Assignment 9

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AI1110

Outline

Problem Statement

Papoulis Pillai Probability Random Variables and Stochastic Processes
Exercise : 5-3

If the random variable x is $N(0, c^2)$ and $g(x)$ is a function defined below, find and sketch the distribution and the density of the random variable $y = g(x)$.

$$g(x) = \begin{cases} x + c & x < -c \\ 0 & -c \leq x \leq c \\ x - c & x > c \end{cases} \tag{1}$$

Distribution and density functions

Probability density function

Probability density function $f_X(x)$ for a Normal or Gaussian random variable $N(\mu, \sigma^2)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty \leq x \leq \infty \tag{2}$$

Probability distribution function or Distribution function

If $f_X(x)$ is probability density function of a random variable X, then it's corresponding Probability distribution function $F_X(x)$ is

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \tag{3}$$

Solution

Given,
The random variable $X = N(0, c^2)$

Comparing $N(0, c^2)$ and $N(\mu, \sigma^2)$

$$\mu = 0 \tag{4}$$
$$\sigma = c \tag{5}$$

Therefore, from (2), (4) and (5)

$$f_X(x) = \frac{1}{\sqrt{2\pi c^2}} e^{\frac{-x^2}{2c^2}}, -\infty \leq x \leq \infty \tag{6}$$

Let's find probability distribution function of the random variable $Y = g(x)$

If $y > 0$

$$F_Y(y) = P(Y \leq y) = P(X - c \leq y) \tag{7}$$
$$= P(X \leq y + c) = F_X(y + c) \tag{8}$$
$$= \int_{-\infty}^{y+c} f_X(t)\, dt \tag{9}$$
$$\implies F_Y(y) = \frac{1}{\sqrt{2\pi c^2}} \int_{-\infty}^{y+c} e^{\frac{-t^2}{2c^2}}\, dt \tag{10}$$
$$= \frac{1}{c\sqrt{2\pi}} \frac{\sqrt{\pi c}}{\sqrt{2}} \left(erf\left(\frac{y+c}{\sqrt{2c}}\right) + 1 \right) \tag{11}$$
$$= \frac{1}{2\sqrt{c}} \left(erf\left(\frac{y+c}{\sqrt{2c}}\right) + 1 \right) \tag{12}$$

If $y \leq 0$

$$F_Y(y) = P(Y \leq y) = P(X + c \leq y) \tag{13}$$
$$= P(X \leq y - c) = F_X(y - c) \tag{14}$$
$$= \int_{-\infty}^{y-c} f_X(t)\, dt \tag{15}$$
$$\implies F_Y(y) = \frac{1}{\sqrt{2\pi c^2}} \int_{-\infty}^{y-c} e^{\frac{-t^2}{2c^2}}\, dt \tag{16}$$
$$= \frac{1}{c\sqrt{2\pi}} \frac{\sqrt{\pi c}}{\sqrt{2}} \left(erf\left(\frac{y-c}{\sqrt{2c}} \right) + 1 \right) \tag{17}$$
$$= \frac{1}{2\sqrt{c}} \left(erf\left(\frac{y-c}{\sqrt{2c}} \right) + 1 \right) \tag{18}$$

Therefore,

$$F_Y(y) = \begin{cases} \frac{1}{2\sqrt{c}}\left(erf\left(\frac{y-c}{\sqrt{2c}}\right)+1\right) & y \leq 0 \\ \frac{1}{2\sqrt{c}}\left(erf\left(\frac{y+c}{\sqrt{2c}}\right)+1\right) & y > 0 \end{cases} \tag{19}$$

As

$$f_Y(y) = \frac{d\left(F_Y(y)\right)}{dy} \tag{20}$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi c^2}}e^{-\frac{(y-c)^2}{2c^2}} & y \leq 0 \\ \frac{1}{\sqrt{2\pi c^2}}e^{-\frac{(y+c)^2}{2c^2}} & y > 0 \end{cases} \tag{21}$$

Probability density function

If $y < 0$

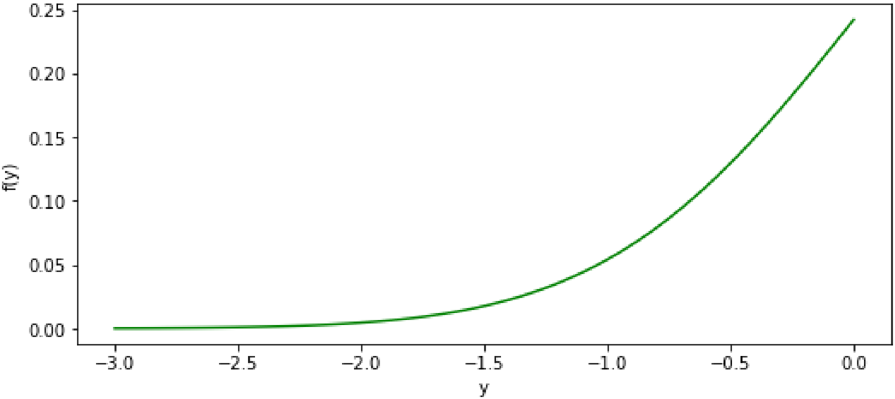

Figure 0: $f_Y(y)$

Probability density function

If $y \geq 0$

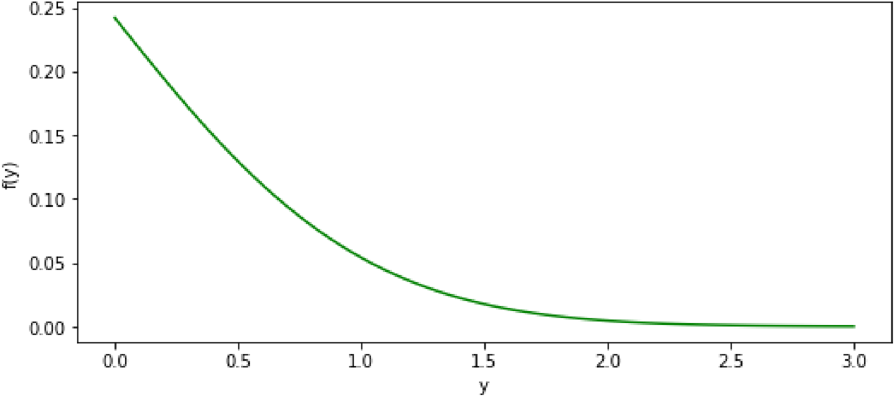

Figure 0: $f_Y(y)$

Probability distribution function

If $y \leq 0$

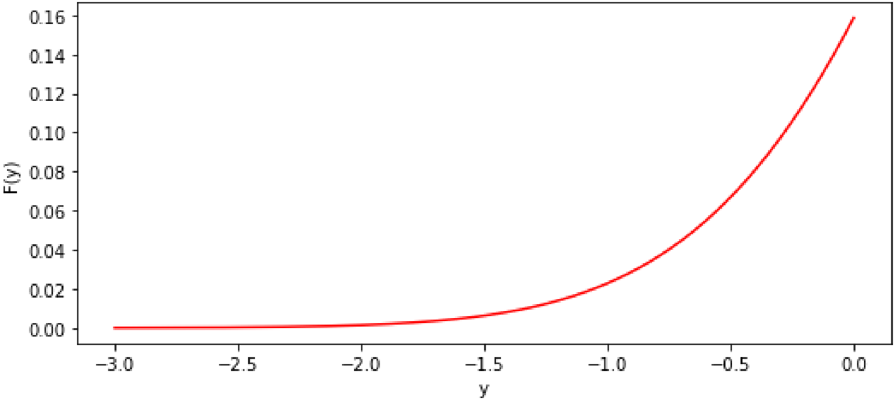

Figure 0: $F_Y(y)$

Probability distribution function

If $y \geq 0$

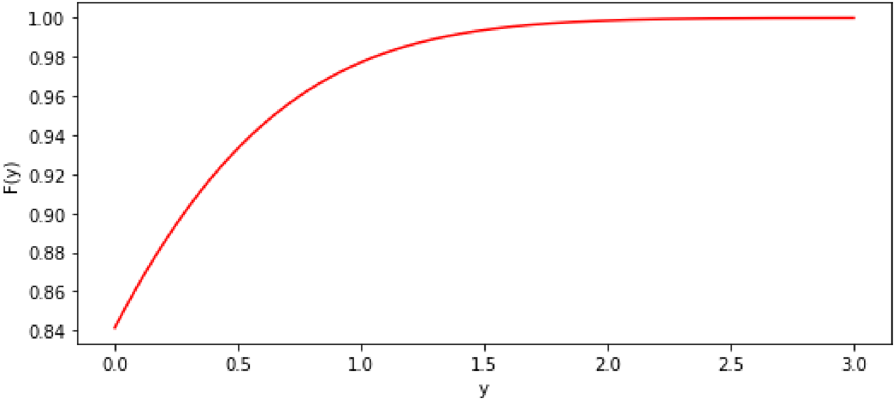

Figure 0: $F_Y(y)$