5.13.44

EE25BTECH11018-Darisy Sreetej

Question: Let

$$\mathbf{P_1} = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{P_4} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{P_5} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{P_6} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\text{and} \quad \mathbf{X} = \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{P_k}^\top.$$

Where $\mathbf{P_k}^\top$ denotes the transpose of matrix $\mathbf{P_k}$.Then which of the following options is/are correct?

1) $\mathbf{X}$ is a symmetric matrix
2) The sum of diagonal elements of $\mathbf{X}$ is 18
3) $\mathbf{X} - 30\mathbf{I}$ is an invertible matrix
4) If $\mathbf{X}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then α is 30

Solution:
From the question, $\mathbf{P_1}^\top = \mathbf{P_1}, \quad \mathbf{P_2}^\top = \mathbf{P_2}, \quad \mathbf{P_3}^\top = \mathbf{P_3}, \quad \mathbf{P_4}^\top = \mathbf{P_5}, \quad \mathbf{P_5}^\top = \mathbf{P_4}, \quad \mathbf{P_6}^\top = \mathbf{P_6}$ and Let

$$\mathbf{Q} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix} \tag{4.1}$$

and $\mathbf{Q}^\top = \mathbf{Q}$

Now,

$$\mathbf{X} = (\mathbf{P_1}Q\mathbf{P_1}^\top) + (\mathbf{P_2}Q\mathbf{P_2}^\top) + (\mathbf{P_3}Q\mathbf{P_3}^\top) + (\mathbf{P_4}Q\mathbf{P_4}^\top) + (\mathbf{P_5}Q\mathbf{P_5}^\top) + (\mathbf{P_6}Q\mathbf{P_6}^\top) \tag{4.2}$$

So,

$$\mathbf{X}^\top = (\mathbf{P_1}Q\mathbf{P_1}^\top)^\top + (\mathbf{P_2}Q\mathbf{P_2}^\top)^\top + (\mathbf{P_3}Q\mathbf{P_3}^\top)^\top + (\mathbf{P_4}Q\mathbf{P_4}^\top)^\top + (\mathbf{P_5}Q\mathbf{P_5}^\top)^\top + (\mathbf{P_6}Q\mathbf{P_6}^\top)^\top \tag{4.3}$$

$$= \mathbf{P_1}Q\mathbf{P_1}^\top + \mathbf{P_2}Q\mathbf{P_2}^\top + \mathbf{P_3}Q\mathbf{P_3}^\top + \mathbf{P_4}Q\mathbf{P_4}^\top + \mathbf{P_5}Q\mathbf{P_5}^\top + \mathbf{P_6}Q\mathbf{P_6}^\top \tag{4.4}$$

$$\Rightarrow \mathbf{X}^{\top} = \mathbf{X} \tag{4.5}$$

$$\Rightarrow \mathbf{X} \text{ is a symmetric matrix.}$$

The sum of diagonal entries of $\mathbf{X} = \mathrm{Tr}(\mathbf{X})$:

$$\mathrm{Tr}(\mathbf{X}) = \sum_{i=1}^{6} \mathrm{Tr}(\mathbf{P_i}\mathbf{Q}\mathbf{P_i}^{\top}) = \sum_{i=1}^{6} \mathrm{Tr}(\mathbf{Q}\mathbf{P_i}^{\top}\mathbf{P_i}) \tag{4.6}$$

$$(\because \mathrm{Tr}(ABC) = \mathrm{Tr}(BCA))$$

$$= \sum_{i=1}^{6} \mathrm{Tr}(\mathbf{QI}) \tag{4.7}$$

$$(\because \mathbf{P_i}'s \quad \text{are orthogonal matrices})$$

$$= \sum_{i=1}^{6} \mathrm{Tr}(\mathbf{Q}) \tag{4.8}$$

$$= 6\,\mathrm{Tr}(\mathbf{Q}) \tag{4.9}$$

$$= 6 \times 3 \tag{4.10}$$

$$= 18 \tag{4.11}$$

Now , let

$$\mathbf{R} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ then} \tag{4.12}$$

$$\mathbf{XR} = \sum_{k=1}^{6} \mathbf{P_k}\mathbf{Q}\mathbf{P_k}^{\top}\mathbf{R} = \sum_{k=1}^{6} \mathbf{P_k}\mathbf{Q}\mathbf{P_k}^{\top}\mathbf{R} \tag{4.13}$$

$$= \sum_{k=1}^{6} \mathbf{P_k}(\mathbf{QR}) \qquad [\because \mathbf{P_k}^{\top}\mathbf{R} = \mathbf{R}] \tag{4.14}$$

$$= \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{4.15}$$

$$= \sum_{k=1}^{6} \mathbf{P_k} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{4.16}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix} \tag{4.17}$$

$$\implies \mathbf{XR} = \begin{pmatrix} 30 \\ 30 \\ 30 \end{pmatrix} \tag{4.18}$$

$$\implies \mathbf{XR} = 30\mathbf{R} \tag{4.19}$$

$$\implies \mathbf{X}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 30\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{4.20}$$

Thus , the value of $\alpha = 30$.
From (4.19),

$$(\mathbf{X} - 30\mathbf{I})\mathbf{R} = 0 \implies \left|\mathbf{X} - 30\mathbf{I}\right| = 0 \tag{4.21}$$

$$\text{So, } (\mathbf{X} - 30\mathbf{I}) \text{ is not invertible} \tag{4.22}$$

Hence, options (a), (b) and (d) are correct.