Sheet 2, Exercise 2

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1 Exercise 2

1. Write a function that solves the linear system of equations $\mathbf{M} \cdot \vec{x} = \vec{b}$ using Gaussian elimination.
2. Use the above program to solve the system of equations

$$\begin{pmatrix} 2.0 & 0.1 & -0.2 \\ 0.05 & 4.2 & 0.032 \\ 0.12 & -0.07 & 5.0 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \tag{1}$$

and verify the solution.
3. Solve the following system of equations without pivoting:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & -2 \\ 0 & 3 & 15 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 1 \\ -2 \\ 33 \end{pmatrix}. \tag{2}$$

Record your result, then add pivoting to the implementation. How does the answer compare between pivoting and no pivoting?

We choose to solve this exercise in Fortran 90. We started the code writing a section in which a matrix multiplication is performed in order to get familiar with the language.

In particular we compute the following product:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}.$$

We obtain the result: $\begin{pmatrix} 68 \\ 167 \\ 266 \end{pmatrix}$ which is the correct result. To compute that we define a function performing the multiplication with a basic nested loop.

Then, we pick and solve the following system of equations as a test:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 3 & -2 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 6 \\ -3 \\ -2 \\ 0 \end{pmatrix}. \tag{3}$$

We directly implement pivoting.

Code structure:

- Initialize the matrix and the vector
- Loop over current column elements (index from i to n-1, where n is the vector dimension)
  - If the current pivot is zero (i.e. its absolute value is smaller than tolerance=10^{-6}) then look at below elements of the column and select the one having the largest magnitude

- If the selected element is zero then
  - If i=n-1 then exit (we are done), otherwise cycle (go directly to the next step of the loop)

- Swap the row identified by the new pivot and the row of the old pivot
- Swap the corresponding vector elements
- Perform operations with the rows to form an upper-triangular matrix by means of a loop over the rows (index k identifying a row from i+1 to n)
- Find the solution to the system by performing back substitution with the relations:

$$x_N = \frac{b_N^{(N-1)}}{a_{NN}^{(N-1)}}, \quad \text{for} \quad a_{NN}^{N-1} \neq 0$$

and

$$x_i = \frac{b_i^{(i-1)} - \sum_{k=i+1}^{N} a_{ik}^{(i-1)} x_k}{a_{ii}^{(i-1)}} \quad \text{for} \quad a_{ii}^{(i-1)} \neq 0$$

for the other values.

Solution:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 6 \\ 3 \\ -2 \\ 6 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}. \tag{4}$$

Then, we move on to the system 1. We solve it both with and without pivoting with a code having the structure described above. Removing pivoting from our code means removing the part in which we check the pivot and search for the largest non-vanishing pivot.

For both the procedures we end up with the following system:

$$\begin{pmatrix} 2 & 0.1 & -2 \\ 0 & -4.197 & -3.7 \\ 0 & 0 & 5.013 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 10 \\ -10.750 \\ 11.595 \end{pmatrix}. \tag{5}$$

The solution is $\vec{x} = \begin{pmatrix} 5.104 \\ 2.541 \\ 2.313 \end{pmatrix}$.

Regarding the system 2 we are able to obtain the solution with pivoting only, because vanishing pivots are encountered.

In particular we have the following transformed system:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & -15 \\ 0 & 0 & -2 \end{pmatrix} \cdot \vec{x} = \begin{pmatrix} 1 \\ -33 \\ -4 \end{pmatrix}. \tag{6}$$

whose solution is $\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.