Probability of No Tails in Three Coin Tosses
Z-Transform Approach

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Problem Statement

Question:
Find the probability that no toss results in a tail when a coin is tossed three times (independent tosses).

Theoretical Solution

- The probability of getting a tail (T) in a single toss is:

$$P(T) = \frac{1}{2}$$

- Since the tosses are independent:

$$P(\text{Three Tails}) = P(T) \times P(T) \times P(T)$$

- Substituting $P(T) = \frac{1}{2}$:

$$P(\text{Three Tails}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

PMF for a Single Coin Toss

Probability Mass Function (PMF):

$$P_X(k) = \begin{cases} p, & \text{if } x = 1 \text{ (Tails)} \\ 1 - p, & \text{if } x = 0 \text{ (Heads)} \end{cases}$$

For a fair coin: $p = 0.5$.

Conditions for a Valid PMF:

1. **Non-negativity:** $\forall k, P_X(k) \geq 0$.
2. **Normalization:** $\sum_k P_X(k) = 1$.

For a single coin toss:

$$P_X(0) + P_X(1) = 1. \quad \text{(Normalization condition)}$$

$$\text{Substituting } p = 0.5:$$

$$P_X(0) = 0.5,$$

$$P_X(1) = 0.5.$$

Problem Statement and Z-Transform

Problem Statement: In n coin tosses, let X denote the number of tails, which follows a binomial distribution. The probability mass function (PMF) is:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Z-Transform Definition: The Z-transform of a sequence a_k is defined as:

$$T(z) = \sum_{k=0}^{n} a_k z^k$$

Substituting $a_k = P(X = k)$:

$$T(z) = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} z^k$$

Derivation and Final Result

Combine z^k with p^k:

$$T(z) = \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1-p)^{n-k}$$

Using the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Let $x = pz$ and $y = 1 - p$. Then:

$$T(z) = (pz + (1-p))^n$$

Final Result:

$$T(z) = (p + (1-p)z)^n$$

Z-Transform Definition

Z-Transform for the Number of Tails in n Coin Tosses:

$$T(z) = (p + (1-p)z)^n \quad \text{where:}$$

- p is the probability of tails ($p = 0.5$ for a fair coin),
- $1 - p$ is the probability of heads ($1 - p = 0.5$ for a fair coin),
- n is the number of tosses.

Expansion of $T(z)$

Expansion of $T(z)$ using the Binomial Theorem:

$$T(z) = \sum_{k=0}^{n} \binom{n}{k} p^{n-k} (1-p)^k z^k$$

where:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Interpretation:

- The coefficient of z^k in $T(z)$ gives the probability $P_X(k)$, where X is the number of tails.

Z-Transform for $n = 1$

For $n = 1$:

$$T(z) = p + (1 - p)z$$

For a fair die,

$$P_X(0) = 0.5, P_X(1) = 0.5$$

Probability Mass Function (PMF)

The PMF is computed as:

$$P_X(k) = \binom{n}{k} p^{n-k}(1-p)^k$$

Substitute $p = 0.5$ and $1 - p = 0.5$ for a fair coin:

$$P_X(k) = \binom{n}{k}(0.5)^n$$

Computational Steps

For each $k \in \{0, 1, \ldots, n\}$:

1. Compute the binomial coefficient:

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}$$

2. Multiply by $(0.5)^n$ to compute $P_X(k)$:

$$P_X(k) = \binom{n}{k} \cdot (0.5)^n$$

Result for $n = 3$

Substituting $n = 3$ and expanding:

$$T(z) = (0.5 + 0.5z)^3 = 0.125 + 0.375z + 0.375z^2 + 0.125z^3$$

The PMF values are:

$$P_X(0) = 0.125, \quad P_X(1) = 0.375, \quad P_X(2) = 0.375, \quad P_X(3) = 0.125$$

CDF Conditions

The CDF must satisfy:

1. $F_X(k)$ is non-decreasing.
2. $F_X(k) \in [0, 1]$ for all k.
3. $F_X(n) = 1$, where n is the maximum number of tails.

CDF Calculation

The CDF is given by:

$$F_X(k) = \sum_{i=0}^{k} P_X(i)$$

CDF Computation

Compute the values:

$$F_X(0) = P_X(0) = 0.125$$

$$F_X(1) = P_X(0) + P_X(1) = 0.125 + 0.375 = 0.5$$

$$F_X(2) = F_X(1) + P_X(2) = 0.5 + 0.375 = 0.875$$

$$F_X(3) = F_X(2) + P_X(3) = 0.875 + 0.125 = 1$$

For $n = 3$, the CDF values are:

$$F_X(0) = 0.125, \quad F_X(1) = 0.5, \quad F_X(2) = 0.875, \quad F_X(3) = 1$$

Probability of $X = 3$

Using the CDF, the probability of $X = 3$ is:

$$P_X(3) = F_X(3) - F_X(2)$$

Substituting the values:

$$P_X(3) = 1 - 0.875 = 0.125$$

Conclusion

The probability of getting all three tails is:

$$P(X = 3) = \mathbf{0.125}$$