8.2.20

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question
Find the equation of the conic, that satisfies the given conditions:
Vertex (0,0) passing through (2,3) and axis is along X axis. **Solution**
The general conic is

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

For axis along the x-axis,

$$\mathbf{V} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} D \\ 0 \end{pmatrix} \tag{2}$$

Since the vertex is at the origin,

$$\nabla g(\mathbf{0}) = 2\mathbf{u} = \mathbf{0} \implies \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

If $\det(\mathbf{V}) = AC \neq 0$, then the conic has a center at the origin, which corresponds to an ellipse or hyperbola.
But the problem specifies a single vertex at the origin, not a center, so this case is invalid.

$$\therefore \det(\mathbf{V}) = 0 \quad \implies \quad \text{The conic is a parabola.} \tag{4}$$

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5}$$

For a parabola with axis along the x-axis, vertex at origin, focus $\mathbf{F} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ and directrix $\mathbf{n}^\top \mathbf{x} = c$, we have $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $c = -p$.

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - \mathbf{n}\mathbf{n}^\top = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

$$\mathbf{u} = c\mathbf{n} - \mathbf{F} = (-p) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} -2p \\ 0 \end{pmatrix} \tag{7}$$

$$f = \|\mathbf{F}\|^2 - c^2 = p^2 - (-p)^2 = 0 \tag{8}$$

Thus, the parabola equation becomes:

$$\mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -2p & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{9}$$

$$y^2 - 4px = 0 \tag{10}$$

Since $(2, 3)$ lies on the parabola:

$$3^2 - 4p(2) = 0 \tag{11}$$

$$9 - 8p = 0 \tag{12}$$

$$p = \frac{9}{8} \tag{13}$$

Therefore,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -\frac{9}{4} \\ 0 \end{pmatrix}, \quad f = 0 \tag{14}$$

and the equation of the required parabola is

$$y^2 = \frac{9}{2}x \tag{15}$$