12.37

EE25BTECH11023 - Venkata Sai

Question:
Let $\mathcal{M}$ be the set of 3×3 real symmetric positive definite matrices. Consider S = $\left\{\mathbf{A} \in \mathcal{M} : \mathbf{A}^{50} - \mathbf{A}^{48} = 0\right\}$. The number of elements in S equals
Solution:
If a matrix is symmetric then it is diagonalizable. Hence

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{1}$$

$$\begin{aligned} \mathbf{A}^2 &= \left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)^2 && (2)\\ &= \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} && (3)\\ &= \mathbf{P}\mathbf{D}\mathbf{I}\mathbf{D}\mathbf{P}^{-1} && (4)\\ &= \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1} && (5) \end{aligned}$$

$$\begin{aligned} \mathbf{A}^k &= \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} && (6)\\ \mathbf{A}^{50} &= \mathbf{P}\mathbf{D}^{50}\mathbf{P}^{-1} && (7)\\ \mathbf{A}^{48} &= \mathbf{P}\mathbf{D}^{48}\mathbf{P}^{-1} && (8) \end{aligned}$$

Given

$$\begin{aligned} \mathbf{A}^{50} - \mathbf{A}^{48} &= 0 && (9)\\ \mathbf{P}\mathbf{D}^{50}\mathbf{P}^{-1} - \mathbf{P}\mathbf{D}^{48}\mathbf{P}^{-1} &= 0 && (10)\\ \mathbf{P}\left(\mathbf{D}^{50} - \mathbf{D}^{48}\right)\mathbf{P}^{-1} &= 0 && (11)\\ \implies \left(\mathbf{D}^{50} - \mathbf{D}^{48}\right) &= 0 && (12)\\ \implies \left(\lambda^{50} - \lambda^{48}\right) &= 0 && (13) \end{aligned}$$

where λ are the eigen values

$$\begin{aligned} \lambda^{48}\left(\lambda^2 - 1\right) = 0 \implies \lambda^{48} = 0 \text{ or } \lambda^2 - 1 = 0 && (14)\\ \lambda = 0 \text{ or } \lambda = \pm 1 && (15) \end{aligned}$$

For a positive definite matrix, eigen values must be greater than 0. Hence

$$\lambda = 1 = \lambda_1 = \lambda_2 = \lambda_3 \implies \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{16}$$

$$\mathbf{A} = \mathbf{P}\mathbf{I}\mathbf{P}^{-1} = \mathbf{P}\mathbf{P}^{-1} = \mathbf{I} \tag{17}$$

Hence Number of elements in S is 1