Assignment 15

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June 20, 2022

AI1110

Outline

Problem Statement

Papoulis Pillai Probability Random Variables and Stochastic Processes
Exercise : 12-11

We wish to estimate the mean η of a process $x(t) = \eta + v(t)$, where $R_{vv}(\tau) = 5\delta(\tau)$. Find the 0.95 confidence interval of η ?

Definitions

Mean-Ergodic Processes

The process is said to be mean ergodic if its time average η_T tends to the ensemble average η as $T \to \infty$.

Auto covariance

For a stationary process,

$$C(\tau) = \frac{1}{2T} \int_{-2T}^{2T} C(\tau - \alpha)\left(1 - \frac{|\alpha|}{2T}\right) d\alpha \tag{1}$$

(2)

Where,
$C(\tau)$ is auto covariance of $x(t)$

Variance

We know that for a mean ergodic process

$$\sigma_T^2 = C(0) \text{ and} \tag{3}$$

$$C(\alpha) = C(-\alpha) \tag{4}$$

From (1),

$$\sigma_T^2 = C(0) = \frac{1}{2T}\int_{-2T}^{2T} C(\alpha)\left(1 - \frac{|\alpha|}{2T}\right)d\alpha \tag{5}$$

$$\implies \sigma_T^2 = \frac{1}{T}\int_{0}^{2T} C(\alpha)\left(1 - \frac{\alpha}{2T}\right)d\alpha \tag{6}$$

Solution

Given,

$$x(t) = \eta + v(t) \tag{7}$$

Here,
$x(t)$ is mean ergodic process.
$v(t)$ is white noise with $R_{vv}(\tau) = 5\delta(\tau)$
$C(\tau) = R_{vv}(\tau)$
As,

$$\sigma_T^2 = \frac{1}{2T}\int_{-2T}^{2T} C(\alpha)\left(1 - \frac{\alpha}{2T}\right) d\alpha \tag{8}$$

Therefore,

$$\sigma_T^2 = \frac{1}{2T}\int_{-2T}^{2T} 5\delta(\tau)\left(1 - \frac{\tau}{2T}\right) d\tau = \frac{5}{2T} \tag{9}$$

Therefore,

$$\sigma_T^2 = \frac{5}{2T} \tag{10}$$

We have to find ϵ such that

$$Pr(\eta - \varepsilon \leq \eta_T \leq \eta + \varepsilon) = 0.95 \tag{11}$$

$$\implies Pr(|\eta_T - \varepsilon| \leq \eta) = 0.95 \tag{12}$$

As

$$Pr(|\eta_T - \varepsilon| \geq \eta) = \int_{-\infty}^{-\eta-\varepsilon} f(x)dx + \int_{\eta+\varepsilon}^{\infty} f(x)dx \text{ and} \tag{13}$$

$$\sigma_T^2 = \int_{-\infty}^{\infty} (x-\eta)^2 f(x)dx \tag{14}$$

$$\geq \int_{|\eta_T - \varepsilon| \geq \eta} (x-\eta)^2 f(x)dx \tag{15}$$

$$\geq \varepsilon^2 \int_{|\eta_T - \varepsilon| \geq \eta} f(x)dx \tag{16}$$

Therefore,

$$\sigma_T^2 \geq \varepsilon^2 \times Pr(|\eta_T - \varepsilon| \geq \eta) \tag{17}$$

$$\implies Pr(|\eta_T - \varepsilon| \geq \eta) \leq \frac{\sigma_T^2}{\varepsilon^2} \tag{18}$$

From (12) and (18),

$$Pr(|\eta_T - \varepsilon| \geq \eta) = 1 - 0.95 \leq \frac{\sigma_T^2}{\varepsilon^2} \tag{19}$$

$$\implies \varepsilon^2 \leq \frac{\sigma_T^2}{0.05} \tag{20}$$

Substituting σ_T^2 in (20)

$$\varepsilon^2 \leq \frac{5}{2T \times 0.05} \tag{21}$$

As $\varepsilon > 0$

$$\varepsilon^2 \leq \frac{5}{2T \times 0.05} \tag{22}$$

$$\leq \frac{50}{T} \tag{23}$$

$$\implies \varepsilon \leq \sqrt{\frac{50}{T}} \tag{24}$$

Therefore, $Pr(\eta - \varepsilon \leq \eta_T \leq \eta + \varepsilon) = 0.95 \implies \varepsilon \leq \sqrt{\frac{50}{T}}$