Assignment 10

G HARSHA VARDHAN REDDY (CS21BTECH11017)

May 29, 2022

AI1110

Outline

Problem Statement

Papoulis Pillai Probability Random Variables and Stochastic Processes
Exercise : 5-40

Let x denote the event "the number of failures that precede the *nth* success" so that $x + n$ represents the total number of trials needed to generate n successes. In that case, the event $x = k$ occurs if and only if the last trial results in a success and among the previous $(x + n - 1)$ trials there are $n - 1$ successes (or x failures). This gives an alternate formula for the Pascal (or negative binomial) distribution as follows:

$$P\{X = k\} = \binom{n+k-1}{k} p^n q^k = \binom{-n}{k} p^n (-q)^k, k = 0, 1, 2, \ldots \tag{1}$$

Find Γ_z and show that $\eta_x = \frac{nq}{p}$ and $\sigma_x^2 = \frac{nq}{p^2}$

Pascal or Negative binomial distribution

Pascal distribution

Pascal random variable describes the number of trials until the k^{th} success, which is why it is sometimes called the "k^{th}*order* interarrival time for a Bernoulli process". Let X_k be a k^{th}*order* Pascal random variable. Then its PMF is given by

$$p_{X_k} = \binom{n+k-1}{k} p^n q^k \tag{2}$$

Solution

As

$$\Gamma(z) = \sum_{k=0}^{\infty} p_{X_k} z^k \tag{3}$$

From (1) and (2)

$$\Gamma(z) = \sum_{k=0}^{\infty} \binom{-n}{k} p^n (-q)^k z^k \tag{4}$$

$$= p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-qz)^k \tag{5}$$

$$\implies \Gamma(z) = p^n (1 - qz)^{-n} \tag{6}$$

And We know that mean $(\eta_X) = \Gamma'(1)$ and $\Gamma''(1) + \eta_X = \sigma_X^2 + \eta_X^2$

From (6),

$$\Gamma'(z) = p^n(-q)(-n)(1-qz)^{-(n+1)} \tag{7}$$

$$\implies \Gamma'(1) = p^n nq(1-q)^{-(n+1)} \tag{8}$$

$$\implies \eta_X = \frac{nqp^n}{p^{(n+1)}} = \frac{nq}{p}........(\text{since} p + q = 1) \tag{9}$$

$$\Gamma''(z) = p^n nq(n+1)q(1-qz)^{-(n+2)} \tag{10}$$

$$\implies \Gamma''(1) = p^n n(n+1)q^2(1-q)^{-(n+2)} \tag{11}$$

$$= \frac{n(n+1)p^n q^2}{p^{n+2}} \tag{12}$$

$$\implies \sigma_X^2 + \eta_X^2 = \frac{n(n+1)q^2}{p^2} + \eta_X \tag{13}$$

$$\implies \sigma_X^2 = \frac{n(n+1)q^2}{p^2} + \frac{nq}{p} - \left(\frac{nq}{p}\right)^2 \tag{14}$$

$$= \frac{nq^2 + npq}{p^2} = \frac{nq(p+q)}{p^2} \tag{15}$$

$$\implies \sigma_X^2 = \frac{nq}{p^2} \tag{16}$$