

## 3.1.2

EE24BTECH11063 - Y. Harsha Vardhan Reddy

### Question:

The cost of 2 kg of apples and 1kg of grapes on a day was found to be ₹160. After a month, the cost of 4 kg of apples and 2 kg of grapes is ₹300. Calculate the cost of 1 kg each of apples and grapes using LU decomposition.

### Solution:

First, we rewrite the question as a system of linear equations.

$$x_1 \implies \text{apples} \quad (0.1)$$

$$x_2 \implies \text{grapes} \quad (0.2)$$

$$2x_1 + 1x_2 = 160 \quad (0.3)$$

$$4x_1 + 2x_2 = 300 \quad (0.4)$$

Now, converting into a matrix form, we get:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix} \quad (0.5)$$

$$\mathbf{Ax} = \mathbf{b} \quad (0.6)$$

To solve the above equation, we have to apply LU - Factorization to matrix  $\mathbf{A}$ .

We do so because,

$$\mathbf{A} \rightarrow \mathbf{LU} \quad (0.7)$$

$$\mathbf{L} \rightarrow \text{Lower Triangular Matrix} \quad (0.8)$$

$$\mathbf{U} \rightarrow \text{Upper Triangular Matrix} \quad (0.9)$$

Let  $y = \mathbf{U}x$ , then we can rewrite the above equation as,

$$\mathbf{Ax} = \mathbf{b} \implies \mathbf{LU}x = \mathbf{b} \implies \mathbf{Ly} = \mathbf{b} \quad (0.10)$$

Now, the above equation can be solved using front-substitution since  $\mathbf{L}$  is lower triangular, thus we get the solution vector  $y$ .

Using this we solve for  $x$  in  $y = \mathbf{U}x$  using back-substitution knowing that  $\mathbf{U}$  is upper triangular. LU Factorizing  $\mathbf{A}$ , we get:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (0.11)$$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad (0.12)$$

$$\mathbf{U} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad (0.13)$$

The solution can now be obtained as:

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ 300 \end{pmatrix} \quad (0.14)$$

Solving for  $y$ , we clearly get

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix} \quad (0.15)$$

Now, solving for  $x$  via back substitution,

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 160 \\ -20 \end{pmatrix} \quad (0.16)$$

Clearly,

$$2x_1 + x_2 = 160 \quad (0.17)$$

$$0 = -20 \quad (0.18)$$

Clearly, this is not possible

Hence, it is proven that this equations do not have a solution (Since, lines are parallel)

### **LU Decomposition computation**

The LU decomposition can be efficiently computed using Doolittle's algorithm. This method generates the matrices  $L$  (lower triangular) and  $U$  (upper triangular) such that  $A = LU$ . The elements of these matrices are calculated as follows:

Elements of the  $U$  Matrix:

For each column  $j$ :

$$U_{ij} = A_{ij} \quad \text{if } i = 0, \quad (0.19)$$

$$U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} \quad \text{if } i > 0. \quad (0.20)$$

Elements of the  $L$  Matrix:

For each row  $i$ :

$$L_{ij} = \frac{A_{ij}}{U_{jj}} \quad \text{if } j = 0, \quad (0.21)$$

$$L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}} \quad \text{if } j > 0. \quad (0.22)$$

This systematic approach ensures that the matrix  $A$  is decomposed into  $L$  and  $U$  without requiring row swaps, provided  $A$  is nonsingular.

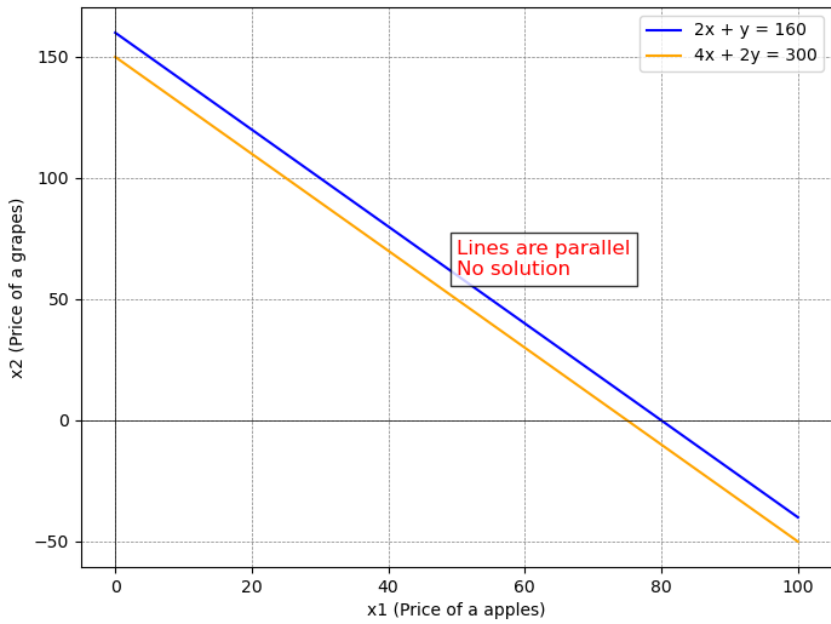


Fig. 0.1: Solution of the system of linear equations