

Assignment

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CONTENTS

1	Uniform Random Numbers	1
2	Central Limit Theorem	2
3	From Uniform to Other	4
4	Triangular Distribution	4
5	Maximal Likelihood	5
6	Gaussian to Other	6
7	Conditional Probability	6

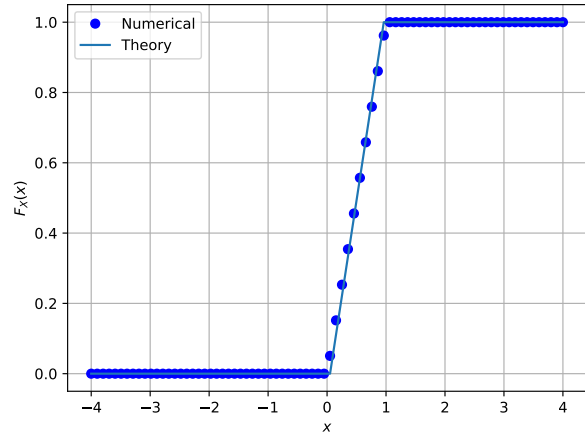


Fig. 1.2: The CDF of U

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
exrand.c
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
coeffs.h
```

Execute the code using

```
gcc ./codes/exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
cdf_plot.py
```

Execute the code using

```
python3 ./codes/cdf_plot.py
```

- 1.3 Find the theoretical expression for $F_U(x)$

Solution:

Given,

U is an uniform random variable

$$\Rightarrow P(U = x) = k \quad \forall x \in [0, 1] \quad (1.2)$$

$$\Rightarrow F_U(x) = \int_0^x k dx \quad (1.3)$$

$$= kx \quad (1.4)$$

As $F_U(1) = 1$

$$k(1) = 1 \Rightarrow k = 1 \quad (1.5)$$

$$\Rightarrow F_U(x) = x \quad \forall x \in [0, 1] \quad (1.6)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the C program.

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
mean_variance.c
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
coeffs.h
```

Exceute the code using

```
gcc ./codes/mean_variance -lm
./a.out
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

Solution: As $F_U(x) = x \forall x \in [0, 1]$

a) $k=1$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.10)$$

$$\Rightarrow E[U] = \int_0^1 x dx \quad (1.11)$$

$$\Rightarrow E[U] = \frac{1}{2} \quad (1.12)$$

b) $k=2$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.13)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \quad (1.14)$$

$$\text{As Variance } \text{Var}[U] = E[U^2] - (E[U])^2 \quad (1.15)$$

$$\text{Var}[U] = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.16)$$

Therefore,

$$\text{Theoritical mean} = \frac{1}{2} \quad (1.17)$$

$$\text{Experimental mean} = 0.500007 \quad (1.18)$$

$$\text{Theoritical Variance} = \frac{1}{12} \quad (1.19)$$

$$\text{Experimental Variance} = 0.083301 \quad (1.20)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
exrand_gaussian.c
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
coeffs.h
```

Exceute the code using

```
gcc ./codes/exrand_gaussian.c -lm
./a.out
```

2.2 Load "gau.dat" in python and plot the empirical CDF of X using the samples in "gau.dat".

What properties does a *CDF* have?

Solution: Consider

$$Q_X(x) = P(X > x) \quad (2.2)$$

$$= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (2.3)$$

$$\text{As } \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2} dx \quad (2.4)$$

$$Q_X(x) = \frac{\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)}{2} \quad (2.5)$$

$$\Rightarrow F_X(x) = 1 - \frac{\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)}{2} \quad (2.6)$$

The *CDF* of X is plotted in Fig. 2.2 using code below

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
cdf_gau_plot.py
```

Exceute the code using

```
python3 ./codes/cdf_gau.py
```

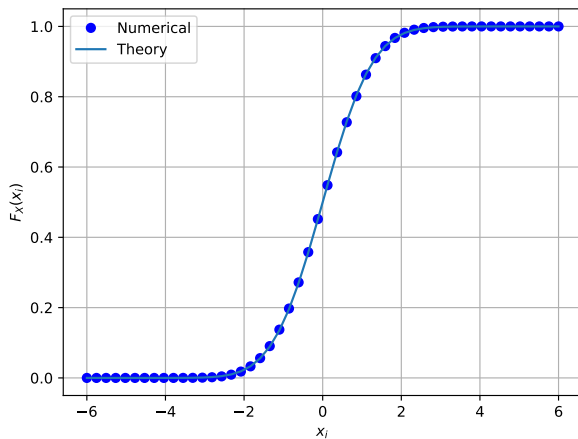


Fig. 2.2: The CDF of X

- a) CDF is monotonic increasing
- b)

$$\lim_{x \rightarrow -\infty} F(x) = 0. \quad (2.7)$$

$$\lim_{x \rightarrow \infty} F(x) = 1 \quad (2.8)$$

2.3 Load *gau.dat* in python and plot the empirical PDF of X using the samples in *gau.dat*. The

PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.9)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
pdf_gau_plot.py
```

Exceute the code using

```
python3 ./codes/pdf_gau_plot.py
```

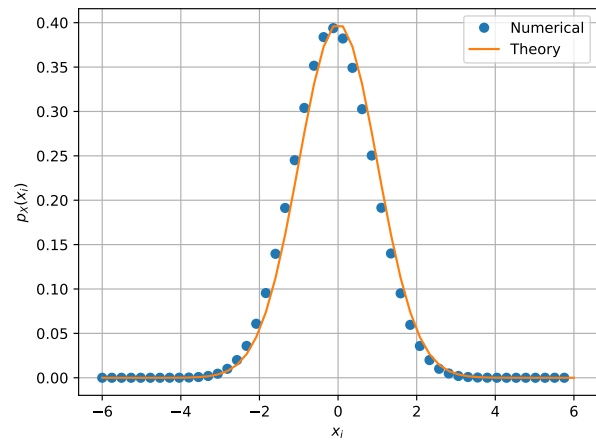


Fig. 2.3: The PDF of X

- a) $\forall x \in \mathbb{R} \ p(x) \geq 0$
- b)

$$\int_{-\infty}^{\infty} p(x) dx = 1 \quad (2.10)$$

2.4 Find the mean and variance of X by writing a C program.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.11)$$

repeat the above exercise theoretically.

Solution: As $dF_X(x) = p_X(x)dx$

a) Mean :

$$E[X] = \int_{-\infty}^{\infty} xp_X(x)dx \quad (2.12)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.13)$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\left(\exp\left(-\frac{x^2}{2}\right)\right) \quad (2.14)$$

$$= 0 \quad (2.15)$$

b) Variance

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.16)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.17)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.18)$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1 \quad (2.19)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
exrand_other.c
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
cdf_other_plot.py
```

Exceute the code using

```
gcc ./codes/exrand_other.c -lm
./a.out
python3 ./codes/cdf_other_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

Given,

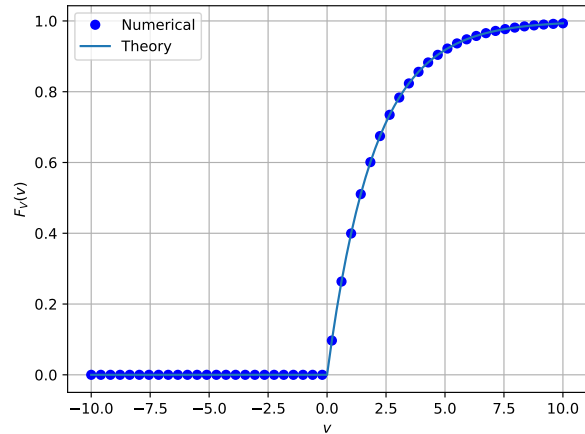


Fig. 3.1: The CDF of V

$V = -2 \ln(1 - U)$ and from (1.6)

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= P\left(U \leq 1 - e^{-\frac{x}{2}}\right) \quad (3.4)$$

$$\Rightarrow F_V(x) = 1 - e^{-\frac{x}{2}} \quad (3.5)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
T_exrand.c
wget https://github.com/
HarshaVardhanReddyG/
A1110_Assignment/blob/main/codes/
coeffs.h
```

Exceute the code using

```
gcc ./codes/T_exrand.c -lm
./a.out
```

4.2 Find the CDF of T .

Solution: The CDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/
HarshaVardhanReddyG/
```

A1110_Assignment/blob/main/codes/
tri_cdf.py

Exceute the code using

```
python3 ./codes/tri_cdf.py
```

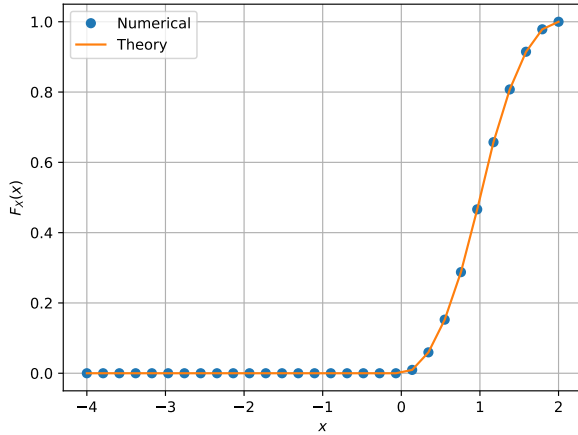


Fig. 4.2: The CDF of T

4.3 Find the PDF of T .

Solution: The PDF of T is plotted in Fig. 4.2 using the code below

```
wget https://github.com/  
HarshaVardhanReddyG/  
A1110_Assignment/blob/main/codes/  
tri_pdf.py
```

Exceute the code using

```
python3 ./codes/tri_pdf.py
```

4.4 Find the theoretical expressions for CDF and PDF of T .

Solution: Given,

$$T = U_1 + U_2 \quad (4.2)$$

$$\Rightarrow p_T(t) = \int_{-\infty}^t p_{U_1}(u_1)p_{U_2}(u_2)du_1 \quad (4.3)$$

$$\text{As } p_{U_1}(u_1) = p_{U_2}(u_2) = p_U(u) \quad (4.4)$$

$$p_T(t) = \int_{-\infty}^t p_U(u)p_U(t-u)du \quad (4.5)$$

i) If $t \leq 1$

$$p_T(t) = \int_0^t p_U(t-u)du \quad (4.6)$$

$$p_T(t) = \int_0^t du = t \quad (4.7)$$

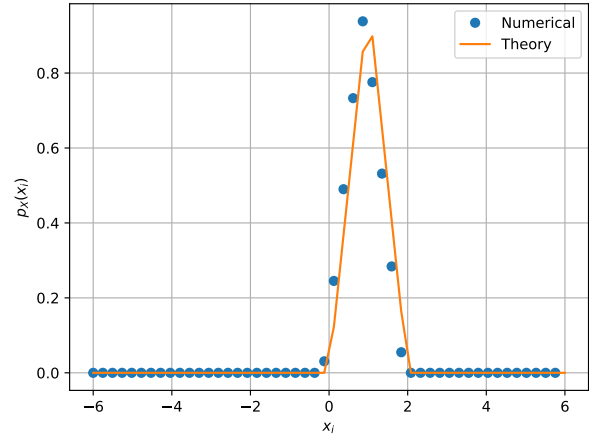


Fig. 4.2: The PDF of T

ii) If $t > 1$

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.8)$$

$$p_T(t) = \int_{t-1}^1 du = 2 - t \quad (4.9)$$

Therefore

$$P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$$

And We know that

$$F_X(x) = \int_{-\infty}^x p_X(x)dx \quad (4.10)$$

Therefore

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases}$$

4.5 Verify your results through a plot.

Solution: Results are verified and shown in (4.2) and (4.2)

5 MAXIMAL LIKELIHOOD

5.1 Generate

$$Y = AX + N \quad (5.1)$$

Where $A = 5dB$, $X_1 \in \{1, -1\}$, is Bernouli and $N \sim \mathcal{N}(0, 1)$

5.2 Plot Y .

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = P(\hat{X} = -1|X = 1), \quad (5.2)$$

$$P_{e|1} = P(\hat{X} = 1|X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verifying by plotting the theoritical P_e

6 GUASSION TO OTHER

6.1 Let $X_1 = \mathcal{N}(0, 1)$, $X_2 = \mathcal{N}(0, 1)$.Plot the PDF and CDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find α

6.3 Plot CDF and PDF of

$$A = \sqrt{V} \quad (6.2)$$

7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = P(\hat{X} = -1|X = 1) \quad (7.1)$$

for

$$Y = AX + N \quad (7.2)$$

7.2