

10.7.53

EE25BTECH11004 - Aditya Appana

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Question

Let C_1 and C_2 be respectively, the parabolas $x^2 = y - 1$ and $y^2 = x - 1$. Let \mathbf{P} be any point on C_1 and \mathbf{Q} be any point on C_2 . Let \mathbf{P}_i and \mathbf{Q}_i be the reflections of \mathbf{P} and \mathbf{Q} respectively with respect to the line $y = x$. Prove that \mathbf{P}_i lies on C_2 , \mathbf{Q}_i lies on C_1 , and $\mathbf{PQ} \geq \min(\mathbf{PP}_i, \mathbf{QQ}_i)$. Hence or otherwise determine points \mathbf{P}_0 and \mathbf{Q}_0 on the parabolas C_1 and C_2 respectively such that $\mathbf{P}_0\mathbf{Q}_0 \leq \mathbf{PQ}$ for all pairs of points (\mathbf{P}, \mathbf{Q}) with \mathbf{P} on C_1 and \mathbf{Q} on C_2 .

Solution

The representation of a conic in vector form is

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

$x^2 = y - 1$ represented in this form is:

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{x} + 1 = 0 \quad (2)$$

$y^2 = x - 1$ represented in this form is:

$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}^T \mathbf{x} + 1 = 0 \quad (3)$$

Given \mathbf{P} lies on (1), and given \mathbf{P}_i is the mirror image of \mathbf{P} with respect to line $y = x$. The mirror image \mathbf{P}_i can therefore be represented as:

$$\mathbf{P}_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} \quad (4)$$

If we want to prove that \mathbf{P}_i lies on C_2 , we need to show that \mathbf{P}_i satisfies C_2 .

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} \right)^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 1 = 0 \quad (5)$$

$$\mathbf{P}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} + 1 = 0 \quad (6)$$

$$\mathbf{P}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{P} + 2 \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}^T \mathbf{P} + 1 = 0 \quad (7)$$

(7) is the same as C_1 , so \mathbf{P}_i satisfies C_2 . In a similar manner, it can be proved that \mathbf{Q}_i lies on C_1 .

We now need to prove $\mathbf{PQ} \geq \min(\mathbf{PP}_1, \mathbf{QQ}_1)$. Take $\mathbf{P}(1, 2)$ and $\mathbf{Q}(5, 2)$.

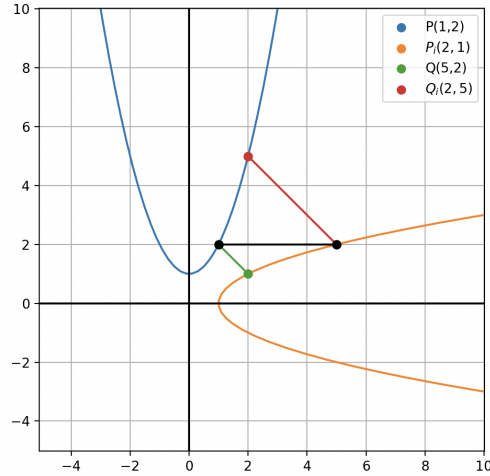


Figure 1: Plot

$$\min(\mathbf{PP}_1, \mathbf{QQ}_1) = \mathbf{PP}_i \quad (8)$$

$$\|\mathbf{PP}_i\| = \|\mathbf{P} - \mathbf{P}_i\| = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{P} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P} \right\| = \sqrt{2} \left\| \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \mathbf{P} \right\| \quad (9)$$

Since the matrix is orthogonal, this equals:

$$\sqrt{2} \|\mathbf{P}\| = \sqrt{2} \times \sqrt{1^2 + 2^2} = \sqrt{10} \quad (10)$$

$$\|\mathbf{PQ}\| = \sqrt{(5-1)^2 + (2-2)^2} = 4 \quad (11)$$

$$\mathbf{PQ} \geq \mathbf{PP}_i \quad (12)$$

We now need to find \mathbf{P}_0 and \mathbf{Q}_0 on the parabolas C_1 and C_2 respectively such that $\mathbf{P}_0\mathbf{Q}_0 \leq \mathbf{PQ}$ for all pairs of points (\mathbf{P}, \mathbf{Q}) with \mathbf{P} on C_1 and \mathbf{Q} on C_2 .

The line of shortest distance will be normal to both parabolas, and it will be perpendicular to the line $y = x$. Therefore the tangents at the point of intersection of line of shortest distance and the parabola will have same slope as $y = x$, $m = 1$.

The equation of tangent to a conic in vector form can be expressed as:

$$\mathbf{m}^T(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \quad (13)$$

Where \mathbf{m} is slope of tangent, and $\mathbf{q} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the point of contact. Substituting the values from (2), we get:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} \right) = 0 \quad (14)$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{q} + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} = 0 \quad (15)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2} = 0 \quad (16)$$

$$x - \frac{1}{2} = 0 \quad (17)$$

$$x = \frac{1}{2} \quad (18)$$

Substituting the value of x in (2), we get $y = \frac{5}{4}$. Therefore:

$$\mathbf{P}_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{4} \end{pmatrix} \quad (19)$$

$$\mathbf{Q}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{P}_0 = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{2} \end{pmatrix} \quad (20)$$