10.6.8

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question

Construct a pair of tangents to a circle of radius 4cm from a point P lying outside the circle at a distance of 6cm from the centre. (10, 2023)

Solution

Let the center of the circle be at origin and point P be at distance 6 from center along x-axis.

$$O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

$$P = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{2}$$

The equation of the circle $x^2 + y^2 = 16$ can be written as:

$$\mathbf{x}^\top \mathbf{x} - 16 = 0 \tag{3}$$

The parameters of the circle are:

$$\mathbf{V} = \mathbf{I}, \quad \mathbf{u} = \mathbf{0}, \quad f = -16 \tag{4}$$

Let the point of contact be $\mathbf{q}$ and

$$\mathbf{q}^\top \mathbf{q} = 16 \tag{5}$$

From the condition of tangency we get:

$$\mathbf{q}^\top (\mathbf{q} - P) = 0 \tag{6}$$

$$P^\top \mathbf{q} = \mathbf{q}^\top \mathbf{q} \tag{7}$$

$$P^\top \mathbf{q} = 16 \tag{8}$$

Let the tangent equation passing through P be:

$$\mathbf{x} = P + k\mathbf{m} \tag{9}$$

where $\mathbf{m}$ is the direction vector of the tangent.

Substituting into the circle equation:

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - 16 \tag{10}$$

$$(P + k\mathbf{m})^\top (P + k\mathbf{m}) - 16 = 0 \tag{11}$$

$$k^2 \mathbf{m}^\top \mathbf{m} + 2kP^\top \mathbf{m} + P^\top P - 16 = 0 \tag{12}$$

$$k^2 \mathbf{m}^\top \mathbf{m} + 2kP^\top \mathbf{m} + g(P) = 0 \tag{13}$$

As the tangent intersects the conic at only one point (the point of contact), the discriminant for the quadratic in k is equal to 0:

$$4(P^\top \mathbf{m})^2 - 4\mathbf{m}^\top \mathbf{m} \cdot g(P) = 0 \tag{14}$$

$$(P^\top \mathbf{m})^2 - g(P)\mathbf{m}^\top \mathbf{m} = 0 \tag{15}$$

Since $P = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$:

$$g(P) = P^\top P - 16 = 36 - 16 = 20 \tag{16}$$

The discriminant condition becomes:

$$\mathbf{m}^\top Q \mathbf{m} = 0 \tag{17}$$

where

$$Q = \begin{pmatrix} -(P^\top P) & 0 \\ 0 & g(P) \end{pmatrix} = \begin{pmatrix} -36 & 0 \\ 0 & 20 \end{pmatrix} \tag{18}$$

Eigenvalue Decomposition of Q:
As Q is a diagonal matrix, the eigenvalues are the diagonal entries:

$$\lambda_1 = -36, \quad \lambda_2 = 20 \tag{19}$$

Applying eigenvalue decomposition for Q:

$$Q = XDX^\top \tag{20}$$

where

$$D = \begin{pmatrix} -36 & 0 \\ 0 & 20 \end{pmatrix} \tag{21}$$

X is an orthogonal matrix whose columns are the corresponding normalized eigenvectors of Q. As Q is a diagonal matrix:

$$X = \mathbf{I} \tag{22}$$

From $\mathbf{m}^\top Q \mathbf{m} = 0$:

$$\mathbf{m}^\top XDX^\top \mathbf{m} = 0 \tag{23}$$

Let $\mathbf{z} = X^\top \mathbf{m}$. Then:

$$\mathbf{z}^\top D \mathbf{z} = 0 \tag{24}$$

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} -36 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \tag{25}$$

$$-36z_1^2 + 20z_2^2 = 0 \tag{26}$$

$$\frac{z_1^2}{z_2^2} = \frac{20}{36} = \frac{5}{9} \tag{27}$$

$$\frac{z_1}{z_2} = \pm\frac{\sqrt{5}}{3} \tag{28}$$

Solving for $\mathbf{m}$:

$$\mathbf{I}\mathbf{m} = \mathbf{z} \tag{29}$$

$$\mathbf{m} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{30}$$

From $z_1/z_2 = \pm\sqrt{5}/3$, the direction vectors for the tangents can be expressed as:

$$\mathbf{m}_1 = \begin{pmatrix} \sqrt{5} \\ 3 \end{pmatrix}, \quad \mathbf{m}_2 = \begin{pmatrix} \sqrt{5} \\ -3 \end{pmatrix} \tag{31}$$

Finding Points of Contact:
Using $P^\top \mathbf{q} = 16$:

$$\begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 16 \tag{32}$$

$$6q_1 = 16 \tag{33}$$

$$q_1 = \frac{8}{3} \tag{34}$$

From $q_1^2 + q_2^2 = 16$:

$$\left(\frac{8}{3}\right)^2 + q_2^2 = 16 \tag{35}$$

$$q_2^2 = 16 - \frac{64}{9} = \frac{80}{9} \tag{36}$$

$$q_2 = \pm\frac{4\sqrt{5}}{3} \tag{37}$$

Therefore, the points of contact are:

$$\mathbf{q}_1 = \begin{pmatrix} \frac{8}{3} \\ \frac{4\sqrt{5}}{3} \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \frac{8}{3} \\ -\frac{4\sqrt{5}}{3} \end{pmatrix} \tag{38}$$

Equations of Tangents:
The tangent at $\mathbf{q}$ is given by: $\mathbf{q}^\top \mathbf{x} = 16$.
Tangent 1 at $\mathbf{q}_1$:

$$\begin{pmatrix} \frac{8}{3} & \frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{39}$$

$$\frac{8}{3}x + \frac{4\sqrt{5}}{3}y = 16 \tag{40}$$

$$2x + \sqrt{5}y = 12 \tag{41}$$

Tangent 2 at $\mathbf{q}_2$:

$$\begin{pmatrix} \frac{8}{3} & -\frac{4\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 16 \tag{42}$$

$$2x - \sqrt{5}y = 12 \tag{43}$$

The equations of the pair of tangents are:

$$\boxed{2x + \sqrt{5}y = 12 \quad \text{and} \quad 2x - \sqrt{5}y = 12} \tag{44}$$