9-9.7-1.3

EE24BTECH11063 - Y. Harsha Vardhan Reddy

Question:
Solve the differential equation:

$$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

Solution:

Theoritical solution:
The given differential equation is of degree 4 and cannot be solved using known methods. Let us compute the solution of the given differential equation computationally
Computational Solution: Euler's method
By the first principle of derivative,

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} \tag{0.1}$$

$$y(x+h) = y(x) + h(y'(x)), h \to 0 \tag{0.2}$$

For a m^{th} order differential equation,

Let

$$y_1 = y \ , \ y_2 = y' \ , \ y_3 = y'' \ , \ \dots \ , \ y_m = y^{m-1} \tag{0.3}$$

then we obtain the system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \\ y_m' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \tag{0.4}$$

Here, f is described by the given differential equation. The initial conditions $y_1(x_0) = K_1$, $y_2(x_0) = K_2$, $\dots$, $y_m(x_0) = K_m$.

Representing the system in Euler's form (using first principle of derivative),

$$\begin{pmatrix} y_1(x+h) \\ y_2(x+h) \\ \vdots \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) + hy_2(x) \\ y_2(x) + hy_3(x) \\ \vdots \\ y_m(x) + hf(x, y_1, y_2 \ldots y_m) \end{pmatrix} \tag{0.5}$$

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{0.6}$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x, y_1, y_2, \ldots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{0.7}$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x, y_1, y_2, \ldots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \tag{0.8}$$

Generalizing the system into an iterative format for plotting $y(x)$,

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ \vdots \\ y_{m,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{m,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ \vdots \\ f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n}) \end{pmatrix} \tag{0.9}$$

$$\mathbf{y_{n+1}} = \mathbf{y_n} + h \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n}, \text{ where } \mathbf{y_n} = \begin{pmatrix} y_{1,n}(x_n) \\ y_{2,n}(x_n) \\ \vdots \\ y_{m,n}(x_n) \end{pmatrix} \tag{0.10}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y_n} \tag{0.11}$$

$$x_{n+1} = x_n + h \tag{0.12}$$

Here, the vector $\mathbf{y_n}$ is not to be confused with y_k which is the $(k-1)^{\text{th}}$ derivative of $y(x)$. The given differential equation can be represented as,

$$y'''' - \sin(y''') = 0 \tag{0.13}$$

$$y'''' = \sin(y''') \tag{0.14}$$

We see that $m = 4$, thus,

$$y_5 = y'''' = \sin(y''') = \sin(y_4) \tag{0.15}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \sin(y_4) \end{pmatrix} \tag{0.16}$$

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ y_{4,n+1} \\ y_{4,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ y_{3,n} \\ y_{4,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ y_{4,n} \\ \sin(y_{4,n}) \end{pmatrix} \tag{0.17}$$

$$\mathbf{y_{n+1}} = \begin{pmatrix} 1 & h & 0 & 0 \\ 0 & 1 & h & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 + \sin y_{3,n} \end{pmatrix} \mathbf{y_n} \tag{0.18}$$

Iteratively plotting the above system taking intial conditions as

$$x_0 = 0 \ , \ y_{1,0} = 0 \ , \ y_{2,0} = 0 \ , \ y_{3,0} = 0 \ , \ y_{4,0} = 1 \tag{0.19}$$

we get the following plot 0.1.

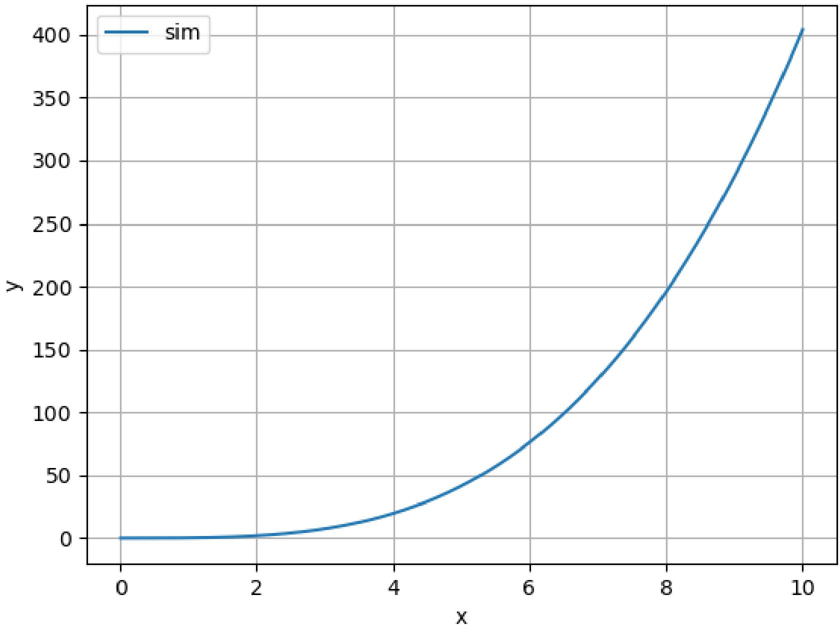

Fig. 0.1