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Question : Let $\mathbf{A}$ be an $n \times n$ real matrix. Consider the following statements:

1) If $\mathbf{A}$ is symmetric, then there exists $c \geq 0$ such that $\mathbf{A} + c\mathbf{I}_n$ is symmetric and positive definite, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.
2) If $\mathbf{A}$ is symmetric and positive definite, then there exists a symmetric and positive definite $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$.
   Which of the above statements is/are true?

a) Only (I) b) Only (II) c) Both (I) and (II) d) Neither (I) nor (II)

Solution :

Name	Description
$\mathbf{A}$	Matrix

Table : Matrix

Checking statement (**I**)
If $\mathbf{A}$ is symmetric, its eigenvalues are real. Let the minimum eigenvalue of $\mathbf{A}$ be $\lambda_{\min}$.Then choose $c > -\lambda_{\min}$.
The Eigen values of $\mathbf{A}$ are given as :

$$\left|\mathbf{A} - \lambda_i \mathbf{I}\right| = 0 \tag{1}$$

The Eigen values of $\mathbf{A} + c\mathbf{I}_n$ are given as :

$$\left|\mathbf{A} - (\lambda_k - c)\mathbf{I}\right| = 0 \tag{2}$$

$$\lambda_k = \lambda_i + c \tag{3}$$

$$\lambda_i + c > 0 \tag{4}$$

Since $\lambda_i + c > 0$ for all i, $\mathbf{A} + c\mathbf{I}_n$ is positive definite and symmetric. Hence, statement (**I**) is **true**.
Checking statement (**II**)
If $\mathbf{A}$ is symmetric and positive definite, then it can be diagonalized as:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \tag{5}$$

where $\mathbf{P}$ is orthogonal and $\mathbf{D}$ is a diagonal matrix with positive entries (since $\mathbf{A}$ is positive definite). Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top \tag{6}$$

Then,

$$\mathbf{B}^2 = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top\mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top = \mathbf{P}\mathbf{D}\mathbf{P}^\top = \mathbf{A} \tag{7}$$

Hence, $\mathbf{B}$ is symmetric and positive definite. Therefore, statement (**II**) is also **true**.
Final Answer: (c) Both (I) and (II)

Examples

Example a for (I)

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

To find eigenvalues, evaluate:

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \tag{9}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \tag{10}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \xleftrightarrow{R_2 \to R_2 + \frac{1}{\lambda} R_1} \begin{vmatrix} -\lambda & 1 \\ 0 & \frac{1-\lambda^2}{\lambda} \end{vmatrix} = 0 \tag{11}$$

$$\lambda^2 - 1 = 0 \tag{12}$$

$$\lambda = \pm 1 \tag{13}$$

$$\lambda_1 = 1, \lambda_2 = -1 \tag{14}$$

The minimum eigenvalue $\lambda_{\min} = -1$. Choose $c = 2$. Then:

$$\lambda_1 + c = 3, \quad \lambda_2 + c = 1 \tag{15}$$

All eigenvalues are positive, so $\mathbf{A} + 2\mathbf{I}$ is symmetric positive definite.

Example b for (I)

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{16}$$

Compute eigenvalues from $\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$:

$$\begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \tag{17}$$

$$(-2-\lambda)(1-\lambda)(3-\lambda) = 0 \tag{18}$$

Hence

$$\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = 3 \tag{19}$$

Choose $c = 3$ so $c > -\lambda_{\min} = 2$. Then $\lambda_i + c > 0$ for all i. Thus $\mathbf{A} + 3\mathbf{I}$ is symmetric positive definite.

Example a for (II)

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \tag{20}$$

Find eigenvalues from $\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$:

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \tag{21}$$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} \xleftrightarrow{R_2 \to R_2 - \frac{2}{5-\lambda} R_1} \begin{vmatrix} 5-\lambda & 2 \\ 0 & (5-\lambda) - \frac{4}{5-\lambda} \end{vmatrix} = 0 \tag{22}$$

$$(5-\lambda)^2 - 4 = 0 \tag{23}$$

$$\lambda^2 - 10\lambda + 21 = 0 \tag{24}$$

$$\lambda = 5 \pm 2 \tag{25}$$

$$\lambda_1 = 7,\ \lambda_2 = 3 \tag{26}$$

For eigenvectors:

$$(\mathbf{A} - 7\mathbf{I})\mathbf{v} = 0, \quad (\mathbf{A} - 3\mathbf{I})\mathbf{v} = 0 \tag{27}$$

They correspond to $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Form $\mathbf{P}$ and $\mathbf{D}$:

$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \tag{28}$$

Then

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^\top, \quad \mathbf{B}^2 = \mathbf{A} \tag{29}$$

Hence verified for 2×2.

Example b for (II)

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{30}$$

Compute $\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0$:

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \tag{31}$$

$$(2-\lambda)(3-\lambda)(4-\lambda) = 0 \tag{32}$$

$$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 4 \tag{33}$$

Diagonalization:

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\top}, \quad \mathbf{P} = \mathbf{I}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{34}$$

Define

$$\mathbf{B} = \mathbf{P}\mathbf{D}^{1/2}\mathbf{P}^{\top} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{35}$$

Then

$$\mathbf{B}^2 = \mathbf{A} \tag{36}$$

Hence **B** is symmetric positive definite.

Conclusion: In all four examples, both statements (I) and (II) hold true.