$$I = \iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$$

over $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$

$\Downarrow$ Ellipsoide

**Sol:** Using Ellipsoidal polar coordinates

put $x = a r \sin\theta \cos\phi$

$y = b r \sin\theta \sin\phi$

$z = c r \cos\theta$

$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$

In the 1st octant of the Ellipsoide $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$r$ varies from 0 to 1

$\theta$ —"— 0 to $\pi/2$

$\phi$ —"— 0 to $\pi/2$

$$I = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \sqrt{1 - r^2}\; abc\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 8abc \int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^2 \sqrt{1 - r^2}\, dr$$

$$= (8abc)\, (\phi)_0^{\pi/2}\, (-\cos\theta)_0^{\pi/2}\; \frac{1}{2} \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= (8abc)\left(\frac{\pi}{2}\right)(1)\, \frac{1}{2}\, \frac{\Gamma\frac{3}{2}\Gamma\frac{3}{2}}{\Gamma 3} = \pi abc\, \frac{1}{2}\Gamma\frac{1}{2}\, \frac{1}{2}\Gamma\frac{1}{2} = \frac{\pi^2 abc}{4}$$

---

put $r^2 = t$

$\Rightarrow r = t^{1/2}$

$dr = \frac{1}{2} t^{-1/2}\, dt$

| $r$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | 1 |

$$\int_0^1 r^2 \sqrt{1 - r^2}\, dr$$

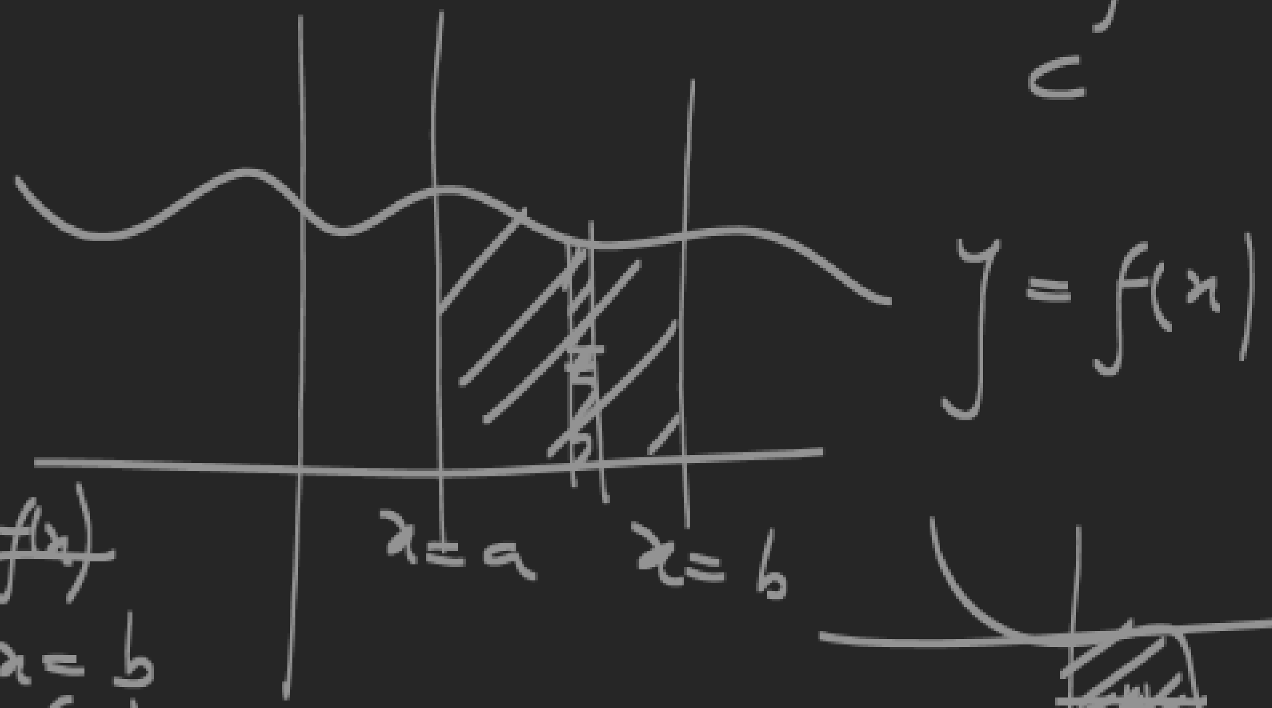
$$= \int_0^1 t\, (1-t)^{1/2}\, \frac{1}{2} t^{-1/2}\, dt$$

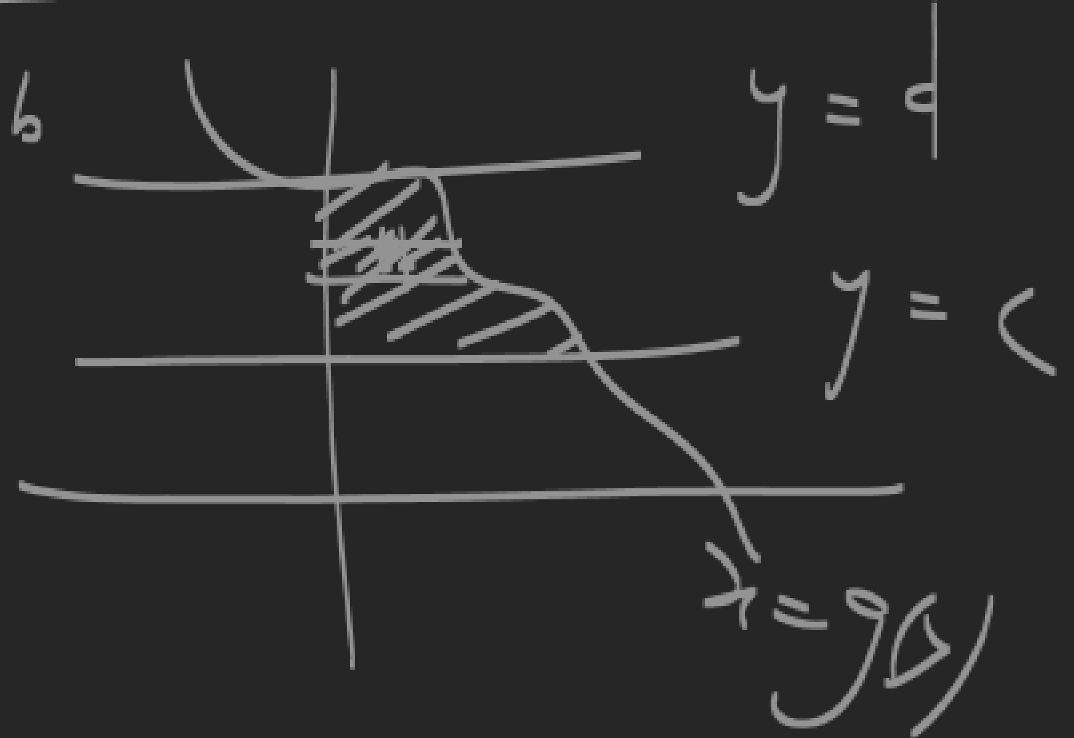
$$= \frac{1}{2} \int_0^1 t^{1/2} (1-t)^{1/2}\, dt$$

$$\underline{\text{Area}} = \iint dx\,dy$$

$$\text{Area} = \int_a^b y\,dx$$

$$= \int_c^d x\,dy$$




$$\int_a^b dx \qquad \int_a^b f(x)\,dx \qquad \int_a^b \left[y\right]_0^{f(x)} dx$$

$y = 0$ to $y = f(x)$

$x = a$ to $x = b$

$$\int_a^b \int_{y=0}^{f(x)} dx\,dy$$

$\boxed{y^2 = 4 - x}$ , $\boxed{y^2 = 4 - 4x}$

$\Downarrow$ parabola

$\boxed{y^2 = -(x-4)}$

parabola
vertex $(4,0)$
AOS $(y=0)$
opening (on LHS)

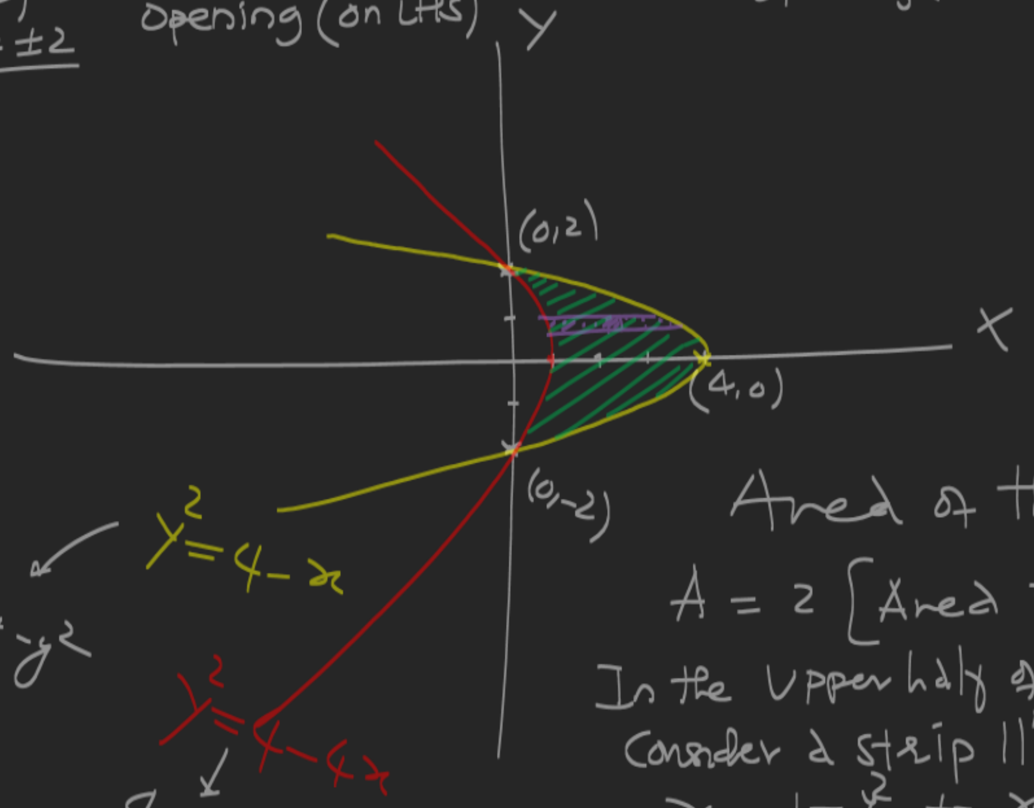
if $x=0$
$y^2=4$
$y=\pm 2$

$\Downarrow$ parabola

$\boxed{y^2 = -4(x-1)}$

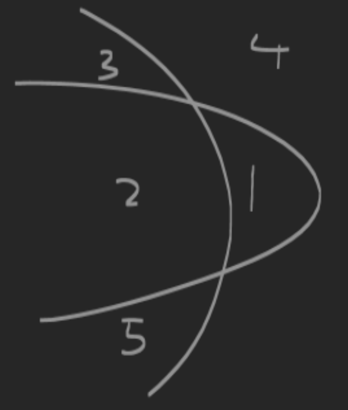
parabola
vertex $(1,0)$
AOS $(y=0)$
opening (on LHS)

if $x=0$
$y^2=4$
$y=\pm 2$




$x = 4 - y^2$

$4x = 4 - y^2$
$x = 1 - \frac{y^2}{4}$




Area of the shaded Region $(A) = \int\int dx\,dy$

$A = 2$ [Area in the upper half]

In the upper half of the shaded Region
Consider a strip ||'el to x-axis
$x = 1 - \frac{y^2}{4}$ to $x = 4 - y^2$
$y = 0$ to $y = 2$

$$A = 2\int_{y=0}^{2}\int_{x=1-\frac{y^2}{4}}^{4-y^2} dx\,dy$$

$$A = 2\int_0^2 [x]_{1-\frac{y^2}{4}}^{4-y^2}\,dy$$

$$= 2\int_0^2 \left((4-y^2) - \left(1-\frac{y^2}{4}\right)\right) dy$$

$-1+\frac{1}{4}$

$\frac{-4+1}{4} = -\frac{3}{4}$

$$= 2\int_0^2 \left(3 - \frac{3}{4}y^2\right) dy$$

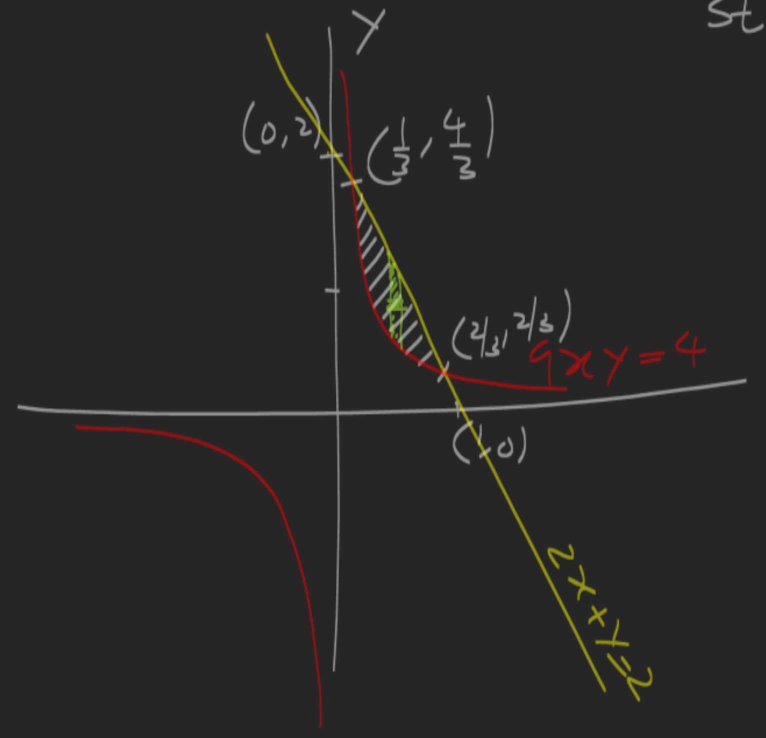
$$= 2(3)\int_0^2 \left(1 - \frac{y^2}{4}\right) dy$$

$$= 6\left[y - \frac{1}{4}\frac{y^3}{3}\right]_0^2$$

$$= 6\left[2 - \frac{1}{12}(8)\right] = 6\left[2 - \frac{2}{3}\right] = 6\left[\frac{4}{3}\right] = 8 \text{ sq. units}$$

$9xy = 4$, $\boxed{2x + y = 2}$

St Line passing th $(0,2)$, $(1,0)$




WBCM

$y = 2-2x$
$y = \frac{4}{9x}$

$\frac{4}{9x} = 2-2x$

$\frac{2}{9x} = 1-x$

$2 = 9x - 9x^2$

$9x^2 - 9x + 2 = 0$

$9x^2 - 6x - 3x + 2 = 0$

$3x(3x-2) - 1(3x-2) = 0$

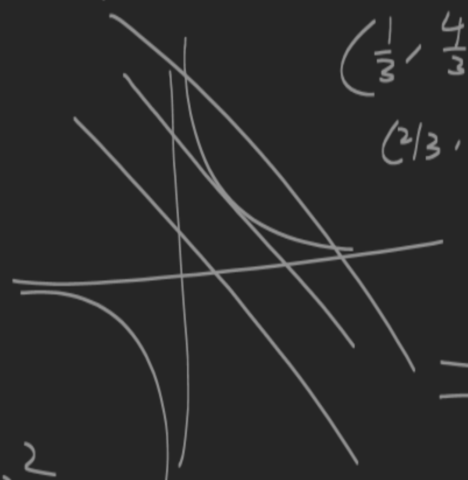
$(3x-2)(3x-1) = 0$

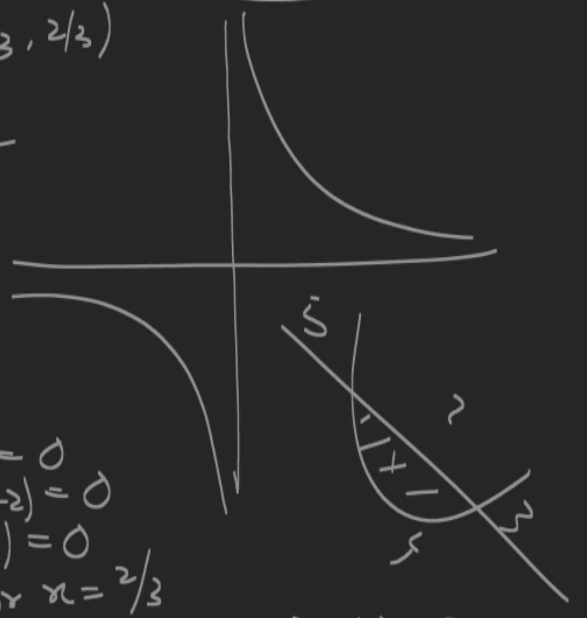
| | |
|---|---|
| $x = \frac{1}{3}$ | or $x = \frac{2}{3}$ |
| $y = 2 - 2\left(\frac{1}{3}\right)$ | $y = 2 - 2\left(\frac{2}{3}\right)$ |
| $= 2 - \frac{2}{3}$ | $= 2 - \frac{4}{3}$ |
| $= \frac{4}{3}$ | $= \frac{2}{3}$ |
| $\left(\frac{1}{3}, \frac{4}{3}\right)$ | $\left(\frac{2}{3}, \frac{2}{3}\right)$ |

$\left(\frac{1}{3}, \frac{4}{3}\right)$

$(2/3, 2/3)$

$xy = k^2$





$2x + y = 2$

$y = 2 - 2x$

Area of the shaded Region

$(A) = \int\int dx\,dy$

In the shaded Region
Consider a strip ||'el to y-axis

$y = \frac{4}{9x}$ to $y = 2-2x$

$x = \frac{1}{3}$ to $x = 2/3$

$$A = \int_{x=1/3}^{2/3} \int_{y=\frac{4}{9x}}^{2-2x} dx\,dy$$

$$A = \int_{1/3}^{2/3} \left[ y \right]_{\frac{4}{9x}}^{2-2x} dx$$

$$= \int_{1/3}^{2/3} \left(2 - 2x - \frac{4}{9x}\right) dx$$

$$= \left(2x - x^2 - \frac{4}{9}\log x\right)_{1/3}^{2/3}$$

$$= 2\left(\frac{1}{3}\right) - \left(\frac{4}{9} - \frac{1}{9}\right) - \frac{4}{9}\left(\log(2/3) - \log(1/3)\right)$$

$$= \frac{2}{3} - \frac{1}{3} - \frac{4}{9} \log\left(\frac{2/3}{1/3}\right)$$

$$A = \left(\frac{1}{3} - \frac{4}{9}\log(2)\right) \text{ sq. units}$$

05

$y = x^2 - 6x + 3$, $\boxed{y = 2x - 9}$

$\Downarrow$

St. Line passing th $(0, -9)$, $\left(\frac{9}{2}, 0\right)$

$= x^2 - 6x + 9 - 9 + 3$

$= (x-3)^2 - 6$

$(y+6) = (x-3)^2$

$(x-3)^2 = (y+6)$

$\Longrightarrow$ parabola

Vertex $(3, -6)$

AOS $(x = 3)$

Opening $\uparrow$

$y = 0$

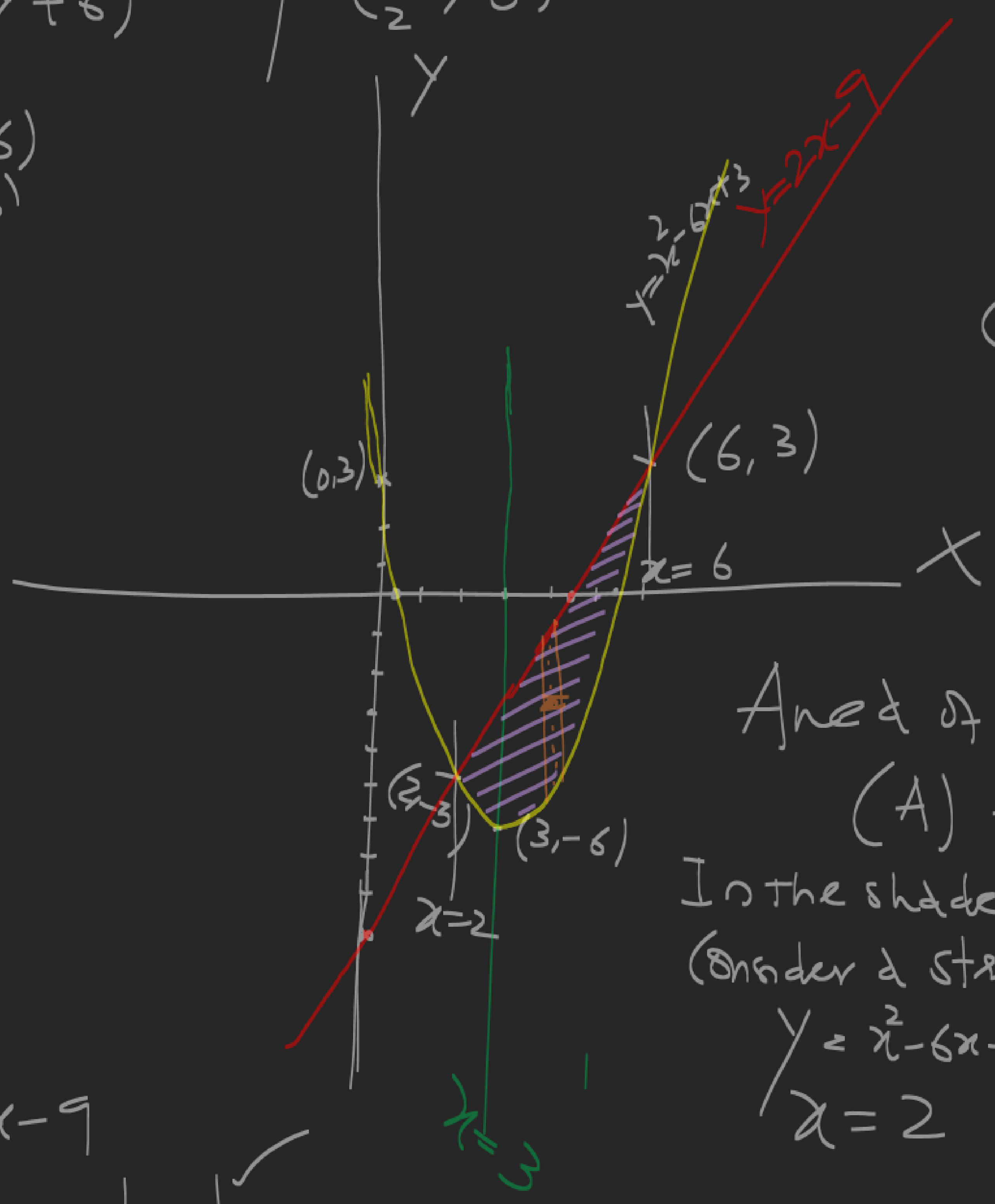
$(x-3)^2 = 6$

$x - 3 = \pm\sqrt{6}$

$x = 3 \pm \sqrt{6}$

$x = 0$

$\Longrightarrow y = 3$





WBCM

$y = x^2 - 6x + 3$, $y = 2x - 9$

$x^2 - 6x + 3 = 2x - 9$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

$x = 2$, $x = 6$

$\Downarrow$ $\qquad \Downarrow$

$y = -5$ $\quad$ $y = 3$

$(2, -5)$

$(6, 3)$

Area of the shaded Region

$$(A) = \iint dx\,dy$$

In the shaded Region

Consider a strip ll'el to yaxis

$y = x^2 - 6x + 3$ to $y = 2x - 9$

$x = 2$ to $x = 6$

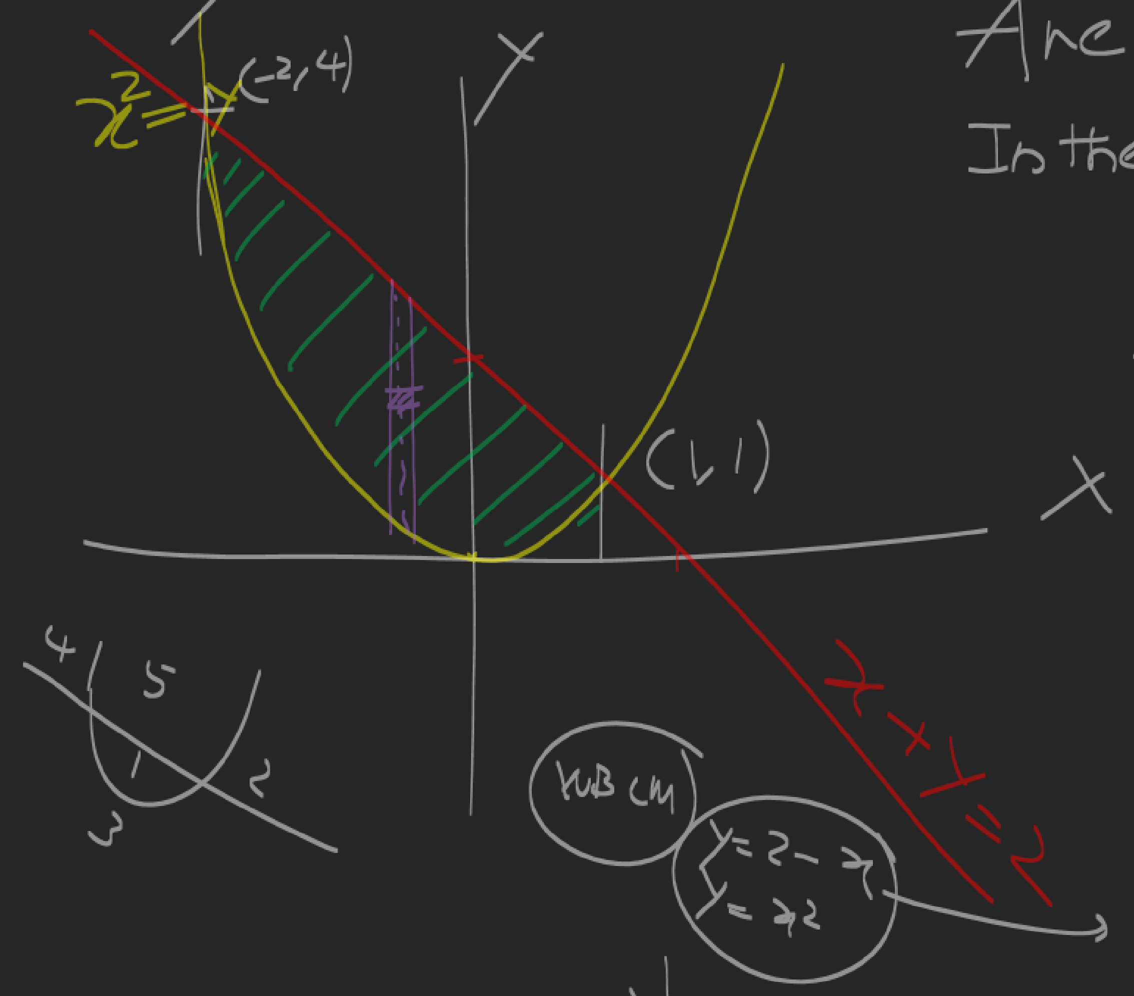
$$A = \int_{x=2}^{6} \int_{y=x^2-6x+3}^{2x-9} dx\,dy$$

$$= \int_2^6 \left[y\right]_{x^2-6x+3}^{2x-9} dx = \int_2^6 \left((2x-9) - (x^2-6x+3)\right) dx$$

$$= \int_2^6 (-x^2 + 8x - 12)\, dx = \left(-\frac{x^3}{3} + 8\frac{x^2}{2} - 12x\right)_2^6 = -\frac{1}{3}(208) + 4(32) - 12(4)$$

$$= 80 - \frac{208}{3} = \frac{240 - 208}{3} = \frac{32}{3} \text{ sq. units}$$

$y = x^2, \quad x + y = 2$




Area of the shaded Region (A) $= \iint dx\,dy$

In the shaded Region, Consider a strip ll'el to y axis

$y = x^2$ to $y = 2 - x$

$x = -2$ to $x = 1$

$$A = \int_{x=-2}^{1} \int_{y=x^2}^{2-x} dx\,dy = \int_{-2}^{1} [y]_{x^2}^{2-x} dx$$

$$= \int_{-2}^{1} (2 - x - x^2)\,dx$$

$x^2 = 2 - x$ | $(x+2)(x-1) = 0$ | (1, 1)

$x^2 + x - 2 = 0$ | $x = 1, -2$ | (-2, 4)

$y = 1 \quad y = 4$

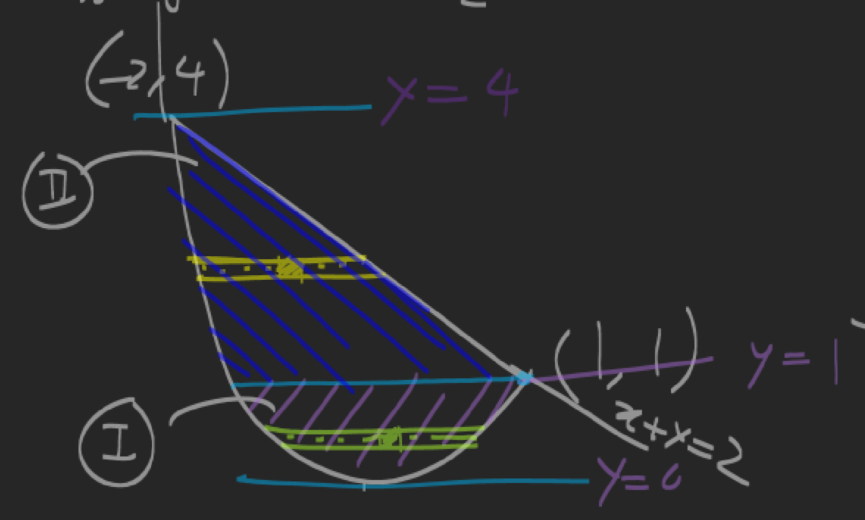
$$A = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-2}^{1}$$

$x = \pm\sqrt{y}$

$x^2 = y$

$$= 2(3) - \frac{1}{2}(1-4) - \frac{1}{3}(1+8)$$

$$= 6 + \frac{3}{2} - 3 = 3 + \frac{3}{2} = \frac{9}{2} \text{ sq.units}$$




Region(II) $x = -\sqrt{y}$ to $x = 2 - y$

$y = 1$ to $y = 4$

Region(I) $x = -\sqrt{y}$ to $x = \sqrt{y}$

$y = 0$ to $y = 1$

$$\text{Area }(A) = \int_{y=0}^{1} \int_{x=-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{y=1}^{4} \int_{x=-\sqrt{y}}^{2-y} dx\,dy$$