10.6.8 - Eigenvector Method

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Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Problem Setup

Let the center of the circle be at origin and point P be at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

The equation of the circle $x^2 + y^2 = 16$ can be written as:

$$\vec{x}^\top \vec{V} \vec{x} + 2\vec{u}^\top \vec{x} + f = 0 \tag{3}$$

where

$$\vec{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -16 \tag{4}$$

Step 1: Eigenvalue Decomposition

The eigenvalues of $\vec{V}$ satisfy:

$$\det(\vec{V} - \lambda \vec{I}) = 0 \tag{5}$$

$$(1 - \lambda)^2 = 0 \tag{6}$$

Eigenvalues: $\lambda_1 = \lambda_2 = 1$

The corresponding orthonormal eigenvectors are:

$$\vec{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

These form the eigenvector matrix:

$$\vec{P} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

Step 2: Semi-axes from Eigenvalues

The semi-axes of the circle along eigenvector directions:

$$a = b = \sqrt{\frac{-f}{\lambda_1}} = \sqrt{\frac{16}{1}} = 4 \tag{9}$$

Step 3: Tangency Conditions

For tangents from external point P to the circle, the contact point $\vec{q}$ must satisfy:

(a) $\vec{q}$ lies on circle: $\vec{q}^\top \vec{V}\vec{q} + f = 0$

(b) Tangent passes through P: $(\vec{V}\vec{q})^\top P + f = 0$

From condition (b) with $\vec{V} = \vec{I}$:

$$\vec{q}^\top P + f = 0 \tag{10}$$

$$\begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 16 \tag{11}$$

$$q_1 = \frac{8}{3} \tag{12}$$

Step 4: Finding Contact Points

From condition (a):

$$q_1^2 + q_2^2 = 16 \tag{13}$$

$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \tag{14}$$

$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{15}$$

Therefore, the two contact points are:

$$\vec{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix}, \quad \vec{q}_2 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{16}$$

Step 5: Eigenvector Representation

Express contact points as linear combinations of eigenvectors:

$$\vec{q}_1 = \frac{8}{3}\vec{p}_1 + \frac{4\sqrt{5}}{3}\vec{p}_2 \tag{17}$$

$$= \frac{8}{3}\begin{pmatrix}1\\0\end{pmatrix} + \frac{4\sqrt{5}}{3}\begin{pmatrix}0\\1\end{pmatrix} \tag{18}$$

$$= \begin{pmatrix}\frac{8}{3}\\ \frac{4\sqrt{5}}{3}\end{pmatrix} \tag{19}$$

$$\vec{q}_2 = \frac{8}{3}\vec{p}_1 - \frac{4\sqrt{5}}{3}\vec{p}_2 \tag{20}$$

$$= \begin{pmatrix}\frac{8}{3}\\ -\frac{4\sqrt{5}}{3}\end{pmatrix} \tag{21}$$

Step 6: Tangent Equations

The tangent at $\vec{q}$ is given by: $(\vec{V}\vec{q})^\top\vec{x} + f = 0$

Tangent 1 at $\vec{q}_1$:

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{22}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{23}$$

$$2x + \sqrt{5}y = 12 \tag{24}$$

Tangent 2 at $\vec{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 16 = 0 \tag{25}$$

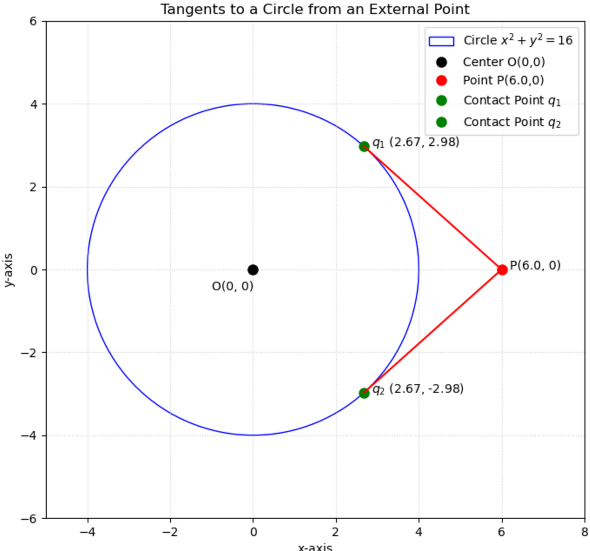
$$2x - \sqrt{5}y = 12 \tag{26}$$

Final Answer

The equations of the pair of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{27}$$

Plot



Tangents to a Circle from an External Point
Circle $x^2 + y^2 = 16$
Center O(0,0)
Point P(6.0,0)
Contact Point q_1
Contact Point q_2
q_1 (2.67, 2.98)
q_2 (2.67, -2.98)
P(6.0, 0)
O(0, 0)
y-axis
x-axis
6
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
8