Control Systems

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Contents

***Abstract*—The objective of this manual is to introduce control system design at an elementary level.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/ketan/codes
```

1 Polar Plot

1.1 Introduction

1.1.1. Sketch the polar plot of

$$G(s) = \frac{1}{(s^2)(s+1)(s+2)}. \tag{1.1.1.1}$$

Solution: Substituting $s = \jmath\omega$ in (1.1.1.1),

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Now the magnitude will be

$$r = |G(\jmath\omega)| = \frac{1}{(\omega^2)(\sqrt{1+\omega^2})(\sqrt{1+4\omega^2})} \tag{1.1.1.2}$$

$$\theta = \angle G(\jmath\omega) = -\tan^{-1}(0) - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.1.3}$$

$$= 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.1.4}$$

The polar plot is the (r, θ) plot for $\omega \in (0, \infty)$. The following python code generates the polar plot in Fig. 1.1.1

```
codes/ee18btech11028.py
```

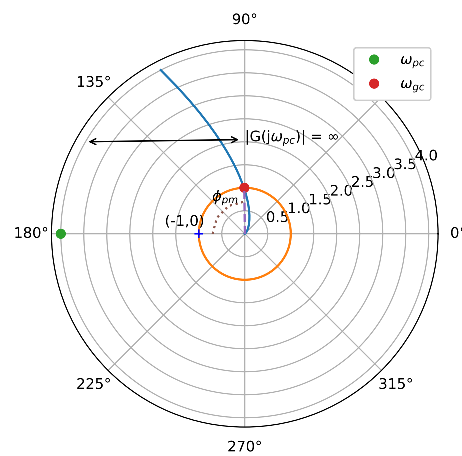

Fig. 1.1.1

The location of $(-1, 0)$ with respect to the polar plot provides information regarding the stability of the system.

- If $(-1, 0)$ is not enclosed, then it is stable.
- If $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 1.1.1, the point $(-1, 0)$ is enclosed by the polar plot, which implies system is not stable. The polar plot also provides info on the GM and PM, which can then be used for determining the stability of the system.

- If the $GM > 1 \cap PM > 0$, then the control system is **stable**.
- If the $GM = 1 \cap PM = 0$, then the control system is **marginally stable**.
- If the $GM < 1 \cup PM < 0$, then the control system is **unstable**.

Therefore, our system is unstable $\because GM < 1 \cap PM < 0$.

1.2 Example

1.2.1. Sketch the Polar Plot of

$$G(s) = \frac{1}{s(1+s^2)} \tag{1.2.1.1}$$

Solution: From (1.2.1.1),

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1-\omega^2)} \tag{1.2.1.2}$$

$$\left|G(\jmath\omega)\right| = \frac{1}{\left|\omega(1-\omega^2)\right|} \tag{1.2.1.3}$$

$$\angle G(\jmath\omega) = \begin{cases} \frac{\pi}{2} & \omega > 1 \\ -\frac{\pi}{2} & 0 < \omega < 1 \end{cases} \tag{1.2.1.4}$$

The corresponding polar plot is generated in Fig. 1.2.1 using

codes/ee18btech11023.py

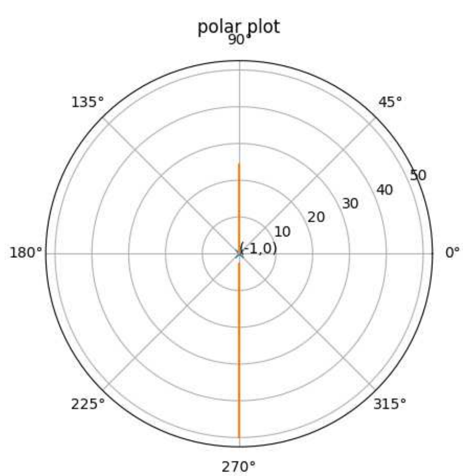

Fig. 1.2.1

In Fig. 1.2.1, (–1,0) is exactly on the polar plot. Hence, the system is marginally stable.

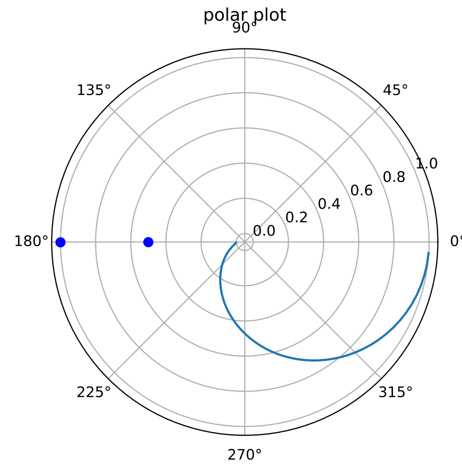

Fig. 1.3.1

1.3 Example

1.3.1. Sketch the Polar Plot for

$$G(s) = \frac{1}{(1+s)(1+2s)} \tag{1.3.1.1}$$

Solution: The following code generates Fig. 1.3.1

codes/ee18btech11012.py

The polar plot is to the right of $(-1, 0)$. Hence the closed loop system is stable.

1.4 Example

1.4.1. Plot the polar plot of

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}. \tag{1.4.1.1}$$

Solution:
The following python code generates the polar plot in Fig. 1.4.1

codes/ee18btech11033.py

$\because$ (-1,0) is on the right side of the polar plot, the system is unstable.

1.5 Example

1.1. Sketch the direct polar plot for a unity feedback system with open loop transfer function

$$G(s) = \frac{1}{s(1+s)^2} \tag{1.1.1}$$

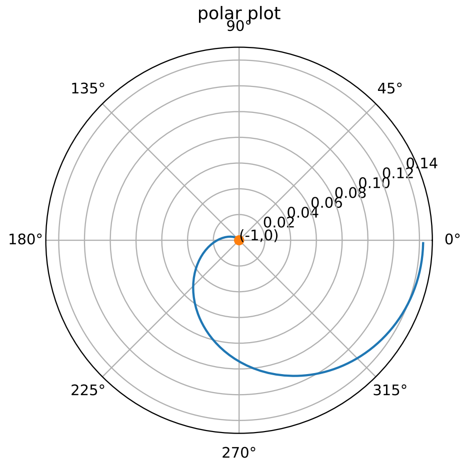

Fig. 1.4.1

Solution: The polar plot is obtained by plotting (r, ϕ)

$$r = |H(\jmath\omega)||G(\jmath\omega)| \tag{1.1.2}$$

$$\phi = \angle H(\jmath\omega)G(\jmath\omega), 0 < \omega < \infty \tag{1.1.3}$$

The following code plots the polar plot in Fig. 1.1

```
codes/ee18btech11002/polarplot.py
```

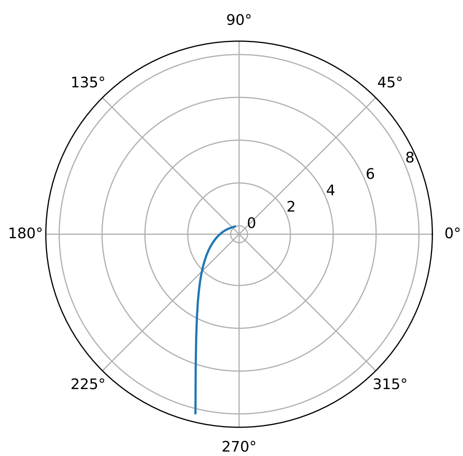

Fig. 1.1: Polar Plot

1.2. Sketch the inverse polar plot for (1.1.1)
Solution: The above code plots the polar plot in Fig. 1.2 by plotting $\left(\frac{1}{r}r, -\phi\right)$

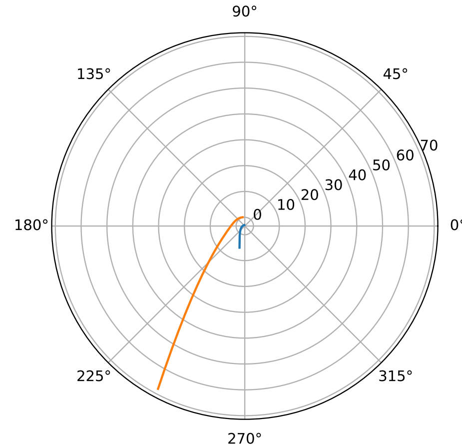

Fig. 1.2: Inverse Polar Plot

1.6 Example

1.1. Plot the polar plot of

$$G(s) = \frac{100(s+5)}{s(s+3)(s^2+4)}. \tag{1.1.1}$$

Solution: The following python code generates the polar plot in Fig. 1.1

```
codes/ee18btech11042.py
```

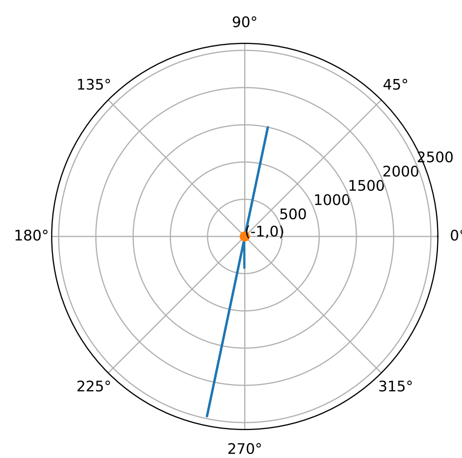

Fig. 1.1

Since (-1,0) is on the polar plot, the above system is marginally stable.

1.7 Example

1.7.1. Sketch the Polar Plot of

$$G(s) = \frac{\left(1 + \frac{s}{29}\right)(1 + 0.0025s)}{(s^3)(1 + 0.005s)(1 + 0.001s)} \tag{1.7.1.1}$$

Solution: The following code generates the polar plot in Fig. 1.7.1

```
codes/ee18btech11029.py
```

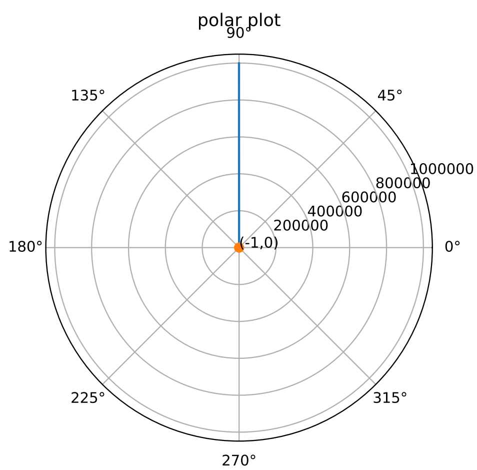

Fig. 1.7.1

- The polar plots use open loop transfer function to determine the stability and hence reference point is shifted to $(-1, 0)$
- If $(-1, 0)$ is left of the polar plot or $(-1, 0)$ is not enclosed, then it is stable
- If $(-1, 0)$ is on right side of the polar plot or $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 1.7.1, $(-1, 0)$ is on the polar plot so the system is marginally stable.

2 Bode Plot

2.1 Gain and Phase Margin

2.1. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{75(1 + 0.2s)}{s(s^2 + 16s + 100)} \tag{2.1.1}$$

Also compute gain margin and phase margin .
Solution: From (2.1.1), we have

$$G(\jmath\omega) = \frac{75(1 + 0.2\jmath\omega)}{\jmath\omega\left((\jmath\omega)^2 + 16\jmath\omega + 100\right)} \tag{2.1.2}$$

poles = 0 , -8-6j , -8+6j
zeros = -5
Gain and phase plots are shown in Fig. 2.1

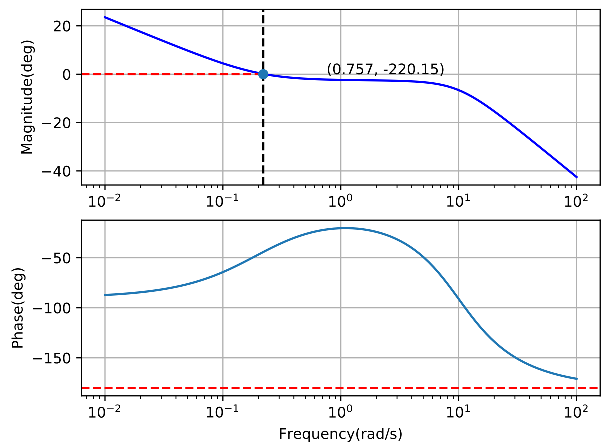

Fig. 2.1: a

The following code plots Fig. 2.1

```
codes/ee18btech11049.py
```

2.2. Find $\angle G(\jmath\omega) + 180^\circ$, where ω is frequency when gain = 1 . This is known as *phase margin* (PM)
Solution: Solving

$$\left|G(\jmath\omega)\right| = \frac{75\sqrt{\omega^2 + 25}}{\omega\sqrt{(\omega + 6)^2 + 64}\sqrt{(\omega - 6)^2 + 64}} = 1, \tag{2.2.1}$$

or from Fig. 2.1, the gain crossover frequency

$$\implies \omega_{gc} = 0.757 \tag{2.2.2}$$

$$\angle G\left(\jmath\omega_{gc}\right) = -88.3 \tag{2.2.3}$$

$$\implies PM = 91.7 \tag{2.2.4}$$

2.3. Find $-G(\jmath\omega)$ db , where ω is frequency when phase = -180° . This is known as *gain margin* (GM)
Solution: From Fig. 2.1 ,we can say that phase never crosses -180° . So , the gain margin is *infinite*. Which means we can add any gain,

and the equivalent closed loop system never becomes unstable.

2.2 *Example*

2.1. Sketch the Bode Magnitude and Phase plot for the following system. Also compute the gain margin and the phase margin.

$$G(s) = \frac{10}{s(1+0.5s)(1+.01s)} \tag{2.1.1}$$

Solution: The Bode magnitude and phase plot are available in Fig. 2.1 and generated by

```
codes/ee18btech11048.py
```

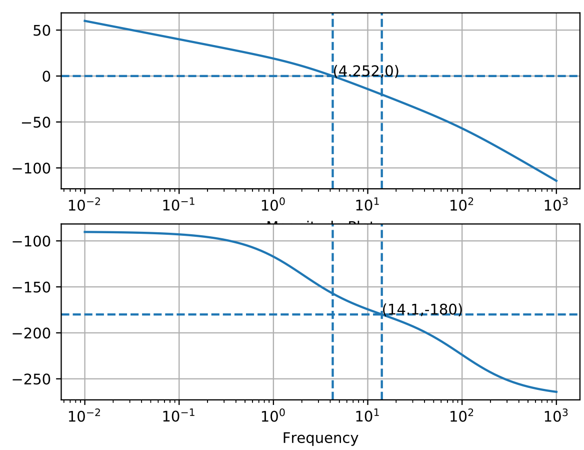

Fig. 2.1: Graphs

The pole-zero locations are available in Table 2.1.

Zeros	Poles
-	0
	-2
	-100

TABLE 2.1: Zeros and Poles

The *Gain* and *Phase* of (2.1.2) are

$$\left|G(\jmath\omega)\right| = \frac{100}{\omega\sqrt{(0.5\omega)^2+1}\sqrt{(0.01\omega)^2+1}} \tag{2.1.2}$$

$$\angle G(\jmath\omega) = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{100}\right) \tag{2.1.3}$$

Hence,

$$\left|G(\jmath\omega_{gc})\right| = 1 \tag{2.1.4}$$
$$\implies \omega_{gc} = 4.25 \tag{2.1.5}$$
$$\angle G(\jmath\omega_{gc}) = -157.2 \tag{2.1.6}$$
$$\implies PM = 22.8 \tag{2.1.7}$$

Similarly,

$$\angle G(\jmath\omega_{pc}) = -180^\circ \tag{2.1.8}$$
$$\implies \omega_{pc} = 14.1 \tag{2.1.9}$$
$$\implies -\left|G(\jmath\omega_{pc})\right| = -20.2dB \tag{2.1.10}$$
$$\implies GM = 20.2dB \tag{2.1.11}$$

2.3 *Example*

2.1. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)} \tag{2.1.1}$$

Also compute gain margin and phase margin . **Solution:** Substituting $s = \jmath\omega$ in (2.1.1) and assuming K =1,

$$G(\jmath\omega) = \frac{(\jmath\omega)^2}{(1+0.2\jmath\omega)(1+0.02\jmath\omega)} \tag{2.1.2}$$

The corner frequencies are

$$\omega_{c1} = 1/0.2 = 5 \tag{2.1.3}$$
$$\omega_{c2} = 1/0.02 = 50 \tag{2.1.4}$$

2.2. Magnitude Plot Calculation.
Solution:

$$20\log\left|G(\jmath\omega)\right| = 20\log\left|(\jmath\omega)^2\right| - 20\log\left|(1+0.2\jmath\omega)\right| - 20\log\left|(1+0.02\jmath\omega)\right| \tag{2.2.1}$$

The various values of G($\jmath\omega$) are available in Table 2.2, in the increasing order of their corner frequencies also slope contributed by each term and the change in slope at the corner frequency. The pase

TERM	Corner Freq	Slope	Slope change
$(\jmath\omega)^2$	$--$	+40	$--$
$\frac{1}{1+\jmath 0.2}$	$\omega_{c1} = \frac{1}{0.2}$	-20	40-20=20
$\frac{1}{1+\jmath 0.02}$	$\omega_{c2} = \frac{1}{0.02}$	-20	20-20=0

TABLE 2.2: Magnitude

$$\begin{aligned}\phi = \angle G(\jmath\omega) = 180^\circ \\ - tan^{-1}(0.2\omega) - tan^{-1}(0.02\omega)\end{aligned} \tag{2.2.2}$$

The phase angle of G($\jmath\omega$) are calculated for various value of ω in Table 2.2. The magnitude

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(0.02\omega)$	$\phi = \angle G(j\omega)$
0.5	5.7	0.6	174
1	11.3	1.1	168
2	21.8	2.3	156
5	45	5.7	130
10	63.4	11.3	106
50	84.3	45	50

TABLE 2.2: Phase

and phase plot are generated in Fig. 2.2 using

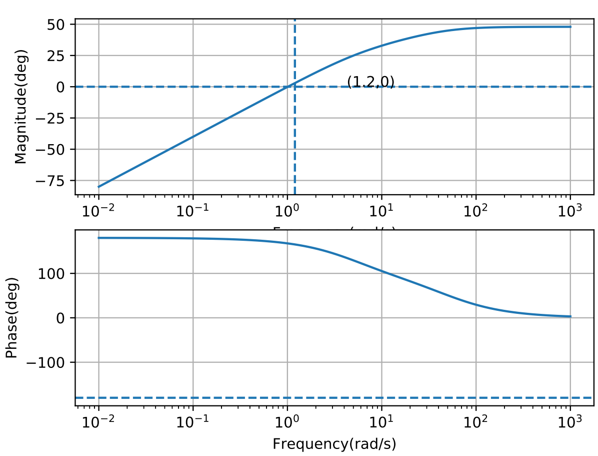

Fig. 2.2: Graphs

the following python code

```
codes/es17btech11002.py
```

$\because$ the gain crossover frequency is 2 and the corresponding gain At $\omega = 2$ is 13dB,

$$20 \log K = -13db \tag{2.2.3}$$

$$\implies K = 0.65 \tag{2.2.4}$$

Solving (2.1.2) or from Fig. 2.2, the gain crossover frequency,

$$\omega_{gc} = 1.2 \tag{2.2.5}$$

$$\implies PM = 344.8 \tag{2.2.6}$$

From Fig. 2.2 ,we can say that phase never crosses -180°. So, the gain margin is *infinite*. Which means we can add any gain, and the equivalent closed loop system never goes unstable.

3 PID Controller

3.1 Introduction

3.1.1. Tabulate the transfer functions of a PID controller and its variants.
Solution: See Table 3.1.1.

Controller	Gain
PID	$K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$
PD	$K_p(1 + T_d s)$
PI	$K_p\left(1 + \frac{1}{T_i s}\right)$

TABLE 3.1.1

3.1.2. For a unity Feedback system

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{3.1.2.1}$$

Design a PD Controller with $K_v = 2$ and Phase Margin 30°
Solution: The gain after cascading the PD Controller with G(s) is

$$G_c(s) = \frac{K_p(1+T_d s)K}{s(s+2)(s+4)(s+6)} \tag{3.1.2.2}$$

Choosing $K_p = 1$ in ,

$$K_v = \lim_{s\to 0} sG_c(s) = 2 \tag{3.1.2.3}$$

$$\implies K = 96 \tag{3.1.2.4}$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\begin{aligned}\tan^{-1}(T_d\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) \\ - \tan^{-1}\left(\frac{\omega}{6}\right) = -60^\circ\end{aligned} \tag{3.1.2.5}$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_d^2\omega^2+1}}{\omega\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{3.1.2.6}$$

By Hit and Trial, one of the best combinations is

$$\omega = 4 \tag{3.1.2.7}$$

$$T_d = 1.884 \tag{3.1.2.8}$$

We get a Phase Margin of 30.31°

3.1.3. Verify using a Python Plot

Solution: The following code plots Fig. 3.1.3

```
codes/ee18btech11021/EE18BTECH11021_3.
    py
```

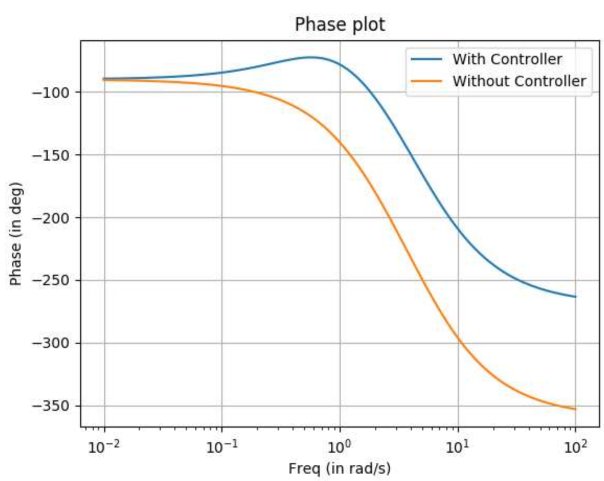

Fig. 3.1.3

3.1.4. Design a PI Controller with $K_v = \infty$ and Phase Margin 30°

Solution: From Table 3.1.1, the open loop gain in this case is

$$G_1(s) = \frac{K_p\left(1+\frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{3.1.4.1}$$

Choose $K_pK = 96$. Then

$$G_1(s) = \frac{96(T_i s+1)}{T_i s^2(s+2)(s+4)(s+6)} \tag{3.1.4.2}$$

For Phase Margin 30°, at Gain Crossover Frequency ω

$$\tan^{-1}(T_i\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \tag{3.1.4.3}$$

and

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_i^2\omega^2+1}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{3.1.4.4}$$

By Hit and Trial, one of the best combinations is

$$\omega = 0.75 \tag{3.1.4.5}$$

$$T_i = 2.713 \tag{3.1.4.6}$$

We get a Phase Margin of 25.53°

3.1.5. Verify using a Python Plot

Solution: The following code plots Fig. 3.1.5.

```
codes/ee18btech11021/EE18BTECH11021_4.
    py
```

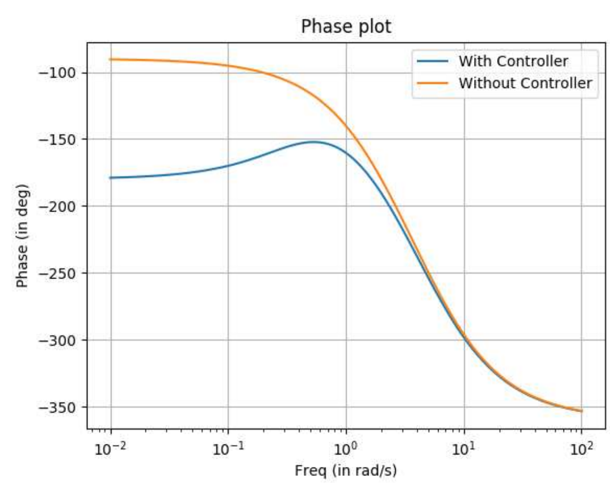

Fig. 3.1.5

3.1.6. Design a PID Controller with $K_v = \infty$ and Phase Margin 30°

Solution:

$$G_1(s) = \frac{K_p\left(1+T_d s+\frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{3.1.6.1}$$

Choose $K_pK = 96$. The open loop gain is

$$G_1(s) = \frac{96(T_iT_ds^2 + T_is + 1)}{T_is^2(s+2)(s+4)(s+6)} \quad (3.1.6.2)$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\tan^{-1}\left(\frac{T_i\omega}{1 - TiT_dw^2}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \quad (3.1.6.3)$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{(1 - TiT_d\omega^2)^2 + T_i^2}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \quad (3.1.6.4)$$

By Hit and Trial, one of the best combinations is

$$\omega = 1 \quad (3.1.6.5)$$
$$T_i = 1.738 \quad (3.1.6.6)$$
$$T_d = 0.4 \quad (3.1.6.7)$$

We get a Phase Margin of 30°

3.1.7. Verify using a Python Plot

Solution: The following code plots Fig. 3.1.7

```
codes/ee18btech11021/EE18BTECH11021_5.py
```

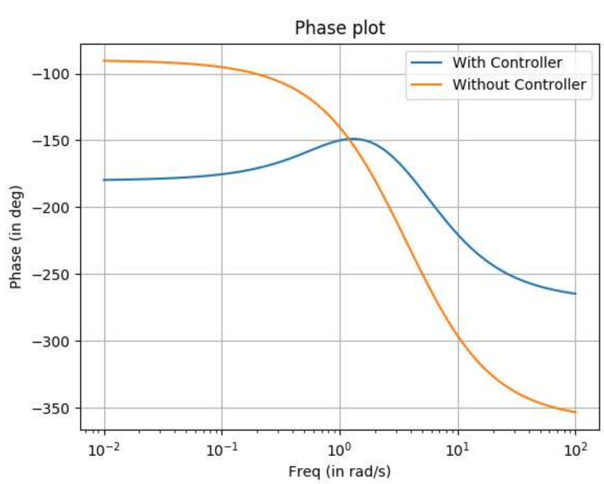

Fig. 3.1.7