Probability Assignment

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Question: Let $\{X_n\}_{n\geq1}$ and Let $\{Y_n\}_{n\geq1}$ be two sequences of random variables and X and Y be two random variables, all of them defined on the same probability space. Which one of the following statements is true?

(A) If $\{X_n\}_{n\geq1}$ converges in distribution to a real constant c, then $\{X_n\}_{n\geq1}$ converges in probability to c.
(B) If $\{X_n\}_{n\geq1}$ converges in probability to X, then $\{X_n\}_{n\geq1}$ converges in 3^{rd} mean to X.
(C) If $\{X_n\}_{n\geq1}$ converges in distribution to X and $\{Y_n\}_{n\geq1}$ converges in distribution to Y, then $\{X_n+Y_n\}_{n\geq1}$ converges in distribution to $X+Y$.
(D) If $\{E(X_n)\}_{n\geq1}$ converges to $E(X)$, then $\{X_n\}_{n\geq1}$ converges in 1^{st} mean to X.

Solution:

1) X_n converges in distribution to X, $X_n \xrightarrow{d} X$, then for all x,
$$lim_{n\to\infty}F_{X_n}(x) = F_X(x) \tag{1}$$
2) X_n converges in probability to X, $X_n \xrightarrow{p} X$, then for all $\epsilon > 0$,
$$lim_{n\to\infty}\Pr(|X_n - X| > \epsilon) = 0 \tag{2}$$
3) X_n converges in p^{th} mean to X, then we have
$$lim_{n\to\infty}E(|X_n - X|^p) = 0 \tag{3}$$

(A) For $\epsilon > 0$, B be defined as
$$B = \{x : |x - c| \geq \epsilon\} \tag{4}$$
Now,
$$\Pr(|X_n - c| \geq \epsilon) = \Pr(X_n \in B) \tag{5}$$
Using Portmanteau Lemma, if $X_n \xrightarrow{d} c$, we have
$$\limsup_{n\to\infty}\Pr(X_n \in B) \leq \Pr(c \in B) \tag{6}$$
$$\leq \Pr(|0-0| \geq \epsilon) \tag{7}$$
$$\leq \Pr(0 \geq \epsilon) \tag{8}$$
$$\leq 0 \tag{9}$$
$$= 0 \tag{10}$$
$$lim_{n\to\infty}\Pr(|X_n - c| > \epsilon) = 0 \tag{11}$$
From (2), $X_n \xrightarrow{p} c$. So, we have
$$X_n \xrightarrow{d} c \implies X_n \xrightarrow{p} c \tag{12}$$
Option (A) is correct.

(B) Statement (B) is may or may not correct. Counter Example: Consider
$$\{X_n\}_{n\geq1} = \begin{cases} 0 & \text{for probability } 1-\frac{1}{n} \\ n & \text{for probability } \frac{1}{n} \end{cases} \tag{13}$$
X_n converges in probability to $X = 0$. Since as n increases, the factor $1-\frac{1}{n}$ increases. So, the area under curve $|X_n - X|$ decreases, at some point we get
$$lim_{n\to\infty}|X_n - X| < \epsilon \tag{14}$$
There exists an $\epsilon > 0$, so,
$$lim_{n\to\infty}\Pr(|X_n - X| > \epsilon) = 0 \tag{15}$$
But X_n does not converges in 3^{rd} mean to $X = 0$.
$$lim_{n\to\infty}E(|X_n - X|^3) = lim_{n\to\infty}E(X_n^3) \tag{16}$$
$$= lim_{n\to\infty}0^3\left(1-\frac{1}{n}\right) + n^3\left(\frac{1}{n}\right) \tag{17}$$
$$= lim_{n\to\infty}n^2 \neq 0 \tag{18}$$

(C) Statement (C) is may or may not correct. Counter Example: Consider
$$Z \sim \mathcal{N}(0,1) \tag{19}$$
Let $\{X_n\}_{n\geq1}$ and $\{Y_n\}_{n\geq1}$ be sequences of random variables such that they both converge in distribution as Z and $(-1)^nZ$. So, we have
$$F_{X_n+Y_n}(x) = \Pr(X_n + Y_n \leq x) \tag{20}$$
$$= \Pr(Z + (-1)^nZ \leq x) \tag{21}$$
For n is even
$$F_{X_n+Y_n}(x) = \Pr(2Z \leq x) \tag{22}$$
$$= \Pr\left(Z \leq \frac{x}{2}\right) \tag{23}$$
$$= Q\left(\frac{x}{2}\right) \tag{24}$$

For n is odd

$$F_{X_n+Y_n}(x) = \Pr(0 \leq x) \tag{25}$$
$$= \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = H(x) \tag{26}$$

So, on generalizing

$$F_{X_n+Y_n}(x) = \begin{cases} Q\left(\frac{x}{2}\right) & \text{if } n \text{ is even} \\ H(x) & \text{if } n \text{ is odd} \end{cases} \tag{27}$$

$lim_{n\to\infty} F_{X_n+Y_n}(x)$ oscillate between $Q\left(\frac{x}{2}\right)$ and $H(x)$. This doesnot imply convergence.

(D) Statement (D) is may or may not correct. Counter Example: Consider

$$\{X_n\}_{n\geq 1} = \begin{cases} 0 & \text{for probability } 1 - \frac{1}{n} \\ n & \text{for probability } \frac{1}{n} \end{cases} \tag{28}$$

$$lim_{n\to\infty} E(X_n) = 0\left(1 - \frac{1}{n}\right) + n\left(\frac{1}{n}\right) = 1 \tag{29}$$

As $n \to \infty$, $E(X_n)$ converges to $E(X) = 1$. From (28),

$$lim_{n\to\infty} X_n = 0 = X \tag{30}$$

To find 1^{st} mean convergennce of X_n. From (29)

$$lim_{n\to\infty} E(|X_n - X|) = lim_{n\to\infty} E(X_n) \tag{31}$$
$$= 1 \neq 0 \tag{32}$$

So, X_n does not converges in 1^{st} mean to X.