1

Consider a random process $X(t)$ where each realization of the process is defined as $X(t) = A\cos(\omega t + \phi)$, where A and ω are constants, and ϕ is uniformly distributed in $[0, 2\pi)$. Find the mean $E[X(t)]$ of this process.

Solution:
Since ϕ is uniformly distributed, the probability density function (PDF) of ϕ is $f_\phi(\phi) = \frac{1}{2\pi}$. The mean is:

$$E[X(t)] = E[A\cos(\omega t + \phi)] = A\int_0^{2\pi} \cos(\omega t + \phi) \cdot \frac{1}{2\pi} d\phi$$

Using the fact that $\int_0^{2\pi} \cos(\omega t + \phi) d\phi = 0$, we get:

$$E[X(t)] = 0$$

2

For the random process $Y(t) = Be^{j\theta}$ where B is constant and θ is uniformly distributed over $[0, 2\pi)$, find $E[Y(t)]$.

Solution:
Since θ is uniformly distributed, the mean of $Y(t)$ is:

$$E[Y(t)] = B\int_0^{2\pi} e^{j\theta} \cdot \frac{1}{2\pi} d\theta$$

By solving this integral, we get:

$$E[Y(t)] = B \cdot 0 = 0$$

3

If $X(t) = Z + Wt$, where Z and W are independent random variables with means $E[Z] = 3$ and $E[W] = 2$, find $E[X(t)]$ as a function of t.

Solution:

The expectation is:

$$E[X(t)] = E[Z + Wt] = E[Z] + E[W] \cdot t = 3 + 2t$$

4

Problem 1:

Given a random process $X(t) = A\sin(\omega t) + B\cos(\omega t)$, where A and B are zero-mean random variables with variances σ_A^2 and σ_B^2 respectively, find $E[X(t)^2]$.

Solution:

$$E[X(t)^2] = E[(A\sin(\omega t) + B\cos(\omega t))^2]$$

Expanding and using $E[A^2] = \sigma_A^2$ and $E[B^2] = \sigma_B^2$:

$$E[X(t)^2] = \sigma_A^2 \sin^2(\omega t) + \sigma_B^2 \cos^2(\omega t)$$

5

For a random process $Y(t) = C\sin(\omega t)$ where C is a random variable with $E[C] = 0$ and $\mathrm{Var}(C) = 4$, find the variance $\mathrm{Var}(Y(t))$.

Solution:

$$\mathrm{Var}(Y(t)) = E[Y(t)^2] - (E[Y(t)])^2$$

Since $E[Y(t)] = 0$, we get:

$$\mathrm{Var}(Y(t)) = E[C^2 \sin^2(\omega t)] = 4\sin^2(\omega t)$$

6

Let $Z(t) = X(t) + Y(t)$ where $X(t)$ and $Y(t)$ are independent random processes with variances $\sigma_X^2 = 1$ and $\sigma_Y^2 = 2$, respectively. Find the variance $\mathrm{Var}(Z(t))$.

Solution:

$$\mathrm{Var}(Z(t)) = \mathrm{Var}(X(t)) + \mathrm{Var}(Y(t)) = 1 + 2 = 3$$

7

Consider a random process $X(t) = A\cos(\omega t + \phi)$, where ϕ is uniformly distributed on $[0, 2\pi)$. Determine if $X(t)$ is wide-sense stationary.

Solution:

1. **Define Wide-Sense Stationarity (WSS):** A process is WSS if:
   - Its mean $E[X(t)]$ is constant and does not depend on t.
   - Its autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$ depends only on the time difference $t_1 - t_2$, not on the individual times t_1 or t_2.
2. **Calculate $E[X(t)]$:** Since ϕ is uniformly distributed over $[0, 2\pi)$, the expected value of $\cos(\omega t + \phi)$ over one period is zero:

   $$E[X(t)] = A \cdot E[\cos(\omega t + \phi)] = A \cdot 0 = 0$$

   Therefore, $E[X(t)]$ is constant.
3. **Check the Autocorrelation Function $R_X(t_1, t_2)$:** The autocorrelation function is:

   $$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

   Expanding $X(t) = A\cos(\omega t + \phi)$, we get:

   $$R_X(t_1, t_2) = A^2 \cdot E[\cos(\omega t_1 + \phi)\cos(\omega t_2 + \phi)]$$

   Using trigonometric identities, we find that $R_X(t_1, t_2)$ depends only on $t_1 - t_2$, satisfying WSS.
4. **Conclusion:**
   Since both conditions of WSS are met, $X(t)$ is wide-sense stationary.

8

For a process $Y(t) = Z + Wt$, where Z and W are independent random variables, check if $Y(t)$ is wide-sense stationary.

Solution:

1. **Calculate $E[Y(t)]$:**

$$E[Y(t)] = E[Z + Wt] = E[Z] + E[W]t$$

   Since $E[Y(t)]$ depends on t, it is not constant over time.

2. **Conclusion:**
   Since $E[Y(t)]$ depends on t, $Y(t)$ does not satisfy the condition for WSS and is not wide-sense stationary.

9

If $X(t) = A\cos(\omega t)$ with constant A and ω, is $X(t)$ strictly stationary?

Solution:

1. **Define Strict Stationarity:** A process is strictly stationary if the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n))$ is identical to $(X(t_1 + \tau), X(t_2 + \tau), \ldots, X(t_n + \tau))$ for any time shift τ and any set of times $t_1, t_2, \ldots, t_n$.

2. **Examine the Distribution:**
   Since A and ω are constants, $X(t)$ is deterministic, meaning the values of $X(t)$ do not change with a shift in time. The statistical properties remain unchanged over time.

3. **Conclusion:**
   All joint distributions are invariant under time shift, so $X(t)$ is strictly stationary.

10

Question:
Given a process $X(t) = A\cos(\omega t)$ where A is a random variable with $E[A] = 3$. Is $X(t)$ mean ergodic?

Solution:

1. **Define Mean Ergodicity:** A process is mean ergodic if the time average of each realization converges to the ensemble mean.
2. **Calculate the Time Average of $X(t)$:** Since $A\cos(\omega t)$ is a function with a mean value equal to $E[A] = 3$ over time, the time average equals the ensemble average.
3. **Conclusion:**
   $X(t)$ is mean ergodic.

11

Question:
If $Y(t) = B + Wt$ where B is a random variable with $E[B] = 2$ and W is deterministic, is $Y(t)$ mean ergodic?

Solution:

1. **Calculate the Time Average of $Y(t)$:** The time average of $Y(t) = B + Wt$ changes as t increases due to the Wt term. Therefore, it does not converge to a constant value.
2. **Conclusion:**
   Since the time average does not converge to the ensemble mean, $Y(t)$ is not mean ergodic.

Consider $Z(t) = A\sin(\omega t)$, where A is a random variable with $E[A^2] = 5$. We need to determine if $Z(t)$ is correlation ergodic.

Solution Steps:

Step 1: Define Correlation Ergodicity

A random process is **correlation ergodic** if its **time-averaged autocorrelation** equals its **ensemble-averaged autocorrelation**. Mathematically:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)Z(t+\tau)\,dt = E[Z(t)Z(t+\tau)]$$

where τ is a time lag.

Step 2: Calculate the Time-Averaged Autocorrelation

Since $Z(t) = A\sin(\omega t)$, let's calculate $Z(t)Z(t+\tau)$ and then find its time average.

1. **Express $Z(t)Z(t+\tau)$:**

$$Z(t)Z(t+\tau) = A\sin(\omega t)\cdot A\sin(\omega(t+\tau)) = A^2\sin(\omega t)\sin(\omega(t+\tau))$$

2. **Use Trigonometric Identities to Simplify:** Expanding $\sin(\omega t)\sin(\omega(t+\tau))$ using the product-to-sum identity:

$$\sin(\omega t)\sin(\omega(t+\tau)) = \frac{1}{2}\left[\cos(\omega\tau) - \cos(2\omega t + \omega\tau)\right]$$

Therefore:

$$Z(t)Z(t+\tau) = \frac{A^2}{2}\left[\cos(\omega\tau) - \cos(2\omega t + \omega\tau)\right]$$

3. **Take the Time Average:** Now, we time-average this expression over a long period T:

$$\frac{1}{T}\int_0^T Z(t)Z(t+\tau)\,dt = \frac{A^2}{2}\frac{1}{T}\int_0^T \left[\cos(\omega\tau) - \cos(2\omega t + \omega\tau)\right]dt$$

Separating the two terms:

$$\frac{A^2}{2}\left(\cos(\omega\tau) - \frac{1}{T}\int_0^T \cos(2\omega t + \omega\tau)\,dt\right)$$

- **First Term:** The first part, $\cos(\omega\tau)$, is independent of t and remains as it is.
- **Second Term:** The second part, $\frac{1}{T}\int_0^T \cos(2\omega t + \omega\tau)\,dt$, is the time average of a cosine function with frequency 2ω. Over a long period $T \to \infty$, this integral will average out to zero because cosine oscillates symmetrically around zero over its period.

So, we get:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t)Z(t+\tau)\,dt = \frac{A^2}{2}\cos(\omega\tau)$$

ep 3: Calculate the Ensemble-Averaged Autocorrelation

ne ensemble-averaged autocorrelation is $E[Z(t)Z(t+\tau)]$.

1. **Calculate $E[Z(t)Z(t+\tau)]$:** Since $Z(t) = A\sin(\omega t)$,

$$E[Z(t)Z(t+\tau)] = E\left[A^2 \sin(\omega t)\sin(\omega(t+\tau))\right]$$

Using the same trigonometric identity and simplifications as above, we get:

$$E[Z(t)Z(t+\tau)] = \frac{E[A^2]}{2}\cos(\omega\tau)$$

Given that $E[A^2] = 5$:

$$E[Z(t)Z(t+\tau)] = \frac{5}{2}\cos(\omega\tau)$$