Problem I

1. Consider Taylor's expansion:

$$y(x) = y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{6}y'''(x_0) + \dots \quad \text{— (1)}$$

The RK3 method approximates y_{n+1} using:

$$y_{n+1} = y_n + h[\alpha_0 f(x_n, y_n) + \alpha_1 f(x_n + A_1 h, y_n + \beta_1 h) + \alpha_2 f(x_n + A_2 h, y_n + \beta_2 h) + \dots]$$

or $y_{n+1} = y_n + [\alpha_0 k_0 + \alpha_1 k_1 + \alpha_2 k_2 + \dots + \alpha_m k_m]$

where,

$$\left.\begin{aligned} k_0 &= hf(x_n, y_n) \\ k_1 &= hf(x_n + A_1 h, y_n + B_{10}k_0) \\ k_2 &= hf(x_n + A_2 h, y_n + B_{20}k_0 + B_{21}k_1) \\ \dots &= \dots \\ k_m &= hf(x_n + A_m h, y_n + B_{m0}k_0 + B_{m1}k_1 + \dots + B_{m\,m-1}k_{m-1}) \end{aligned}\right\} \text{— (2)}$$

let $y'(x) = \frac{dy}{dx} = f(x,y)$

$$y''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}f(x,y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \equiv f_x + f\cdot f_y = f'(x_n, y_n)$$

eq^n (1): $y_{n+1} = y_n + hf_n + \frac{h^2}{2}(f_x + f f_y)_{x=x_n}$ — (3)

Another possible recurrence relation can be: $y_{n+1} = y_n + ak_0 + bk_1$ — (4)

where, $k_0 = hf(x_n, y_n)$

$k_1 = hf(x_n + Ah, y_n + Bk_0)$

$$y_{n+1} = y_n + ahf(x_n, y_n) + bhf[x_n + Ah, y_n + Bhf(x_n, y_n)] \quad \text{— (5)}$$

and, $f[x_n + Ah, y_n + Bhf(x_n, y_n)] \approx [f + (Ah)f_x + (Bfh)f_y]_{x=x_n}$

eq (5): $y_{n+1} = y_n + ahf_n + bh[f_n + (Ah)f_x + (Bfh)f_y]$

$$= y_n + h(a+b)f_n + h^2(Abf_x + Bbff_y)_n$$

Comparing with (1): $a+b=1$, $A\cdot b = \frac{1}{2}$, $Bb = \frac{1}{2}$

let $a = \frac{1}{2} \Rightarrow b = \frac{1}{2} \Rightarrow A = B = 1$

$\therefore$ eq (4): $y_{n+1} = y_n + \frac{1}{2}k_0 + \frac{1}{2}k_1 \Rightarrow$ ~~$y_n + \frac{1}{2}(k_0+k_1)$~~ $y_{n+1} = y_n + \frac{1}{2}(k_0+k_1)$ where $k_0 = hf(x_n, y_n)$, $k_1 = hf(x_n+h, y_n+k_0)$

Proceeding to RK3:

expanding eq^{n}s. (2):

$$k_0 = hf$$
$$k_1 = h[f + hA_1 f_x + hB_{10} f_y f + O(h^2)]$$
$$k_2 = h[f + hA_2 f_x + (hB_{20} f_y f + hB_{21} f_y k_1) + O(h^2)]$$

$$\Rightarrow k_2 = h\{f + hA_2 f_x + [B_{20}hf_y f + h^2 B_{21} f_y(f + hA_1 f_x + hB_{10} f_y f)]\}$$

$$= h\{f + hA_2 f_x + hf_y[B_{20}f + B_{21}h(f + hA_1 f_x + hB_{10} f_y f)]\} \quad \text{— (6)}$$

Considering the recurrence relation: $y_{n+1} = y_n + ak_0 + bk_1 + ck_2$

$$\Rightarrow y_{n+1} = y_n + [af + b(f + A_1 hf_x + hB_{10} f_y f) + c(f + hA_2 f_x + hf_y(B_{20}f + B_{21}(f + hA_1 f_x + hB_{10} f_y f)))]h$$

$$= y_n + [(a+b+c)f + (hbA_1 + hcA_2)f_x + (bB_{10}h + hcB_{20} + hcB_{21})f_y f + h^2 cB_{21}A_1 f_y f_x + \underbrace{h^2 B_{21}B_{10} f_y^2 f}_{\text{neglecting this term}}]h$$

Comparing the coefficients with eq (1):

$a + b + c = 1$ — (coeff. of f).

$bA_1 + cA_2 = \frac{1}{2}$ — (coeff. of f_x)

$bB_{10} + c(B_{20} + B_{21}) = \frac{1}{2}$ — (coeff. of $f_y f$)

$cB_{21}A_1 = \frac{1}{6}$ — (coeff. of $f_y f_x$)

let $A_1 = \frac{1}{2}$, $B_{10} = \frac{1}{2}$, $A_2 = \frac{3}{4}$, $B_{20} = 0$, $B_{21} = \frac{3}{4}$

$\Rightarrow b \cdot \frac{1}{2} + c \cdot \frac{3}{4} = \frac{1}{2}$, $b \cdot \frac{1}{2} + c\left(0 + \frac{3}{4}\right) = \frac{1}{2}$, $c \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{1}{6}$ $\Rightarrow \boxed{c = \frac{4}{9}}$

$\Rightarrow b \cdot \frac{1}{2} = \frac{1}{2} - \frac{3}{4} \cdot \frac{4}{9}$ $\Rightarrow \boxed{b = \frac{1}{3}}$

and $a + b + c = 1$ $\Rightarrow a = 1 - \frac{1}{3} - \frac{4}{9}$ $\Rightarrow \boxed{a = \frac{2}{9}}$

Now our recurrence relation becomes (eq (6)'):

$y_{n+1} = y_n + \frac{2}{9} k_0 + \frac{1}{3} k_1 + \frac{4}{9} k_2$ $\Rightarrow \boxed{y_{n+1} = y_n + \frac{1}{9}(2k_0 + 3k_1 + 4k_2)}$

where, $k_0 = h f(x, y)$
$k_1 = h f\left(x + \frac{1}{2} h, y + \frac{1}{2} k_0\right)$
$k_2 = h f\left(x + \frac{3}{4} h, y + \frac{3}{4} k_1\right)$

3. For $y' = f(x)$, RK4 takes the form:

$k_1 = h f(x_n)$
$k_2 = h f\left(x_n + \frac{h}{2}\right)$
$k_3 = h f\left(x_n + \frac{h}{2}\right)$
$k_4 = h f(x_n + h)$

Thus
$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) + O(h^5)$$
$$= y_n + \frac{h}{6}\left[f(x_n) + 2f\left(x_n + \frac{h}{2}\right) + 2f\left(x_n + \frac{h}{2}\right) + f(x_n + h)\right] + O(h^5)$$
$$= y_n + \frac{h}{6}\left[f(x_n) + 4f\left(x_n + \frac{h}{2}\right) + f(x_n + h)\right] + O(h^5)$$

$$\Rightarrow \boxed{y_{n+1} - y_n = \int_{x_n}^{x_n + h} f(x)\, dx = \frac{h}{6}\left[f(x_n) + 4f\left(x_n + \frac{h}{2}\right) + f(x_n + h)\right] + O(h^5)}$$

which is exactly the Simpson's method.