Matrices in Geometry 8.4.38

EE25BTECH11035 - Kushal B N

Question: Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval

1) $(5, \infty)$
2) $(\frac{3}{2}, 3]$
3) $(2, 3]$
4) $(1, \frac{3}{2}]$

Given:
Hyperbola $\sec^2(\theta)x^2 - \csc^2(\theta)y^2 = 1$ for which $e > 2$
Solution:

Comparing to the general conic form $\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0$, we get

$$\mathbf{V} = \begin{pmatrix} \sec^2(\theta) & 0 \\ 0 & \csc^2(\theta) \end{pmatrix} \tag{1}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

$$f = -1 \tag{3}$$

Now, here as $\mathbf{V}$ is a diagonal matrix, its eigenvalues are its diagonal entries, that is,

$$\lambda_1 = \frac{1}{\sin^2\theta} \tag{4}$$

$$\lambda_2 = \frac{1}{\cos^2\theta} \tag{5}$$

Now, the eccentricity of the hyperbola

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{6}$$

$$\implies e^2 = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{7}$$

As $0 < \theta < \frac{\pi}{2}$, $\cos\theta$ is positive and so,

$$e = \frac{1}{\cos\theta} \tag{8}$$

Now, as $e > 2$

$$\frac{1}{\cos\theta} > 2 \implies \cos\theta < \frac{1}{2} \tag{9}$$

$$\implies \frac{\pi}{3} < \theta < \frac{\pi}{2} \tag{10}$$

Length of the latus rectum,

$$l = \frac{2\sqrt{|f_0\lambda_1|}}{|\lambda_2|} \tag{11}$$

where,

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{12}$$

as $\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we get

$$f_0 = -f = -(-1) = 1 \tag{13}$$

Substituting these values into (11), we get

$$l = \frac{2\sqrt{\left|1.\sec^2\theta\right|}}{\left|\csc^2\theta\right|} \tag{14}$$

In the interval $0 < \theta < \frac{\pi}{2}$, $\sec\theta$ is positive and so

$$\implies l = \frac{2\sin^2\theta}{\cos\theta} \tag{15}$$

Now from (10), we get

$$\boxed{l \in (3, \infty)} \tag{16}$$

Final Answer:
The length of the latus rectum for the given hyperbola lies in the interval $(3, \infty)$.

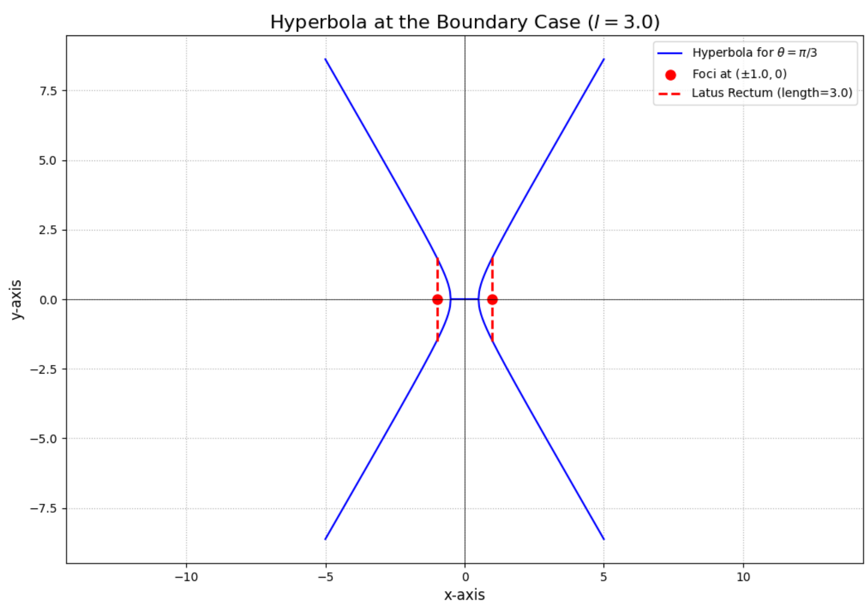

Fig. 1: Plot for 8.4.38