4.13.17

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Question:
Three distinct points A, B and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point:

1) $\left(\frac{5}{4}, 0\right)$
2) $\left(\frac{5}{2}, 0\right)$
3) $\left(\frac{5}{3}, 0\right)$
4) $(0, 0)$

Solution: let $\mathbf{F_1}$, $\mathbf{F_2}$ be the vectors such that:

Point	Vector
$\mathbf{F_1}$	$\begin{pmatrix}1\\0\end{pmatrix}$
$\mathbf{F_2}$	$\begin{pmatrix}-1\\0\end{pmatrix}$

TABLE 4: Variables used

Let $\mathbf{P}$ be any vector in the plane of $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$.
given,

$$\frac{\|\mathbf{PF_1}\|}{\|\mathbf{PF_2}\|} = \frac{1}{3} \tag{4.1}$$

$$\frac{\sqrt{(\mathbf{P}-\mathbf{F_1})^\top(\mathbf{P}-\mathbf{F_1})}}{\sqrt{(\mathbf{P}-\mathbf{F_2})^\top(\mathbf{P}-\mathbf{F_2})}} = \frac{1}{3} \tag{4.2}$$

Squaring on both sides

$$9(\mathbf{P}-\mathbf{F_1})^\top(\mathbf{P}-\mathbf{F_1}) = (\mathbf{P}-\mathbf{F_2})^\top(\mathbf{P}-\mathbf{F_2}) \tag{4.3}$$

$$9(\mathbf{P}^\top\mathbf{P} - \mathbf{P}^\top\mathbf{F_1} - \mathbf{F_1}^\top\mathbf{P} + \mathbf{F_1}^\top\mathbf{F_1}) = \mathbf{P}^\top\mathbf{P} - \mathbf{P}^\top\mathbf{F_2} - \mathbf{F_2}^\top\mathbf{P} + \mathbf{F_2}^\top\mathbf{F_2} \tag{4.4}$$

$$\text{as } \mathbf{P}^\top\mathbf{F_1} = \mathbf{F_1}^\top\mathbf{P} \tag{4.5}$$

$$\text{and } \mathbf{P}^\top\mathbf{F_2} = \mathbf{F_2}^\top\mathbf{P} \tag{4.6}$$

$$9(\mathbf{P}^\top\mathbf{P} - 2\mathbf{P}^\top\mathbf{F_1} + \mathbf{F_1}^\top\mathbf{F_1}) = \mathbf{P}^\top\mathbf{P} - 2\mathbf{P}^\top\mathbf{F_2} + \mathbf{F_2}^\top\mathbf{F_2} \tag{4.7}$$

$$8\mathbf{P}^\top\mathbf{P} - 2\mathbf{P}^\top(9\mathbf{F_1} - \mathbf{F_2}) + 9\mathbf{F_1}^\top\mathbf{F_1} - \mathbf{F_2}^\top\mathbf{F_2} = 0 \tag{4.8}$$

$$\mathbf{P}^\top\mathbf{P} - \frac{1}{4}\mathbf{P}^\top(9\mathbf{F_1} - \mathbf{F_2}) + \frac{9}{8}\mathbf{F_1}^\top\mathbf{F_1} - \frac{1}{8}\mathbf{F_2}^\top\mathbf{F_2} = 0 \tag{4.9}$$

This can be compared with general equation of circle :

$$\|\mathbf{P} - \mathbf{C}\| = r \tag{4.10}$$

$$\tag{4.11}$$

where , $\mathbf{P}$ = any point on the circle
$\mathbf{C}$ = Center of circle
r = radius of circle

$$(\mathbf{P} - \mathbf{C})^\top(\mathbf{P} - \mathbf{C}) = r^2 \tag{4.12}$$

$$\mathbf{P}^\top\mathbf{P} - 2\mathbf{P}^\top\mathbf{C} + \mathbf{C}^\top\mathbf{C} = r^2 \tag{4.13}$$

Substituting $\mathbf{P}$, $\mathbf{F_1}$ and $\mathbf{F_2}$
Center of circle = $\left(\frac{5}{4}, 0\right)$
Hence, the circumcenter of the triangle is $\left(\frac{5}{4}, 0\right)$

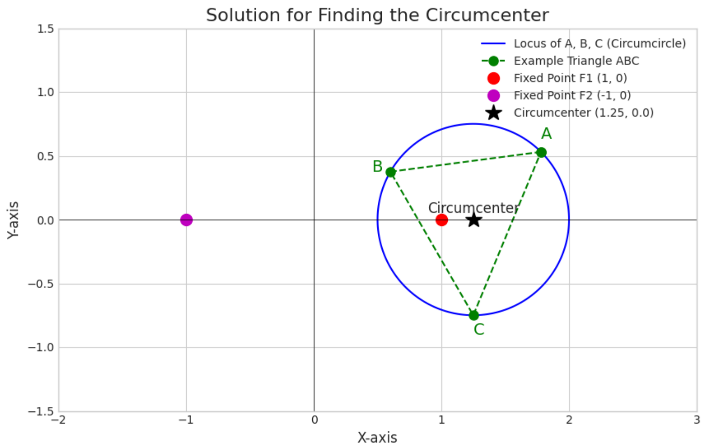

Fig. 4.1: 4.13.17