10.7.72

EE25BTECH11023 - Venkata Sai

Question:
The chords of contact of the pair of tangents drawn from each point on the line $2x+y=4$ to $x^2+y^2=1$ pass through the point ...
Solution:
The general equation of conic

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

The chord of contact of tangents from an external point $\mathbf{q}$ is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{2}$$

Given circle in matrix form

$$x^2 + y^2 = 1 \tag{3}$$

$$x^2 + y^2 - 1 = 0 \tag{4}$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{5}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -1 \tag{6}$$

Given line

$$2x + y = 4 \tag{7}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{8}$$

As $\mathbf{q}$ satisfies (8)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{q} = 4 \tag{9}$$

From (2) and (6)

$$(\mathbf{I}\mathbf{q} + \mathbf{0})^\top \mathbf{x} + \mathbf{0}^\top \mathbf{q} - 1 = 0 \tag{10}$$

$$(\mathbf{I}\mathbf{q})^\top \mathbf{x} - 1 = 0 \tag{11}$$

$$\mathbf{q}^\top \mathbf{x} - 1 = 0 \tag{12}$$

$$\mathbf{q}^\top \mathbf{x} = 1 \implies \mathbf{x}^\top \mathbf{q} = 1 \tag{13}$$

From (9) and (13)

$$\mathbf{x}^\top = k\begin{pmatrix}2 & 1\end{pmatrix} \tag{14}$$

$$\left(k\begin{pmatrix}2 & 1\end{pmatrix}\right)^\top \mathbf{q} = 1 \implies k\begin{pmatrix}2 & 1\end{pmatrix}^\top \mathbf{q} = 1 \tag{15}$$

$$k\,(4) = 1 \implies k = \frac{1}{4} \tag{16}$$

$$\mathbf{x}^\top = \frac{1}{4}\begin{pmatrix}2 & 1\end{pmatrix} \implies \mathbf{x}^\top = \begin{pmatrix}\frac{1}{2} & \frac{1}{4}\end{pmatrix} \tag{17}$$

$$\mathbf{x} = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{4}\end{pmatrix} \tag{18}$$

Hence the chords of contact pass through the point $\begin{pmatrix}\frac{1}{2}\\ \frac{1}{4}\end{pmatrix}$

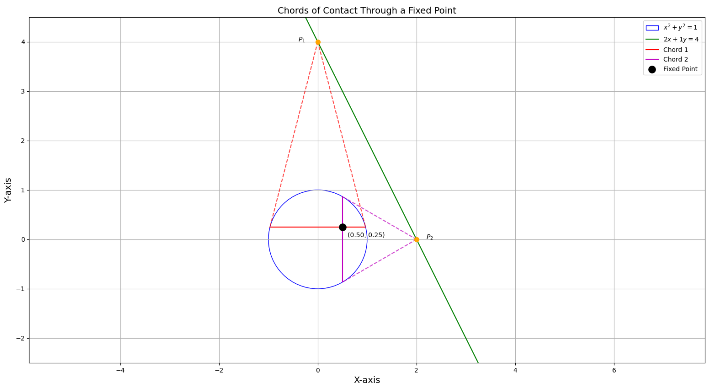

Fig. 0.1