Solving linear equations using LU Decomposition

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Problem Statement

The cost of 2 kg of apples and 1kg of grapes on a day was found to be 160. After a month, the cost of 4 kg of apples and 2 kg of grapes is 300. Calculate the cost of 1 kg each of apples and grapes using LU decomposition.

Equations for Cost of Fruits

The given problem can be represented mathematically as:

$$2x + y = 160,$$
$$4x + 2y = 300.$$

Where:

- x is the cost of 1 kg of apples,
- y is the cost of 1 kg of grapes.

Theoretical Solution

It can be observed that the equations are parallel but not co-incident

Therefore, the equations are in-consistent(no solution)

LU Decomposition Computation: Doolittle's Algorithm

Purpose: The LU decomposition is used to factorize a matrix A into:

$$A = L \cdot U,$$

where:

- L: A lower triangular matrix,
- U: An upper triangular matrix.

Steps to Compute:

1. For each column j, compute the elements of U:

$$U_{ij} = \begin{cases} A_{ij}, & \text{if } i = 0, \\ A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj}, & \text{if } i > 0. \end{cases}$$

2. For each row i, compute the elements of L:

$$L_{ij} = \begin{cases} \frac{A_{ij}}{U_{jj}}, & \text{if } j = 0, \\ \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}}, & \text{if } j > 0. \end{cases}$$

Problem Representation in Matrix Form

The given system of equations is represented as:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix},$$

or in compact form:

$$A\mathbf{x} = \mathbf{b}.$$

To solve this equation, we apply LU decomposition:

$$A \to L \cdot U,$$

where:

- L: Lower triangular matrix,
- U: Upper triangular matrix.

LU Decomposition Example

Given the matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}. \quad \text{(0.11 - 0.13)}$$

Solving the System

The solution can now be obtained as:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix}. \tag{0.14}$$

Solving for y, we clearly get:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix}. \tag{0.15}$$

Back Substitution

Now, solving for x via back substitution:

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix}. \tag{0.16}$$

We get

$$0 = -20$$

Inconsistency in Linear Systems

- **Inconsistent Systems:** If a system of linear equations is inconsistent (e.g., parallel lines), the coefficient matrix A is singular ($\det(A) = 0$).
- **LU Decomposition and Redundancy:** During LU decomposition, a row of zeros (except possibly in the augmented column) in the U matrix indicates a redundant equation. This equation doesn't provide new information.
- **No Solution:** A non-zero value in the last column of the augmented matrix's corresponding row (where U has all zeros) signifies an inconsistent system with no solution.

Example:

$$\begin{aligned} 2x_1 + x_2 &= 160 \\ 0 &= -20 \end{aligned}$$

The second equation is clearly false, indicating no solution. This corresponds to parallel lines.

Resulting Plot