Sheet 5

Harsh Solanki, Pietro Dalbosco, Rakhshanda Naureen Ansari

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1 Problem I

Prove the central difference relation

$$y(x_0)'' = y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi).$$

To prove this relation we have to combine in a proper way the following two Taylor expansions:

$$y(x_0 + h) \equiv y_1 = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{(4)} + \frac{h^5}{120} y_0^{(5)} + \frac{h^6}{720} y_0^{(6)} + ... \tag{1}$$

$$y(x_0 - h) \equiv y_{-1} = y_0 - hy_0' + \frac{h^2}{2} y_0'' - \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{(4)} - \frac{h^5}{120} y_0^{(5)} + \frac{h^6}{720} y_0^{(6)} + ... \tag{2}$$

We get rid of the orders higher than the fourth one and take the sum of 1 and 2:

$$y_1 + y_{-1} = 2y_0 + h^2 y_0'' + \frac{h^4}{12} y_0^{(4)} \implies y_0'' = \frac{y_1 - 2y_0 + y_{-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi).$$

By considering the truncated series of both relation (1) and (2), and following the algebraic steps to prove our sought relation we get:

$$\sum_{i=4}^{\infty} \frac{h^i}{i!} f_0^{(i)} \implies E = -\sum_{i=4}^{\infty} \frac{h^{i-2}}{i!} f_0^{(i)}. \tag{3}$$

where i is even. But still we are not sure if we were asked to do this :(

2 Problem II

Prove the central difference relation

$$y(x_0)'' = y_0'' = \frac{-y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2}}{12h^2} + O(h^4).$$

To prove this relation we also have to take into account the following two Taylor expansions:

$$y(x_0 + 2h) \equiv y_1 = y_0 + 2hy_0' + 2h^2 y_0'' + \frac{4}{3} h^3 y_0''' + \frac{2}{3} h^4 y_0^{(4)} + \frac{4}{15} h^5 y_0^{(5)} + \frac{4}{45} h^6 y_0^{(6)} + ... \tag{4}$$

$$y(x_0 - 2h) \equiv y_{-1} = y_0 - 2hy_0' + 2h^2 y_0'' - \frac{4}{3} h^3 y_0''' + \frac{2}{3} h^4 y_0^{(4)} - \frac{h^5}{120} y_0^{(5)} + \frac{4}{45} h^6 y_0^{(6)} + ... \tag{5}$$

To obtain the sought relation we have to compute $16 \times (\text{eq.1} + \text{eq.2}) - (\text{eq.4} + \text{eq.5})$ up to the sixth order:

$$\begin{aligned} &16(y_1 + y_{-1}) - (y_2 + y_{-2}) = 30y_0 + 12h^2 y_0'' - \frac{6}{45} h^6 y_0^{(6)} \\ &\implies y_0'' = \frac{-y_2 + 16y_1 - 30y_0 + 16y_{-1} - y_{-2}}{12h^2} + \frac{h^4}{90} y_0^{(6)}. \end{aligned} \tag{6}$$

By considering the truncated series of all the four Taylor expansions involved, and following the algebraic steps to prove our sought relation we get:

$$16 \sum_{i=6}^{\infty} \frac{h^i}{i!} f_0^{(i)} - \sum_{i=6}^{\infty} \frac{(2h)^i}{i!} f_0^{(i)} \implies E = -\frac{4}{3} \sum_{i=6}^{\infty} \frac{h^{i-2}}{i!} f_0^{(i)} + \frac{1}{12} \sum_{i=6}^{\infty} \frac{2^i h^{i-2}}{i!} f_0^{(i)}. \tag{7}$$

where i is even.