12.869

EE25BTECH11023 - Venkata Sai

Question:

For $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, quadratic form

$$\mathbf{Q}(\mathbf{X}) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \tag{1}$$

Let $\mathbf{M}$ be symmetric matrix of Q. For $\mathbf{Y} \in \mathbb{R}^3$? non-zero define

$$a_n = \frac{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n+1} \mathbf{Y}}{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n} \mathbf{Y}} \tag{2}$$

Then $\lim_{n\to\infty} a_n = \dots$

Solution:

Given

$$\mathbf{Q}(\mathbf{X}) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \tag{3}$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

$\mathbf{M}$ is a symmetric matrix of Q

$$\mathbf{Q} = \mathbf{X}^\top \mathbf{M} \mathbf{X} \tag{5}$$

$$\mathbf{Q} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1M_{11} + x_2M_{21} + x_3M_{13} \\ x_1M_{12} + x_2M_{22} + x_3M_{23} \\ x_1M_{13} + x_2M_{23} + x_3M_{33} \end{pmatrix} \tag{7}$$

$$\begin{aligned} = x_1^2M_{11} + x_1x_2M_{12} + x_1x_3M_{13} + x_2x_1M_{12} + x_2^2M_{22} \\ + x_2x_3M_{23} + x_3x_1M_{13} + x_3x_2M_{23} + x_3^2M_{33} \end{aligned} \tag{8}$$

$$= x_1^2M_{11} + 2x_1x_2M_{12} + 2x_1x_3M_{13} + 2x_2x_3M_{23} + x_2^2M_{22} + x_3^2M_{33} \tag{9}$$

On comparing

$$M_{11} = 2, M_{12} = \frac{4}{2} = 2, M_{13} = \frac{2}{2} = 1, M_{23} = \frac{2}{2} = 1, M_{22} = 2, M_{33} = 3 \tag{10}$$

$$\mathbf{M} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \tag{11}$$

$$\mathbf{M} + \mathbf{I_3} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \mathbf{A} \tag{12}$$

For eigen values of $\mathbf{A}$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{13}$$

$$\left|\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0 \tag{14}$$

$$\left|\begin{pmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{pmatrix}\right| = 0 \tag{15}$$

$$(3-\lambda)\left((3-\lambda)(4-\lambda) - 1\right) - 2\left(2(4-\lambda) - 1\right) + 1\left(2 - (3-\lambda)\right) = 0 \tag{16}$$

$$(3-\lambda)\left(\lambda^2 - 7\lambda + 12 - 1\right) - 2(8 - 2\lambda - 1) + 1(2 + \lambda - 3) = 0 \tag{17}$$

$$(3-\lambda)\left(\lambda^2 - 7\lambda + 11\right) - 2(7 - 2\lambda) + 1(\lambda - 1) = 0 \tag{18}$$

$$\left(3\lambda^2 - 21\lambda + 33 - \lambda^3 + 7\lambda^2 - 11\lambda\right) - 14 + 4\lambda + \lambda - 1 = 0 \tag{19}$$

$$-\lambda^3 - 10\lambda^2 - 32\lambda + 33 - 14 + 5\lambda - 1 = 0 \tag{20}$$

$$\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0 \tag{21}$$

$$(\lambda - 1)\left(\lambda^2 - 9\lambda + 18\right) = 0 \tag{22}$$

$$(\lambda - 1)(\lambda - 3)(\lambda - 6) = 0 \tag{23}$$

The eigen values of $\mathbf{A}$ are 1,3,6
Given

$$a_n = \frac{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n+1} \mathbf{Y}}{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n} \mathbf{Y}} \tag{24}$$

$$a_n = \frac{\mathbf{Y}^\top \mathbf{A}^{n+1} \mathbf{Y}}{\mathbf{Y}^\top \mathbf{A}^{n} \mathbf{Y}} \tag{25}$$

As $\mathbf{A}$ is symmetric

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \text{ where } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{26}$$

$$\mathbf{A}^n = \mathbf{P} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix} \mathbf{P}^\top \tag{27}$$

$$\mathbf{Y}^\top \mathbf{A}^n \mathbf{Y} = \left(\mathbf{P}^\top \mathbf{Y}\right)^\top \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix} \mathbf{P}^\top \mathbf{Y} = \mathbf{v}^\top \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix} \mathbf{v} \tag{28}$$

where

$$\mathbf{v} = \mathbf{P}^\top \mathbf{Y} \tag{29}$$

$$\mathbf{Y}^\top \mathbf{A}^n \mathbf{Y} = v_1^2 (1)^n + v_2^2 (3)^n + v_3^2 (6)^n \tag{30}$$

which will be of the form

$$a_n = \frac{v_1^2 (1)^{n+1} + v_2^2 (3)^{n+1} + v_3^2 (6)^{n+1}}{v_1^2 (1)^n + v_2^2 (3)^n + v_3^2 (6)^n} \tag{31}$$

$$a_n = \frac{v_1^2 (1)^n \, 1 + v_2^2 (3)^n \, 3 + v_3^2 (6)^n \, 6}{v_1^2 (1)^n + v_2^2 (3)^n + v_3^2 (6)^n} \tag{32}$$

$$a_n = \frac{6^n \left(v_1^2 \left(\frac{1}{6}\right)^n 1 + v_2^2 \left(\frac{3}{6}\right)^n 3 + v_3^2 \left(\frac{6}{6}\right)^n 6\right)}{6^n \left(v_1^2 \left(\frac{1}{6}\right)^n + v_2^2 \left(\frac{3}{6}\right)^n + v_3^2 \left(\frac{6}{6}\right)^n\right)} \tag{33}$$

$$lim_{n\to\infty} a_n = \frac{0 + 0 + v_3^2 (6)}{0 + 0 + v_3^2} = 6 \tag{34}$$

where as $n \to \infty$

$$\frac{1}{6} \to 0, \frac{3}{6} \to 0 \tag{35}$$

Hence

$$lim_{n\to\infty} a_n = 6 \tag{36}$$