

Assignment 9

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AI1110

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Problem Statement

Papoulis Pillai Probability Random Variables and Stochastic Processes Exercise : 5-3

If the random variable x is $N(0, c^2)$ and $g(x)$ is a function defined below, find and sketch the distribution and the density of the random variable $y = g(x)$.

$$g(x) = \begin{cases} x + c & x < -c \\ 0 & -c \leq x \leq c \\ x - c & x > c \end{cases} \quad (1)$$

Distribution and density functions

Probability density function

Probability density function $f_X(x)$ for a Normal or Gaussian random variable $N(\mu, \sigma^2)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty \leq x \leq \infty \quad (2)$$

Probability distribution function or Distribution function

If $f_X(x)$ is probability density function of a random variable X , then its corresponding Probability distribution function $F_X(x)$ is

$$F_X(x) = \int_{-\infty}^x f_X(x) dx \quad (3)$$

Solution

Given,

The random variable $X = N(0, c^2)$

Comparing $N(0, c^2)$ and $N(\mu, \sigma^2)$

$$\mu = 0 \quad (4)$$

$$\sigma = c \quad (5)$$

Therefore, from (2), (4) and (5)

$$f_X(x) = \frac{1}{\sqrt{2\pi c^2}} e^{\frac{-x^2}{2c^2}}, -\infty \leq x \leq \infty \quad (6)$$

Let's find probability distribution function of the random variable $Y = g(x)$

If $y > 0$

$$F_Y(y) = P(Y \leq y) = P(X - c \leq y) \quad (7)$$

$$= P(X \leq y + c) = F_X(y + c) \quad (8)$$

$$= \int_{-\infty}^{y+c} f_X(t) dt \quad (9)$$

$$\Rightarrow F_Y(y) = \frac{1}{\sqrt{2\pi c^2}} \int_{-\infty}^{y+c} e^{\frac{-t^2}{2c^2}} dt \quad (10)$$

$$= \frac{1}{c \sqrt{2\pi}} \frac{\sqrt{\pi c}}{\sqrt{2}} \left(\operatorname{erf} \left(\frac{y+c}{\sqrt{2c}} \right) + 1 \right) \quad (11)$$

$$= \frac{1}{2 \sqrt{c}} \left(\operatorname{erf} \left(\frac{y+c}{\sqrt{2c}} \right) + 1 \right) \quad (12)$$

If $y \leq 0$

$$F_Y(y) = P(Y \leq y) = P(X + c \leq y) \quad (13)$$

$$= P(X \leq y - c) = F_X(y - c) \quad (14)$$

$$= \int_{-\infty}^{y-c} f_X(t) dt \quad (15)$$

$$\Rightarrow F_Y(y) = \frac{1}{\sqrt{2\pi c^2}} \int_{-\infty}^{y-c} e^{-\frac{t^2}{2c^2}} dt \quad (16)$$

$$= \frac{1}{c \sqrt{2\pi}} \frac{\sqrt{\pi c}}{\sqrt{2}} \left(\operatorname{erf}\left(\frac{y-c}{\sqrt{2c}}\right) + 1 \right) \quad (17)$$

$$= \frac{1}{2\sqrt{c}} \left(\operatorname{erf}\left(\frac{y-c}{\sqrt{2c}}\right) + 1 \right) \quad (18)$$

Therefore,

$$F_Y(y) = \begin{cases} \frac{1}{2\sqrt{c}} \left(\operatorname{erf} \left(\frac{y-c}{\sqrt{2c}} \right) + 1 \right) & y \leq 0 \\ \frac{1}{2\sqrt{c}} \left(\operatorname{erf} \left(\frac{y+c}{\sqrt{2c}} \right) + 1 \right) & y > 0 \end{cases} \quad (19)$$

As

$$f_Y(y) = \frac{d(F_Y(y))}{dy} \quad (20)$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi c^2}} e^{-\frac{(y-c)^2}{2c^2}} & y \leq 0 \\ \frac{1}{\sqrt{2\pi c^2}} e^{-\frac{(y+c)^2}{2c^2}} & y > 0 \end{cases} \quad (21)$$

Probability density function

If $y < 0$

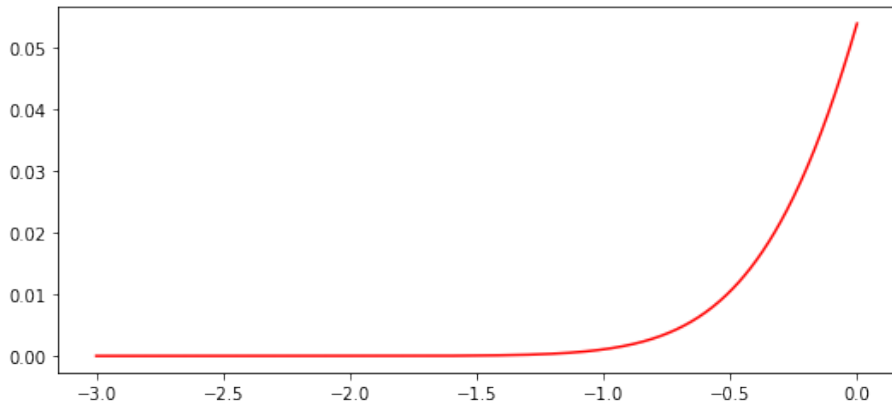


Figure 0: $f_Y(y)$

Probability density function

If $y \geq 0$

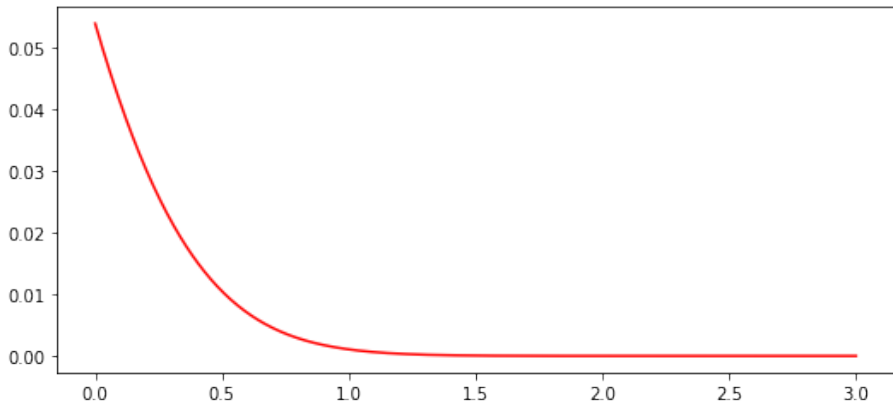


Figure 0: $f_Y(y)$

Probability distribution function

If $y \leq 0$

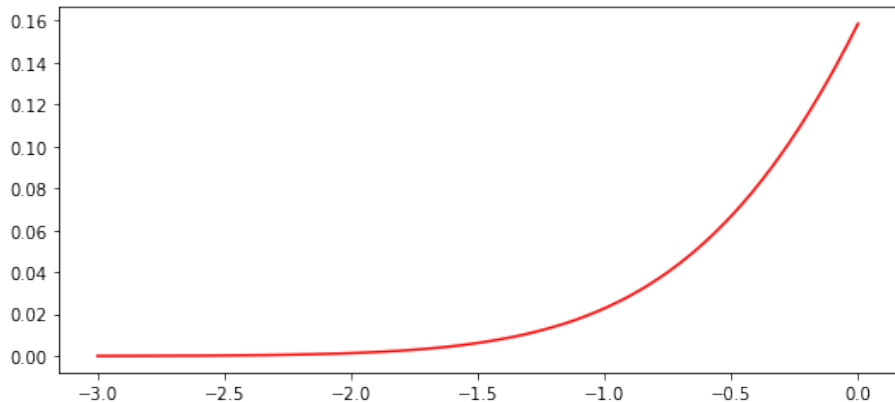


Figure 0: $f_Y(y)$

Probability distribution function

If $y \geq 0$

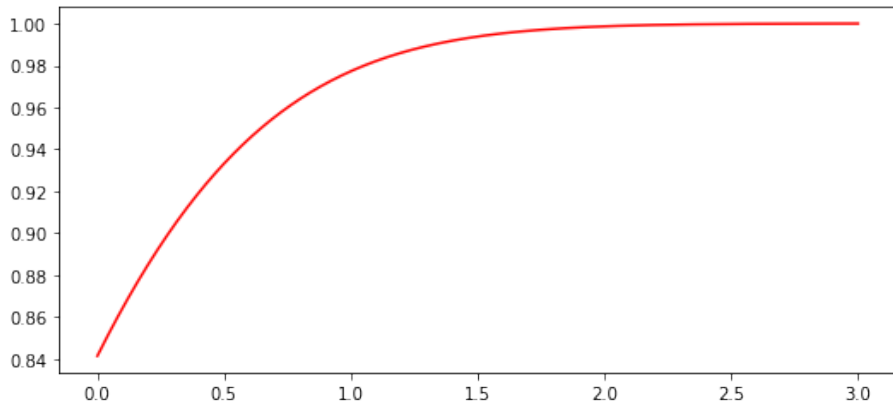


Figure 0: $f_Y(y)$