Multiple Integrals

Double integration:

Ex: In order of appearance

$$\int_0^1\int_0^2 xy\,dx\,dy$$

$$= \int_0^1 \left[y\left(\frac{x^2}{2}\right)\Big|_0^2 \right] dy$$

$$= \int_0^1 \frac{y}{2}(4)\,dy = 2\int_0^1 y\,dy = 2\left(\frac{y^2}{2}\right)\Big|_0^1 = (y^2)\Big|_0^1 = 1$$

Ex: In order of given limits

$$\int_{x=0}^{1}\int_{y=0}^{2} (x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{2}(x+y)\,dy\right]dx$$

$$= \int_{x=0}^{1}\left\{ x(y)\Big|_0^2 + \left(\frac{y^2}{2}\right)\Big|_0^2 \right\}dx$$

$$= \int_{x=0}^{1}(2x+2)\,dx$$

$$= 2\left(\frac{x^2}{2}\right)\Big|_0^1 + 2x\Big|_0^1$$

$$= 1 + 2(1)$$

$$= 1+2$$

$$= 3$$

Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$

$= \int_0^3 \int_1^2 (xy(1+x+y)\,dy)\cdot dx$

$= \int_0^3 \int_1^2 (xy + x^2y + xy^2)\cdot dy \cdot dx$

$= \int_0^3 \left[x\cdot y^2/2 + x^2 y^2/2 + x\, y^3/3 \right]_1^2 \cdot dx$

$= \int_0^3 x(y^2/2)\Big|_1^2 + x^2(y^2/2)\Big|_1^2 + x(y^3/3)\Big|_1^2 \; dx$

$= \int_0^3 \left[x/2 (3) + x^2/2 (3) + x/3 (7) \right] dx$

$= 3/2 (x^2/2)\Big|_0^3 + 3/2 (x^3/3)\Big|_0^3 + 7/3 (x^2/2)\Big|_0^3$

$= 3/4 (9) + 3/6 (27) + 7/6 (9)$

$= \dfrac{81 + 162 + 126}{12} = \dfrac{369}{12} = \dfrac{123}{4}$

81
162
126
$12 \overline{)369}\,(3$
36
009
84

Q. Evaluate $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2)\,dy\,dx$

$= \int_{x=0}^{a} \left[\int_{y=0}^{b} (x^2+y^2)\,dy \right] \cdot dx$

$= \int_{x=0}^{a} \left[x^2(y) + y^3/3 \right]_0^b \cdot dx$

$= \int_{x=0}^{a} x^2 b + b^3/3 \;.dx$

$= \left[b\cdot\dfrac{x^3}{3} + b^3/3 \cdot x \right]_0^a$

$= b\cdot a^3/3 + a\, b^3/3 = \dfrac{ab^3 + ba^3}{3} = \dfrac{ab(a^2+b^2)}{3}$

Evaluate $\int_0^3 \int_1^2 \underbrace{xy(1+x+y)}\,dy\,dx.$

$= \int_0^3 \int_1^2 (xy(1+x+y)\,dy)\cdot dx$

$= \int_0^3 \int_1^2 (xy + x^2y + xy^2)\cdot dy \cdot dx$

$= \int_0^3 \left[x\cdot\frac{y^2}{2} + x^2\frac{y^2}{2} + x\,\frac{y^3}{3}\right]_1^2 \cdot dx$

$= \int_0^3 x\left(\frac{y^2}{2}\right)\Big|_1^2 + x^2\left(\frac{y^2}{2}\right)\Big|_1^2 + x\left(\frac{y^3}{3}\right)\Big|_1^2 \cdot dx$

$= \int_0^3 \left[\frac{x}{2}(3) + \frac{x^2}{2}(3) + \frac{x}{3}(7)\right]dx$

$= \frac{3}{2}\left(\frac{x^2}{2}\right)\Big|_0^3 + \frac{3}{2}\left(\frac{x^3}{3}\right)\Big|_0^3 + \frac{7}{3}\left(\frac{x^2}{2}\right)\Big|_0^3$

$= \frac{3}{4}(9) + \frac{3}{6}(27) + \frac{7}{6}(9)$

$= \frac{81+162+126}{12} = \frac{369}{12} = \frac{123}{4}$

81
162
126
$12\overline{)369}(3$
36
009
84
12×7
14

Evaluate $\int_{x=0}^{a} \int_{y=0}^{b} (x^2+y^2)\,dy\,dx$

$= \int_{x=0}^{a} \left[\int_{y=0}^{b} (x^2+y^2)\,dy\right]\cdot dx$

$= \int_{x=0}^{a} \left[x^2(y) + \frac{y^3}{3}\right]_0^b \cdot dx$

$= \int_{x=0}^{a} x^2b + \frac{b^3}{3} \cdot dx$

$= \left[b\cdot\frac{x^3}{3} + \frac{b^3}{3}\cdot x\right]_0^a$

$= b\,\frac{a^3}{3} + a\,\frac{b^3}{3} = \frac{ab^3 + ba^3}{3} = \frac{ab(a^2+b^2)}{3}$

(4) Evaluate $\int_0^2\int_0^x e^{x+y}dy.dx$

$$\Rightarrow \int_{x=0}^{2}\left[\int_{y=0}^{x} e^x e^y\,dy\right]dx$$

$$\Rightarrow \int_{x=0}^{2} \left[e^x(e^y)\Big|_0^x\right].dx$$

$$\Rightarrow \int_{x=0}^{2} e^x[e^x - e^0]dx$$

$$\Rightarrow \int_{x=0}^{2}(e^{2x}-e^x)dx$$

$$\Rightarrow \left(\frac{e^{2x}}{2}-e^x\right)\Big|_0^2$$

$$\Rightarrow \left(\frac{e^4}{2}-e^2\right)-\left(\frac{1}{2}-1\right)$$

$$\Rightarrow \frac{e^4-2e^2}{2}-\left(-\frac{1}{2}\right)$$

$$\Rightarrow \frac{e^4-2e^2}{2}+\frac{1}{2}$$

$$\Rightarrow \frac{e^4-2e^2+1}{2} = \frac{(e^2-1)^2}{2} \;//$$

Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$

$$\Rightarrow \int_{x=0}^{5}\left[\int_{y=0}^{x^2} x^3+xy^2.\,dy\right].dx$$

$$\Rightarrow \int_{x=0}^{5}\left[x^3y\right]\Big|_0^{x^2}+\left[x.\frac{y^3}{3}\right]\Big|_0^{x^2}.dx$$

$$\Rightarrow \int_{x=0}^{5}\left[x^5+\frac{x}{3}x^6\right].dx$$

$$\Rightarrow \left[\frac{x^6}{6}+\frac{1}{3}\frac{x^8}{8}\right]_0^5 \rightarrow \frac{5^6}{6}+\frac{5^8}{24} = 5^6\left(\frac{1}{6}+\frac{25}{24}\right)$$

$$= 5^6\left(\frac{4+25}{24}\right) = 5^6\left(\frac{2}{2}\right.$$

Evaluate $\iint xy(x+y)\,dx\,dy$. ~~Evalual~~ over the region bounded by $y = x^2$ & $y = x$.

$\Rightarrow \iint [xy(x+y)\,dx]\,dy$

$y = x^2$

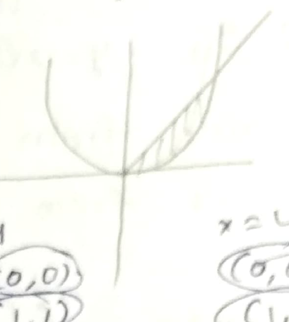
$x = 0 \Rightarrow y = 0$

$x = 1, \Rightarrow y = 1$

$x = 2 \Rightarrow y = 4$

$\Rightarrow (0,0)$

$(1,1)$



$x = y$

$(0,0)$

$(1,1)$

$(2,2)$

$\Rightarrow \iint [x^2y + xy^2\,dx]\,dy$

$\Rightarrow \int_{x=0}^{1} \left[\int_{y=x^2}^{x} (x^2y + xy^2)\,dy \right] dx$

x varies 0 to 1

y varies x^2 to x

$= \int_{x=0}^{1} \left[x^2 \left(\frac{y^2}{2}\right)\Big|_{x^2}^{x} + x\left(\frac{y^3}{3}\Big|_{x^2}^{x}\right) \right] dx$

$= \int_{x=0}^{1} \left\{ \frac{x^2}{2}(x^2 - x^4) + \frac{x}{3}(x^3 - x^6) \right\} dx$

$= \int_{x=0}^{1} \left(\frac{x^4}{2} - \frac{x^6}{2} + \frac{x^4}{3} - \frac{x^7}{3} \right) dx$

$= \left(\frac{x^5}{10} - \frac{x^7}{14} + \frac{x^5}{15} - \frac{x^8}{24} \right)\Big|_0^1$

$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24} = \frac{3}{56}.$

Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$.

$x^2 = 4y$

$x^2 = 4(2\sqrt{x})$

$x^2 = 8\sqrt{x}$

$x^4 = 64x$

$x^4 - 64x = 0$

$x(x^3 - 64) = 0$

$x = 0, x^3 - 64 = 0$

$x^3 = 64$

$\boxed{x = 4}$

$\boxed{x = 0}$

$x^2 = 4y$

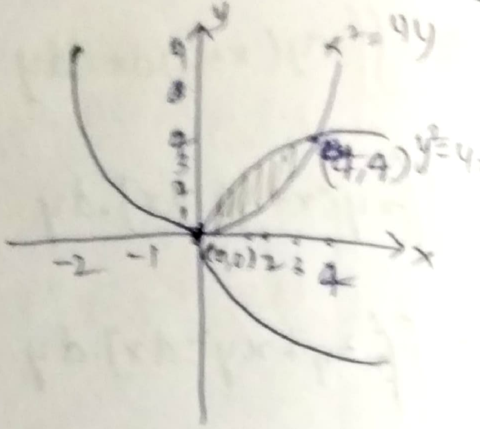
$16 = 4y$

$\boxed{y = 4}$

$x^2 = 4y \quad y^2 = 4x$

$x = 2\sqrt{y} \quad x = y^2/4$

$y = x^2/4 \quad y = 2\sqrt{x}$



x varies from 0 to 4

y varies from $x^2/4$ to $2\sqrt{x}$

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\left[\int_{y=x^2/4}^{2\sqrt{x}} y\,dy\right].dx$$

$$= \int_{x=0}^{4} \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} .dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(4x - \frac{x^4}{16}\right).dx$$

$$= \frac{1}{2}\left\{\cancel{4}^2\left(\frac{x^2}{\cancel{2}}\right) - \frac{x^5}{80}\right\}\Bigg|_0^4$$

$$= \frac{1}{2}\left\{2(4^2) - \frac{4^5}{80}\right\}$$

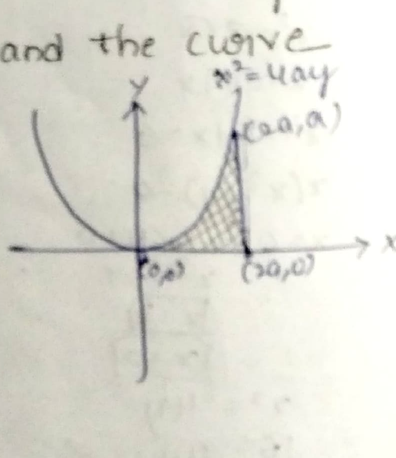
$$= \frac{16}{2}\left\{2 - \frac{\cancel{64}^4}{\cancel{80}_5}\right\}$$

$$= \frac{16}{2}\left\{\frac{10-4}{5}\right\}$$

$$= \frac{\cancel{16}^8}{\cancel{2}}\left\{6/5\right\}$$

$$= 48/5.$$

Evaluate $\iint_R xy \cdot ~~(\ldots)~~ dx\,dy$, where R is the region bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.



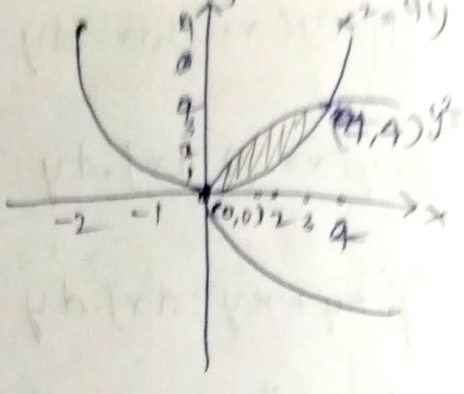
$x^2 = 4ay$

$y = x^2/4a$

$\cancel{4}a^{\cancel{2}} = \cancel{4}\cancel{a}y$

$\boxed{y = a}$

$\boxed{x = 2a}$



$x^2=4y \qquad y^2=4x$

$x=2\sqrt{y} \qquad x=y^2/4$

$y=x^2/4 \qquad y=2\sqrt{x}$

x varies from 0 to 4

y varies from $x^2/4$ to $2\sqrt{x}$

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\left[\int_{y=x^2/4}^{2\sqrt{x}} y\,dy\right].dx$$

$$= \int_{x=0}^{4}\left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}}.dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(4x-\frac{x^4}{16}\right).dx$$

$$= \frac{1}{2}\left\{\cancel{4}^{2}\left(\frac{x^2}{\cancel{2}}\right)-\frac{x^5}{80}\right\}\Big|_0^4$$

$$= \frac{1}{2}\left\{2(4^2)-\frac{4^5}{80}\right\}$$

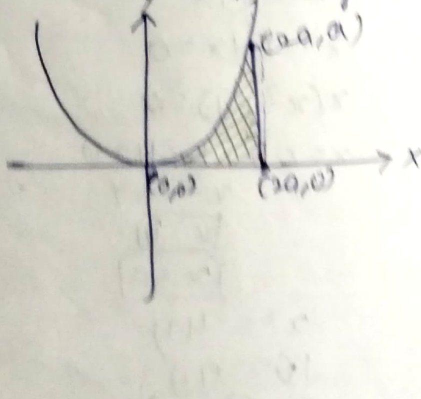
$$= \frac{16}{2}\left\{2-\frac{\cancel{64}^{4}}{\cancel{80}_{5}}\right\}$$

$$= \frac{16}{2}\left\{\frac{10-4}{5}\right\}$$

$$= \frac{\cancel{16}^{8}}{\cancel{2}}\left\{\frac{6}{5}\right\}$$

$$= 48/5.$$

Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$.



$x^2=4ay$

$y=x^2/4a$

$\cancel{4}a^{\cancel{2}}=\cancel{4}a y$

$\boxed{y=a}$

$\boxed{x=2a}$

x varies from 0 to $2a$

y varies from 0 to $x^2/4a$

$$\iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x\left(\frac{y^2}{2}\right)\Big|_0^{x^2/4a} .dx$$

$$= \int_{x=0}^{2a} \frac{x}{2}\left[\frac{x^4}{16a^2}\right].dx$$

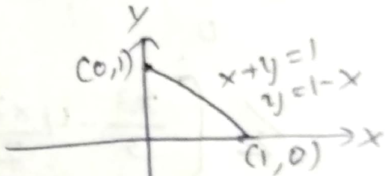
$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]\Big|_0^{2a}$$

$$= \frac{1}{\cancel{32}a^{\cancel{2}}}\ \frac{\cancel{64}^{2}a^{\cancel{6}\,4}}{\cancel{6}_3}$$

$$= a^4/3 \text{ units.}$$

3 | 7, 11, 3, 12
4 | 2, 4, 1, 1
2, 1, 1, 1
12 + 0 − 16 + 2
24
4/24

Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region in the positive quadrant for which, $\boxed{x+y \le 1}$



$x+y=1$

$x=0, y=1$ $\quad$ x varies from 0 to 1

$y=0, x=0$ $\quad$ y varies from 0 to $1-x$

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx = \int_{x=0} \left[x^2(y) + y^3/3\right]\Big|_0^{1-x} .dx$$

$$= \int_{x=0}^{1}\left[x^2(1-x) + \frac{(1-x)^3}{3}\right]dx$$

Let $1-x=t \Rightarrow -dx = dt$

$x=0 \to t=1$

$x=1 \Rightarrow t=0$

$$\Rightarrow \int_{t=1}^{0} (1-t)^2(t) + t^3/3\ (dt)$$

$$\Rightarrow \int_{t=0}^{1} (1+t^2-2t)(t) + t^3/3 \cdot dt$$

$$\Rightarrow \int_{t=0}^{1} t + t^3 - 2t^2 + t^3/3 \cdot dt$$

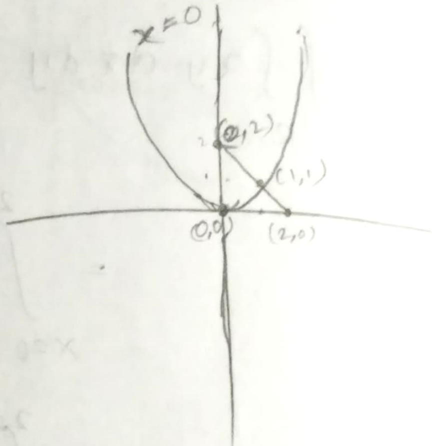
$$\rightarrow t^2/2 + t^4/4 - 2t^3/3 + t^4/12\Big|_0^1$$

$$= \frac{1}{2} + 1/4 - 2/3 + 1/12$$

$$= \frac{12+6-16+2}{24}$$

$$= 4/24 = 1/6$$

Evaluate $\iint_R y\,dx\,dy$ where R is the domain bounded by y axis, the curve $y=x^2$ and the line $x+y=2$ in the 1st quadrant.



$x+y=2$

$0,2$

$2,0$

$1,1$

$y=x^2$

$(0,0)$

$(1,1)$

x varies from 0 to 1

y varies from x^2 to $2-x$

$$\int_0^1 \int_{x^2}^{2-x} y\,dy\,.dx$$

$$\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} .dx$$

$$\frac{1}{2}\int_0^1 ((2-x)^2 - x^4).dx$$

$$\frac{1}{2}\int_0^1 x^2+4-4x-x^4.dx$$

$$\frac{1}{2}\left[\frac{x^3}{3}-4\frac{x^2}{2}-\frac{x^5}{5}+4x\right]_0^1$$

$$\frac{1}{2}\left[\frac{1}{3}-\frac{4}{2}-\frac{1}{5}+4\right]$$

$$\frac{1}{2}\left[\frac{10-60-6+120}{30}\right] = \frac{1}{2}\left[\frac{130-66}{30}\right] = \frac{\cancel{32}\,\cancel{64}}{\cancel{60}\,\cancel{30}}\ \frac{16}{15}$$

$$= \frac{16}{15}$$

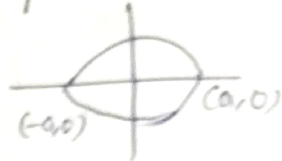
$4.\int dx$

$3,2,5$

$130 - 66 = 64$

$66 + 64 = 130$

Evaluate $\iint (x^2+y^2)\,dx\,dy$, $x^2/a^2 + y^2/b^2 = 1$ over the area bounde by the ellipse



x varies from $-a$ to a

Given that $x^2/a^2 + y^2/b^2 = 1$

$y^2/b^2 = 1 - x^2/a^2 \Rightarrow y^2 = b^2(1 - x^2/a^2)$

$y = \pm b\sqrt{1 - x^2/a^2}$

∴ y varies from $-b\sqrt{1 - x^2/a^2}$ to $b\sqrt{1-x^2/a^2}$

$$\iint (x^2+y^2)\,dx\,dy = \int_{x=-a}^{a}\int_{y=-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} (x^2+y^2)\,dy\,dx$$

$$= 2\int_{x=-a}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}} (x^2+y^2)\,dy\,dx$$

$$= 2\int_{x=-a}^{a} \left.(x^2 y) + y^3/3\right|_0^{b\sqrt{1-x^2/a^2}} .\,dx$$

$$= 2\int_{-a}^{a}\left[x^2\left(b\sqrt{1-x^2/a^2}\right) + \frac{1}{3}\left(b\sqrt{1-x^2/a^2}\right)^3\right].dx$$

$$= 2\int_{x=-a}^{a}\left[x^2\, b/a\sqrt{a^2-x^2} + \frac{b^3}{3a^3}\left(\sqrt{a^2-x^2}\right)^3\right]dx$$

$$= 4\int_{x=0}^{a}\left[x^2\frac{b}{a}(a^2-x^2)^{1/2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

let $x = a\sin\theta$ $\quad a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$

$dx = a\cos\theta\,d\theta$

$$\Rightarrow 4\int_{\theta=0}^{\pi/2}\left\{\frac{b}{a}\left[a^2\sin^2\theta \cdot a\cos\theta\right] + b^3/3a^3\left(a^3\cos^3\theta\right)\right\} a\cos\theta\,d\theta$$

$$\Rightarrow 4\left\{\frac{ba^4}{a}\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta + \int_{\theta=0}^{\pi/2}\frac{b^3}{a}a(\cos^4\theta)\,d\theta\right\}$$

$$\Rightarrow 4\left\{ba^3\left(\tfrac14\cdot\tfrac12\cdot\pi/2\right) + ab^3/3\left(\tfrac34\cdot\tfrac12\cdot\pi/2\right)\right\}$$

$\int_0^{\pi/2}\sin^m x\cos^n x\,dx$, where m, n even

$$\Rightarrow 4\left\{ba^3(\pi/16) + ab^3(3\pi/16)\right\}$$

$$\Rightarrow 4ab\pi/16\,(a^2+b^2)$$

$\dfrac{x^{-1/2+1}}{1/2+1}$ $\quad \dfrac{x^{1/2}}{3/2}$ $\quad \dfrac{\sqrt{x}}{3/2}$ $\quad \dfrac{2\sqrt{x}}{3}$

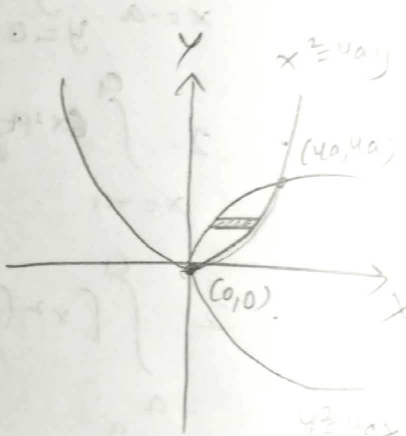
Change of order of integration:—

Working rule: $\int_{x=a}^{b}\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\,dx$

Draw the region of integration by drawing the curves $y=f_1(x)$ & $y=f_2(x)$ and the lines $x=a, x=b$. If these curves and lines intersect then draw straight lines parallel to x-axis to get various subregions. In each of these subregions, draw elementary strips parallel to x-axis and obtain the limits for x in terms of y and then the limits for y as constants.

1Q) Change the order of integration & evaluate

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$



Sol: Given limits are $x: 0 \to 4a$

y limits are $x^2/4a$ & $2\sqrt{ax}$

$y = x^2/4a$ & $y = 2\sqrt{ax}$

$x^2 = 4ay \to (1)$

$y^2 = 4ax \to (2)$

By changing the order of integration we must fix y. For a fixed y, x varies from $y^2/4a$ to $2\sqrt{ay}$ & y varies from 0 to 4a. After changing the order of

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$\Rightarrow \int_{y=0}^{4a} (x)\Big|_{y^2/4a}^{2\sqrt{ay}} \cdot dy$$

$$\Rightarrow \int_{y=0}^{4a} \left(2\sqrt{ay} - y^2/4a\right) dy$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left(y^{3/2}\right)\Big|_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)\Big|_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{1}{12a}(4a)^3$$

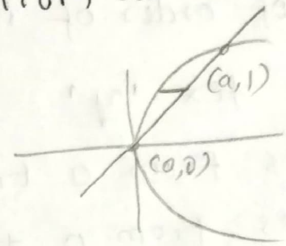
$$= \frac{4}{3}a^2\times 8 - \frac{1}{\cancel{12a}_{3}}\cancel{64a^3}^{16\,a^2}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

29) Change the order of integration and evaluate

$$\int_0^a\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy.$$



$y = x/a$ $\quad$ $y=\sqrt{x/a}$

$x = ay$ $\quad$ $y^2 = x/a$

$x = ay^2$

$\frac{a^3}{12} - \frac{a^3}{21} + \frac{a}{4} - \frac{a}{5}$

4 | 12, 21, 4, 5
3 | 3, 21, 4, 5
 | 1, 7, 4, 5

28 × 60

Given that for a fixed 'x'

y varies from x/a to $\sqrt{x/a}$

$y = x/a$ & $y = \sqrt{x/a}$

$\boxed{x = ay}$ & $y^2 = x/a \Rightarrow \boxed{x = ay^2}$

$\cancel{a}y^2 = \cancel{a}y$

$y^2 - y = 0$

$y(y-1) = 0$

$y = 0$ or 1

By changing the order of integration

fix 'y' & x varies from ay^2 to ay

y varies from 0 to 1

$$\int_0^a\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy = \int_{y=0}^{1}\int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy = \int_{y=0}^{1}\left[\frac{x^3}{3} + y^2(x)\right]_{ay^2}^{ay} dy$$

$$= \int_{y=0}^{1}\left(\frac{1}{3}\left[(ay)^3 - (ay^2)^3\right] + y^2(ay - ay^2)\right) dy$$

$$= \int_{y=0}^{1} \frac{a^3y^3}{3} - \frac{a^3y^6}{3} + ay^3 - ay^4\,dy = \left[\frac{a^3}{3}\left(\frac{y^4}{4}\right) - \frac{a^3}{3}\left(\frac{y^7}{7}\right) + a\left(\frac{y^4}{4}\right) - a\left(\frac{y^5}{5}\right)\right]_0^1$$

$$= \frac{a^3}{12} - \frac{a^3}{21} + \frac{a}{4} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20}$$

3) Evaluate by changing the order of integration

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$$

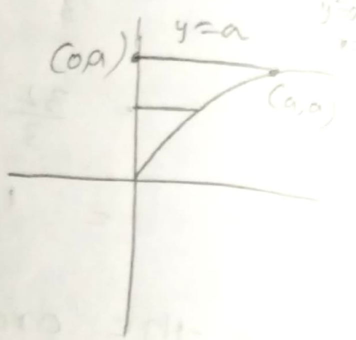
For a fix 'x', $x: 0 \to a$

y limits are from $\sqrt{ax}$ to a

$y=\sqrt{ax}$ & $y=a$ $\qquad y^2=ax$

$y^2=ax$ & $y=a$ $\qquad a^{\cancel{2}}=\cancel{a}x$

$\boxed{x=a}$



By change of order of integration,

for a fix 'y'

x varies from 0 to y^2/a.

y varies from 0 to a.

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}} = \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2\,dx}{\sqrt{y^4-a^2x^2}}\,dy =$$

$$= \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2\,dx}{\sqrt{a^2\left(\frac{y^4}{a^2}-x^2\right)}}\,dy$$

$$= \frac{1}{a}\int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2\,dx}{\sqrt{\left(\frac{y^2}{a}\right)^2-x^2}}\,.dy$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2\left[\sin^{-1}\left(\frac{x}{y^2/a}\right)\right]_0^{y^2/a}.dy$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2\left[\sin^{-1}(1)-\sin^{-1}(0)\right]dy$$

$$= \frac{1}{a}\int_{y=0}^{a} y^2\left(\frac{\pi}{2}-0\right)dy$$

$$= \frac{\pi}{2a}\left(\frac{y^3}{3}\right)\Big|_0^a = \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}$$

4) Change the order of integration and evaluate

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy.$$

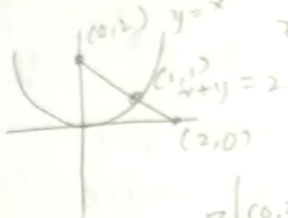
y = x^2

y = 2-x

x = 2-y

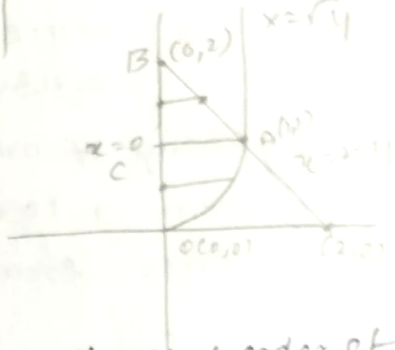
x = √y

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$$



for a fix 'x'

y limits are from x^2 to $2-x$



Given that $y = x^2$ & $y = 2-x$

$(0,0)$ $\quad (x,y) = (2,0)$

$(1,1)$ $\quad (x,y) = (0,2)$

$(-2,4)$ $\quad (x,y) = (1,1)$

change of order of integration

for fix 'y'

x varies from 0 to √y,

y : 0 to 1

x varies from 0 to 2-y,

y : 1 to 2

$$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy \Rightarrow \iint_{OAB} xy\,dx\,dy = \iint_{OAC} + \iint_{CAB}$$

$$\Rightarrow \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$\Rightarrow \int_{y=0}^{1} y\left(x^2/2\right)\Big|_0^{\sqrt{y}} dy + \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)\Big|_0^{2-y} \cdot dy$$

$$\Rightarrow \int_{y=0}^{1} \left[y/2\,(y)\right] \cdot dy + \int_1^2 y/2\,(2-y)^2 \cdot dy$$

$$\Rightarrow \int_{y=0}^{1} \frac{y^2}{2} \cdot dy + \int_1^2 y/2\,(4+y^2-4y) \cdot dy$$

5/8

$$= \left[\frac{1}{2} \cdot \frac{y^3}{3}\right]\Big|_0^1 + \frac{4}{2}\left(\frac{y^2}{2}\right)\Big|_1^2 + \frac{1}{2}\left(\frac{y^4}{4}\right)\Big|_1^2 - \frac{4}{2}\left(\frac{y^3}{3}\right)\Big|_1^2$$

$$= \frac{1}{6}(y^3)_0^1 + (y^2)_1^2 + \frac{1}{8}(y^4)_1^2 - \frac{2}{3}(y^3)_1^2$$

$$= \frac{1}{6}(1-0) + (4-1) + \frac{1}{8}(16-1) - \frac{2}{3}(8-1)$$

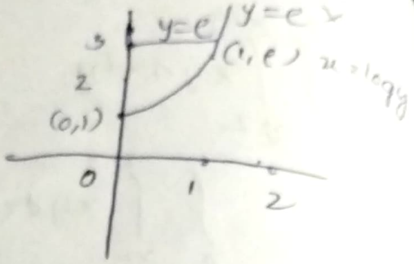
$$= \frac{1}{6} + 3 + \frac{1}{8}(15) - \frac{2}{3}(7) = \frac{4+72+45-112}{24} = \frac{121-112}{24}$$

$$= \frac{9}{24} = 3/8$$

2×3 / 12

14×8 / 112

(5) Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by change of order of integration



Given that x is 'fix'

x limits are from 0 to 1

$y = e^x \quad y = e$

if $x=0 \Rightarrow y=1$

$x=1 \Rightarrow y=e$

$= 2.718$

By change of order of integration for a fix 'y' limits are from 1 to e.

x varies from $x=0$ to $x = \log y$

$$\int_0^1 \int_{e^x}^{e} \frac{dy\,dx}{\log y} = \int_{y=1}^{e} \int_{x=0}^{\log y} \frac{dx}{\log y}\, dy = \int_{y=1}^{e} \frac{1}{\log y} (x) \Big|_0^{\log y} dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y} (\log y)\, dy = \int_{y=1}^{e} dy = (y) \Big|_1^{e}$$

$$= e - 1$$

$$= 2.718 - 1$$

$$= 1.718.$$

By change of order of integration $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy$.

Show that $\int_0^\infty \frac{\sin px}{x} \cdot dx = \pi/2$.

Given that $\int_0^\infty \left[\int_0^\infty e^{-xy} \sin px \, dx \right] dy \quad a = y \quad b = p$

$$\int e^{-ax} \sin bx\, dx = \frac{-e^{-ax}}{a^2+b^2} [a \sin bx + b \cos bx]$$

$$\rightarrow \int_0^\infty -\left[\frac{e^{-xy}}{(-y)^2+(p)^2} (+y \sin px + p \cos px) \right] \Big|_0^\infty dy$$

$$\rightarrow -\int_0^\infty \frac{[0 - 1(0+p)]}{y^2+p^2} dy$$

$$\Rightarrow -\int_0^\infty \frac{-p}{y^2+p^2} dy = p \int_0^\infty \frac{1}{p^2+y^2} dy = \frac{1}{p} \cdot p \operatorname{Tan}^{-1}(y/p) \Big|_0^\infty = \operatorname{Tan}^{-1}(\infty) - \operatorname{Tan}^{-1}(0)$$

$$= \pi/2 - 0 = \pi/2$$

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \pi/2 \longrightarrow ①$$

By changing order of integration

$$\int e^{ax} = \frac{e^{-ax}}{-a}$$

$$\rightarrow \int_0^\infty \int_0^\infty e^{-xy} \sin px \, dy \, dx$$

$$\Rightarrow \int_0^\infty \sin px \cdot \left(\frac{e^{-xy}}{-x}\right)\Big|_0^\infty dx \Rightarrow -\int_0^\infty \frac{\sin px (0-1)}{x} . dx$$

$$\Rightarrow \int_0^\infty \frac{\sin px}{x} dx \longrightarrow ②$$

from ① & ②

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \frac{\sin px}{x} dx = \pi/2 .$$

$$\therefore \int_0^\infty \frac{\sin px}{x} . dx = \pi/2$$

Hence proved.

Changing into polar co-ordinates:-

we have to consider

To change into polar co-ordinates

$x = r\cos\theta$ and $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Evaluate the double integration $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2) dy\, dx$, by changing into polar coordinates.

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2) dy\, dx$$

Given that $0 \le x \le \sqrt{a^2-y^2}$ & $0 \le y \le a$

Let $x = r\cos\theta$ $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2\sin^2\theta + r^2\cos^2\theta$

$= r^2(\sin^2\theta + \cos^2\theta)$

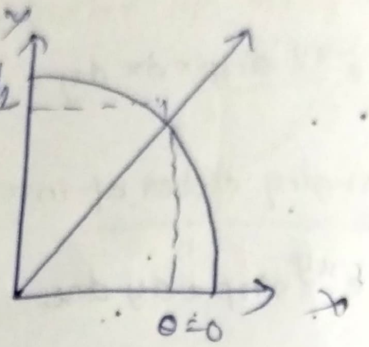
$= r^2(1) \rightarrow ①$

$x = \sqrt{a^2-y^2}$

$x^2 = a^2 - y^2$

$\Rightarrow x^2+y^2 = a^2 \longrightarrow ②$

from ① & ②



$$x^2+y^2=a^2$$

$$x^2+y^2=r^2$$

$$r^2=a^2$$

$$\boxed{r=a}$$

r is from 0 to a

θ varies from 0 to $\pi/2$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 r\,dr\,d\theta$$

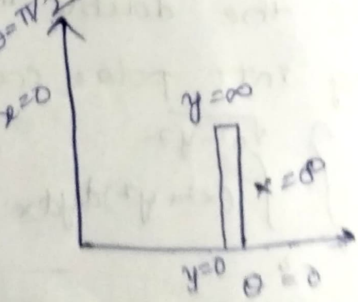
$$\Rightarrow \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3\,dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4}\right)\Big|_0^a .\,d\theta = \int_{\theta=0}^{\pi/2} \frac{a^4}{4}.\,d\theta$$

$$\Rightarrow \frac{a^4}{4} \int_{\theta=0}^{\pi/2} .\,d\theta = \frac{a^4}{4}(\theta)\Big|_0^{\pi/2}$$

$$\Rightarrow \frac{a^4}{4}\left(\frac{\pi}{2}\right) = \frac{\pi a^4}{8}$$

2. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar co-ordinates.



$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Given that $0 \le x \le \infty$, $0 \le y \le \infty$

Let $x = r\cos\theta$ & $y = r\sin\theta$ $\quad x^2+y^2 = r^2 \rightarrow ①$

x varies from 0 to ∞

y varies from 0 to $\pi/2$

$dx\,dy = r\,dr\,d\theta$

$$\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{\infty} e^{-r^2} r\,dr\right]d\theta$$

Let $r^2 = t$

$2r\,dr = dt$

$r\,dr = dt/2$

As $r \to 0 \Rightarrow t \to 0$

$r \to \infty \Rightarrow t \to \infty$

$$\int_{\theta=0}^{\pi/2}\left[\int_{t=0}^{\infty} e^{-t}\frac{dt}{2}\right]d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}(-e^{-t})\Big|_0^{\infty} d\theta = -\frac{1}{2}\int_0^{\pi/2}(0-1)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{1}{2}(\theta)\Big|_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$$

$= \pi/4$

3. By changing into polar coordinates evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the angular region between the circles, $x^2+y^2=a^2$ & $x^2+y^2=b^2$

Let $x = r\cos\theta$ & $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$x^2+y^2 = r^2 \to (1)$

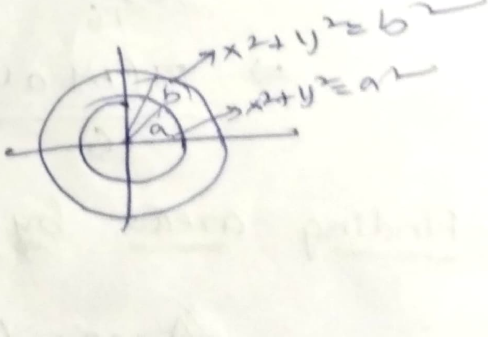
$dx\,dy = r\,dr\,d\theta$

Given that $x^2+y^2=a^2$ & $x^2+y^2=b^2$

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$ & $r^2\cos^2\theta + r^2\sin^2\theta = b^2$

$r^2 = a^2$ & $r^2 = b^2$

$\therefore r = a$ & b



r varies from a to b

θ varies from 0 to 2π

$$\int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2\sin^2\theta + r^2\cos^2\theta}\cdot r\,dr\,d\theta$$

$$\int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^{4}(\cos^2\theta\sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=a}^{b} r^3\cos^2\theta\sin^2\theta\, dr\, d\theta$$

$$\Rightarrow \int_{\theta=0}^{2\pi} \sin^2\theta\cos^2\theta \left(r^4/4\right)\Big|_a^b d\theta$$

$$\Rightarrow \frac{b^4-a^4}{4}\int_{\theta=0}^{2\pi} \sin^2\theta\cos^2\theta\, d\theta$$

$$\rightarrow \frac{b^4-a^4}{16}\int_{\theta=0}^{2\pi} 4\sin^2\theta\cos^2\theta\, .d\theta$$

$2\sin\theta\cos\theta = \sin 2\theta$

$(\sin 2\theta)^2 = 4\sin^2\theta\cos^2\theta$

$$\Rightarrow \frac{b^4-a^4}{16}\int_{0}^{2\pi} (\sin 2\theta)^2 . d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2 2\theta = \frac{1-\cos 2(2\theta)}{2}$

$= \frac{1-\cos 4\theta}{2}$

$$\Rightarrow \frac{b^4-a^4}{16}\int_{0}^{2\pi} \sin^2 2\theta . d\theta$$

$$\Rightarrow \frac{b^4-a^4}{16}\int_{0}^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$\Rightarrow \frac{b^4-a^4}{16}\int_{0}^{2\pi} \frac{1}{2} . d\theta - \frac{b^4-a^4}{16}\int_{0}^{2\pi} \cos 4\theta . d\theta$$

$$\Rightarrow \frac{b^4-a^4}{32}(\theta)\Big|_0^{2\pi} - \frac{b^4-a^4}{16}\left(+\frac{\sin 4\theta}{4}\right)\Big|_0^{2\pi}$$

$$\Rightarrow \frac{b^4-a^4}{\cancel{32}\,16}(\cancel{2}\pi) - \frac{b^4-a^4}{64}(+\sin 8\pi)$$

$$\Rightarrow \frac{\pi(b^4-a^4)}{16}$$

Finding areas by double integrals:-

Area $= \iint dx\,dy$

① Find the area of a plate in the form of a quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$

Given that $x^2/a^2 + y^2/b^2 = 1$

$$y^2/b^2 = 1 - x^2/a^2$$

$$y^2 = b^2(1 - x^2/a^2)$$

$$y^2 = b^2/a^2 (a^2 - x^2)$$

$$y = b/a \sqrt{a^2-x^2}$$

y varies from 0 to $b/a \sqrt{a^2-x^2}$

x varies from 0 to a

Area of a plate $= \iint dx\,dy$

$$\Rightarrow \int_{x=0}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} dy\,dx = \int_{x=0}^{a} (y)\Big|_0^{b/a\sqrt{a^2-x^2}} \cdot dx = \int_{x=0}^{a} b/a \sqrt{a^2-x^2}\,dx$$

$$\Rightarrow b/a \int_{x=0}^{a} \sqrt{a^2-x^2}\,dx$$

$$\Rightarrow b/a \left[\frac{x}{a}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]\Bigg|_0^a$$

$$\Rightarrow b/a \left\{ a^2/2 (\pi/2) - 0 \right]$$

$$= \frac{\pi}{4} ab$$

② Show that the area of the parabolas $y^2 = 4ax$ & $x^2 = 4ay$ is $16a^2/3$.

Given that $y^2 = 4ax$ ① & $x^2 = 4ay$ ②

from ① & ②

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\frac{x^4}{16a^2} = 4ax$$

$$x^{\cancel{4}} = 64a^3 \cancel{x}$$

$$x^3 = 64a^3$$

$$\boxed{x = 4a} \rightarrow ③$$

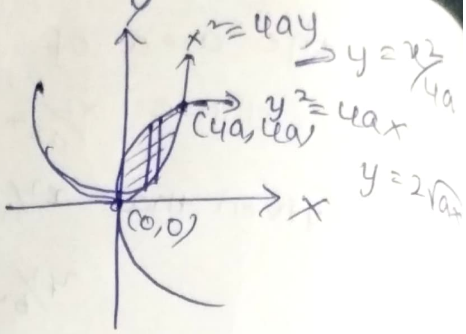
③ in ②

$$16a^2 = 4ay$$

$$\boxed{y = 4a}$$

y varies from $x^2/4a$ to $2\sqrt{ax}$

x varies from 0 to $4a$



$$\iint dx\,dy = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$$\Rightarrow \int_{x=0}^{4a} (y)\Big|_{x^2/4a}^{2\sqrt{ax}} \cdot dx$$

$$\rightarrow \int_{x=0}^{4a} (2\sqrt{ax} - x^2/4a)\,dx$$

$$\Rightarrow 2\sqrt{a}\left(\frac{2}{3}x^{3/2}\right)\Big|_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)\Big|_0^{4a}$$

$$\Rightarrow \frac{4}{3}a^2(8) - \frac{1}{12\cancel{a}}(64a^{\cancel{3}})$$

$$\Rightarrow \frac{32}{3}a^2 - \frac{16a^2}{3}$$

$$\Rightarrow \frac{16a^2}{3} \text{ units}$$

$\therefore$ Hence proved

Triple Integrals:

Ex: Order of appearance

$$\Rightarrow \int_0^1 \int_1^2 \int_2^3 xyz\,dx\,dy\,dz$$

$$\Rightarrow \int_{z=0}^{1} \int_{y=1}^{2} \int_{x=2}^{3} xyz\,dx\,dy\,dz$$

$$\Rightarrow \int_{z=0}^{1} \int_{y=1}^{2} yz\left(\frac{x^2}{2}\right)\Big|_2^3 dy\,dz$$

$$\rightarrow \frac{1}{2} \int_{z=0}^{1} \int_{y=1}^{2} yz(9-4)\,dy\,dz$$

$$\Rightarrow \frac{5}{2}\int_{z=0}^{1}\int_{y=1}^{2} yz\,dy\,dz$$

$$\Rightarrow \frac{5}{2}\int_{z=0}^{1} z\left(\frac{y^2}{2}\right)\Big|_1^2 dz$$

$$\Rightarrow \frac{5}{4}\int_{z=0}^{1} z(3)\cdot dz$$

$$\Rightarrow \frac{15}{4}\int_{z=0}^{1} z\cdot dz$$

$$\Rightarrow \frac{15}{4}\left(\frac{z^2}{2}\right)\Big|_0^1$$

$$\Rightarrow \frac{15}{4}\left(\frac{1}{2}-0\right)$$

$$= \frac{15}{8} \;//.$$

Ex: In order of given limits.

$$\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$$

$$\Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\,dy\,dx\,dz \Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\left[x(y)\Big|_{x-z}^{x+z} + \left(\frac{y^2}{2}\right)\Big|_{x-z}^{x+z} + z(y)\Big|_{x-z}^{x+z}\right] dx\,dz$$

$$\Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\left\{x[2z] + \frac{1}{2}\left[(x+z)^2-(x-z)^2\right] + z[2z]\right\} dx\,dz$$

$$\Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\left[2xz + \frac{\cancel{4}^{2}xz}{\cancel{2}} + 2z^2\right] dx\,dz$$

$$\Rightarrow \int_{z=-1}^{1}\int_{x=0}^{z}\left[4xz + 2z^2\right] dx\,dz \Rightarrow \int_{z=-1}^{1}\left\{4z\left(\frac{x^2}{2}\right)\Big|_0^z + 2z^2(x)\Big|_0^z\right\} dz$$

$$\Rightarrow \int_{z=-1}^{1}(2z^3+2z^3)\,dz = \int_{z=-1}^{1} 4z^3\,dz = 4\left(\frac{z^4}{4}\right)\Big|_{-1}^{1}$$

$$\Rightarrow \frac{4}{4}\left(1-(-1)^4\right) = 1\,(1-1) = 1(0)$$

$$= 0.$$

Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$

$$\int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} (e^{x+y+z}) \, dz \, dy \, dx$$

$$\Rightarrow \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} (e^{x+y+z}) \, dz \cdot dy \cdot dx$$

$$\Rightarrow \int_{x=0}^{a} \int_{y=0}^{x} (e^{x+y+z}) \Big|_0^{x+y} dy \, dx$$

$$\rightarrow \int_{x=0}^{a} \int_{y=0}^{x} (e^{2x+2y} - e^{x+y}) \, dy \, dx$$

$$\Rightarrow \int_{x=0}^{a} \left\{ \left(\frac{e^{2x+2y}}{2} \right) \Big|_0^x - (e^{x+y}) \Big|_0^x \right\} dx \qquad - \frac{e^{2x}}{2} - e^{2x}$$

$$\Rightarrow \int_{x=0}^{a} \left\{ \frac{1}{2} \left[e^{4x} - e^{2x} \right] - \left[e^{2x} - e^{x} \right] \right\} dx$$

$$\rightarrow \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right) dx = \left(\frac{e^{4x}}{8} \right) \Big|_0^a - \frac{3e^{2x}}{4} \Big|_0^a + (e^{x}) \Big|_0^a$$

$$\Rightarrow \frac{(e^{4a} - 1)}{8} - \frac{3}{4} (e^{2a} - 1) + (e^{a} - 1)$$

$$\Rightarrow \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} \right) + \left(-\frac{1}{8} + \frac{3}{4} - 1 \right)$$

$$\Rightarrow \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} + \left(\frac{-1 + 6 - 8}{8} \right)$$

$$\Rightarrow \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

Evaluate $\int_0 \int_0 \int_0 xyz\,dz\,dy\,dx$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$$

$$\Rightarrow \int_{x=0}^{1} x\,dx \int_{y=0}^{\sqrt{1-x^2}} y\,dy \int_{z=0}^{\sqrt{1-x^2-y^2}} z\,dz$$

$$\Rightarrow \int_{x=0}^{1} x\,dx \int_{y=0}^{\sqrt{1-x^2}} y\,dy \left(\frac{z^2}{2}\right)\Big|_0^{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} x\,dx \int_{y=0}^{\sqrt{1-x^2}} y(1-x^2-y^2)\,dy$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} x\,dx \int_{y=0}^{\sqrt{1-x^2}} (y-yx^2-y^3)\,dy$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} x\,dx \left[\left(\frac{y^2}{2}\right)\Big|_0^{\sqrt{1-x^2}} - x^2\left(\frac{y^2}{2}\right)\Big|_0^{\sqrt{1-x^2}} - \left(\frac{y^4}{4}\right)\Big|_0^{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} x\,dx \left[\frac{1}{2}(1-x^2) - \frac{x^2}{2}(1-x^2) - \frac{1}{4}(1-x^2)^2\right]$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1} x\left[\frac{1}{2} - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^4}{2} - \frac{1}{4}(1+x^4-2x^2)\right]dx$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{1}\left(\frac{x}{2} - \frac{x^3}{2} + \frac{x^5}{4} - \frac{x}{4}\right)dx$$

$$\Rightarrow \frac{1}{2}\left[\left(\frac{x^2}{4}\right)\Big|_0^1 - \left(\frac{x^4}{8}\right)\Big|_0^1 + \left(\frac{x^6}{24}\right)\Big|_0^1 - \left(\frac{x^2}{8}\right)\Big|_0^1\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{4} - \frac{1}{8} + \frac{1}{24} - \frac{1}{8}\right] = \frac{1}{2}\left[\frac{\cancel{6}-\cancel{3}+1-\cancel{3}}{24}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{24}\right]$$

$$= \frac{1}{48}$$

Finding volumes by using Triple integrals:

Volume (V) $= \iiint dx\,dy\,dz$

Ex1: Calculate the volume of the solid bounded by the planes

$x=0, y=0, x+y+z=a, z=0.$

Given that $x=0, y=0, z=0$

$x+y+z=a \rightarrow ①$

$z=a-x-y$

$z: 0 \rightarrow a-x-y$

put $z=0$ in eq ①

$x+y=a$

$y=a-x$

y varies from 0 to $a-x$

put $y=0, z=0$ in eq ①

$\boxed{x=a}$

x varies from 0 to a

volume $= \iiint dx\,dy\,dz$

$$\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dz\,dy\,dx$$

$$\int_{x=0}^{a}\int_{y=0}^{a-x} (z)\Big|_{0}^{a-x-y} dy\,dx$$

$$\int_{x=0}^{a}\int_{y=0}^{a-x} (a-x-y)\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{a}\left[a(y)\Big|_{0}^{a-x} - x(y)\Big|_{0}^{a-x} - (y^2/2)\Big|_{0}^{a-x}\right]dx$$

$$\Rightarrow \int_{x=0}^{a}\left[a(a-x) - x(a-x) - \tfrac{1}{2}(a-x)^2\right]dx$$

$$\Rightarrow \int_{x=0}^{a}\left[a^2 - ax - \cancel{ax} + x^2 - \frac{a^2}{2} - \frac{x^2}{2} + \cancel{ax}\right]dx$$

$$\Rightarrow \int_{x=0}^{a}\left(\frac{a^2}{2} - ax + \frac{x^2}{2}\right)dx$$

$$\Rightarrow \frac{a^2}{2}(x)\Big|_0^a - a\left(\frac{x^2}{2}\right)\Big|_0^a + \left(\frac{x^3}{6}\right)\Big|_0^a$$

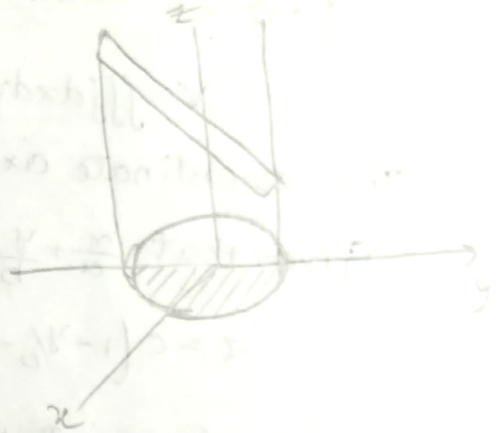
$$\Rightarrow \frac{a^2}{2}(a) - a\left(\frac{a^2}{2}\right) + \frac{a^3}{6}$$

$$\Rightarrow \frac{a^3}{2} - \frac{a^3}{2} + \frac{a^3}{6}$$

$$= a^3/6$$

2. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ & $z=0$.

Volume of solid cylinder with $\oint$ as base bounded by the given surface with generators. parallel to the z-axis is equal to $\iint z\,dx\,dy$ integrated over the region 's'.



Given that $x^2+y^2=4$

$y+z=4$

$z=0$ ①

$$\iiint dx\,dy\,dz = \iint z\,dx\,dy$$

$x^2 = 4-y^2$

$x = \pm\sqrt{4-y^2}$

Put $x=0$ in Eq ①

$y^2=4 \Rightarrow y=\pm 2$

$y+z=4$

$\boxed{z=4-y} \rightarrow$ ②

$$V = \iint z\,dx\,dy$$

$$= \int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{y=-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{y=-2}^{2} (4x-yx)\Big|_0^{\sqrt{4-y^2}}\,dy$$

$$\Rightarrow 2\int_{y=-2}^{2}\left[4(\sqrt{4-y^2}) - y\sqrt{4-y^2}\right]dy$$

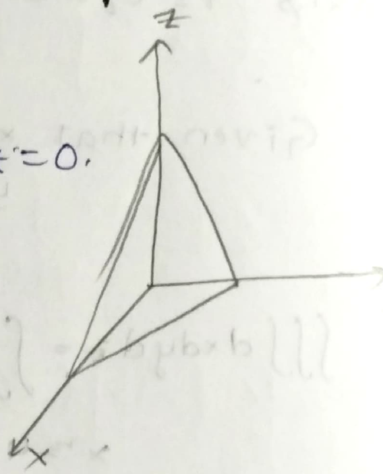
$$\Rightarrow 2\left\{8\int_{y=0}^{2}\sqrt{4-y^2}\,dy - 2\int_{y=0}^{2} y\sqrt{4-y^2}\,dy\right\}$$

$$\Rightarrow 16\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}(y/2)\right]\Big|_0^2$$

$$\Rightarrow 16\left[\cancel{2}(\pi/\cancel{2}) - 0\right]$$

$$\Rightarrow 16\pi$$

Find the volume of tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes by triple integrals.



$$V = \iiint dx\,dy\,dz$$

The co-ordinate axis are $x=0, y=0, z=0$.

Given that $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow ①$

$$z = c(1-y/b-x/a)$$

z varies from 0 to $c(1-x/a-y/b)$

put $z=0$ in eq ①

$$x/a + y/b = 1$$

$$y = b(1-x/a)$$

$\therefore y$ varies from $0 \rightarrow b(1-x/a)$

Put y & $z=0$ in eq ②

$$\frac{x}{a}=1 \Rightarrow x=a$$

$$x: 0 \rightarrow a$$

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} dz\,dy\,dx.$$

$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c(1-x/a-y/b)\,dy\,dx$$

$$\rightarrow c\int_{x=0}^{a}\left[(y) - x/a(y) - 1/b(y^2/2)\right]\Big|_0^{b(1-x/a)} \cdot dx$$

$$= c\int_{x=0}^{a}\left[b(1-x/a)-\frac{x}{a}\,b\,(1-x/a)-\frac{1}{2b}\left(b(1-x/a)\right)^2\right]dx$$

$$= c\int_{x=0}^{a}\left[b-\frac{xb}{a}-\frac{xb}{a}+\frac{x^2b}{a^2}-\frac{b}{2}\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\right]dx$$

$$= c\left[b(x)-\frac{2b}{a}\left(\frac{x^2}{2}\right)+\frac{b}{a^2}\left(\frac{x^3}{3}\right)-b/2\,(x)-\frac{b}{2a^2}\left(\frac{x^3}{3}\right)+\frac{b}{a}\left(\frac{x^2}{2}\right)\right]_0^a$$

$$= c\left[ab-\frac{2b}{a}\left(\frac{a^2}{2}\right)+\frac{b}{a^2}\left(\frac{a^3}{3}\right)-\frac{b}{2}(a)-\frac{b}{2a^2}\left(\frac{a^3}{3}\right)+\frac{b}{a}\left(\frac{a^2}{2}\right)\right]$$

$$= c\left[ab-ba+\frac{ab}{3}-\frac{ab}{2}-\frac{ab}{6}+\frac{ab}{2}\right]$$

$$= c\left[\frac{ab}{2}-\frac{ab}{6}\right]$$

$$= c\left[\frac{3ab-ab}{6}\right]$$

$$= c\left[\frac{\cancel{2}ab}{\cancel{6}3}\right]$$

$$= abc/3 \,//.$$

Find the volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The solid figure $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 equal parts by three co-ordinate planes. Hence the volume of the solid bounded by $x=0, y=0, z=0$ and surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

For a fixed point (x,y) on the xy plane

" z varies from $0 \to c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ "

Hence the required volume $(V) = 8\iiint dx\,dy\,dz$

$x=0,\ y=0,\ z=0$

Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$\therefore z$ varies from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

put $z=0$ in eq ①

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y^2=b^2(1-x^2/a^2)$$

$$y=\pm b\sqrt{1-x^2/a^2}$$

y varies from 0 to $b\sqrt{1-x^2/a^2}$

put y & $z=0$ in eq ①

$$x^2/a^2=1 \Rightarrow x^2=a^2$$

$$x=\pm a$$

$\therefore x$ varies from 0 to a

$$V=8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}}\int_{z=0}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx.$$

$$=8\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}}\left(c\sqrt{1-x^2/a^2-y^2/b^2}\right)dy\,dx$$

$$\Rightarrow 8c\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\left[\sqrt{\left(1-\frac{x^2}{a^2}\right)-y^2/b^2}\right]dy\,dx$$

$$\Rightarrow 8c\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}}\left[\sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\right]dy\,dx$$

$$\Rightarrow \frac{8c}{b}\int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-x^2/a^2}}\sqrt{b^2(1-x^2/a^2)-y^2}.dy.dx$$

Let $b\sqrt{1-x^2/a^2}=p$

$$p^2=b^2(1-x^2/a^2)$$

$$\Rightarrow \frac{8c}{b}\int_{x=0}^{a}\int_{y=0}^{p}\left(\sqrt{p^2-y^2}\right)dy.dx$$

$\sqrt{p^2-y^2} = \sqrt{p^2-p^2\sin^2\theta} = p\sqrt{\cos^2\theta} = p\cos\theta$

$dy = p\cos\theta\, d\theta$

$y=0$	$y=p$
$0=p\sin\theta$	$\cancel{p}=\cancel{p}\sin\theta$
$\boxed{\theta=0}$	$\sin\theta=1$
	$\boxed{\theta=\pi/2}$

$$\Rightarrow \frac{8c}{b}\int_{x=0}^{a}\left[\int_{\theta=0}^{\pi/2} p\cos\theta\, p\cos\theta\, d\theta\right]dx$$

$$\Rightarrow \frac{8c}{b}\int_{x=0}^{a}\left[\int_{\theta=0}^{\pi/2} p^2\cos^2\theta\, d\theta\right]dx$$

$$\Rightarrow \frac{8c}{b}\int_{x=0}^{a} p^2 \cdot \frac{1}{2}(\pi/2)\, dx \qquad \left[\because \int_0^{\pi/2}\cos^n\theta . d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\pi/2\right]$$

$$\Rightarrow \frac{\overset{2}{\cancel{8}}c}{b}\cdot \pi/\cancel{4}\int_{x=0}^{a} p^2 . dx$$

$$\Rightarrow \frac{2\pi c}{b}\int_{x=0}^{a} b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$\Rightarrow \frac{2\pi c}{\cancel{b}}\cdot\frac{b\cancel{{}^2}}{a^2}\int_{x=0}^{a}(a^2-x^2)\,dx$$

$$\Rightarrow \frac{2\pi c b}{a^2}\left[\int_{x=0}^{a} a^2 . dx - \int_{x=0}^{a} x^2 . dx\right]$$

$$\Rightarrow \frac{2\pi bc}{a^2}\ a^2(x)\Big|_0^a - \left(\frac{x^3}{3}\right)\Big|_0^a$$

$$\Rightarrow \frac{2\pi bc}{a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$\Rightarrow \frac{2\pi bc}{\cancel{a^2}}\left[\frac{2a\cancel{{}^3}}{3}\right]$$

$$\Rightarrow \frac{4\pi abc}{3} \,//.$$