Problem 12.349

ee25btech11023-Venkata Sai

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Problem

Let $T_1, T_2 : \mathbb{R}_5 \to \mathbb{R}_3$ be linear transformations such that $\text{rank}(T_1) = 3$ and $\text{nullity}(T_2) = 3$. Let $T_3 : \mathbb{R}_3 \to \mathbb{R}_3$ be a linear transformation such that $T_3 \circ T_1 = T_2$. Then $\text{rank}(T_3)$ is ... (MA 2014)

Given

According to Rank-Nullity theorem,
For a linear transformation $T : \mathbb{R}_m \to \mathbb{R}_n$

$$\text{rank}\,(T) + \text{nullity}\,(T) = \dim\,(\text{domain}) \tag{3.1}$$

where $\dim \mathbb{R}_m$ is the dimension of the domain i.e vector space $\mathbb{R}_m$

Given $T_2 : \mathbb{R}_5 \to \mathbb{R}_3$ and nullity(T_2)=3

$$\text{rank}\,(T_2) + \text{nullity}\,(T_2) = \dim \mathbb{R}_5 \tag{3.2}$$

$$\text{rank}\,(T_2) + 3 = 5 \tag{3.3}$$

$$\text{rank}\,(T_2) = 2 \tag{3.4}$$

Given $T_1 : \mathbb{R}_5 \to \mathbb{R}_3$ and rank(T_1)=3

$$\dim\,(\text{Co-domain}) = 3 \tag{3.5}$$

$$\text{rank}\,(T_1) = \dim\,(\text{Co-domain}) \tag{3.6}$$

Conclusion

It is onto and hence

$$\dim(\operatorname{Im}(T_1)) = \dim(\text{Co-domain}) \tag{3.7}$$
$$\dim(\operatorname{Im}(T_1)) = 3 \implies \operatorname{Im}(T_1) = \mathbb{R}_3 \tag{3.8}$$

where $\operatorname{Im}(T)$ is the Image space of the linear transformation T
Given $T_3 : \mathbb{R}_3 \to \mathbb{R}_3$

$$T_3 \circ T_1 = T_2 \tag{3.9}$$
$$(T_3 \circ T_1)(\mathbb{R}_5) = \operatorname{Im}(T_2) \tag{3.10}$$
$$T_3(T_1(R_5)) = \operatorname{Im}(T_2) \tag{3.11}$$
$$T_3(\operatorname{Im}(T_1)) = \operatorname{Im}(T_2) \tag{3.12}$$
$$T_3(\mathbb{R}_3) = \operatorname{Im}(T_2) \tag{3.13}$$
$$\operatorname{Im}(T_3) = \operatorname{Im}(T_2) \tag{3.14}$$
$$\implies \operatorname{rank}(T_3) = \operatorname{rank}(T_2) \tag{3.15}$$

From (4)

$$\operatorname{rank}(T_3) = 2 \tag{3.16}$$