Probability Assignment

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Question: Let $\{X_n\}_{n\geq1}$ and Let $\{Y_n\}_{n\geq1}$ be two sequences of random variables and X and Y be two random variables, all of them defined on the same probability space. Which one of the following statements is true?

(A) If $\{X_n\}_{n\geq1}$ converges in distribution to a real constant c, then $\{X_n\}_{n\geq1}$ converges in probability to c.
(B) If $\{X_n\}_{n\geq1}$ converges in probability to X, then $\{X_n\}_{n\geq1}$ converges in 3^{rd} mean to X.
(C) If $\{X_n\}_{n\geq1}$ converges in distribution to X and $\{Y_n\}_{n\geq1}$ converges in distribution to Y, then $\{X_n+Y_n\}_{n\geq1}$ converges in distribution to $X+Y$.
(D) If $\{E(X_n)\}_{n\geq1}$ converges to $E(X)$, then $\{X_n\}_{n\geq1}$ converges in 1^{st} mean to X.

Solution:

X_n converges in distribution to X, $X_n \xrightarrow{d} X$, then for all x,

$$F_{X_n}(x) \to F_X(x) \tag{1}$$

X_n converges in probability to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,

$$lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \tag{2}$$

For $\epsilon > 0$, B be defined as

$$B = \{x : |x-c| \geq \epsilon\} \tag{3}$$

Now,

$$\Pr(|X_n - c| \geq \epsilon) = \Pr(X_n \in B) \tag{4}$$

Using Portmanteau Lemma, if $X_n \xrightarrow{d} c$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \Pr(X_n \in B) &\leq \Pr(c \in B) && (5)\\
&\leq \Pr(|0-0| \geq \epsilon) && (6)\\
&\leq \Pr(0 \geq \epsilon) && (7)\\
&\leq 0 && (8)\\
&= 0 && (9)\\
lim_{n\to\infty} \Pr(|X_n - c| > \epsilon) = 0 & && (10)
\end{aligned}$$

From (2), $X_n \xrightarrow{p} c$. So, we have

$$X_n \xrightarrow{d} c \implies X_n \xrightarrow{p} c \tag{11}$$

Option (A) is correct.