Problem Sheets [1]

1. For a real number n the function $\log^*(n)$ is defined as follows: $\log^*(n)$ is the smallest natural number i so that after applying logarithm function (base 2) i times on n we get a number less than or equal to 1. E.g. $\log^*(2^2)$ is 2 because $\log(\log(2^2)) = 1 \leq 1$. $\log^*(2^{2^2})$ is 3 because $\log\log(\log(2^{2^2})) = 1 \leq 1$.
   Either prove or disprove:

   (a) $\log(\log^*(n)) = O(\log^*(\log(n)))$.

   (b) $\log^*(\log(n)) = O(\log(\log^*(n)))$.

2. Take the following list of functions and arrange them in ascending order of growth rate (with proof). That is, if function $g(n)$ immediately follows function $f(n)$ in your list, then it should be the case that $f(n)$ is $O(g(n))$.

   (a) $g_1(n) = 2^{\sqrt{\log n}}$.

   (b) $g_2(n) = 2^n$.

   (c) $g_3(n) = n^{\frac{4}{3}}$.

   (d) $g_4(n) = n(\log n)^3$.

   (e) $g_5(n) = n^{\log n}$.

   (f) $g_6(n) = 2^{2^n}$.

   (g) $g_7(n) = 2^{n^2}$.

3. Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give a proof or counterexample:

   (a) $\log_2 f(n)$ is $O(\log_2 g(n))$.

   (b) $2^{f(n)}$ is $O(2^{g(n)})$.

   (c) $f(n)^2$ is $O(g(n)^2)$.

4. Prove that

   $$\Theta(n-1) + \Theta(n) = \Theta(n).$$

   Does it follow that

   $$\Theta(n) = \Theta(n) - \Theta(n-1)?$$

   Justify your answer.

5. Let $f(n)$ and $g(n)$ be two nonnegative functions, and suppose there is a constant $c > 0$ for which $\lim_{n\to\infty} f(n)/g(n) = c$ then $f(n) = \Theta(g(n))$.

6. If $f(n) \in o(g(n))$, then $f(n)$ is in $O(g(n)) \setminus \Omega(g(n))$.

7. Suppose $a > 1$ and $b \neq 0$ are constants, with $|b| < a$. Prove that $a^n + b^n = \Theta(a^n)$.

8. Use mathematical induction to prove that, for all integers $k > 1$ and some $c > 0$, $\log^k n = O(n^c)$.

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