Control System Design for Mechanical Systems Using Contraction Theory

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***Abstract*—Contraction theory is a recently developed control system tool based on an exact differential analysis of convergence. After establishing new combination properties of contracting systems, this paper derives new controller and observer designs for mechanical systems such as aircraft and underwater vehicles. The classical nonlinear strap down algorithm is also shown to be marginally contracting. The relative simplicity of the approach stems from its effective exploitation of the systems' structural specificities.**

***Index Terms*—Contraction theory, mechanical systems, modular systems, nonlinear systems, stability theory.**

I. Introduction

Nonlinear system analysis has been very successfully applied to particular classes of systems and problems [7], [15], [8], [22], [21], [18]. In an attempt to systematically generalize its range of application, [11] derived a body of new results, referred to as *contraction analysis*, using elementary tools from continuum mechanics and differential geometry. This paper exploits these results to derive new controller and observer designs for classes of mechanical systems.

After a brief review of the basic results of [11] in Section II, Section III studies combination properties of contracting systems, which will be systematically exploited in the following developments. Section IV discusses nonlinear mechanical observer and controller design examples in the context of aircraft and underwater vehicles.

II. Contraction Analysis

Stability analysis using differential approximation is the basis of all linear control system design. What is new in contraction analysis is that differential stability analysis can be made *exact* and in turn yield global results on the nonlinear system. In this section, we summarize the basic results of [11], to which the reader is referred for more details.

We consider general deterministic systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) \tag{1}$$

where $\boldsymbol{f}$ is an $n \times 1$ nonlinear vector function and $\boldsymbol{x}$ is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $\boldsymbol{u}(\boldsymbol{x}, t)$. All quantities are assumed to be real and smooth, by which it is meant that any required derivative or partial derivative exists and is continuous.

The plant equation (1) can be thought of as an n-dimensional fluid flow, where $\dot{\boldsymbol{x}}$ is the n-dimensional "velocity" vector at the n-dimensional position $\boldsymbol{x}$ and time t. Assuming as we do that $\boldsymbol{f}(\boldsymbol{x}, t)$ is continuously differentiable, (1) yields the exact differential relation

$$\delta\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, t)\delta\boldsymbol{x} \tag{2}$$

where $\delta\boldsymbol{x}$ is a virtual displacement—recall that a virtual displacement is an infinitesimal displacement *at fixed time*. Note that virtual displacements, pervasive in physics and in the calculus of variations, are also well-defined mathematical objects [1], [20]. In particular, if one views the position of the system at time t as a smooth function of the initial condition $\boldsymbol{x}_o$ and of time, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_o, t)$, then one simply has $\delta\boldsymbol{x} = (\partial\boldsymbol{x}/\partial\boldsymbol{x}_o)\, d\boldsymbol{x}_o$.

The line vector $\delta\boldsymbol{x}$ can also be expressed using the differential coordinate transformation

$$\delta z = \boldsymbol{\Theta}\delta\boldsymbol{x} \tag{3}$$

where $\boldsymbol{\Theta}(\boldsymbol{x}, t)$ is a square matrix. This leads to

$$\delta z^T \delta z = \delta\boldsymbol{x}^T \boldsymbol{M} \delta\boldsymbol{x} \tag{4}$$

where $\boldsymbol{M}(\boldsymbol{x}, t) = \boldsymbol{\Theta}^T\boldsymbol{\Theta}$ represents a symmetric and continuously differentiable *metric*—formally, (4) defines a Riemann space [13]. Since (3) is in general not integrable, we cannot expect to find explicit new coordinates $z(\boldsymbol{x}, t)$, but δz and $\delta z^T\delta z$ can always be defined, which is all we need. We shall require $\boldsymbol{M}$ to be uniformly positive definite so that exponential convergence of δz to $\boldsymbol{0}$ also implies exponential convergence of $\delta\boldsymbol{x}$ to $\boldsymbol{0}$. Distance between two points P_1 and P_2 with respect to the metric $\boldsymbol{M}$ is defined as the shortest path length (i.e., the smallest path integral $\int_{P_1}^{P_2} \|\delta z\|$) between these two points. Accordingly, a ball of center $\boldsymbol{c}$ and radius R is defined as the set of all points whose distance to $\boldsymbol{c}$ with respect to $\boldsymbol{M}$ is strictly less than R.

Computing the dynamics of δz (or "virtual" dynamics)

$$\frac{d}{dt}\delta z = \boldsymbol{F}\delta z \qquad \text{where } \boldsymbol{F} = \left(\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)\boldsymbol{\Theta}^{-1} \tag{5}$$

we can state the following definition and main result [11].

Definition 1: Given the system equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$, a region of the state space is called a contraction region with respect to a uniformly positive definite metric $\boldsymbol{M}(\boldsymbol{x}, t) = \boldsymbol{\Theta}^T\boldsymbol{\Theta}$ if $\boldsymbol{F}$ in (5) is uniformly negative definite in that region.

Regions where $\boldsymbol{F}$ is negative semidefinite are called semi-contracting, and regions where $\boldsymbol{F}$ is skew-symmetric are called indifferent.

Theorem 1: Given the system equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$, any trajectory, which starts in a ball of constant radius with respect to the metric $\boldsymbol{M}(\boldsymbol{x}, t)$, centered at a given trajectory and contained at all times in a contraction region with respect to $\boldsymbol{M}(\boldsymbol{x}, t)$, remains in that ball and converges exponentially to this trajectory.

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region with respect to the metric $\boldsymbol{M}(\boldsymbol{x}, t)$.

The generality of contraction analysis, as compared to related classical results [9], [5], [6], stems from its use of pure differential analysis, and specifically of a pure *differential coordinate transformation*, leading to a necessary and sufficient characterization of exponential convergence. Indeed, it can be shown conversely that the existence of a uniformly positive definite metric with respect to which the whole state space is a contraction region is actually a necessary condition for global exponential convergence. In the linear time-invariant case, a system is globally contracting if and only if it is strictly stable, with $\boldsymbol{F}$ simply being a normal Jordan form of the system and $\boldsymbol{\Theta}$ the coordinate transformation to that form.

III. Combinations of Contracting Systems

The classical passivity formalism [19] analyzes combinations of systems of the form

$$\dot{V}_i = \boldsymbol{y}_i^T \boldsymbol{u}_i - g_i$$

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with positive V_i, input $\boldsymbol{u}_i$ and output $\boldsymbol{y}_i$, and $g_i \geq 0$ (see, e.g., [21]). Parallel or feedback combinations lead to the augmented Lyapunov dynamics (for zero overall input)

$$\frac{d}{dt}\sum_i V_i = -\sum_i g_i.$$

In geometric terms, this simply corresponds to constructing a total length $\sum_i V_i$ out of length elements V_i. This section discusses related differential closure properties of contracting systems. Results from [11] are first summarized in Section III-A (detailed proofs can be found in [10]), while Sections III-B and III-C derive new system combination properties.

A. Basic Combinations

Parallel Combination: Consider two systems of the same dimension, and their associated virtual dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_i(\boldsymbol{x}, t) \qquad \frac{d}{dt}\delta z = \boldsymbol{F}_i \delta z \qquad i = 1, 2$$

and connect them in a parallel combination. If both systems are contracting *in the same metric*, so is any uniformly positive superposition ($\exists\, \alpha > 0,\ \forall t \geq 0,\ \alpha_i(t) \geq \alpha$)

$$(\alpha_1(t)\boldsymbol{F}_1 + \alpha_2(t)\boldsymbol{F}_2). \tag{6}$$

Furthermore, if the metric does not depend explicitly on time, then the direct superposition

$$\dot{\boldsymbol{x}} = \alpha_1(t)\boldsymbol{f}_1(\boldsymbol{x}, t) + \alpha_2(t)\boldsymbol{f}_2(\boldsymbol{x}, t)$$

is contracting with the same metric.

Feedback Combination: Consider two systems, of possibly different dimensions

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$$
$$\dot{\boldsymbol{x}}_2 = \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$$

and connect them in the feedback combination

$$\frac{d}{dt}\begin{pmatrix}\delta z_1\\ \delta z_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{F}_1 & \boldsymbol{G}\\ -\boldsymbol{G}^T & \boldsymbol{F}_2\end{pmatrix}\begin{pmatrix}\delta z_1\\ \delta z_2\end{pmatrix}.$$

Then the augmented system is contracting if the individual plants are contracting.

Hierarchical Combination: Consider a smooth virtual dynamics of the form

$$\frac{d}{dt}\begin{pmatrix}\delta z_1\\ \delta z_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{F}_{11} & \boldsymbol{0}\\ \boldsymbol{F}_{21} & \boldsymbol{F}_{22}\end{pmatrix}\begin{pmatrix}\delta z_1\\ \delta z_2\end{pmatrix}.$$

The first equation does not depend on the second, so that exponential convergence of δz_1 to zero can be concluded for uniformly negative definite $\boldsymbol{F}_{11}$. In turn, for *bounded* $\boldsymbol{F}_{21}$, $\boldsymbol{F}_{21}\delta z_1$ represents an exponentially decaying disturbance in the second equation. Thus, uniform negative definitiveness of $\boldsymbol{F}_{22}$ implies exponential convergence of δz_2 to zero, so that the augmented system is contracting as well.

By recursion, the result can be extended to systems similarly partitioned in more than two equations. Variations can also be derived in the case that some subsystems are only indifferent rather than contracting. For instance, if $\boldsymbol{F}_{11}$ is skew-symmetric rather than negative definite, then the dynamics in δz_1 is indifferent. Hence, for bounded $\boldsymbol{F}_{21}$, $\boldsymbol{F}_{21}\delta z_1$ represents a bounded disturbance in the second equation. A skew-symmetric $\boldsymbol{F}_{22}$ then leads to a bound on $\|\delta z_2\|$ which increases linearly with time. By analogy with linear analysis, we will call such systems marginally contracting.

Translation and Scaling: It is straightforward to show that if $\boldsymbol{f}(\boldsymbol{x}, t)$ defines a contracting dynamics with respect to a *constant* $\boldsymbol{\Theta}$, so does any scaled and translated version $\boldsymbol{f}(a(t)\boldsymbol{x} - b(t), t)$, where $a(t)$ and $b(t)$ are arbitrary differentiable functions and $a(t)$ is uniformly positive definite. This property, combined with the parallel combination property above, can allow contracting dynamics to be used as wavelet-like basis functions in problems of dynamic approximation, estimation, and adaptive control.

B. Adaptation

It is straightforward to incorporate adaptive techniques in contraction-based designs if part of the system's uncertainty consists of unknown but constant (or slowly-varying) parameters $\boldsymbol{a}$. For instance, consider a closed-loop plant error dynamics

$$\dot{\tilde{z}} = \boldsymbol{f}(z, t) - \boldsymbol{f}(z_d, t) + \boldsymbol{W}(z, t)\boldsymbol{a} - \boldsymbol{W}(z, t)\hat{\boldsymbol{a}}$$

with parameter estimate vector $\hat{\boldsymbol{a}}$, state vector z, desired state vector z_d, and $\tilde{z} = z - z_d$. Letting $\tilde{\boldsymbol{a}} = \hat{\boldsymbol{a}} - \boldsymbol{a}$, and choosing the parameter adaptation

$$\dot{\tilde{\boldsymbol{a}}} = \dot{\hat{\boldsymbol{a}}} = -\boldsymbol{W}^T(z, t)\tilde{z}$$

Barbalat's lemma [21] and the Lyapunov-like analysis

$$\dot{V} = \frac{d}{dt}\left(\tilde{z}^T\tilde{z} + \tilde{\boldsymbol{a}}^T\tilde{\boldsymbol{a}}\right) = 2\tilde{z}^T\int_o^1 \frac{\partial \boldsymbol{f}}{\partial z}(z_d + \lambda\tilde{z})\, d\lambda\, \tilde{z}$$

show asymptotic convergence of $\tilde{z}$ to zero for uniformly negative definite $\partial\boldsymbol{f}/\partial z$ and bounded $\ddot{V}$.

C. Time-Delayed Transmission Channels

Many practical applications involve multiple systems with time-delayed feedback connections, due to transmission or computation delays. In robotics for instance, this is the case in force-reflecting teleoperation, underwater vehicle control through acoustic transmissions, and hand–eye coordination. Similar questions occur in routing and scheduling of large communication networks.

Consider two such contracting systems, of possibly different dimensions (Fig. 1)

$$\dot{z}_i = \boldsymbol{f}_i(z_i, t) + \boldsymbol{G}_i\tau_i \qquad i = 1, 2$$

where the $\boldsymbol{G}_i$ are constant and τ_1 and τ_2 have the same dimension. While delays are inherent to the system physics or the computational limitations, the designer is free to choose which variables are actually transmitted. Directly inspired by the use of wave variables in force-reflecting teleoperation [17], define intermediate variables

$$\boldsymbol{u}_i = \boldsymbol{G}_i^T z_i + \boldsymbol{K}\tau_i \qquad i = 1, 2$$
$$\boldsymbol{y}_i = \boldsymbol{G}_i^T z_i - \boldsymbol{K}\tau_i \qquad i = 1, 2$$

where $\boldsymbol{K}$ is a constant symmetric positive definite matrix, and transmit these in place of the more obvious $\boldsymbol{G}_i^T z_i$

$$\boldsymbol{u}_1(t) = \boldsymbol{y}_2(t - T_2) \qquad \boldsymbol{u}_2(t) = \boldsymbol{y}_1(t - T_1).$$

The rate of change of differential length can then be computed similarly to [17]

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1,2}\left(\delta z_i^T\delta z_i + \frac{1}{2}\int_{t-T_i}^t \delta\boldsymbol{y}_i^T\boldsymbol{K}^{-1}\delta\boldsymbol{y}_i\, d\tau\right) = \sum_{i=1,2}\delta z_i\,\frac{\partial \boldsymbol{f}_i}{\partial z_i}\,\delta z_i.$$

We have used

$$\sum_{i=1,2}\left(\int_{t-T_i}^t \delta\boldsymbol{y}_i^T\boldsymbol{K}^{-1}\delta\boldsymbol{y}_i\, d\tau - \int_{-T_i}^0 \delta\boldsymbol{y}_i^T\boldsymbol{K}^{-1}\delta\boldsymbol{y}_i(\tau)\, d\tau\right) = \sum_{i=1,2}\int_o^t\left(\delta\boldsymbol{y}_i^T\boldsymbol{K}^{-1}\delta\boldsymbol{y}_i(\tau) - \delta\boldsymbol{y}_i^T\boldsymbol{K}^{-1}\delta\boldsymbol{y}_i(\tau - T_i)\right) d\tau$$

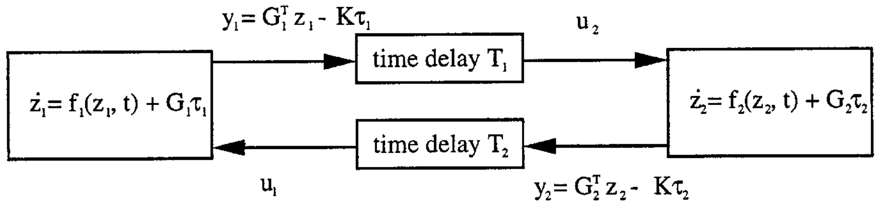

Fig. 1. Time-delayed transmission channel.

$$= \sum_{i=1,2} \int_o^t \left(\delta \boldsymbol{y}_i^T \boldsymbol{K}^{-1} \delta \boldsymbol{y}_i(\tau) - \delta \boldsymbol{u}_i^T \boldsymbol{K}^{-1} \delta \boldsymbol{u}_i(\tau) \right) d\tau$$
$$= -4 \sum_{i=1,2} \int_o^t \delta z_i^T \boldsymbol{G}_i \delta \boldsymbol{\tau}_i \, d\tau$$

where the time arguments apply to the whole dot-products and $\boldsymbol{y}_1 = \boldsymbol{y}_2 = 0, \forall t \leq 0$.

Integrating the above from time 0 to time t leads to

$$\frac{1}{2} \sum_{i=1,2} \left(\delta z_i^T \delta z_i(t) - \delta z_i^T \delta z_i(0) \right) \leq \sum_{i=1,2} \int_o^t \delta z_i \frac{\partial \boldsymbol{f}_i}{\partial z_i} \delta z_i \, d\tau$$

since $\sum_{i=1,2} \int_{t-T}^t \delta \boldsymbol{y}_i^T \boldsymbol{K}^{-1} \delta \boldsymbol{y}_i \, d\tau$ is always positive. Furthermore, Barbalat's lemma shows asymptotic convergence of the δz_i to zero.

The derivation can be extended straightforwardly to feedback loops composed of more than two systems. In the special case of no delays, the above reduces to $\boldsymbol{G}_1^T z_1 = \boldsymbol{G}_2^T z_2$ and $\tau_1 = -\tau_2$. Finally, some extra flexibility in the choice of intermediate transmission variables can be obtained by noticing that contraction is preserved through any orthonormal coordinate change on the z_i, and that the inputs τ_i can be redefined through any constant invertible transformation. In the case that the individual plants are autonomous, the system thus tends towards a unique equilibrium [11], which is independent of the delays. Furthermore, if constant external inputs are introduced, the system may be viewed as performing, in a distributed fashion, the associated algebraic computations $\dot{z}_i = 0, \forall i$, with constraints of the form $\boldsymbol{u}_i = \boldsymbol{y}_j$.

It is interesting to remark (generalizing the discussion on motion primitives in [11]) that biological systems, through the processes of evolution and development, are themselves accumulations of simpler "stable" elements (e.g., [3]). The discussion on the preservation of contraction through system combinations, together with the above result on transmission delays (by analogy with nerve transmission delays, see, e.g., [16]), show that contracting systems may be attractive candidates as basic building blocks for biological models or robots.

IV. Mechanical Systems

This section presents immediate applications of the above discussion to hierarchical, unconstrained mechanical control and estimation problems. Corresponding extensive simulations can be found in [12] and [10].

A. Strap Down Algorithm

We first illustrate the use of differential coordinate changes. Consider the Euler dynamics of a rigid body, with Euler angles $\boldsymbol{x} = (\psi, \theta, \phi)^T$ and rotation vector $\omega = (\omega_1, \omega_2, \omega_3)^T$ in body coordinates [4]

$$\dot{\boldsymbol{x}} = \boldsymbol{H}^{-1} \omega \tag{7}$$

with

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \cos\theta \sin\psi \\ 0 & -\sin\psi & \cos\theta\cos\psi \end{pmatrix}.$$

Analyzing the differential angular displacement $\delta z = \boldsymbol{\Theta} \delta \boldsymbol{x}$ in inertial coordinates [4], with

$$\boldsymbol{\Theta} = \boldsymbol{A}\boldsymbol{H}$$

where $\boldsymbol{A}$ is the orthonormal transformation matrix from body coordinates to inertial coordinates, as shown at the bottom of this page, leads after a straightforward but tedious calculation to the generalized Jacobian

$$\boldsymbol{F} = \boldsymbol{0}.$$

Thus, the Euler dynamics (7) is indifferent. Note that this derivation entirely relies on the use of differential coordinates δz, since z itself does not exist.

In inertial navigation, the classical strap down algorithm measures the body turn rate ω and the inertial acceleration in body coordinates and then combines (7) with

$$\dot{\boldsymbol{v}} = \boldsymbol{A} \qquad \dot{\boldsymbol{r}} = \boldsymbol{v} \tag{8}$$

where $\boldsymbol{r}$ are inertial coordinates, and $\boldsymbol{v}$ the corresponding velocities. Recognizing that (7), (8) represent a hierarchy of three indifferent systems, and noticing that $\partial \boldsymbol{A} / \partial \boldsymbol{x}$ is bounded, this shows that the classical strap down algorithm is marginally contracting. This result extends the well-known analysis in the linearized case to the complete nonlinear dynamics.

Note that adding to (8) a linear spring, representing a combination of gravity and barometric feedback, then leads to a linear translational oscillation (Schuler oscillation) driven by $\boldsymbol{A}$.

B. An Aircraft Controller

Fig. 2 describes a simplified model of the longitudinal motion of a high-performance aircraft, possibly at high angle of attack

$$\ddot{\theta} = \frac{1}{I}(M + \eta)$$
$$\dot{\boldsymbol{u}} = \begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix} \boldsymbol{u} + \frac{1}{m} \boldsymbol{f}_A + \frac{1}{m} \begin{pmatrix} \xi \\ 0 \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ -\cos\theta \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} \cos\theta\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \cos\theta\sin\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{pmatrix}$$

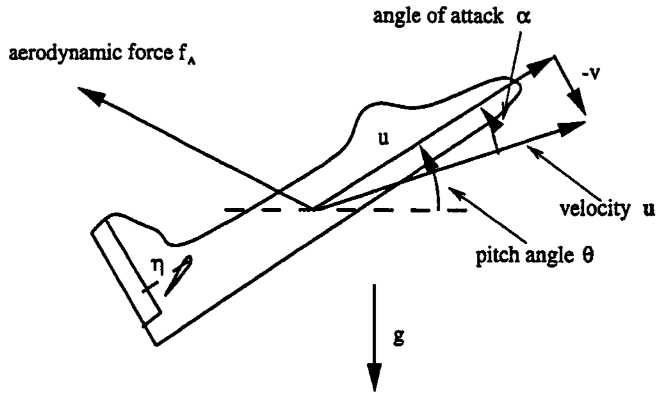

Fig. 2. Longitudinal aircraft dynamics.

with body-fixed velocity $\boldsymbol{u}$, pitch angle θ, mass m, inertia I, gravity constant g, constant thrust ξ, elevator torque η and external torque $M(\boldsymbol{u}, \theta, \dot{\theta})$ around the center of mass, and the aerodynamic force $\boldsymbol{f}_A(\boldsymbol{u}, \eta)$. The dynamic structure is hierarchical, reflecting the physics of the system. A simple rotational controller

$$\eta = -M + I\left(\dot{\theta}_d - \dot{\theta} + \theta_d - \theta + \ddot{\theta}_d\right)$$

guarantees exponential convergence of θ to θ_d. In turn, the virtual velocity dynamics

$$\delta\dot{\boldsymbol{u}} = \left[\begin{pmatrix} 0 & \dot{\theta} \\ -\dot{\theta} & 0 \end{pmatrix} + \frac{1}{m}\frac{\partial \boldsymbol{f}_A}{\partial \boldsymbol{u}}\right]\delta\boldsymbol{u}$$

is contracting under the mild physical assumption that $\partial\boldsymbol{f}_A/\partial\boldsymbol{u}$ is uniformly negative definite—this control-motivated requirement may actually guide aerodynamic design for high angle of attack aircraft. The convergence rate may be improved with a thrust controller. Exponential convergence to a desired trajectory in $\alpha_d(t) = -\arctan(v_d(t)/u_d(t))$ or $\boldsymbol{u}_d(t)$ can be guaranteed by selecting a corresponding time-varying $\theta_d(t)$.

Example 1: The aerodynamic force in body fixed coordinates can be computed from the lift and drag forces

$$\boldsymbol{f}_A = \begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}\begin{pmatrix} L \\ D \end{pmatrix}$$

with effective angle of attack $\alpha = -\arctan(v/u)$, and the lift and drag forces $L(\boldsymbol{u}, \eta)$ and $D(\boldsymbol{u}, \eta)$. A reasonable approximation for the α/π periodic lift force and $(2\alpha/\pi)$ periodic drag force is, with $\boldsymbol{u} = [u\ v]^T$

$$L = \frac{\rho S}{2}(u^2+v^2)c_{L\max}\sin\alpha\cos\alpha = -\frac{\rho S}{2}c_L uv$$
$$D = \frac{\rho S}{2}(u^2+v^2)(c_o + c_{i\max}\sin^2\alpha)$$

with air density ρ, wing area S, maximal lift coefficient $c_{L\max} > 0$, frictional drag coefficients $c_o > 0$, and maximal induced drag coefficient $c_{i\max} > 0$. It is then straightforward to compute the corresponding Jacobian $\partial\boldsymbol{f}_A/\partial\boldsymbol{u}$ as a function of the angle of attack α.

The eigenvalues of the symmetric part of $\partial\boldsymbol{f}_A/\partial\boldsymbol{u}$ divided by $(1/2)\rho S\sqrt{u^2+v^2}$ are illustrated in Figs. 3 and 4 as functions of $|\alpha|$ for a typical high-performance aircraft (see [10] for details). Since both eigenvalues are uniformly negative, the system is naturally contracting. Note that the contraction behavior increases with the energy dissipation at high angle of attack.

Consider now the rotational dynamics

$$I\ddot{\theta} = -\tfrac{1}{2}\rho S\sqrt{u^2+v^2}\,c_q\bar{c}^2\dot{\theta} + \eta$$

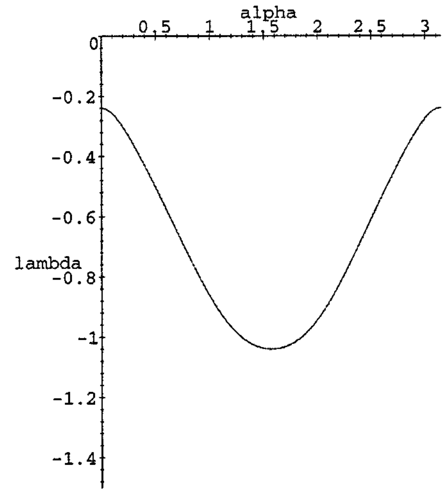

Fig. 3. Natural contraction behavior of aircraft. First eigenvalue.

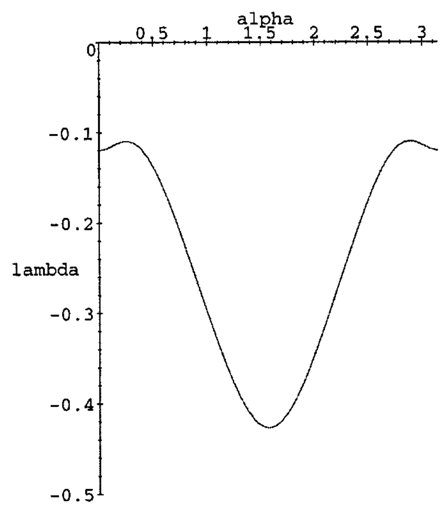

Fig. 4. Natural contraction behavior of aircraft. Second eigenvalue.

with $\bar{c}$ the reference length, and assume that the rotational inertia I and damping coefficient c_q are unknown. We can first design an adaptive stabilization controller [21] with sliding variable $s = \dot{\theta} + \lambda(\theta - 2\pi)$

$$\eta = -\lambda\hat{I}\dot{\theta} + \tfrac{1}{2}\rho S\sqrt{u^2+v^2}\,\hat{c}_q\bar{c}^2\dot{\theta} - K_D s$$
$$\dot{\hat{I}} = \gamma_I\lambda\dot{\theta}s \qquad \dot{\hat{c}}_q = \gamma_{c_q}\tfrac{1}{2}\rho S\sqrt{u^2+v^2}\,\bar{c}^2\dot{\theta}s$$

with λ, K_D, γ_I, and γ_{c_q} strictly positive constants. Asymptotic convergence of θ to 2π can be shown with the Lyapunov function candidate $V = (1/2)Is^2 + (1/2)\gamma_I(I-\hat{I})^2 + (1/2)\gamma_{c_q}(c_q - \hat{c}_q)^2$ whose time derivative is $\dot{V} = -K_D s^2$ and $\ddot{V}$ is bounded. The hierarchical structure of the system then implies asymptotic convergence of $\boldsymbol{u}$.eurofighter.

Note that in this approach nonminimum phase characteristics are irrelevant to the stability discussion, but rather they simply affect the planning of the desired trajectory. Additional unknown aerodynamic parameters appearing linearly can be adapted upon similarly to Section III-B. Also note that any stable rotational controller for the three-dimensional case also guarantees contraction behavior for the translational motion since for free-moving objects inertia forces always correspond to a skew-symmetric Jacobian. Similar derivations can be performed for depth control of underwater vehicles and planar control of ship motions.

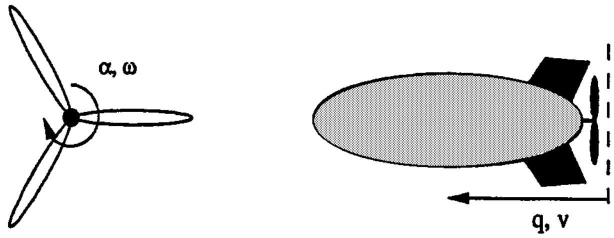

Fig. 5. Underwater vehicle.

C. Time-Delayed Underwater Vehicle Controller

As an application of Section III-C, consider the simple underwater vehicle model in Fig. 5

$$\begin{pmatrix} \dot{\omega} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \tau_\omega - c\omega|\omega| - 10\omega \\ -10v|v| + \omega|\omega| \end{pmatrix}$$

with unknown drag coefficient $c > 0$, and assume that the underwater vehicle is controlled over a time-delayed sonar transmission channel, with master dynamics

$$\dot{x} = -10x + \tau_x + r_d$$

where $r_d(t)$ implicitly specifies the desired vehicle velocity $v_d(t)$ (the numbers in this example are arbitrary and are simply meant to illustrate the procedure). Similarly to Section III-C, we use intermediate variables

$$u_x = x + \tau_x \qquad u_\omega = \omega + \tau_\omega$$
$$y_x = x - \tau_x \qquad y_\omega = \omega - \tau_\omega$$

and transmit these

$$u_x = y_\omega(t - T) \qquad u_\omega = y_x(t - T).$$

The open-loop term τ_d can be computed from the desired vehicle velocity $v_d(t)$ and its derivatives, assumed to be specified with a time advance of T, using a backward recursion through the system, namely $\omega_d(t) = \sqrt{|\dot{v}_d + 10v_d|v_d||}\text{sign}(\dot{v}_d + 10v_d|v_d|)$, $\tau_{\omega d}(t) = \dot{\omega}_d + \hat{c}\omega_d|\omega_d| + 10\omega_d$, $y_{\omega d}(t) = \omega_d - \tau_{\omega d}$, $u_{\omega d}(t) = \omega_d + \tau_{\omega d}$, $x_d(t) = (1/2)(y_{\omega d}(t-T) + u_{\omega d}(t+T))$, $\tau_{xd}(t) = (1/2)(y_{\omega d}(t-T) - u_{\omega d}(t+T))$, $r_d(t) = \dot{x}_d + 10x_d - \tau_{xd}$, where an omitted time index corresponds to time t. Based on Section III-B, the unknown drag coefficient $\hat{c}$ can be adapted locally at the vehicle site in a straightforward fashion. Alternatively, it can also be adapted remotely at the master site

$$\dot{\hat{c}} = -\gamma W(x - x_d)$$

where γ is a strictly positive constant and W is the gain of $\hat{c}$ in the x dynamics

$$\begin{aligned} W = \frac{1}{2}\frac{d}{dt}\left(\omega_d(t+T)|\omega_d(t+T)| - \omega_d(t-T)|\omega(t-T)|\right) \\ + 5\left(\omega_d(t+T)|\omega_d(t+T)| - \omega_d(t-T)|\omega(t-T)|\right) \\ + \frac{1}{2}\left(\omega_d(t+T)|\omega_d(t+T)| + \omega_d(t-T)|\omega(t-T)|\right). \end{aligned}$$

Using the hierarchical structure of the underwater vehicle, we can analyze the contraction behavior of the propeller dynamics using the results of Sections III-B and III-C

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\delta x^2 + \delta\omega^2 + \frac{1}{\gamma}\delta\hat{c}^2 + \frac{1}{2}\int_{t-T}^{t}(\delta y_x^2 + \delta y_\omega^2)\,d\tau\right) \\ = -10\delta x^2 - (10 + c|\omega|)\delta\omega^2 \end{aligned}$$

which guarantees asymptotic tracking convergence of x and ω. Since the velocity dynamics is contracting for $v \neq 0$ [which is true in finite time for the given $r_d(t)$], with $(\partial\dot{v}/\partial v) = -10|v|$, asymptotic convergence of v and hence of the total system can be concluded.

Note that local feedback loops, say a depth stabilization controller, do not affect this stability discussion (as long as they preserve the slave's contraction). Also, while in this simple application the option of remote (rather than local) adaptation may seem of mostly theoretical interest, in others it may be of more fundamental importance. This is the case for instance in more computationally involved implementations (high number of degrees of freedom, so-called "remote-brain" robotic applications), or in models of physiological motor control.

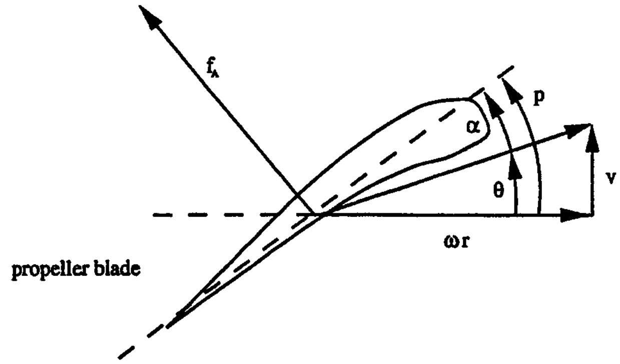

Fig. 6. Propeller geometry.

D. Underwater Vehicle Observer

Let us illustrate observer design for mechanical systems on the underwater vehicle of Fig. 5, but now using the recent detailed hydrodynamic model of [23]

$$\begin{aligned} m\dot{v} &= T - c_D v|v| \\ I\dot{\omega} &= -k_1\omega + k_2 U - Q \end{aligned}$$

with propeller velocity ω, vehicle velocity v, vehicle mass m, effective propeller mass I, vehicle drag coefficient c_D, motor back-emf k_1, motor gain k_2, vehicle thrust T, and propeller drag Q. One can write

$$\begin{aligned} T &= L\cos\theta - D\sin\theta \\ L &= 500(v^2 + r^2\omega^2)c_{L\max}\sin\alpha\cos\alpha \\ Q &= L\sin\theta + D\cos\theta \\ D &= 500(v^2 + r^2\omega^2)(c_o + c_{i\max}\sin^2\alpha) \end{aligned}$$

with L the blade lift, D the blade drag, $\alpha = p - \theta$ the angle of attack, p the blade angle, $\theta = \arctan(v/r\omega)$ the pitch angle, and r the effective propeller radius r (Fig. 6). Using mass as the metric, the Jacobian of this dynamics is

$$\begin{aligned} \boldsymbol{F} = &\begin{pmatrix} -10\,000|v| & 0 \\ 0 & -0.01 \end{pmatrix} \\ &+ \begin{pmatrix} \cos p & \sin p \\ -\sin p & \cos p \end{pmatrix} \frac{\partial \boldsymbol{f}_A}{\partial \boldsymbol{u}} \begin{pmatrix} \cos p & -\sin p \\ \sin p & \cos p \end{pmatrix} \end{aligned}$$

where $\boldsymbol{f}_A$ is defined exactly as in Example 1 of Section IV-B as a function of α, L, and D. As in that example, the eigenvalues of the symmetric part of $\partial\boldsymbol{f}_A/\partial\boldsymbol{u}$ can be shown to be uniformly negative, with contraction behavior increasing with energy dissipation at high angles of attack.

Assuming that the vehicle position q and propeller angle α are measured, while the velocities v and ω must be estimated, the reduced-order observer

$$\begin{aligned} m\dot{\overline{v}} &= T - c_D\hat{v}|\hat{v}| - k_q\hat{v} \qquad \hat{v} = \overline{v} + k_q q \\ I\dot{\overline{\omega}} &= -k_1\hat{\omega} + k_2 U - Q - k_\alpha\hat{\omega} \qquad \hat{\omega} = \overline{\omega} + k_\alpha\alpha \end{aligned}$$

leads to the exponentially convergent observer dynamics

$$m\dot{\hat{v}} = T - c_D \hat{v}|\hat{v}| - k_q(\hat{v} - \dot{q})$$
$$I\dot{\hat{\omega}} = -k_1\hat{\omega} + k_2 U - Q - k_\alpha(\hat{\omega} - \dot{\alpha})$$

hence exploiting and augmenting the natural contraction properties of the system.

REFERENCES

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. New York: Springer Verlag, 1978.
[2] R. Brockhaus, *Flight Control*. New York: Springer Verlag, 1994.
[3] R. Dawkins, *The Selfish Gene*. New York: Penguin, 1979.
[4] H. Goldstein, *Classical Mechanics*. New York: Addison-Wesley, 1980.
[5] W. Hahn, *Stability of Motion*. New York: Springer Verlag, 1967.
[6] P. Hartmann, *Ordinary Differential Equations*, 2nd ed. Boston, MA: Birkhauser, 1982.
[7] A. Isidori, *Nonlinear Control Systems*, 3rd ed: Springer Verlag, 1995.
[8] H. Khalil, *Nonlinear Systems*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[9] N. N. Krasovskii, *Problems of the Theory of Stability of Motion* (in English translation by Stanford Univ. Press, 1963). Moscow, Russia: Mir, 1959.
[10] W. Lohmiller, “Contraction analysis for nonlinear systems,” Ph.D. dissertation, Dep. Mechanical Eng., M.I.T., 1998.
[11] W. Lohmiller and J. J. E. Slotine, “On contraction analysis for nonlinear systems,” *Automatica*, vol. 34, no. 6, 1998.
[12] ——, “Contraction analysis of mechanical systems,” M.I.T. Nonlinear Systems Laboratory, Rep. NSL-980 501, 1998.
[13] D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*. New York: Dover, 1989.
[14] D. G. Luenberger, *Introduction to Dynamic Systems*. New York: Wiley, 1979.
[15] R. Marino and T. Tomei, *Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[16] S. Massaquoi and J. J. E. Slotine, “The intermediate cerebellum may function as a wave variable processor,” *Neuroscience Lett.*, vol. 215, 1996.
[17] G. Niemeyer and J. J. E. Slotine, “Stable adaptive teleoperation,” *IEEE J. Oceanic Eng.*, vol. 16, pp. 152–162, Jan. 1991.
[18] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer Verlag, 1990.
[19] V. M. Popov, *Hyperstability of Control Systems*. New York: Springer-Verlag, 1973.
[20] L. Schwartz, *Analyze*. Paris, France: Hermann, 1993.
[21] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991.
[22] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1992.
[23] L. Withcomb and D. Yoerger, “Preliminary experiments in the dynamical control of marine thrusters—Part 1: Dynamical modeling,” *IEEE J. Oceanic Eng.*, vol. 23, 1998.

Exponential Stabilization of a Class of Unstable Bilinear Systems

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***Abstract*—This paper considers the control design of bilinear systems with multiplicative control inputs. Previous control designs for such systems normally assume that the open-loop bilinear system is (neutrally) stable. In this paper, a new nonlinear control design is proposed for open-loop *unstable* bilinear systems. The new control stabilizes the bilinear system globally and exponentially *if* a sufficient stability condition, which can be checked by off-line computer simulations in advance of the control, is satisfied.**

***Index Terms*—Bilinear system, exponential stability, global stability, multiplicative control, nonlinear control.**

I. INTRODUCTION

Bilinear systems have been of great interest in recent years. This interest arises from the fact that many real-world systems can be adequately approximated by a bilinear model. Real-world examples include engineering applications in nuclear, thermal, and chemical processes, and nonengineering applications in biology, socio-economics, immunology, and so on. Detailed reviews of bilinear systems and their control designs can be found in [1] and [2]. For a bilinear system whose control input is both multiplicative and additive [2], one can use linear state feedback control [3] to obtain *local* asymptotical stability. Other control designs, such as the bang-bang control [4] or the optimal control [5], [6], obtain *global* asymptotic stability, but they all assume that the open-loop system is either stable or neutrally stable. When the open-loop system is unstable, it is difficult to obtain global asymptotical stability except when *independent* additive and multiplicative control inputs [7] exist.

This paper considers the control design for bilinear systems with multiplicative control inputs only. For such bilinear systems, it has been shown that *quadratic* state feedback control [8]–[10] can achieve global asymptotical stabilization, and *normalized* quadratic state feedback control [11] achieves global exponential stabilization. However, they also restrict the open-loop system to be stable or neutrally stable. In this paper, an attempt is made to find a nonlinear control, based on the normalized quadratic state feedback control design in [11], that can achieve global exponential stabilization for certain open-loop unstable bilinear systems. Our results show that the proposed new control will stabilize the system if a sufficient stability condition, which can be checked by off-line computer simulations in advance of the control, is satisfied.

II. NONLINEAR CONTROL

Consider bilinear systems with multiplicative control inputs

$$\dot{x}(t) = Ax(t) + u(t)Nx(t), \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t)$ is a scalar control input, and $A \in R^{n\times n}$ and $N \in R^{n\times n}$ are constant square matrices. For simplicity, only the single-input case is treated; the results in this paper, however, can be easily extended to the multi-input case.

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