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MATHS Bit Bank

UNIT-4

1) Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$

A) $\frac{\partial z}{\partial x} = 3x^2 - 3ay$; $\frac{\partial z}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 z}{\partial x^2} = 3(2)x = 6x$; $\frac{\partial^2 z}{\partial y^2} = 6y$

2) If $v = (x^2+y^2+z^2)^{-1/2}$ show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$.

A) $\frac{\partial v}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$\frac{\partial v}{\partial y} = \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cancel{2}(y)$

$\frac{\partial v}{\partial z} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}(\cancel{2}z)$

$\frac{\partial^2 v}{\partial x^2} = -\left[x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) + (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)(1)\right]$

$= -(x^2+y^2+z^2)^{-5/2}(-3x^2+x^2+y^2+z^2)$

$= -(x^2+y^2+z^2)^{-5/2}(y^2+z^2-2x^2)$

$\frac{\partial^2 v}{\partial y^2} = -(x^2+y^2+z^2)^{-5/2}(z^2+x^2-2y^2)$

$\frac{\partial^2 v}{\partial z^2} = -(x^2+y^2+z^2)^{-5/2}(x^2+y^2-2z^2)$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -(x^2+y^2+z^2)^{5/2}(y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2)$

$= 0$

3) If $u = \log(x^3+y^3+z^3-3xyz)$ then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

A) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$

$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$

$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3zx)$

$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z} = t$

$$\frac{\partial u}{\partial t} \left| \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \cdot t = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)\right.$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2} = RHS.$$

6) 4) If $u = x^y$ then show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

A) $\log u = \log x^y \Rightarrow \log u = y \log x.$

W.R.T x, $\frac{1}{u} = \frac{y}{x}.$

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{y}{x} \Rightarrow \frac{\partial u}{\partial x} = \frac{uy}{x} = \frac{x^y y}{x} = x^{y-1}y$$

7) WRT y, $\frac{1}{u}\frac{\partial u}{\partial y} = \log x$

A) $\frac{\partial u}{\partial y} = u \log x = x^y \log x.$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial u}{\partial x}\left[\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right]$$

$$= \frac{\partial u}{\partial x}\left[y x^{y-1}(\log x) + x^y\left(\frac{1}{x}\right)\right]$$

$$= \frac{\partial u}{\partial x}\left[\frac{y}{x} x^y (\log x) + x^y\left(\frac{1}{x}\right)\right]$$

$$= \frac{\partial u}{\partial x}\left[\frac{x^y}{x}(y\log x + 1)\right] = \frac{\partial u}{\partial x}\left[\underset{u}{x^{y-1}}\underset{v}{(y\log x+1)}\right]$$

$$= (y-1)x^{y-2}(y\log x + 1) + x^{y-1}\left(\frac{y}{x}\right) = x^{y-2}(y-1(y\log x+1)+y)$$

$$= x^{y-2}(y-1(y\log x + 1)+y)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right)$$

$$= \frac{\partial u}{\partial x}\left(x^{y-1}(1) + y(x^{y-1}\log x)\right)$$

$$= \frac{\partial u}{\partial x}\left(x^{y-1}(1+y\log x)\right) = (y-1)x^{y-2}(y\log x + 1) + x^{y-1}\left(\frac{y}{x}\right)$$

$$= x^{y-2}(y + y-1(y\log x+1))$$

Hence proved.

5) If $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ then find total derivative $\frac{du}{dt}$

A) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right); \quad \frac{\partial x}{\partial t} = e^t; \quad \frac{\partial y}{\partial t} = 2t; \quad \frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right)\left(-\frac{2}{y^2}\right)$$

$$\frac{\partial u}{\partial t} = \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)e^t + \cos\left(\frac{x}{y}\right)\left(-\frac{2}{y^2}\right)(2t)$$

$$\frac{\partial u}{\partial t} = \cos\left(\frac{e^t}{t^2}\right)\left[\frac{e^t}{t^2} - \frac{2}{t^{4/3}}(2t)\right]$$

$$= \cos\left(\frac{e^t}{t^2}\right) \cancel{\frac{te^t - 4}{t^3}}$$

6) If $z = u^2+v^2$, $u = at^2$, $V = 2at$ find total derivative of $\frac{dz}{dt}$

A) $\frac{dz}{du} = 2u$; $\frac{dz}{dV} = 2V$; $\frac{du}{dt} = 2at$; $\frac{dV}{dt} = 2a$

$$\frac{dz}{dt} = \frac{dz}{du}\cdot\frac{du}{dt} + \frac{dz}{dV}\cdot\frac{dV}{dt}$$

$$= 2u \cdot 2at + 2V \cdot 2a$$

$$= 4uat + 4aV.$$

7) If $u = \tan^{-1}\left(\frac{y}{x}\right)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ find total derivate $\frac{du}{dt}$

A) $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{\cancel{x^2}}{x^2+y^2}\left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{dx}{dt} = e^t + e^{-t}, \quad \frac{dy}{dt} = e^t - e^{-t}$$

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t+e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t})$$

8) If $u = x\log(xy)$, $x^3+y^3+3axy = 1$ then find $\frac{du}{dx}$

A) $u = x(\log x + \log y)$

$$u = x\log x + x\log y$$

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{x}\right) + \log x + \log y$$

$$\frac{\partial u}{\partial x} = 1 + \log x + \log y \quad -①$$

$$\frac{\partial u}{\partial y} = x\left(\frac{1}{y}\right)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$f(x,y) = x^3+y^3+3axy = 1$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y \qquad \frac{\partial f}{\partial y} = 3y^2+3x.$$

$$\frac{dy}{dx} = \frac{-(\partial f/\partial x)}{(\partial f/\partial y)} = \frac{-(x^2+y)}{y^2+x}$$

$$\frac{du}{dx} = (1+\log x + \log y) + \left(\frac{x}{y}\right)\left(-\frac{(x^2+y^2)}{y^2+x}\right).$$

9) If $u = F(x-y,\ y-z,\ z-x)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

A) let $r = y-z$ $\quad s = z-x$ $\quad t = x-y$

$\frac{dr}{dx} = 0$ $\quad \frac{ds}{dx} = 1$ $\quad \frac{dt}{dx} = +1$

$\frac{dr}{dy} = 1$ $\quad \frac{ds}{dy} = 0$ $\quad \frac{dt}{dy} = -1$

$\frac{dr}{dz} = -1$ $\quad \frac{ds}{dz} = -1$ $\quad \frac{dt}{dz} = 0.$

$$\frac{du}{dx} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{dx} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \quad -(1)$$

$$\frac{du}{dy} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad -(2)$$

$$\frac{du}{dz} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad -(3)$$

(1)+(2)+(3) $\quad \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} = 0.$

10) If $z = f(x,y)$, $x = e^u\cos v$, $y = e^u \sin v$ prove that $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$.

A) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{dy}{du}$ $\cancel{+\frac{\partial z}{\partial}}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}.$$

$x = e^u\cos v$ | $y = e^u \sin v$

$\frac{\partial x}{\partial u} = e^u\cos v$ | $\frac{\partial y}{\partial u} = e^u \sin v$

$\frac{\partial x}{\partial v} = -e^u \sin v$ | $\frac{\partial y}{\partial v} = e^u\cos v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^u\cos v) + \frac{\partial z}{\partial y}(e^u \sin v) \quad -(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^u\sin v) + \frac{\partial z}{\partial y}(e^u\cos v) \quad -(2)$$

$x(2) + y(1) \Rightarrow x\frac{\partial z}{\partial x}\ -e^u\sin v + x\frac{dz}{dy}\ e^u\cos v + y\frac{dz}{dx}\ e^u\cos v + y\frac{dz}{dy}\ e^u\sin v$

$$\frac{\partial z}{\partial x}(-xe^u\sin v + ye^u\cos v) + \frac{\partial z}{\partial y}(xe^u\cos v + ye^u\sin v)$$

$\frac{\partial z}{\partial x}(-e^{2u}\cos v\sin v + e^{2u}(\cos v \sin v)) + \frac{\partial z}{\partial y}(e^{2u}\cos^2 v + e^{2u}\sin^2 v)$

$= e^{2u}\frac{dz}{dy}(1) = e^{2u}\frac{dz}{dy}.$

11) If $z = f(x,y)$ and $x = e^{u}+e^{-v}, y = e^{-u}-e^{v}$ Prove that

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

A) $\frac{dz}{du} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$

$\frac{dz}{dv} = \frac{\partial z}{\partial x}\frac{dx}{dv} + \frac{\partial z}{\partial y}\frac{dy}{dv}$

$x = e^{u}+e^{-v}$ $\quad\quad$ $y = e^{-u}-e^{v}$

$\frac{\partial x}{\partial u} = e^{u}$ $\quad\quad$ $\frac{\partial y}{\partial u} = -e^{-u}$

$\frac{\partial x}{\partial v} = -e^{-v}$ $\quad\quad$ $\frac{\partial y}{\partial v} = -e^{v}$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}e^{u} + \frac{\partial z}{\partial y}(-e^{-u})$ — ①

$\frac{dz}{dv} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$ — ②

① - ② $\Rightarrow \frac{\partial z}{\partial x}\cdot(e^{u}+e^{-v}) - \frac{\partial z}{\partial y}(e^{-u}-e^{v}) = \frac{\partial z}{\partial x}(x) - \frac{\partial z}{\partial y}(y).$

12) Expand $e^{x}\log(1+y)$ in powers of x, y.

r) $f(x) = e^{x}\log(1+y)$ $\quad a=0, b=0.$

$f(0,0) = e^{0}\log(1+0) = 0.$

$f_x = e^{x}\log(1+y) \Rightarrow f_x(0,0) = 0.$

$f_y = e^{x}\left(\frac{1}{1+y}\right) \Rightarrow f_y(0,0) = 1$

$f_{xx} = e^{x}\log(1+y) \Rightarrow f_{xx}(0,0) = 0$

$f_{xy} = \frac{e^{x}}{1+y} \Rightarrow f_{xy}(0,0) = 1$

$f_{yy} = \frac{e^{x}}{(1+y)^2} \Rightarrow f_{yy}(0,0) = -1.$

$f_{xxx} = \frac{e^{x}\log(1+y)}{\cancel{1+y}} \Rightarrow f_{xxx}(0,0) = 0.$

$f_{xxy} = \frac{e^{x}}{1+y} \Rightarrow f_{xxy}(0,0) = 1$

$f_{xyy} = -\frac{e^{x}}{(1+y)^2} \Rightarrow f_{xyy}(0,0) = -1$

$f_{yyy} = \frac{2e^{x}}{(1+y)^3} \Rightarrow f_{yyy}(0,0) = 2.$

$$f(x,y) = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right] + \cdots$$ (6)

$$e^x \log(1+y) = y + \frac{1}{2}\left[2xy - y^2\right] + \frac{1}{6}\left[3x^2y - 3xy^2 + 2y^3\right].$$

13) Expand $x^2y + 3y - 2$ in powers of $x-1, y+2$.

A) $f(x,y) = x^2y + 3y - 2 \qquad a=1 \quad b=-2$

$f(1,-2) = 1(-2) + 3(-2) - 2 = -10$

$f_x = 2xy \Rightarrow f_x(1,-2) = -4$

$f_y = x^2 + 3 \Rightarrow f_y(1,-2) = 4$

$f_{xx} = 2y \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy} = 2x \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy} = 0 \Rightarrow f_{yy}(1,-2) = 0. \quad , f_{xxx} = 0 \Rightarrow f_{xxx}(1,-2) = 0.$

$f_{xxy} = 2 \Rightarrow f_{xxy}(1,-2) = 2$

$f_{xyy} = 0 \Rightarrow f_{xyy}(1,-2) = 0$

$f_{yyy} = 0 \Rightarrow f_{yyy}(1,-2) = 0$

$$f(x,y) = f(1,-2) + (x-1)f_x(1,-2) + (y+2)f_y(1,-2) + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)\right]$$
$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2(y+2) f_{xxy}(1,-2) + 3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2)\right] + \cdots$$

$$x^2y + 3y - 2 = -10 + (x-1)(-4) + (y+2)(4) + \frac{1}{2}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + 0\right] + \frac{1}{6}\left[3(x-1)^2(y+2)(4)\right]$$

$$x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + \cdots$$

14) Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in powers of $x-1, y-1$.

A) $a=1, b=1. \qquad f(1,1) = \tan^{-1}(1/1) = \pi/4.$

$$f_x = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = -1/2$$

$$f_y = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} \Rightarrow f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{-y(-2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{(x^2+y^2)(1) - (-y)(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-x(2y)}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{-1}{2}$$

$$f_{xxx} = \frac{(x^2+y^2)(2y)-(2xy)(2x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2yx^2+2y^3-8x^2y}{(x^2+y^2)^3} = \frac{2y^3-6x^2y}{(x^2+y^2)^3} \Rightarrow f_{xxx}(1,1) = \frac{-1}{2}$$

$$f_{xxy} = \frac{(x^2+y^2)^2(-2x)-(y^2-x^2)(2(x^2+y^2))(2x)}{(x^2+y^2)^4} = \frac{-6y^2x-2x^3}{(x^2+y^2)^3} \Rightarrow f_{xxy}(1,1) = \frac{-1}{2}$$

$$f_{xyy} = \frac{(x^2+y^2)^2(-2y)-(-2xy)\,2(x^2+y^2)(2x)}{(x^2+y^2)^4} \Rightarrow f_{xyy}^{(1,1)} = \frac{1}{2}$$

$$f_{yyy} = \frac{(x^2+y^2)^2(-2x)-(-2xy)\,2(x^2+y^2)(2y)}{(x^2+y^2)^4} \Rightarrow f_{yyy}(1,1) = \frac{1}{2}$$

$$f(x,y) = f(1,1) + (x-1)f_x(1,1) + (y-1)f_y(1,1) + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\right]$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right] + \frac{1}{6}\left[(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\left(\frac{1}{2}\right) + (y-1)^3\left(\frac{1}{2}\right)\right]$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} + \frac{x+1}{2} + \frac{y-1}{2} + \frac{1}{4}\left[(x-1)^2-(y-1)^2\right] + \frac{1}{12}\left[-(x-1)^3 - 3(x-1)^2(y-1) + 3(x-1)(y-1)^2 + (y-1)^3\right]$$

15) If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$ show that Jacobian of y_1, y_2, y_3 w.r.t. x_1, x_2, x_3 is 4.

A) $\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}$ $\quad \frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$ $\quad \frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$

$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$ $\quad \frac{\partial y_2}{\partial x_2} = -\frac{x_1 x_3}{x_2^2}$ $\quad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$, $\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$, $\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$ $\quad \frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$

$$J\left(\frac{y_1\ y_2\ y_3}{x_1\ x_2\ x_3}\right) = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & x_3/x_1 & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix} = \frac{1}{x_1}\cdot\frac{1}{x_2}\cdot\frac{1}{x_3}\begin{vmatrix} -\frac{x_2x_3}{x_1} & x_3 & x_2 \\ x_3 & -\frac{x_1x_3}{x_2} & x_1 \\ x_2 & x_1 & -\frac{x_1x_2}{x_3} \end{vmatrix}$$

$$= \frac{1}{x_1}\frac{1}{x_2}\frac{1}{x_3}(0+2x_1x_2x_3+2x_1x_2x_3) = \underline{4}.$$

16) If $u = x^2 - y^2$ $v = 2xy$ $x = r\cos\theta$ $y = r\sin\theta$ Find Jacobian of u, v wrt r, θ.

A) $U = x^2 - y^2$ $J = ?$

$V = 2xy$

$x = r\cos\theta$, $y = r\sin\theta$.

$v = 2(r\cos\theta)(r\sin\theta) = 2r^2\sin\theta\cos\theta = r^2\sin 2\theta$.

$U = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta)$

$\frac{\partial u}{\partial r} = 2r(\cos^2\theta - \sin^2\theta)$ | $\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$

$\frac{\partial v}{\partial r} = 2r(\sin 2\theta)$ | $\frac{\partial u}{\partial \theta} = r^2(2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta)$

$= -r^2\, 4\sin\theta\cos\theta$

$$= \begin{vmatrix} 2r\cos^2\theta - 2\sin^2\theta r & -4r^2\sin\theta\cos\theta \\ 4r\sin\theta\cos\theta & 2r^2(1-2\sin^2\theta) \end{vmatrix} \Rightarrow \begin{matrix} 2r \\ 2r \end{matrix} \begin{vmatrix} \cos^2\theta - \sin^2\theta & -2r\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & r(\cos^2\theta - \sin^2\theta) \end{vmatrix}$$

$= r(\cos^4\theta - \cos^2\theta\sin^2\theta - \cos^2\theta\sin^2\theta + 8\sin^4\theta) + 4r(\sin^2\theta\cos^2\theta)$

$= 2r\sin^2\theta\cos^2\theta \Rightarrow 8r^3\sin^2\theta\cos^2\theta = J$.

17) If $U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}x + \sin^{-1}y$. show that u, v are functionally related and also find the relation between them.

A) $U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}x + \sin^{-1}y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = \sqrt{1-y^2} + \frac{y(-2x)}{2\sqrt{1-x^2}}$; $\frac{\partial u}{\partial y} = \frac{x(-2y)}{2\sqrt{1-y^2}} + \sqrt{1-x^2}$

$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$

$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-x^2}}$ $\frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-y^2}}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$\therefore u, v$ are functionally related.

$v = \sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$v = \sin^{-1} 0$