9.2.15

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included above the x axis.

Solution

General Equation of Conic in Matrix form is given by :

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

First,The curve $y = \sqrt{a^2 - x^2}$ can be Rearranged as :

$$y^2 = a^2 - x^2 \implies x^2 + y^2 - a^2 = 0 \tag{2}$$

Using this, The specific conic parameters for this curve:

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -a^2 \tag{3}$$

Next, the area is bounded below by the x-axis (the line $y = 0$). The parameters for this line are:

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Using the formula given in the book to solve point of intersection of line with a given curve:

$$\kappa = \frac{1}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^\top \mathbf{V}\mathbf{m})} \right)$$

where $g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f$. We calculate each component:

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{5}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \tag{6}$$

$$g(\mathbf{h}) = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} - a^2 = 0 + 0 - a^2 = -a^2 \tag{7}$$

Substituting these results back into the formula for κ:

$$\kappa = \frac{1}{1} \left(-0 \pm \sqrt{(0)^2 - (-a^2)(1)} \right) = \pm \sqrt{a^2} = \pm a$$

The solutions are $\kappa_1 = a$ and $\kappa_2 = -a$, confirming the intersections at $x = \pm a$.

Hence,The area can be given by :

$$A = \int_{-a}^{a} \sqrt{a^2 - x^2} dx \tag{8}$$

This integral is solved using trigonometric substitution. Let $x = a \sin \theta$, so $dx = a \cos \theta \, d\theta$. The limits become $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot (a \cos \theta \, d\theta) \tag{9}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 \cos^2 \theta} \cdot (a \cos \theta \, d\theta) \tag{10}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a \cos \theta)(a \cos \theta \, d\theta) \tag{11}$$

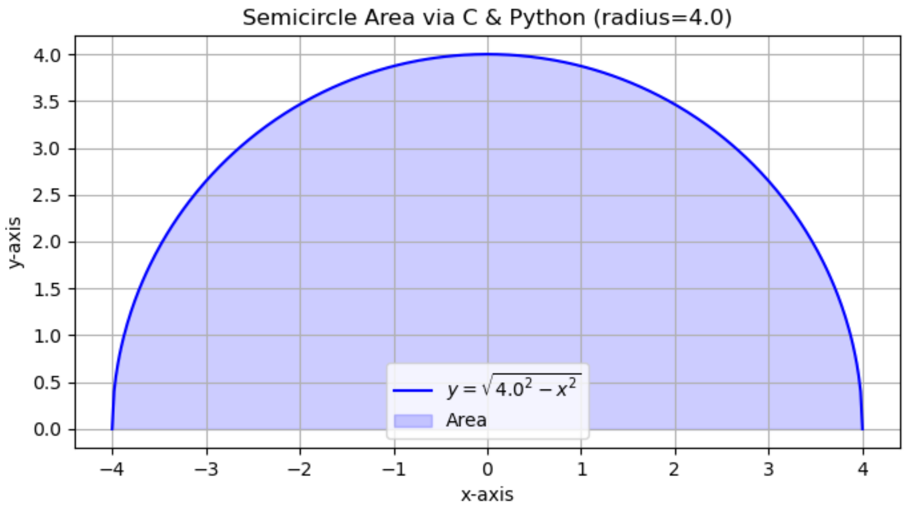
$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \tag{12}$$

$$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} \, d\theta \tag{13}$$

$$= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin(2\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tag{14}$$

$$= \frac{a^2}{2} \left(\left[\frac{\pi}{2} + \frac{1}{2} \sin(\pi) \right] - \left[0 + \frac{1}{2} \sin(-\frac{\pi}{2}) \right] \right) \tag{15}$$

$$= \frac{a^2}{2} \left(\frac{\pi}{2} \right) = \frac{\pi a^2}{2} \tag{16}$$



Semicircle Area via C & Python (radius=4.0)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y-axis
−4
−3
−2
−1
0
1
2
3
4
x-axis
$y = \sqrt{4.0^2 - x^2}$
Area

Fig. 1