12.66

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Question

The matrix

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$$

is:

1. orthogonal
2. symmetric
3. anti-symmetric
4. unitary

(PH 2014)

Solution - Given

Given matrix:

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{1}$$

Solution - Check 1: Symmetric

Check 1: Symmetric ($A = A^\top$)
The transpose of A is:

$$A^\top = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{2}$$

Since $A^\top \neq A$ (the off-diagonal elements are different: $1+i \neq 1-i$),

A is NOT symmetric.

Solution - Check 2: Anti-symmetric

Check 2: Anti-symmetric ($A = -A^\top$)
For anti-symmetric:

$$-A^\top = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{3}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -1+i \\ -1-i & 1 \end{pmatrix} \tag{4}$$

Since $A \neq -A^\top$,

***A* is NOT anti-symmetric.**

Check 3: Orthogonal ($AA^\top = I$)
Note: For real matrices, orthogonal means $AA^\top = I$. However, A contains complex entries.

$$AA^\top = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{5}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix} \tag{6}$$

Solution - Check 3: Orthogonal (Part 2)

Computing the $(1,1)$ entry:

$$(AA^\top)_{11} = \frac{1}{3}\,[1 \cdot 1 + (1+i)(1+i)] \tag{7}$$

$$= \frac{1}{3}\left[1 + 1 + 2i + i^2\right] \tag{8}$$

$$= \frac{1}{3}[1 + 2i] \neq 1 \tag{9}$$

Since this is complex (not real), $AA^\top \neq I$,

A **is NOT orthogonal.**

Solution - Check 4: Unitary (Part 1)

Check 4: Unitary ($A\overline{A^\top} = I$)
For a unitary matrix, we need $A\overline{A^\top} = I$, where $\overline{A^\top}$ is the conjugate transpose.
The conjugate transpose is:

$$\overline{A^\top} = \overline{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}} \tag{10}$$

$$= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{11}$$

Notice that $\overline{A^\top} = A$ (the matrix is Hermitian!)

Solution - Check 4: Unitary (Part 2)

Let's verify unitarity:

$$A\overline{A^\top} = \frac{1}{3}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{12}$$

Computing each entry:

$$(A\overline{A^\top})_{11} = \frac{1}{3}\left[1 \cdot 1 + (1+i)(1-i)\right] \tag{13}$$

$$= \frac{1}{3}\left[1 + 1 - i^2\right] \tag{14}$$

$$= \frac{1}{3}[1 + 1 + 1] = 1 \tag{15}$$

Solution - Check 4: Unitary (Part 3)

Entry $(1,2)$**:**

$$(A\overline{A^\top})_{12} = \frac{1}{3}\left[1 \cdot (1+i) + (1+i)(-1)\right] = 0 \tag{16}$$

Entry $(2,1)$**:**

$$(A\overline{A^\top})_{21} = \frac{1}{3}\left[(1-i) \cdot 1 + (-1)(1-i)\right] = 0 \tag{17}$$

Entry $(2,2)$**:**

$$(A\overline{A^\top})_{22} = \frac{1}{3}\left[(1-i)(1+i) + (-1)(-1)\right] \tag{18}$$

$$= \frac{1}{3}[2+1] = 1 \tag{19}$$

Solution - Final Answer

Therefore:

$$A\overline{A^{\top}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{20}$$

A **is UNITARY.**

$\boxed{\text{Option 4: The matrix } A \text{ is unitary}}$