Matgeo Presentation - Problem 12.797

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Problem Statement

Let $\mathbf{A}$ be an $n \times n$ real matrix. Consider the following statements:

(I) If $\mathbf{A}$ is symmetric, then there exists $c \geq 0$ such that $\mathbf{A} + c\mathbf{I}_n$ is symmetric and positive definite, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

(II) If $\mathbf{A}$ is symmetric and positive definite, then there exists a symmetric and positive definite $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$.

Which of the above statements is/are true?

(a) Only (I)
(b) Only (II)
(c) Both (I) and (II)
(d) Neither (I) nor (II)

Data

Name	**Description**
A	Matrix

Table : Matrix

Solution

Checking statement (**I**)

If **A** is symmetric, its eigenvalues are real. Let the minimum eigenvalue of **A** be λ_{min}.Then choose $c > -\lambda_{\text{min}}$.

The Eigen values of **A** are given as :

$$\left|\mathbf{A} - \lambda_i \mathbf{I}\right| = 0 \tag{0.1}$$

The Eigen values of $\mathbf{A} + c\mathbf{I}_n$ are given as :

$$\left|\mathbf{A} - (\lambda_k - c)\mathbf{I}\right| = 0 \tag{0.2}$$

$$\lambda_k = \lambda_i + c \tag{0.3}$$

$$\lambda_i + c > 0 \tag{0.4}$$

Since $\lambda_i + c > 0$ for all i, $\mathbf{A} + c\mathbf{I}_n$ is positive definite and symmetric.

Hence, statement (**I**) is **true**.

Solution

Checking statement (**II**)

If **A** is symmetric and positive definite, then it can be diagonalized as:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{0.5}$$

where **P** is orthogonal and **D** is a diagonal matrix with positive entries (since **A** is positive definite). Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top \tag{0.6}$$

Then,

$$\mathbf{B}^2 = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top\mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top = \mathbf{P}\mathbf{D}\mathbf{P}^\top = \mathbf{A} \tag{0.7}$$

Hence, **B** is symmetric and positive definite. Therefore, statement (**II**) is also **true**.

Final Answer: (c) Both (I) and (II)

Solution

Examples

Example a for (I)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{0.8}$$

To find eigenvalues, evaluate:

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \tag{0.9}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \tag{0.10}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{1}{\lambda} R_1} \begin{vmatrix} -\lambda & 1 \\ 0 & \frac{1-\lambda^2}{\lambda} \end{vmatrix} = 0 \tag{0.11}$$

$$\lambda^2 - 1 = 0 \tag{0.12}$$

$$\lambda = \pm 1 \tag{0.13}$$

$$\lambda_1 = 1, \lambda_2 = -1 \tag{0.14}$$

Solution

The minimum eigenvalue $\lambda_{\min} = -1$. Choose $c = 2$. Then:

$$\lambda_1 + c = 3, \quad \lambda_2 + c = 1 \tag{0.15}$$

All eigenvalues are positive, so $\mathbf{A} + 2\mathbf{I}$ is symmetric positive definite.

Example b for (I)

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{0.16}$$

Compute eigenvalues from $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$:

$$\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \tag{0.17}$$

$$(-2-\lambda)(1-\lambda)(3-\lambda) = 0 \tag{0.18}$$

Hence

$$\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = 3 \tag{0.19}$$

Solution

Choose $c = 3$ so $c > -\lambda_{\text{min}} = 2$. Then $\lambda_i + c > 0$ for all i. Thus $\mathbf{A} + 3\mathbf{I}$ is symmetric positive definite.

Example a for (II)

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \tag{0.20}$$

Find eigenvalues from $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$:

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \tag{0.21}$$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} \xleftrightarrow{R_2 \to R_2 - \frac{2}{5-\lambda} R_1} \begin{vmatrix} 5-\lambda & 2 \\ 0 & (5-\lambda) - \frac{4}{5-\lambda} \end{vmatrix} = 0 \tag{0.22}$$

$$(5-\lambda)^2 - 4 = 0 \tag{0.23}$$

$$\lambda^2 - 10\lambda + 21 = 0 \tag{0.24}$$

$$\lambda_1 = 7,\ \lambda_2 = 3 \tag{0.25}$$

Solution

For eigenvectors:

$$(\mathbf{A} - 7\mathbf{I})\mathbf{v} = 0, \quad (\mathbf{A} - 3\mathbf{I})\mathbf{v} = 0 \tag{0.26}$$

They correspond to $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Form **P** and **D**:

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \tag{0.27}$$

Then

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top, \quad \mathbf{B}^2 = \mathbf{A} \tag{0.28}$$

Hence verified for 2×2.
Example b for (II)

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{0.29}$$

Solution

Compute $\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \tag{0.30}$$

$$(2-\lambda)(3-\lambda)(4-\lambda) = 0 \tag{0.31}$$

$$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 4 \tag{0.32}$$

Diagonalization:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top, \quad \mathbf{P} = \mathbf{I}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{0.33}$$

Solution

Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{0.34}$$

Then

$$\mathbf{B}^2 = \mathbf{A} \tag{0.35}$$

Hence **B** is symmetric positive definite.
Conclusion: In all four examples, both statements (I) and (II) hold true.