8.4.23

EE25BTECH11020 - Darsh Pankaj Gajare

Question:
The curve described parametrically by $x = t^2 + t + 1$ and $y = t^2 - t + 1$ represents:

1) a pair of straight lines
2) an ellipse
3) a parabola
4) a hyperbola

Solution:

TABLE I

$\mathbf{x}$	$\begin{pmatrix} x \\ y \end{pmatrix}$
$\mathbf{a}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\mathbf{b}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$\mathbf{c}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The parametric form can be written as

$$\mathbf{x} = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}. \tag{1}$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^\top \begin{pmatrix} t^2 \\ t \end{pmatrix} + \mathbf{c} \tag{2}$$

$$\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} t^2 \\ t \end{pmatrix} + \mathbf{c} \tag{3}$$

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) \tag{4}$$

$$\text{Let } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5}$$

$$\mathbf{M} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{z} = \mathbf{x} - \mathbf{c}. \tag{6}$$

$$\text{Then } \begin{pmatrix} t^2 \\ t \end{pmatrix} = \mathbf{Mz}, \quad t^2 = \mathbf{e}_1^\top \mathbf{Mz}, \quad t = \mathbf{e}_2^\top \mathbf{Mz}. \tag{7}$$

$$\text{Eliminate } t: \quad \mathbf{e}_1^\top \mathbf{Mz} = \left(\mathbf{e}_2^\top \mathbf{Mz}\right)^2. \tag{8}$$

$$\text{Define } \mathbf{w} = \mathbf{Mz} \ \Rightarrow \ \mathbf{e}_1^\top \mathbf{w} = \left(\mathbf{e}_2^\top \mathbf{w}\right)^2. \tag{9}$$

$$\text{In matrix form:}\quad \mathbf{z}^\top\mathbf{M}^\top\mathbf{e}_1\mathbf{e}_1^\top\mathbf{M}\mathbf{z} - \mathbf{z}^\top\mathbf{M}^\top\mathbf{e}_2\mathbf{e}_2^\top\mathbf{M}\mathbf{z} = 0. \tag{10}$$

$$\text{Let } \mathbf{E} = \mathbf{e}_1\mathbf{e}_1^\top - \mathbf{e}_2\mathbf{e}_2^\top = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \ \mathbf{Q} = \mathbf{M}^\top\mathbf{E}\mathbf{M}. \tag{11}$$

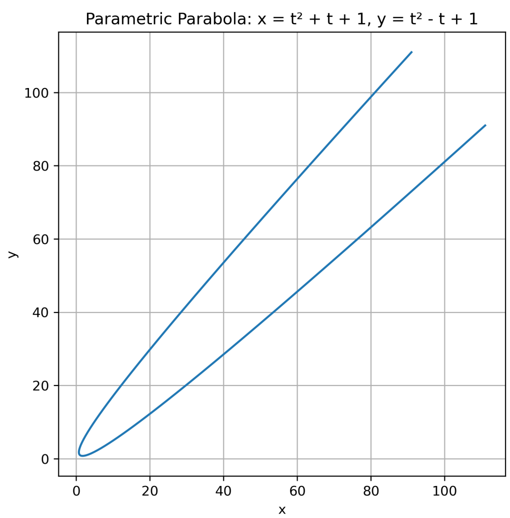
$$\mathbf{Q} = \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

$$\mathbf{z}^\top\mathbf{Q}\mathbf{z} = 0, \quad \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{13}$$

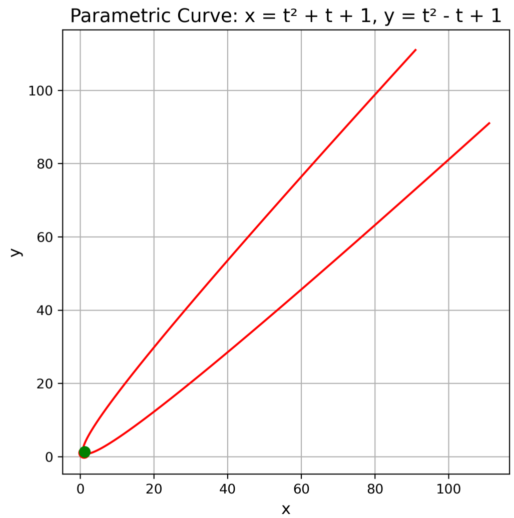
$$(x-1)(y-1) = \frac{1}{2}(y-x)^2 \iff (x-y)^2 = 2(x+y-2)\,. \tag{14}$$

$$\text{Thus the conic is a parabola.} \tag{15}$$

Plot using C libraries:



Plot using Python:


Parametric Curve: x = t² + t + 1, y = t² - t + 1
0
20
40
60
80
100
x
y