Probability Assignment

EE22BTECH11022-G.SAI HARSHITH*

Question: Let X be a positive valued continuous random variable with finite mean μ. If $Y = [X]$, the largest integer less than or equal to X, then which of the following statements is/are true?

(A) $\Pr(Y \le \mu) \le \Pr(X \le \mu)$ for all $\mu \ge 0$
(B) $\Pr(Y \ge \mu) \le \Pr(X \ge \mu)$ for all $\mu \ge 0$
(C) E(X) < E(Y)
(D) E(X) > E(Y)

Solution: Given that X is a positive valued random variable and $Y = [X]$.So,

$$X = Y + Z \tag{1}$$

Here, Z is an uniform distrubtion.

$$Z \sim U[0,1] \tag{2}$$
$$F_Z(x) = x \tag{3}$$
$$E(Z) = \frac{1}{2} \tag{4}$$

Consider

1)

$$\begin{aligned}
\Pr(Y \le \mu) &= \Pr(X - Z \le \mu) &(5)\\
&= \Pr(Z \ge X - \mu) &(6)\\
&= E(1 - F_Z(X - \mu)) &(7)\\
&= E(1 - X + \mu) &(8)\\
&= 1 - E(X) + \mu &(9)\\
&= 1 &(10)
\end{aligned}$$

From option (A), we have $1 \le \Pr(X \le \mu)$. Option (A) is wrong since probability can't be greater than 1.

2)

$$\begin{aligned}
\Pr(Y \ge \mu) &= \Pr(X - Z \ge \mu) &(11)\\
&= \Pr(Z \le X - \mu) &(12)\\
&= E(F_Z(X - \mu)) &(13)\\
&= E(X - \mu) &(14)\\
&= E(X) - \mu &(15)\\
&= 0 &(16)
\end{aligned}$$

From option B, we have $\Pr(X \le \mu) \ge 0$. Option (B) is correct.

3)

$$\begin{aligned}
E(Y) &= E(X - Z) &(17)\\
&= E(X) - E(Z) &(18)\\
&= \mu - \frac{1}{2} &(19)\\
&= E(X) - \frac{1}{2} &(20)
\end{aligned}$$

$E(X) > E(Y)$. Option (D) is correct and (C) is wrong.

Verification using Gaussian:

Let X be a postive valued gaussian random variable with mean μ.

$$X \sim \mathcal{N}(\mu, \sigma) \tag{21}$$
$$Y = [X] \tag{22}$$

Consider X alwyas positive

1)

$$\begin{aligned}
\Pr(Y \le \mu) &= \Pr([X] \le \mu) &(23)\\
&= \Pr(0 \le X < [\mu] + 1) &(24)\\
&= \int_0^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx &(25)\\
&= \int_0^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&\quad + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx &(26)\\
&\approx \Pr(X \le \mu) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx &(27)
\end{aligned}$$

So, $\Pr(Y \le \mu) \ge \Pr(X \le \mu)$. Option (A) wrong.

2) For integer μ

$$\begin{aligned}
\Pr(Y \ge \mu) &= 1 - \Pr(Y < \mu) &(28)\\
&= 1 - \Pr([X] < \mu) &(29)\\
&= 1 - \Pr(X < \mu) &(30)\\
&= \Pr(X \ge \mu) &(31)
\end{aligned}$$

For non-integer μ

$$\begin{aligned}
\Pr(Y \geq \mu) &= 1 - \Pr(Y < \mu) && (32)\\
&= 1 - \Pr([X] < \mu) && (33)\\
&= 1 - \Pr(0 \leq X < [\mu] + 1) && (34)\\
&= 1 - \left(\int_0^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) && (35)\\
&= 1 - \left(\int_0^{\mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) && (36)\\
&\approx 1 - \left(\Pr(X < \mu) + \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) && (37)\\
&= \Pr(X \geq \mu) - \int_{\mu}^{[\mu]+1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx && (38)
\end{aligned}$$

So, $\Pr(Y \geq \mu) \leq \Pr(X \geq \mu)$. Option (B) correct.