x

9-9.5-11

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
For the following differential equation, find the particular solution satisfying the given condition:
$(x + y)\,dy + (x - y)\,dx = 0;\ y = 1$ when $x = 1$
Solution:
First let us solve the given differential equation theoritically and then do it computationally and verify if they are equal

$$(x + y)\ dy + (x - y)\ dx = 0 \tag{0.1}$$

or

$$\frac{dy}{dx} = \frac{y - x}{y + x} \tag{0.2}$$

Let

$$F(x, y) = \frac{y - x}{y + x} \tag{0.3}$$

then

$$F(\alpha x, \alpha y) = \alpha^0 \times \frac{y - x}{y + x} \tag{0.4}$$

Therefore, this is a homogeneous equation in x and y
Let

$$y = vx \tag{0.5}$$

or

$$v + x\frac{dv}{dx} = \frac{v - 1}{v + 1} \tag{0.6}$$

or

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v \tag{0.7}$$

or

$$x\frac{dv}{dx} = -\frac{v^2 + 1}{v + 1} \tag{0.8}$$

or

$$\frac{v + 1}{v^2 + 1} \cdot dv = -\frac{dx}{x} \tag{0.9}$$

Integrating on Both sides,

$$\int \frac{v}{v^2+1} \cdot dv + \int \frac{1}{v^2+1} \cdot dv = -\int \frac{dx}{x} \tag{0.10}$$

or

$$\frac{1}{2} \ln(v^2+1) + \tan^{-1}(v) = -\ln x + c \tag{0.11}$$

By sustituting $v = \frac{y}{x}$,

$$\frac{1}{2} \ln\left(\frac{y^2}{x^2} + 1\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln x + c \tag{0.12}$$

where, c is the constant of integration By substituting $x = 1$ and $y = 1$ in **??**,

$$c = \frac{1}{2} \ln 2 + \frac{\pi}{4} \tag{0.13}$$

Final equation,

$$\frac{1}{2} \ln\left(\frac{y^2}{x^2} + 1\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\ln x + \frac{1}{2} \ln 2 + \frac{\pi}{4} \tag{0.14}$$

Now let us verify this computationally From definition of $\frac{dy}{dx}$,

$$y_{n+1} = y_n + \frac{dy}{dx} \cdot h \tag{0.15}$$

(where h is small number tending to zero) From the differential equation given,

$$\frac{dy}{dx} = \frac{y_n - x_n}{y_n + x_n} \tag{0.16}$$

By substituting **??** in **??**,

$$y_{n+1} = y_n + \left(\frac{y_n - x_n}{y_n + x_n}\right) \cdot h \tag{0.17}$$

By taking $x_1 = 1$ and $y_1 = 1$ and $h = 0.04$ going till $x = 3$ by iterating through the loop and finding $y_2, y_3, \cdots, y_{500}$ and plotting the graph the implicit function **??** we can verify if the function we got by solving the differential equation mathematically.