12.234

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Question:
Consider the set of vectors in three-dimensional real vector space $\mathbb{R}^3$,

$$S = \{(1, 1, 1), (1, -1, 1), (1, 1, -1)\}.$$

Which one of the following statements is true?

a) S is not a linearly independent set.
b) S is a basis for $\mathbb{R}^3$.
c) The vectors in S are orthogonal.
d) An orthogonal set of vectors cannot be generated from S.

Solution:
let the vectors in S be:

Point	**Vector**
$\mathbf{v_1}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
$\mathbf{v_2}$	$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
$\mathbf{v_3}$	$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

TABLE 0: Variables used

Let A be the matrix with its columns as vectors of S

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{0.1}$$

Option a:
The column vectors of a matrix A are **linearly independent** if and only if the equation

$$A\mathbf{x} = \mathbf{0} \tag{0.2}$$

$$\text{has only the trivial solution} \tag{0.3}$$

$$(\mathbf{x} = \mathbf{0}) \tag{0.4}$$

We can find the solution by guassian elimination of A.

$$\text{Perform row operations to get it into row echelon form:} \tag{0.5}$$

$$\tag{0.6}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{0.7}$$

$$\tag{0.8}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow[R_2 \to R_2/-2]{R_3 \to R_3/-2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.9}$$

$$\tag{0.10}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.11}$$

The reduced row echelon form of A is the Identity matrix I.

$\therefore$ The only possible solution is the trivialsolution: $\mathbf{x} = 0$
$\therefore$ Vectors are linearly independent

Option b:
Since there are 3 linearly independent vectors in $\mathbb{R}^3$
they form a basis for $\mathbb{R}^3$

Option c:
Let the vector be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

$$\mathbf{v_1}^\top \mathbf{v_2} \neq 0 \tag{0.12}$$
$$\mathbf{v_1}^\top \mathbf{v_3} \neq 0 \tag{0.13}$$
$$\mathbf{v_2}^\top \mathbf{v_3} \neq 0 \tag{0.14}$$

$\therefore$ These vectors are not orthogonal

Option d:
Applying Gram-Schmidt process:
let the orthogonal vectors be $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ generated from $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{0.15}$$

$$\mathbf{u_2} = \mathbf{v_2} - (\mathbf{u_1^\top v_2})\hat{\mathbf{u}}_1 \tag{0.16}$$

$$\mathbf{u_2} = \mathbf{v_2} - \left(\frac{\mathbf{v_2}^\top \mathbf{u_1}}{\mathbf{u_1}^\top \mathbf{u_1}}\right)\mathbf{u_1} \tag{0.17}$$

$$= \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix} \tag{0.18}$$

$$\mathbf{u_3} = \mathbf{v_3} - (\hat{\mathbf{u}}_2^\top \mathbf{v_3})\hat{\mathbf{u}}_2 \tag{0.19}$$

$$= \mathbf{v_3} - \left(\frac{\mathbf{u_2}^\top \mathbf{v_3}}{\mathbf{u_2}^\top \mathbf{u_2}}\right)\mathbf{u_2} \tag{0.20}$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.21}$$

$$\mathbf{u_1}^\top \mathbf{u_2} = 0 \tag{0.22}$$

$$\mathbf{u_2}^\top \mathbf{u_3} = 0 \tag{0.23}$$

$$\mathbf{u_1}^\top \mathbf{u_3} = 0 \tag{0.24}$$

$\therefore$ an orthogonal set of vectors can be generated from S.
$\therefore$ Options b and d are correct.