

5.13.45

EE25BTECH11019 - Darji Vivek M.

Question:

Let $x \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix}, \quad R = P Q P^{-1}.$$

Then which of the following options is/are correct?

1) $\det \mathbf{R} = \det \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} + 8$ for all $x \in \mathbb{R}$.

2) For $x = 1$, there exists a unit vector $\alpha \mathbf{i} + \beta \mathbf{j} + \gamma \mathbf{k}$ such that $\mathbf{R} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

3) There exists a real number x such that $\mathbf{PQ} = \mathbf{QP}$.

4) For $x = 0$, if $\mathbf{R} = \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, then $a + b = 5$.

Solution:

Matrix Method:

Given

$$\mathbf{R} = \mathbf{PQP}^{-1}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix}$$

Since ^(a) $|\mathbf{R}| = |\mathbf{Q}|$,

$$\begin{aligned} |\mathbf{Q}| &= 2 \begin{vmatrix} 4 & 0 \\ x & 5 \end{vmatrix} - x \begin{vmatrix} 0 & 0 \\ x & 5 \end{vmatrix} + x \begin{vmatrix} 0 & 4 \\ x & x \end{vmatrix} \\ &= 2(20) - x(0) + x(0 - 4x) \Rightarrow |\mathbf{Q}| = 40 - 4x^2 \\ &\Rightarrow |\mathbf{R}| = 40 - 4x^2 \end{aligned}$$

Hence, option (a) is **false**.

For ^(b) $x = 1$,

$$|\mathbf{R}| = 40 - 4(1)^2 = 36 \neq 0$$

Since $|\mathbf{R}| \neq 0$, \mathbf{R} is invertible, so $\mathbf{R} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only the trivial solution. Hence, no unit vector exists \Rightarrow (b) **false**.

(c)

$$\mathbf{PQ} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} = \begin{pmatrix} 2+x & 2x+4 & x+5 \\ 2x & 8+2x & 10 \\ 3x & 3x & 15 \end{pmatrix}$$

$$\mathbf{QP} = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2+2x & 2+5x \\ 0 & 8 & 8 \\ x & 3x & 3x+15 \end{pmatrix}$$

Comparing entries, no x satisfies $\mathbf{PQ} = \mathbf{QP}$. Hence, (c) **false**.

(d)
For $x = 0$,

$$\mathbf{Q} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow \text{Eigenvalues of } \mathbf{R} = \{2, 4, 5\}$$

If $\mathbf{R} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 6 \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, then 6 must be an eigenvalue, which is not possible. Hence, (d) **false**.

All options are incorrect.