Matrices in Geometry 10.5.3

EE25BTECH11035 - Kushal B N

Question: Draw a pair of tangents to a circle of radius $5cm$ which are inclined to each other at an angle of $60°$.

Solution:

Let the center be the origin. Then the circle with radius $5cm$ is

$$\mathbf{C} : \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , f = -25 \tag{2}$$

Let the tangents be drawn from an external point $\mathbf{h}$. Line segment from O to h bisects the angle between the tangents and it forms two right angled triangles. So that, we have

$$\sin \frac{60°}{2} = \frac{r}{\|\mathbf{h}\|} = \frac{5}{\|\mathbf{h}\|} \tag{3}$$

$$\implies \|\mathbf{h}\| = \frac{5}{\sin 30°} = 10 \tag{4}$$

So the point h should be at a distance of $10cm$ from the origin(centre). Let the point be

$$\mathbf{h} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \tag{5}$$

Calculating the matrix Σ

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \tag{6}$$

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f = \|\mathbf{h}\|^2 + f = 100 - 25 = 75 \tag{7}$$

$$\Sigma = \mathbf{h}\mathbf{h}^\top - g(\mathbf{h})\mathbf{V} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \begin{pmatrix} 10 & 0 \end{pmatrix} - 75 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

$$\implies \Sigma = \begin{pmatrix} 25 & 0 \\ 0 & -75 \end{pmatrix} \tag{9}$$

The eigen values of the matrix Σ are

$$\lambda_1 = 25 \tag{10}$$

$$\lambda_2 = -75 \tag{11}$$

The direction vectors for the tangents

$$\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} = \begin{pmatrix} 5\sqrt{3} \\ \pm 5 \end{pmatrix} \tag{12}$$

$$\implies \mathbf{m}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} , \mathbf{m}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \tag{13}$$

As the tangents pass through $\mathbf{h}$ with direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$, using the normal form of the line $\mathbf{n}^\top(\mathbf{x} - \mathbf{h}) = 0$. Their equations are

$$\begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right) = 0 \tag{14}$$

$$\begin{pmatrix} 1 & \sqrt{3} \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right) = 0 \tag{15}$$

Plot:

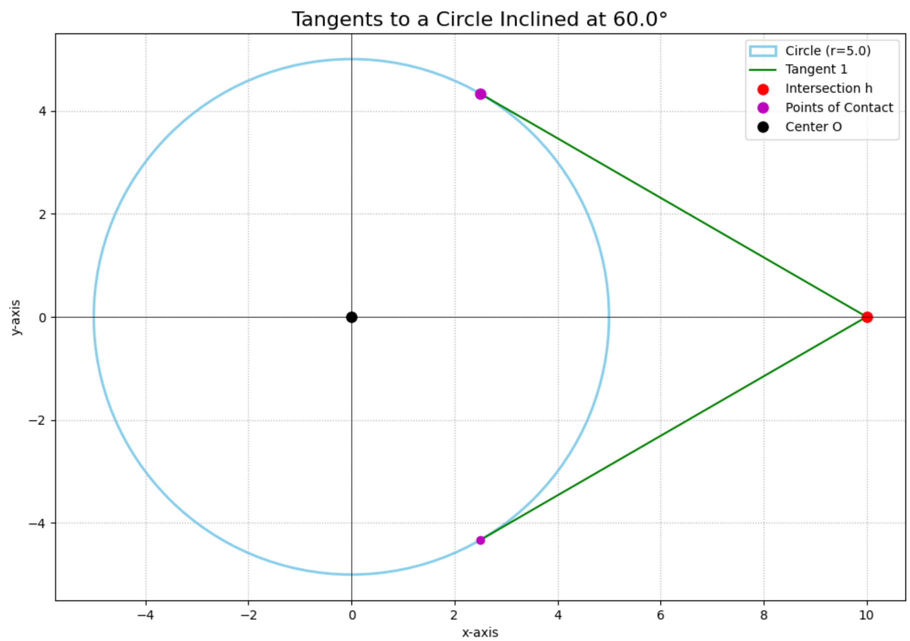

Fig. 1: Plot for 10.5.3