9-9.5-11

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
For the following differential equation, find the particular solution satisfying the given condition:
$(x + y)\ dy + (x - y)\ dx = 0;\ \ y = 1$ when $x = 1$
Solution:
First let us solve the given differential equation theoritically and then do it computationally and verify if they are equal

$$(x + y)\ \ dy + (x - y)\ \ dx = 0 \tag{0.1}$$

or

$$\frac{dy}{dx} = \frac{y - x}{y + x} \tag{0.2}$$

Let

$$F(x, y) = \frac{y - x}{y + x} \tag{0.3}$$

then

$$F(\alpha x, \alpha y) = \alpha^0 \times \frac{y - x}{y + x} \tag{0.4}$$

Therefore, this is a homogeneous equation in x and y
Let

$$y = vx \tag{0.5}$$

or

$$v + x\frac{dv}{dx} = \frac{v - 1}{v + 1} \tag{0.6}$$

or

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v \tag{0.7}$$

or

$$x\frac{dv}{dx} = -\frac{v^2 + 1}{v + 1} \tag{0.8}$$

or

$$\frac{v + 1}{v^2 + 1} \cdot dv = -\frac{dx}{x} \tag{0.9}$$

Integrating on Both sides,

$$\int \frac{v}{v^2+1} \cdot dv + \int \frac{1}{v^2+1} \cdot dv = -\int \frac{dx}{x} \tag{0.10}$$

or

$$\frac{1}{2}\ln(v^2+1) + \tan^{-1}(v) = -\ln x + c \tag{0.11}$$

By sustituting $v = \frac{y}{x}$,

$$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln x + c \tag{0.12}$$

(where, c is the constant of integration) By substituting $x = 1$ and $y = 1$,

$$c = \frac{1}{2}\ln 2 + \frac{\pi}{4} \tag{0.13}$$

Final equation,

$$\frac{1}{2}\ln\left(\frac{y^2}{x^2}+1\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln x + \frac{1}{2}\ln 2 + \frac{\pi}{4} \tag{0.14}$$

Now let us verify this computationally From definition of $\frac{dy}{dx}$,

$$y_{n+1} = y_n + \frac{dy}{dx} \cdot h \tag{0.15}$$

(where h is small number tending to zero) From the differential equation given,

$$\frac{dy}{dx} = \frac{y_n - x_n}{y_n + x_n} \tag{0.16}$$

By substituting,

$$y_{n+1} = y_n + \left(\frac{y_n - x_n}{y_n + x_n}\right) \cdot h \tag{0.17}$$

By taking $x_1 = 1$ and $y_1 = 1$ and $h = 0.04$ going till $x = 3$ by iterating through the loop and finding $y_2, y_3, \cdots, y_{500}$ and plotting the graph the implicit function we can verify if the function we got by solving the differential equation mathematically. The comparision between theoretical and simulated graphs is shown in the following page 0.1

Clearly, both theoretical and simulated curves are coinciding and Hence the theoretical solution is verified.

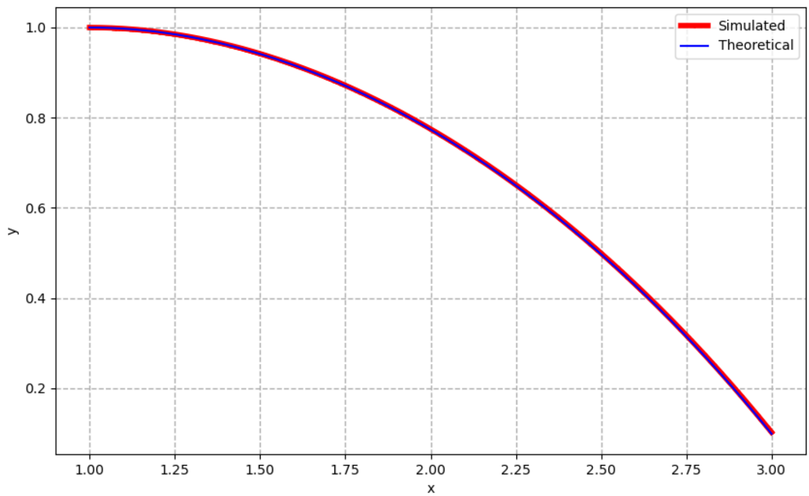
Simulated
Theoretical
1.0
0.8
0.6
0.4
0.2
y
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
x

Fig. 0.1