Probability Assignment

EE22BTECH11022-G.SAI HARSHITH*

Question: Let X be a positive valued continuous random variable with finite mean μ. If $Y = [X]$, the largest integer less than or equal to X, then which of the following statements is/are true?

(A) $\Pr(Y \le \mu) \le \Pr(X \le \mu)$ for all $\mu \ge 0$
(B) $\Pr(Y \ge \mu) \le \Pr(X \ge \mu)$ for all $\mu \ge 0$
(C) E(X) < E(Y)
(D) E(X) > E(Y)

Solution: Given that X is a positive valued random variable and $Y = [X]$.So,

$$X = Y + Z \tag{1}$$

Here, Z is an uniform distrubtion.

$$Z \sim U[0,1) \tag{2}$$
$$F_Z(x) = x \tag{3}$$
$$E(Z) = \frac{1}{2} \tag{4}$$

Consider

$$\begin{aligned}\Pr(Y \le \mu) &= \Pr(X - Z \le \mu) && (5)\\ &= \Pr(Z \ge X - \mu) && (6)\\ &= E(1 - F_Z(X - \mu)) && (7)\\ &= E(1 - X + \mu) && (8)\\ &= 1 - E(X) + \mu && (9)\\ &= 1 && (10)\end{aligned}$$

From option (A), we have $1 \le \Pr(X \le \mu)$. Option (A) is wrong since probability can't be greater than 1.

$$\begin{aligned}\Pr(Y \ge \mu) &= \Pr(X - Z \ge \mu) && (11)\\ &= \Pr(Z \le X - \mu) && (12)\\ &= E(F_Z(X - \mu)) && (13)\\ &= E(X - \mu) && (14)\\ &= E(X) - \mu && (15)\\ &= 0 && (16)\end{aligned}$$

From option B, we have $\Pr(X \le \mu) \ge 0$. Option (B) is correct.

$$\begin{aligned}E(Y) &= E(X - Z) && (17)\\ &= E(X) - E(Z) && (18)\\ &= \mu - \frac{1}{2} && (19)\\ &= E(X) - \frac{1}{2} && (20)\end{aligned}$$

$E(X) > E(Y)$. Option (D) is correct and (C) is wrong.

Verification:

As uniform distribution is a continuous distributiion. Let X be uniform distribution. Let 0 < x < 1.

$$X \sim U[0,1) \tag{21}$$
$$\mu = E(X) = \frac{1}{2} \tag{22}$$
$$p_X(x) = 1 \tag{23}$$
$$\begin{aligned}F_X(x) &= \Pr(X \le x) && (24)\\ &= x && (25)\end{aligned}$$

GIven $Y = [X]$, So, $Y = 0$.

$$Y = 0 \tag{26}$$
$$\begin{aligned}E(Y) &= 0 p_Y(0) && (27)\\ &= 0 && (28)\end{aligned}$$

From (22) and (28),

$$E(X) > E(Y) \tag{29}$$

Now,

$$\Pr(Y \le \mu) = \Pr\left(Y \le \frac{1}{2}\right) \tag{30}$$
$$= \Pr(Y = 0) \tag{31}$$
$$= 1 \tag{32}$$
$$\Pr(X \le \mu) = \Pr\left(X \le \frac{1}{2}\right) \tag{33}$$
$$= F_X\left(\frac{1}{2}\right) \tag{34}$$
$$= \frac{1}{2} \tag{35}$$
$$\Pr(Y \ge \mu) = \Pr\left(Y \ge \frac{1}{2}\right) \tag{36}$$
$$= 0 \tag{37}$$
$$\Pr(X \ge \mu) = \Pr\left(X \ge \frac{1}{2}\right) \tag{38}$$
$$= 1 - \Pr\left(X \le \frac{1}{2}\right) \tag{39}$$
$$= 1 - F_X\left(\frac{1}{2}\right) \tag{40}$$
$$= 1 - \frac{1}{2} \tag{41}$$
$$= \frac{1}{2} \tag{42}$$

From (32) and (35),

$$\Pr(Y \le \mu) \ge \Pr(X \le \mu) \tag{43}$$

From (37) and (42),

$$\Pr(Y \ge \mu) \le \Pr(X \ge \mu) \tag{44}$$

Option (B) and (D) are correct.