2.5.25

EE25BTECH11059 - Vaishnavi Ramkrishna Anantheertha

Question: Let $\mathbb{R}^3$ denote the three-dimensional space. Take two points $P = (1, 2, 3)$ and $Q = (4, 2, 7)$. Let $\text{dist}(X, Y)$ denote the distance between two points X and Y in $\mathbb{R}^3$.

Let
$$S = \{X \in \mathbb{R}^3 : (\text{dist}(X, P))^2 - (\text{dist}(X, Q))^2 = 50\}$$
and
$$T = \{Y \in \mathbb{R}^3 : (\text{dist}(Y, Q))^2 - (\text{dist}(Y, P))^2 = 50\}.$$

Then which of the following statements are TRUE?

(a) There is a triangle whose area is 1 and all of whose vertices are from S.
(b) There are two distinct points L and M in T such that each point on the line segment LM is also in T.
(c) There are infinitely many rectangles of perimeter 48, two of whose vertices are from S and the other two vertices are from T.
(d) There is a square of perimeter 48, two of whose vertices are from S and the other two vertices are from T.

.

Solution:

Variable	Value
P	$(1, 2, 3)$
Q	$(4, 2, 7)$

TABLE 0: Variables Used

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{0.1}$$

$$\mathbf{Q} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} \tag{0.2}$$

$$\text{dist}(X, P))^2 - (\text{dist}(X, Q))^2 = 50 \tag{0.3}$$

$$(\|\mathbf{X} - \mathbf{P}\|_2)^2 - (\|\mathbf{X} - \mathbf{Q}\|_2)^2 = 50 \tag{0.4}$$

$$\mathbf{X}^{\mathbf{T}}\mathbf{X} - 2\mathbf{P}^{\mathbf{T}}\mathbf{X} + \mathbf{P}^{\mathbf{T}}\mathbf{P} - \mathbf{X}^{\mathbf{T}}\mathbf{X} + 2\mathbf{Q}^{\mathbf{T}}\mathbf{X} - \mathbf{Q}^{\mathbf{T}}\mathbf{Q} = 50 \tag{0.5}$$

$$2(\mathbf{Q}^{\mathbf{T}} - \mathbf{P}^{\mathbf{T}})\mathbf{X} + \mathbf{P}^{\mathbf{T}}\mathbf{P} - \mathbf{Q}^{\mathbf{T}}\mathbf{Q} = 50 \tag{0.6}$$

(0.6)is the eq of plane S

$$\text{dist}(Y, Q))^2 - (\text{dist}(Y, P))^2 = 50 \tag{0.7}$$

$$(\|\mathbf{Y} - \mathbf{Q}\|_2)^2 - (\|\mathbf{Y} - \mathbf{P}\|_2)^2 = 50 \tag{0.8}$$

$$\mathbf{Y}^{\mathbf{T}}\mathbf{Y} - 2\mathbf{Q}^{\mathbf{T}}\mathbf{Y} + \mathbf{Q}^{\mathbf{T}}\mathbf{Q} - \mathbf{Y}^{\mathbf{T}}\mathbf{Y} + 2\mathbf{P}^{\mathbf{T}}\mathbf{Y} - \mathbf{P}^{\mathbf{T}}\mathbf{P} = 50 \tag{0.9}$$

$$2(\mathbf{P}^{\mathbf{T}} - \mathbf{Q}^{\mathbf{T}})\mathbf{Y} + \mathbf{Q}^{\mathbf{T}}\mathbf{Q} - \mathbf{P}^{\mathbf{T}}\mathbf{P} = 50 \tag{0.10}$$

(0.10) is the eq of plane T
S And T are parallel planes
S is an infinite plane. In any plane one can choose three non collinear points whose triangle has unit area.Therefore a) is correct

let eq of T be $\mathbf{n}^{\mathbf{T}}\mathbf{X} = c$
let $\mathbf{A}, \mathbf{B}$ be two points on T

$$\mathbf{n}^{\mathbf{T}}\mathbf{A} = c \tag{0.11}$$

$$\mathbf{n}^{\mathbf{T}}\mathbf{B} = c \tag{0.12}$$

any point on line AB can be written as

$$\mathbf{C} = (1 - k)\mathbf{A} + k\mathbf{B} \tag{0.13}$$

$$\mathbf{n}^{\mathbf{T}}[(1 - k)\mathbf{A} + k\mathbf{B}] = \mathbf{n}^{\mathbf{T}}\mathbf{A} + k[\mathbf{n}^{\mathbf{T}}\mathbf{B} - \mathbf{n}^{\mathbf{T}}\mathbf{A}] = c \tag{0.14}$$

$$\mathbf{n}^{\mathbf{T}}\mathbf{C} = c \tag{0.15}$$

Hence C lies on T
Therefore b) is correct
distance between S and T=$\frac{|c_1 - c_2|}{|n|} = 10$
let $\mathbf{A}, \mathbf{B}$ lie on S and $\mathbf{A}', \mathbf{B}'$ where

$$\|\mathbf{A} - \mathbf{B}\|_2 = \|\mathbf{A}' - \mathbf{B}'\|_2 = d(say) \tag{0.16}$$

$$2(d + 10) = 48 \tag{0.17}$$

$$d = 14 \tag{0.18}$$

There are infinitely many ways to pick two points in plane S that are 14 units apart (because the plane is infinite).
Therefore c) is correct

The distance between the planes S and T is 10. A square with perimeter 48 has side length 12. If two opposite vertices lie on S and the other two on T, then each side of the square must connect the two planes. Since the vertical separation is 10, and the side is

12, the square can be tilted so that part of the side length is vertical (10 units) and the rest is horizontal: ($\sqrt{12^2 - 10^2} = \sqrt{44}$ units)

Hence d) is correct

Refer to Figure

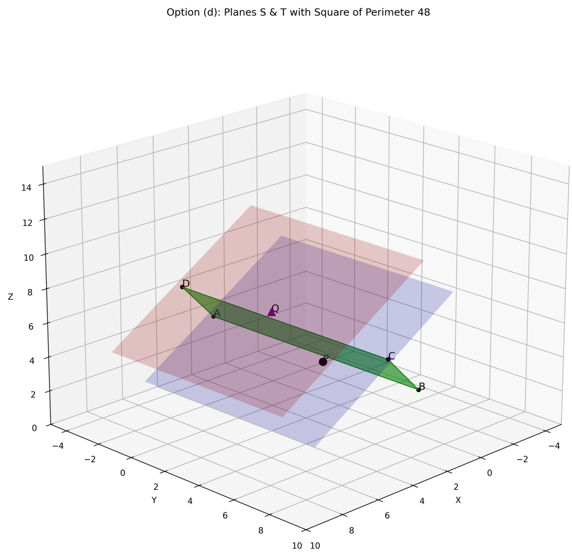

Fig. 0.1