Problem I :

Simpson's 1/3 rule : for $\boxed{n=2}$ : $\displaystyle\int_{a=x_0}^{b=x_2} f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx = \frac{h}{3}(f_0+4f_1+f_2)$

for $\boxed{n=4}$ : $\displaystyle\int_{a=x_0}^{b=x_4} f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx + \int_{x_2}^{x_4} P_2(x)\,dx = \frac{h}{3}(f_0+4f_1+2f_2+4f_3+f_4)$

Romberg integration method : $A = I_1 + ch^2$ and $A = I_2 + c(kh^2)$

Thus $A = \dfrac{k^2 I_1 - I_2}{k^2-1}$ and for $k=\frac{1}{2}$ : $A = \dfrac{4I_2 - I_1}{3}$

where,

for $\boxed{n=2}$ : $I_1 = 2h\left(\dfrac{f_0+f_2}{2}\right) = h(f_0+f_2)$ ; $I_2 = h\left(\dfrac{f_0+f_1}{2}\right) + h\left(\dfrac{f_1+f_2}{2}\right) = \dfrac{h}{2}(f_0+2f_1+f_2)$

$$\Rightarrow A = \frac{4I_2-I_1}{3} = \frac{4\frac{h}{2}(f_0+2f_1+f_2) - h(f_0+f_2)}{3} = \frac{h}{3}(f_0+4f_1+f_2) \quad \text{— (1a)}$$

if we half the step size $h \to h/2$ :

$I_1 = \frac{h}{2}(f_0+f_2)$ ; $I_2 = \frac{h}{4}(f_0+2f_1+f_2)$

$$\Rightarrow A = \frac{4I_2-I_1}{3} = \frac{4\frac{h}{4}(f_0+2f_1+f_2) - \frac{h}{2}(f_0+f_2)}{3} = \frac{h}{6}(f_0+4f_1+f_2) \quad \text{— (1b)}$$

for $\boxed{n=4}$ : $I_1 = 2h\left(\dfrac{f_0+f_2}{2}\right) + 2h\left(\dfrac{f_2+f_4}{2}\right) = h(f_0+2f_2+f_4)$

$I_2 = \frac{h}{2}(f_0+f_1) + h\left(\dfrac{f_1+f_2}{2}\right) + h\left(\dfrac{f_2+f_3}{2}\right) + h\left(\dfrac{f_3+f_4}{2}\right) = \dfrac{h}{2}(f_0+2f_1+2f_2+2f_3+f_4)$

$$\Rightarrow A = \frac{4I_2-I_1}{3} = \frac{2h(f_0+2f_1+2f_2+2f_3+f_4) - h(f_0+2f_2+f_4)}{3} = \frac{h}{3}(f_0+4f_1+2f_2+4f_3+f_4) \quad \text{— (2a)}$$

if we half the step size $h \to h/2$ :

$I_1 = \frac{h}{2}(f_0+2f_2+f_4)$ ; $I_2 = \frac{h}{4}(f_0+2f_1+2f_2+2f_3+f_4)$

$$\Rightarrow A = \frac{4I_2-I_1}{3} = \frac{h(f_0+2f_1+2f_2+2f_3+f_4) - \frac{h}{2}(f_0+2f_2+f_4)}{3} = \frac{h}{6}(f_0+4f_1+2f_2+4f_3+f_4) \quad \text{— (2b)}$$

So the formula that we implement from Newton-Cotes method is :

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0+4f_1+f_2)$$

or in general : $$\int_{x_i}^{x_{i+2}} f(x)\,dx = \frac{h}{3}(f_i+4f_{i+1}+f_{i+2})$$