Problem 10.7.72

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Problem

The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to $x^2 + y^2 = 1$ pass through the point ...

Equation

The general equation of conic

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{3.1}$$

The chord of contact of tangents from an external point $\mathbf{q}$ is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{3.2}$$

Given circle in matrix form

$$x^2 + y^2 = 1 \tag{3.3}$$

$$x^2 + y^2 - 1 = 0 \tag{3.4}$$

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - 1 = 0 \tag{3.5}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}, \mathbf{u} = \mathbf{0}, f = -1 \tag{3.6}$$

Simplify

Given line

$$2x + y = 4 \tag{3.7}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{x} = 4 \tag{3.8}$$

As $\mathbf{q}$ satisfies (8)

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \mathbf{q} = 4 \tag{3.9}$$

From (2) and (6)

$$(\mathbf{lq} + \mathbf{0})^\top \mathbf{x} + \mathbf{0}^\top \mathbf{q} - 1 = 0 \tag{3.10}$$

$$(\mathbf{lq})^\top \mathbf{x} - 1 = 0 \tag{3.11}$$

$$\mathbf{q}^\top \mathbf{x} - 1 = 0 \tag{3.12}$$

$$\mathbf{q}^\top \mathbf{x} = 1 \implies \mathbf{x}^\top \mathbf{q} = 1 \tag{3.13}$$

From (9) and (13)

Conclusion

$$\mathbf{x}^\top = k\begin{pmatrix}2 & 1\end{pmatrix} \tag{3.14}$$

$$\left(k\begin{pmatrix}2 & 1\end{pmatrix}\right)^\top \mathbf{q} = 1 \implies k\begin{pmatrix}2 & 1\end{pmatrix}^\top \mathbf{q} = 1 \tag{3.15}$$

$$k\,(4) = 1 \implies k = \frac{1}{4} \tag{3.16}$$

$$\mathbf{x}^\top = \frac{1}{4}\begin{pmatrix}2 & 1\end{pmatrix} \implies \mathbf{x}^\top = \begin{pmatrix}\frac{1}{2} & \frac{1}{4}\end{pmatrix} \tag{3.17}$$

$$\mathbf{x} = \begin{pmatrix}\frac{1}{2} \\ \frac{1}{4}\end{pmatrix} \tag{3.18}$$

Hence the chords of contact pass through the point $\begin{pmatrix}\frac{1}{2} \\ \frac{1}{4}\end{pmatrix}$

Plot

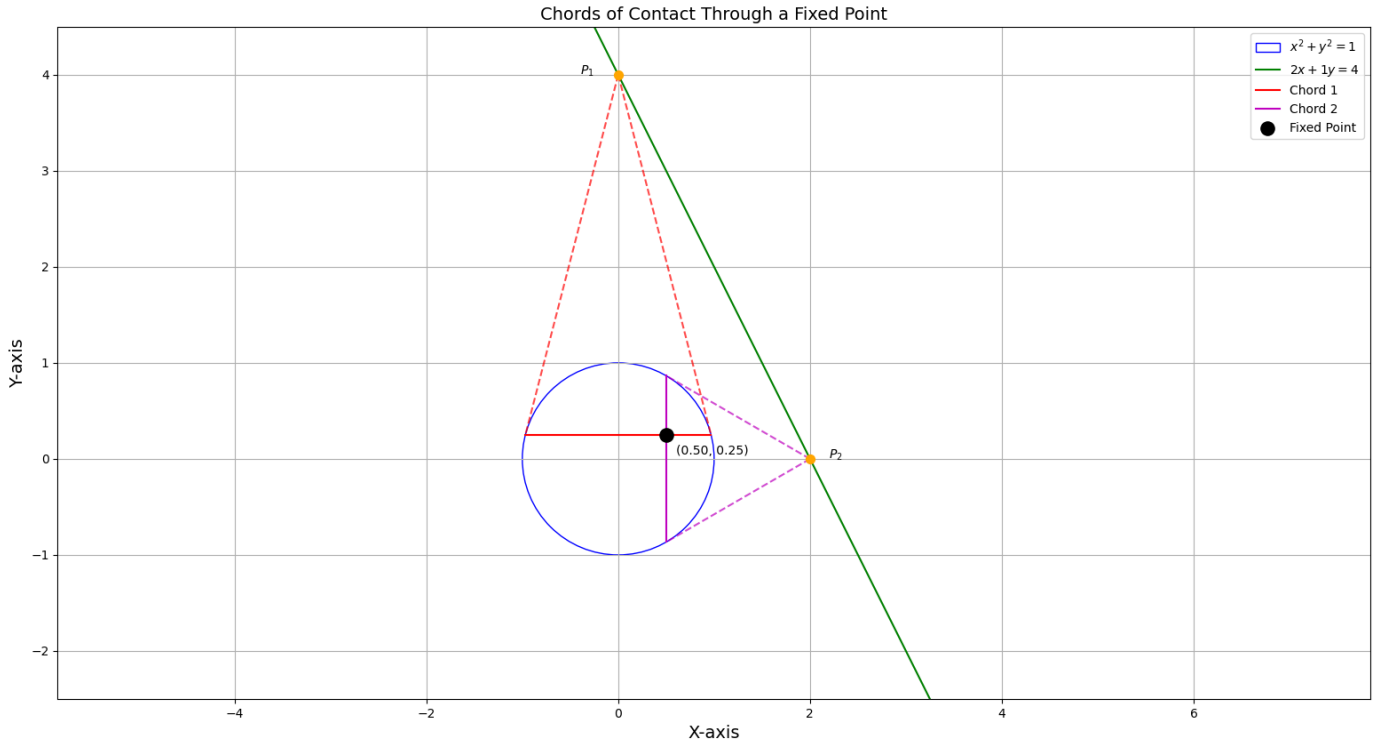
Chords of Contact Through a Fixed Point
$x^2 + y^2 = 1$
$2x + 1y = 4$
Chord 1
Chord 2
Fixed Point
P_1
P_2
(0.50, 0.25)
Y-axis
X-axis
4
3
2
1
0
−1
−2
−4
−2
0
2
4
6

Figure:

C Code

```c
void get_problem_data(double* out_data) {
    out_data[0] = 2.0;
    out_data[1] = 1.0;
    out_data[2] = -4.0;
    out_data[3] = 1.0;
}
```

Python Code for Solving

```
import ctypes
import numpy as np
import sympy as sp

def get_plot_data():

    lib = ctypes.CDLL('./code.so')
    out_data = (ctypes.c_double * 4)()
    lib.get_problem_data(out_data)
    a, b, c, r = list(out_data)

    x, y, x0, y0 = sp.symbols('x y x0 y0')
    y0_expr = sp.solve(a*x0 + b*y0 + c, y0)[0]
    chord_on_line = (x*x0 + y*y0 - r**2).subs(y0, y0_expr)
    poly = sp.Poly(chord_on_line, x0)

    eq1 = poly.coeff_monomial(x0)
    eq2 = poly.coeff_monomial(1)
```

Python Code for Solving

```
sol = sp.solve([eq1, eq2], [x, y])
    p_fixed = np.array([float(sol[x]), float(sol[y])])

    p1 = np.array([0.0, 4.0])
    p2 = np.array([2.0, 0.0])

    t1 = np.array([np.sqrt(15)/4, 1/4])
    t2 = np.array([-np.sqrt(15)/4, 1/4])
    t3 = np.array([1/2, np.sqrt(3)/2])
    t4 = np.array([1/2, -np.sqrt(3)/2])

    return {
        p_fixed: p_fixed,
        p1: p1, t1: t1, t2: t2,
        p2: p2, t3: t3, t4: t4,
        line_coeffs: (a, b, c)
    }
```

Python Code for Plotting

```
# Code by /sdcard/github/matgeo/codes/CoordGeoVV Sharma
# September 12, 2023
# Revised July 21, 2024
# Released under GNU GPL
# Section Formula
import sys
sys.path.insert(0, '/workspaces/urban-potato/matgeo/codes/
    CoordGeo/')
import matplotlib.pyplot as plt
import numpy as np
from call import get_plot_data

data = get_plot_data()
P = data[p_fixed]
P1, T1, T2 = data[p1], data[t1], data[t2]
P2, T3, T4 = data[p2], data[t3], data[t4]
a, b, c = data[line_coeffs]

fig, ax = plt.subplots(figsize=(9, 9))
```

Python Code for Plotting

```
circle = plt.Circle((0, 0), 1, color='blue', fill=False, label='
    $x^2+y^2=1$')

ax.add_patch(circle)
x_vals = np.array([-1, 4])
y_vals = (-a * x_vals - c) / b
ax.plot(x_vals, y_vals, 'g-', label=f'${int(a)}x+{int(b)}y={int(-
    c)}$')
ax.plot([P1[0], T1[0]], [P1[1], T1[1]], 'r--', alpha=0.7)
ax.plot([P1[0], T2[0]], [P1[1], T2[1]], 'r--', alpha=0.7)
ax.plot([T1[0], T2[0]], [T1[1], T2[1]], 'r-', label='Chord 1')

ax.plot([P2[0], T3[0]], [P2[1], T3[1]], 'm--', alpha=0.7)
ax.plot([P2[0], T4[0]], [P2[1], T4[1]], 'm--', alpha=0.7)
ax.plot([T3[0], T4[0]], [T3[1], T4[1]], 'm-', label='Chord 2')

ax.scatter([P1[0], P2[0]], [P1[1], P2[1]], color='orange', s=50,
    zorder=5)
```

Python Code for Plotting

```
ax.text(P1[0] - 0.4, P1[1], '$P_1$')
ax.text(P2[0] + 0.2, P2[1], '$P_2$')

ax.scatter(P[0], P[1], color='black', s=120, zorder=5, label='
    Fixed Point')
ax.text(P[0] + 0.1, P[1] - 0.2, f'({P[0]:.2f}, {P[1]:.2f})')

ax.set_title('Chords of Contact Through a Fixed Point')
ax.set_xlabel('X-axis'); ax.set_ylabel('Y-axis')
ax.text(P1[0] - 0.4, P1[1], '$P_1$')
ax.text(P2[0] + 0.2, P2[1], '$P_2$')

ax.scatter(P[0], P[1], color='black', s=120, zorder=5, label='
    Fixed Point')
ax.text(P[0] + 0.1, P[1] - 0.2, f'({P[0]:.2f}, {P[1]:.2f})')

ax.set_title('Chords of Contact Through a Fixed Point')
ax.set_xlabel('X-axis'); ax.set_ylabel('Y-axis')
```