Problem 2.10.47

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Problem

The value of a so that the volume of parallelopiped formed by $\hat{i} + a\hat{j} + \hat{k}, \hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum is

1. -3
2. 3
3. $\frac{1}{\sqrt{3}}$
4. $\sqrt{3}$

Formula

Volume of the parallelopiped

$$V = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r})$$

Obtaining Volume

The Volume of the parallelopiped formed by $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is ,

$$V = \mathbf{p} \cdot (\mathbf{q} \times \mathbf{r}) \tag{1.1}$$

$$V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} \tag{1.2}$$

$$V = a^3 - a + 1 \tag{1.3}$$

Finding 'a' for minimum volume

Now , consider

$$f(a) = a^3 - a + 1 \tag{1.4}$$
$$f'(a) = 3a^2 + 1 \tag{1.5}$$

$$\text{Set } f'(a) = 0 \Rightarrow a^2 = \frac{1}{\sqrt{3}} \Rightarrow a = \frac{1}{\sqrt{3}} or - \frac{1}{\sqrt{3}}$$

$$\text{Second derivative } f''(a) = 6a \tag{1.6}$$
$$\text{At } a = \frac{1}{\sqrt{3}}, f'' > 0 \Rightarrow minimum \tag{1.7}$$
$$\text{At } a = -\frac{1}{\sqrt{3}}, f'' < 0 \Rightarrow maximum \tag{1.8}$$

Therefore , $a = \frac{1}{\sqrt{3}}$ for which the Volume of the parallelopiped becomes minimum.

Plot

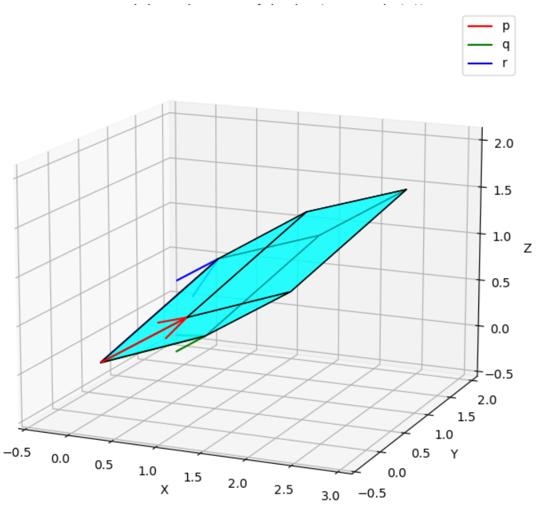

Figure: Parallelopiped with Vectors $\mathbf{p}, \mathbf{q}, \mathbf{r}$ for which $a = \frac{1}{\sqrt{3}}$(Volume is minimum)

C Code

```c
#include <stdio.h>
#include <math.h>
// Function to calculate determinant of 3x3 matrix
double determinant(double a) {
    double mat[3][3] = {
        {1, a, 1},
        {0, 1, a},
        {a, 0, 1}
    };
    double det = mat[0][0]*(mat[1][1]*mat[2][2] - mat[1][2]*mat
        [2][1])
             - mat[0][1]*(mat[1][0]*mat[2][2] - mat[1][2]*mat
                [2][0])
             + mat[0][2]*(mat[1][0]*mat[2][1] - mat[1][1]*mat
                [2][0]);

    return det; }
```

C code

```c
// Function f(a) = a^3 - a + 1
double f(double a) {
    return (a*a*a - a + 1);
}
// Function to check if a given 'a' is a local minimum
int isLocalMinimum(double a) {
    double secondDerivative = 6*a;
    return (secondDerivative > 0); // local min if f''(a) > 0
}
int main() {
    double options[4] = {-3, 3, 1.0/sqrt(3), sqrt(3)};
    int i;

    printf("Checking all options:\n");
```

C code

```c
  for (i = 0; i < 4; i++) {
        double a = options[i];
        double vol = determinant(a);

        printf("a = %lf, Determinant = %lf, f(a) = %lf", a, vol,
            f(a));

        if (fabs(a - 1.0/sqrt(3)) < 1e-6 && isLocalMinimum(a)) {

        printf("\n");
    }
    printf("\nTherefore, the local minimum occurs at a = 1/sqrt
        (3).\n");
    return 0;
}
```

Python Code for Solving

```python
import ctypes
import math
# Load the compiled shared library
lib = ctypes.CDLL("./volume.so")

# Declare function signatures
lib.determinant.argtypes = [ctypes.c_double]
lib.determinant.restype = ctypes.c_double
lib.f.argtypes = [ctypes.c_double]
lib.f.restype = ctypes.c_double
lib.isLocalMinimum.argtypes = [ctypes.c_double]
lib.isLocalMinimum.restype = ctypes.c_int

# Options to check
options = [-3, 3, 1.0/math.sqrt(3), math.sqrt(3)]
print("Checking all options:\n")
```

Python Code for Solving

```
for a in options:
    det_val = lib.determinant(a)
    f_val = lib.f(a)
    is_min = lib.isLocalMinimum(a)
    print(f"a = {a:.6f}, Determinant = {det_val:.6f}, f(a) = {
        f_val:.6f}", end="")
print("\nTherefore, the local minimum occurs at a = 1/sqrt(3).")
```

Python Code for Plotting

```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection

# Define vectors for a = 1/sqrt(3)
a = 1/np.sqrt(3)
p = np.array([1, 0, a])
q = np.array([a, 1, 0])
r = np.array([1, a, 1])

# Parallelepiped vertices (8 corners)
O = np.array([0, 0, 0]) # origin
P = p
Q = q
R = r
PQ = p + q
PR = p + r
QR = q + r
PQR = p + q + r
```

Python Code for Plotting

```python
vertices = [O, P, Q, PQ, R, PR, QR, PQR]

# Faces of parallelepiped (each face is a list of 4 vertices)
faces = [
    [O, P, PQ, Q],
    [O, P, PR, R],
    [O, Q, QR, R],
    [P, PQ, PQR, PR],
    [Q, PQ, PQR, QR],
    [R, PR, PQR, QR]
]

# Plot
fig = plt.figure(figsize=(8, 8))
ax = fig.add_subplot(111, projection='3d')

# Draw faces
ax.add_collection3d(Poly3DCollection(faces, facecolors='cyan',
                                     edgecolors='black', alpha=0.6))
```

Python Code for Plotting

```
# Draw vectors p, q, r
ax.quiver(0, 0, 0, *p, color='r', label='p')
ax.quiver(0, 0, 0, *q, color='g', label='q')
ax.quiver(0, 0, 0, *r, color='b', label='r')

# Set limits
all_points = np.array(vertices)
ax.set_xlim([np.min(all_points[:,0])-0.5, np.max(all_points[:,0])
    +0.5])
ax.set_ylim([np.min(all_points[:,1])-0.5, np.max(all_points[:,1])
    +0.5])
ax.set_zlim([np.min(all_points[:,2])-0.5, np.max(all_points[:,2])
    +0.5])
ax.set_xlabel("X")
ax.set_ylabel("Y")
ax.set_zlabel("Z")
ax.set_title("Parallelopiped formed by p, q, r (a = 1/sqrt(3))")
ax.legend()
plt.show()
```