10.3.26

EE25BTECH11023 - Venkata Sai

Question:
Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.
Solution:
Given curve

$$y = \sqrt{4x-3} - 1 \tag{1}$$

$$y + 1 = \sqrt{4x-3} \implies (y+1)^2 = 4x - 3 \tag{2}$$

$$y^2 + 2y + 1 = 4x - 3 \tag{3}$$

$$y^2 - 4x + 2y + 4 = 0 \tag{4}$$

Equation (4) in matrix form

$$y^2 + 2(-2x + y) + 4 = 0 \tag{5}$$

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{6}$$

The general equation of conic

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{7}$$

On comparing (6) with (7)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, f = 4 \tag{8}$$

Given slope

$$m = \frac{2}{3} \tag{9}$$

The normal vector to the given tangent is

$$\mathbf{n} = \begin{pmatrix} -m \\ 1 \end{pmatrix} \implies \mathbf{n} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} \tag{10}$$

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \tag{11}$$

$$\left| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \implies \left| \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \tag{12}$$

$$\left| \begin{pmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \right| = 0 \implies (-\lambda)(1-\lambda) = 0 \tag{13}$$

$$\lambda_1 = 0 \text{ and } \lambda_2 = 1 \tag{14}$$

Finding eigen vector for $\lambda_1 = 0$

$$(\mathbf{V} - \lambda\mathbf{I})\,\mathbf{p} = \mathbf{0} \tag{15}$$

$$\begin{pmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

$$0 = 0, y = 0 \implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{17}$$

For a given normal vector $\mathbf{n}$, the point of contact $\mathbf{q}$ for a given curve is given by the matrix equation

$$\begin{pmatrix} (\mathbf{u} + \kappa\mathbf{n})^\top \\ \mathbf{V} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -f \\ \kappa\mathbf{n} - \mathbf{u} \end{pmatrix} \quad \text{where } \kappa = \frac{\mathbf{p_1}^\top\mathbf{u}}{\mathbf{p_1}^\top\mathbf{n}} \tag{18}$$

$$\kappa = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}} = \frac{-2}{-\frac{2}{3}} = 3 \tag{19}$$

From (8)

$$\begin{pmatrix} \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}\right)^\top \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 3\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} \begin{pmatrix} -4 \\ 4 \end{pmatrix}^\top \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{pmatrix} \implies \begin{pmatrix} \begin{pmatrix} -4 & 4 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{q} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \tag{22}$$

Taking augmented matrix

$$\left(\begin{array}{cc|c} -4 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \to R_1 - 4R_2} \left(\begin{array}{cc|c} -4 & 0 & -12 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \to -\frac{1}{4}R_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right) \tag{23}$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{24}$$

Hence the point of contact is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

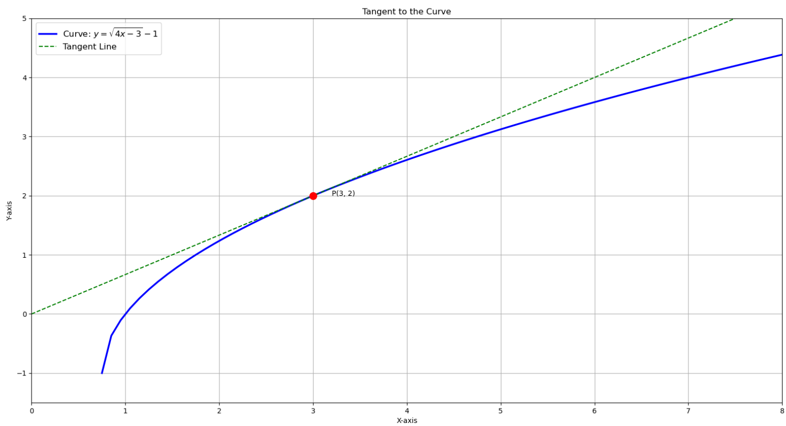
Tangent to the Curve
Curve: $y = \sqrt{4x-3} - 1$
Tangent Line
P(3, 2)
X-axis
Y-axis
0
1
2
3
4
5
6
7
8
−1
0
1
2
3
4
5

Fig. 0.1