UGC/MATH Dec 2018 104

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Question

Problem UGC/MATH Dec 2018 104

Let $X_1, X_2, \cdots$ be i.i.d. $N(0,1)$ random variables.Let $S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1$.Which of the following statements are correct?

- (A) $\frac{S_n - n}{\sqrt{2}} \sim N(0,1)$ for all $n \geq 1$
- (B) For all $\epsilon > 0$,$\Pr\left(\left|\frac{S_n}{n} - 2\right| > \epsilon\right) \to 0$ as $n \to \infty$
- (C) $\frac{S_n}{n} \to 1$ with probability 1
- (D) $\Pr(S_n \leq n + \sqrt{n}x) \to \Pr(Y \leq x) \forall x \in R$,where $Y \sim N(0,2)$

Some definitions

Definition (Almost sure convergence)

A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge **almost surely** or **with probability 1** to X if

$$\Pr(\omega | X_n(\omega) \to X(\omega)) = 1 \tag{1}$$

Notation:

$$X_n \xrightarrow{a.s} X \tag{2}$$

Some definitions

Definition (Convergence in probability)

A sequence of random variables $\{X_n\}_{n \in N}$ is said to converge in probability to X if

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0, \forall \epsilon > 0 \tag{3}$$

Notation:

$$X_n \xrightarrow{i.p} X \tag{4}$$

Some theorems

Theorem (Central limit theorem)

The Central limit theorem states that the distribution of the sample approximates a normal distribution as the sample size becomes larger,given that all the samples are equal in size,regardless of the distribution of the individual samples.

Theorem (Weak law of large numbers)

Let $X_1, X_2, \cdots$ *be i.i.d random variables with same expectation*(μ) *and finite variance*(σ^2)*.Let* $S_n = X_1 + X_2 + \cdots X_n$*, Then as* $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{i.p} \mu, \tag{5}$$

in probability

Some theorems

Theorem (Strong law of large numbers)

Let $X_1, X_2, \cdots$ *be i.i.d random variables with same expectation*(μ) *and finite variance*(σ^2)*.Let* $S_n = X_1 + X_2 + \cdots X_n$*, Then as* $n \to \infty$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu, \tag{6}$$

almost surely.

Solution

We know that a random variable following normal distribution with mean μ and variance σ^2 is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{7}$$

Given $X_1, X_2, \cdots$ follow normal distribution with mean 0 and variance 1.Hence pdf of $X_1, X_2, \cdots$ will be

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, i \in \{1, 2, \cdots\} \tag{8}$$

Since $E(X)$ is 0, We can write

$$E(X_i^2) = Var(X_i), \forall i \in \{1, 2, \cdots\} \tag{9}$$
$$= 1 \tag{10}$$

Option A

As $X_1, X_2, \cdots$ are i.i.d random variables therefore $X_1^2, X_2^2, \cdots$ are also identical and independent.

$$E\left(\frac{S_n - n}{\sqrt{2}}\right) = E\left(\frac{\sum_{i=1}^{n}\left(X_i^2 - 1\right)}{\sqrt{2}}\right) \tag{11}$$

$$= \frac{\sum_{i=1}^{n} E(X_i^2 - 1)}{\sqrt{2}} \tag{12}$$

$$E(X_i^2 - 1) = \int_{-\infty}^{\infty} (x^2 - 1) f_{X_i}(x) dx \tag{13}$$

$$= Var(X) - \int_{-\infty}^{\infty} f_{X_i}(x) dx \tag{14}$$

$$= 0 \tag{15}$$

$$\implies E\left(\frac{S_n - n}{\sqrt{2}}\right) = 0 \tag{16}$$

Now consider,

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = Var\left(\frac{\sum_{i=1}^{n}(X_i^2 - 1)}{\sqrt{2}}\right) \tag{17}$$

$$= \frac{\sum_{i=1}^{n} Var(X_i^2 - 1)}{\sqrt{2}} \tag{18}$$

$$Var(X_i^2 - 1) = \int_{-\infty}^{\infty} (x^2 - 1)^2 f_{X_i}(x) dx \tag{19}$$

$$= \int_{-\infty}^{\infty} (x^4 + 1 - 2x^2) f_{X_i}(x) dx \tag{20}$$

$$= \int_{-\infty}^{\infty} (x^4) f_{X_i}(x) dx + 1 - 2Var(X) \tag{21}$$

$$= \int_{-\infty}^{\infty} (x^4) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - 1 \tag{22}$$

On simplifying we get ,

$$Var(X_i^2 - 1) = 2 \tag{23}$$

$$Var\left(\frac{S_n - n}{\sqrt{2}}\right) = n\sqrt{2} \tag{24}$$

Hence from Central limit theorem as $n \to \infty$

$$\left(\frac{S_n - n}{\sqrt{2}}\right) \sim N(0, n\sqrt{2}) \tag{25}$$

Hence **Option A is false.**

Option B

Given,

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{26}$$

Hence from theorem of Weak law of large numbers we can write

$$\frac{S_n}{n} \xrightarrow{i.p} E(X^2) \tag{27}$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{28}$$

$$\implies \frac{S_n}{n} \xrightarrow{i.p} 1 \tag{29}$$

From definition of convergence in probability we can write,

$$\implies \Pr\left(\left|\frac{S_n}{n} - 1\right| > \epsilon\right) \to 0, \forall \epsilon > 0 \tag{30}$$

Hence **Option B is false .**

Option C

Given,

$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 . \forall n \geq 1 \tag{31}$$

Hence from theorem of Strong law of large numbers we can write

$$\frac{S_n}{n} \xrightarrow{i.p} Var(X) \tag{32}$$

$$\implies \frac{S_n}{n} \xrightarrow{a.s} 1 \tag{33}$$

almost surely.From definition of convergence with probability we can write,

$$\frac{S_n}{n} \xrightarrow{w.p.1} 1 \tag{34}$$

Hence **Option C is true**.

Option D

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = E\left(\frac{\sum_{i=1}^{n}\left(X_i^2 - 1\right)}{\sqrt{n}}\right) \tag{35}$$

$$= \frac{\sum_{i=1}^{n} E(X_i^2 - 1)}{\sqrt{n}} \tag{36}$$

From (15),

$$E\left(\frac{S_n - n}{\sqrt{n}}\right) = 0 \tag{37}$$

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = Var\left(\frac{\sum_{i=1}^{n}(X_i^2 - 1)}{\sqrt{n}}\right) \tag{38}$$

$$= \frac{\sum_{i=1}^{n} Var(X_i^2 - 1)}{\sqrt{n}}. \tag{39}$$

From (23).

$$Var\left(\frac{S_n - n}{\sqrt{n}}\right) = 2\sqrt{n} \tag{40}$$

From Central limit theorem we can write,

$$\left(\frac{S_n - n}{\sqrt{n}}\right) \sim N(0, 2\sqrt{n}) \tag{41}$$

$$\Pr\left(S_n \leq n + \sqrt{n}x\right) = \Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) \tag{42}$$

$$\Pr\left(\frac{S_n - n}{\sqrt{n}} \leq x\right) \to \Pr(Y \leq x) \tag{43}$$

where $Y \sim N(0, 2\sqrt{n})$ as $n \to \infty$ Hence **Option D is false.**