Python 9 assignment
The Digital Fourier Transform

EE16B014
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Abstract

In this report we are going to discuss about Digital Fourier Transform of periodic signals by using 'fft' command we will plot the magnitude and phase spectrum and compare with the original function.We will also see how the plot varies by varying the sampling period.

1 Fourier Transform:

We can determine function's time domain from its frequency domain and viceversa.
The relation for $F(j\omega), f(t)$:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

When the function is periodic the Fourier transform becomes Fourier series as shown below($t_0$can also be taken as 0):

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$$
$$c_n = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} f(t)e^{-jnt}dt$$

If $f(t)$ is discontinuous ,say $f[n]$ then we define Z transform of function as shown below:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

By just replacing z with $e^{j\theta}$we get:

$$F(e^{j\theta}) = \sum_{n=-\infty}^{\infty} f[n] e^{-jn\theta}$$

The $F(z)$ looks like a Fourier series with coefficients as $f[n]$.
$F(e^{j\theta})$ is also called as Digital Spectrum of the samples $f[n]$ and also called as the DTFT of f[n].

If the $f[n]$ is periodic with period N,i.e:

$$f[n+N] = f[n] \forall n$$

The then DTFT of $f[n]$ also becomes periodic with same period N:

$$F[n] = \sum_{n=0}^{N-1} f[n] exp\left(-2\pi \frac{nk}{N} j\right) = \sum_{n=0}^{N-1} f[n] W^{-n}$$
$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] W^{-nk}$$

Here $W = e^{\frac{-2\pi j}{N}}$ and $F[k]$ is same as $F(e^{j\theta})$ where $\theta = \frac{2\pi k}{N}$ so we can write $F\left(e^{j2\pi k/N}\right)$ can be written as sum of the periodic repetitions:

$$F\left(e^{j2\pi k/N}\right) = \sum_{n=0}^{N-1} f[n] exp\left(-2\pi \frac{nk}{N} j\right) + \ldots$$

The repetitions are just helpful in building up the impulses,We can say that DFT is a sampled version of the DTFT.

2 Examples:

There are actually two commands available in python for doing forward Fourier transform and inverse Fourier transform as shown below:

```
numpy.fft.fft()
numpy.fft.ifft()
```

Now lets take an example function $y = sin(5x)$ which can be written as:

$$y = sin(x) = \frac{e^{j5x} - e^{-j5x}}{2j}$$

So the Fourier transform spectrum would look like:

$$Y(j\omega) = \frac{1}{2j}\left[\delta(\omega - 5) - \delta(\omega + 5)\right]$$

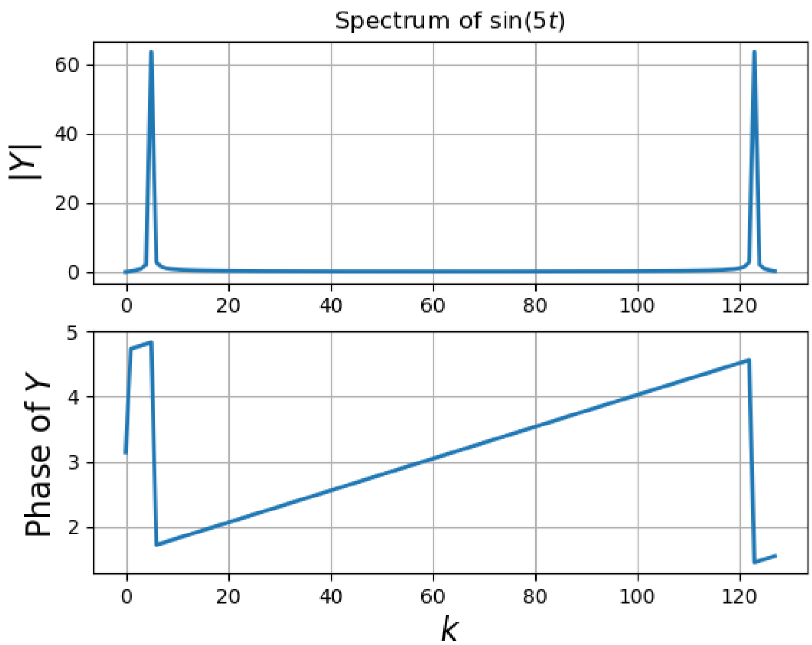
Now we get the spectrum of $sin(5x)$:

```
x=linspace(0,2*pi,128)
y=sin(5*x)
Y=fft(y)
```

Now we will plot the spectrum stored in Y:

```
figure()
subplot(2,1,1)
plot(abs(Y),lw=2)
grid(True)

subplot(2,1,2)
plot(unwrap(angle(Y)),lw=2)
grid(True)
show()
```

Here is the plot:



Its not exactly the plot we are expecting to get, we will do small changes in the code as shown below:

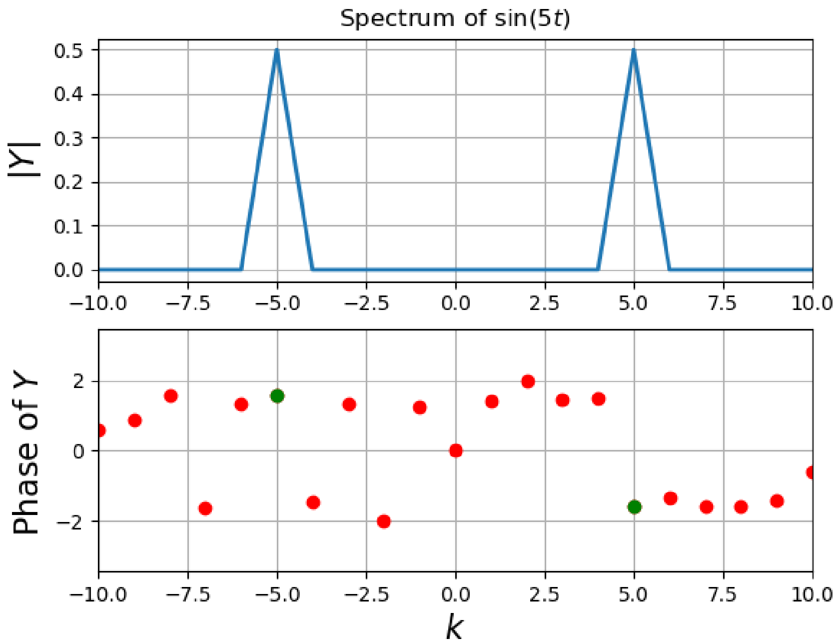
```
y=sin(5*x)
Y=fftshift(fft(y))/128.0
w=linspace(-64,63,128)
```

And the code for plotting the graph is:

```
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\sin(5t)$")
grid(True)
subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
```

```
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
savefig("fig9-2.png")
show()
```

Here we are plotting the graph of phases for magnitude greater than10^{-3} with separate colour:



So we got peaks at $-5, 5$ as derived from the equations and phase as $\frac{\pi}{2}$, $\frac{-\pi}{2}$ respectively.

Now lets take another example $f(t) = (1 + 0.1cost)cos10t$, In this we stretch the spectrum by keeping sampling time constant.
Solving the Fourier spectrum of given $f(t)$:

$$f(t) = \left(1 + 0.1\left(\frac{e^{0.1tj} + e^{-0.1tj}}{2}\right)\right)\left(\frac{e^{10tj} + e^{-10tj}}{2}\right)$$
$$f(t) = 0.5\left(e^{10tj} + e^{-10tj}\right) + 0.025\left(e^{11tj} + e^{-11tj} + e^{9tj} + e^{-9tj}\right)$$

So we should get 6 peaks at:

- $\omega = 10, -10$ with amplitude 0.5 with phase=0;
- $\omega = 11, -11, 9, -9$ with amplitude 0.025 with phase=0;

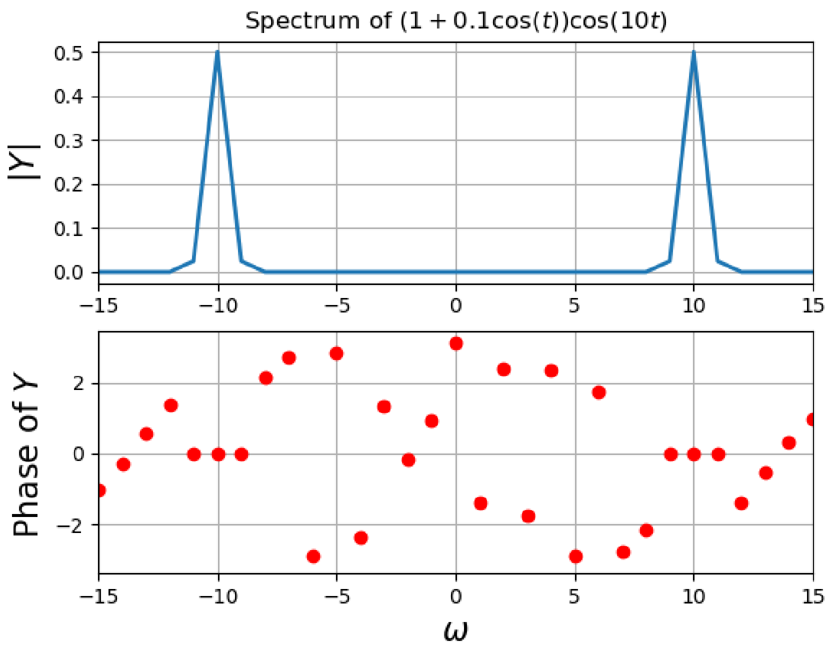
, now lets take sampling period as 128:

```
t=linspace(0,2*pi,129);
t=t[:-1]
y=(1+0.1*cos(t))*cos(10*t)
Y=fftshift(fft(y))/128.0
w=linspace(-64,63,128)
```

Lets plot the graph for this :

```
figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-15,15])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\left(1+0.1\cos\left(t\right)\right)\cos\left(1
grid(True)

subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
xlim([-15,15])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$\omega$",size=16)
grid(True)
savefig("fig9-4.png")
show()
```

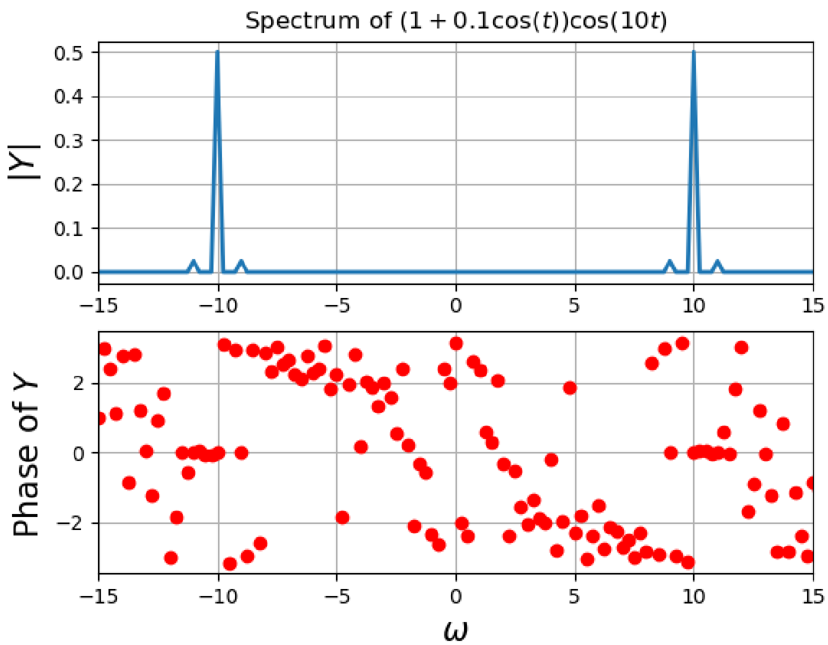
The graph:



In the above graph we can't see the six peaks, only two are clearly visible,So Now lets make a small change in no of points taken,by increasing the interval of sampling by keeping sampling time fixed:

```
t=linspace(-4*pi,4*pi,513);
t=t[:-1]
y=(1+0.1*cos(t))*cos(10*t)
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513);
w=w[:-1]
```

We can see the six peaks clearly



Here we have just increased no.of points of ω to see the peaks clearly.

3 Spectrum's of $sin^3(t), cos^3(t)$:

Now first we will discuss the spectrum of $sin^3(t)$:
$sin^3(t)$ can be written as :

$$y = sin^3(t) = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)^3 = -\frac{1}{8j}\left(e^{3jt} - 3e^{jt} + 3e^{-jt} - e^{-3jt}\right)$$

So the Fourier transform is given by:

$$Y(j\omega) = -\frac{1}{8j}\left(\left[\delta(\omega-3) - \delta(\omega+3)\right] + 3\left[\delta(\omega+1) - \delta(\omega-1)\right]\right)$$

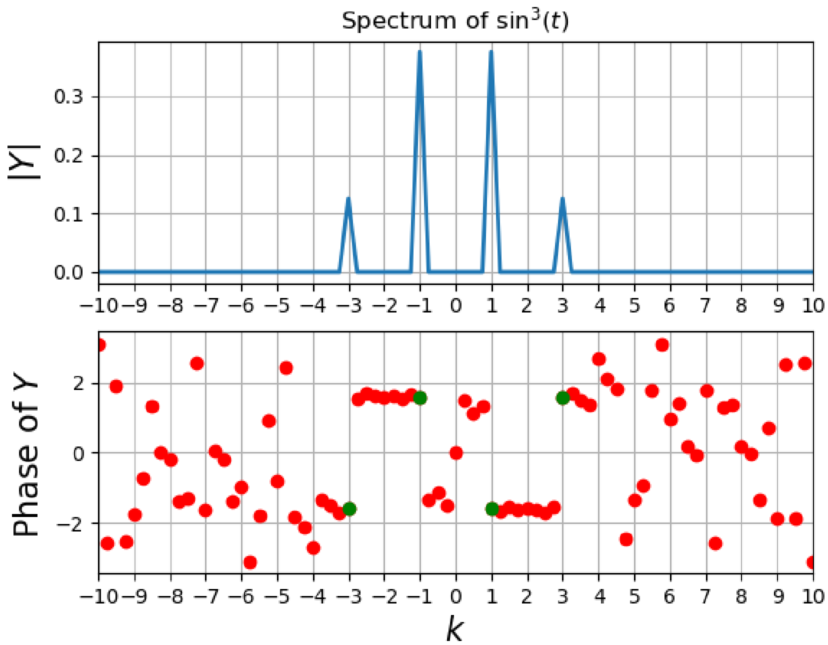
So there will be four peaks at:

- $\omega = 3, -3$ with magnitude $\frac{1}{8}$ with phase $+\frac{\pi}{2}, -\frac{\pi}{2}$ respectively;
- $\omega = -1, 1$ with magnitude $\frac{3}{8}$ with phase $+\frac{\pi}{2}, -\frac{\pi}{2}$ respectively;

Okay now lets see the code for plotting the spectrum:

```
y=sin(x)**3
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513)
w=w[:-1]

figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
ylim([0,0.75])
title(r"Spectrum of $\sin^3(5t)$")
grid(True)
xticks(np.arange(-10, 11, 1.0))
yticks(np.arange(0,0.837,0.1875))

subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
xticks(np.arange(-10, 11, 1.0))
show()
```

Here we can see the plot:



There are four poles as we discussed having high magnitude at $+3, -3$ and more amplitude at $+1, -1$.In the plot green dots represent the phase at the points required.

Now lets take a look at $cos^3(t)$:

$$y = cos^3(t) = \left(\frac{e^{jt} + e^{-jt}}{2}\right)^3 = \frac{1}{8}\left(e^{3jt} + 3e^{jt} + 3e^{-jt} + e^{-3jt}\right)$$

The Fourier transform looks like:

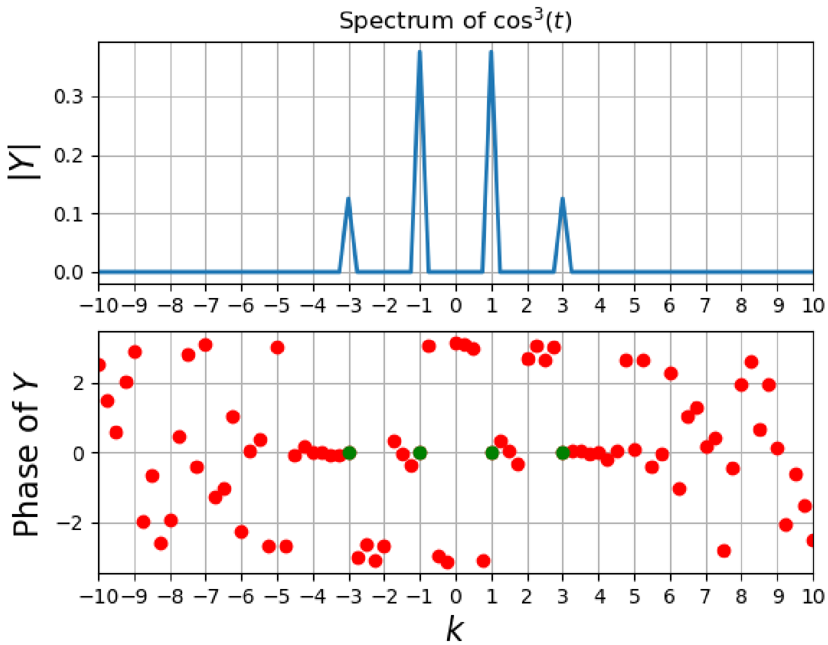
$$Y(j\omega) = \frac{1}{8}\left(\left[\delta(\omega - 3) + \delta(\omega + 3)\right] + 3\left[\delta(\omega + 1) + \delta(\omega - 1)\right]\right)$$

So there are four peaks at:

- $\omega = 3, -3$ having magnitude as $\frac{1}{8}$ with phase at each peak as 0;
- $\omega = 1, -1$ having magnitude as $\frac{3}{8}$ with phase at each peak as 0.

```
from pylab import*
x=linspace(-4*pi,4*pi,513);
x=x[:-1]

y=cos(x)**3
Y=fftshift(fft(y))/512.0
w=linspace(-64,64,513)
w=w[:-1]

figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\cos^3(t)$")
grid(True)
xticks(np.arange(-10, 11, 1.0))


subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-3)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
xticks(np.arange(-10, 11, 1.0))
show()
```

Now lets see the plot:



The graph is showing the peaks at $-3, 3, -1, 1$ with the expected amplitudes of $\frac{1}{8}, \frac{3}{8}$ and all the four has phase as 0.

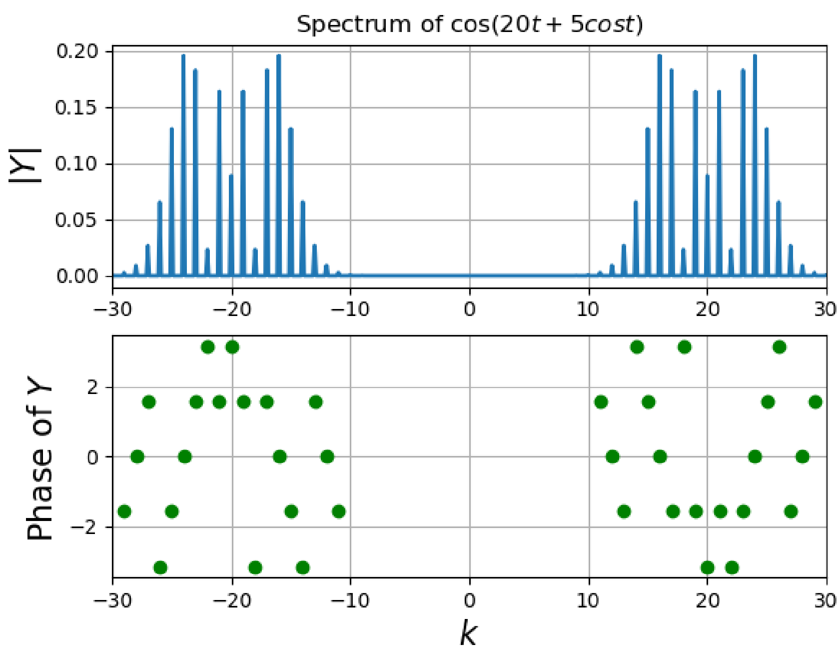
Spectrum of $cos(20t + 5cos(t))$:

Here we can't calculate the Fourier transform by hand,so we will directly use the code and see what happens :

```
from pylab import*
x=linspace(-8*pi,8*pi,1025);
x=x[:-1]

y=cos(20*x+5*cos(x))
Y=fftshift(fft(y))/1024.0
w=linspace(-64,64,1025)
w=w[:-1]

figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
```

```
xlim([-30,30])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $\cos(20t+5cost)$")
grid(True)


subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-6)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-30,30])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
savefig("fig9-2.png")
show()
```

Here we will plot the phase plot of those having magnitude greater than 10^{-3}.



Here we can clearly tell one thing that the phase is an odd function and magnitude is even function.

Spectrum of $exp(\frac{-t^2}{2})$:

Lets calculate the Fourier transform

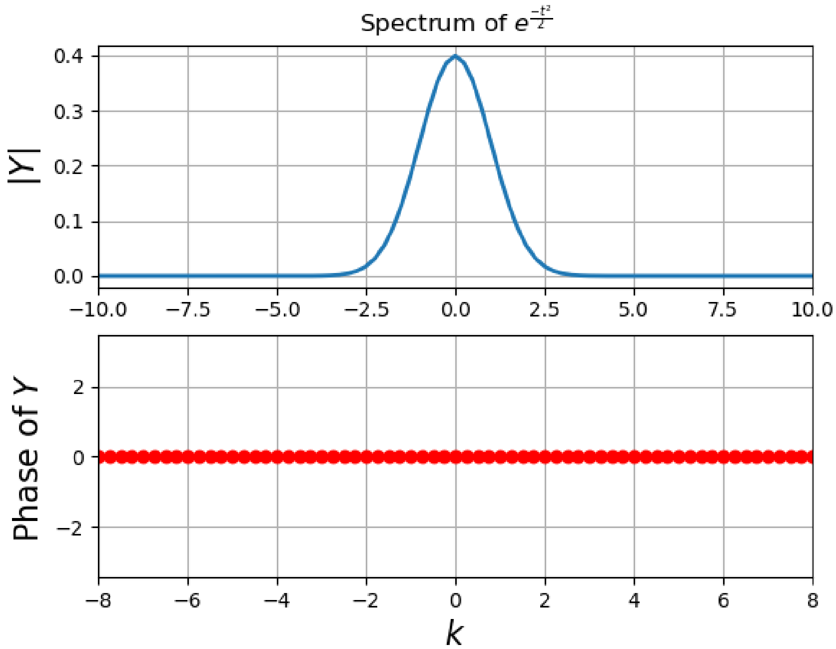
$$y = e^{\frac{-t^2}{2}}$$
$$Y(j\omega) = \frac{1}{\sqrt{2\pi}}e^{\frac{-\omega^2}{2}}$$

which means the Fourier transforms looks same as the given function with a multiplication factor of $\frac{1}{\sqrt{2\pi}}$.

```
from pylab import*
x=linspace(-4*pi,4*pi,513);
x=x[:-1]


y=e**(-x*x/2.0)
Y=4*fftshift(fft(fftshift(y)))/512
w=linspace(-64,64,513)
w=w[:-1]

figure()
subplot(2,1,1)
plot(w,abs(Y),lw=2)
xlim([-10,10])
ylabel(r"$|Y|$",size=16)
title(r"Spectrum of $e^\frac{-t^2}{2}}$")
grid(True)


subplot(2,1,2)
plot(w,angle(Y),'ro',lw=2)
ii=where(abs(Y)>1e-6)
plot(w[ii],angle(Y[ii]),'go',lw=2)
xlim([-10,10])
ylabel(r"Phase of $Y$",size=16)
xlabel(r"$k$",size=16)
grid(True)
```

```
savefig("fig9-2.png")
show()    #/sqrt(2*pi)
```

I have approximated the spectrum to six digits :



Here we can see the maximum amplitude is nothing but $\frac{1}{\sqrt{2\pi}}$.Since the input is a *even symmetric function* phase is zero everywhere in the spectrum.