

GATE-2023 EC Q.25

EE23BTECH11214 - Harsha Vardhan Kumar

Question: In the context of signals and systems, determine the phase cross-over frequency of the open-loop transfer function

$$G(s) = \frac{k \cdot s \cdot (1 + sT_1) \cdot (1 + sT_2)}{s}$$

with positive constants k, T_1, T_2 are positive constants. The phase crossover frequency, in rad/s, is [EC, GATE-2023]

- (a) $\frac{1}{\sqrt{T_1 T_2}}$
- (b) $\frac{1}{T_1 T_2}$
- (c) $\frac{1}{T_1 \sqrt{T_2}}$
- (d) $\frac{1}{\sqrt{T_2 T_1}}$

Solution: finding the frequency at which the phase of the transfer function becomes -180° or π radians.

The phase of $G(s)$

$$\begin{aligned} \angle G(s) &= \angle(k s (1 + sT_1)(1 + sT_2)) - \angle s \quad (1) \\ &= \angle k s + \angle(1 + sT_1) + \angle(1 + sT_2) - \angle s \end{aligned}$$

The phase contribution of each term can be calculated as follows:

$$\angle k s = \angle k + \angle s = 0 + \frac{\pi}{2} = \frac{\pi}{2} \text{ radians} \quad (2)$$

$$\angle(1 + sT_1) = \tan^{-1}(0) + \tan^{-1}(sT_1) = \tan^{-1}(sT_1)$$

$$\angle(1 + sT_2) = \tan^{-1}(0) + \tan^{-1}(sT_2) = \tan^{-1}(sT_2)$$

$$\angle s = \frac{\pi}{2} \text{ radians}$$

So, the total phase of $G(s)$ becomes:

$$\begin{aligned} \angle G(s) &= \frac{\pi}{2} + \tan^{-1}(sT_1) + \tan^{-1}(sT_2) - \frac{\pi}{2} \\ &= \tan^{-1}(sT_1) + \tan^{-1}(sT_2) \quad (3) \end{aligned}$$

Now, to find the phase cross-over frequency, we need to find the frequency at which the phase angle $\angle G(s)$ equals $-\pi$ radians.

$$\tan^{-1}(j\omega T_1) + \tan^{-1}(j\omega T_2) = -\pi \quad (4)$$

$$\tan^{-1}(j\omega T_1) + \tan^{-1}(j\omega T_2) = -\frac{\pi}{2} \quad (5)$$

$$\tan^{-1}(j\omega T_1) = -\frac{\pi}{2} - \tan^{-1}(j\omega T_2) \quad (6)$$

$$j\omega T_1 = \tan\left(-\frac{\pi}{2} - \tan^{-1}(j\omega T_2)\right) \quad (7)$$

$$j\omega T_1 = -\frac{1}{\tan(\tan^{-1}(j\omega T_2))} \quad (8)$$

$$j\omega T_1 = -\frac{1}{j\omega T_2} \quad (9)$$

$$\omega T_1 = \frac{1}{\omega T_2} \quad (10)$$

$$\omega^2 = \frac{1}{T_1 T_2} \quad (11)$$

$$\omega = \frac{1}{\sqrt{T_1 T_2}} \quad (12)$$

So, the phase cross-over frequency is $\frac{1}{\sqrt{T_1 T_2}}$. Therefore, the correct option is:

$$\boxed{\frac{1}{\sqrt{T_1 T_2}}}$$