Matrices in Geometry 7.4.42

EE25BTECH11035 - Kushal B N

Question:
Find the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$.

Given:
Circle C: $x^2 + y^2 - \frac{1+\sqrt{2}a}{2}x - \frac{1-\sqrt{2}a}{2}y = 0$
Point $\mathbf{P} = \begin{pmatrix} \frac{1+\sqrt{2}a}{2} \\ \frac{1-\sqrt{2}a}{2} \end{pmatrix}$
Line L: $\mathbf{n}_L^\top \mathbf{x} = 0$, where $\mathbf{n}_L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution:
The center of the circle is

$$\mathbf{c} = \frac{1}{4}\begin{pmatrix} 1 + \sqrt{2}a \\ 1 - \sqrt{2}a \end{pmatrix} \tag{1}$$

By observation,

$$\mathbf{P} = 2\mathbf{c} \tag{2}$$

The locus of midpoints, $\mathbf{M}$, of chords from $\mathbf{P}$ is given by,

$$(\mathbf{M} - \mathbf{c})^\top (\mathbf{P} - \mathbf{M}) = 0 \tag{3}$$

The midpoint $\mathbf{M}$ lies on the line L, the direction vector of which is

$$\mathbf{m}_L = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4}$$

$$\implies \mathbf{M} = \lambda \mathbf{m}_L \tag{5}$$

Substituting this into equation (3), we get

$$(\lambda \mathbf{m}_L - \mathbf{c})^\top (\mathbf{P} - \lambda \mathbf{m}_L) = 0 \tag{6}$$

$$\implies (\mathbf{m}_L^\top \mathbf{m}_L)\lambda^2 - \mathbf{m}_L^\top (\mathbf{P} + \mathbf{c})\lambda + \mathbf{c}^\top \mathbf{P} = 0 \tag{7}$$

For two distinct chords, the discriminant $\Delta = b^2 - 4ac > 0$.

$$\Delta = \left(\mathbf{m}_L^\top (\mathbf{P} + \mathbf{c})\right)^2 - 4\left(\mathbf{m}_L^\top \mathbf{m}_L\right)\left(\mathbf{c}^\top \mathbf{P}\right) > 0 \tag{8}$$

Here,

$$\mathbf{m}_L^\top \mathbf{m}_L = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \tag{9}$$

$$\mathbf{m}_L^\top (\mathbf{P} + \mathbf{c}) = \mathbf{m}_L^\top (3\mathbf{c}) = 3\begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 1 + \sqrt{2}a \\ 1 - \sqrt{2}a \end{pmatrix} = \frac{3\sqrt{2}a}{2} \tag{10}$$

$$\mathbf{c}^\top \mathbf{P} = 2\|\mathbf{c}\|^2 = \frac{1 + 2a^2}{4} \tag{11}$$

Substituting into the inequality (8):

$$\left(\frac{3\sqrt{2}a}{2}\right)^2 - 4(2)\left(\frac{1+2a^2}{4}\right) > 0 \tag{12}$$

$$\frac{9a^2}{2} - 2(1+2a^2) > 0 \tag{13}$$

$$\frac{a^2}{2} > 2 \implies a^2 > 4 \tag{14}$$

$$\implies a > 2 \quad \text{or} \quad a < -2 \tag{15}$$

Final Answer:
The intervals of values for a are $(-\infty, -2) \cup (2, \infty)$.

$$\boxed{a \in (-\infty, -2) \cup (2, \infty)} \tag{16}$$

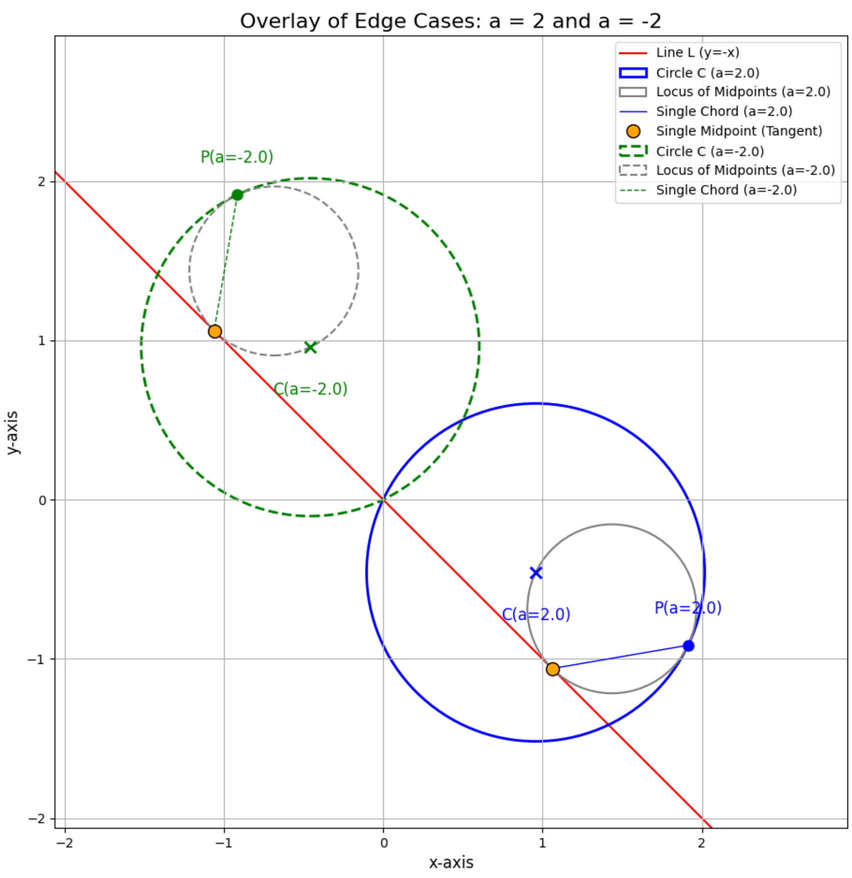

Fig. 1: Plot for 7.4.42