12.557

EE25BTECH11023 - Venkata Sai

Question:
Let $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix}$.Then the trace of $\mathbf{A}^{1000}$ equals
Solution:
Given

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \tag{1}$$

$$\tag{2}$$

To find eigen values

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{3}$$

$$\left| \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0 \tag{4}$$

$$\left| \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \tag{5}$$

$$\left| \begin{pmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{pmatrix} \right| = 0 \tag{6}$$

$$(5-\lambda)(-4-\lambda) + 3(6) = 0 \tag{7}$$

$$\lambda^2 + 4\lambda - 5\lambda - 20 + 18 = 0 \tag{8}$$

$$\lambda^2 - \lambda - 2 = 0 \tag{9}$$

$$(\lambda - 2)(\lambda + 1) = 0 \tag{10}$$

$$\lambda_1 = 2 \text{ (and) } \lambda_2 = -1 \tag{11}$$

For a given matrix **A**

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{12}$$

$$\mathbf{A}^2 = \left(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\right)^2 \tag{13}$$

$$= \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{14}$$

$$= \mathbf{P}\mathbf{D}\mathbf{I}\mathbf{D}\mathbf{P}^{-1} \tag{15}$$

$$= \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1} \tag{16}$$

where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

$$\tag{18}$$

$$\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1} \tag{19}$$

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}\right) \tag{20}$$

$$= \text{trace}\left(\left(\mathbf{P}\mathbf{D}^k\right)\mathbf{P}^{-1}\right) \tag{21}$$

Since trace(**AB**)=trace(**BA**)

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\left(\mathbf{P}\mathbf{D}^k\right)\mathbf{P}^{-1}\right) \tag{22}$$

$$= \text{trace}\left(\mathbf{P}^{-1}\left(\mathbf{P}\mathbf{D}^k\right)\right) \tag{23}$$

$$\text{trace}\left(\mathbf{A}^k\right) = \text{trace}\left(\mathbf{I}\mathbf{D}^k\right) = \text{trace}\left(\mathbf{D}^k\right) \tag{24}$$

$$\text{trace}\left(\mathbf{A}^{1000}\right) = \text{trace}\left(\mathbf{D}^{1000}\right) \tag{25}$$

$$= \text{trace}\begin{pmatrix} 2^{1000} & 0 \\ 0 & (-1)^{1000} \end{pmatrix} \tag{26}$$

$$= 2^{1000} + 1 \tag{27}$$