Homework 2

1) $f(x) = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$

Take gradient of $f(x)$

$$\frac{df(x)}{dx_1} = 4x_1 - 4x_2 = 0 \quad \text{— (1)}$$

$$\frac{df(x)}{dx_2} = -4x_1 + 3x_2 + 1 = 0 \quad \text{— (2)}$$

Solving Eq^n (1) & (2) to get the values of x_1 & x_2

we get $x_1 = 1$ & $x_2 = 1$

Stationary Point

Hessian

$$\frac{d^2f(x)}{dx_1^2} = 4 \quad -4$$

$$\frac{d^2f(x)}{dx_2^2} = -4 \quad 3$$

$$H(x) = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

$\det |\lambda I - A| = 0$

$$(\lambda-4)(\lambda-3) - 16 = 0$$
$$\lambda^2 - 7\lambda - 2 = 0$$

Clearly eigen values are positive & negative & hence Hession is indefinite.

$\therefore$ Stationary Point is Saddle point

Taylor's Expansion

$$f(x) = f(x_0) + g_{x_0}(x - x_0) + \frac{1}{2}(x - x_0)^T H_{x_0}(x - x_0)$$

here $g_{x_0} = 0$

$$\Rightarrow f(x) - f(x_0) = \frac{1}{2}(x - x_0)^T H_{x_0}(x - x_0)$$

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad f(x_0) = 0.5$$

$$f(x) - 0.5 = \frac{1}{2}[x_1 - 1, \ x_2 - 1] \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 2 \end{bmatrix}$$

$$f(x) - 0.5 = \frac{1}{2}\begin{bmatrix} x_1 - 1, & x_2 - 1 \end{bmatrix}\begin{bmatrix} 4x_1 - 4 - 4x_2 + 8 \\ -4x_1 + 4 + 3x_2 - 6 \end{bmatrix}$$

$$= \frac{1}{2}\Big[4x_1^2 - 4x_1x_2 + 4x_1 + -4x_2x_1 - 2x_2 + 3x_2^2 + \cancel{-4x_1} + 4x_2 + 4 \cancel{+4x_1} - 3x_2 + 2\Big]$$

$$0 \geq \frac{1}{2}\left[4x_1^2 + 3x_2^2 - 8x_1x_2 + 4x_1 - x_2 - 2\right]$$

2a) Eq^{n} of Plane

$$x_1 + 2x_2 + 3x_3 = 1 \quad \text{in } R^3$$

Point $(-1, 0, 1)$

$$\text{Min } (x_1+1)^2 + x_2^2 + (x_3-1)^2$$

Unconstrained

Take $x_1 = 1 - 2x_2 - 3x_3$

$$\text{Min } (1-2x_2-3x_3+1)^2 + x_2^2 + (x_3-1)^2$$

$$\frac{df(x)}{dx_2} = -2(2-2x_2-3x_3)\cdot 2 + 2x_2$$

$$\Rightarrow 10x_2 + 12x_3 - 8 = 0$$

$$\Rightarrow 5x_2 + 6x_3 - 4 = 0 \quad \text{—} \quad \boxed{1}$$

$$\frac{df(x)}{dx_3} = -6(2-2x_2-3x_3) + 2(x_3-1)$$

$$= 20x_3 + 12x_2 - 14 = 0$$

$$= 6x_2 + 10x_3 - 7 = 0 \quad \text{—} \quad \boxed{2}$$

2 − 1

$$x_2 + 4x_3 - 3 = 0$$

$$x_2 = 3 - 4x_3$$

$$\Rightarrow 15 - 20x_3 + 6x_3 - 4 = 0$$

$$\boxed{x_3 = \frac{11}{14}}$$

$$\Rightarrow 5x_2 - 6 \times \frac{11}{14} - 4 = 0$$

$$\boxed{x_2 = -\frac{1}{7}}$$

$$x_1 = 1 - 2x_2 - 3x_3 \quad \boxed{x_1 = -\frac{15}{14}}$$

$$H = \begin{matrix} \frac{d^2 f(x)}{dx_2^2} & \frac{d^2 f(x)}{dx_2 x_3} \\ \frac{d^2 f(x)}{dx_2 x_3} & \frac{d^2 f(x)}{dx_3^2} \end{matrix}$$

$$H = \begin{bmatrix} 10 & 12 \\ 12 & 20 \end{bmatrix}$$

$$\det |\lambda I - A| = 0$$

$$(\lambda - 10)(\lambda - 20) - 144 = 0$$

$$\lambda^2 - 30\lambda + 56 = 0$$

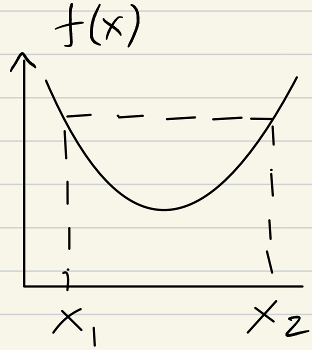
$$\lambda = 2 \quad \text{or} \quad \lambda = 28$$

Hessian is positive definite

$\therefore$ It is Convex

3 a) For any points x_1 & $x_2 \in X$
A Convex funct^n is based on Convex set

$\lambda x_1 + (1-\lambda) x_2$



$f: X \to R$ is Convex
iff X is a convex set
& $\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

|||ly

$$g(\lambda x_1 + (1-\lambda) x_2) \leq \lambda g(x_1) + (1-\lambda) g(x_2)$$

$\therefore$

$$\boxed{\begin{gathered} af(\lambda x_1 + (1-\lambda) x_2) + bg(\lambda x_1 + (1-\lambda) x_2) \\ \leq \\ \lambda(af(x_1) + bg(x_1)) + (1-\lambda)(af(x_2) + bg(x_2)) \end{gathered}}$$

$\therefore$ $af(x) + bg(x)$ is convex $\quad \forall a, b > 0$

3b) Take $j(x) = f(g(x))$

Differentiate $j(x)$ wrt x

$$\frac{dj(x)}{dx} = f'(g(x)) \cdot g'(x)$$

Differentiate again [use chain Rule]

$$\frac{d^2 j(x)}{dx^2} = f''(g(x))\, g'(x) \cdot g'(x) + f'(g(x))\, g''(x)$$

$$= \underline{f''(g(x))\, g'^2(x) + f'(g(x))\, g''(x)}$$

The condition for the function to be convex is given by

$$\boxed{f''(g')^2 + f' g'' > 0}$$

4) To Show that $f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)$

For a Convex funcⁿ $f(x) : \mathcal{X} \to R$
$x_0, x_1 \in \mathcal{X}$

Take

Lemma

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1)$$

where $\lambda \in [0, 1]$

$$\Rightarrow f(\lambda x_0 + x_1 - \lambda x_1) \leq \lambda f(x_0) - \lambda f(x_1) + f(x_1)$$

$$\Rightarrow f(x_1 + \lambda(x_0 - x_1)) \leq f(x_1) + \lambda(f(x_0) - f(x_1))$$

$$\Rightarrow \frac{f(x_1)}{\lambda} \geq \frac{f(x_1 + \lambda(x_0 - x_1))}{\lambda} + f(x_1) - f(x_0)$$

$$\Rightarrow f(x_0) \geq f(x_1) + \frac{f(x_1 + \lambda(x_0 - x_1)) - f(x_1)}{\lambda}$$

$$\lim_{t \to 0} \frac{f(x_1 + \lambda(x_0 - x_1)) - f(x_1)}{\lambda} = f'(x_1)(x_0 - x_1)$$

$$f(x_0) \geq f(x_1) + f'(x_1)(x_0 - x_1)$$

Take

$$y = \lambda x_0 + (1-\lambda) x_1 \quad \forall \lambda \in [0,1]$$

We get

$$f(x_0) \geqslant f(y) + f'(y)(x_0 - y)$$

&

$$f(x_1) \geqslant f(y) + f'(y)(x_1 - y)$$

$$\therefore \lambda f(x_0) + (1-\lambda) f(x_1) \geqslant f(y)$$

Thus

$$\boxed{\lambda f(x_0) + (1-\lambda) f(x_1) \geqslant f(\lambda x_0 + (1-\lambda) x_1)}$$

5) n lamps & m mirrors

$I_t \equiv$ Target Intensity

$a_k^T p \equiv$ Reflection Intensity level

Objective : To min error b/w I_t & $a_k^T p$

$\Rightarrow$ formulation :

$$\min_p \sum_k (a_k^T p_i - I_t)^2$$

$$0 \leq p_i \leq p_{max} \quad \forall \; i = 1, 2, \ldots, n$$

gradient : $2(a_k^T p_i - I_t)\, a_k$

$$\Rightarrow 2 a_k^T p_i\, a_k - 2 a_k I$$

Hession : $2 a_k^T a_k \geq 0$

$\Rightarrow$ Use Lemma,

if $d^T H d \geq 0$ where $d \neq 0$
meaning H is p.s.d

Multiplying $d^T d$ on both sides

$$\Rightarrow d^T H d = \sum_k 2\, d^T a_k^T a_k d$$

$\Rightarrow \sum_{k} 2 U_k$

$\therefore$ J_+ is a Convex Problem

$\Rightarrow$ For Unique Sol^n, it should strictly convex

$\therefore$ Hessian should be positive definite

$\Rightarrow \sum_{k} 2 d^T a_k a_k^T d = \sum_{k} 2 U_k^2 > 0$

We cannot find d for all $U_k^2 = 0$

Meaning

No. of Mirrors > No. of Lamps

which is necessary for a Unique Sol^n

$\Rightarrow$ At half of lamps emitting light,
No. of Mirrors > No. of lamps

$\therefore$ J_+ has a Unique Sol^n