8.4.22

EE25BTECH11019 - Darji Vivek M.

Question:

The radius of the circle passing through the foci of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

and having its centre at $(0, 3)$ is -

1) 4
2) 3
3) $\sqrt{\frac{1}{2}}$
4) $\frac{7}{2}$

Solution:

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Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. The given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \qquad \Longleftrightarrow \qquad \mathbf{x}^\top \mathbf{V} \mathbf{x} = 1, \qquad \mathbf{V} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}.$$

The eigenvalues (principal values) of $\mathbf{V}$ are its diagonal entries; take them in increasing order

$$\lambda_1 = \frac{1}{16}, \qquad \lambda_2 = \frac{1}{9}.$$

Hence the squared semi-axes (principal-form relations) are

$$a^2 = \frac{1}{\lambda_1} = 16, \qquad b^2 = \frac{1}{\lambda_2} = 9,$$

so $a = 4,\ b = 3$.

Using the formula for eccentricity.

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{1 - \frac{1/16}{1/9}} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}.$$

The focal distance (from the centre) is $c = ae$, therefore

$$c = 4 \cdot \frac{\sqrt{7}}{4} = \sqrt{7}.$$

Since $\mathbf{V}$ is diagonal with λ_1 along the x-direction, the principal axis unit vector is.

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the foci (in vector form) are

$$\mathbf{F}_1 = c\mathbf{n} = \begin{pmatrix} \sqrt{7} \\ 0 \end{pmatrix}, \qquad \mathbf{F}_2 = -c\mathbf{n} = \begin{pmatrix} -\sqrt{7} \\ 0 \end{pmatrix}.$$

The required circle has centre $C = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and passes through either focus, so the radius is

$$R = \|\mathbf{F}_1 - \mathbf{C}\| = \sqrt{(\sqrt{7} - 0)^2 + (0 - 3)^2} = \sqrt{7 + 9} = \sqrt{16} = 4.$$

Therefore the radius of the required circle is $\boxed{4}$.