

Matrices in Geometry - 10.5.3

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Problem Statement

Draw a pair of tangents to a circle of radius 5 cm which are inclined to each other at an angle of 60° .

Solution

Let the center be the origin. Then the circle with radius 5 5cm iw

$$\mathbf{C} : \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = -25 \quad (2)$$

Let the tangents be drawn from an external point \mathbf{h} . Line segment from O to h bisects the angle between the tangents and it forms two right angled triangles. So that, we have

$$\sin \frac{60^\circ}{2} = \frac{r}{\|\mathbf{h}\|} = \frac{5}{\|\mathbf{h}\|} \quad (3)$$

$$\implies \|\mathbf{h}\| = \frac{5}{\sin 30^\circ} = 10 \quad (4)$$

Solution

So the point h should be at a distance of $10cm$ from the origin(centre).

Let the point be

$$\mathbf{h} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} \quad (5)$$

Calculating the matrix Σ

$$\Sigma = (\mathbf{V}\mathbf{h} + \mathbf{u})(\mathbf{V}\mathbf{h} + \mathbf{u})^\top - g(\mathbf{h})\mathbf{V} \quad (6)$$

$$g(\mathbf{h}) = \mathbf{h}^\top \mathbf{V}\mathbf{h} + 2\mathbf{u}^\top \mathbf{h} + f = \|\mathbf{h}\|^2 + f = 100 - 25 = 75 \quad (7)$$

$$\Sigma = \mathbf{h}\mathbf{h}^\top - g(\mathbf{h})\mathbf{V} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} (10 \ 0) - 75 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (8)$$

$$\implies \Sigma = \begin{pmatrix} 25 & 0 \\ 0 & -75 \end{pmatrix} \quad (9)$$

Solution

The eigen values of the matrix Σ are

$$\lambda_1 = 25 \quad (10)$$

$$\lambda_2 = -75 \quad (11)$$

The direction vectors for the tangents

$$\begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm \sqrt{|\lambda_1|} \end{pmatrix} = \begin{pmatrix} 5\sqrt{3} \\ \pm 5 \end{pmatrix} \quad (12)$$

$$\Rightarrow \mathbf{m}_1 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \quad (13)$$

Solution

As the tangents pass through \mathbf{h} with direction vectors \mathbf{m}_1 and \mathbf{m}_2 , using the normal form of the line $\mathbf{n}^\top(\mathbf{x} - \mathbf{h}) = 0$. Their equations are

$$(1 \quad -\sqrt{3}) \left(\mathbf{x} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right) = 0 \quad (14)$$

$$(1 \quad \sqrt{3}) \left(\mathbf{x} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right) = 0 \quad (15)$$

Plot for 10.5.3

