2.9.4

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Question

If $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{a} \cdot \mathbf{b} = 1$, and $\mathbf{a} \times \mathbf{b} = \hat{j} - \hat{k}$, then find $|\mathbf{b}|$.

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Solution

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{a}^\top \mathbf{b} = 1 \tag{1}$$

$$\mathbf{a}^\top (\mathbf{a} \times \mathbf{b}) = 0 \tag{2}$$

And the key identity:

$$\begin{pmatrix} \mathbf{a}^\top \\ (\mathbf{a} \times \mathbf{b})^\top \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4}$$

Solution

Let $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

From the two equations:

$$b_1 + b_2 + b_3 = 1 \tag{5}$$

$$b_2 - b_3 = 0 \implies b_2 = b_3 \tag{6}$$

Substituting $b_2 = b_3$ into the first equation:

$$b_1 + b_2 + b_2 = 1 \tag{7}$$

$$b_1 + 2b_2 = 1 \tag{8}$$

$$b_1 = 1 - 2b_2 \tag{9}$$

$$\mathbf{b} = \begin{pmatrix} 1 - 2\lambda \\ \lambda \\ \lambda \end{pmatrix} \tag{10}$$

where $\lambda = b_2$.

Solution

Therefore:

$$|\mathbf{b}|^2 = (1-2\lambda)^2 + \lambda^2 + \lambda^2 \tag{11}$$
$$= 1 - 4\lambda + 4\lambda^2 + \lambda^2 + \lambda^2 \tag{12}$$
$$= 1 - 4\lambda + 6\lambda^2 \tag{13}$$

$$|\mathbf{b}| = \sqrt{1 - 4\lambda + 6\lambda^2} \tag{14}$$

Answer: $|\mathbf{b}| = \sqrt{1 - 4\lambda + 6\lambda^2}$ where λ is a parameter.