

Claim Let  $N(i, j)$  denote the # of nodes in recursion tree (i.e., # of rec. calls). Then  $N(i, j) = 2 \binom{i+j}{j} - 1$

Pf. Observe  $N(i, j) = N(i-1, j) + N(i, j-1) + 1$

Thus, by IH, we have:

$$\begin{aligned} N(i, j) &= 2 \binom{i+j-1}{j} - 1 \\ &\quad + 2 \binom{i+j-1}{j-1} - 1 + 1 \\ &= 2 \left[ \binom{i+j-1}{j} + \binom{i+j-1}{j-1} \right] - 1 \end{aligned}$$

Noting that  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ ,

$$N(i, j) = 2 \binom{i+j}{j} - 1 \quad \square$$

Claim.  $\binom{2n}{n} \geq \frac{4^n}{2n+1}$

$$\begin{aligned} \text{Pf. } \binom{2(n+1)}{n+1} &= \binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} \\ &= 2 \binom{2n+1}{n+1} = 2 \left[ \binom{2n}{n+1} + \binom{2n}{n} \right] \\ &= 2 \left[ \binom{2n}{n} \frac{n}{n+1} + \binom{2n}{n} \right] = 2 \binom{2n}{n} \frac{2n+1}{n+1} \end{aligned}$$

$$\text{IH} \Rightarrow \geq 2 \frac{4^n}{2n+1} \cdot \frac{2n+1}{n+1} = 4^n \cdot \frac{2}{n+1}$$

$$\geq 4^n \cdot \frac{4}{2n+3} = \frac{4^{n+1}}{2(n+1)+1} \quad \square$$