# 12.141

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**Question:**
Let $\mathbf{A}$ be a 3×3 matrix. Suppose that the eigenvalues of $\mathbf{A}$ are -1, 0, 1 with respective eigenvectors $(1,-1,0)^\top, (1,1,-2)^\top$ and $(1,1,1)^\top$ . Then 6$\mathbf{A}$ equals

1) $\begin{pmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$
2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
3) $\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{pmatrix}$
4) $\begin{pmatrix} -3 & 9 & 0 \\ 9 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

**Solution:**
For an invertible matrix $\mathbf{P}$

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{1}$$

Given eigen values are

$$\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1 \tag{2}$$

Given eigen vectors are

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3}$$

where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \tag{5}$$

$$|\mathbf{P}| = 1\,(1+2) - 1\,(-1+0) + 1\,(2+0) = 3 + 1 + 2 = 6 \neq 0 \tag{6}$$

$$\mathbf{P}\mathbf{P}^{-1} = \mathbf{I} \tag{7}$$

Augmented matrix of $\left(\mathbf{P} \mid \mathbf{I}\right)$ is given by

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{2}(R_1+R_2)} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \tag{8}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \tag{9}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \tag{10}$$

$$\mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \tag{11}$$

$$6\mathbf{A} = 6\mathbf{PDP}^{-1} = (\mathbf{PD})\left(6\mathbf{P}^{-1}\right) \tag{12}$$

$$6\mathbf{A} = \left(\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\left(6\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\right) \tag{13}$$

$$= \left(\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 3 & -3 & 0 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -3 & 0 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} -3+2 & 3+2 & 0+2 \\ 3+2 & -3+2 & 0+2 \\ 0+2 & 0+2 & 0+2 \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix} \tag{16}$$