8.4.23

EE25BTECH11020 - Darsh Pankaj Gajare

October 11, 2025

Question:
The curve described parametrically by $x = t^2 + t + 1$ and $y = t^2 - t + 1$ represents:

(A) a pair of straight lines

(B) an ellipse

(C) a parabola

(D) a hyperbola

Solution:

Table

x	$\begin{pmatrix} x \\ y \end{pmatrix}$
a	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
b	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
c	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The parametric form can be written as

$$\mathbf{x} = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}. \tag{0.1}$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^\top \begin{pmatrix} t^2 \\ t \end{pmatrix} + \mathbf{c} \tag{0.2}$$

$$\mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} t^2 \\ t \end{pmatrix} + \mathbf{c} \tag{0.3}$$

$$\begin{pmatrix} t^2 \\ t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) \tag{0.4}$$

$$\text{Let } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{0.5}$$

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\ \mathbf{z} = \mathbf{x} - \mathbf{c}. \tag{0.6}$$

$$\text{Then}\ \begin{pmatrix}t^2\\ t\end{pmatrix} = \mathbf{Mz}, \quad t^2 = \mathbf{e}_1^\top \mathbf{Mz}, \quad t = \mathbf{e}_2^\top \mathbf{Mz}. \tag{0.7}$$

$$\text{Eliminate } t: \quad \mathbf{e}_1^\top \mathbf{Mz} = \left(\mathbf{e}_2^\top \mathbf{Mz}\right)^2. \tag{0.8}$$

$$\text{Define } \mathbf{w} = \mathbf{Mz} \ \Rightarrow\ \mathbf{e}_1^\top \mathbf{w} = \left(\mathbf{e}_2^\top \mathbf{w}\right)^2. \tag{0.9}$$

$$\text{In matrix form:}\ \ \mathbf{z}^\top \mathbf{M}^\top \mathbf{e}_1 \mathbf{e}_1^\top \mathbf{Mz} - \mathbf{z}^\top \mathbf{M}^\top \mathbf{e}_2 \mathbf{e}_2^\top \mathbf{Mz} = 0. \tag{0.10}$$

$$\text{Let } \mathbf{E} = \mathbf{e}_1\mathbf{e}_1^\top - \mathbf{e}_2\mathbf{e}_2^\top = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \mathbf{Q} = \mathbf{M}^\top \mathbf{EM}. \tag{0.11}$$

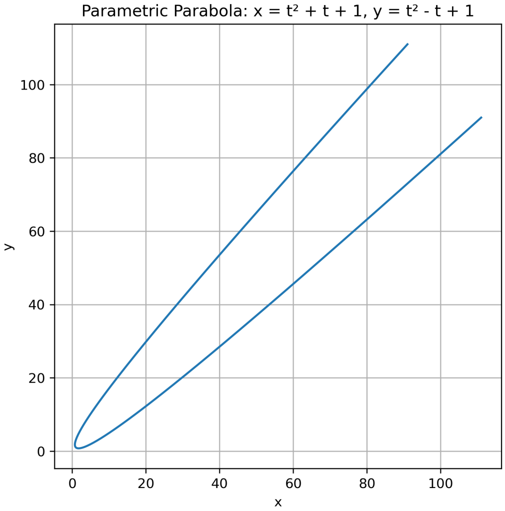
$$\mathbf{Q} = \left(\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right)^{\top}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{0.12}$$

$$\mathbf{z}^{\top}\mathbf{Q}\mathbf{z} = 0, \quad \mathbf{z} = \begin{pmatrix}x\\ y\end{pmatrix} - \begin{pmatrix}1\\ 1\end{pmatrix}. \tag{0.13}$$

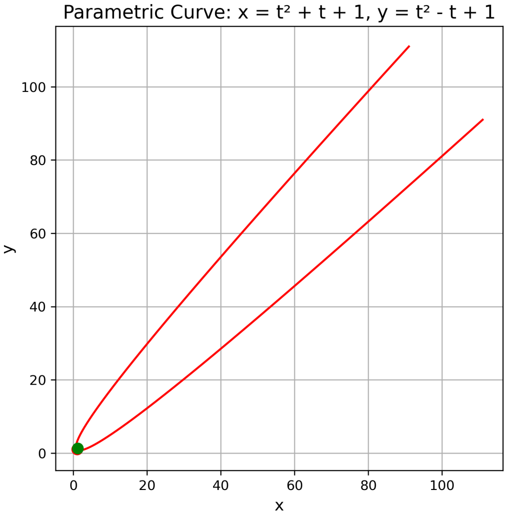
$$(x-1)(y-1) = \frac{1}{2}(y-x)^2 \iff (x-y)^2 = 2(x+y-2). \tag{0.14}$$

$$\text{Thus the conic is a parabola.} \tag{0.15}$$

Since $\Delta = 0$ the conic is a parabola.



Plot using C libraries:



Plot using Python: