Probability Assignment

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Question: Let X be a positive valued continuous random variable with finite mean μ. If $Y = [X]$, the largest integer less than or equal to X, then which of the following statements is/are true?

(A) $\Pr(Y \leq \mu) \leq \Pr(X \leq \mu)$ for all $\mu \geq 0$
(B) $\Pr(Y \geq \mu) \leq \Pr(X \geq \mu)$ for all $\mu \geq 0$
(C) E(X) < E(Y)
(D) E(X) > E(Y)

Solution: Given that X is a positive valued random variable and $Y = [X]$.So,

$$X = Y + Z \tag{1}$$

Here, Z is an uniform distrubtion.

$$Z \sim U[0,1] \tag{2}$$

$$F_Z(x) = x \tag{3}$$

$$E(Z) = \frac{1}{2} \tag{4}$$

Consider

1)

$$\begin{aligned}
\Pr(Y \leq \mu) &= \Pr(X - Z \leq \mu) && (5)\\
&= \Pr(Z \geq X - \mu) && (6)\\
&= E(1 - F_Z(X - \mu)) && (7)\\
&= E(1 - X + \mu) && (8)\\
&= 1 - E(X) + \mu && (9)\\
&= 1 && (10)
\end{aligned}$$

From option (A), we have $1 \leq \Pr(X \leq \mu)$. Option (A) is wrong since probability can't be greater than 1.

2)

$$\begin{aligned}
\Pr(Y \geq \mu) &= \Pr(X - Z \geq \mu) && (11)\\
&= \Pr(Z \leq X - \mu) && (12)\\
&= E(F_Z(X - \mu)) && (13)\\
&= E(X - \mu) && (14)\\
&= E(X) - \mu && (15)\\
&= 0 && (16)
\end{aligned}$$

From option B, we have $\Pr(X \leq \mu) \geq 0$. Option (B) is correct.

3)

$$\begin{aligned}
E(Y) &= E(X - Z) && (17)\\
&= E(X) - E(Z) && (18)\\
&= \mu - \frac{1}{2} && (19)\\
&= E(X) - \frac{1}{2} && (20)
\end{aligned}$$

$E(X) > E(Y)$. Option (D) is correct and (C) is wrong.

Verification using Gaussian:
Let X be a postive valued gaussian random variable with mean μ.

$$X \sim \mathcal{N}(\mu, \sigma) \tag{21}$$

$$Y = [X] \tag{22}$$

Consider X alwyas positive

1)

$$\begin{aligned}
\Pr(Y \leq \mu) &= \Pr([X] \leq \mu) && (23)\\
&= \Pr(0 \leq X < [\mu] + 1) && (24)\\
&= \int_0^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx && (25)\\
&= \int_0^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&\quad + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx && (26)\\
&\approx \Pr(X \leq \mu) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx && (27)
\end{aligned}$$

So, $\Pr(Y \leq \mu) \geq \Pr(X \leq \mu)$. Option (A) wrong.

2) For integer μ

$$\begin{aligned}
\Pr(Y \geq \mu) &= 1 - \Pr(Y < \mu) && (28)\\
&= 1 - \Pr([X] < \mu) && (29)\\
&= 1 - \Pr(X < \mu) && (30)\\
&= \Pr(X \geq \mu) && (31)
\end{aligned}$$

For non-integer μ

$$\Pr(Y \geq \mu) = 1 - \Pr(Y < \mu) \tag{32}$$
$$= 1 - \Pr([X] < \mu) \tag{33}$$
$$= 1 - \Pr(0 \leq X < [\mu] + 1) \tag{34}$$
$$= 1 - \left(\int_0^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) \tag{35}$$
$$= 1 - \left(\int_0^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) \tag{36}$$
$$\approx 1 - \left(\Pr(X < \mu) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) \tag{37}$$
$$= \Pr(X \geq \mu) - \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \tag{38}$$

So, $\Pr(Y \geq \mu) \leq \Pr(X \geq \mu)$. Option (B) correct.

Steps for Simulation:

1) Taking n samples, Generate n positive Gaussian random variable(X) samples.
2) Generate n samples of $Y = [X]$ by floor to every sample of X.
3) Find number of samples of X where $X \leq \mu$ and $X \geq \mu$ and divide with n to get $\Pr(X \leq \mu)$ and $\Pr(X \geq \mu)$ respectively.
4) Find number of samples of Y where $Y \leq \mu$ and $Y \geq \mu$ and divide with n to get $\Pr(Y \leq \mu)$ and $\Pr(Y \geq \mu)$ respectively.
5) Sum the n samples of X and Y and divide with n to get $E(X)$ and $E(Y)$.