10.7.69

EE25BTECH11020 - Darsh Pankaj Gajare

Question:

Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from the point $(4, 5)$ with a pair of radii form a quadrilateral with area_____.

Solution:

$$\text{Conic:} \quad \mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad f = -11 \tag{2}$$

Matrix equation of a line through P:

$$\mathbf{x} = \mathbf{P} + t\mathbf{m}, \quad \mathbf{P} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} 1 \\ k \end{pmatrix} \tag{3}$$

Substitute into the conic:

$$(\mathbf{P} + t\mathbf{m})^\top V (\mathbf{P} + t\mathbf{m}) + 2\mathbf{u}^\top (\mathbf{P} + t\mathbf{m}) + f = 0 \tag{4}$$

$$\left(k^2 + 1\right) t^2 + (8k + 4)\, t + 4 = 0 \tag{5}$$

Tangency from P$\Rightarrow$ double root in t

$$(8k + 4)^2 - 4 \cdot \left(k^2 + 1\right) \cdot 4 = 0 \tag{6}$$

$$k = 0, -\frac{4}{3} \tag{7}$$

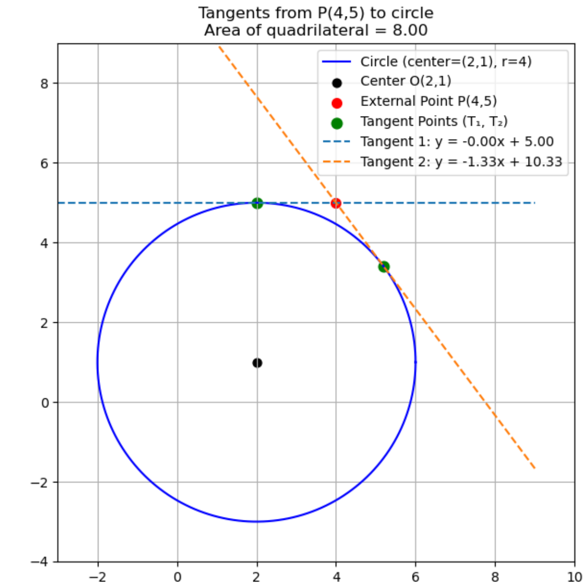
$$\text{For each } k: \quad t = -\frac{8k + 4}{2(k^2 + 1)} \tag{8}$$

$$\text{Thus contact points } A = \mathbf{P} + t\mathbf{m} \text{ are } A_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{26}{5} \\ \frac{17}{5} \end{pmatrix} \tag{9}$$

$$C = -\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{10}$$

$$area\,(A_2CA_1P) = \frac{1}{2}\left(\|(\mathbf{A_1} - \mathbf{C}) \times (\mathbf{P} - \mathbf{C})\| + \|(\mathbf{A_2} - \mathbf{C}) \times (\mathbf{P} - \mathbf{C})\|\right) = 8 \tag{11}$$

Plot using C libraries:



Plot using Python: