8.4.22

EE25BTECH11019 – Darji Vivek M.

Question

Question:

The radius of the circle passing through the foci of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

and having its centre at $(0, 3)$ is -

1. 4
2. 3
3. $\sqrt{\frac{1}{2}}$
4. $\frac{7}{2}$

Solution

Use the matrix form (matrix method). Let $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. The ellipse is

$$\mathbf{x}^\top \mathbf{V}\,\mathbf{x} = 1, \qquad \mathbf{V} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}.$$

Eigenvalues of $\mathbf{V}$ (diagonal entries) are

$$\lambda_1 = \frac{1}{16}, \qquad \lambda_2 = \frac{1}{9}.$$

For the principal-form ellipse $\mathbf{x}^\top \mathbf{V}\mathbf{x} = 1$ the semi-axes satisfy

$$a^2 = \frac{1}{\lambda_1} = 16, \qquad b^2 = \frac{1}{\lambda_2} = 9.$$

Using the formula for eccentricity.

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \sqrt{1 - \frac{1/16}{1/9}} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}.$$

Solution

The focal distance (from the centre) is $c = ae$, therefore

$$c = 4 \cdot \frac{\sqrt{7}}{4} = \sqrt{7}.$$

Since $\mathbf{V}$ is diagonal with λ_1 along the x-direction, the principal axis unit vector is.

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the foci (in vector form) are

$$\mathbf{F}_1 = c\mathbf{n} = \begin{pmatrix} \sqrt{7} \\ 0 \end{pmatrix}, \qquad \mathbf{F}_2 = -c\mathbf{n} = \begin{pmatrix} -\sqrt{7} \\ 0 \end{pmatrix}.$$

Solution

The required circle has centre $\mathbf{C} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ and passes through, say, $\mathbf{F_1}$.

Therefore its radius is

$$R = \|\mathbf{F_1} - \mathbf{C}\| = \sqrt{\left(\sqrt{7} - 0\right)^2 + \left(0 - 3\right)^2} = \sqrt{7 + 9} = \sqrt{16} = \boxed{4}.$$

C code

```
#include <stdio.h>
#include <math.h>

float circle_radius() {
    // Ellipse: x^2/16 + y^2/9 = 1
    float a2 = 16.0;   // a^2
    float b2 = 9.0;    // b^2
    // Focal distance from origin: c = sqrt(a^2 - b^2)
    float c = sqrt(a2 - b2); // sqrt(7)
    // Foci: ( sqrt (7), 0)
    // Centre of circle: (0, 3)
    // Radius = distance between (sqrt(7), 0) and (0,
        3)
    float radius = sqrt((c - 0)*(c - 0) + (0 - 3)*(0 -
         3));
    return radius; // should be 4
}
```

Python

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load C shared library
lib = ctypes.CDLL("./15.so")
# Set return type for circle_radius()
lib.circle_radius.restype = ctypes.c_float
# Call the C function
r = lib.circle_radius()
print("Radius of the circle =", r)
# -------------- Plotting --------------
# Ellipse: x^2/16 + y^2/9 = 1
a = 4
b = 3
x = np.linspace(-a, a, 400)
y_top = b * np.sqrt(1 - (x**2 / a**2))
y_bottom = -y_top
```

Python

```
# Foci
c = np.sqrt(a**2 - b**2)
F1 = (c, 0)
F2 = (-c, 0)

# Circle centered at (0, 3) passing through (sqrt(7),
    0)
theta = np.linspace(0, 2*np.pi, 200)
x_circ = r * np.cos(theta)
y_circ = 3 + r * np.sin(theta)

# Plot ellipse
plt.plot(x, y_top, 'b', label="Ellipse")
plt.plot(x, y_bottom, 'b')

# Plot circle
plt.plot(x_circ, y_circ, 'r', label="Circle")
```

Python

```
# Plot foci and center
plt.scatter([F1[0], F2[0]], [F1[1], F2[1]], color='
    green', label="Foci")
plt.scatter(0, 3, color='black', label="Circle center
    (0,3)")

# Equal aspect ratio
plt.gca().set_aspect('equal', adjustable='box')
plt.xlabel("x")
plt.ylabel("y")
plt.title("Ellipse and Circle passing through Foci")
plt.grid(True)
plt.legend()
plt.show()
```

Pyhton plot

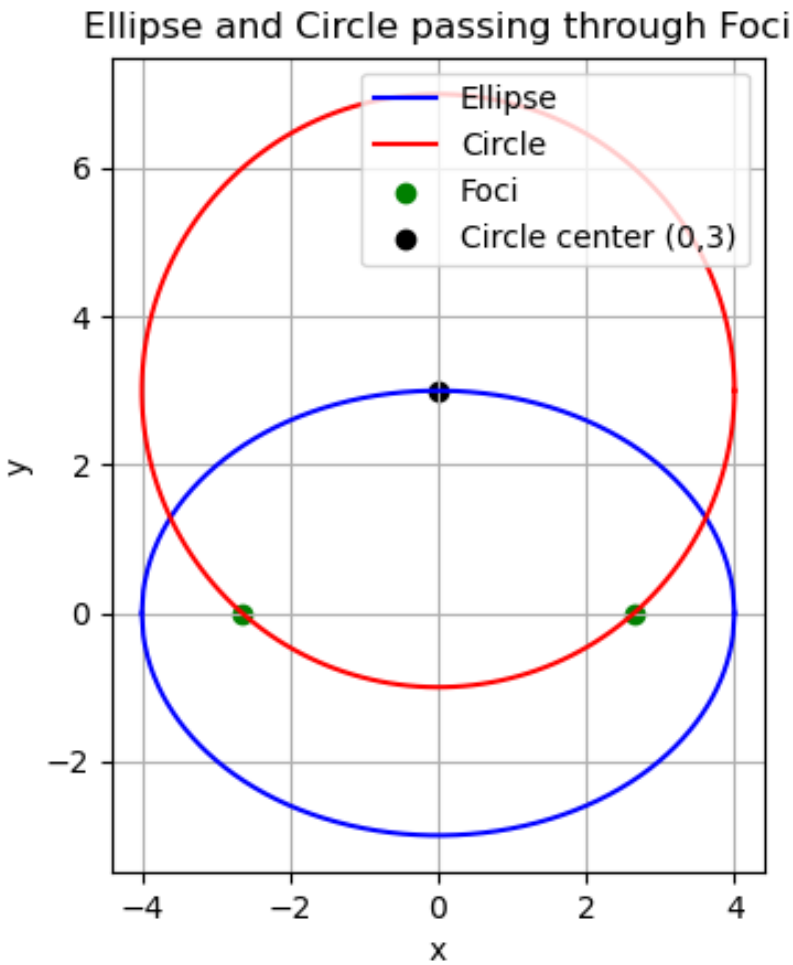

Figure: plot if p=2,q=2