5.13.44

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Question

Let

$$\mathbf{P_1} = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{P_4} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{P_5} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P_6} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\text{and} \quad \mathbf{X} = \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P_k}^\top.$$

Where $\mathbf{P_k}^\top$ denotes the transpose of matrix $\mathbf{P_k}$.Then which of the following options is/are correct?

Question

1. $\mathbf{X}$ is a symmetric matrix
2. The sum of diagonal elements of $\mathbf{X}$ is 18
3. $\mathbf{X} - 30\mathbf{I}$ is an invertible matrix
4. If $\mathbf{X}\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$, then α is 30

Solution

From the question, $\mathbf{P_1}^\top = \mathbf{P_1}, \quad \mathbf{P_2}^\top = \mathbf{P_2}, \quad \mathbf{P_3}^\top = \mathbf{P_3}, \quad \mathbf{P_4}^\top = \mathbf{P_5}, \quad \mathbf{P_5}^\top = \mathbf{P_4}, \quad \mathbf{P_6}^\top = \mathbf{P_6}$ and Let

$$\mathbf{Q} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \tag{1}$$

and $\mathbf{Q}^\top = \mathbf{Q}$

Now,

$$\mathbf{X} = (\mathbf{P_1} Q \mathbf{P_1}^\top) + (\mathbf{P_2} Q \mathbf{P_2}^\top) + (\mathbf{P_3} Q \mathbf{P_3}^\top) + (\mathbf{P_4} Q \mathbf{P_4}^\top) + (\mathbf{P_5} Q \mathbf{P_5}^\top) + \tag{2}$$

$$(\mathbf{P_6} Q \mathbf{P_6}^\top)$$

So,

$$\mathbf{X}^\top = (\mathbf{P_1}Q\mathbf{P_1}^\top)^\top + (\mathbf{P_2}Q\mathbf{P_2}^\top)^\top + (\mathbf{P_3}Q\mathbf{P_3}^\top)^\top + (\mathbf{P_4}Q\mathbf{P_4}^\top)^\top + \tag{3}$$

$$(\mathbf{P_5}Q\mathbf{P_5}^\top)^\top + (\mathbf{P_6}Q\mathbf{P_6}^\top)^\top$$

$$= \mathbf{P_1}Q\mathbf{P_1}^\top + \mathbf{P_2}Q\mathbf{P_2}^\top + \mathbf{P_3}Q\mathbf{P_3}^\top + \mathbf{P_4}Q\mathbf{P_4}^\top + \mathbf{P_5}Q\mathbf{P_5}^\top + \mathbf{P_6}Q\mathbf{P_6}^\top \tag{4}$$

$$\Rightarrow \mathbf{X}^\top = \mathbf{X} \tag{5}$$

$$\Rightarrow \mathbf{X} \text{ is a symmetric matrix.}$$

The sum of diagonal entries of $\mathbf{X} = \text{Tr}(\mathbf{X})$:

$$\text{Tr}(\mathbf{X}) = \sum_{i=1}^{6} \text{Tr}(\mathbf{P_i}\mathbf{Q}\mathbf{P_i}^\top) = \sum_{i=1}^{6} \text{Tr}(\mathbf{Q}\mathbf{P_i}^\top\mathbf{P_i}) \tag{6}$$

$$(\because \operatorname{Tr}(ABC) = \operatorname{Tr}(BCA))$$

$$= \sum_{i=1}^{6} \operatorname{Tr}(\mathbf{QI}) \tag{7}$$

$$(\because \mathbf{P_i}'s \quad \text{are orthogonal matrices})$$

$$= \sum_{i=1}^{6} \operatorname{Tr}(\mathbf{Q}) \tag{8}$$

$$= 6 \operatorname{Tr}(\mathbf{Q}) \tag{9}$$

$$= 6 \times 3 \tag{10}$$

$$= 18 \tag{11}$$

Now , let

$$\mathbf{R} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ then} \tag{12}$$

$$\mathbf{XR} = \sum_{k=1}^{6} \mathbf{P_k} \mathbf{Q} \mathbf{P_k}^\top \mathbf{R} = \sum_{k=1}^{6} \mathbf{P_k} \mathbf{Q} \mathbf{P_k}^\top \mathbf{R} \tag{13}$$

$$= \sum_{k=1}^{6} \mathbf{P_k} (\mathbf{QR}) \qquad [\because \mathbf{P_k}^\top \mathbf{R} = \mathbf{R}] \tag{14}$$

$$= \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{15}$$

$$= \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{16}$$

$$\tag{17}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{18}$$

$$\implies \mathbf{XR} = \begin{pmatrix} 30 \\ 30 \\ 30 \end{pmatrix} \tag{19}$$

$$\implies \mathbf{XR} = 30\mathbf{R} \tag{20}$$

$$\implies \mathbf{X} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 30 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{21}$$

Thus , the value of $\alpha = 30$.
From (4.19),

$$(\mathbf{X} - 30\mathbf{I})\mathbf{R} = 0 \implies \left|\mathbf{X} - 30\mathbf{I}\right| = 0 \tag{22}$$

$$\text{So, } (\mathbf{X} - 30\mathbf{I}) \text{ is not invertible} \tag{23}$$

Conclusion

Hence, options (a), (b) and (d) are correct.

C code

```
#include <stdio.h>
#include <math.h>
#define N 3

void compute_X(double X[N][N]) {
    for (int i=0; i<N; ++i)
        for (int j=0; j<N; ++j)
            X[i][j] = (i==j) ? 6.0 : 12.0;
}

double trace(double X[N][N]) {
    double t = 0.0;
    for (int i=0; i<N; ++i)
        t += X[i][i];
    return t;
}
```

C Code

```c
int is_symmetric(double X[N][N], double tol) {
    for (int i=0; i<N; ++i)
        for (int j=i+1; j<N; ++j)
            if (fabs(X[i][j]-X[j][i])>tol)
                return 0;
    return 1;
}

void mat_vec_mul(double X[N][N], double v[N], double y[N]) {
    for (int i=0; i<N; ++i){
        y[i]=0;
        for (int j=0; j<N; ++j)
            y[i]+=X[i][j]*v[j];
    }
}
```

C Code

```c
void subtract_scalar_I(double X[N][N], double scalar, double Y[N
    ][N]) {
    for (int i=0;i<N;++i)
        for (int j=0;j<N;++j)
            Y[i][j] = X[i][j] - (i==j ? scalar : 0);
}

double determinant(double X[N][N]) {
    return X[0][0]*(X[1][1]*X[2][2]-X[1][2]*X[2][1])
         - X[0][1]*(X[1][0]*X[2][2]-X[1][2]*X[2][0])
         + X[0][2]*(X[1][0]*X[2][1]-X[1][1]*X[2][0]);
}
```

Python + C code

```python
import ctypes
import numpy as np

# Load shared library
lib = ctypes.CDLL("./libmatrix_X.so")

# Define types
N = 3
DoubleArray3 = ctypes.c_double * N
DoubleMatrix3 = (DoubleArray3 * N)

# Function signatures
lib.compute_X.argtypes = [DoubleMatrix3]
lib.trace.argtypes = [DoubleMatrix3]
lib.trace.restype = ctypes.c_double
lib.is_symmetric.argtypes = [DoubleMatrix3, ctypes.c_double]
lib.is_symmetric.restype = ctypes.c_int
lib.mat_vec_mul.argtypes = [DoubleMatrix3, DoubleArray3,
    DoubleArray3]
```

Python + C code

```
lib.subtract_scalar_I.argtypes = [DoubleMatrix3, ctypes.c_double,
     DoubleMatrix3]
lib.determinant.argtypes = [DoubleMatrix3]
lib.determinant.restype = ctypes.c_double

# Initialize matrices
X = DoubleMatrix3()
lib.compute_X(X)

# Convert to numpy for easier viewing
X_np = np.array([[X[i][j] for j in range(N)] for i in range(N)])
print("Matrix X =\n", X_np)

# Compute trace
trace_val = lib.trace(X)
print("\nTrace(X) =", trace_val)
```

Python + C code

```
    # Check symmetry
sym = lib.is_symmetric(X, 1e-9)
print("Symmetric:", "Yes" if sym else "No")

# Compute  for X*[1 1 1]^T
v = DoubleArray3(1.0, 1.0, 1.0)
y = DoubleArray3()
lib.mat_vec_mul(X, v, y)
y_vals = [y[i] for i in range(N)]
alpha = y_vals[0]
print("\nX*[1 1 1]^T =", y_vals)
print("Alpha =", alpha)

# Compute determinant of (X - 30I)
Y = DoubleMatrix3()
lib.subtract_scalar_I(X, 30.0, Y)
det_val = lib.determinant(Y)
```

Python + C code

```
print("\nDeterminant of (X - 30I):", det_val)
if abs(det_val) < 1e-9:
    print("=> (X - 30I) is NOT invertible (singular)")
else:
    print("=> (X - 30I) is invertible")
```