

## 5.13.30

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### Question

Let  $\mathbf{A}$  be a square matrix all of whose entries are integers. Then which of the following is true?

- A) If  $\det(\mathbf{A}) \neq \pm 1$ , then  $\mathbf{A}^{-1}$  exists but all its entries are not necessarily integers
- B) If  $\det(\mathbf{A}) \neq \pm 1$ , then  $\mathbf{A}^{-1}$  exists and all its entries are non-integers
- C) If  $\det(\mathbf{A}) = \pm 1$ , then  $\mathbf{A}^{-1}$  exists but all its entries are integers
- D) If  $\det(\mathbf{A}) = \pm 1$ , then  $\mathbf{A}^{-1}$  need not exist

### Solution

We will proceed by checking each option.

A) If  $\det(\mathbf{A}) \neq \pm 1$ , then  $\mathbf{A}^{-1}$  exists but all its entries are not necessarily integers

Let us take a square matrix  $\mathbf{A}$  having all integer entries. Let rows  $R_1$  and  $R_2$  be equal. By performing row operation  $R_1 \rightarrow R_1 - R_2$ , all elements in  $R_1$  become 0. Therefore,  $|\mathbf{A}| = 0$ . We know that if  $|\mathbf{A}| = 0$ ,  $\mathbf{A}^{-1}$  does not exist. Therefore, this option is wrong.

For example, consider a matrix  $\mathbf{A} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix}$ .  $\mathbf{A}$  has only integer entries.

$$\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 1 & 4 & 2 \end{pmatrix}$$

Since  $R_1$  consists of only 0's,  $|\mathbf{A}| = 0$ . Hence  $\mathbf{A}$  is not invertible.

B) If  $\det(\mathbf{A}) \neq \pm 1$ , then  $\mathbf{A}^{-1}$  exists and all its entries are non-integers

For example, consider a matrix  $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ .  $|\mathbf{A}| = 2$

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}.$$

This is a counterexample to the statement; hence, option **B** is wrong.

D) If  $\det(\mathbf{A}) = \pm 1$ , then  $\mathbf{A}^{-1}$  need not exist

We know that if  $|\mathbf{A}| \neq 0$ ,  $\mathbf{A}^{-1}$  exists. By this logic, this option is wrong.

For example, consider a matrix  $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ .  $|\mathbf{A}| = 1$ , and since this is an orthogonal matrix,

$$\mathbf{A}^{-1} = \mathbf{A}^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\mathbf{A}^{-1}$  exists, which is a contradiction to the statement in option **D**. Therefore, the correct answer is C).