Problem 12.37

ee25btech11023-Venkata Sai

October 3, 2025

1 Problem

2 Solution
- Condition
- Conclusion

Problem

Let $\mathcal{M}$ be the set of 3×3 real symmetric positive definite matrices. Consider $\mathsf{S} = \{\mathbf{A} \in \mathcal{M} : \mathbf{A}^{50} - \mathbf{A}^{48} = 0\}$. The number of elements in S equals

Condition

If a matrix is symmetric then it is diagonalizable. Hence

$$\mathbf{A} = \mathbf{PDP}^{-1} \tag{3.1}$$

$$\mathbf{A}^2 = \left(\mathbf{PDP}^{-1}\right)^2 \tag{3.2}$$

$$= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \tag{3.3}$$

$$= \mathbf{PDIDP}^{-1} \tag{3.4}$$

$$= \mathbf{PD}^2\mathbf{P}^{-1} \tag{3.5}$$

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1} \tag{3.6}$$

$$\mathbf{A}^{50} = \mathbf{PD}^{50}\mathbf{P}^{-1} \tag{3.7}$$

$$\mathbf{A}^{48} = \mathbf{PD}^{48}\mathbf{P}^{-1} \tag{3.8}$$

Given

$$\mathbf{A}^{50} - \mathbf{A}^{48} = 0 \tag{3.9}$$

$$\mathbf{PD}^{50}\mathbf{P}^{-1} - \mathbf{PD}^{48}\mathbf{P}^{-1} = 0 \tag{3.10}$$

Conclusion

$$\mathbf{P}\left(\mathbf{D}^{50}-\mathbf{D}^{48}\right)\mathbf{P}^{-1}=0 \tag{3.11}$$

$$\implies \left(\mathbf{D}^{50}-\mathbf{D}^{48}\right)=0 \tag{3.12}$$

$$\implies \left(\lambda^{50}-\lambda^{48}\right)=0 \tag{3.13}$$

where λ are the eigen values

$$\lambda^{48}\left(\lambda^{2}-1\right)=0 \implies \lambda^{48}=0 \text{ or } \lambda^{2}-1=0 \tag{3.14}$$

$$\lambda=0 \text{ or } \lambda=\pm 1 \tag{3.15}$$

For a positive definite matrix, eigen values must be greater than 0. Hence

$$\lambda=1=\lambda_1=\lambda_2=\lambda_3 \implies \mathbf{D}=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I} \tag{3.16}$$

$$\mathbf{A}=\mathbf{PIP}^{-1}=\mathbf{PP}^{-1}=\mathbf{I} \tag{3.17}$$