4.13.36

EE25BTECH11031 - Sai Sreevallabh

Question:

Let PQR be a right angled isosceles triangle, right at $P(2,1)$. If the equation of the line QR is $2x+y=3$, then the equation representing the pair of lines PQ and PR is

1) $3x^2-3y^2+8xy+20x+10y+25=0$
2) $3x^2-3y^2+8xy-20x-10y+25=0$
3) $3x^2-3y^2+8xy+10x+15y+20=0$
4) $3x^2-3y^2-8xy-10x-15y-20=0$

Solution:

Given point is $\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and given line can be written as

$$\mathbf{n}^\top \mathbf{x} = c \tag{4.1}$$

where, $\mathbf{n} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $c = 3$.

Parametric form of line through $\mathbf{P}$ is

$$\mathbf{r} = \mathbf{P} + \lambda \mathbf{m} \tag{4.2}$$

Using this, we can represent points Q and R as

$$\mathbf{Q} = \mathbf{P} + \lambda_1 \mathbf{m_1} \tag{4.3}$$

$$\mathbf{R} = \mathbf{P} + \lambda_2 \mathbf{m_2} \tag{4.4}$$

where, $\mathbf{m_1} = \begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ and $\mathbf{m_2} = \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$ are direction vectors of lines $\mathbf{Q} - \mathbf{P}$ and $\mathbf{R} - \mathbf{P}$, while m_1 and m_2 are the respective slopes.

Given that the lines are perpendicular,

$$\mathbf{m_1}^\top \mathbf{m_2} = 0 \tag{4.5}$$

$$\implies m_1 m_2 = -1 \tag{4.6}$$

Substituting equation (4.3) in (4.1)

$$\mathbf{n}^\top (\mathbf{P} + \lambda_1 \mathbf{m_1}) = c \tag{4.7}$$

$$\implies \lambda_1 = \frac{c - \mathbf{n}^\top \mathbf{P}}{\mathbf{n}^\top \mathbf{m_1}} \tag{4.8}$$

Substituting the values, we get

$$\lambda_1 = \frac{-2}{2+m_1} \tag{4.9}$$

Similarly, substituting equation (4.4) in (4.1)

$$\lambda_2 = \frac{c-\mathbf{n}^\top\mathbf{P}}{\mathbf{n}^\top\mathbf{m_2}} \tag{4.10}$$

Substituting values,

$$\lambda_2 = \frac{-2}{2+m_2} \tag{4.11}$$

$$\implies \lambda_2 = \frac{-2m_1}{2m_1-1} \tag{4.12}$$

Let $\mathbf{M}$ be the midpoint of $\mathbf{Q}-\mathbf{R}$:

$$\mathbf{M} = \frac{\mathbf{Q}+\mathbf{R}}{2} \tag{4.13}$$

Since $\triangle PQR$ is isosceles,

$$\mathbf{P}-\mathbf{M} \perp \mathbf{Q}-\mathbf{R} \tag{4.14}$$

$$\implies \left(\mathbf{P}-\frac{\mathbf{Q}+\mathbf{R}}{2}\right)^\top(\mathbf{Q}-\mathbf{R}) = 0 \tag{4.15}$$

Substituting values from (4.3), (4.4), we get

$$(\lambda_1\mathbf{m_1}+\lambda_2\mathbf{m_2})^\top(\lambda_1\mathbf{m_1}-\lambda_2\mathbf{m_2}) = 0 \tag{4.16}$$

$$\lambda_1^2\mathbf{m_1}^\top\mathbf{m_1} = \lambda_2\mathbf{m_2}^\top\mathbf{m_2} \tag{4.17}$$

$$\lambda_1^2\left(1+m_1^2\right) = \lambda_2\left(1+\frac{1}{m_1^2}\right) \tag{4.18}$$

$$|\lambda_1| = \left|\frac{\lambda_2}{m_1}\right| \tag{4.19}$$

Substituting values of λ_1 and λ_2 from (4.9) and (4.12)

$$\left|\frac{2}{2+m_1}\right| = \left|\frac{2}{2m_1-1}\right| \tag{4.20}$$

Solving the above, we get

$$m_1 = 3 \text{ or } m_1 = \frac{-1}{3} \tag{4.21}$$

Correspondingly,

$$m_2 = \frac{-1}{3} \text{ or } m_2 = 3 \tag{4.22}$$

So, the equations of the two required lines are

$$3x - y - 5 = 0 \quad \text{and} \quad x + 3y - 5 = 0 \tag{4.23}$$

$\therefore$ Multiplying the above two equations, we get the pair of straight lines to be

$$3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$$