12.482

Shriyansh Chawda-EE25BTECH11052 October 11, 2025

Question

Consider the matrix $\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$. The normalized eigen-vector corresponding to the smallest eigen-value of the matrix $\mathbf{M}$ is:

(PE 2016)

1. $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$
2. $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$
3. $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$
4. $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Solution

Given matrix:

$$\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \tag{1}$$

The characteristic equation is:

$$\det(\mathbf{M} - \lambda \mathbf{I}) = 0 \tag{2}$$

$$\det \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} = 0 \tag{3}$$

$$(5-\lambda)(5-\lambda) - (3)(3) = 0 \tag{4}$$

$$(5-\lambda)^2 - 9 = 0 \tag{5}$$

$$\lambda^2 - 10\lambda + 16 = 0 \tag{6}$$

Using the quadratic formula:

Solution

$$\lambda = \frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm \sqrt{36}}{2} = \frac{10 \pm 6}{2} \tag{7}$$

$$\lambda_1 = 8 \quad \text{(largest eigenvalue)} \tag{8}$$

$$\lambda_2 = 2 \quad \text{(smallest eigenvalue)} \tag{9}$$

For $\lambda = 2$, solve $(\mathbf{M} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

$$3v_1 + 3v_2 = 0 \tag{12}$$

$$v_1 + v_2 = 0 \implies v_1 = -v_2 \tag{13}$$

Solution

Let $v_2 = t$, then $v_1 = -t$.
The general eigenvector is:

$$\mathbf{v} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot (-1) \tag{14}$$

We can choose:

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{15}$$

The normalized eigenvector is:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{16}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}^\top \mathbf{v}} = \sqrt{2} \tag{17}$$

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{18}$$

$$\boxed{\text{The correct answer is (c) } \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}}$$