9) $f(x) = \tan^{-1}x$.

$f(x)$ is continuous in $[a, b]$

$f'(x) = \frac{1}{1+x^2}$

$f(x)$ is differentiable in (a, b)

There exists a point $c \in (a, b)$

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\boxed{\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}} \quad \text{(1)}$$

~~$a < c < b \Rightarrow a^2 < c^2 < b^2$~~

~~$1+a^2 < 1+c^2 < 1+b^2 \Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$~~

~~$\frac{1}{1+a^2}$~~

$\therefore a < c < b$

$$\frac{\tan^{-1}b - \tan^{-1}a}{b-a} = \frac{1}{1+c^2}$$

Now $c > a \Rightarrow c^2 > a^2$

$$1+c^2 > 1+a^2 \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+a^2} \quad \text{(2)}$$

and $c < b \Rightarrow c^2 < b^2 \Rightarrow 1+c^2 < 1+b^2$

$$\Rightarrow \frac{1}{1+c^2} > \frac{1}{1+b^2} \quad \text{(3)}$$

from 2, 3 we get

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

Now, take $a=1$ $b=\frac{4}{3}$.

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1}$$

$$\frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{\frac{1}{3}}{2}.$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{1}{6} \Rightarrow \frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan\frac{\pi}{4} < \frac{1}{6}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

10) $f(x) = \sin^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f(x)$ is continuous $\forall\ 0 < x < 1$

$f'(x)$ is differential $\forall\ x \in (0,1)$

there exists a point $c \in (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$$\frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a}$$

$a < c < b \Rightarrow a^2 < c^2 < b^2$

$-a^2 > -c^2 > -b^2$

$1-a^2 > 1-c^2 > 1-b^2$

$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\frac{1}{\sqrt{1-a^2}} > \frac{1}{\sqrt{1-c^2}} > \frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}} > \frac{\sin^{-1}b - \sin^{-1}a}{b-a} > \frac{1}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} > \sin^{-1}b - \sin^{-1}a > \frac{b-a}{\sqrt{1-b^2}}.$$

~~ii) $f(x) = \log\left(\frac{b}{a}\right) =$~~

ii) $f(x) = \log x$.

$f(x)$ is continuous $\forall x \in [a, b]$

$f'(x) = \frac{1}{x}$

$f'(x)$ is differentiable $\forall x \in (a, b)$

there exists a point $c \in (a, b)$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log b - \log a}{b-a} \quad (1)$$

$\cancel{a < c < b} \Rightarrow \cancel{\frac{1}{a} < \frac{1}{c} < \frac{1}{b}}$.

$a < b < c \Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$.

$$\frac{1}{b} < \frac{\log b - \log a}{b-a} < \frac{1}{a}$$

$$\frac{b-a}{b} < \log b - \log a < \frac{b-a}{a}$$

put $a = 3$ $b = 4$

$$\frac{1}{4} < \log\left(\frac{4}{3}\right) < \frac{1}{3}$$

UNIT-3

Q1) Given $f(x) = \frac{\sin x}{e^x}$ in $(0, \pi)$

A) e^x is exists for all x in $[0, \pi]$

12) Given $f(x) = e^x$ $g(x) = e^{-x}$ $[a, b]$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$f'(x) = e^x$

$g'(x) = -e^{-x}$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^{a}}{e^{-b} - e^{-a}}$$

$$-e^c \cdot e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} \Rightarrow -e^{2c} = \frac{e^b - e^a}{e^a - e^b} e^a e^b$$

$$\cancel{-} e^{2c} = \frac{\cancel{-(e^a - e^b)}}{\cancel{e^a - e^b}} e^a e^b$$

$e^{2c} = e^a e^b \Rightarrow 2c = a + b$

$c = \frac{a+b}{2}$

$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right) \cdot \cos\left(\frac{C-D}{2}\right)$

13) $f(x) = \sin x$ $g(x) = \cos x$ $[a, b]$

$f'(x) = \cos x$. $g'(x) = -\sin x$

$$\frac{\cos c}{-\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$\boxed{\begin{aligned}\cos C - \cos D &= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \\ \sin C - \sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{aligned}}$$

$$-\cot c = \frac{\sin b - \sin a}{\cos b - \cos a} = \frac{2\cos\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)}{2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{a-b}{2}\right)}$$

$\cot c = \cot\left(\frac{a+b}{2}\right)$ $2c = a+b$

$c = \frac{a+b}{2}$

14) $f(x) = \log_e^x \qquad g(x) = \frac{1}{x} \qquad [1, e]$

$f'(x) = \frac{1}{x} \qquad g'(x) = \frac{-1}{x^2}$

$$\frac{\frac{1}{c}}{-\frac{1}{c^2}} = \frac{\log e - \log 1}{\cancel{\log} -\frac{1}{e^2} + \frac{1}{1^2}}$$

$$-\frac{c^{\cancel{2}}}{c} = \frac{\log e - 0}{-\frac{1}{e^2} + 1}$$

?

$$-c = \frac{\log e}{-\frac{1+e^2}{e^2}} \Rightarrow \cancel{-}c = \cancel{-}\frac{\log_e e}{1+e^2} \cdot e^2$$

$$c = \frac{e^2 \cdot \log_e e}{1+e^2} = \frac{e^2}{1+e^2}$$

16) $f(x) = \tan x$
$f(0) = \tan 0$
$= 0.$

$f'(x) = \sec^2 x$
$= 1 + \tan^2 x$
$f'(0) = 1 + 0 = 1$

$f''(x) = 2\sec^2 x \tan x$
$f''(0) = 0$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$
$f'''(0) = 2$

$f^4(x) = 16\tan x + 40\tan^3 x + 24\tan^5 x.$
$f^4(0) = 0.$

$f^5(x) = 16(1+\tan^2 x) + 120\tan^2 x \sec^2 x + 120\tan^4 x \sec^2 x.$
$f^5(0) = 16.$

$\cancel{f(x)}$
$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + \frac{x^5}{5!} f^5(0)$

$f(x) = x + \frac{x^3}{3!} + \frac{2x^5}{15}$

17) $f(x) = e^{\sin x}$ $\quad y = e^{\sin x}$ $\quad f(0) = e^{\sin 0} = e^0 = 1$

$y_1 = f'(x) = e^{\sin x} \cos x \Rightarrow f'(0) = 1$

$y_2 = f''(x) = -e^{\sin x} \sin x + \cos x \, e^{\sin x} \cos x$ $\quad = -y\sin x + y_1 \cos x.$ $\quad \cancel{(\cos^2 x e^{\sin x})}$

$f''(0) = -e^{\sin 0} \sin 0 + \cos 0 \, e^{\sin 0} \cos 0$

$= 0 + 1 = 1.$

$y_3 = f'''(x) = -[e^{\sin x} \cos x + \sin x \, e^{\sin x} \cos x] + \cos^2 x \, e^{\sin x} \cos x - 2e^{\sin x} \cos x \sin x.$

$-[y \cos x + \sin x \, y_1] + \cos x \cdot y_2 - \sin x \, y_1$

$= f'''(0) = 0.$

$y_4 = f^4(x) = -[y(-\sin x) + \cos y_1 + \sin y_2 + \cos x \, y_1] + \cos x \, y_3 - \sin x \, y_2 -$
$[\sin x \, y_2 + \cos x \, y_3]$

$f^4(0) = -3.$

$\cancel{f(x)} =$
$e^{\sin x} = 1 + x + \frac{x^2}{2!} + \frac{x^4}{4!}(-3)$

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18) $\log(1+\sin^2 x) = f(x)$:

$\left(\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots\right)$. $\left(\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}\right.$

$f(x) = \sin^2 x - \frac{\sin^4 x}{2} + \frac{\sin^6 x}{3} \cdots$

$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^2 - \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^4 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right.$

$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^2 - \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^4 + \left(\cancel{\frac{x^6}{3}} \cdots\right) \frac{1}{3}(x)^6$

$= \left(x^2 - \frac{x^6}{(3!)^2} - \frac{2x^4}{3!} + \frac{2x^6}{5!} + \cdots\right) - \frac{1}{2}\left(x^4 - \frac{4x^6}{3!} + \cdots\right) + \left(\frac{x^6}{3} + \cdots\right)$

$= x^2 - x^4\left(\frac{2}{3!} + \frac{1}{2}\right) + x^6\left(\frac{1}{(3!)^2} + \frac{2}{5!} + \frac{2}{3!} + \frac{1}{3}\right) + \cdots$

$= x^2 - x^4\left(\frac{2}{6} + \frac{1}{2}\right) + x^6\left(\frac{1}{36} + \frac{2}{120} + \frac{2}{6} + \frac{1}{3}\right) + \cdots$

$= x^2 - x^4\left(\frac{5}{6}\right) + \frac{256}{360} x^6 + \cdots$ $\quad \frac{10+6+120+120}{360}$

$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \cdots$ $\quad x-1$ $\quad x-a \quad a=1$

19) $f(x) = \log_e x$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$

$f'''(x) = \frac{2}{x^3}$, $f^{iv}(x) = \frac{-6}{x^4}$

$f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f^{4}(x) = -6$

$\log_e x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{4}(1)$

$\log_e x = 0 + (x-1) - (x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

Put $x = 1.1$

$$\log_e^{1.1} = (0.1) - (0.1)^2 + \frac{(0.1)^3}{3} - \frac{1}{4}(0.1)^4$$

$$\log_e^{1.1} = 0.1 - 0.005 + 0.000003 - 0.000025 + \ldots$$

$$\log_e^{1.1} = 0.094978$$

$$\log_e^{1.1} = 0.0949$$

~~2Q) $f(x) = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$~~

21) $f(x) = \log(1+x)$. $\Rightarrow f(0) = \log(1+0) = 0$.

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1+0} = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = \frac{-1}{(1+0)^2} = -1$$

$$f'''(x) = +\frac{2}{(1+x)^3} \Rightarrow f'''(0) = \frac{2}{(1+0)^3} = 2$$

$$f^{4}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{4}(0) = \frac{-6}{(1+0)^4} = -6$$

$-\{(1+x)^{-2}\}$

$= -[2(1+x)^{-3}]$

$(1+x)^{-3}$

$2 \cdot 3(1+x)^{-4}$

$$f(x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \frac{x^4}{4!} f^{4}(a)$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$$

3×2 $(4\times3\times2\times1)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\log\sqrt{\frac{1+x}{1-x}} = \log\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\{\log(1+x) - \log(1-x)\}$$

wrt

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$= \frac{1}{2}\left\{\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right]\right\}$$

$$= \frac{1}{2}\left[x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}}\right]$$

$$= \frac{1}{2}\left[2x + 2\frac{x^3}{3}\right] = \frac{1}{\cancel{2}}\cancel{2}\left(x + \frac{x^3}{3}\right)$$

$$\log\left(\sqrt{\frac{1+x}{1-x}}\right) = x + \frac{x^3}{3}.$$

20) $f(x) = \log(1+x)$ $\quad 0<\theta<1$

$f(0) = \log(1+0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = \frac{1}{1+0} = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad \cancel{f'''(0) = 2}$

$f'''(\theta x) = \frac{2}{(1+\theta x)^3}$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x)$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{\cancel{6}_3}\left(\frac{\cancel{2}}{(1+\theta x)^3}\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$$

Given $x>0, \theta>0$

$\theta x > 0$

$1+\theta x > 1 \Rightarrow (1+\theta x)^3 > 1$

$\frac{1}{(1+\theta x)^3} < 1$

$$\frac{x^3}{3(1+\theta x^3)} < \frac{x^3}{3}$$

$$-\frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < \frac{x^3}{3} - \frac{x^2}{2}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$