7.4.25

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Question: The locus of the centre of a circle, which touches the circle is $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis, is given by the equation:

1) $x^2 - 6x - 10y + 14 = 0$
2) $x^2 - 10x - 6y + 14 = 0$
3) $y^2 - 6x - 10y + 14 = 0$
4) $y^2 - 10x - 6y + 14 = 0$

Solution:
Given circle equation is ,

$$x^2 + y^2 - 6x - 6y + 14 = 0$$

can be represented as

$$\|\mathbf{x}\|^2 + 2\begin{pmatrix}-3\\-3\end{pmatrix}^\top \mathbf{x} + 14 = 0 \tag{4.1}$$

The centre of circle is $\mathbf{c}_1 = \begin{pmatrix}3\\3\end{pmatrix}$ and radius $r = 2 \left(\because f = 14, \quad \mathbf{u} = \begin{pmatrix}-3\\-3\end{pmatrix}\right)$

Let the centre of the moving circle be $\mathbf{c} = \begin{pmatrix}h\\k\end{pmatrix}$

As the circle touches $X-$axis , Distance of a point from x-axis is given by

$$R = |\mathbf{n}^\top \mathbf{c}| \tag{4.2}$$

where $\mathbf{n}$ is the unit vector normal to x-axis

$$\mathbf{n} = \begin{pmatrix}1\\0\end{pmatrix} \tag{4.3}$$

Distance between their centers equal to sum of their radius

$$\|\mathbf{c} - \mathbf{c_1}\| = R \pm r \tag{4.4}$$

$$\|\mathbf{c} - \mathbf{c_1}\| = |\mathbf{n}^\top \mathbf{c}| \pm r \tag{4.5}$$

$$\|\mathbf{c} - \mathbf{c_1}\|^2 = \left(|\mathbf{n}^\top \mathbf{c}| \pm r\right)^2 \tag{4.6}$$

$$(\mathbf{c} - \mathbf{c_1})(\mathbf{c} - \mathbf{c_1})^\top = \left(|\mathbf{n}^\top \mathbf{c}| \pm r\right)^2 \tag{4.7}$$

$$\mathbf{c}^\top\mathbf{c} + \mathbf{c_1}\mathbf{c_1}^\top - \mathbf{c_1}^\top\mathbf{c} - \mathbf{c}^\top\mathbf{c_1} = \left(|\mathbf{n}^\top\mathbf{c}|\right)^2 \pm 2r|\mathbf{n}^\top\mathbf{c}| + r^2 \tag{4.8}$$

$$\mathbf{c}^\top\mathbf{c} + \mathbf{c_1}\mathbf{c_1}^\top - \mathbf{c_1}^\top\mathbf{c} - \mathbf{c}^\top\mathbf{c_1} = \left(\mathbf{n}^\top\mathbf{c}\right)^\top\left(\mathbf{n}^\top\mathbf{c}\right) \pm 2r|\mathbf{n}^\top\mathbf{c}| + r^2 \tag{4.9}$$

$$\mathbf{c}^\top\mathbf{c} + \|\mathbf{c_1}\|^2 - 2\mathbf{c_1}^\top\mathbf{c} = \left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right) \pm 2r|\mathbf{n}^\top\mathbf{c}| + r^2 \tag{4.10}$$

$$\mathbf{c}^\top\mathbf{c} + 18 = \left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right) + 2\mathbf{n}^\top\mathbf{c} \pm 2r|\mathbf{n}^\top\mathbf{c}| + r^2 + 2\mathbf{c_1}^\top\mathbf{c} \tag{4.11}$$

$$\mathbf{c}^\top\mathbf{c} + 14 = \left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right) + 2\mathbf{n}^\top\mathbf{c} \pm 4|\mathbf{n}^\top\mathbf{c}| + 2\mathbf{c_1}^\top\mathbf{c} \quad \text{(Since r=2)} \tag{4.12}$$

Case 1 : (External Tangency)

$$\mathbf{c}^\top\mathbf{c} + 14 = \left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right) + 2\mathbf{n}^\top\mathbf{c} + 4|\mathbf{n}^\top\mathbf{c}| + 2\mathbf{c_1}^\top\mathbf{c} \tag{4.13}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 14 = \begin{pmatrix} x & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} + 4\left(\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) + 2\left(\begin{pmatrix} 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \tag{4.14}$$

$$x^2 + y^2 + 14 = x^2 + 4x + 6x + 6y \tag{4.15}$$

$$y^2 - 10x - 6y + 14 = 0 \tag{4.16}$$

Case 2 : (Internal Tangency)

$$\mathbf{c}^\top\mathbf{c} + 14 = \left(\mathbf{c}^\top\mathbf{n}\mathbf{n}^\top\mathbf{c}\right) + 2\mathbf{n}^\top\mathbf{c} - 4|\mathbf{n}^\top\mathbf{c}| + 2\mathbf{c_1}^\top\mathbf{c} \tag{4.17}$$

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 14 = \begin{pmatrix} x & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} - 4\left(\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) + 2\left(\begin{pmatrix} 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \tag{4.18}$$

$$x^2 + y^2 + 14 = x^2 - 4x + 6x + 6y \tag{4.19}$$

$$y^2 - 2x - 6y + 14 = 0 \tag{4.20}$$

Therefore , the locus of the centre of a circle is

$$y^2 - 10x - 6y + 14 = 0 \quad \text{(from the options)}$$

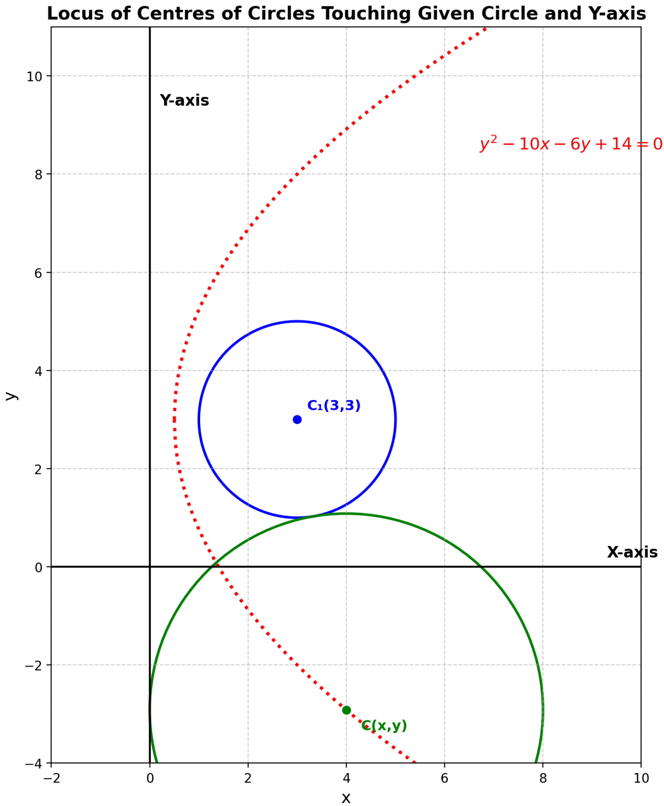
Locus of Centres of Circles Touching Given Circle and Y-axis
Y-axis
$y^2 - 10x - 6y + 14 = 0$
C₁(3,3)
X-axis
C(x,y)
x
y
−2
0
2
4
6
8
10
−4
−2
0
2
4
6
8
10

Fig. 4.1