8.2.7

EE25BTECH11043 - Nishid Khandagre

Question: Find the coordinates of the focus, vertex, eccentricity, axis of the conic section, the equation of the directrix and the length of the latus rectum. $16x^2 + y^2 = 16$

Solution:

We use an affine transformation to convert the conic equation to its standard form.

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.1}$$

The symmetric matrix $\mathbf{V}$ is spectrally decomposed to align axes with eigenvectors.

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^\top, \ \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \mathbf{P}^\top \mathbf{P} = \mathbf{I} \tag{0.2}$$

Substituting the decomposition into the conic equation.

$$\mathbf{x}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.3}$$

A rotation

$$\mathbf{x_r} = \mathbf{P}^\top \mathbf{x} \tag{0.4}$$

aligns the conic with the coordinate axes.

$$\mathbf{x} = \mathbf{P}\mathbf{x_r} \tag{0.5}$$

Applying the rotation to the conic equation.

$$(\mathbf{P}\mathbf{x_r})^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top (\mathbf{P}\mathbf{x_r}) + 2\mathbf{u}^\top (\mathbf{P}\mathbf{x_r}) + f = 0 \tag{0.6}$$

$$\mathbf{x_r}^\top \mathbf{P}^\top \mathbf{P}\mathbf{D}\mathbf{P}^\top \mathbf{P}\mathbf{x_r} + 2\left(\mathbf{P}^\top \mathbf{u}\right)^\top \mathbf{x_r} + f = 0 \tag{0.7}$$

$$\mathbf{x_r}^\top \mathbf{D}\mathbf{x_r} + 2\mathbf{u_r}^\top \mathbf{x_r} + f = 0 \tag{0.8}$$

A translation

$$\mathbf{x_c} = \mathbf{x_r} + \mathbf{D}^{-1}\mathbf{u_r} \tag{0.9}$$

moves the conic's center to the origin.

$$f_c = f - \mathbf{u_r}^\top \mathbf{D}^{-1}\mathbf{u_r} \tag{0.10}$$

The center of the conic in the original coordinates is

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{0.11}$$

$$\mathbf{c} = -\left(\mathbf{P}\mathbf{D}\mathbf{P}^\top\right)^{-1}\mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{P}^\top \mathbf{u} = -\mathbf{P}\mathbf{D}^{-1}\mathbf{u_r} \tag{0.12}$$

The complete transformation from original to centered coordinates is

$$\mathbf{x_c} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \tag{0.13}$$

$$\mathbf{x_c} = \mathbf{P}^\top \mathbf{x} + \mathbf{D}^{-1}\mathbf{u_r} = \mathbf{P}^\top \mathbf{x} - \mathbf{P}^\top \mathbf{c} = \mathbf{P}^\top (\mathbf{x} - \mathbf{c}) \tag{0.14}$$

$$\implies \mathbf{x} = \mathbf{P}\mathbf{x_c} + \mathbf{c} \tag{0.15}$$

The given conic equation

$$16x^2 + y^2 = 16 \tag{0.16}$$

$$\frac{16x^2}{16} + \frac{y^2}{16} = \frac{16}{16} \tag{0.17}$$

$$\frac{x^2}{1} + \frac{y^2}{16} = 1 \tag{0.18}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f = -1 \tag{0.19}$$

The major axis corresponds to smaller eigenvalue.

$$\lambda_1 = \frac{1}{16}, \ \lambda_2 = 1, \ \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.20}$$

Applying the rotation to find the canonical coordinates.

$$\mathbf{x_c} = \mathbf{P}^\top \mathbf{x} \implies \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \tag{0.21}$$

The standard form of the ellipse in canonical coordinates.

$$\frac{x_c^2}{-f/\lambda_1} + \frac{y_c^2}{-f/\lambda_2} = 1 \tag{0.22}$$

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \frac{\sqrt{15}}{4} \tag{0.23}$$

$$\mathbf{f_c} = \pm \sqrt{\frac{f(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2}} \mathbf{e_1} = \pm \sqrt{15} \mathbf{e_1} \tag{0.24}$$

$$\mathbf{v_c} = \pm \sqrt{\frac{-f}{\lambda_1}} \mathbf{e_1} = \pm 4 \mathbf{e_1} \tag{0.25}$$

$$\mathbf{d_c} : \mathbf{e_1}^\top \mathbf{x_c} = \pm \sqrt{\frac{-f\lambda_2}{\lambda_1 (\lambda_2 - \lambda_1)}} \pm \frac{16}{\sqrt{15}} \tag{0.26}$$

$$L = \frac{-2f}{\lambda_2} \sqrt{\frac{\lambda_1}{-f}} = \frac{1}{2} \tag{0.27}$$

Transforming properties back to the original coordinate system using (0.15)

$$\mathbf{f} = \mathbf{P}\left(\pm\sqrt{15}\mathbf{e_1}\right) = \pm\sqrt{15}\mathbf{e_2} \tag{0.28}$$

$$\mathbf{v} = \mathbf{P}\left(\pm 4\mathbf{e_1}\right) = \pm 4\mathbf{e_2} \tag{0.29}$$

$$\mathbf{d} : \mathbf{e_2}^\top \mathbf{x} = \pm\frac{16}{\sqrt{15}} \implies \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = \pm\frac{16}{\sqrt{15}} \tag{0.30}$$

Property	**Value**
Eccentricity	$\frac{\sqrt{15}}{4}$
Axis	$x = 0$
Vertices	$(0, \pm 4)$
Foci	$\left(0, \pm\sqrt{15}\right)$
Directrices	$y = \pm\frac{16}{\sqrt{15}}$
Latus Rectum	$\frac{1}{2}$

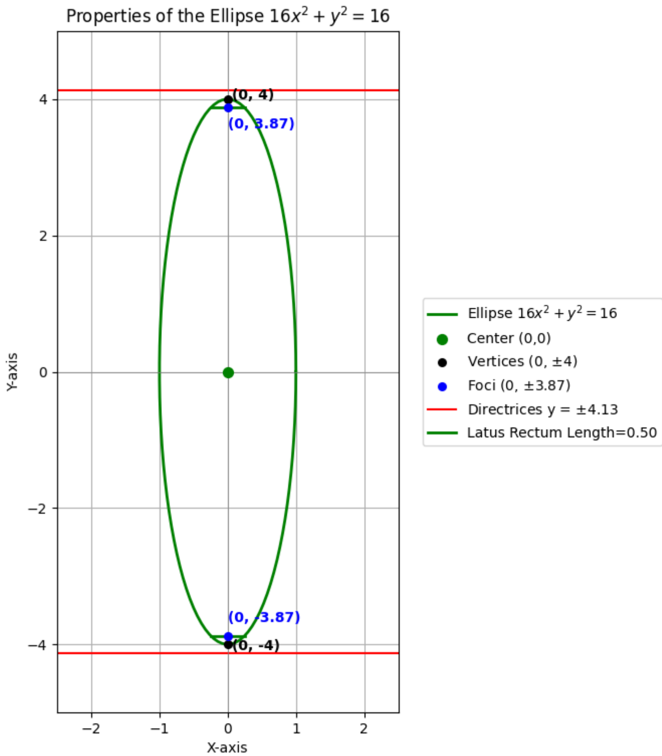

Fig. 0.1