12.482

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:
Consider the matrix $\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$. The normalized eigen-vector corresponding to the smallest eigen-value of the matrix $\mathbf{M}$ is:

1) $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ 2) $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$ 3) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ 4) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Solution
Given matrix:

$$\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \tag{1}$$

The characteristic equation is:

$$\det(\mathbf{M} - \lambda\mathbf{I}) = 0 \tag{2}$$
$$\det\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} = 0 \tag{3}$$
$$(5-\lambda)(5-\lambda) - (3)(3) = 0 \tag{4}$$
$$(5-\lambda)^2 - 9 = 0 \tag{5}$$
$$\lambda^2 - 10\lambda + 16 = 0 \tag{6}$$

Using the quadratic formula:

$$\lambda = \frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm \sqrt{36}}{2} = \frac{10 \pm 6}{2} \tag{7}$$
$$\lambda_1 = 8 \quad \text{(largest eigenvalue)} \tag{8}$$
$$\lambda_2 = 2 \quad \text{(smallest eigenvalue)} \tag{9}$$

For $\lambda = 2$, solve $(\mathbf{M} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$
$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$
$$3v_1 + 3v_2 = 0 \tag{12}$$
$$v_1 + v_2 = 0 \implies v_1 = -v_2 \tag{13}$$

Let $v_2 = t$, then $v_1 = -t$.
The general eigenvector is:

$$\mathbf{v} = t\begin{pmatrix}-1\\1\end{pmatrix} = t\begin{pmatrix}1\\-1\end{pmatrix}\cdot(-1) \tag{14}$$

$$\mathbf{v} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{15}$$

The normalized eigenvector is:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||} \tag{16}$$

$$||\mathbf{v}|| = \sqrt{\mathbf{v}^\top\mathbf{v}} = \sqrt{2} \tag{17}$$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix} \tag{18}$$

$$\boxed{\text{The correct answer is (c) } \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}}$$