Assignment

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Contents

1 Uniform Random Numbers

et U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    exrand.c
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    coeffs.h
```

Exceute the code using

```
gcc ./codes/exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

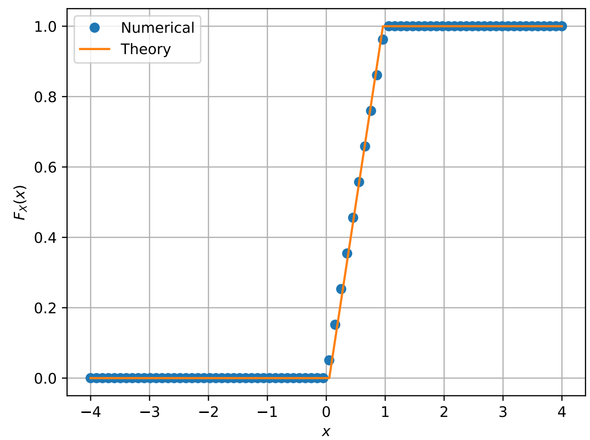

Fig. 1.2: The CDF of U

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    cdf_plot.py
```

Exceute the code using

```
python3 ./codes/cdf_plot.py
```

1.3 Find the theoretical expression for $F_U(x)$
Solution:
Given,
U is an uniform random variable

$$\implies P(U = x) = k \qquad \forall x \in [0, 1] \tag{1.2}$$

$$\implies F_U(x) = \int_0^x k dx \tag{1.3}$$

$$= kx \tag{1.4}$$

As $F_U(1) = 1$

$$k(1) = 1 \implies k = 1 \tag{1.5}$$

$$\implies F_U(x) = x \qquad \forall x \in [0, 1] \tag{1.6}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    mean_variance.c
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    coeffs.h
```

Exceute the code using

```
gcc ./codes/mean_variance -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution: As $F_U(x) = x \; \forall x \in [0,1]$

a) k=1

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.10}$$

$$\implies E[U] = \int_{0}^{1} x dx \tag{1.11}$$

$$\implies E[U] = \frac{1}{2} \tag{1.12}$$

b) k=2

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.13}$$

$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{1.14}$$

As Variance $Var[U] = E\left[U^2\right] - (E[U])^2$ (1.15)

$$Var[U] = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.16}$$

Therefore,

$$\text{Theoritical mean} = \frac{1}{2} \tag{1.17}$$

$$\text{Experimental mean} = 0.500007 \tag{1.18}$$

$$\text{Theoritical Variance} = \frac{1}{12} \tag{1.19}$$

$$\text{Experimental Variance} = 0.083301 \tag{1.20}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    exrand_gaussian.c
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    coeffs.h
```

Exceute the code using

```
gcc ./codes/exrand_gaussian.c -lm
./a.out
```

2.2 Load ”gau.dat” in python and plot the empirical CDF of X using the samples in ”gau.dat”.

What properties does a CDF have?
Solution: Consider

$$Q(x) = P(X > x) \tag{2.2}$$
$$= 1 - F_X(x) \tag{2.3}$$
$$= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.4}$$
$$\text{As } erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2} dx \tag{2.5}$$
$$Q(x) = \frac{erfc\left(\frac{x}{\sqrt{2}}\right)}{2} \tag{2.6}$$
$$\implies F_X(x) = 1 - \frac{erfc\left(\frac{x}{\sqrt{2}}\right)}{2} \tag{2.7}$$

The CDF of X is plotted in Fig. 2.2 using code below

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    cdf_gau_plot.py
```

Exceute the code using

```
python3 ./codes/cdf_gau.py
```

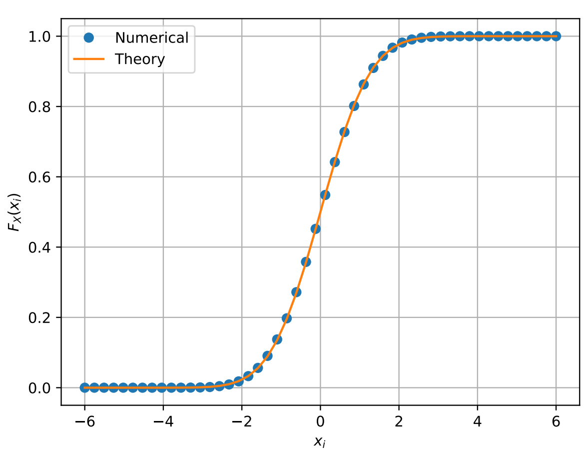

Fig. 2.2: The CDF of X

a) CDF is monotonic increasing
b)
$$\lim_{x\to-\infty} F(x) = 0. \tag{2.8}$$
$$\lim_{x\to\infty} F(x) = 1 \tag{2.9}$$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.10}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    pdf_gau_plot.py
```

Exceute the code using

```
python3 ./codes/pdf_gau_plot.py
```

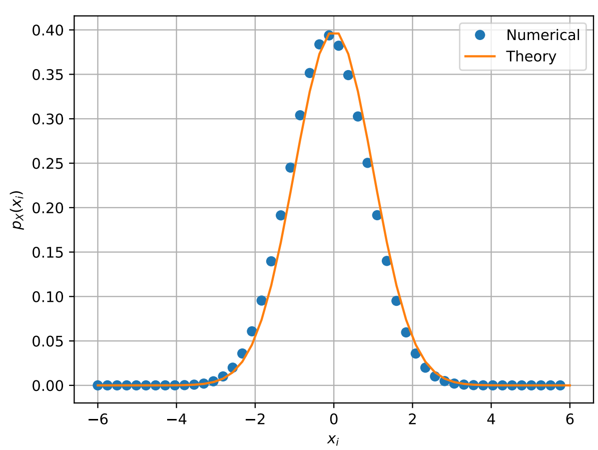

Fig. 2.3: The PDF of X

a) $\forall x \in \mathbb{R}\ p(x) \geq 0$
b)
$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{2.11}$$

2.4 Find the mean and variance of X by writing a C program.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.12}$$

repeat the above exercise theoretically.
Solution: As $dF_X(x) = p_X(x)dx$

a) Mean :

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.13}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x exp\left(-\frac{x^2}{2}\right) dx \tag{2.14}$$
$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\left(exp\left(-\frac{x^2}{2}\right)\right) \tag{2.15}$$
$$= 0 \tag{2.16}$$

b) Variance

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.17}$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 exp\left(-\frac{x^2}{2}\right) dx \tag{2.18}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 exp\left(-\frac{x^2}{2}\right) dx \tag{2.19}$$
$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1 \tag{2.20}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    exrand_other.c
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    cdf_other_plot.py
```

Exceute the code using

```
gcc ./codes/exrand_other.c −lm
./a.out
python3 ./codes/cdf_other_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:
Given,

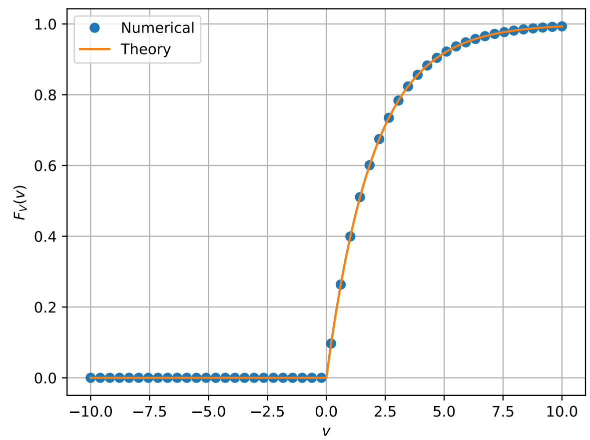

Fig. 3.1: The CDF of V

$V = -2\ln(1-U)$ and from (1.6)

$$F_V(x) = P(V \le x) \tag{3.2}$$
$$= P(-2\ln(1-U) \le x) \tag{3.3}$$
$$= P\left(U \le 1 - e^{-\frac{x}{2}}\right) \tag{3.4}$$
$$\implies F_V(x) = 1 - e^{-\frac{x}{2}} \tag{3.5}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    T_exrand.c
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    coeffs.h
```

Exceute the code using

```
gcc ./codes/T_exrand.c −lm
./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/
    HarshaVardhanReddyG/
```

A1110_Assignment/blob/main/codes/
tri_cdf.py

Exceute the code using

```
python3 ./codes/tri_cdf.py
```

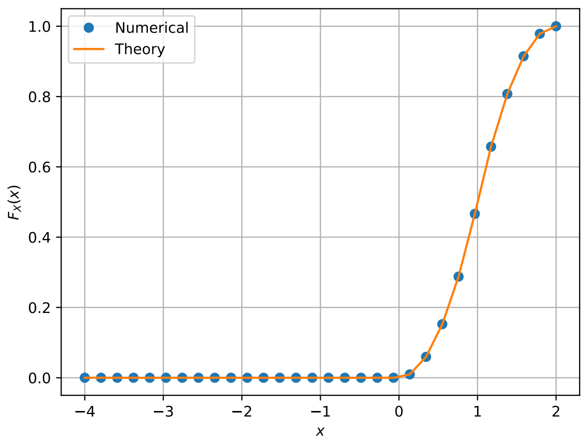

Fig. 4.2: The CDF of T

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/
    HarshaVardhanReddyG/
    A1110_Assignment/blob/main/codes/
    tri_pdf.py
```

Exceute the code using

```
python3 ./codes/tri_pdf.py
```

4.4 Find the theoritical expressions for CDF and PDF of T.
Solution: Given,

$$T = U_1 + U_2 \tag{4.2}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U_1}(u_1) p_{U_2}(u_2) du_1 \tag{4.3}$$

$$\text{As } p_{U_1}(u_1) = p_{U_2}(u_2) = p_U(u) \tag{4.4}$$

$$p_T(t) = \int_{-\infty}^{t} p_U(u) p_U(t-u) du \tag{4.5}$$

i) If $t <= 1$

$$p_T(t) = \int_0^t p_U(t-u) du \tag{4.6}$$

$$p_T(t) = \int_0^t du = t \tag{4.7}$$

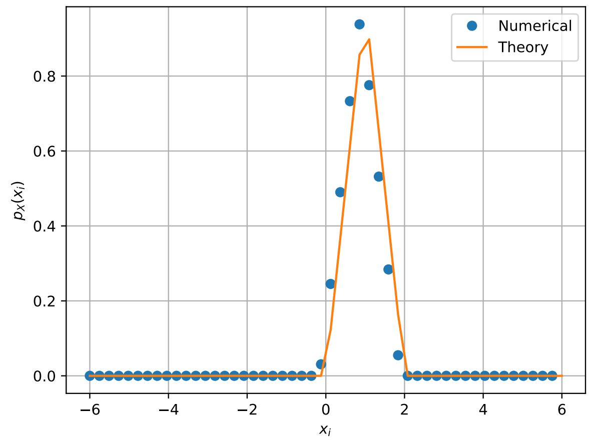

Fig. 4.2: The PDF of T

ii) If $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u) du \tag{4.8}$$

$$p_T(t) = \int_{t-1}^1 du = 2 - t \tag{4.9}$$

Therefore

$$P_T(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

And We know that

$$F_X(x) = \int_{-\infty}^{x} p_X(x) dx \tag{4.10}$$

Therefore

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot.
Solution: Results are verified and shown in (4.2) and (4.2)

5 Maximal Likelihood

5.1 Generate

$$Y = AX + N \tag{5.1}$$

Where $A = 5dB$, $X_1\{1, -1\}$, is Bernouli and $N \sim \mathcal{N}(0, 1)$

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = P(\hat{X} = -1|X = 1), \tag{5.2}$$
$$P_{e|1} = P(\hat{X} = 1|X = -1) \tag{5.3}$$

5.5 Find P_e.

5.6 Verifying by plotting the theoritical P_e

6 Guassion to Other

6.1 Let $X_1 = \mathcal{N}(0, 1)$, $X_2 = \mathcal{N}(0, 1)$ .Plot the PDF and CDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find α

6.3 Plot CDF and PDF of

$$A = \sqrt{V} \tag{6.2}$$

7 Conditional Probability

7.1 Plot

$$P_e = P(\hat{X} = -1|X = 1) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

7.2