

Matrices in Geometry - 7.4.42

EE25BTECH11035 Kushal B N

Problem Statement

Find the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}a)x - (1 - \sqrt{2}a)y = 0$.

Solution

Given, Circle C: $x^2 + y^2 - \frac{1+\sqrt{2}a}{2}x - \frac{1-\sqrt{2}a}{2}y = 0$

$$\text{Point } \mathbf{P} = \begin{pmatrix} \frac{1+\sqrt{2}a}{2} \\ \frac{1-\sqrt{2}a}{2} \end{pmatrix}$$

Line L: $\mathbf{n}_L^\top \mathbf{x} = 0$, where $\mathbf{n}_L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The center of the circle is

$$\mathbf{c} = \frac{1}{4} \begin{pmatrix} 1 + \sqrt{2}a \\ 1 - \sqrt{2}a \end{pmatrix} \quad (1)$$

By observation,

$$\mathbf{P} = 2\mathbf{c} \quad (2)$$

The locus of midpoints, \mathbf{M} , of chords from \mathbf{P} is given by,

$$(\mathbf{M} - \mathbf{c})^\top (\mathbf{P} - \mathbf{M}) = 0 \quad (3)$$

Solution

The midpoint \mathbf{M} lies on the line L , the direction vector of which is

$$\mathbf{m}_L = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (4)$$

$$\implies \mathbf{M} = \lambda \mathbf{m}_L \quad (5)$$

Substituting this into equation (??), we get

$$(\lambda \mathbf{m}_L - \mathbf{c})^\top (\mathbf{P} - \lambda \mathbf{m}_L) = 0 \quad (6)$$

$$\implies (\mathbf{m}_L^\top \mathbf{m}_L) \lambda^2 - \mathbf{m}_L^\top (\mathbf{P} + \mathbf{c}) \lambda + \mathbf{c}^\top \mathbf{P} = 0 \quad (7)$$

Solution

For two distinct chords, the discriminant $\Delta = b^2 - 4ac > 0$.

$$\Delta = \left(\mathbf{m}_L^\top (\mathbf{P} + \mathbf{c}) \right)^2 - 4 \left(\mathbf{m}_L^\top \mathbf{m}_L \right) \left(\mathbf{c}^\top \mathbf{P} \right) > 0 \quad (8)$$

Here,

$$\mathbf{m}_L^\top \mathbf{m}_L = (1 \quad -1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \quad (9)$$

$$\mathbf{m}_L^\top (\mathbf{P} + \mathbf{c}) = \mathbf{m}_L^\top (3\mathbf{c}) = 3 (1 \quad -1) \frac{1}{4} \begin{pmatrix} 1 + \sqrt{2}a \\ 1 - \sqrt{2}a \end{pmatrix} = \frac{3\sqrt{2}a}{2} \quad (10)$$

$$\mathbf{c}^\top \mathbf{P} = 2 \|\mathbf{c}\|^2 = \frac{1 + 2a^2}{4} \quad (11)$$

Solution

Substituting into the inequality (??):

$$\left(\frac{3\sqrt{2}a}{2}\right)^2 - 4(2)\left(\frac{1+2a^2}{4}\right) > 0 \quad (12)$$

$$\frac{9a^2}{2} - 2(1+2a^2) > 0 \quad (13)$$

$$\frac{a^2}{2} > 2 \implies a^2 > 4 \quad (14)$$

$$\implies a > 2 \quad \text{or} \quad a < -2 \quad (15)$$

Conclusion

The intervals of values for a are $(-\infty, -2) \cup (2, \infty)$.

$$\boxed{a \in (-\infty, -2) \cup (2, \infty)} \quad (16)$$

Plot

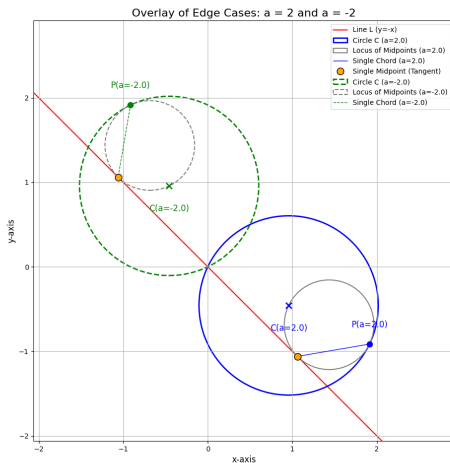


Figure: Plot for 7.4.42