10.3.1.3

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹300. Calculate the cost of 1 kg each of apples and grapes using LU decomposition.
Solution:
First, we rewrite the question as a system of linear equations.

$$x_1 \implies \text{apples} \tag{0.1}$$
$$x_2 \implies \text{grapes} \tag{0.2}$$

$$2x_1 + 1x_2 = 160 \tag{0.3}$$
$$4x_1 + 2x_2 = 300 \tag{0.4}$$

Now, converting into a matrix form, we get:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix} \tag{0.5}$$
$$\mathbf{A}x = \mathbf{b} \tag{0.6}$$

To solve the above equation, we have to apply LU - Factorization to matrix **A**.
We do so because,

$$A \to LU \tag{0.7}$$
$$L \to \text{Lower Triangular Matrix} \tag{0.8}$$
$$U \to \text{Upper Triangular Matrix} \tag{0.9}$$

Let $y = \mathbf{U}x$, then we can rewrite the above equation as,

$$\mathbf{A}x = \mathbf{b} \implies \mathbf{LU}x = \mathbf{b} \implies \mathbf{L}y = \mathbf{b} \tag{0.10}$$

Now, the above equation can be solved using front-substitution since **L** is lower triangular, thus we get the solution vector y.
Using this we solve for x in $y = \mathbf{U}x$ using back-substitution knowing that **U** is upper triangular. LU Factorizing **A**, we get:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \tag{0.11}$$
$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{0.12}$$
$$\mathbf{U} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \tag{0.13}$$

The solution can now be obtained as:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix} \tag{0.14}$$

Solving for y, we clearly get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix} \tag{0.15}$$

Now, solving for x via back substitution,

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix} \tag{0.16}$$

- If the system is inconsistent (such as parallel, non-coincident lines), the matrix A will be singular, meaning its determinant is zero.
- During LU Decomposition, you might encounter a row in the matrix U that has all zeros except in the last column (the augmented column from the right-hand side vector **b**). This indicates that you have a row of zeros, which corresponds to a situation where one of the equations is redundant (i.e., it doesn't provide any new information).
- If the augmented matrix has a non-zero value in the corresponding position in the last column (i.e., the equation is inconsistent), then the system has no solution.

Clearly,

$$2x_1 + x_2 = 160 \tag{0.17}$$

$$0 = -20 \tag{0.18}$$

Clearly, this is not possible
Hence, it is proven that this equations do not have a solution(Since, lines are parallel)
LU Decomposition computation
The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices L (lower triangular) and U (upper triangular) such that $A = LU$. The elements of these matrices are calculated as follows:
Elements of the U Matrix:
For each column j:

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \tag{0.19}$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \tag{0.20}$$

Elements of the L Matrix:
For each row i:

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \tag{0.21}$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \tag{0.22}$$

This systematic approach ensures that the matrix A is decomposed into L and U without requiring row swaps, provided A is nonsingular.

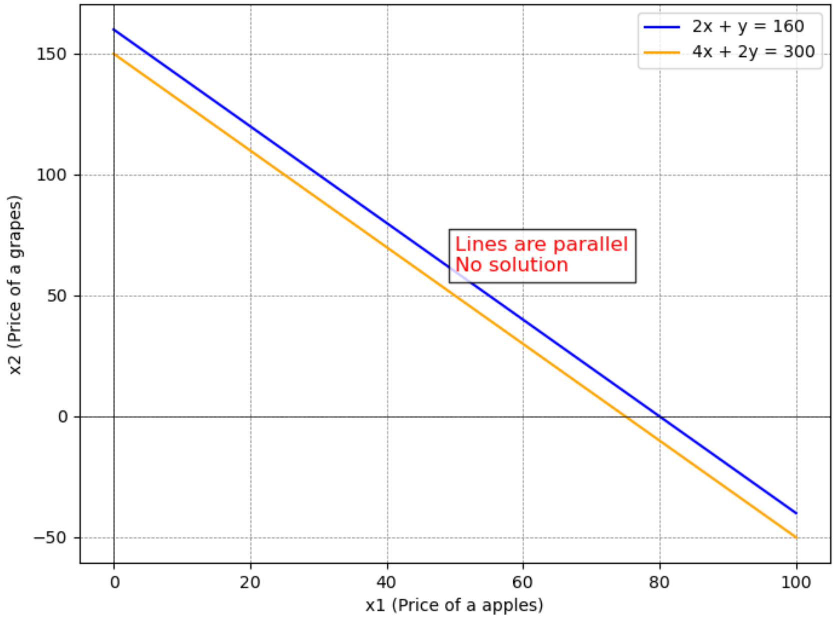

Fig. 0.1: Solution of the system of linear equations