

Assignment 6

CS20BTECH11028

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1 PROBLEM UGC/MATH DEC 2018 104

Let X_1, X_2, \dots be i.i.d. $N(0, 1)$ random variables. Let $S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1$. Which of the following statements are correct?

- (A) $\frac{S_n - n}{\sqrt{2}} \sim N(0, 1)$ for all $n \geq 1$
- (B) For all $\epsilon > 0, \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0$ as $n \rightarrow \infty$
- (C) $\frac{S_n}{n} \rightarrow 1$ with probability 1
- (D) $\Pr(S_n \leq n + \sqrt{n}x) \rightarrow \Pr(Y \leq x) \forall x \in \mathbb{R}$, where $Y \sim N(0, 2)$

2 SOLUTION

Definition 2.1 (Almost sure convergence). A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to converge almost surely or with probability 1 (denoted by a.s or w.p 1) to X if

$$\Pr(\omega | X_n(\omega) \rightarrow X(\omega)) = 1 \quad (2.0.1)$$

Definition 2.2 (Convergence in probability). A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to converge in probability (denoted by i.p) to X if

$$\lim_{n \rightarrow \infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \quad (2.0.2)$$

Theorem 2.1 (Weak law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \quad (2.0.3)$$

in probability

Theorem 2.2 (Strong law of large numbers). Let X_1, X_2, \dots be i.i.d random variables with same expectation(μ) and finite variance(σ^2). Let $S_n = X_1 + X_2 + \dots + X_n$, Then as $n \rightarrow \infty$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \quad (2.0.4)$$

almost surely.

Theorem 2.3 (Central limit theorem). The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger, given that all the samples are equal in size, regardless of the distribution of the individual samples.

Given X_1, X_2, \dots follow normal distribution with mean 0 and variance 1.

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \dots\} \quad (2.0.5)$$

As X_1, X_2, \dots are i.i.d random variables therefore X_1^2, X_2^2, \dots are also identical and independent. We can write

$$E(X^2) = \text{Var}(X) \quad (2.0.6)$$

(A)

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (2.0.7)$$

$$= \frac{\sum_i E(X_i^2 - 1)}{\sqrt{2}} \quad (2.0.8)$$

From (2.0.6) we can write

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \quad (2.0.9)$$

$$\text{Var}\left(\frac{S_n - n}{\sqrt{2}}\right) = \text{Var}\left(\frac{\sum_i (X_i^2 - 1)}{\sqrt{2}}\right) \quad (2.0.10)$$

$$= \frac{\sum_i \text{Var}(X_i^2 - 1)}{\sqrt{2}} \quad (2.0.11)$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (X_i^2 - 1)^2 f_{X_i}(x) dx \quad (2.0.12)$$

$$= \int_{-\infty}^{\infty} (X_i^4 + 1 - 2X_i^2) f_{X_i}(x) dx \quad (2.0.13)$$

$$= 2 \quad (2.0.14)$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n \sqrt{2} \quad (2.0.15)$$

Hence from theorem 2.2 as $n \rightarrow \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n \sqrt{2}) \quad (2.0.16)$$

Hence **Option A is false.**

(B) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (2.0.17)$$

Hence from theorem 2.1 we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \quad (2.0.18)$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \quad (2.0.19)$$

in probability. From definition 2.2 we can write,

$$\implies \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \rightarrow 0, \forall \epsilon > 0 \quad (2.0.20)$$

Hence **Option B is false .**

(C) Given

$$S_n = X_1^2 + X_2^2 + \dots + X_n^2, \forall n \geq 1 \quad (2.0.21)$$

Hence from theorem ?? we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \quad (2.0.22)$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \quad (2.0.23)$$

almost surely. From definition 2.1 we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \quad (2.0.24)$$

with probability 1. Hence **Option C is true.**

(D) Consider,

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \quad (2.0.25)$$

using (2.0.6) and (2.0.8).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = \frac{2n}{\sqrt{n}} \quad (2.0.26)$$

$$= 2\sqrt{n}. \quad (2.0.27)$$

using (2.0.14). From theorem 2.3 we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \quad (2.0.28)$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) = \Pr(S_n \leq n + \sqrt{n}x) \quad (2.0.29)$$

Hence using (2.0.28), **Option D is false.**