Unit-5
Multiple Integrals

1. Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Sol:

$$I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy = \int_{x=0}^{5}\left[\int_{y=0}^{x^2}(x^3+xy^2)\,dy\right]dx$$

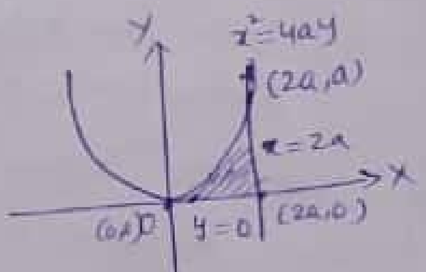
$$= \int_{x=0}^{5}\left[x^3y + x\frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_{x=0}^{5}\left[x^5+\frac{x^7}{3}\right]dx$$

$$= \left[\frac{x^6}{6}+\frac{x^8}{24}\right]_0^5 = \frac{5^6}{6}+\frac{5^8}{24} = 5^6\left[\frac{1}{6}+\frac{25}{24}\right]$$

$$= 5^6\left(\frac{29}{24}\right)$$

2. Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis & ordinate $x=2a$ & the curve $x^2=4ay$



Sol: G.T A is the domain bounded by x-axis i.e, $y=0$ & ordinate $x=2a$ & the curve $x^2=4ay$

$4a^2 = 4ay \Rightarrow y=a$

P.O.I is $(2a,a)$

x limits are 0 to 2a, y limits are $y=0$ to $y=\frac{x^2}{4a}$

$$I = \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx = \int_{x=0}^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx = \int_{x=0}^{2a}\frac{x}{2}\left[\frac{x^4}{16a^2}\right]dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\,\frac{64a^6}{6\,3} = \frac{a^4}{3}$$

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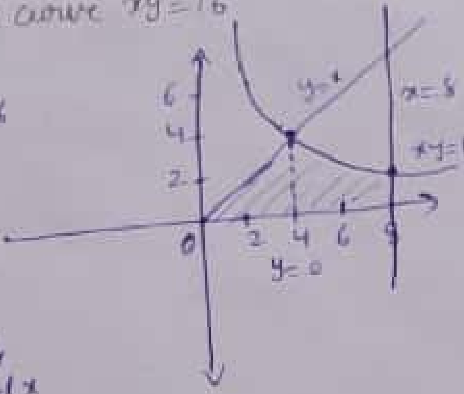
Q. changing the order of integration ... $\int^{4a} \int^{2\sqrt{ax}} dy\,dx$

3) Evaluate $\iint_R x^2 dx dy$ where R is the region in the first quadrant bounded by the lines $x=y, y=0, x=8$ & the curve $xy=16$

Sol: Given lines $x=y, y=0, x=8$ & $xy=16$



$$\iint_R x^2 dx dy = \int_{x=0}^{4}\int_{y=0}^{y=x} x^2 dx dy + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2 dx dy$$

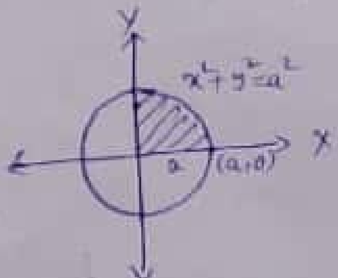
$$= \int_{x=0}^{4} x^2[y]_0^x dx + \int_{x=4}^{8} x^2[y]_0^{16/x} dx$$

$$= \int_{x=0}^{4} x^3 dx + \int_{x=4}^{8} 16x\,dx = \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64+8(64-16)$$

$$= 64+384 = 448$$

4) find $\iint xy\,dx dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$ by changing into polar coordinates.

Sol: Given circle $x^2+y^2=a^2$



Given region x varies from 0 to a

y varies from 0 to $\sqrt{a^2-x^2}$

Changing into polar coordinates

by putting $x=r\cos\theta$, $y=r\sin\theta$

$x^2+y^2=r^2$ & $dx dy = r dr d\theta$

r varies from 0 to a & θ varies from 0 to $\pi/2$

$$\iint xy\,dx dy = \int_{r=0}^{a}\int_{\theta=0}^{\pi/2} (r\cos\theta\, r\sin\theta)\, r dr d\theta$$

$$= \left[\frac{r^4}{4}\right]_0^a \frac{1}{2}\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= \frac{a^4}{4}\cdot\frac{1}{2}\left(\frac{-\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$= \frac{a^4}{16}(-\cos\pi + \cos 0) = \frac{a^4}{16}(1+1) = \frac{a^4}{8}$$

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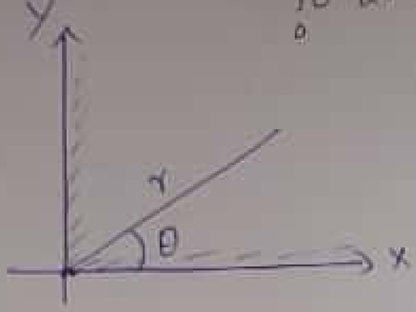
5) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing into polar coordinates. Hence show that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$



Sol: Since both x & y vary from 0 to ∞ the region of integration is the 1st quadrant of the xy-plane.

Changing into polar coordinates by putting
$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$
& $x^2+y^2 = r^2$. In the region of integration
r varies from 0 to ∞ & θ varies from 0 to $\pi/2$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

put $t = r^2 \Rightarrow dt = 2r\,dr$. If $r = 0$ then $t = 0$
If $r = \infty$ then $t = \infty$

$$= \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\frac{dt}{2}\,d\theta = \int_{\theta=0}^{\pi/2} \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_0^\infty d\theta$$

$$= \frac{-1}{2}\int_{\theta=0}^{\pi/2}(0-1)\,d\theta = \frac{-1}{2}[-\theta]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{\pi}{4}$$

Also $I = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \left[\int_0^\infty e^{-x^2}dx\right]^2$

$$\frac{\pi}{4} = \left[\int_0^\infty e^{-x^2}dx\right]^2 \Rightarrow \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$ //

6) S.T the area b/w the parabolas $y^2 = 4ax$, $x^2 = 4ay$ is $\frac{16a^2}{3}$

Sol: Given parabolas $y^2 = 4ax$ —(1), $x^2 = 4ay$ —(2)

From (1) $y = 2\sqrt{ax}$

From (2) $x^2 = 4a(2\sqrt{ax})$
$\Rightarrow x^2 = 8a\sqrt{ax}$
S.O.B.S
$\Rightarrow x^4 = 64a^2 ax$

$\Rightarrow x^4 - 64a^3x = 0$
$\Rightarrow x(x^3 - 64a^3) = 0$
$\Rightarrow x = 0,\ x^3 = (4a)^3$
$x = 4a$

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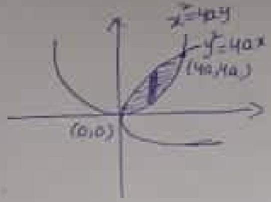
If $x=0$ then $y=0$

$x=4a$ then $y=4a$

point of intersections are $(0,0)$ & $(4a,4a)$



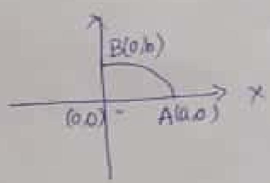
$$\text{Area} = \iint dA = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dx\,dy$$

$$= \int_{x=0}^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx = \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,4a^{3/2} - \frac{1}{12a}\,\cancel{64}^{16}a^3 = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq. units}$$

7. Find the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Sol: x limits are 0 to a

y limits are 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\text{Area} = \iint dA = \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_{x=0}^{a} [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx = \int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0\right]$$

$$= \frac{b}{a}\cdot\frac{a^2}{2}\,\frac{\pi}{2} = \frac{\pi ab}{4} \text{ sq. units}$$

8. By changing the order of integration evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

Sol:

x limits are 0 to $4a$

y limits are $\frac{x^2}{4a}$ to $2\sqrt{ax}$

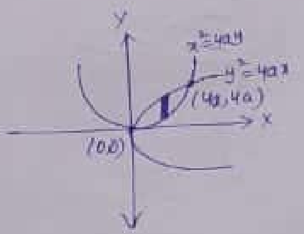
Let us draw the curves $y=\frac{x^2}{4a}$ & $y=2\sqrt{ax}$

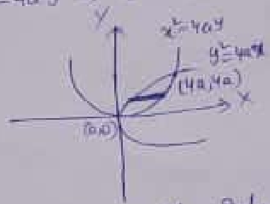
$x^2=4ay$ & $y^2=4ax$

5/1 FN

10/1 FN

24/1 AN





After changing the order of integration

x limits are $\frac{y^2}{4a}$ to $2\sqrt{ay}$

y limits are 0 to $4a$

Hence the given integral is equal to $\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$

$$= \int_{y=0}^{4a} \left[x\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_{y=0}^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{1}{4a}\cdot\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{1}{\cancel{12a}_{3}}\,\cancel{64}^{16}a^3$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

9. Change the order of integration in $I=\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ & hence evaluate

Sol:

G.T $I=\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$

x limits $0\to 1$, y limits $x^2 \to 2-x$

Let us draw the curves $y=x^2$ —(1) & $y=2-x$ —(2)

$x+y=2$

$\frac{x}{2}+\frac{y}{2}=1$

P.O.I is (a,a)

x limits 0 to y^2/a

y limits 0 to a

$$I = \int_{y=0}^{a}\int_{x=0}^{y^2/a} \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$$

$$= \int y^2 \Big/ \int^{y^2/a} \frac{1/a}{\ldots}$$

From (1) & (2)

$$x^2 = 2-x$$

$\Rightarrow x^2+x-2=0$

$\Rightarrow x^2+2x-x-2=0$

$\Rightarrow x(x+2)-1(x+2)=0$

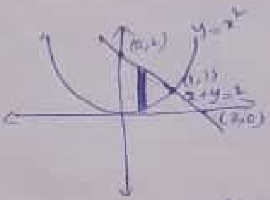
$\Rightarrow (x-1)(x+2)=0$

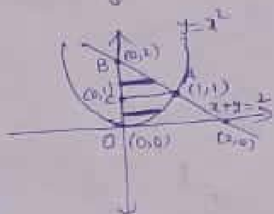
$\Rightarrow x = 1, -2$

If $x=1$ then $y=1$

If $x=-2$ then $y=4$

P.O.I's are $(1,1), (-2,4)$





$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \iint_{OACO} xy\,dx\,dy + \iint_{CABC} xy\,dx\,dy$$

In OACO

x limits are 0 to $\sqrt{y}$

y limits are 0 to 1

In CABC

x limits are 0 to $2-y$

y limits are 1 to 2

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_{y=0}^{1} y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_{y=1}^{2} y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2\,dy + \frac{1}{2}\int_{y=1}^{2} y(4+y^2-4y)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[4\frac{y^2}{2}+\frac{y^4}{4}-4\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{6}+\frac{1}{2}\left[2(4-1)+\frac{1}{4}(16-1)-\frac{4}{3}(8-1)\right]$$

$$= \frac{1}{6}+\frac{1}{2}\left[6+\frac{15}{4}-\frac{28}{3}\right] = \frac{1}{6}+\frac{1}{2}\cdot\frac{72+45-112}{12} = \frac{1}{6}+\frac{5}{24}$$

$$= \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$$

10. Change the order of integration & hence evaluate $I = \int_0^a\int_{\sqrt{ax}}^{a} \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$

Sol. Given integral $I = \int_0^a\int_{\sqrt{ax}}^{a} \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$

$x: 0 \to a \quad y: \sqrt{ax} \to a$

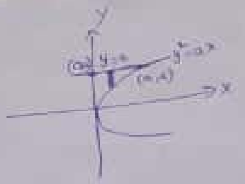
Let us draw the curves $y=\sqrt{ax}$ —(1) $y=a$ —(2)

from (1) & (2) $y^2=ax \Rightarrow a^2=ax \Rightarrow x=a$

P.O.I is (a, a)

x limits 0 to y^2/a

y limits 0 to a



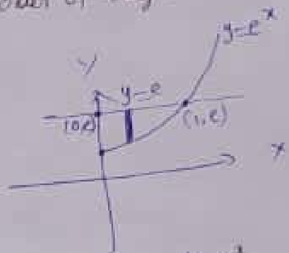
$$I = \int_{y=0}^{a} \int_{x=0}^{y^2/a} \frac{y^2\, dx\, dy}{\sqrt{y^4 - a^2x^2}}$$

$$= \int_{y=0}^{a} y^2 \left[\int_{x=0}^{y^2/a} \frac{1/a}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}\, dx \right] dy = \frac{1}{a} \int_{y=0}^{a} y^2 \left[\sin^{-1}\left(\frac{x}{y^2/a}\right) \right]_0^{y^2/a} dy$$

$$= \frac{1}{a} \int_{y=0}^{a} y^2 \left(\sin^{-1}(1) - \sin^{-1} 0 \right) dy = \frac{\pi}{2a} \int_{y=0}^{a} y^2\, dy = \frac{\pi}{2a} \left[\frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{2a} \cdot \frac{a^3}{3} = \frac{\pi a^2}{6}$$

11. Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dy\, dx}{\log y}$ by changing the order of integration



Sol:

$$I = \int_0^1 \int_{e^x}^{e} \frac{dy\, dx}{\log y}$$

$x: 0 \to 1, \quad y: e^x \to e$

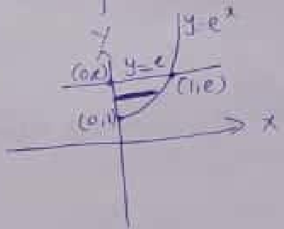
i.e., $y = e^x, \ y = e$

$\Rightarrow e = e^x$

$\Rightarrow e^{x-1} = e^0$

$\Rightarrow x - 1 = 0$

$\Rightarrow x = 1$



P.O.I $(1, e)$

y limits are 1 to e, x limits are 0 to $\log y$

$$I = \int_{y=1}^{e} \int_{x=0}^{\log y} \frac{dx\, dy}{\log y} = \int_{y=1}^{e} [x]_0^{\log y} \frac{dy}{\log y} = \int_{y=1}^{e} \log y \frac{dy}{\log y}$$

$$= [y]_1^e = e - 1$$

12. By changing order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy$, show that $\int_0^\infty \frac{\sin px}{p}\, dx = \frac{\pi}{2}$

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy = \int_{y=0}^{\infty} \left(\int_{x=0}^{\infty} e^{-xy} \sin px\, dx \right) dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{2(x+y)} - e^{2y}\, dy\, dx = \int_{x=0} \int_{y=0} e$$

$$= \int_0^\infty \left[\frac{e^{-xy}}{(-y)^2+p^2}(-y\sin px - p\cos px)\right]_0^\infty dy \qquad \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$= \int_0^\infty \frac{-1}{y^2+p^2}(0-1(p))\,dy$$

$$= \int_0^\infty \frac{p}{y^2+p^2}\,dy = p\left[\frac{1}{p}\tan^{-1}\frac{y}{p}\right]_0^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}0$$

$$= \frac{\pi}{2} \quad \text{--- (1)}$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy}\sin px\,dx\,dy = \int_{x=0}^\infty \sin px \left\{\int_{y=0}^\infty e^{-xy}dy\right\} dx$$

$$= \int_0^\infty \sin px\left[\frac{e^{-xy}}{-x}\right]_{y=0}^\infty dx$$

$$= \int_0^\infty \frac{\sin px}{-x}(0-1)\,dx$$

$$= \int_0^\infty \frac{\sin px}{x}\,dx \quad \text{--- (2)}$$

from (1) & (2), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$

13. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

Sol: Given integral $I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left\{\left(x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right)-\left(x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right)\right\} dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}(2xz+2xz+2z^2)\,dx\,dz = \int_{z=-1}^{1}\int_{x=0}^{z}(4xz+2z^2)\,dx\,dz$$

$$= 2\int_{z=-1}^{1}\int_{x=0}^{z}(2xz+z^2)\,dx\,dz = 2\int_{z=-1}^{1}\left[2z\frac{x^2}{2}+z^2x\right]_0^z dz$$

$$= 2\int_{z=-1}^{1}(z^3+z^3)\,dz = 4\int_{z=-1}^{1} z^3 dz = 4\left(\frac{z^4}{4}\right)_{-1}^{1} = 1-1=0$$

14. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol:

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2}(1-x^2-y^2)\,dy\,dx = \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}(xy-x^3y-xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{4}\int_{x=0}^{1}\left[x(1-x^2)-x^3(1-x^2)-\frac{x}{2}(1-x^2)^2\right]dx$$

$$= \frac{1}{4}\int_{x=0}^{1}\left(x-x^3-\cancel{x^3}+x^5-\frac{x}{2}-\frac{x^5}{2}+\cancel{x^3}\right)dx$$

$$= \frac{1}{4}\int_{x=0}^{1}\left(\frac{x}{2}-x^3+\frac{x^5}{2}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{4}-\frac{x^4}{4}+\frac{x^6}{12}\right]_0^1$$

$$= \frac{1}{4}\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{4}}+\frac{1}{12}\right) = \frac{1}{48}$$

15. Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

Sol:

$$I = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y+z}\,dz\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x} e^{x+y}\left[e^z\right]_0^{x+y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{x} e^{2(x+y)}-e^{x+y}\,dy\,dx = \int_{x=0}^{a}\int_{y=0}^{x} e^{2x}.e^{2y}\,dy\,dx - \int_{x=0}^{a}\int_{y=0}^{x} e^x.e^y\,dy\,dx$$

Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$= \int_{x=0}^{a} e^{2x}\left[\frac{e^{2y}}{2}\right]_0^x dx - \int_{x=0}^{a} e^{x}\left[e^{y}\right]_0^x dx = \frac{1}{2}\int_{x=0}^{a} e^{2x}(e^{2x}-1)dx - \int_{x=0}^{a} e^{x}(e^{x}-1)dx$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{4} - \frac{e^{2x}}{2}\right]_0^a - \left[\frac{e^{2x}}{2} - e^{x}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{e^{4a}}{4} - \frac{e^{2a}}{2} - \frac{1}{4} + \frac{1}{2}\right] - \left\{\frac{e^{2a}}{2} - e^{a} - \frac{1}{2} + 1\right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}$$

16. Calculate volume of the solid bounded by the planes $x=0, y=0, x+y+z=a, z=0$

Sol: Given planes $x=0, y=0, z=0, x+y+z=a$

$z = a-x-y$

If $z=0$ then $y=a-x$

If $y=0, z=0$ then $x=a$

$x: 0$ to a, $y: 0$ to $a-x$, $z: 0$ to $a-x-y$

The required volume $= \iiint_V dv = \iiint_V dx\,dy\,dz$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}\left[z\right]_0^{a-x-y} dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\,dy\,dx$$

$$= \int_{x=0}^{a}\left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a}\left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right]dx$$

$$= \int_{x=0}^{a}\frac{(a-x)^2}{2}dx = \frac{1}{2}\int_{x=0}^{a}(a^2+x^2-2ax)dx = \frac{1}{2}\left[a^2x + \frac{x^3}{3} - \cancel{2}a\frac{x^2}{\cancel{2}}\right]_0^a$$

$$= \frac{1}{2}\left[a^3 + \frac{a^3}{3} - a^3\right] = \frac{a^3}{6} \text{ cubic units}$$

17. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol: Given ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

If $z=0$ then $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

If $y=0, z=0$ then $x^2=a^2 \Rightarrow x=\pm a$

$x: -a$ to a , $y: -\frac{b}{a}\sqrt{a^2-x^2}$ to $+\frac{b}{a}\sqrt{a^2-x^2}$

$z: -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ to $+c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

The required volume $=\iiint_V dv = \iiint_V dx\,dy\,dz$

$$=\int_{x=-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}\int_{z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$=8\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$=8\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$=8c\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$$

$$=8c\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\,dx$$

$$=\frac{8c}{b}\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}}\cancel{\sqrt{\left(\sqrt{\frac{b^2}{a}\left(1-\frac{x^2}{a^2}\right)}\right)^2-y^2}\,dy\,dx}\quad\sqrt{\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2-y^2}\,dy\,dx$$

$$\cancel{=\frac{8c}{b}\int_{x=0}^{a}\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+\frac{b^2}{2}\left(1-\frac{x^2}{a^2}\right)\sin^{-1}\left(\frac{y}{b^2}\right.}$$

$$=\frac{8c}{b}\int_{x=0}^{a}\left[\frac{y}{2}\sqrt{\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2-y^2}+\frac{b^2}{2a^2}(a^2-x^2)\sin^{-1}\frac{y}{\frac{b}{a}\sqrt{a^2-x^2}}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx$$

19. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$

Sol: Given cylinder $x^2+y^2=4$ & the plane $y+z=4,\ z=0$

$$= \frac{8c}{b}\int_{x=0}^{a} \frac{b^2}{2a^2}(a^2-x^2)\sin^{-1} 1 \, dx$$

$$= \left(\frac{\cancel{8}^2 c b^{\cancel{2}}}{\cancel{2}\cancel{b}a^2}\right)\frac{\pi}{2}\int_{x=0}^{a}(a^2-x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi cb}{a^2}\left(a^3-\frac{a^3}{3}\right)$$

$$= \frac{2\pi bc}{\cancel{a^2}}\times\frac{2a^{\cancel{3}}}{3} = \frac{4\pi abc}{3} \text{ cubic units}$$

y $\int_{z=0}^{4-y} dz$

x

18 Find the volume of the tetrahedron bounded by the coordinates planes & the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol: Given planes $x=0, y=0, z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$z = c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

If $z=0$ then $y=b\left(1-\frac{x}{a}\right)$

If $y=0, z=0$ then $x=a$

x: 0 to a, y: 0 to $b\left(1-\frac{x}{a}\right)$, z: 0 to $c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

The required volume of tetrahedron $= \iiint_V dv = \iiint_V dx\,dy\,dz$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}[z]_0^{c(1-\frac{x}{a}-\frac{y}{b})}dy\,dx = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$= c\int_{x=0}^{a}\left[\left(1-\frac{x}{a}\right)y-\frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})}dx = c\int_{x=0}^{a}\left[b\left(1-\frac{x}{a}\right)^2-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$= \frac{cb}{2}\int_{x=0}^{a}\left(1-\frac{x}{a}\right)^2dx = \frac{cb}{2}(-a)\int_{x=0}^{a}\left(\frac{-1}{a}\right)\left(1-\frac{x}{a}\right)^2dx$$

$$= -\frac{abc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3}\right]_0^a = -\frac{abc}{6}(0-1) = \frac{abc}{6} \text{ cubic units.}$$

$x^2)\big)\big]\,d$

x

c units

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19. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4, z=0$

Sol: Given cylinder $x^2+y^2=4$ & the plane $y+z=4, z=0$

$z=4-y$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

If $y=0$ then $x^2=4 \Rightarrow x=\pm 2$

$x: -2$ to 2, $y: -\sqrt{4-x^2}$ to $\pm\sqrt{4-x^2}$, $z: 0$ to $4-y$

$$\text{Volume} = \iiint_V dv = \iiint_V dx\,dy\,dz = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{4-y} dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y}\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx$$

$$= \int_{x=-2}^{2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2}\left[4\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right)-\frac{1}{2}\left((4-x^2)-(4-x^2)\right)\right]dx$$

$$= 8\int_{x=-2}^{2}\sqrt{4-x^2}\,dx = 16\int_{x=0}^{2}\sqrt{4-x^2}\,dx$$

$$= 16\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 16\left[2\sin^{-1}1\right] = 32\left(\frac{\pi}{2}\right) = 16\pi \text{ cubic units.}$$

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