Probability Assignment

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Question: The frequency response $H(f)$ of of a linear time-invariant syatem has magnitude as shown in Fig. 0
Statement 1: The system is necessarily a pure delay system for inputs which are bandlimited to $-a \leq f \leq a$.
Statement 2: For any wide-sense stationary input process with power spectral density $S_X(f)$, the output power spectral density $S_Y(f)$ obeys $S_X(f) = S_Y(f)$ for $-a \leq f \leq a$.

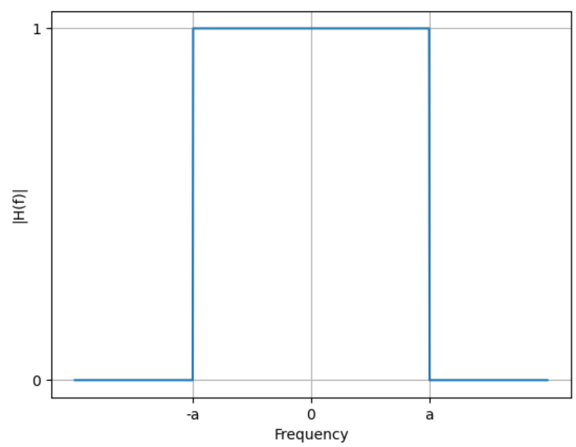

Fig. 0. $|H(f)|$ vs frequency

Solution: A system where output signal is a delayed version of input signal with no other transformations or operations is called a pure delay syatem.

1) Let us consider a pure delay LTI system with x(t) and y(t) as input and output signals in time domian. Let T_d be delay between input and output. So,

$$y(t) = x(t - T_d) \tag{1}$$

Appling fourier transform,

$$\int_{-\infty}^{\infty} y(t)e^{-2\pi fjwt}dt = \int_{-\infty}^{\infty} x(t - T_d)e^{-2\pi fjt}dt \tag{2}$$

$$= \int_{-\infty}^{\infty} x(t)e^{-2\pi fj(t+T_d)}d(t + T_d) \tag{3}$$

$$= e^{-2\pi fjT_d} \int_{-\infty}^{\infty} x(t)e^{-2\pi fjt}dt \tag{4}$$

$$Y(f) = e^{-2\pi fjT_d}X(f) \tag{5}$$

Here $Y(f)$ and $X(f)$ are output and input signals in frequency domian. Let $H(f)$ be

$$H(f) = \frac{Y(f)}{X(f)} \tag{6}$$

$$= e^{-2\pi fjT_d} \tag{7}$$

$$= (1)e^{-2\pi fjT_d} \tag{8}$$

Comparing (8) with

$$H(f) = |H(f)|e^{j\angle H(f)} \tag{9}$$

$$|H(f)| = 1 \tag{10}$$

$$\angle H(f) = -2\pi fT_d \tag{11}$$

As input is bandlimited in $-a \leq f \leq a$.We have

$$|H(f)| = \begin{cases} 1 & \text{if } -a \leq f \leq a \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

From (12) and Fig. 0, $|H(f)|$ is same.But from (11), $\angle H(f)$ is directly propotional to frequency and time delay. The system will acts as pure delay sytem. Now, if we take f^2 as frequency, the system doesn't act as pure delay system.
Example : Consider

$$H(f) = e^{-2\pi f^2 jT_d} \tag{13}$$

Appling inverse fourier transform to get time responce.

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{2\pi jft}df \tag{14}$$

$$= \sqrt{\frac{T_d}{it}}e^{-(\frac{i\pi t}{T_d})} \tag{15}$$

But $h(t)$ should be delta function that is $\delta(t-T_d)$ to get $x(t-T_d)$ when convoluted with $x(t)$. Therfore system doesn't necessarily be pure delay even $|H(f)|$ is same. So, Statement 1 is incorrect.

2) For wide-sense stationary LTI sytem,

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{16}$$

From (12), for $-a \leq f \leq a$,

$$S_Y(f) = (1)^2 S_X(f) \tag{17}$$

$$S_Y(f) = S_X(f) \tag{18}$$

Statement 2 is correct.