Matrices in Geometry 10.7.84

EE25BTECH11035 - Kushal B N

Question: Let $2x^2 + y^2 - 3xy = 0$ be the equation of pair of tangents drawn from the origin $\mathbf{O}$ to a circle of radius 3 with the centre in the first quadrant. If $\mathbf{A}$ is one of the points of contact, find the length of OA. (JEE 2001)

Given:
The equation of pair of tangents

$$y^2 - 3xy + 2x^2 = 0 \tag{1}$$

Radius $r = 3$ **Solution:**
Factorising the given equation of pair of tangents

$$(y - x)(y - 2x) = 0 \tag{2}$$

The direction vectors of the two tangent lines from the origin

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3}$$

$$\mathbf{m}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4}$$

Now, let 2ϕ be the angle between the two tangent lines.

$$\mathbf{m}_1^\top \mathbf{m}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3 \tag{5}$$

$$\|\mathbf{m}_1\| = \sqrt{\mathbf{m}_1^\top \mathbf{m}_1} = \sqrt{2} \tag{6}$$

$$\|\mathbf{m}_2\| = \sqrt{\mathbf{m}_2^\top \mathbf{m}_2} = \sqrt{5} \tag{7}$$

$$\cos 2\phi = \frac{\mathbf{m}_1^\top \mathbf{m}_2}{\|\mathbf{m}_1\| \|\mathbf{m}_2\|} = \frac{3}{\sqrt{2}\sqrt{5}} = \frac{3}{\sqrt{10}} \tag{8}$$

$$\implies \tan 2\phi = \frac{\sqrt{1 - \cos^2 2\phi}}{\cos 2\phi} = \frac{1/\sqrt{10}}{3\sqrt{10}} = \frac{1}{3} \tag{9}$$

Let the centre of the circle be $\mathbf{C}$, so that in the right-angled triangle ΔOAC, we have (as $AC = r = 3$)

$$\tan \phi = \frac{AC}{OA} = \frac{3}{OA} \tag{10}$$

$$\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} \tag{11}$$

Let $t = \tan \phi$

$$\frac{1}{3} = \frac{2t}{1 - t^2} \tag{12}$$

$$\implies t^2 + 6t - 1 = 0 \tag{13}$$

Solving the above quadratic equation using the quadractic formula, we get

$$t = -3 \pm \sqrt{10} \tag{14}$$

Now, the angle being acute implies that $\tan\phi > 0$, so

$$\tan\phi = \sqrt{10} - 3 \tag{15}$$

$$\implies OA = \frac{3}{\tan\phi} = \frac{3}{\sqrt{10} - 3} \tag{16}$$

Rationalising the denominator, we get

$$\boxed{OA = 9 + 3\sqrt{10}} \tag{17}$$

Final Answer:
$\therefore$ The length of OA is equal to $9 + 3\sqrt{10}$ units.

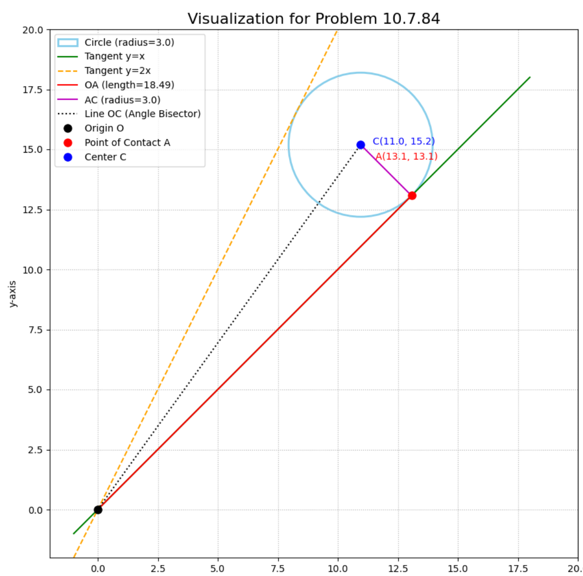

Fig. 1: Plot for 10.7.84