5.13.30

Aditya Appana - EE25BTECH11004

September 30,2025

Let $\mathbf{A}$ be a square matrix all of whose entries are integers. Then which of the following is true?

1. If $det(\mathbf{A}) \neq \pm 1$, then $\mathbf{A}^{-1}$ exists but all its entries are not necessarily integers
2. If $det(\mathbf{A}) \neq \pm 1$, then $\mathbf{A}^{-1}$ exists and all its entries are non-integers
3. If $det(\mathbf{A}) = \pm 1$, then $\mathbf{A}^{-1}$ exists but all its entries are integers
4. If $det(\mathbf{A}) = \pm 1$, then $\mathbf{A}^{-1}$ need not exist

Solution

We will proceed by checking each option.

A)

Let us take a square matrix $\mathbf{A}$ having all integer entries. Let rows R_1 and R_2 be equal. By performing row operation $R_1 \to R_1 - R_2$, all elements in R_1 become 0. Therefore, $|\mathbf{A}| = 0$. We know that if $|\mathbf{A}| = 0$, $\mathbf{A}^{-1}$ does not exist. Therefore, this option is wrong.

Solution

For example, consider a matrix $\mathbf{A} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix}$. $\mathbf{A}$ has only integer entries.

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix} \xrightarrow{\mathrm{R}_1 \to \mathrm{R}_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix}$$

Since R_1 consists of only 0's, $|\mathbf{A}| = 0$. Hence $\mathbf{A}$ is not invertible.

B)

For example, consider a matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$. $|\mathbf{A}| = 2$

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}.$$

This is a counterexample to the statement; hence, option **B** is wrong.

Solution

D)

We know that if $|\mathbf{A}| \neq 0$, $\mathbf{A}^{-1}$ exists. By this logic, this option is wrong.

For example, consider a matrix $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. $|\mathbf{A}| = 1$, and since this is an orthogonal matrix,

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathsf{T}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\mathbf{A}^{-1}$ exists, which is a contradiction to the statement in option **D**.
Therefore, the correct answer is C).