2.9.4

EE25BTECH11052 - Shriyansh Kalpesh Chawda

Question:
If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{a} \cdot \mathbf{b} = 1$, and $\mathbf{a} \times \mathbf{b} = \hat{j} - \hat{k}$, then find $|\mathbf{b}|$. (12, 2022)
Solution:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{a}^\top \mathbf{b} = 1 \tag{0.1}$$

$$\mathbf{a}^\top (\mathbf{a} \times \mathbf{b}) = 0 \tag{0.2}$$

And the key identity:

$$\begin{pmatrix} \mathbf{a}^\top \\ (\mathbf{a} \times \mathbf{b})^\top \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.3}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.4}$$

Forming augmented matrix and find its Reduced Row Echelon Form (RREF).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \tag{0.5}$$

From the RREF, we get the equations:

$$b_1 + 2b_3 = 1 \tag{0.6}$$

$$b_2 - b_3 = 0 \implies b_2 = b_3 \tag{0.7}$$

Let $b_2 = \lambda$,

$$b_1 = 1 - 2\lambda \tag{0.8}$$

Thus, the vector $\mathbf{b}$ is:

$$\mathbf{b} = \begin{pmatrix} 1 - 2\lambda \\ \lambda \\ \lambda \end{pmatrix} \tag{0.9}$$

$$|\mathbf{b}|^2 = (1 - 2\lambda)^2 + \lambda^2 + \lambda^2 \tag{0.10}$$

$$|\mathbf{b}| = \sqrt{1 - 4\lambda + 6\lambda^2} \tag{0.11}$$