac{1}{2}\rho S\sqrt{u^2+v^2}\hat{c}_q \bar{c}^2\dot{\theta} - K_D s \dot{\hat{I}} = \gamma_I \lambda \dot{\theta} s$$

$$\dot{\hat{c}}_q = \gamma_{c_q}\frac{1}{2}\rho S\sqrt{u^2+v^2}\bar{c}^2\dot{\theta} s$$



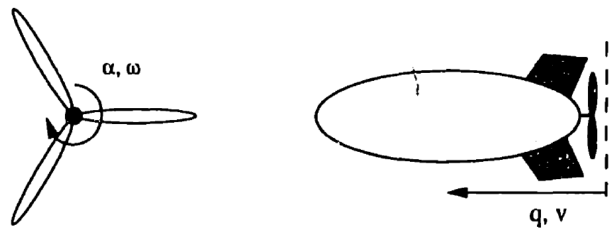
c. Time-Delayed Underwater Vehicle Controller

Consider the simple underwater vehicle model below

$$\begin{pmatrix} \dot{\omega} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \tau_\omega - c\omega|\omega| - 10\omega \\ -10v|v| + \omega|\omega| \end{pmatrix}$$

with unknown drag coefficient $c = 3$ and assume that the underwater vehicle is controlled over a time-delayed sonar transmission channel, with

$$\dot{x} = -10x - (\tau_x - \tau_d)$$



where τ_x connects the master dynamics x to the vehicle dynamics, and τ_d implicitly specifies the desired vehicle velocity $v_d(t)$. Similarly we use wave variables

$$u_x = x + \tau_x$$
$$v_x = x - \tau_x$$
$$u_\omega = \omega + \tau_\omega$$
$$v_\omega = \omega - \tau_\omega$$

and transmit these

$$u_\omega = u_x(t-T)$$
$$v_x = v_\omega(t-T)$$

The open-loop term τ_d can be computed from the desired vehicle velocity $v_d(t)$ with $\omega_d + \tau_{\omega d}, v_{\omega d}(t) = \omega_d - \tau_{\omega d}, x_d(t) = \frac{1}{2}\left(u_{\omega d}(t+T) + v_{\omega d}(t-T)\right), \tau_{xd}(t) = \frac{1}{2}\left(u_{\omega d}(t+T) - v_{\omega d}(t-T)\right), \tau_d(t) = \dot{x}d + 10x_d + \tau xd$, where an omitted time index corresponds to time t. The unknown drag coefficient $\hat{c}$ can be adapted with

$$\dot{\hat{c}} = -\gamma W\,(x - x_d)$$

where $\gamma = 10^{-4}$ and

$$W = \frac{1}{2}\frac{d}{dt}\left(\omega_d(t+T)\,|\omega_d(t+T)| - \omega_d(t-T)|\omega(t-T)|\right)$$
$$+ 5\left(\omega_d(t+T)\,|\omega_d(t+T)| - \omega_d(t-T)|\omega(t-T)|\right)$$
$$+ \frac{1}{2}\left(\omega_d(t+T)\,|\omega_d(t+T)| + \omega_d(t-T)|\omega(t-T)|\right)$$

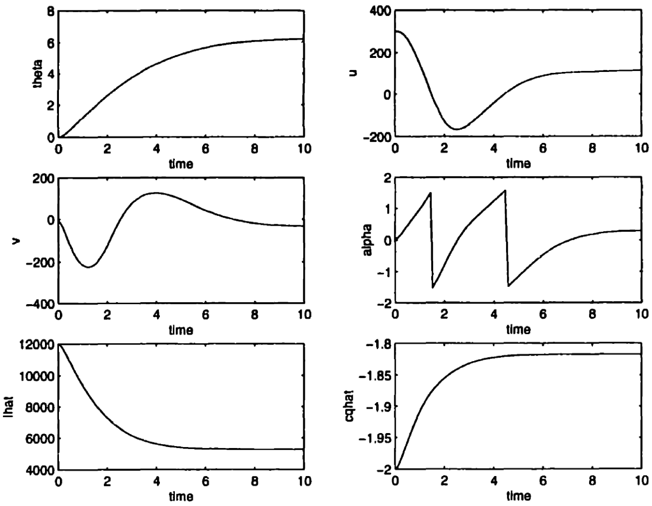
is the gain of $\hat{c}$ in the x dynamics. Due to the hierarchical structure of the underwater vehicle let us first analyze the contraction behavior of the propeller dynamics

$$\frac{1}{2}\frac{d}{dt}\left(\delta x^2+\delta\omega^2+\frac{1}{\gamma}\delta\hat{c}^2+\frac{1}{2}\int_{t-T}^{t}\left(\delta u_1^2+\delta v_2^2\right)d\tau\right)=-10\delta x^2-\delta\omega^2(10+c|\omega|)$$

which guarantees asymptotic tracking convergence of x and ω for positive unknown c. Since the velocity dynamics is contracting with $\frac{\partial\dot{v}}{\partial v}=-10|v|$ for $v\neq 0$ (which is left in finite time for the following open-loop input $\tau_d(t)$) asymptotic convergence of v and hence of the total system can be concluded.

Note that local feedback loops, as say an altitude stabilization controller, are irrelevant to the stability discussion since they are not transmitted over the timedelayed channel.

System responses for the initial conditions $\omega(0)=0, v(0)=0, x(0)=0, \hat{c}(0)=2$, and $v_\omega(0)=v_x(0)=u_\omega(0)=u_x(0)$ in the complete transmission channel. The dashed line represents the desired velocity and the solid line the actual system response. Note that the sharp wave refection attenuate asymtotically.



d. Under Water Vehicle Observer

Consider again the underwater vehicle in the figure below:

$$\begin{pmatrix}\dot{\omega}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}\tau-3\omega|\omega|\\ -10v|v|+\omega|\omega|\end{pmatrix}$$
$$\begin{pmatrix}\dot{\alpha}\\ \dot{q}\end{pmatrix}=\begin{pmatrix}\omega\\ v\end{pmatrix}$$

with propeller velocity ω, vehicle velocity v, measured propeller position α, measured vehicle position q, propeller thrust $\omega|\omega|$, propeller drag $-3\omega|\omega|$, vehicle drag $-10v|v|$, and torque input to the propeller $\tau(t)$. The system dynamics is heavily damped for large $|\omega|$ and $|v|$. However, this natural damping is ineffective at low velocities. This suggests using a coordinate error feedback in the reduced-order

$$\begin{pmatrix}\dot{\omega}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}\tau(t)-3\hat{\omega}|\hat{\omega}|-k_\omega\hat{\omega}\\ -10\hat{v}|\hat{v}|+\hat{\omega}|\hat{\omega}|-k_v\hat{v}\end{pmatrix}$$
$$\begin{pmatrix}\hat{\omega}\\ \hat{v}\end{pmatrix}=\begin{pmatrix}\bar{\omega}+k_\omega\alpha\\ \bar{v}+k_v q\end{pmatrix}$$

where

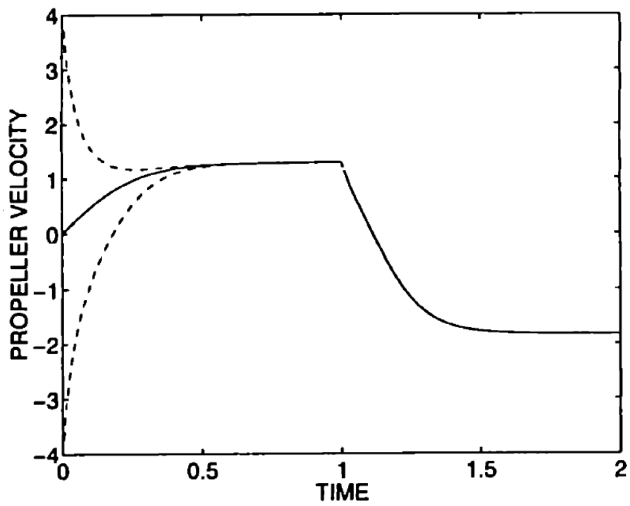
k$_\omega$ and k_v are strictly positive constants. This leads to the hierarchical dynamics

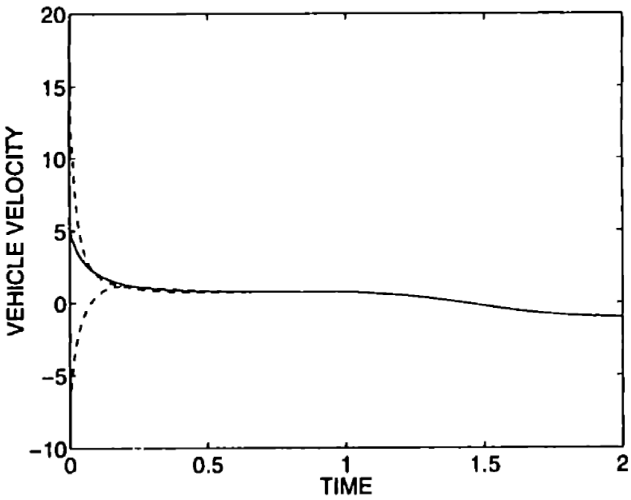
$$\begin{pmatrix}\dot{\hat{\omega}}\\ \dot{\hat{v}}\end{pmatrix}=\begin{pmatrix}\tau(t)-3\hat{\omega}|\hat{\omega}|-k_\omega(\hat{\omega}-\dot{\alpha})\\ -10\hat{v}|\hat{v}|+\hat{\omega}|\hat{\omega}|-k_v(\hat{v}-\dot{q})\end{pmatrix}$$

The uniform negative definiteness of $\frac{\partial\dot{\omega}}{\partial\hat{\omega}}=(-3|\hat{\omega}|-k_\omega)$ and $\frac{\partial\dot{\hat{v}}}{\partial\hat{v}}=(-10|\hat{v}|-k_v)$ (which is implied by our choice of strictly positive constants k_v and k_ω) guarantees exponential convergence to the actual system trajectory, which is indeed a particular solution. System responses to the input

$$\tau=\begin{pmatrix}5 & \text{for } 0\leq t<1\\ -10 & \text{for } 1\leq t<2\end{pmatrix}$$

with initial conditions $\omega(0)=0, \hat{\omega}(0)=4$ or $-4, v(0)=5, \hat{v}(0)=-10$ or 20 and feedback gains $k_v=k_\omega=5$. The solid line represents the actual plant and the dashed lines the observer estimates.

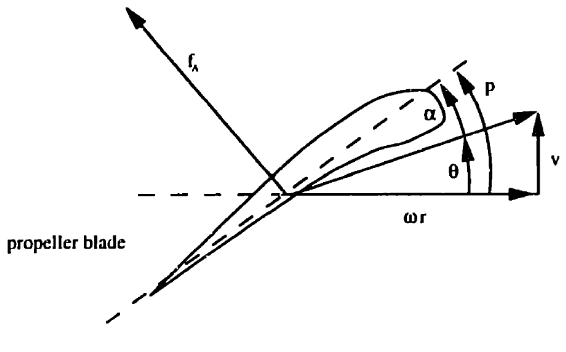




Let us illustrate observer design for mechanical systems on the underwater vehicle, but now using the recent detailed hydrodynamic model.

$$
\begin{aligned}
m\dot{v} &= T - c_D v|v| \\
I\dot{\omega} &= -k_1\omega + k_2 U - Q
\end{aligned}
$$

with propeller velocity ω, vehicle velocity v, vehicle mass $m = 400$ kg, propeller inertia $I = 0.01\mathrm{kgm}^2$, vehicle drag coefficient $c_D = 10^4\mathrm{Ns}^2/\mathrm{m}^2$, motor back-emf $k_1 = 0.01\mathrm{Nms}$, motor gain $k_2 = 0.1\mathrm{Nm/V}$, vehicle thrust T, and propeller drag Q. The vehicle thrust and propeller drag are Figure : Propeller geometry



$$
\begin{aligned}
T &= L\cos\theta - D\sin\theta \\
Q &= L\sin\theta + D\cos\theta
\end{aligned}
$$

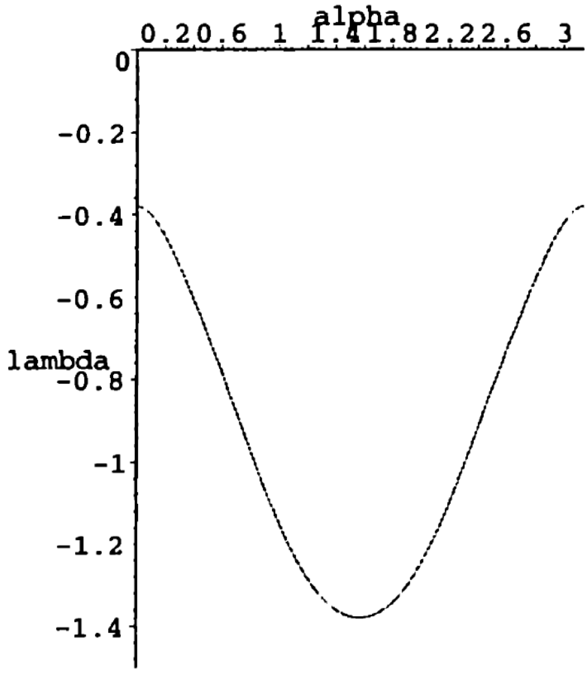
where ihe blade lift L and drag forces D can in turn be expressed as

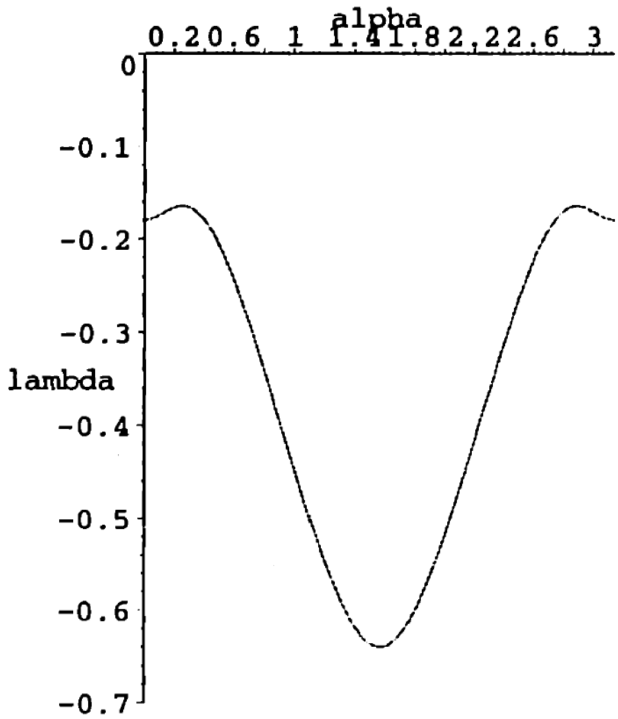
$$
\begin{aligned}
L &= 500\left(v^2 + r^2\omega^2\right) c_{L\max}\sin\alpha\cos\alpha \\
D &= 500\left(v^2 + r^2\omega^2\right)\left(c_o + c_{\mathrm{imax}}\ \sin^2\alpha\right)
\end{aligned}
$$

with angle of attack $\alpha = p - \theta$, blade angle $p = 0.4$rad, pitch angle $\theta = \arctan\frac{v}{r\omega}$, and effective propeller radius $r = 0.1$ m . Using inertia as the metric, the Jacobian of this dynamics is

$$
\mathbf{F} = \begin{pmatrix} -10000|v| & 0 \\ 0 & -0.01 \end{pmatrix} + \begin{pmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{pmatrix} \frac{\partial \mathbf{f}_A}{\partial \mathbf{u}} \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix}
$$

where $\frac{\partial \mathbf{f}A}{\partial \mathbf{u}}$ is given. The eigenvalues of the symmetric part of $\frac{\partial \mathrm{f}_A}{\partial \mathrm{u}}$ divided by $500\sqrt{v^2+0.01\omega^2}$ are illustrated in the figure, for $c_o = 0.1, c\mathrm{imax}\ = 2.5$,





Assuming that the vehicle position q and propeller angle α

measured, the following coordinate error feedback observer.

$$
\begin{aligned}
m\dot{\bar{v}} &= T - c_D\hat{v}|\hat{v}| - k_q\hat{v} \\
I\dot{\bar{\omega}} &= -k_1\hat{\omega} + k_2U - Q - k_\alpha\hat{\omega} \\
\hat{v} &= \bar{v} + k_q q \\
\hat{\omega} &= \bar{\omega} + k_\alpha \alpha
\end{aligned}
$$

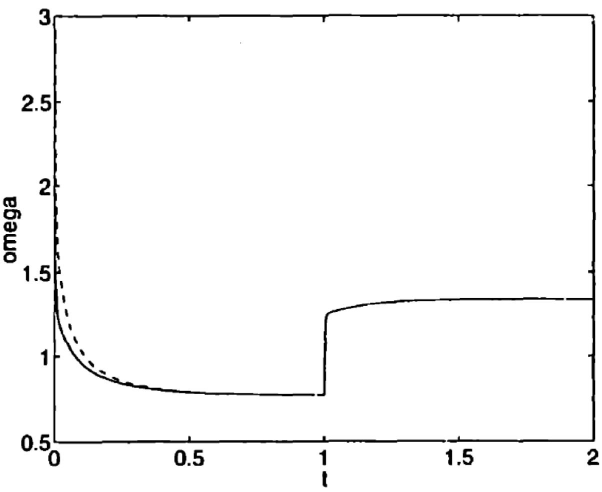
leads to the exponentially convergent observer dynamics

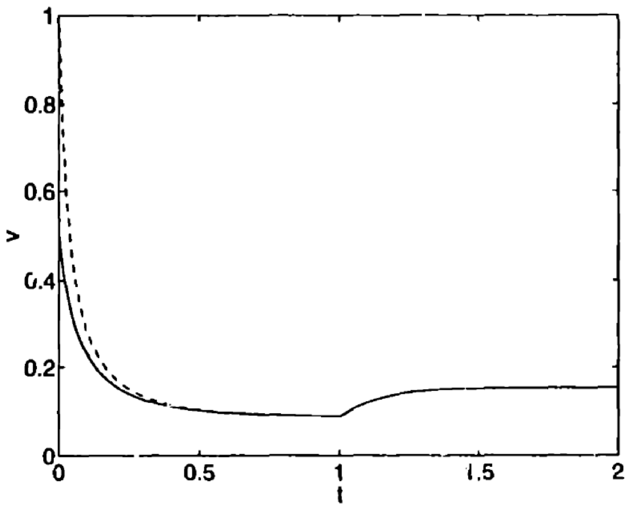
$$
\begin{aligned}
m\dot{\hat{v}} &= T - c_D\hat{v}|\hat{v}| - k_q(\hat{v} - \dot{q}) \\
I\dot{\hat{\omega}} &= -k_1\hat{\omega} + k_2U - Q - k_\alpha(\hat{\omega} - \dot{\alpha})
\end{aligned}
$$

hence augmenting the natural contraction behavior of the system. System responses to the input

$$
U = \begin{pmatrix} 100\ \mathrm{V} & \text{for } 0 \le t < 1 \\ 300\ \mathrm{V} & \text{for } 1 \le t < 2 \end{pmatrix}
$$

with initial conditions $v(0) = 0.5\ \mathrm{m/s}, \hat{v}(0) = 1\ \mathrm{m/s}, \omega(0) = 21/\mathrm{s}, \hat{\omega}(0) = 31/\mathrm{s}$ and feedback gains $k_q = 1\mathrm{Ns/m}$ and $k_\alpha = 1\mathrm{Nms}$. The solid line represents the actual plant and the dashed lines the observer estimates.





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