Matrices in Geometry - 9.4.48

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Problem Statement

Find two consecutive odd positive integers sum of whose squares is 290.

Solution

Let the two consecutive odd positive integers be n and $(n+2)$, so that we get,

$$n^2 + (n+2)^2 = 290 \tag{1}$$

$$\implies 2n^2 + 2n - 143 = 0 \tag{2}$$

Representing this equation as a conic section

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}, f = -143 \tag{3}$$

We need to find intersection points with $y = 0$, that is, the X-axis.

$$\mathbf{x} = \mathbf{h} + k\mathbf{m}, \mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Solution

Substituting $\mathbf{x} = k\mathbf{m}$

$$k^2\mathbf{m}^\top \mathbf{V}\mathbf{m} + 2k\mathbf{u}^\top \mathbf{m} + f = 0 \tag{5}$$

$$\implies k = \frac{1}{2}\left[-2\mathbf{u}^\top \mathbf{m} \pm \sqrt{4(\mathbf{u}^\top \mathbf{m})^2 - 4f\mathbf{m}^\top \mathbf{V}\mathbf{m}}\right] \tag{6}$$

$$\implies k = -\mathbf{u}^\top \mathbf{m} \pm \sqrt{(\mathbf{u}^\top m)^2 - f\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{7}$$

$$\mathbf{u}^\top \mathbf{m} = \begin{pmatrix} 1 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{8}$$

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{9}$$

$$k = -1 \pm \sqrt{(1)^2 - (-143)(1)} \tag{10}$$

Solution

$$k = -1 \pm \sqrt{144} \tag{11}$$

$$\implies k = -1 \pm 12 \implies \boxed{k = 11 \text{ OR } k = -13} \tag{12}$$

Substituting k into $\mathbf{x}$, we get

$$\mathbf{x} = \begin{pmatrix} 11 \\ 0 \end{pmatrix} \text{ OR } \mathbf{x} = \begin{pmatrix} -13 \\ 0 \end{pmatrix} \tag{13}$$

This implies that the roots of the equation are 11 and -13. So, we have

$$\implies \boxed{n = 11} \tag{14}$$

Final Answer

$\therefore$ The two consecutive odd positive integers whose sum of squares is 290 are 13 and 11.

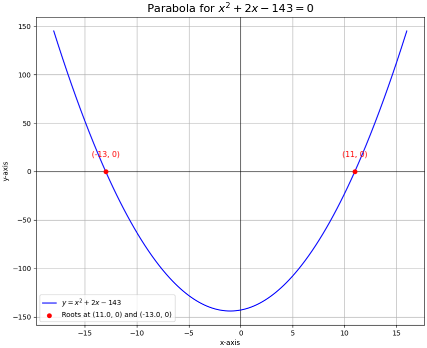

Figure: Plot for 9.4.48