Problem 12.869

ee25btech11023-Venkata Sai

October 11, 2025

1. Problem
2. Solution
   - Equation
   - Finding the vector A
   - Condition
   - Finding eigen values
   - Simplify
   - Conclusion

Problem

For $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, quadratic form

$$\mathbf{Q}(\mathbf{X}) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \tag{2.1}$$

Let $\mathbf{M}$ be symmetric matrix of Q. For $\mathbf{Y} \in \mathbb{R}^3$? non-zero define

$$a_n = \frac{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n+1} \mathbf{Y}}{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n} \mathbf{Y}} \tag{2.2}$$

Then $\lim_{n\to\infty} a_n = \ldots$

Equation

Given

$$\mathbf{Q}\left(\mathbf{X}\right) = 2x_1^2 + 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 \tag{3.1}$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.2}$$

M is a symmetric matrix of Q

$$\mathbf{Q} = \mathbf{X}^\top \mathbf{M}\mathbf{X} \tag{3.3}$$

$$\mathbf{Q} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.4}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} x_1 M_{11} + x_2 M_{21} + x_3 M_{13} \\ x_1 M_{12} + x_2 M_{22} + x_3 M_{23} \\ x_1 M_{13} + x_2 M_{23} + x_3 M_{33} \end{pmatrix} \tag{3.5}$$

Finding $\mathbf{A}$

$$
\begin{aligned}
&= x_1^2 M_{11} + x_1 x_2 M_{12} + x_1 x_3 M_{13} + x_2 x_1 M_{12} + x_2^2 M_{22} \\
&\quad + x_2 x_3 M_{23} + x_3 x_1 M_{13} + x_3 x_2 M_{23} + x_3^2 M_{33}
\end{aligned} \tag{3.6}
$$

$$
= x_1^2 M_{11} + 2x_1 x_2 M_{12} + 2x_1 x_3 M_{13} + 2x_2 x_3 M_{23} + x_2^2 M_{22} + x_3^2 M_{33} \tag{3.7}
$$

On comparing

$$
M_{11} = 2, M_{12} = \frac{4}{2} = 2, M_{13} = \frac{2}{2} = 1, M_{23} = \frac{2}{2} = 1, M_{22} = 2, M_{33} = 3 \tag{3.8}
$$

$$
\mathbf{M} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \tag{3.9}
$$

$$
\mathbf{M} + \mathbf{I_3} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \mathbf{A} \tag{3.10}
$$

Condition

For eigen values of **A**

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \tag{3.11}$$

$$\left|\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0 \tag{3.12}$$

$$\left|\begin{pmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{pmatrix}\right| = 0 \tag{3.13}$$

$$(3-\lambda)\left((3-\lambda)(4-\lambda)-1\right) - 2\left(2(4-\lambda)-1\right) + 1\left(2-(3-\lambda)\right) = 0 \tag{3.14}$$

$$(3-\lambda)\left(\lambda^2 - 7\lambda + 12 - 1\right) - 2(8 - 2\lambda - 1) + 1(2 + \lambda - 3) = 0 \tag{3.15}$$

$$(3-\lambda)\left(\lambda^2 - 7\lambda + 11\right) - 2(7 - 2\lambda) + 1(\lambda - 1) = 0 \tag{3.16}$$

Finding eigen values

$$\left(3\lambda^2 - 21\lambda + 33 - \lambda^3 + 7\lambda^2 - 11\lambda\right) - 14 + 4\lambda + \lambda - 1 = 0 \tag{3.17}$$

$$-\lambda^3 - 10\lambda^2 - 32\lambda + 33 - 14 + 5\lambda - 1 = 0 \tag{3.18}$$

$$\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0 \tag{3.19}$$

$$(\lambda - 1)\left(\lambda^2 - 9\lambda + 18\right) = 0 \tag{3.20}$$

$$(\lambda - 1)(\lambda - 3)(\lambda - 6) = 0 \tag{3.21}$$

The eigen values of $\mathbf{A}$ are 1,3,6

Given

$$a_n = \frac{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n+1} \mathbf{Y}}{\mathbf{Y}^\top (\mathbf{M} + \mathbf{I}_3)^{n} \mathbf{Y}} \tag{3.22}$$

$$a_n = \frac{\mathbf{Y}^\top \mathbf{A}^{n+1} \mathbf{Y}}{\mathbf{Y}^\top \mathbf{A}^{n} \mathbf{Y}} \tag{3.23}$$

As $\mathbf{A}$ is symmetric

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top \text{ where } \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{3.24}$$

Simplify a_n

$$\mathbf{A}^n = \mathbf{P}\begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix}\mathbf{P}^\top \tag{3.25}$$

$$\mathbf{Y}^\top\mathbf{A}^n\mathbf{Y} = \left(\mathbf{P}^\top\mathbf{Y}\right)^\top \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix}\mathbf{P}^\top\mathbf{Y} = \mathbf{v}^\top \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 6^n \end{pmatrix}\mathbf{v} \tag{3.26}$$

where

$$\mathbf{v} = \mathbf{P}^\top\mathbf{Y} \tag{3.27}$$

$$\mathbf{Y}^\top\mathbf{A}^n\mathbf{Y} = v_1^2\left(1\right)^n + v_2^2\left(3\right)^n + v_3^2\left(6\right)^n \tag{3.28}$$

which will be of the form

$$a_n = \frac{v_1^2\left(1\right)^{n+1} + v_2^2\left(3\right)^{n+1} + v_3^2\left(6\right)^{n+1}}{v_1^2\left(1\right)^n + v_2^2\left(3\right)^n + v_3^2\left(6\right)^n} \tag{3.29}$$

Conclusion

$$a_n = \frac{v_1^2 (1)^n 1 + v_2^2 (3)^n 3 + v_3^2 (6)^n 6}{v_1^2 (1)^n + v_2^2 (3)^n + v_3^2 (6)^n} \tag{3.30}$$

$$a_n = \frac{6^n \left(v_1^2 \left(\frac{1}{6}\right)^n 1 + v_2^2 \left(\frac{3}{6}\right)^n 3 + v_3^2 \left(\frac{6}{6}\right)^n 6 \right)}{6^n \left(v_1^2 \left(\frac{1}{6}\right)^n + v_2^2 \left(\frac{3}{6}\right)^n + v_3^2 \left(\frac{6}{6}\right)^n \right)} \tag{3.31}$$

$$lim_{n\to\infty} a_n = \frac{0 + 0 + v_3^2 (6)}{0 + 0 + v_3^2} = 6 \tag{3.32}$$

where as $n \to \infty$

$$\frac{1}{6} \to 0, \frac{3}{6} \to 0 \tag{3.33}$$

Hence

$$lim_{n\to\infty} a_n = 6 \tag{3.34}$$