11.16.3.3.4

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
Find the probability that no toss results in a tail when coin is tossed thrice(independent tosses)
Solution:
Theoretical solution:
To calculate the probability of getting all three tails when a coin is tossed three times:

- The probability of getting a tail (T) in a single toss is:
$$P(T) = \frac{1}{2}.$$
- Since the tosses are independent, the probability of getting three tails (TTT) is given by:
$$P(\text{Three Tails}) = P(T) \times P(T) \times P(T).$$
- Substituting $P(T) = \frac{1}{2}$, we have:
$$P(\text{Three Tails}) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Thus, the probability of getting all three tails is:
$$\boxed{\frac{1}{8}}$$

Computational solution:

Z-Transform Computational Method for Coin Toss PMF

PMF for a Single Coin Toss

For a single coin toss, the probability mass function (PMF) is:
$$P_X(k) = \begin{cases} p, & \text{if } x = 1 \text{ (Tails)} \\ 1 - p, & \text{if } x = 0 \text{ (Heads)} \end{cases}$$
where $p = 0.5$ for a fair coin.

Conditions for PMF

A valid PMF must satisfy the following conditions:

1) Non-negativity: $\forall k, P_X(k) \geq 0$.
2) Normalization: $\sum_k P_X(k) = 1$.

For a single coin toss:

$$P_X(0) + P_X(1) = 1. \tag{2.1}$$

Substituting $p = 0.5$:

$$P_X(0) = 0.5, \quad P_X(1) = 0.5. \tag{2.2}$$

Z-Transform Expansion

The Z-transform for the number of tails in n coin tosses is given by:

$$T(z) = (p + (1 - p)z)^n \tag{2.3}$$

where:

- p is the probability of tails ($p = 0.5$ for a fair coin),
- $1 - p$ is the probability of heads ($1 - p = 0.5$ for a fair coin),
- n is the number of tosses ($n = 3$).

For $n = 1$,

$$T(z) = (p + (1 - p)z) \tag{2.4}$$

For a fair die,

$$P_X(0) = 0.5, \quad P_X(1) = 0.5, \tag{2.5}$$

Expansion of $T(z)$

Expand the expression $(p + (1 - p)z)^n$ using the binomial theorem:

$$T(z) = \sum_{k=0}^{n} \binom{n}{k} p^{n-k} (1 - p)^k z^k \tag{2.6}$$

where:

- $\binom{n}{k}$ is the binomial coefficient, computed as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.7}$$

- The coefficient of z^k in $T(z)$ gives the probability $P_X(k)$, where X is the number of tails.

Probability Mass Function (PMF)

The PMF is computed as:

$$P_X(k) = \binom{n}{k} p^{n-k} (1 - p)^k \tag{2.8}$$

Substitute $p = 0.5$ and $1 - p = 0.5$ for a fair coin:

$$P_X(k) = \binom{n}{k} (0.5)^n \tag{2.9}$$

Computational Steps

For each $k \in \{0, 1, \ldots, n\}$:

1) Compute the binomial coefficient:

$$\binom{n}{k} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \tag{1.1}$$

2) Multiply by $(0.5)^n$ to compute $P_X(k)$:

$$P_X(k) = \binom{n}{k} \cdot (0.5)^n \tag{2.1}$$

Result for $n = 3$

Substituting $n = 3$ and expanding:

$$T(z) = (0.5 + 0.5z)^3 = 0.125 + 0.375z + 0.375z^2 + 0.125z^3 \tag{2.2}$$

The PMF values are:

$$P_X(0) = 0.125, \quad P_X(1) = 0.375, \quad P_X(2) = 0.375, \quad P_X(3) = 0.125 \tag{2.3}$$

The CDF must satisfy:

1) $F_X(k)$ is non-decreasing.
2) $F_X(k) \in [0, 1]$ for all k.
3) $F_X(n) = 1$, where n is the maximum number of tails.

CDF Calculation

The CDF is:

$$F_X(k) = \sum_{i=0}^{k} P_X(i). \tag{3.1}$$

Compute the values:

$$F_X(0) = P_X(0) = 0.125, \tag{3.2}$$

$$F_X(1) = P_X(0) + P_X(1) = 0.125 + 0.375 = 0.5, \tag{3.3}$$

$$F_X(2) = F_X(1) + P_X(2) = 0.5 + 0.375 = 0.875, \tag{3.4}$$

$$F_X(3) = F_X(2) + P_X(3) = 0.875 + 0.125 = 1. \tag{3.5}$$

For $n = 3$, the CDF values are:

$$F_X(0) = 0.125, \quad F_X(1) = 0.5, \quad F_X(2) = 0.875, \quad F_X(3) = 1. \tag{3.6}$$

Probability of $X = 3$

Using the CDF, the probability of $X = 3$ is:

$$P_X(3) = F_X(3) - F_X(2). \tag{3.7}$$

Substitute the values:

$$P_X(3) = 1 - 0.875 = 0.125. \tag{3.8}$$

Conclusion

The probability of getting all three tails is:

$$P(X = 3) = \boxed{0.125}.$$