Q: Define Inner Product?

~~V is/a vector space~~ Let V be a vector space.

An inner product on V is a function that takes ~~every~~ each ordered pair (u, v) of element of V to a number in F has the following properties.

1. $(v, v) \geq 0$ for all $v \in V$
2. $(v, v) = 0$ if and only if $v = 0$
3. $(u+v, w) = (u, w) + (v, w)$ for all $u, v, w \in V$
4. $(\lambda u, v) = \lambda (u, v)$ for all $\lambda \in F$ and $u, v \in V$
5. $(u, v) = \overline{(v, u)}$ for all $u, v \in V$

If these properties exist we say V is inner product. Inner product is also called as inner product space.

2. Define Norm?

V is a inner product.

Let $v \in V$

The Norm of v is denoted by $\|v\| = \sqrt{(v, v)}$

$$\|v\|^2 = (v, v)$$

3. In the inner product V prove the following

1. For each fixed $v \in V$ the function takes $u \in V$ to (u, v) is a linear map from V to F

Sol: We know that,

from definition of inner product

for $v \in V$

① property gives

$$u \to (u, v)$$

i.e. linearity exists

② $(0,v) = 0$ for every $v \in V$

② We know that

$$0 \in V$$

$$0 \to (0, v)$$

from this $(0, v) = 0$

③ $(v, 0) = 0$ for every $v \in V$

Using definition of inner product

$$(v, 0) = \overline{(0, v)}$$

$$= \overline{0}$$

$$= 0$$

4 $(u, v+w) = (u,v) + (u,w)$ for all $u, v, w \in V$

4 $(u, v+w) = \overline{(v+w, u)}$

$$= \overline{(v,u) + (w,u)}$$

$$= \overline{(v,u)} + \overline{(w,u)}$$

$$= (u,v) + (u,w)$$

5. $(u, \lambda v) = \bar{\lambda}(u, v)$ for all $\lambda \in F$ and $u, v \in V$

5. $(u, \lambda v) = \overline{(\lambda \cdot v, u)}$

$$= \overline{\lambda (v, u)}$$

$$= \bar{\lambda}\, \overline{(v,u)}$$

$$= \bar{\lambda}\, (u, v)$$

Q: State and prove PYTHAGOREAN THEOREM and define orthogonal vectors

V is the inner product.

$u, v \in V$

If $(u,v)=0$ then we can say that two vectors u and v are orthogonal

If u and v are orthogonal then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Proof:

V is a inner product

$u, v \in V$

u, v are orthogonal

By definition $(u,v)=0$

$$\|u+v\|^2 = (u+v, u+v)$$

$$= (u, u+v) + (v, u+v)$$

$$= (u,u) + (u,v) + (v,u) + (v,v)$$

$$= \|u\|^2 + 0 + \overline{(u,v)} + \|v\|^2$$

$$= \|u\|^2 + \bar{0} + \|v\|^2$$

$$= \|u\|^2 + \|v\|^2$$

Q: State and prove orthogonal decomposition?

ORTHOGONAL DECOMPOSITION:

V is a Inner product, $v \neq 0$,

$u, v \in V$, set $c = \frac{(u,v)}{\|v\|^2}$ and $w = u - \frac{(u,v)}{\|v\|^2} v$

Then $u = cv + w$ and $(w, v) = 0$

Proof:

Given that V is a Inner product

$v \neq 0$

$u, v \in V$

$$c = \frac{u, v}{\|v\|^2}$$

$$w = u - \frac{(u,v)}{\|v\|^2 \neq 0} \cdot v$$

$$cv + w = \frac{(u,v)}{\|v\|^2} \cdot v + u - \frac{(u,v)}{\|v\|^2} \cdot v$$

$$cv + w = u$$

$$u = cv + w$$

$$(w, v) = \left(u - \frac{(u,v)}{\|v\|^2} \cdot v, v\right)$$

$$= (u,v) - \left(\frac{(u,v)}{\|v\|^2} v, v\right)$$

$$= (u,v) - \frac{(u,v)}{\|v\|^2} (v,v)$$

$$= (u,v) - \frac{(u,v)}{\cancel{\|v\|^2}} \cancel{\|v\|^2}$$

$$= (u,v) - (u,v)$$

$$= 0$$

$$\therefore (w, v) = 0$$

Q: State and prove CAUCHY - SCHWARZ INEQUALITY

V is a Inner product,

$u, v \in V$

Then $|(u,v)| \leq \|u\| \, \|v\|$

Proof:

V is a inner product, $u, v \in V$

Taking $u = \frac{(u,v)}{\|v\|^2} \cdot v + w$

$$\|u\|^2 = (u,u)$$

$$= \left\| \frac{(u,v)}{\|v\|^2} \cdot v + w \right\|^2$$

By applying pythagorean law

$$\|u\|^2 = \left\| \frac{(u,v)}{\|v\|^2} \cdot v \right\|^2 + \|w\|^2$$

(u,v) is a scalar

If we take

$$\|\lambda v\|^2 = (\lambda v, \lambda v)$$

$$= \lambda (v, \lambda v)$$

$$= \lambda \overline{(\lambda v, v)}$$

$$= \lambda \overline{\lambda (v,v)}$$

$$= \lambda \bar{\lambda} \, \overline{(v,v)}$$

$$= |\lambda|^2 \|v\|^2$$

$$\frac{|(u,v)|^2 \, \|v\|^2}{\|v\|^4} + \|w\|^2$$

$$\frac{|(u,v)|^2}{\|v\|^2} + \|w\|^2$$

$$\geq \frac{|(u,v)|^2}{\|v\|^2}$$

$$\|u\|^2 \|v\|^2 \geq |(u,v)|^2$$

$$\|u\| \, \|v\| \geq |(u,v)|$$

$$|(u,v)| \leq \|u\| \, \|v\|$$

Q: State and prove TRIANGLE INEQUALITY ?

V is a inner product

$u, v \in V$

Then $\|u+v\| \leq \|u\| + \|v\|$

Proof :

Given that V is a inner product

$u, v \in V$

$$\|u+v\|^2 = (u+v, u+v)$$

$$= (u, u+v), (v, u+v)$$

$$= (u,u) + (u,v) + (v,u) + (v,v)$$

$$= \|u\|^2 + \|v\|^2 + (u,v) + \overline{(u,v)}$$

$$z = x+iy$$

$$\bar{z} = \overline{x+iy} = x-iy$$

$$z + \bar{z} = x+iy+x-iy = 2x$$

$= \|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\,(u,v)$

$\leq \|u\|^2 + \|v\|^2 + 2|(u,v)|$

$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\,\|v\|$

$|z| = \sqrt{x^2+y^2}$

$\mathrm{Re}\,z$

$x < \sqrt{x^2+y^2}$

Real $\leq |z|$

$= \|\cancel{u\|^2+}\|$

$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$\|u+v\| \leq \|u\| + \|v\|$

Q: State & prove parallelogram equality

V is a inner product

$u,v \in V$

$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

Proof:
Given that V is a inner product

$u,v \in V$

L.H.S

$= \|u+v\|^2 + \|u-v\|^2$

$= (u+v, u+v) + (u-v, u-v)$

$= (u, u+v) + (v, u+v) + (u, u-v) - (v, u-v)$

$= (u,u) + \cancel{(u,v)} + \cancel{(v,u)} + (v,v) + (u,u) - \cancel{(u,v)} - \cancel{(v,u)} + (v,v)$

$= 2\|u\|^2 + 2\|v\|^2$

$= 2(\|u\|^2 + \|v\|^2)$

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(1) Define (a) Random experiment

(b) Probability

(c) Sample Space

(d) Probability Axiomatic Approach

(e) ~~Ad~~

(2) Addition theorem on probability

(3) Define Conditional probability

(4) Multiplication theorem of probability

(5) Boole's Inequality

(6) Baye's theorem

1. Random Experiment

Also known as trial

An experiment is conducted any number of times, there is a set of all possible outcomes are known, but result is not certain, that type of experiment is called random experiment or trial.

Ex: 1. Tossing a coin $\{H, T\}$

2. Throwing a die $\{1, 2, 3, 4, 5, 6\}$

b. Probability:

In a random experiment assume that A is an event, the favourable no. of cases to the event A is m (say), in the experiment total outcomes are n (say),

Now, we define the event probability

$$P(A) = \frac{m}{n}$$

In the experiment event doesn't have favourable results number $= n - m$

Assume that event $\bar{A}$ is not favourable to the event A.

$\therefore$ We say that $P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$

$\therefore P(A) + P(\bar{A}) = 1$

We know that

$m \leq n$

$\Rightarrow 0 \leq m \leq n$

$\Rightarrow 0 \leq \frac{m}{n} \leq 1$

$\Rightarrow 0 \leq P(A) \leq 1$

$\therefore$ Any event probability always lies between 0 and 1.

(c.) Sample space:

The set of all possible simple events in a trial is called a sample space. Every element of a sample space is called a sample point.

Ex: Two coins are tossed.

Sample space $S = \{HH, HT, TH, TT\}$

(d) Probability axiomatic approach:

Assume that S is a sample space. If we take any probability function satisfies the following axioms is called probability axiomatic approach. ① $P(E) \geq 0$ for every $E \subset S$

② $P(S) = 1$

③ $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

where $E_1 \subset S$, $E_2 \subset S$, $E_1 \cap E_2 = \phi$

Q: in a sample space S, prove that $P(\phi) = 0$, $P(\overline{E}) = 1 - P(E)$

A: Given that S is a sample space

E is any event of S

$\therefore E \subset S$

We know that $E = E \cup \phi$

$$P(E) = P(E \cup \phi)$$

We know that $E \cap \phi = \phi$

By definition $P(E) = P(E \cup \phi) = P(E) + P(\phi)$

$$P(E) = P(E) + P(\phi)$$

$$P(\phi) = P(E) - P(E)$$

$$P(\phi) = 0$$

We know that S is a sample space.

Assume that E is any event in S.

$E \subset S$

We know that not favourable results to Event E is event $\overline{E}$.

$\overline{E} \subset S$ but $E \cup \overline{E} = S$

$$E \cap \overline{E} = \phi$$

$$\cancel{P(E) \cup P(\overline{E}) = P(S)}$$

$$P(E \cup \overline{E}) = P(S)$$

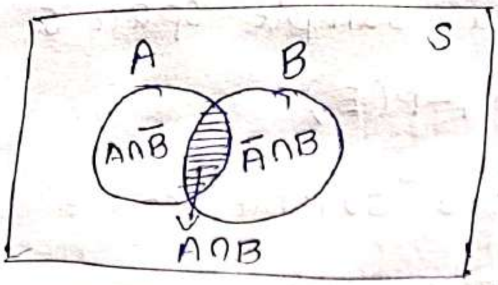
$$P(E) + P(\overline{E}) = 1$$

$$P(\overline{E}) = 1 - P(E)$$

Q. ADDITION THEOREM ON PROBABILITY

In a sample space S, A and B are any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ is called addition theorem on probability

Proof :



S is a sample space, A and B are two events in s. By observing the diagram

$$(A\cap\overline{B})\cap(A\cap B)=\phi$$

$$(A\cap\overline{B})\cup(A\cap B)=A$$

~~∴ By definition~~

Also, $A\cap(\overline{A}\cap B)=\phi$

$$A\cup(\overline{A}\cap B)=A\cup B$$

∴ By definition

$$P(A\cup B)=P(A)+P(\overline{A}\cap B)$$
$$=P(A)+P(\overline{A}\cap B)+P(A\cap B)-P(A\cap B)$$
$$=P(A)+P((\overline{A}\cap B)\cup(A\cap B))-P(A\cap B)$$
$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Hence the theorem is proved.

3. Define conditional Probability

Define Conditional Probability?

Sol: Let S be a sample space.
Let A, E are any two events in S.
The event E happening is exist after the happening of event A is called conditional event and is denoted by E/A. This conditional event probability is defined as $P(E/A) = P(E \cap A)/P(A)$ where $P(A) > 0$.

Q: <u>Multiplication Theorem of Probability:</u>

In a random experiment E_1, E_2 are any two events such that $P(E_1) \neq 0, P(E_2) \neq 0$

Then $P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$.

$$P(E_2 \cap E_1) = P(E_2) \cdot P\left(\frac{E_1}{E_2}\right)$$

Proof: In a random experiment S is a sample space. E_1, E_2 are any two events in S.

$\therefore$ $E_1 \subset S, E_2 \subset S$

$$P(E_1) \neq 0, \; P(E_2) \neq 0$$

Assume that E_2 happening is depending on E_1.

$\therefore$ Condition event is E_2/E_1

By definition $P(E_2/E_1) = \dfrac{P(E_1 \cap E_2)}{P(E_1)}$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1)$$

Assume that E_1 happening is depending on E_2

We say that conditional event is E_1/E_2

By definition $P(E_1/E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)}$

$$P(E_1 \cap E_2) = P(E_2)\, P(E_1/E_2)$$

Q: State and Prove Boole's inequality

For n events $A_1, A_2, \dots A_n$ we have

(1) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

(2) $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

Proof:

Given that

$A_1, A_2, \dots A_n$ are n events

Assume that given statement is

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1) \longrightarrow (1)$$

To prove this result, by using mathematical induction

We know that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P(A_1 \cup A_2) \leq 1$$

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$P(A_1) + P(A_2) - 1 \leq P(A_1 \cap A_2)$$

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$$

$$P\left(\bigcap_{i=1}^{2} A_i\right) \geq \sum_{i=1}^{2} P(A_i) - (2-1) \longrightarrow (2)$$

By observing eq (2) we say that

$\therefore$ Statement is true for $n=2$

Assume that given statement is true for $n=k$.

$$P\left(\bigcap_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{k} P(A_i) - (k-1) \longrightarrow (3)$$

$$P\left(\bigcap_{i=1}^{k+1} A_i\right) = P\left(\bigcap_{i=1}^{k} A_i \cap A_{k+1}\right)$$

$$\geq P\left(\bigcap_{i=1}^{k} A_i\right) + P(A_{k+1}) - 1$$

$$\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1$$

$$= \sum_{i=1}^{k+1} P(A_i) - k + 1 - 1$$

$$P\left(\bigcap_{i=1}^{k+1} A_i\right) \geq \sum_{i=1}^{k+1} P(A_i) - k \longrightarrow (4)$$

Put $n = k+1$ in eq (1)

We get eq (4)

$\therefore$ eq (1) is true for $n = k+1$

Using mathematical induction, given statement is true for all possible values of n.

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Assume that given statement is

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \longrightarrow (1)$$

We know that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

But $P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2)$

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

$$P\left(\bigcup_{i=1}^{2} A_i\right) \le \sum_{i=1}^{2} P(A_i) \longrightarrow (2)$$

In the eq (1) put $n=2$ we get eq (2)

$\therefore$ Given statement is true for $n=2$

Assume given statement is true for $n=k$

$$P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i) \longrightarrow (3)$$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

$$\le P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$$

$$\le \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$$

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) \le \sum_{i=1}^{k+1} P(A_i) \longrightarrow (4)$$

Put $n=k+1$ in eq (1)

We get eq (4)

$\therefore$ Given statement is true for $n=k+1$

Using mathematical induction given statement is true for all possible values of n.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

Q: State & Prove Baye's theorem ;

If $E_1, E_2, \ldots E_n$ are mutually disjoint events with $P(E_i) \neq 0$ for each i then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ with $P(A) > 0$ we have

$$P\left(E_i/A\right) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

Proof :

Given that $E_1, E_2, \ldots E_n$ are mutually disjoint events

$$E_i \cap E_j = \phi \quad \text{where } i \neq j$$

Given that $P(E_i) \neq 0$

for $i = 1, 2, \ldots n$

Given that A is an arbitrary event

$$A \subset \bigcup_{i=1}^{n} E_i \quad \text{and} \quad P(A) > 0$$

$$A \cap \bigcup_{i=1}^{n} E_i = A$$

$$\bigcup_{i=1}^{n} (A \cap E_i) = A$$

$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap E_i)\right)$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$= \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)$$

We know that by definition:

$$P(E_i/A) = P(A \cap E_i) / P(A)$$

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)}$$