1

Probability Assignment

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Question: Let X be a positive valued continuous random variable with finite mean μ . If Y = [X], the largest integer less than or equal to X, then which of the following statements is/are true?

- (A) $Pr(Y \le \mu) \le Pr(X \le \mu)$ for all $\mu \ge 0$
- (B) $Pr(Y \ge \mu) \le Pr(X \ge \mu)$ for all $\mu \ge 0$
- (C) E(X) < E(Y)
- (D) E(X) > E(Y)

Solution: Given that X is a positive valued random variable and Y = [X]. So,

$$X = Y + Z \tag{1}$$

Here, Z is an uniform distrubtion.

$$Z \sim U[0,1) \tag{2}$$

$$F_Z(x) = x \tag{3}$$

$$E(Z) = \frac{1}{2} \tag{4}$$

Consider

1)

$$Pr(Y \le \mu) = Pr(X - Z \le \mu)$$

$$= Pr(Z \ge X - \mu)$$
(6)

$$= E(1 - F_Z(X - \mu)) \tag{7}$$

$$= E(1 - X + \mu) \tag{8}$$

$$=1-E(X)+\mu\tag{9}$$

$$= 1 \tag{10}$$

From option (A), we have $1 \le \Pr(X \le \mu)$. Option (A) is wrong since probability can't be greater than 1.

2)

$$\Pr(Y \ge \mu) = \Pr(X - Z \ge \mu) \tag{11}$$

$$= \Pr\left(Z \le X - \mu\right) \tag{12}$$

$$= E(F_Z(X - \mu)) \tag{13}$$

$$= E(X - \mu) \tag{14}$$

$$= E(X) - \mu \tag{15}$$

$$=0 (16)$$

From option B, we have $Pr(X \le \mu) \ge 0$. Option (B) is correct.

3)

$$E(Y) = E(X - Z) \tag{17}$$

$$= E(X) - E(Z) \tag{18}$$

$$=\mu-\frac{1}{2}\tag{19}$$

$$= E(X) - \frac{1}{2} \tag{20}$$

E(X) > E(Y). Option (D) is correct and (C) is wrong.

Verification using Gaussian:

Let X be a postive valued gaussian random variable with mean μ .

$$X \sim \mathcal{N}(\mu, \sigma)$$
 (21)

$$Y = [X] \tag{22}$$

Consider X alwyas positive

1)

$$\Pr(Y \le \mu) = \Pr([X] \le \mu) \tag{23}$$

$$= \Pr(0 \le X < [\mu] + 1) \tag{24}$$

$$= \int_{0}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx \qquad (25)$$

$$=\int_0^\mu \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (26)$$

$$\approx \Pr\left(X \le \mu\right) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx$$
(27)

So, $\Pr(Y \le \mu) \ge \Pr(X \le \mu)$. Option (A) wrong.

2) For integer μ

$$\Pr(Y \ge \mu) = 1 - \Pr(Y < \mu)$$
 (28)

$$= 1 - \Pr([X] < \mu)$$
 (29)

$$= 1 - \Pr(X < \mu)$$
 (30)

$$= \Pr(X \ge \mu) \tag{31}$$

For non-integer μ

$$\Pr(Y \ge \mu) = 1 - \Pr(Y < \mu)$$

$$= 1 - \Pr([X] < \mu)$$

$$= 1 - \Pr(0 \le X < [\mu] + 1)$$

$$= 1 - \left(\int_{0}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx\right)$$

$$= 1 - \left(\int_{0}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx\right)$$

$$\approx 1 - \left(\Pr(X < \mu) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx\right)$$

$$= \Pr(X \ge \mu) - \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^{2}} dx$$

$$(38)$$

So, $Pr(Y \ge \mu) \le Pr(X \ge \mu)$. Option (B) correct.

Steps for Simulation:

- 1) Taking n samples, Generate n positive Gaussian random variable(*X*) samples.
- 2) Generate n samples of Y = [X] by floor to every sample of X.
- 3) Find number of samples of X where $X \le \mu$ and $X \ge \mu$ and divide with n to get $\Pr(X \le \mu)$ and $\Pr(X \ge \mu)$ respectively.
- 4) Find number of samples of Y where $Y \le \mu$ and $Y \ge \mu$ and divide with n to get $\Pr(Y \le \mu)$ and $\Pr(Y \ge \mu)$ respectively.
- 5) Sum the n samples of X and Y and divide with n to get E(X) and E(Y).