# Control Systems

G V V Sharma*

Contents



*Abstract*—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. 

## 1 Mason's Gain Formula

1.1. The Block diagram of a system is illustrated in the figure shown, where $X(s)$ is the input and $Y(s)$ is the output. Draw the equivalent signal flow graph.

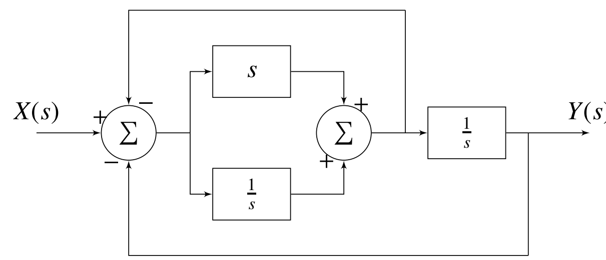


Fig. 1.1.1: Block Diagram

**Solution:** The signal flow graph of the block diagram in Fig. 1.1.1 is available in Fig. 1.1.2

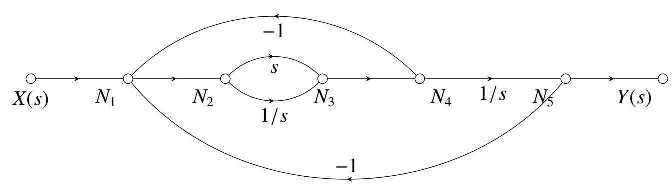


Fig. 1.1.2: Signal Flow Graph

1.2. Draw all the forward paths in Fig. 1.1.2 and compute the respective gains.
**Solution:** The forward paths are available in Figs. 1.2.3 and 1.2.4. The respective gains are

$$P_1 = s\left(\frac{1}{s}\right) = 1 \tag{1.2.1}$$

$$P_2 = (1/s)(1/s) = 1/s^2 \tag{1.2.2}$$

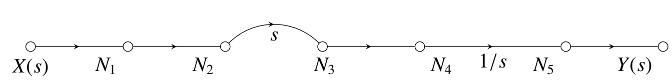


Fig. 1.2.3: $P_1$

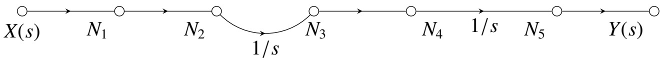


Fig. 1.2.4: $P_2$

1.3. Draw all the loops in Fig. 1.1.2 and calculate the respective gains.
**Solution:** The loops are available in Figs. 1.3.5-1.3.8 and the corresponding gains are

$$L_1 = (-1)(s) = -s \tag{1.3.1}$$

$$L_2 = s\left(\frac{1}{s}\right)(-1) = -1 \tag{1.3.2}$$

$$L_3 = \left(\frac{1}{s}\right)(-1) = -\frac{1}{s} \tag{1.3.3}$$

$$L_4 = \left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(-1) = -\frac{1}{s^2} \tag{1.3.4}$$

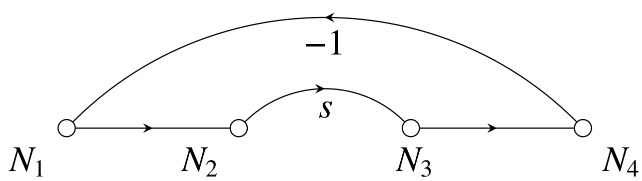


Fig. 1.3.5: $L_1$

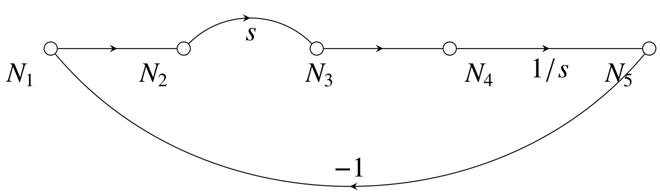


Fig. 1.3.6: $L_2$

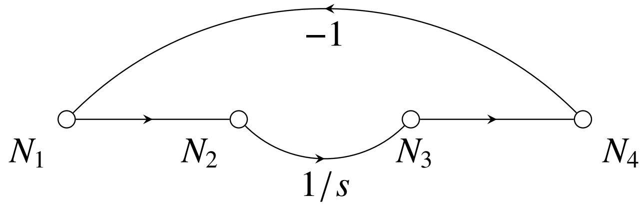


Fig. 1.3.7: $L_3$

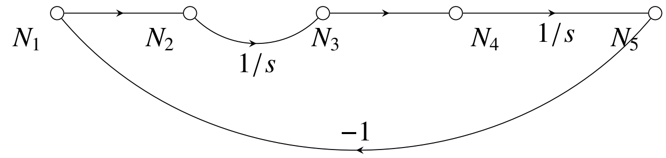


Fig. 1.3.8: $L_4$

1.4. State Mason's Gain formula and explain the parameters through a table.
**Solution:** According to Mason's Gain Formula,

$$T = \frac{Y(s)}{X(s)} \tag{1.4.1}$$

$$= \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \tag{1.4.2}$$

where the parameters are described in Table 1.4

| Variable | Description |
|---|---|
| $P_i$ | $i$th forward path |
| $L_j$ | $j$th loop |
| $\Delta$ | $1 - \sum_{L_i} + \sum_{L_i \cap L_j = \phi} L_i L_j - \sum_{L_i \cap L_j \cap L_k = \phi} L_i L_j L_k + \ldots$ |
| $\Delta_i$ | $1 - \sum_{L_k \cap P_i = \phi} L_k + \sum_{L_k \cap L_j \cap P_i = \phi} L_k L_j - \ldots$ |

TABLE 1.4

1.5. List the parameters in Table 1.4 for Fig. 1.1.2.
**Solution:** The parameters are available in Table 1.5

<table>
<tr><th>Path</th><th>Value</th><th>Parameter</th><th>Value</th><th>Remarks</th></tr>
<tr><td>$P_1$</td><td>1</td><td>$\Delta_1$</td><td>1</td><td>All loops intersect with $P_1$</td></tr>
<tr><td>$P_2$</td><td>$\frac{1}{s^2}$</td><td>$\Delta_2$</td><td>1</td><td>All loops intersect with $P_2$</td></tr>
<tr><td>$L_1$</td><td>$-s$</td><td rowspan="4">$\Delta$</td><td rowspan="4">$1 - \sum_i L_i$</td><td rowspan="4">All loops intersect</td></tr>
<tr><td>$L_2$</td><td>$-1$</td></tr>
<tr><td>$L_3$</td><td>-$\frac{1}{s}$</td></tr>
<tr><td>$L_4$</td><td>-$\frac{1}{s^2}$</td></tr>
</table>

TABLE 1.5

1.6. Find the transfer function using Mason's Gain Formula.

**Solution:** From (1.4.2) and 1.5,

$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{1.6.1}$$

$$= \frac{1 + \frac{1}{s^2}}{1 - (-s - 1 - \frac{1}{s} - \frac{1}{s^2})} \tag{1.6.2}$$

$$= \frac{s^2 + 1}{s^3 + 2s^2 + s + 1} \tag{1.6.3}$$

after simplification.

1.7. Write a program to compute Mason's gain formula, given the branch nodes and gains for each path.

# 2 Bode Plot

## 2.1 Introduction

2.1. For an LTI system, the Bode plot for its gain defined as

$$G(s) = 20\log|H(s)| \tag{2.1.1}$$

is as illustrated in the Fig. 2.1. Express $G(f)$ in terms of $f$.

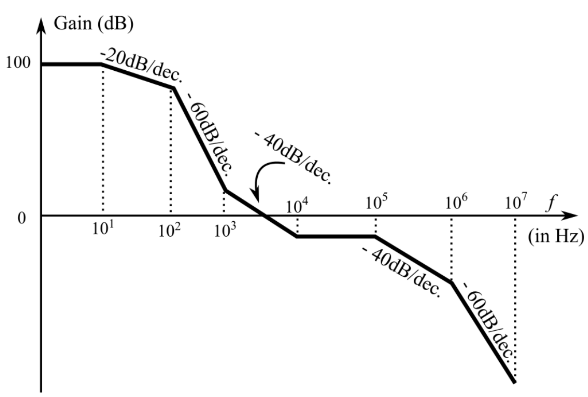


Fig. 2.1

**Solution:**

$$G(f) = \begin{cases} 100 & 0 < f < 10^1 \\ 120 - 20\log(f) & 10 < f < 10^2 \\ 200 - 60\log(f) & 10^2 < f < 10^3 \\ 140 - 40\log(f) & 10^3 < f < 10^4 \\ -20 & 10^4 < f < 10^5 \\ 180 - 40\log(f) & 10^5 < f < 10^6 \\ 300 - 60\log(f) & 10^6 < f < 10^7 \end{cases} \tag{2.1.2}$$

2.2. Express the slope of $G(f)$ in terms of $f$.
**Solution:** The desired slope is

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} \tag{2.2.1}$$

$$\nabla G(f) = \begin{cases} 0 & 0 < f < 10^1 \\ -20 & 10 < f < 10^2 \\ -60 & 10^2 < f < 10^3 \\ -40 & 10^3 < f < 10^4 \\ 0 & 10^4 < f < 10^5 \\ -40 & 10^5 < f < 10^6 \\ -60 & 10^6 < f < 10^7 \end{cases} \tag{2.2.2}$$

2.3. Express the change of slope of $G(f)$ in terms of $f$.
**Solution:**
$\Delta(\nabla G(f))$ = Change of slope G(f) at f

$$\Delta(\nabla G(f)) = \begin{cases} -20 & f = 10^1 \\ -40 & f = 10^2 \\ +20 & f = 10^3 \\ +40 & f = 10^4 \\ -40 & f = 10^5 \\ -20 & f = 10^6 \end{cases} \tag{2.3.1}$$

2.4. Tabulate the poles and zeros of $H(s)$ using (2.3.1).
**Solution:** Table 2.4 provides the details.

| $f$ **(Hz)** | $\Delta(\nabla G(f))$ | **Pole** | **Zero** |
|---|---|---|---|
| $10^1$ | -20 | 1 | 0 |
| $10^2$ | -40 | 2 | 0 |
| $10^3$ | 20 | 0 | 1 |
| $10^4$ | 40 | 0 | 2 |
| $10^5$ | -40 | 2 | 0 |
| $10^6$ | -20 | 1 | 0 |
| **Total** | | **6** | **3** |

TABLE 2.4

2.5. Obtain the transfer function of $H(s)$.
**Solution:** From Table 2.4,

$$H(s) = \frac{K(s + j2\pi 10^3)(s + j2\pi 10^4)^2}{(s + j2\pi 10^1)(s + j2\pi 10^2)^2(s + j2\pi 10^5)^2(s + j2\pi 10^6)} \tag{2.5.1}$$

2.6. Justify the above results.
**Solution:** Let us consider a generalized transfer

gain

$$H(s) = k\frac{(s-z_1)(s-z_2)...(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)....(s-p_{n-1})(s-p_n)} \tag{2.6.1}$$

The gain

$$\begin{aligned} G(f) &= 20\log|H(s)| \\ &= 20\log|k| + 20\log|s-z_1| \\ &+ 20\log|s-z_2| + \cdots + 20\log|s-z_m| \\ &- 20\log|s-p_1| - 20\log|s-p_2| \\ &- \cdots - 20\log|s-z_n| \end{aligned} \tag{2.6.2}$$

Substituting $s = \jmath\omega$, for real $z_1$

$$20\log|s-z_1| = 20\log\left|\sqrt{\omega^2+z_1^2}\right| \tag{2.6.3}$$

$$= \begin{cases} 20\log|z_1|, & \omega \ll z_1 \\ 20\log|\omega|, & \omega \gg z_1 \end{cases} \tag{2.6.4}$$

Taking the derivative,

$$\frac{d\left(20\log|s-z_1|\right)}{d\left(\log|\omega|\right)} = \begin{cases} 0, & \omega \ll z_1 \\ 20, & \omega \gg z_1 \end{cases} \tag{2.6.5}$$

Thus, when a zero is encountered, the gradient of $H(\jmath\omega)$ jumps by +20 in the log scale. When a pole is encountered, the gradient falls by -20. Note that this is a very loose justification, but works well in practice.

2.7. Obtain the Bode plot and the slope plot for $H(s)$ and verify with Fig. 2.1
**Solution:** Bode Plot of obtained Transfer Function is Fig. **??**, obtained from (2.5.1), is a close reconstruction of Fig. **??**.

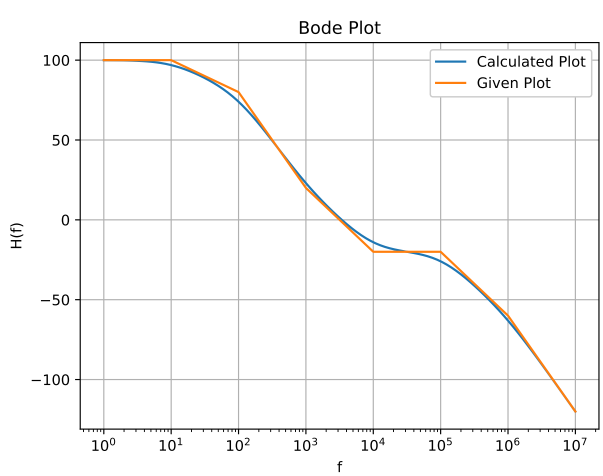


Fig. 2.7

### 2.2 *Example*

2.2.1. The asymptotic Bode magnitude plot of minimum phase transfer function G(s) is show in Fig. 2.2.1 . Express $20\log\left|G(\jmath\omega)\right|$ as a function

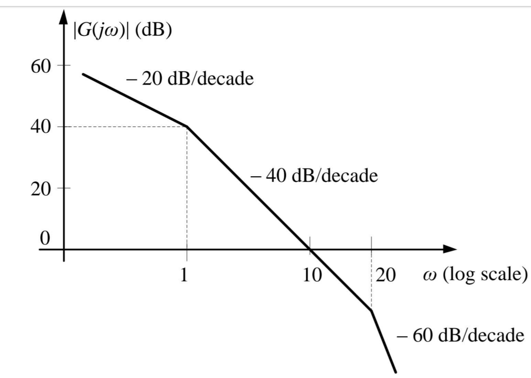


Fig. 2.2.1

of $\omega$ using Fig. 2.2.1.
**Solution:** The desired expression (in dB) is

$$|G(j\omega)| = \begin{cases} 60 - 20(\log(\omega) - \log(0.1)) & 0.1 < \omega < 1 \\ 80 - 40(\log(\omega) - \log(0.1)) & 1 < \omega < 20 \\ 126.02 - 60(\log(\omega) - \log(0.1)) & 20 < \omega \end{cases} \tag{2.2.1.1}$$

2.2.2. Express the slope of $20\log\left|G(\jmath\omega)\right|$ as a function of $\omega$.
**Solution:** The desired slope is

$$\nabla 20\log|G(j\omega)| = \begin{cases} -20 & \omega < 1 \\ -40 & 1 < \omega < 20 \\ -60 & 20 < \omega \end{cases} \tag{2.2.2.1}$$

2.2.3. Express the change of slope of $20\log\left|G(\jmath\omega)\right|$ as a function of $\omega$.
**Solution:**

$$\Delta(\nabla 20\log|G(j\omega)|) = \begin{cases} -20 & \omega = 0 \\ -20 & \omega = 1 \\ -20 & \omega = 20 \end{cases} \tag{2.2.3.1}$$

2.2.4. Find the poles and zeros of $G(s)$.
**Solution:** From (2.2.3.1), the poles are located at 0,1,20. There are no zeros.

2.2.5. Find $G(s)$
**Solution:**

$$G(s) = \frac{k}{s(1+s)(20+s)} \tag{2.2.5.1}$$

2.2.6. Obtain the Bode plot of $G(s)$ through a python code and compare with the line plot of the expression that you obtained in Problem 2.2.1
**Solution:** Fig. 2.2.6 shows the Bode plot of the transfer function obtained. The **Line plot** is the approximation of the **caluculated bode plot**.

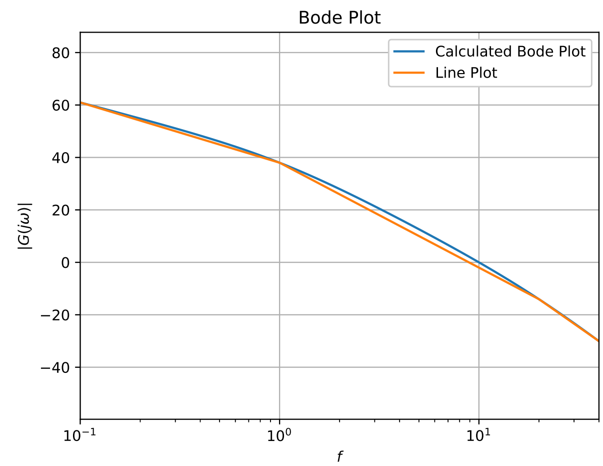


Fig. 2.2.6

2.2.7. Verify if at very high frequency ($\omega \to \infty$), the phase angle $\angle G(j\omega) = -3\pi/2$.
**Solution:** Phase $\phi$ is the sum of all the phases corresponding to each pole and zero.

$$\Rightarrow G(j\omega) = \frac{k}{j\omega(1 + j\omega)(20 + j\omega)} \tag{2.2.7.1}$$

$$\Longrightarrow \phi = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{20}\right) \tag{2.2.7.2}$$

$$= -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{20}\right) \tag{2.2.7.3}$$

$$\Longrightarrow \lim_{\omega \to \infty} \phi = -3\pi/2 \tag{2.2.7.4}$$

## 3 Second order System

### 3.1 Damping

3.1.1. List the different kinds of damping for a second order system defined by

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2} \tag{3.1.1.1}$$

where $\omega$ is the natural frequency and $\zeta$ is the damping factor.
**Solution:** The details are available in Table 3.1.1

| **Damping Ratio** | **Damping Type** |
|---|---|
| $\zeta > 1$ | Overdamped |
| $\zeta = 1$ | Critically Damped |
| $0 < \zeta < 1$ | Underdamped |
| $\zeta = 0$ | Undamped |

TABLE 3.1.1

3.1.2. Classify the following second-order systems according to damping.
a) $H(s) = \frac{15}{s^2+5s+15}$
b) $H(s) = \frac{25}{s^2+10s+25}$
c) $H(s) = \frac{35}{s^2+18s+35}$
**Solution:** For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{3.1.2.1}$$

$$\omega^2 = 25, 2\zeta\omega = 10 \tag{3.1.2.2}$$

$$\Longrightarrow \omega = 1, \zeta = 1 \tag{3.1.2.3}$$

and the system is critically damped. Similarly, the damping factors for other systems in Problem 3.1.2 are calculated and listed in Table 3.1.2

| **H(s)** | $\omega$ | $\zeta$ | **Damping Type** |
|---|---|---|---|
| $\frac{35}{s^2+18s+35}$ | $\sqrt{35}$ | $\sqrt{\frac{81}{35}} > 1$ | Overdamped |
| $\frac{25}{s^2+10s+25}$ | 5 | 1 | Critically Damped |
| $\frac{15}{s^2+5s+15}$ | $\sqrt{15}$ | $\sqrt{\frac{5}{12}} < 1$ | Underdamped |

TABLE 3.1.2

3.1.3. Find the step response of each $H(s)$ in Table 3.1.2.
**Solution:**
a) For

$$H(s) = \frac{15}{s^2 + 5s + 15}, \tag{3.1.3.1}$$

the step response is

$$y(t) = 25te^{-5t}u(t) \tag{3.1.3.2}$$

b) For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{3.1.3.3}$$

the step response is

$$y(t) = \frac{30}{\sqrt{35}} e^{\frac{-5t}{2}} \sin\left(\frac{\sqrt{35}}{2} t\right) u(t) \quad (3.1.3.4)$$

c) For

$$H(s) = \frac{35}{s^2 + 18s + 35}, \quad (3.1.3.5)$$

the step response is

$$y(t) = \frac{35}{2\sqrt{46}} \left[e^{(-9+\sqrt{46})t} - e^{(-9-\sqrt{46})t}\right] u(t) \quad (3.1.3.6)$$

3.1.4. Illustrate the effect of damping by plotting the step responses in (3.1.3.2)-(3.1.3.6)
**Solution:** The following code

```
codes/ee18btech11012.py
```

plots the desired graphs in Fig. **??**.
using a Python code to sketch the response.

*3.2 Example*

3.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \quad (3.1.1)$$

Is the system stable?
**Solution:** The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \quad (3.1.2)$$

are at

$$s = -1 \quad (3.1.3)$$

i.e., the left half of s-plane. Hence the system is stable.

3.2. Find and sketch the step response $c(t)$ of the system.
**Solution:** For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1 + 2s + s^2}\right] \quad (3.2.1)$$

$$= \frac{1}{s(1+s)^2} \quad (3.2.2)$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \quad (3.2.3)$$

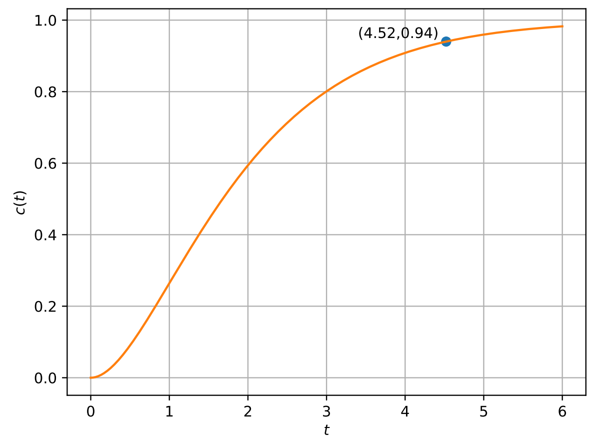


Fig. 3.2

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \quad (3.2.4)$$

$$= \left(1 - e^{-t} - te^{-t}\right) u(t) \quad (3.2.5)$$

The following code plots $c(t)$ in Fig. 3.2

```
codes/ee18btech11002/plot.py
```

3.3. Find the steady state response of the system using the final value theorem. Verify using 3.2.5
**Solution:** To know the steady response value of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \quad (3.3.1)$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1 + s + s^2}\right) = \frac{1}{1 + 0 + 0} = 1 \quad (3.3.2)$$

Using 3.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \quad (3.3.3)$$

$$= (1 - 0 - 0) = 1 \quad (3.3.4)$$

3.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
**Solution:** Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \quad (3.4.1)$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{3.4.2}$$

## 4 Routh Hurwitz Criterion

### *4.1 Routh Array*

4.1.1. Generate the Routh array for the polynomial,

$$f(s) = s^7 + s^6 + 7s^5 + 14s^4 + 31s^3 + 73s^2 + 25s + 200 \tag{4.1.1.1}$$

**Solution:**

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \end{vmatrix} \tag{4.1.1.2}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \end{vmatrix} \tag{4.1.1.3}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 0 & 0 & 0 & \end{vmatrix} \tag{4.1.1.4}$$

When such a case is encountered, we take the derivative of the expression formed the the coefficients above it i.e derivative of $8s^4+48s^2+200$.

$$\frac{d}{dx}(8s^4 + 48s^2 + 200) = 32s^3 + 96s$$

The coefficients of obtained expression are placed in the table.

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \end{vmatrix} \tag{4.1.1.5}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \end{vmatrix} \tag{4.1.1.6}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \end{vmatrix} \tag{4.1.1.7}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \\ 200 & & & \end{vmatrix} \tag{4.1.1.8}$$

So, the above one is the Routh-Hurwitz Table.

4.1.2. Find the number of roots of the polynomial in the right half of the $s$-plane.
**Solution:** The number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column. From 4.1.1.8, the polynomial in (4.1.1.1) has 4 roots lie on right-side of Imaginary Axis.

4.1.3. Write a Python code for generating each stage of the Routh Table.
**Solution:** The following code

```
codes/ee18btech11014/ee18btech11014.py
```

generates the various stages.

4.1.4. Find the roots of the polynomial in in (4.1.1.1) and verify that 4 roots are in the right half $s$-plane.
**Solution:** The following code generates the necessary roots.

```
codes/ee18btech11014/Roots.py
```

*4.2 Marginal Stability*

4.2.1. Consider a unity feedback system as shown in Fig. 4.2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2+3s+2} \tag{4.2.1.1}$$

where k greater than 0. Find its closed loop transfer function.

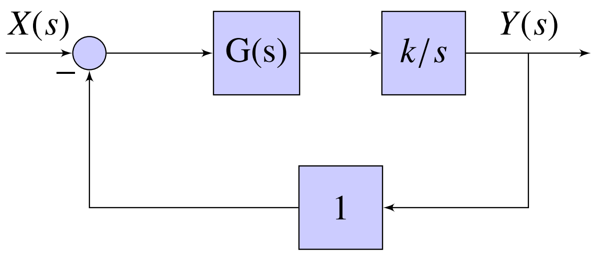


Fig. 4.2.1

**Solution:** $\because H(s) = 1$ in Fig. 4.2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{4.2.1.2}$$

$$\implies T(s) = \frac{k}{s^3+3s^2+2s} \tag{4.2.1.3}$$

4.2.2. Find the *characteristic* equation for $G(s)$.
**Solution:** The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{4.2.2.1}$$

$$\implies 1 + \left[\frac{k}{s^3+3s^2+2s}\right] = 0 \tag{4.2.2.2}$$

$$\text{or, } s^3+3s^2+2s+k = 0 \tag{4.2.2.3}$$

4.2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on $\jmath\omega$ axis.
**Solution:** This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0s^n + a_1s^{n-1} + \ldots.. + a_{n-1}s + a_n = 0 \tag{4.2.3.1}$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix}\begin{vmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{vmatrix} \tag{4.2.3.2}$$

where

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \tag{4.2.3.3}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \tag{4.2.3.4}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \tag{4.2.3.5}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \tag{4.2.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 4.2.2.1,

$$s^3+3s^2+2s+k = 0 \tag{4.2.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{4.2.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{4.2.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{4.2.3.10}$$

4.2.4. Repeat the above using the determinant method.
**Solution:** The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2\cdots & & \\ 0 & a_1 & a_3\cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{pmatrix} \tag{4.2.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{4.2.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{4.2.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{4.2.4.4}$$

$$\dots \tag{4.2.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (4.2.2.1),

$$D_1 = 1 \neq 0 \tag{4.2.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} \quad = k - 6 = 0 \implies k = 6 \tag{4.2.4.7}$$

4.2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.
**Solution:** The following code verifies the stability using the tabular method

```
codes/ee18btech11005_2.py
```

and the following one verifies using the determinant method.

```
codes/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infinity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.
- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the right side of s-plane.That means it has sign changes.

### *4.3 Stability*

4.3.1. The characteristic equation of linear time invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \tag{4.3.1.1}$$

Find the condition for the system to be BIBO stable using the Routh Array.
**solution**

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \tag{4.3.1.2}$$

The Routh hurwitz criterion:-

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 3 & k \\ 3 & 1 & 0 \\ \frac{8}{3} & k & 0 \\ \frac{\frac{8}{3}-3k}{\frac{8}{3}} & 0 & 0 \\ k & 0 & 0 \end{vmatrix} \tag{4.3.1.3}$$

From the above array, the given system is stable if

$$k > 0$$
$$\frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0 \tag{4.3.1.4}$$

$$\implies 0 < k < \frac{8}{9} \tag{4.3.1.5}$$

4.3.2. Modify the Python code in Problem 4.2.5 to verify your solution by choosing two different values of $k$.
**Solution:** The following code

```
codes/ee18btech11008.py
```

provides the necessary soution for $k = 0.5, 3$.

- $k = 0.5 < \frac{8}{9}$ has no sign changes in first column of its routh array.So the system is stable.
- $k = 3 > \frac{8}{9}$ has 2 sign changes in first column of its routh array.So the system is unstable.

## 5 State-Space Model

### *5.1 Controllability and Observabiity*

5.1. State the general model of a state space system specifying the dimensions of the matrices and

vectors.
**Solution:** The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{5.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{5.1.2}$$

with parameters listed in Table 5.1.

| Variable | Size | Description |
|---|---|---|
| **u** | $p \times 1$ | input(control) vector |
| **y** | $q \times 1$ | output vector |
| **x** | $n \times 1$ | state vector |
| **A** | $n \times n$ | state or system matrix |
| **B** | $n \times p$ | input matrix |
| **C** | $q \times n$ | output matrix |
| **D** | $q \times p$ | feedthrough matrix |

TABLE 5.1

5.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
**Solution:** Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5.2.1}$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \tag{5.2.2}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{5.2.3}$$

$$+ (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{5.2.4}$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + D\mathbf{I}\mathbf{U}(s) \tag{5.2.5}$$

Substituting from (5.2.4) in the above,

$$\mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{5.2.6}$$

5.3. Find $H(s)$ for a SISO (single input single output) system.
**Solution:**

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I} \tag{5.3.1}$$

5.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.4.1}$$

$$D = 0 \tag{5.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.3}$$

find **A** and **C** such that the state-space realization is in *controllable canonical form.*
**Solution:**

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{5.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{5.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{5.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{5.4.7}$$

so the above equation can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.4.8}$$

Letting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.4.9}$$

$$\mathbf{X}_1 = \begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} \tag{5.4.10}$$

$$\mathbf{X} = \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix}, \tag{5.4.11}$$

$$\mathbf{X}_1(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{5.4.12}$$

$$Y = \mathbf{C}\mathbf{X}_1(s) \tag{5.4.13}$$

where

$$\mathbf{C} = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \tag{5.4.14}$$

5.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form*.
**Solution:** Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}, \tag{5.5.1}$$
$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.5.2}$$
$$\implies U(s) = Y(s)(s^3 + 3s^2 + 2s + 1) \tag{5.5.3}$$
$$\text{or, } Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.4}$$

Let

$$X_1(s) = Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.5}$$
$$X_2(s) = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{5.5.6}$$
$$X_3(s) = s^{-1}(U(s) - Y(s)) \tag{5.5.7}$$

$$\begin{aligned} & sX_1(s) = -3Y(s) + X_2(s) \\ \implies & sX_2(s) = -2Y(s) + X_3(s) \\ & sX_3(s) = U(s) - Y(s) \end{aligned} \tag{5.5.8}$$

Substituting $Y = X_1(s)$ the above,

$$sX_1(s) = -3X_1(s) + X_2(s) \tag{5.5.9}$$
$$sX_2(s) = -2X_1(s) + X_3(s) \tag{5.5.10}$$
$$sX_3(s) = U(s) - X_1(s) \tag{5.5.11}$$

which can be expressed as

$$\begin{pmatrix} sX_1(s) \\ sX_2(s) \\ sX_3(s) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.5.12}$$

$$\text{or, } \begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \\ Y(s) &= \mathbf{B}\mathbf{X}(s) \end{aligned} \tag{5.5.13}$$

where

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.5.14}$$

$$\mathbf{B} = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \tag{5.5.15}$$

5.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
**Solution:** The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

5.7. Theoretically, show that eigenvaues of **A** are the poles of $H(s)$.
**Solution:** As we know that the characteristic equation is det(sI-A)

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.7.1}$$
$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{5.7.2}$$

$$\implies \left| s\mathbf{I} - \mathbf{A} \right| = s(s^2 + 3s + 2) + 1(1) \tag{5.7.3}$$
$$= s^3 + 3s^2 + 2s + 1 \tag{5.7.4}$$

which is the denominator of $H(s)$ in (5.4.1)

### 5.2 *Second Order System*

5.2.1. Consider a state-variable model of a system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} r \tag{5.2.1.1}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.2.1.2}$$

where y is the output, and r is the input.

5.2.2. List the various state matrices in (5.2.1.1)

5.2.3. Find the the system transfer function $H(s)$.
**Solution:** From (**??**) and , (5.3.1), the transfer function for the state space model is

$$H(s) = C(sI - A)^{-1}B + D \tag{5.2.3.1}$$
$$= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + 2\beta & 1 \\ -\alpha & s \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{s(s + 2\beta) + \alpha} \tag{5.2.3.2}$$
$$= \frac{b_1(s + 2\beta) + b_2}{s^2 + 2s\beta + \alpha} \tag{5.2.3.3}$$
$$\implies H(s) = \frac{b_1 s}{s^2 + 2s\beta + \alpha} + \frac{2b_1\beta + b_2}{s^2 + 2s\beta + \alpha} \tag{5.2.3.4}$$

5.2.4. Find the Damping ratio $\zeta$ and the Undamped natural frequency $\omega_n$ of the system.

**Solution:** Generally for a second order system the transfer function is given by 3.1.1.1

$$H(s) = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \tag{5.2.4.1}$$

Comparing the denominator of the above with (5.2.3.4),

$$2\zeta\omega_n = 2\beta, \tag{5.2.4.2}$$

$$\omega_n^2 = \alpha \tag{5.2.4.3}$$

$$\implies \zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha} \tag{5.2.4.4}$$

5.2.5. Using Table 3.1.1, explain how the damping conditions depend upon $\alpha$ and $\beta$.

## 6 Nyquist Plot

6.1. The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{\pi e^{-0.25s}}{s} \tag{6.1.1}$$

6.2. Find $\text{Re}\{G(\jmath\omega)\}$ and $\text{Im}\{G(\jmath\omega)\}$.
**Solution:** From (6.1.1),

$$G(j\omega) = \frac{\pi}{\omega}(-\sin 0.25\omega - j\cos 0.25\omega) \tag{6.2.1}$$

$$\implies \text{Re}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-\sin 0.25\omega) \tag{6.2.2}$$

$$\text{Im}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-j\cos 0.25\omega) \tag{6.2.3}$$

6.3. Sketch the Nyquist plot.
**Solution:** The Nyquist plot is a graph of $\text{Re}\{G(\jmath\omega)\}$ vs $\text{Im}\{G(\jmath\omega)\}$. The following python code generates the Nyquist plot in Fig. 6.3

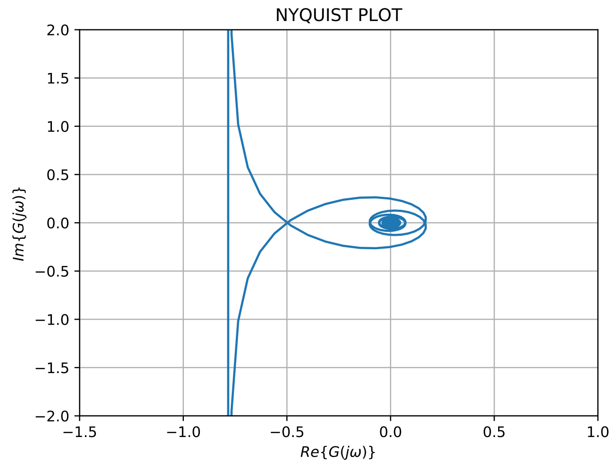


Fig. 6.3

6.4. Find the point at which the Nyquist plot of G(s) passes through the negative real axis
**Solution:** Nyquist plot cuts the negative real axis at $\omega$ for which

$$\angle G(\jmath\omega) = -\pi \tag{6.4.1}$$

From (6.1.1),

$$G(\jmath\omega) = \frac{\pi e^{-\frac{\jmath\omega}{4}}}{\jmath\omega} = \frac{\pi e^{-\jmath\left(\frac{\omega}{4}+\frac{\pi}{2}\right)}}{\omega} \tag{6.4.2}$$

$$\implies \angle G(\jmath\omega) = -\left(\frac{\omega}{4} + \frac{\pi}{2}\right) \tag{6.4.3}$$

From (6.4.3) and (6.4.1),

$$\frac{\omega}{4} + \frac{\pi}{2} = \pi \tag{6.4.4}$$

$$\implies \omega = 2\pi \tag{6.4.5}$$

Also, from (6.1.1),

$$\left|G(\jmath\omega)\right| = \frac{\pi}{|\omega|} \tag{6.4.6}$$

$$\implies \left|G(\jmath 2\pi)\right| = \frac{1}{2} \tag{6.4.7}$$

6.5. Use the Nyquist Stability criterion to determine if the system in (6.4.3) is stable.

| Variable | Value | Description |
|---|---|---|
| Z | 0 | Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane |
| P | 0 | Poles of $G(s)H(s)$ in right half of s plane |
| N | 0 | No of clockwise encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot |

TABLE 6.5

**Solution:** Consider Table 6.5. According to the Nyquist stability criterion,

a) If the open-loop transfer function $G(s)$ has a zero pole of multiplicity $l$, then the Nyquist plot has a discontinuity at $\omega$ =0. During further analysis it should be assumed that the phasor travels l times clock-wise along a semicircle of infinite radius. After applying this rule, the zero poles should be neglected, i.e. if there are no other unstable poles, then the open-loop transfer function $G(s)$ should be considered stable.

b) If the open-loop transfer function $G(s)$ is stable, then the closed-loop system is unstable for any encirclement of the point -1. If the open-loop transfer function $G(s)$ is unstable, then there must be one counter clock-wise encirclement of -1 for each pole of $G(s)$ in the right-half of the complex plane.

c) The number of surplus encirclements (N + P greater than 0) is exactly the number of unstable poles of the closed-loop system.

d) However, if the graph happens to pass through the point $-1+\jmath 0$, then deciding upon even the marginal stability of the system becomes difficult and the only conclusion that can be drawn from the graph is that there exist zeros on the $\jmath\omega$ axis.

From (6.1.1), $G(s)$ is stable since it has a single pole at $s = 0$. Further, from Fig. 6.3, the Nyquist plot doesnot encircle $s = -1$. From Theorem 6.5b, we may conclude that the system is stable.

## 7 Gain Margin

7.1. The open loop transfer function of a feedback control system is

$$G(s) = \frac{1}{s(1+2s)(1+s)} \tag{7.1.1}$$

Find the magnitude and phase of $\left|G\left(\jmath\omega\right)\right|$.
**Solution:**

$$G\left(\jmath\omega\right) = \frac{1}{\jmath\omega(1+2\jmath\omega)(1+\jmath\omega)} \tag{7.1.2}$$

$$= \frac{1}{\jmath\omega(1+3\jmath\omega-2\omega^2)} = \frac{1}{\jmath\omega-3\omega^2} \tag{7.1.3}$$

$$= \frac{1}{-3\omega^2+\jmath\omega(1-2\omega^2)} \tag{7.1.4}$$

$$\implies \angle G\left(\jmath\omega\right) = -tan^{-1}\left(\frac{\omega(1-2\omega^2)}{-3\omega^2}\right) \tag{7.1.5}$$

7.2. The frequency at which the phase of open-loop transfer function reaches -180° or +180° depending upon the range of tan inverse function) is defined to be the phase crossover frequency. Find the phase crossover frequency for (7.1.1).
**Solution:** From (7.1.5), at $\omega = \omega_{pc}$

$$\omega(1-2\omega^2) = 0 \tag{7.2.1}$$

$$\implies \omega_{pc} = \frac{1}{\sqrt{2}} \tag{7.2.2}$$

7.3. The gain Margin is given by,

$$GM = -20log_{10}\left|G\left(\jmath\omega_{pc}\right)\right| = 20log_{10}k_g \tag{7.3.1}$$

where

$$k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} \tag{7.3.2}$$

Find the GM for (7.1.5).
**Solution:**

$$\left|G\left(\jmath\omega_{pc}\right)\right| = \frac{1}{\left(\frac{3}{2}\right)} \implies k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} = \frac{3}{2}3.5dB \tag{7.3.3}$$

The greater the Gain Margin (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.

7.4. Obtain the GM from the Bode plot. **Solution:** The following code

```
codes/ee18btech11016.py
```

plots the amplitude and phase of (7.1.1) in Fig. . So,in the above figure, since $20log_{10}(G(\jmath\omega_{pc}))$

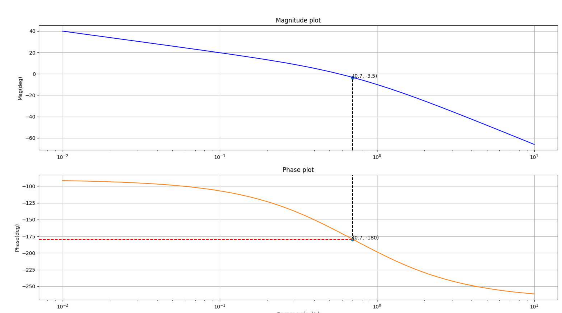


Fig. 7.4

= -3.5dB at $\omega_{pc} = -180°$ so GM = +3.5dB.

7.5. A positive GM indicates closed loop stability with unity feedback. Verify this for (7.1.1).

**Solution:** The characteristic equation is

$$1 + G(s) = 0 \implies 2s^3 + 3s^2 + s + 1 = 0 \tag{7.5.1}$$

Constructing the routh array

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \end{vmatrix} \tag{7.5.2}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{7.5.3}$$

There are no sign changes in the first column of the routh array. $\therefore$ the system is stable.

## 8 Compensators

### 8.1 Phase Lead

8.1.1. Consider a control system with

$$G(s) = \frac{1}{s(3s+1)} \tag{8.1.1.1}$$

Find its phase margin.
**Solution:**

$$G(\jmath\omega) = \frac{1}{(j\omega)(3j\omega+1)} \tag{8.1.1.2}$$

$$\implies \left|G(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} \tag{8.1.1.3}$$

$$\implies \angle G(\jmath\omega) = -\tan^{-1}(3\omega) - 90^\circ \tag{8.1.1.4}$$

At Gain Crossover,

$$\left|G(\jmath\omega)\right| = 1 \tag{8.1.1.5}$$

$$\implies \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} = 1 \tag{8.1.1.6}$$

$$\implies \omega_{gc} = 0.531 \tag{8.1.1.7}$$

$$\implies \angle G(\jmath\omega) = -147.88^\circ \tag{8.1.1.8}$$

$$\implies PM = 32.12^\circ \tag{8.1.1.9}$$

8.1.2. The minimum acceptable PM for a control system is 45°. Design a suitable *lead compensator* for (8.1.1.1).
**Solution:** Let the desired compensator be

$$D(s) = \frac{3(s+\frac{1}{3T})}{(s+\frac{1}{T})} \tag{8.1.2.1}$$

Choosing T = 1,

$$D(s) = \frac{3(s+\frac{1}{3})}{(s+1)} \tag{8.1.2.2}$$

By cascading the Compensator and the Open Loop Transfer Function,

$$G_1(s) = D(s)G(s) \tag{8.1.2.3}$$

$$= \frac{1}{s(3s+1)}\frac{3(s+\frac{1}{3})}{(s+1)} \tag{8.1.2.4}$$

$$\implies G_1(s) = \frac{1}{s(s+1)} \tag{8.1.2.5}$$

$$\implies G_1(\jmath\omega) = \frac{1}{(j\omega)(j\omega+1)} \tag{8.1.2.6}$$

$$\implies \left|G_1(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{\omega^2+1}\right)} \tag{8.1.2.7}$$

$$\angle G_1(\jmath\omega) = -\tan^{-1}(\omega) - 90^\circ \tag{8.1.2.8}$$

At Gain Crossover,

$$\left|G_1(\jmath\omega)\right| = 1 \tag{8.1.2.9}$$

$$\implies \frac{1}{\omega\left(\sqrt{\omega^2+1}\right)} = 1 \tag{8.1.2.10}$$

$$\implies \omega_{gc} = 0.786 \tag{8.1.2.11}$$

$$\implies \angle G(\jmath\omega) = -128.167^\circ \tag{8.1.2.12}$$

$$\implies PM = 51.83^\circ \tag{8.1.2.13}$$

8.1.3. Verify the above improvement in Phase Margin with the help of a Python Code
**Solution:** The following code generates Fig. 8.1.3

```
codes/ee18btech11021.py
```

## 9 Oscillator

9.1. Fig. 9.1 shows a Hartley oscillator built using opamp. Draw the equivalent block diagram.
**Solution:** Fig. shows the block diagram of the Hartley Oscillator in Fig. 9.1.

9.2. Show that the gain of the oscillator is

$$G = \frac{V_{out}}{V_{in}} = \frac{A}{1-AB} \tag{9.2.1}$$

**Solution:** From 9.1

$$A(V_{in} + BV_{out}) = V_{out} \tag{9.2.2}$$

resulting in (9.2.1).

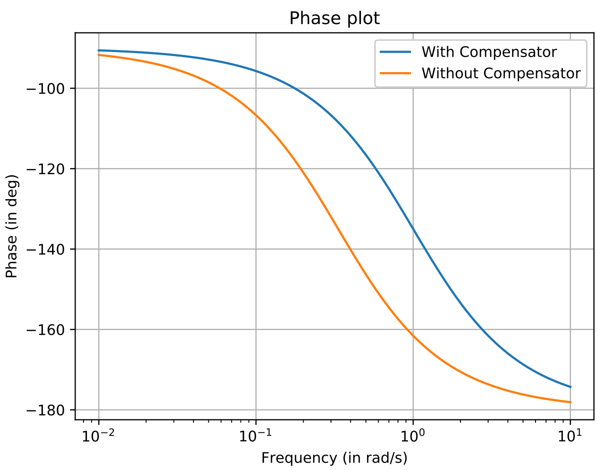


Fig. 8.1.3

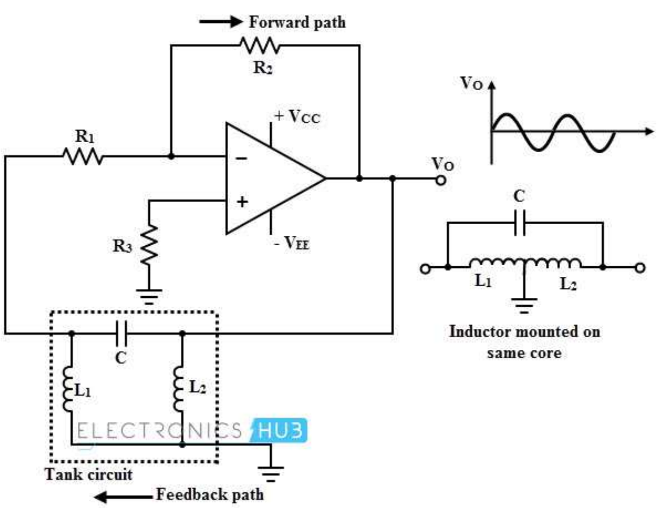


Fig. 9.1: block diagram for oscillator

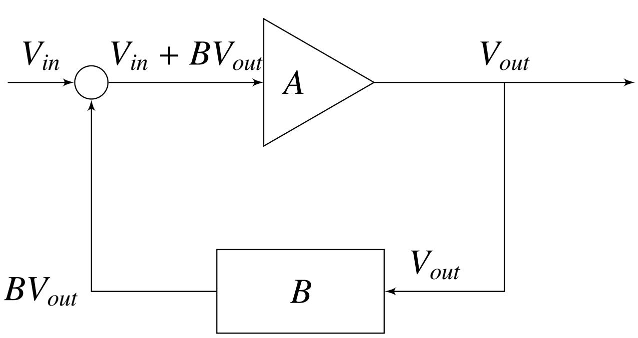


Fig. 9.1: block diagram for oscillator

9.3. State the condition for sustained oscillations. Justify.
9.4. Find $A$ and $B$.
9.5. Find the frequency of oscillation using the condition that $AB = 1$.

Given Below is basic block diagram
Resonant frequency, is the frequency at which oscillator oscillates, it depends on R/L/C components of the circuit it's been fed back through.
Oscillators work because they overcome the losses of their positive feedback circuit either in the form of a capacitor, inductor or both. In other words, an oscillator is a an amplifier which uses positive feedback that generates an output frequency without the use of an input signal.
Oscillators gain can be given as follows:

9.6. Hartley oscillator:
The Hartley oscillator is one of the classical LC feedback circuits,i.e feedback is made of LC components.Below here we can see a general form of any LC-type oscillator:
For any LC oscillator,

$$Z_1 = jX_1 \tag{9.6.1}$$
$$Z_2 = jX_2 \tag{9.6.2}$$
$$Z_3 = jX_3 \tag{9.6.3}$$
$$\tag{9.6.4}$$

We know that feedback gain is B, i.e, $\frac{V_0}{V_f} = B$
Applyting voltage divider rule we get

$$B = \frac{Z_1}{Z_1 + Z_3} \tag{9.6.5}$$

Consider the below circuit

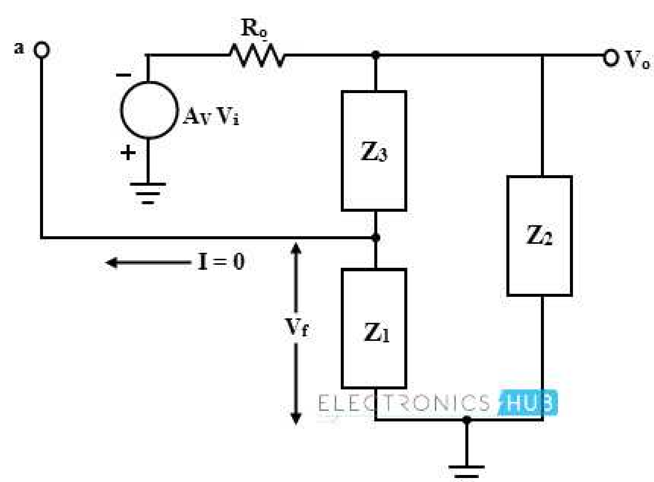


Fig. 9.6: block diagram for oscillator

$$A = \frac{V_o}{V_{in}} = \frac{AZ_L}{R_o + Z_L} \tag{9.6.6}$$
$$where \tag{9.6.7}$$
$$Z_L = \frac{(Z_2 + Z_3)Z_1}{Z_1 + Z_2 + Z_3} \tag{9.6.8}$$
$$\tag{9.6.9}$$

Now,we know that $AB = 1$ for sustained oscillations, putting the the above terms in the equation
on solving,

$$AB = \frac{Z_1 Z_2 A}{(Z_1 + Z_2 + Z_3)A + Z_1(Z_2 + Z_3)} \tag{9.6.10}$$
$$Z_1 = jX_1, Z_2 = jX_2, Z_3 = jX_3 \tag{9.6.11}$$
$$\tag{9.6.12}$$

putting that in we get

$$AB = \frac{AX_1X_2}{X_1(X_2 + X_3) - jR_o(X_1 + X_2 + X_3)} \tag{9.6.13}$$

For hartley oscillator,

$$Z_1 = j\omega L_1(inductor) \tag{9.6.14}$$
$$Z_2 = j\omega L_2(inductor) \tag{9.6.15}$$
$$Z_3 = \frac{1}{j\omega C}(capacitor) \tag{9.6.16}$$
$$\tag{9.6.17}$$

Since, AB is real

$$X_1 + X_2 + X_3 = 0 \tag{9.6.18}$$
$$\tag{9.6.19}$$

For, Hartley oscillator, substituting terms in above equation

$$\omega L_1 + \omega L_2 = \frac{1}{\omega C} \tag{9.6.20}$$
$$\omega = \frac{1}{\sqrt{(L_1 + L_2)(C)}} \tag{9.6.21}$$
$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)(C)}} \tag{9.6.22}$$

$$B = \frac{Z_1}{Z_1 + Z_3} = \frac{Z_1}{Z_2} \tag{9.6.23}$$
$$= \frac{L_1}{L_2} \tag{9.6.24}$$
$$A = \frac{L_2}{L_1} \tag{9.6.25}$$

Given below is circuit for

9.7. For Hartley oscillator frequency generated can be given as

$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)C}} \tag{9.7.1}$$

Taking,

$$L_1 = 1\mu H \tag{9.7.2}$$
$$L_2 = 1\mu H \tag{9.7.3}$$
$$C = 1.2pF \tag{9.7.4}$$

We get f = 103 MHz
Feedback factor for Hartley given by:

$$Feedbackfactor = \frac{L_2}{L_1} = 1 \tag{9.7.5}$$

W.K.T, $AB = 1$
$\therefore$ Minimum amplification Gain = 1

## 10 Phase Margin

10.1. The open loop transfer function of a system is

$$G(s) = \frac{2}{(s+1)(s+2)} \tag{10.1.1}$$

Find its magnitude and phase response.
**Solution:** Substituting $s = \jmath\omega$ in (10.1.1),

$$G(\jmath\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)} \tag{10.1.2}$$
$$\implies \left|G(\jmath\omega)\right| = \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} \tag{10.1.3}$$
$$\angle G(\jmath\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) \tag{10.1.4}$$

10.2. Find $\omega$ for which the gain of (10.1.1) first becomes 1.
**Solution:** From (10.1.3)

$$\left|G\left(\jmath\omega\right)\right| = 1 \tag{10.2.1}$$

$$\implies \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} = 1 \tag{10.2.2}$$

$$\implies \omega_{gc} = 0 \tag{10.2.3}$$

which is the desired frequency.

10.3. Find $\angle G(\jmath\omega_{gc}) + 180°$. This is known as the *phase margin*(PM)
**Solution:** From (10.1.4),

$$\angle G\left(\jmath\omega\right) = 0° \implies PM = 180° \tag{10.3.1}$$

10.4. Verify your result by plotting the gain and phase plots of $G(\jmath\omega)$.
**Solution:** The following code plots Fig. 10.4

```
codes/ee18btech11017.py
```

The Phase plot is as shown,

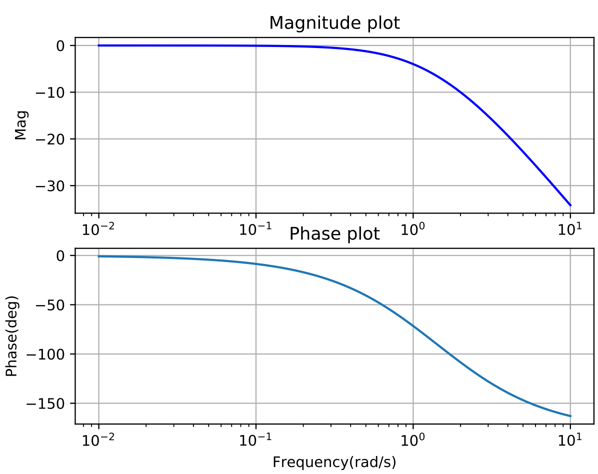


Fig. 10.4

10.5. A positive phase margin for the open loop system indicates a stable closed loop system. (10.3.1) indicates that $G(s)$ with unity feedback is stable. Show that the roots of $1+G(s)$ lie in the left half plane proving closed loop stability.
**Solution:** Let the closed loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)} \tag{10.5.1}$$

Then

$$1 + G(s) = 0 \tag{10.5.2}$$

$$\implies s^2 + 3s + 4 = 0 \tag{10.5.3}$$

$$\text{or } s = -1.5 + 1.3j, -1.5 - 1.3j \tag{10.5.4}$$

Since the roots are in the left half plane, the system is stable.

10.6. Instead of unity feedback, consider a system with

$$H(s) = \frac{50}{s+1} \tag{10.6.1}$$

Compute the open loop phase margin for this system.
**Solution:**

$$\because G(s)H(s) = \frac{100}{(s+1)^2(s+2)}, \tag{10.6.2}$$

the magnitude and phase are

$$\left|G\left(\jmath\omega\right)H\left(\jmath\omega\right)\right| = \frac{10^2}{\sqrt{(\omega^2+1)^2}\sqrt{\omega^2+4}} \tag{10.6.3}$$

$$\angle G\left(\jmath\omega\right)H\left(\jmath\omega\right) = -\tan^{-1}\frac{\omega}{2} - 2\tan^{-1}(\omega) \tag{10.6.4}$$

The gain crossover frequency is given by

$$\frac{10^2}{\sqrt{\omega_{gc}^2+4}\sqrt{(\omega_{gc}^2+10^2)^2}} = 1 \tag{10.6.5}$$

$$\tag{10.6.6}$$

$$\omega_{gc}^6 + 6\omega_{gc}^4 + 9\omega_{gc}^2 - 9996 = 0 \tag{10.6.7}$$

$$\implies \omega_{gc} = 4.42 \tag{10.6.8}$$

From (10.6.4) and (10.6.8), the phase margin is

$$PM = 180° - 2\tan^{-1}(\omega_{gc}) - \tan^{-1}\left(\frac{\omega_{gc}}{2}\right) \tag{10.6.9}$$

$$\implies P.M = -40.15° \tag{10.6.10}$$

10.7. Verify your result through the magnitude and phase plot.
**Solution:** The following code plots Fig. 10.7

```
codes/ee18btech11017_2.py
```

10.8. Since the PM in (10.6.10) is negative, the closed loop system is unstable . Verify this

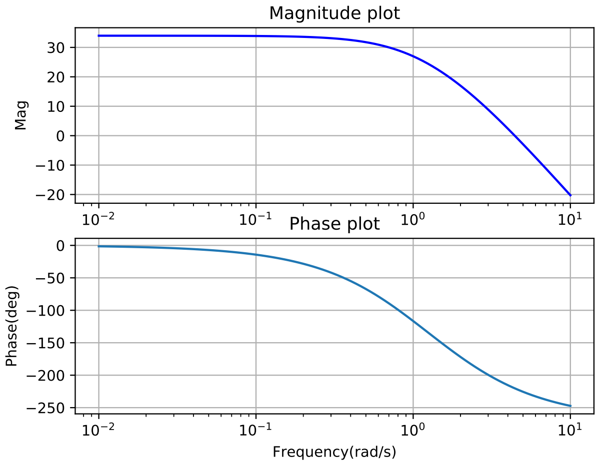


Fig. 10.7

using the Routh-Hurwitz criterion.
**Solution:** The characteristic equation is

$$1 + G(s)H(s) = 0 \quad (10.8.1)$$

$$\implies s^3 + 4s^2 + 5s + 102 = 0 \quad (10.8.2)$$

Constructing the routh array for (10.8.2),

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \end{vmatrix} \quad (10.8.3)$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \\ 102 & 0 & 0 \end{vmatrix} \quad (10.8.4)$$

$\because$ there are two sign changes in the first column of the routh array, two poles lie on right half of s-plane. Therefore,the system is unstable.