10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

(3,0) is the point from which three normals are drawn to the parabola $y^2 = 4x$ which meet the parabola in the points **P**, **Q**, and **R**. Match the following (2006)

	Column I		Column II
a)	Area of $\triangle POR$	a)	2
b)	Radius of circumcircle of $\triangle PQR$	b)	$\frac{5}{2}$
c)	Centroid of $\triangle POR$	c)	$\left(\frac{5}{2}, 0\right)$
d)	Circumcentre of $\triangle PQR$	d)	$\left(\frac{2}{3}, 0\right)$

Solution

In matrix form,the parabola can be written as:

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{1}$$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f = 0 \tag{2}$$

For point h to lie on a normal to the conic, we use formula (10.1.9.1).

Let direction vector $m = \begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ and normal vector $n = \begin{pmatrix} -m_1 \\ 1 \end{pmatrix}$.

Computing required terms::

$$Vh + u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{3}$$

$$\begin{aligned} g(h) &= h^\top V h + 2u^\top h + f \\ &= \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 0 \\ &= 0 - 12 + 0 = -12 \end{aligned} \tag{4}$$

$$m^\top (Vh + u) = \begin{pmatrix} 1 & m_1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = -2 \tag{5}$$

$$n^\top V n = \begin{pmatrix} -m_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = \begin{pmatrix} -m_1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \tag{6}$$

$$m^\top V n = \begin{pmatrix} 1 & m_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -m_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & m_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = m_1 \tag{7}$$

$$n^\top (Vh + u) = \begin{pmatrix} -m_1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = 2m_1 \tag{8}$$

Substituting into (10.1.9.1):: The condition for h to lie on a normal is:

$$\left[m^\top (Vh+u)\right]^2 \left[n^\top Vn\right] - 2\left[m^\top Vn\right]\left[m^\top (Vh+u)\right]\left[n^\top (Vh+u)\right] + g(h)\left[m^\top Vn\right]^2 = 0 \quad (9)$$

Substituting values:

$$(-2)^2(1) - 2(m_1)(-2)(2m_1) + (-12)(m_1)^2 = 0 \quad (10)$$

$$4 + 8m_1^2 - 12m_1^2 = 0 \quad (11)$$

$$4 - 4m_1^2 = 0 \implies m_1^2 = 1 \implies m_1 = \pm 1 \quad (12)$$

Additionally, $m_1 = 0$ (horizontal normal) is also a solution. Therefore, the three slopes of normals are: $m = 0, 1, -1$
For parabola $y^2 = 4x$ (where $4a = 4 \implies a = 1$), if normal has slope m, the point of contact is:

$$\left(am^2, -2am\right) = \left(m^2, -2m\right) \quad (13)$$

For $m = 0$:

$$R = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (14)$$

For $m = 1$:

$$P = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (15)$$

For $m = -1$:

$$Q = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (16)$$

(a) Area of $\triangle PQR$: Using the determinant formula:

$$\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{pmatrix}\right| \quad (17)$$

$$= \frac{1}{2}\left|\begin{pmatrix} 1 & -2 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right| \quad (18)$$

$$= \frac{1}{2}\left|1 \cdot \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}\right| = \frac{1}{2}|4| = 2 \quad (19)$$

Answer: Column II-a

(b) radius of circumcircle of δPQR: Using the formula:

$$R = \frac{|PQ| \cdot |QR| \cdot |RP|}{4 \cdot \text{Area}} \quad (20)$$

$$|PQ| = \left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 \\ -4 \end{pmatrix} \right\| = 4 \tag{21}$$

$$|QR| = \left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \sqrt{5} \tag{22}$$

$$|RP| = \left\| \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = \sqrt{5} \tag{23}$$

$$R = \frac{4 \cdot \sqrt{5} \cdot \sqrt{5}}{4 \cdot 2} = \frac{5}{2} \tag{24}$$

Answer:Column II-b

(c) Centroid of $\triangle PQR$: The centroid is given by:

$$G = \frac{1}{3} \begin{pmatrix} x_P + x_Q + x_R \\ y_P + y_Q + y_R \end{pmatrix} \tag{25}$$

$$G = \frac{1}{3} \begin{pmatrix} 1 + 1 + 0 \\ -2 + 2 + 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \tag{26}$$

Ans :Column II-d

(d) Circumcentre of $\triangle PQR$: Since the triangle is isosceles with $|QR| = |RP| = \sqrt{5}$ and the points P and Q have the same x-coordinate with opposite y-coordinates, the circumcentre lies on the x-axis by symmetry.

Let the circumcentre be $O = \begin{pmatrix} x_c \\ 0 \end{pmatrix}$.

For circumcentre, the distance to all three vertices must be equal. Using vertices P and R:

$$|OP|^2 = |OR|^2 \tag{27}$$

$$(x_c - 1)^2 + (-2 - 0)^2 = (x_c - 0)^2 + (0 - 0)^2 \tag{28}$$

$$x_c^2 - 2x_c + 1 + 4 = x_c^2 \tag{29}$$

$$-2x_c + 5 = 0 \implies x_c = \frac{5}{2} \tag{30}$$

Answer: Column II-c

Final Matching

Column I	**Column II**
a)	a)
b)	b)
c)	d)
d)	c)

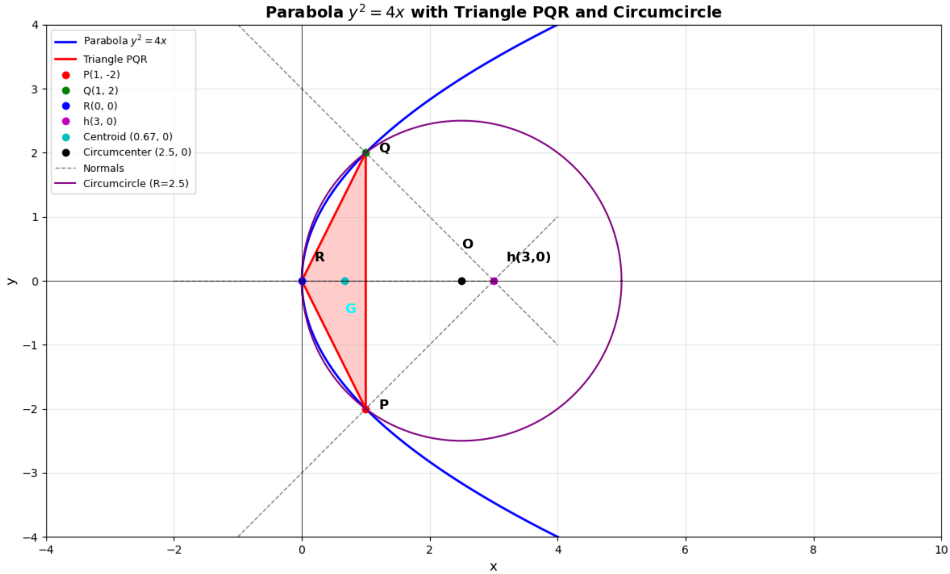
Parabola $y^2 = 4x$ with Triangle PQR and Circumcircle
Parabola $y^2 = 4x$
Triangle PQR
P(1, -2)
Q(1, 2)
R(0, 0)
h(3, 0)
Centroid (0.67, 0)
Circumcenter (2.5, 0)
Normals
Circumcircle (R=2.5)
Q
O
h(3,0)
R
G
P
x
y

Fig. 1