4.10.21

EE25BTECH11019 - Darji Vivek M.

Question:
Prove that the line through $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$.

Solution:

$$\mathbf{A} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 3 \\ 9 \\ 4 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}, \tag{1}$$

$$\mathbf{d}_1 = \mathbf{B} - \mathbf{A} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \quad \mathbf{d}_2 = \mathbf{D} - \mathbf{C} = \begin{pmatrix} -7 \\ -5 \\ 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{P}(\lambda) = \mathbf{A} + \lambda \mathbf{d}_1, \quad \mathbf{Q}(\mu) = \mathbf{C} + \mu \mathbf{d}_2, \tag{3}$$

$$\mathbf{P}(\lambda) = \mathbf{Q}(\mu) \implies \lambda \mathbf{d}_1 - \mu \mathbf{d}_2 = \mathbf{C} - \mathbf{A}, \tag{4}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} 3 \\ 10 \\ 5 \end{pmatrix}. \tag{5}$$

Formulating as a Matrix Equation:

$$\begin{pmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \mathbf{C} - \mathbf{A}. \tag{6}$$

Substituting the values,

$$\begin{pmatrix} 4 & 7 \\ 6 & 5 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 5 \end{pmatrix}. \tag{7}$$

Augmented matrix and row-reduction:

$$\left(\begin{array}{cc|c} 4 & 7 & 3 \\ 6 & 5 & 10 \\ 2 & 0 & 5 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 2 & 0 & 5 \\ 6 & 5 & 10 \\ 4 & 7 & 3 \end{array}\right) \xleftrightarrow[R_2 \leftarrow R_2 - 6R_1,\ R_3 \leftarrow R_3 - 4R_1]{R_1 \leftarrow \frac{1}{2} R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & 5 & -5 \\ 0 & 7 & -7 \end{array}\right)$$

$$\xleftrightarrow[R_3 \leftarrow R_3 - \frac{7}{5} R_2]{R_2 \leftarrow \frac{1}{5} R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

From the reduced system we read off

$$\lambda = \tfrac{5}{2}, \tag{8}$$
$$\mu = -1. \tag{9}$$

Intersection point:

$$\mathbf{P}\left(\tfrac{5}{2}\right) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \tfrac{5}{2}\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 4 \end{pmatrix}, \tag{10}$$
$$\mathbf{Q}(-1) = \begin{pmatrix} 3 \\ 9 \\ 4 \end{pmatrix} + (-1)\begin{pmatrix} -7 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 4 \end{pmatrix}. \tag{11}$$

Therefore, the lines intersect at $\boxed{\begin{pmatrix} 10 \\ 14 \\ 4 \end{pmatrix}}$.

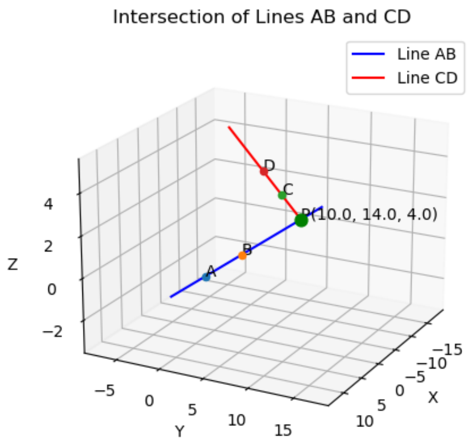

Fig. 0.1: Given 2 lines are Intersecting