Control Systems

G V V Sharma*

Contents

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

***Abstract*—The objective of this manual is to introduce control system design at an elementary level.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/ketan/codes
```

1 Frequency Response Analysis

1.1 Polar Plot

1.1.1. Sketch the Polar Plot for

$$G(s) = \frac{1}{(1+s)(1+2s)} \tag{1.1.1}$$

Solution: The following code generates Fig. 1.1.1

```
codes/ee18btech11012.py
```

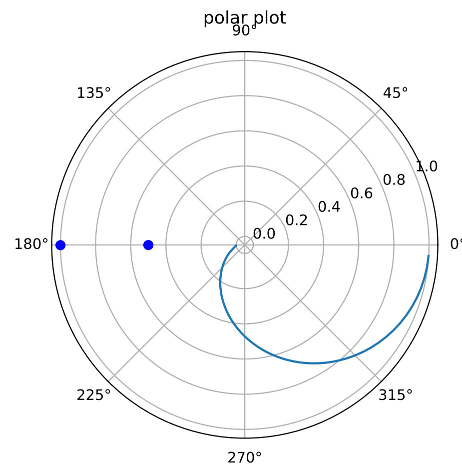


Fig. 1.1.1

The polar plot is to the right of $(-1, 0)$. Hence the closed loop system is stable.

1.1.2. Sketch the polar plot of

$$G(s) = \frac{1}{(s^2)(s+1)(s+2)}. \tag{1.1.2}$$

Solution: Substituting $s = \jmath\omega$ in (1.1.2),
Now the magnitude will be

$$r = |G(\jmath\omega)| = \frac{1}{(\omega^2)(\sqrt{1+\omega^2})(\sqrt{1+4\omega^2})} \tag{1.1.3}$$

$$\theta = \angle G(\jmath\omega) = -\tan^{-1}(0) - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.4}$$

$$= 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.5}$$

The polar plot is the (r, θ) plot for $\omega \in (0, \infty)$. The following python code generates the polar plot in Fig. 1.1.2

```
codes/ee18btech11028.py
```

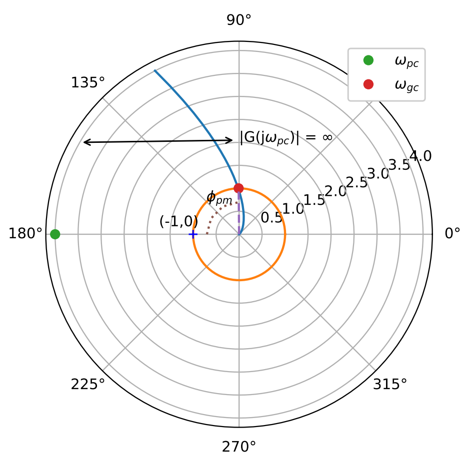


Fig. 1.1.2

The location of $(-1, 0)$ with respect to the polar plot provides information regarding the stability of the system.

- If $(-1, 0)$ is not enclosed, then it is stable.
- If $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 1.1.2, the point $(-1, 0)$ is enclosed by the polar plot, which implies system is not stable. The polar plot also provides info on the GM and PM, which can then be used for determining the stability of the system.

- If the $GM > 1 \cap PM > 0$, then the control system is **stable**.
- If the $GM = 1 \cap PM = 0$, then the control system is **marginally stable**.
- If the $GM < 1 \cup PM < 0$, then the control system is **unstable**.

Therefore, our system is unstable $\because GM < 1 \cap PM < 0$.

1.1.3. Sketch the Polar Plot of

$$G(s) = \frac{1}{s(1+s^2)} \tag{1.1.6}$$

Solution: From (1.1.6),

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1-\omega^2)} \tag{1.1.7}$$

$$|G(\jmath\omega)| = \frac{1}{|\omega(1-\omega^2)|} \tag{1.1.8}$$

$$\angle G(\jmath\omega) = \begin{cases} \frac{\pi}{2} & \omega > 1 \\ -\frac{\pi}{2} & 0 < \omega < 1 \end{cases} \tag{1.1.9}$$

The corresponding polar plot is generated in Fig. 1.1.3 using

```
codes/ee18btech11023.py
```

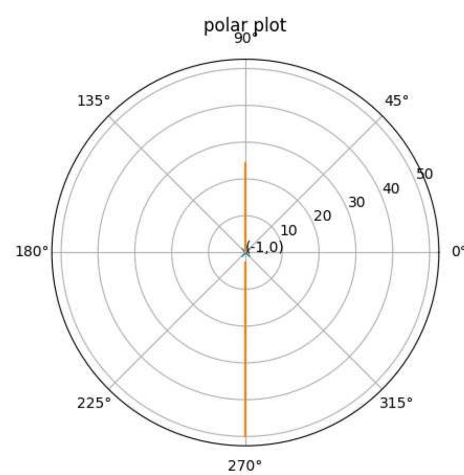


Fig. 1.1.3

In Fig. 1.1.3, (–1,0) is exactly on the polar plot. Hence, the system is marginally stable.

1.1.4. Sketch the Polar Plot of

$$G(s) = \frac{\left(1+\frac{s}{29}\right)(1+0.0025s)}{(s^3)(1+0.005s)(1+0.001s)} \tag{1.1.10}$$

Solution: The following code generates the polar plot in Fig. 2.9.1

```
codes/ee18btech11029.py
```

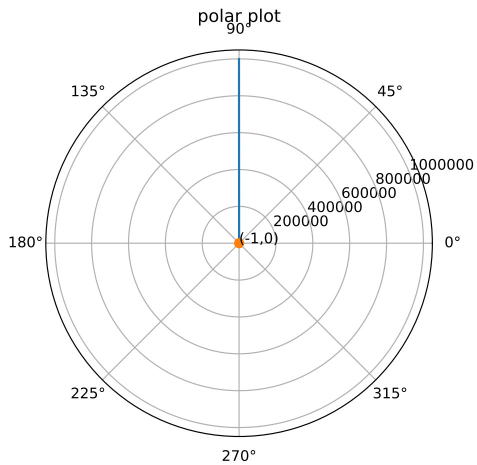


Fig. 1.1.4

- The polar plots use open loop transfer function to determine the stability and hence reference point is shifted to $(-1, 0)$
- If $(-1, 0)$ is left of the polar plot or $(-1, 0)$ is not enclosed, then it is stable
- If $(-1, 0)$ is on right side of the polar plot or $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 2.9.1, $(-1, 0)$ is on the polar plot so the system is marginally stable.

1.1.5. Plot the polar plot of

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}. \tag{1.1.11}$$

Solution:
The following python code generates the polar plot in Fig. 1.1.5

```
codes/ee18btech11033.py
```

∵ (-1,0) is on the right side of the polar plot, the system is unstable.

1.1.6. Plot the polar plot of

$$G(s) = \frac{100(s+5)}{s(s+3)(s^2+4)}. \tag{1.1.12}$$

Solution: The following python code generates the polar plot in Fig. 1.1.6

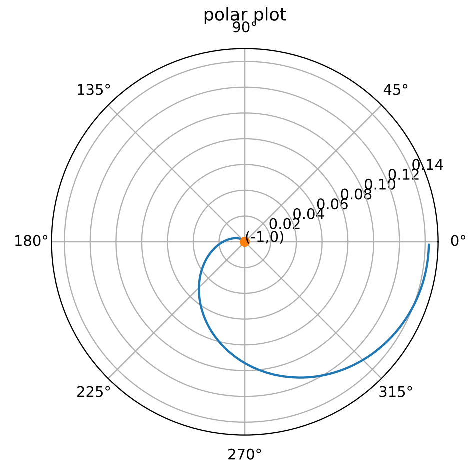


Fig. 1.1.5

```
codes/ee18btech11042.py
```

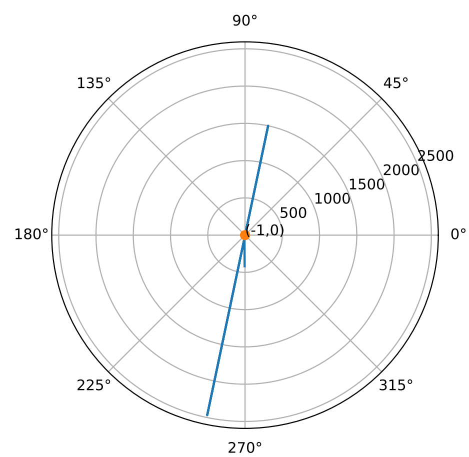


Fig. 1.1.6

Since (-1,0) is on the polar plot, the above system is marginally stable.

1.2 Direct and Inverse Polar Plot

Sketch the direct polar plot for a unity feedback system with open loop transfer function

$$G(s) = \frac{1}{s(1+s)^2} \tag{1.2.1}$$

Solution: The polar plot is obtained by plotting

(r, ϕ)

$$r = |H(\jmath\omega)||G(\jmath\omega)| \tag{1.2.2}$$

$$\phi = \angle H(\jmath\omega)G(\jmath\omega), 0 < \omega < \infty \tag{1.2.3}$$

The following code plots the polar plot in Fig. 1.2.1

```
codes/ee18btech11002/polarplot.py
```

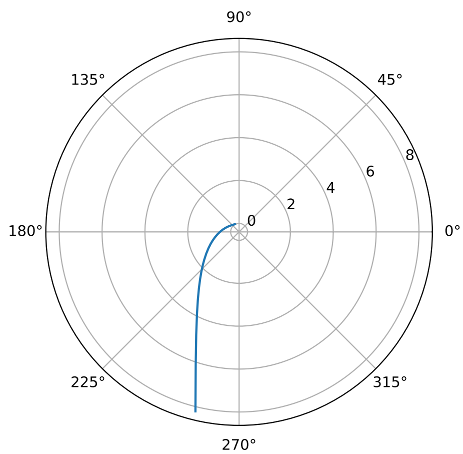


Fig. 1.2.1: Polar Plot

Sketch the inverse polar plot for (1.2.1)

Solution: The above code plots the polar plot in Fig. 1.2.2 by plotting $\left(\frac{1}{r}r, -\phi\right)$

1.3 Bode Plot

1.3.1. Sketch the Bode Magnitude and Phase plot for the following system. Also compute the gain margin and the phase margin.

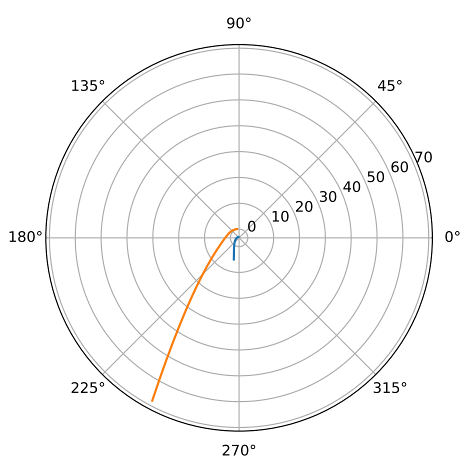


Fig. 1.2.2: Inverse Polar Plot

$$G(s) = \frac{10}{s(1+0.5s)(1+.01s)} \tag{1.3.1}$$

Solution: The Bode magnitude and phase plot are available in Fig. 1.3.1 and generated by

```
codes/ee18btech11048.py
```

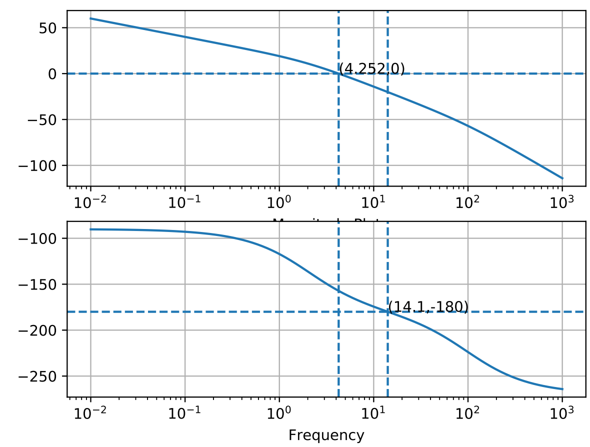


Fig. 1.3.1: Graphs

The pole-zero locations are available in Table 1.3.1.

Zeros	**Poles**
-	0
	-2
	-100

TABLE 1.3.1: Zeros and Poles

The *Gain* and *Phase* of (1.3.2) are

$$\left|G(\jmath\omega)\right| = \frac{100}{\omega\sqrt{(0.5\omega)^2+1}\sqrt{(0.01\omega)^2+1}} \tag{1.3.2}$$

$$\underline{/G(\jmath\omega)} = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{100}\right) \tag{1.3.3}$$

Hence,

$$\left|G\left(\jmath\omega_{gc}\right)\right| = 1 \tag{1.3.4}$$
$$\implies \omega_{gc} = 4.25 \tag{1.3.5}$$
$$\angle G\left(\jmath\omega_{gc}\right) = -157.2 \tag{1.3.6}$$
$$\implies PM = 22.8 \tag{1.3.7}$$

Similarly,

$$\angle G\left(\jmath\omega_{pc}\right) = -180^\circ \tag{1.3.8}$$
$$\implies \omega_{pc} = 14.1 \tag{1.3.9}$$
$$\implies -\left|G\left(\jmath\omega_{pc}\right)\right| = -20.2dB \tag{1.3.10}$$
$$\implies GM = 20.2dB \tag{1.3.11}$$

1.3.2. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} \tag{1.3.12}$$

Also compute gain margin and phase margin .
Solution: From (1.3.12), we have

$$G(\jmath\omega) = \frac{75(1+0.2\jmath\omega)}{\jmath\omega\left((\jmath\omega)^2+16\jmath\omega+100\right)} \tag{1.3.13}$$

poles = 0 , -8-6j , -8+6j
zeros = -5
Gain and phase plots are shown in Fig. 1.3.2

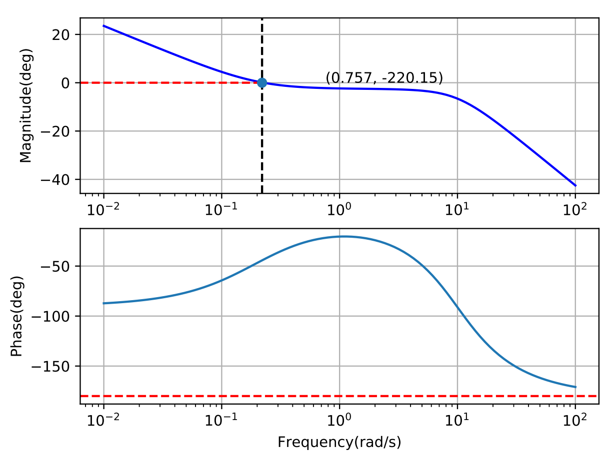


Fig. 1.3.2: a

The following code plots Fig. 1.3.2

```
codes/ee18btech11049.py
```

Solving

$$\left|G(\jmath\omega)\right| = \frac{75\sqrt{\omega^2+25}}{\omega\sqrt{(\omega+6)^2+64}\sqrt{(\omega-6)^2+64}} = 1, \tag{1.3.14}$$

or from Fig. 1.3.2, the gain crossover frequency

$$\implies \omega_{gc} = 0.757 \tag{1.3.15}$$
$$\angle G\left(\jmath\omega_{gc}\right) = -88.3 \tag{1.3.16}$$
$$\implies PM = 91.7 \tag{1.3.17}$$

Solution: From Fig. 1.3.2 ,we can say that phase never crosses -180° . So , the gain margin is *infinite*. Which means we can add any gain, and the equivalent closed loop system never becomes unstable.

1.3.3. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)} \tag{1.3.18}$$

Also compute gain margin and phase margin .
Solution: Substituting $s = \jmath\omega$ in (3.6.2.1) and assuming K =1,

$$G(\jmath\omega) = \frac{(\jmath\omega)^2}{(1+0.2\jmath\omega)(1+0.02\jmath\omega)} \tag{1.3.19}$$

The corner frequencies are

$$\omega_{c1} = 1/0.2 = 5 \tag{1.3.20}$$
$$\omega_{c2} = 1/0.02 = 50 \tag{1.3.21}$$

$$20\log\left|G(\jmath\omega)\right| = 20\log\left|(\jmath\omega)^2\right| - 20\log\left|(1+0.2\jmath\omega)\right| - 20\log\left|(1+0.02\jmath\omega)\right| \tag{1.3.22}$$

The various values of G($\jmath\omega$) are available in Table 3.6.2, in the increasing order of their corner frequencies also slope contributed by each term and the change in slope at the corner frequency. The pase

$$\phi = \angle G(\jmath\omega) = 180^\circ - tan^{-1}(0.2\omega) - tan^{-1}(0.02\omega) \tag{1.3.23}$$

The phase angle of G($\jmath\omega$) are calculated for various value of ω in Table 1.3.3. The magnitude and phase plot are generated in Fig. 1.3.3

TERM	Corner Freq	Slope	Slope change
$(\jmath\omega)^2$	$--$	+40	$--$
$\frac{1}{1+\jmath 0.2}$	$\omega_{c1} = \frac{1}{0.2}$	-20	40-20=20
$\frac{1}{1+\jmath 0.02}$	$\omega_{c2} = \frac{1}{0.02}$	-20	20-20=0

TABLE 1.3.2: Magnitude

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(0.02\omega)$	$\phi = \angle G(j\omega)$
0.5	5.7	0.6	174
1	11.3	1.1	168
2	21.8	2.3	156
5	45	5.7	130
10	63.4	11.3	106
50	84.3	45	50

TABLE 1.3.3: Phase

using the following python code

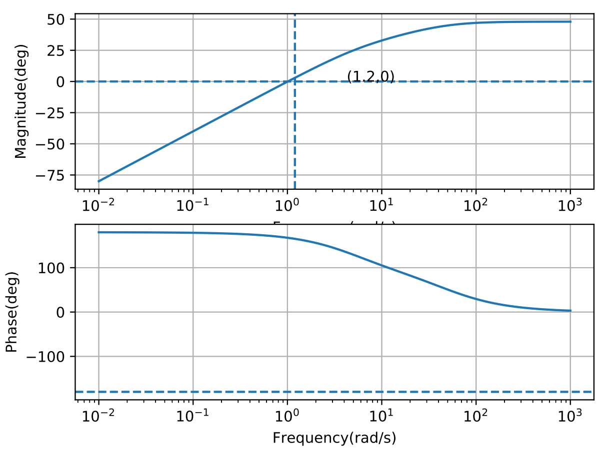


Fig. 1.3.3: Graphs

codes/es17btech11002.py

$\because$ the gain crossover frequency is 2 and the corresponding gain At $\omega = 2$ is 13dB,

$$20 \log K = -13db \tag{1.3.24}$$

$$\implies K = 0.65 \tag{1.3.25}$$

Solving (1.3.19) or from Fig. 1.3.3, the gain crossover frequency,

$$\omega_{gc} = 1.2 \tag{1.3.26}$$

$$\implies PM = 344.8 \tag{1.3.27}$$

From Fig. 1.3.3 ,we can say that phase never crosses $-180°$. So, the gain margin is *infinite*. Which means we can add any gain, and the equivalent closed loop system never goes unstable.

1.3.4. Sketch the Bode magnitude and phase plots for

$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)} \tag{1.3.28}$$

Also compute the gain margin and phase margin.

Solution:

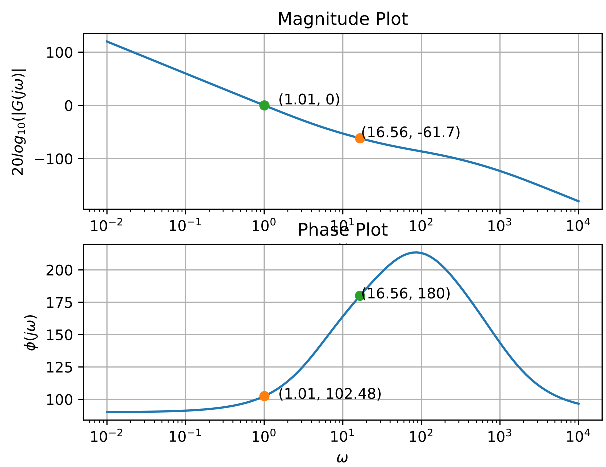


Fig. 1.3.4: Bode plot

From fig. 1.3.4,

$$\omega_{gc} = 16.55,\ \text{Gain Margin} = -61.7dB \tag{1.3.29}$$

$$\omega_{pc} = 1,\ \text{Phase Margin} = -77.52^0 \tag{1.3.30}$$

The program for plotting bode plot and finding phase margin and gain margin -

codes/ee18btech11039.py

1.3.5. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{50(s+3)(s+5)}{s(s+2)(s+4)(s+6)} \tag{1.3.31}$$

The magnitude and phase plot are as follows: Fig1.3.5

The python code to obtain the graphs and results:

codes/ee18btech11031.py

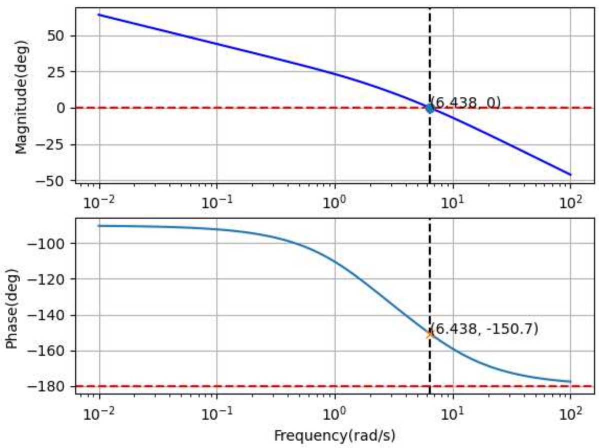


Fig. 1.3.5: Graphs

Gain and *Phase* of Transfer Function

$$G(j\omega) = \frac{50(j\omega+3)(j\omega+5)}{j\omega(j\omega+2)(j\omega+4)(j\omega+6)} \quad (1.3.32)$$

Gain:

$$\frac{100\sqrt{(\omega)^2+9}\sqrt{(\omega)^2+25}}{\omega\sqrt{(\omega)^2+4}\sqrt{(\omega)^2+16}\sqrt{(\omega)^2+36}} \quad (1.3.33)$$

Phase:

$$\tan^{-1}(0)+\tan^{-1}\left(\frac{\omega}{3}\right)+\tan^{-1}\left(\frac{\omega}{5}\right)-\tan^{-1}\left(\frac{\omega}{0}\right)$$
$$-\tan^{-1}\left(\frac{\omega}{2}\right)-\tan^{-1}\left(\frac{\omega}{4}\right)-\tan^{-1}\left(\frac{\omega}{6}\right) \quad (1.3.34)$$

$$PM = \angle G(j\omega_{gc}) + 180^\circ \quad (1.3.35)$$
$$\omega_{gc} = \text{Gain Crossover Frequency} \quad (1.3.36)$$
$$\text{At } \omega_{gc}\, |G(s)| = 1 \quad (1.3.37)$$

Solution:

$$\frac{100\sqrt{\left(\omega_{gc}\right)^2+9}\sqrt{\left(\omega_{gc}\right)^2+25}}{\omega gc\sqrt{\left(\omega_{gc}\right)^2+4}\sqrt{\left(\omega_{gc}\right)^2+16}\sqrt{\left(\omega_{gc}\right)^2+36}} = 1 \quad (1.3.38)$$

Solving Eq. (1.3.38) *or* from Fig 1.3.5 :

$$\implies \omega_{gc} = 6.438 \quad (1.3.39)$$
$$\angle G\left(j\omega_{gc}\right) = -150.725 \quad (1.3.40)$$
$$\implies PM = 29.275 \quad (1.3.41)$$

$$GM = 0 - G(j\omega_{pc})db \quad (1.3.42)$$
$$\omega_{pc} = \text{Phase Crossover Frequency} \quad (1.3.43)$$

$$\text{At } \omega_{pc}, \angle G(s) = -180^\circ \quad (1.3.44)$$

From Fig 1.3.5 ,we can say that phase never crosses -180° . So , the gain margin is *infinite* and from the equation: 1.3.44, ω_{pc} is non-existent.

1.3.6. Sketch the bode magnitude and phase plots for the closed loop (negative feedback) system given by:

$$G(s) = \frac{100(s+2)(s+4)}{s^2-3s+10} \quad (1.3.45)$$

$$H(s) = \frac{1}{s} \quad (1.3.46)$$

Solution: The system can be represented as:

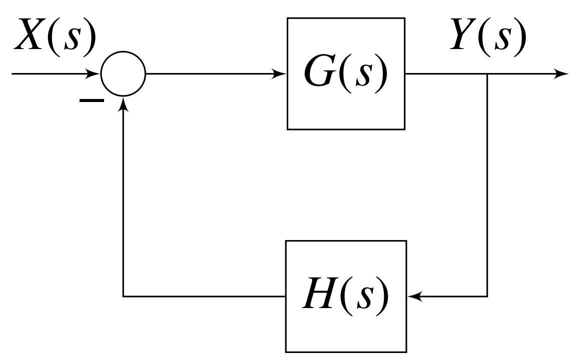


Fig. 1.3.6: Block diagram for the system

The closed loop transfer function of the system is given by:

$$G_m(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s).H(s)} \quad (1.3.47)$$
$$= \frac{100s(s+2)(s+4)}{s^3+97s^2+610s+800} \quad (1.3.48)$$

Evaluate at $s = \jmath\omega$:

$$G_m(\jmath\omega) = \frac{100\jmath\omega(\jmath\omega+2)(\jmath\omega+4)}{(\jmath\omega)^3+97(\jmath\omega)^2+610(\jmath\omega)+800} \quad (1.3.49)$$

$$= \frac{-600\omega^2+\jmath\left(800\omega-100\omega^3\right)}{800-97\omega^2+\jmath(610\omega-\omega^3)} \quad (1.3.50)$$

From (1.3.50):

$$\left|G_m(\jmath\omega)\right| = \frac{\sqrt{\left(600\omega^2\right)^2 + \left(800\omega - 100\omega^3\right)^2}}{\sqrt{\left(800 - 97\omega^2\right)^2 + \left(610\omega - \omega^3\right)^2}} \tag{1.3.51}$$

$$\begin{aligned}\angle G_m(\jmath\omega) = \tan^{-1}\left(\frac{\omega^2 - 8}{6\omega}\right)\\ - \tan^{-1}\left(\frac{610\omega - \omega^3}{800 - 97\omega^2}\right)\end{aligned} \tag{1.3.52}$$

The following code plots the bode magnitude and phase plots in Fig. 1.3.7:

```
codes/ee18btech11045/
    ee18btech11045_bode1.py
```

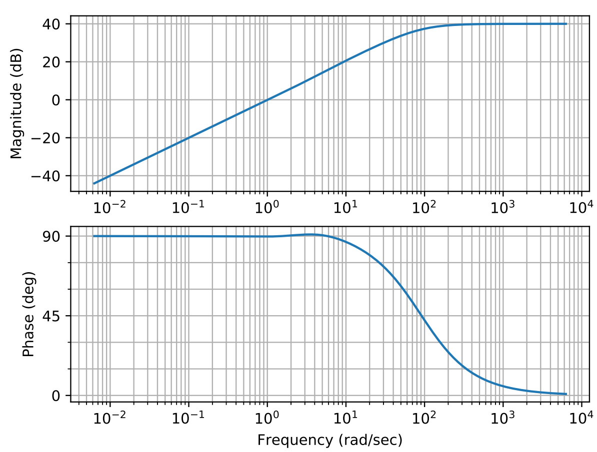


Fig. 1.3.7: Bode plot for $G_m(\jmath\omega)$

$$G(\jmath\omega)H(\jmath\omega) = \left(\frac{100(\jmath\omega+2)(\jmath\omega+4)}{(\jmath\omega)^2 - 3\jmath\omega + 10}\right)\left(\frac{1}{\jmath\omega}\right) \tag{1.3.53}$$

$$= \frac{100\left(-\omega^2 + 8 + \jmath 6\omega\right)}{3\omega^2 + \jmath\left(10\omega - \omega^3\right)} \tag{1.3.54}$$

Using (1.3.54)

$$\left|G(\jmath\omega)H(\jmath\omega)\right| = \frac{100\sqrt{\left(8-\omega^2\right)^2 + (6\omega)^2}}{\sqrt{\left(3\omega^2\right)^2 + \left(10\omega - \omega^3\right)^2}} \tag{1.3.55}$$

$$\begin{aligned}\angle G(\jmath\omega)H(\jmath\omega) = \tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right)\\ - \tan^{-1}\left(\frac{10-\omega^2}{3\omega}\right)\end{aligned} \tag{1.3.56}$$

At the phase crossover frequency ω_{pc}:

$$\left|\angle G(\jmath\omega)H(\jmath\omega)\right| = 180 \tag{1.3.57}$$

$$\implies \tan^{-1}\left(\frac{6\omega_{pc}}{8-\omega_{pc}^2}\right) - \tan^{-1}\left(\frac{10-\omega_{pc}^2}{3\omega_{pc}}\right) = 180 \tag{1.3.58}$$

Solving the above equation:

$$\frac{6\omega_{pc}}{8-\omega_{pc}^2} = \frac{10-\omega_{pc}^2}{3\omega_{pc}} \tag{1.3.59}$$

$$\implies \omega_{pc} = 5.8 rad/sec \tag{1.3.60}$$

$$\left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_pc} = 28.1dB \tag{1.3.61}$$

Gain Margin GM :

$$GM = 0 - \left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_pc} dB \tag{1.3.62}$$

$$= -28.1dB \tag{1.3.63}$$

At the gain crossover frequency ω_{gc}:

$$\left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_{gc}} = 1 \tag{1.3.64}$$

From (1.3.55),

$$10^4\left(\left(8-\omega^2\right)^2 + (6\omega)^2\right) = 9\omega^4 + \left(10\omega - \omega^3\right)^2 \tag{1.3.65}$$

$$\implies \omega_{gc} = 100.15 rad/sec \tag{1.3.66}$$

Substitute ω_{gc} in (1.3.56):

$$\angle G(\jmath\omega)H(\jmath\omega)_{\omega=\omega_{gc}} = 265^\circ \tag{1.3.67}$$

Phase Margin PM:

$$PM = 180^\circ - \angle G(\jmath\omega)H(\jmath\omega)_{\omega=\omega_{gc}} \tag{1.3.68}$$

$$= 180^\circ - 265^\circ = -85^\circ \tag{1.3.69}$$

The following code is used to verify the gain and phase margins:

```
codes/ee18btech11045/
    ee18btech11045_bode2.py
```

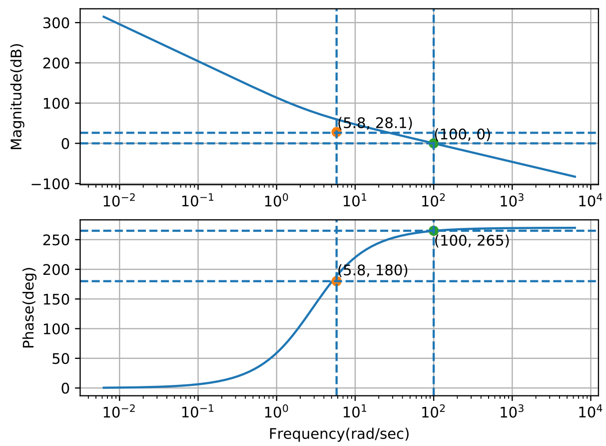


Fig. 1.3.8: Bode plot for $G(\jmath\omega)H(\jmath\omega)$

As both the Gain Margin (GM) and Phase Margin (PM) are found to be negative, the system is unstable.

1.4 *Transient Response from Bode Plot*

1.4.1. Consider the following transfer functions as open-loop transfer functions in two different unity feedback(negative) systems.

$$G(s) = \frac{50(s+3)(s+5)}{s(s+2)(s+4)(s+6)} \tag{1.4.1.1}$$

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} \tag{1.4.1.2}$$

Estimate transient response of these systems from their respective bode plots.

Solution:

a) The dominant pole approximation is used to characterize higher order systems because it is difficult to characterize and analyse systems with order greater than 3.

b) Consider a transfer function.

$$H(s) = K\frac{\alpha\beta}{(s+\alpha)(s+\beta)} \tag{1.4.1.3}$$

It has two poles $-\alpha$ and $-\beta$. If the magnitude of β is very large compared to α (typically if $\frac{|\beta|}{|\alpha|} > 5$) we can approximate for the transfer function assuming s is sufficiently small compared to β as follows.

$$H(s) = K_2\left(\frac{1}{s+\alpha}\right) \tag{1.4.1.4}$$

Note that the value of $H(0)$ should be unchanged for the exact and approximate transfer functions.This is necessary to ensure that the final value of the step response is unchanged.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{1.4.1.5}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sU(s)H(s) = H(0) \tag{1.4.1.6}$$

In order to acheive this we adjust the gain value of the approximated transfer function by equating $H(0)$ values.

$$\implies H(s) = K\frac{\alpha}{(s+\alpha)} \tag{1.4.1.7}$$

c) In terms of poles, the pole closer to the origin is considered as the dominating pole.As considered above,the magnitude of α is small therefore the time constant $\frac{1}{\alpha}$ will be high and reaches equilibrium slowly and vice versa in case of β.Therefore,this approximation assumes that the slowest part of the system dominates the response.The faster parts of the system are ignored.

d) Complex poles along with real poles : In this case the dominant pole(s) can be determined by comparing only the real parts.If the real part of the complex conjugate poles is greater in magnitude than the real pole, the two complex conjugate poles the dominant poles.

e) If the transfer function has zeros along with poles,we have to consider the fact that pole and zero cancel out each other if their respective magnitudes are comparable.

1.4.2. Find the closed loop transfer function of a negative unity feedback system given open loop transfer function $G(s)$.

Solution:

$$T(s) = \frac{G(s)}{1+G(s)} \tag{1.4.2.1}$$

1.4.3. Find the approximate transfer function for the open loop transfer function.

$$G(s) = \frac{50(s+3)(s+5)}{s(s+2)(s+4)(s+6)} \tag{1.4.3.1}$$

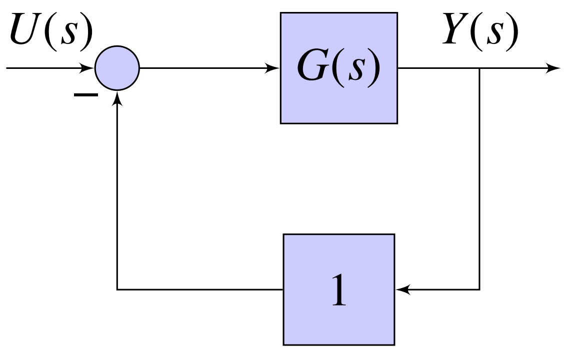


Fig. 1.4.1

Solution: Using equation(1.4.2.1)

$$T(s) = \frac{50(s^2 + 8s + 15)}{s^4 + 12s^3 + 94s^2 + 448s + 750} \tag{1.4.3.2}$$

The following code gives the poles and zeros of the transfer function.

```
codes/ee18btech11047/ee18btech11047_1.py
```

Poles	Zeros
$p_1 = -5.14$	$z_1 = -5$
$p_2 = -3.09$	$z_2 = -3$
$p_3 = -1.87 + 6.60j$	
$p_4 = -1.87 - 6.60j$	

TABLE 1.4.1

The real poles (p_1, p_2) and zeros (z_1, z_2) cancel out each other as mentioned above.So, we are left with the two conjugate poles.The approximated transfer function is

$$T_1(s) = \frac{K_1}{(s - p_3)(s - p_4)} \tag{1.4.3.3}$$

$$T(0) = T_1(0) \tag{1.4.3.4}$$

$$\implies K_1 = p_3 p_4 \tag{1.4.3.5}$$

$$T_1(s) = \frac{47.09}{s^2 + 3.74s + 47.09} \tag{1.4.3.6}$$

1.4.4. Estimate the transient response of the obtained second order system using the respective bode plot.
Solution: The following code generates the bode plot for open loop transfer function.

```
codes/ee18btech11047/ee18btech11047_2.py
```

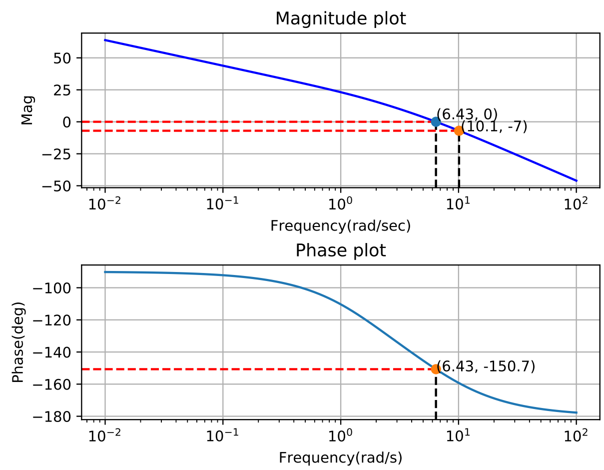


Fig. 1.4.2: 1

The phase margin is

$$\phi_M = 180° - 150.7° \implies \phi_M = 29.3° \tag{1.4.4.1}$$

The closed-loop bandwith, ω_{BW}(-3 dB frequency), equals the frequency at which the open-loop magnitude response is around -7 dB.

$$\omega_{BW} = 10.1 rad/sec \tag{1.4.4.2}$$

Damping ratio: Substitute ϕ_M value from equation (1.4.4.1)

$$\phi_M = tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^2}}}\right) \tag{1.4.4.3}$$

$$\implies \zeta = 0.34 \tag{1.4.4.4}$$

Settling time: Substitute ω_{BW} value from equation(1.4.4.2) and ζ

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{1.4.4.5}$$

$$\implies T_s = 1.65 sec \tag{1.4.4.6}$$

Peak time:

$$T_p = \frac{\pi\zeta T_s}{4\sqrt{1 - \zeta^2}} \tag{1.4.4.7}$$

$$\implies T_p = 0.325 sec \quad (1.4.4.8)$$

Percent overshoot:

$$\%OS = 100e^{-(\frac{\zeta\pi}{\sqrt{1-\zeta^2}})} \quad (1.4.4.9)$$

$$\implies \%OS = 35.1\% \quad (1.4.4.10)$$

Note that the answers will be approximate due to the dominant pole approximation.The following code generates the step response of the system.

codes/ee18btech11047/ee18btech11047_3.py

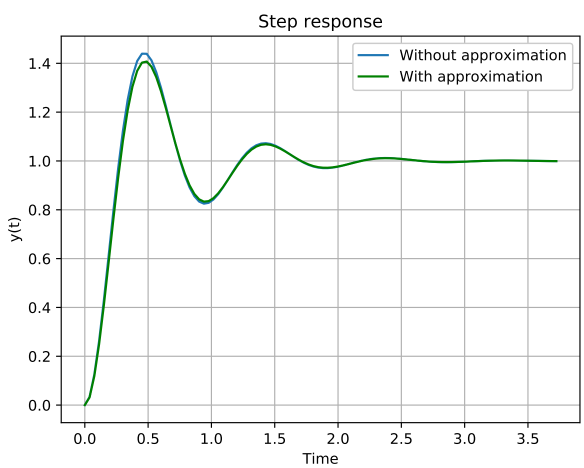


Fig. 1.4.3: 2

1.4.5. Find the approximate transfer function for the open loop transfer function

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} \quad (1.4.5.1)$$

Solution: Using equation (1.4.2.1)

$$T(s) = \frac{75(1+0.2s)}{s^3+16s^2+115s+75} \quad (1.4.5.2)$$

The following code gives the poles and zeros of the transfer function.

codes/ee18btech11047/ee18btech11047_4.py

The real part of the complex conjugate poles is comparable with the zero z_1 of the transfer function.So,they cancel out each other.The approximated transfer function is of first order.

$$T_2(s) = \frac{K_2}{(s-p_1)} \quad (1.4.5.3)$$

Poles	Zeros
$p_1 = -0.72$	$z_1 = -5$
$p_2 = -7.64 + 6.75j$	
$p_3 = -7.63 - 6.75j$	

TABLE 1.4.2

$$T(0) = T_2(s) \quad (1.4.5.4)$$

$$\implies K_2 = p_1 \quad (1.4.5.5)$$

$$T_2(s) = \frac{0.72}{s+0.72} \quad (1.4.5.6)$$

1.4.6. Estimate the transient response of the obtained first order system.
Solution: Time constant: The time constant is the time taken by the step response to rise to 63% of it's final value.

$$T = \frac{1}{|pole|} \quad (1.4.6.1)$$

$$T = \frac{1}{0.72} = 1.388 sec \quad (1.4.6.2)$$

Rise time: Rise time is the time for the waveform to go from 0.1 to 0.9 of it's final value.

$$T_r = \frac{2.2}{|pole|} \quad (1.4.6.3)$$

$$T_r = \frac{2.2}{0.72} = 3.05 sec \quad (1.4.6.4)$$

Settling time: Settling time is defined as the time for the response to reach and stay within, 2% of its final value.

$$T_s = \frac{4}{|pole|} \quad (1.4.6.5)$$

$$T_s = \frac{4}{0.72} = 5.55 sec \quad (1.4.6.6)$$

The following code plots the step response of the system.

codes/ee18btech11047/ee18btech11047_5.py

1.5 Parameter Estimation from Bode Plot

Obtain the transfer function from the frequency response data given below. **Solution:** The following

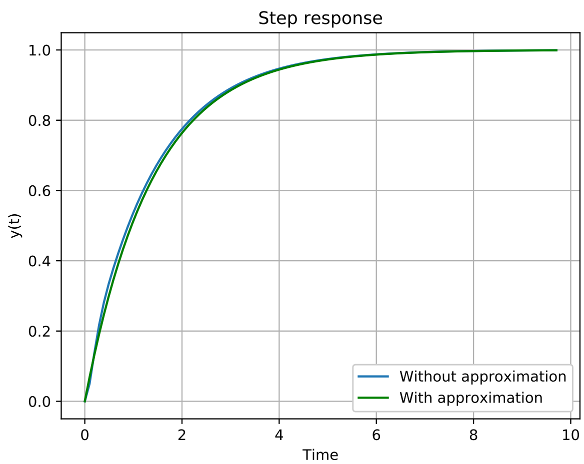


Fig. 1.4.4

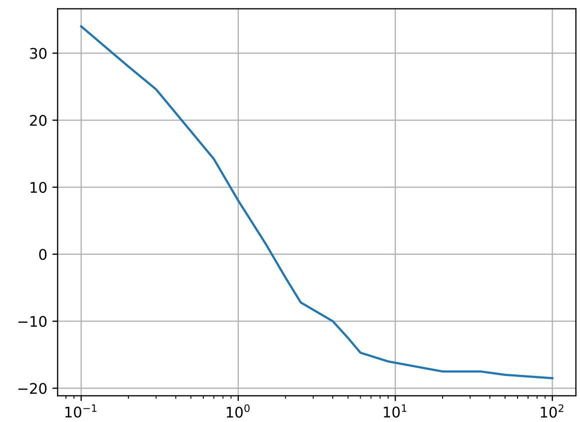


Fig. 1.5.1: 1

Freq(rad/sec)	Mag(dB)
0.1	34
0.2	28
0.3	24.6
0.7	14.2
1.0	8
1.5	1.5
2.0	-3.5
2.5	-7.2
4.0	-10
5.0	-10
6.0	-14.7
9.0	-16.0
20	-17.5
35	-17.5
50	-18
100	-18.5

TABLE 1.5.1

code generates the plot for the given data.

```
codes/ee18btech11006/ee18btech11006_1.py
```

Consider the Transfer function

$$H(s) = \frac{k(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} \tag{1.5.1}$$

Let's draw the magnitude bode plot.

$$\begin{aligned} 20log_{10}|H(s)| = 20log_{10}|s+z_1| + 20log_{10}|s+z_2| \\ - 20log_{10}|s+p_1| - 20log_{10}|s+p_2| \\ + 20log_{10}k \end{aligned} \tag{1.5.2}$$

Now, for the given set of points finding slopes corresponding to every two points and identifying the points at which the slope change occurs by 20dB would give us the poles and zeros respectively. The slope initially is-20dB/decade i.e. pole at 0.1 We can observe that the slope decreases approximately by 20dB/dec at the point 0.7 and increases further by 20dB/dec at 2.5 and 6 giving a slope of almost 0 later on.

$$Poles = 0.1, 0.7 \tag{1.5.3}$$

$$Zeros = 2.5, 6 \tag{1.5.4}$$

Lets plot the magnitude bode plot considering these poles and zeros. The following code generates the plot for the transfer function

$$H(s) = \frac{(s+2.5)(s+6)}{(s+0.1)(s+0.7)} \tag{1.5.5}$$

```
codes/ee18btech11006/ee18btech11006_2.py
```

Now the gain,

$$K = \frac{H(\omega)(given)}{H(\omega)(calculated)} \tag{1.5.6}$$

This value would vary for different frequencies. Considering the average value , K= 0.1778. The following code generates the plot for the transfer

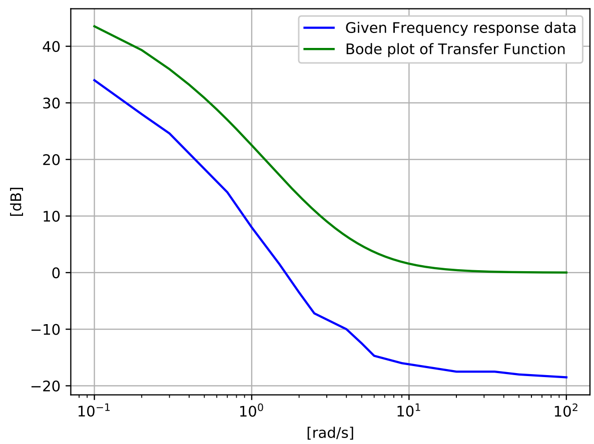


Fig. 1.5.2: 2

function

$$H(s) = \frac{0.1778\,(s+2.5)\,(s+6)}{(s+0.1)\,(s+0.7)} \tag{1.5.7}$$

codes/ee18btech11006/ee18btech11006_3.py

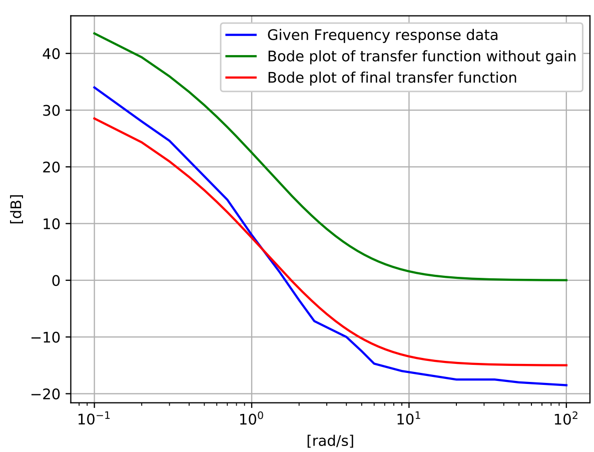


Fig. 1.5.3: 3

Thus, our Transfer function is

$$H(s) = \frac{0.1778\,(s+2.5)\,(s+6)}{(s+0.1)\,(s+0.7)} \tag{1.5.8}$$

2 Stability in Frequency Domain

2.1 Nyquist Criterion

2.1.1. Using Nyquist Criterion, find out whether this system is stable or not.

$$G(s) = \frac{50}{s(s+3)(s+6)} \tag{2.1.1.1}$$

$$H(s) = 1. \tag{2.1.1.2}$$

Nyquist Stability:

$$N = Z - P \tag{2.1.1.3}$$

where Z is number of unstable poles of closed loop transfer function, P is number of unstable poles of open loop transfer function. and N is number of clockwise encirclement of $-1 + j0$. Closed Loop Transfer Function:

$$T(s) = \frac{50}{s^3 + 9s^2 + 18s + 50} \tag{2.1.1.4}$$

$$Z = 0, P = 0 \tag{2.1.1.5}$$

$$N = 0 \tag{2.1.1.6}$$

Thus, system is stable, which can be verified from Nyquist Plot in Fig 2.1.1

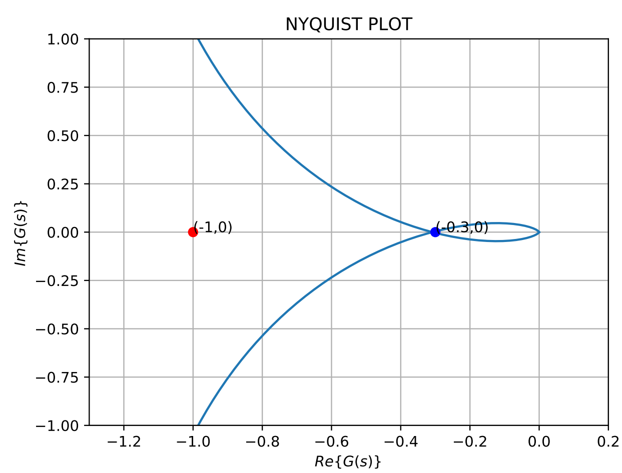


Fig. 2.1.1

The following code generates Fig 2.1.1

codes/ee18btech11050_1.py

2.1.2. Using Nyquist criterion find the range of K for which closed loop system is stable.

$$G(s) = \frac{K}{s(s+6)} \tag{2.1.2.1}$$

$$H(s) = \frac{1}{s+9} \tag{2.1.2.2}$$

Solution: The system flow can be described as,

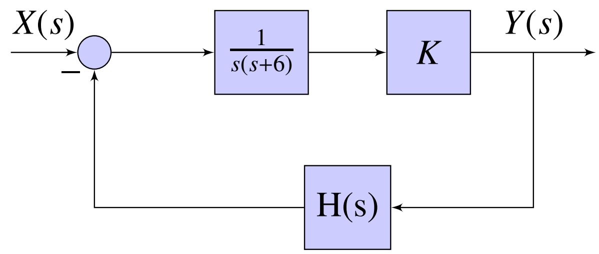


Fig. 2.1.2

$$G_1(s) = \frac{1}{s(s+6)}. \quad (2.1.2.3)$$

For Nyquist plot,

$$\text{Im}\{G_1(\jmath\omega)H(\jmath\omega)\} = \frac{-(54-\omega^2)}{(\omega)(\omega^2+56)(\omega^2+81)} \quad (2.1.2.4)$$

$$\text{Re}\{G_1(\jmath\omega)H(\jmath\omega)\} = \frac{-15\omega}{(\omega)(\omega^2+56)(\omega^2+81)} \quad (2.1.2.5)$$

From (2.2.1.4) and (2.2.1.5)

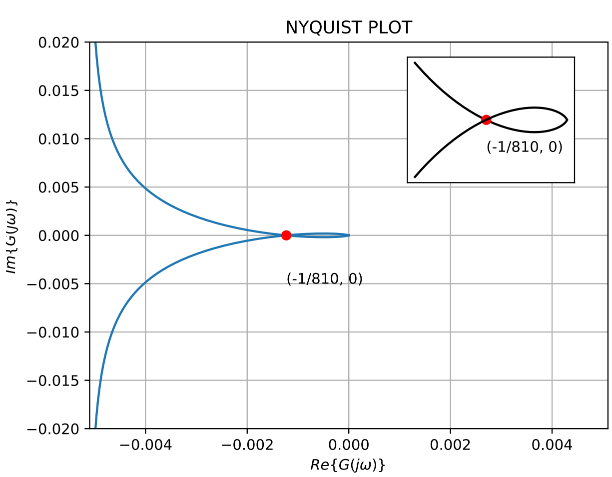


Fig. 2.1.3: Nyquist plot for $G_1(s)H(s)$

Nyquist Stability Criterion:

$$N = Z - P \quad (2.1.2.6)$$

where Z is # unstable poles of closed loop transfer function, P is # unstable poles of open loop transfer function and N is # clockwise encirclement of $(-1/K, 0)$.

For stable system,

$$Z = 0 \quad (2.1.2.7)$$

From (2.2.1.2) and (2.2.1.3),

$$P = 0 \quad (2.1.2.8)$$
$$\implies N = 0 \quad (2.1.2.9)$$

Since, there is a zero at origin, an infinite radius half circle will enclose the right hand side of end points of the Nyquist plot. So for (2.2.1.9),

$$\implies \frac{-1}{K} < \frac{-1}{810} \implies K < 810 \quad (2.1.2.10)$$

And also,

$$K > 0 \quad (2.1.2.11)$$
$$\implies 0 < K < 810 \quad (2.1.2.12)$$

The following python code generates Fig. 2.2.2

```
codes/ee18btech11028_1.py
```

2.1.3. Using Nyquist criterion, find out whether the system below is stable or not

$$G(s) = \frac{41}{s^2(s+3)} \quad (2.1.3.1)$$

$$H(s) = (s+4) \quad (2.1.3.2)$$

Solution: According to the Nyquist criteria the number of unstable closed-loop poles (Z) is equal to the number of unstable open-loop poles (P) plus the number of clockwise encirclements (N) of the point (-1,j0) of the Nyquist plot of G(s)H(s), i.e

$$Z = N + P \quad (2.1.3.3)$$

Open loop transfer function :

$$G(s)H(s) = \frac{41(s+4)}{s^2(s+3)} \quad (2.1.3.4)$$

Closed loop transfer function:

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{41}{s^3+3s^2+41s+164} \quad (2.1.3.5)$$

In Fig.2.1.4 it can be seen that there is a

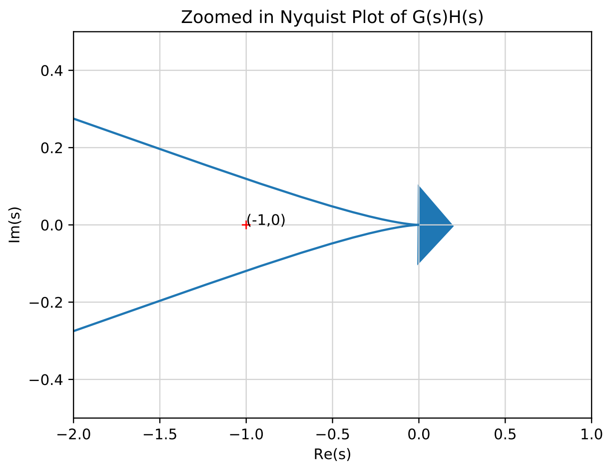


Fig. 2.1.4

clockwise encirclement around (-1+0j). As the open loop transfer function has zero pole of multiplicity 2, therefore it should be assumed that the phasor travels 2 times clock-wise along a semicircle of infinite radius.

N=2, P=0

$$\implies Z = 2 \tag{2.1.3.6}$$

Therefore, The system T(s) is unstable as it has two poles on the right side of the s plane. The following code generates the nyquist plot

```
codes/ee18btech11041.py
```

2.1.4. Using Nyquist criterion, find out whether the following is stable or not.

$$G(s) = \frac{100(s+5)}{s(s^2+4)(s+3)} \tag{2.1.4.1}$$

$$H(s) = 1 \tag{2.1.4.2}$$

Solution: Open loop transfer function (oltf):

$$G(s)H(s) = \frac{100(s+5)}{s(s^2+4)(s+3)} \tag{2.1.4.3}$$

Closed loop transfer function (cltf):

$$\frac{G(s)}{1+G(s)H(s)} \tag{2.1.4.4}$$

Nyquist Stability Criterion can be expressed as:

$$Z = N + P \tag{2.1.4.5}$$

where:

- Z = zeros of $1 + G(s)H(s)$ in RHS of s-plane
- N = number of encirclement of critical point $1{+}0j$ in the clockwise direction.
- P = poles of G(s)H(s) in RHS of s-plane.

The pole-zero plot of equation (2.1.4.3) is fig. (2.1.5) which gives **P = 0**.

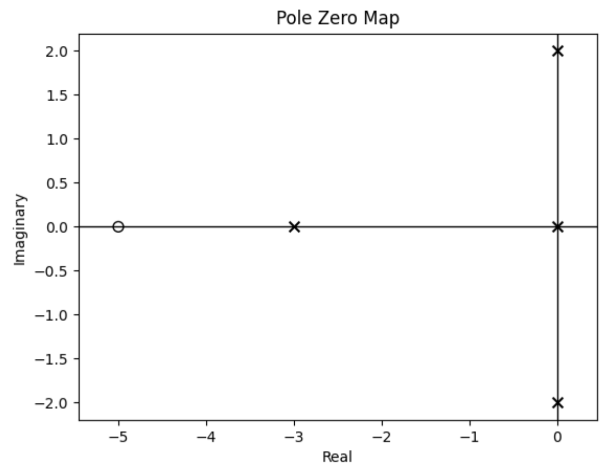


Fig. 2.1.5

- Since the multiplicity of zero pole is 1 (fig.2.1.5), it should be assumed that the phasor travels one time clockwise along a semicircle of infinite radius.
- Same applies for poles at $2j$ and -$2j$.
- Fig. (2.1.6) shows a schematic, the dotted lines are infinite radii semi-circles.
- The point -1+0j is not encircled by the nyquist plot (fig. 2.1.8).

From the nyquist plot (fig. 2.1.8), -1+0j is not encircled by the plot. So from above points, the only clockwise encirclement is considered due to the mentioned poles (zero, $2j$ and -$2j$) with multiplicity of 1.
Therefore, **N=2**
Substituting values of P = 0 and N = 2 in equation (2.1.4.5):

$$\implies Z = 2 \tag{2.1.4.6}$$

This is verified using pole zero plot of 1+G(s)H(s) (fig. 2.1.9). Two zeroes on RHS of s-plane i.e. Z=2.
Since $Z \neq 0$, **the closed loop system is unstable**.
The **open loop system is stable** as there are **no poles on RHS** of s-plane.

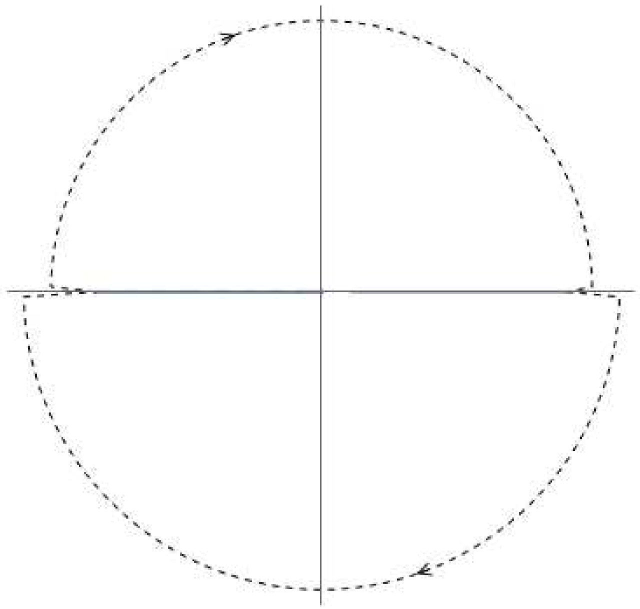


Fig. 2.1.6

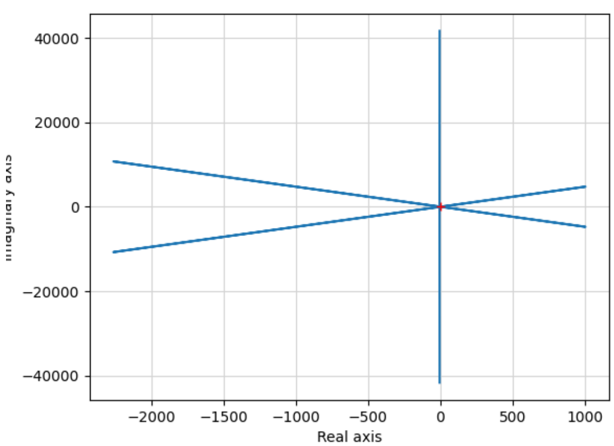


Fig. 2.1.7: Nyquist plot of G(s)H(s)

The following code plots the pole zero plot and the nyquist plot.

codes/ee18btech11025.py

2.1.5. Using Nyquist criterion, find out whether the system below is stable or not.

$$G(s) = \frac{20}{s(s+1)}, H(s) = \frac{s+3}{s+4} \tag{2.1.5.1}$$

Solution: The following python code generates the Nyquist plot in Fig.2.1.10.

codes/ee18btech11011.py

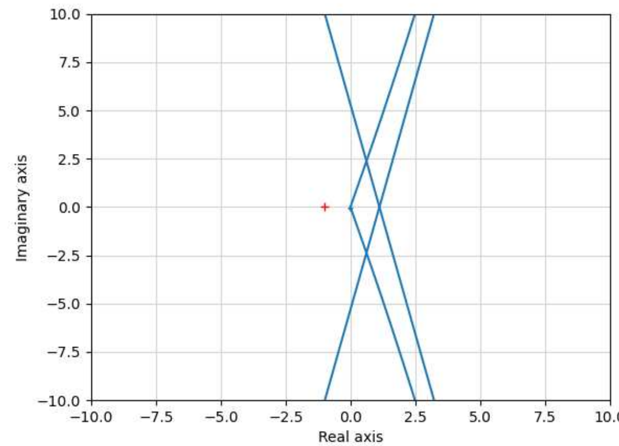


Fig. 2.1.8: Zoomed in

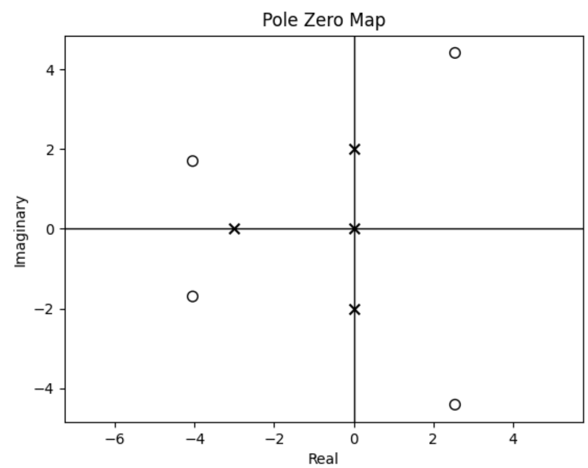


Fig. 2.1.9

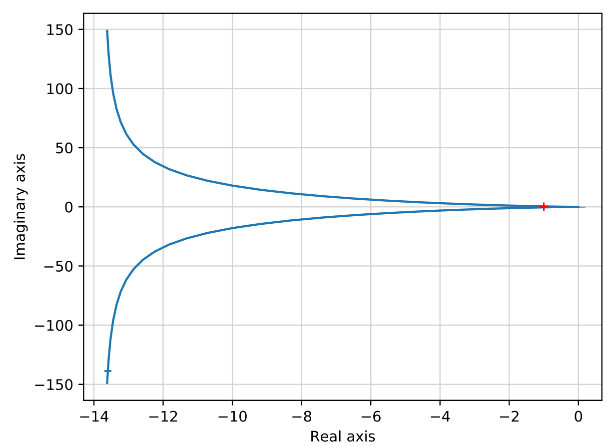


Fig. 2.1.10: Nyquist Plot

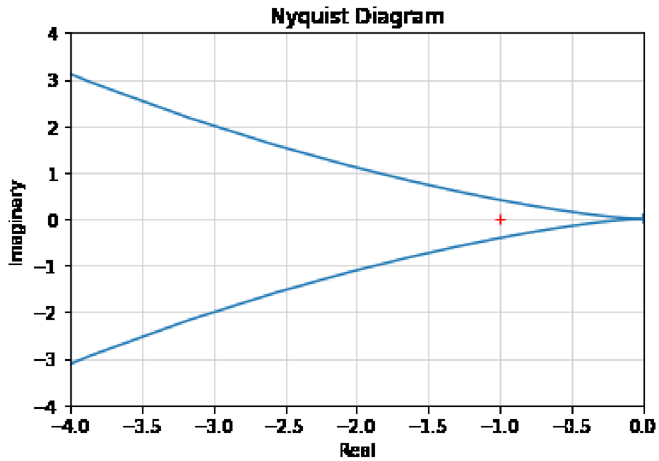


Fig. 2.1.11: Zoomed image

The closed loop system the transfer function will be =

$$\frac{G(s)}{1+G(s)H(s)} \tag{2.1.5.2}$$

$$\implies G(s)H(s) = \frac{20(s+3)}{s(s+1)(s+4)} \tag{2.1.5.3}$$

So it has 3 open-loop poles 0,-1 and -4, therefore P=0.Further we know that N = Z-P , now we know Z = Poles of $\frac{G(s)}{1+G(s)H(s)}$in right half of s plane.To find the poles we can use the following Routh Hurwitz python code.Using this we get Z = 0.

```
codes/ee18btech11011_1.py
```

$$P = 0, Z = 0 \tag{2.1.5.4}$$

$$\implies N = 0 \tag{2.1.5.5}$$

This can also be seen from the Fig. 2.1.10 that the encirclement is counter-clockwise not clockwise.Hence the system is stable.

2.1.6. Using Nyquist criterion, find out the range of K for which the closed loop system will be stable.

$$G(s) = \frac{K(s+2)(s+4)}{s^2-3s+10}; H(s) = \frac{1}{s} \tag{2.1.6.1}$$

Solution: The system flow can be described by Fig. 2.1.12 From (2.1.6.1),

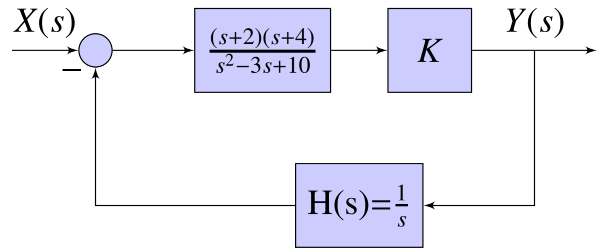


Fig. 2.1.12

$$G(s)H(s) = \frac{K(s+2)(s+4)}{s(s^2-3s+10)} \tag{2.1.6.2}$$

$$G(\jmath\omega)H(\jmath\omega) = \frac{K(j\omega+2)(j\omega+4)}{j\omega((10-\omega^2)-3j\omega)} \tag{2.1.6.3}$$

$$\text{Re}\left\{G(\jmath\omega)H(\jmath\omega)\right\} = \frac{K(84\omega^2-9\omega^4)}{\omega^6-11\omega^4+100\omega^2} \tag{2.1.6.4}$$

$$\text{Im}\left\{G(\jmath\omega)H(\jmath\omega)\right\} = \frac{K(-\omega^5+36\omega^3-80\omega)}{\omega^6-11\omega^4+100\omega^2} \tag{2.1.6.5}$$

The Nyquist plot is a graph of $\text{Re}\left\{G(\jmath\omega)H(\jmath\omega)\right\}$ vs $\text{Im}\left\{G(\jmath\omega)H(\jmath\omega)\right\}$. Let's take K =1 and draw the nyquist plot .

The following python code generates the Nyquist plot in Fig. 2.1.13

```
codes/ee18btech11016.py
```

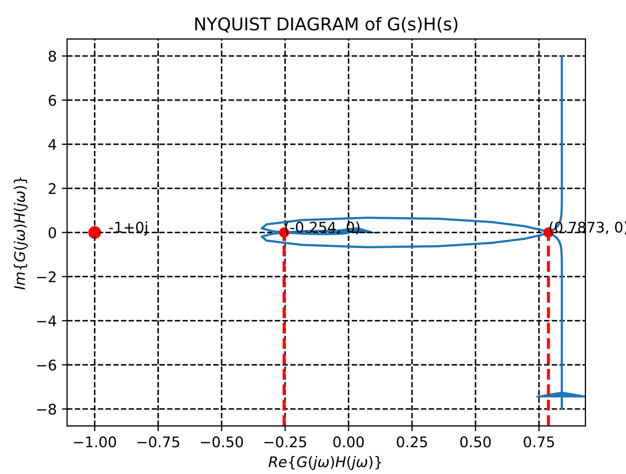


Fig. 2.1.13

Note that this nyquist plot is plotted when K=1.

Nyquist criterion-For the stable system :

$$Z = P + N = 0, \tag{2.1.6.6}$$

Variable	Description
Z	Poles of $\frac{G(s)}{1+G(s)H(s)}$in right half of s plane
N	No of encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot
P	Poles of $G(s)H(s)$ in right half of s plane

TABLE 2.1.1

Since from the equation (2.1.6.2), P = 2 as the number of poles on right hand side of s-plane is equal to 2 .So, for Z to be equal to 0 ,we have to choose the range of K such that N should be equal to -2.
If we consider the Nyquist plot with K term i.e. of equation (2.1.6.2) , then it will cut x-axis at x = -0.254K , x = 0 and at x = 0.7873K (as we have nyquist graph at K=1, now we just need to multiply the intersected coordinates on x-axis by K).

So, we have to make sure that $(-1 + j0)$ should be included in between x = -0.254K to x = 0, because then only N = -2 (as the no. of encirclements are 2 in anticlockwise direction in this case so N=-2)

$$-0.254K < -1 < 0 \tag{2.1.6.7}$$

So,

$$K > \frac{1}{0.254} \tag{2.1.6.8}$$

i.e.

$$K > 3.937 \tag{2.1.6.9}$$

Hence K > 3.937 ensures that the system is stable as no. of poles on the right hand side of s-plane (in this case) is 0.

2.2 *Nyquist Criterion*

2.2.1. Using Nyquist criterion find the range of K for which closed loop system is stable.

$$G(s) = \frac{K}{s(s+6)} \tag{2.2.1.1}$$

$$H(s) = \frac{1}{s+9} \tag{2.2.1.2}$$

Solution: The system flow can be described as,

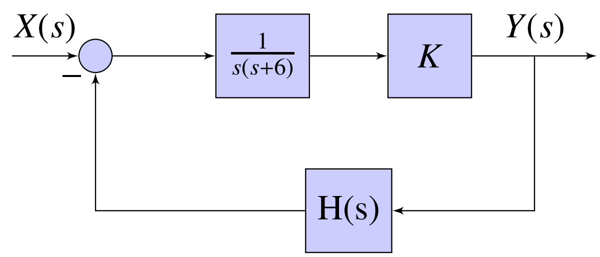


Fig. 2.2.1

$$G_1(s) = \frac{1}{s(s+6)}. \tag{2.2.1.3}$$

For Nyquist plot,

$$\text{Im}\{G_1(\jmath\omega)H(\jmath\omega)\} = \frac{-(54-\omega^2)}{(\omega)(\omega^2+56)(\omega^2+81)} \tag{2.2.1.4}$$

$$\text{Re}\{G_1(\jmath\omega)H(\jmath\omega)\} = \frac{-15\omega}{(\omega)(\omega^2+56)(\omega^2+81)} \tag{2.2.1.5}$$

From (2.2.1.4) and (2.2.1.5)
Nyquist Stability Criterion:

$$N = Z - P \tag{2.2.1.6}$$

where Z is # unstable poles of closed loop transfer function, P is # unstable poles of open loop transfer function and N is # clockwise encirclement of $(-1/K, 0)$.
For stable system,

$$Z = 0 \tag{2.2.1.7}$$

From (2.2.1.2) and (2.2.1.3),

$$P = 0 \tag{2.2.1.8}$$

$$\implies N = 0 \tag{2.2.1.9}$$

Since, there is a zero at origin, an infinite radius

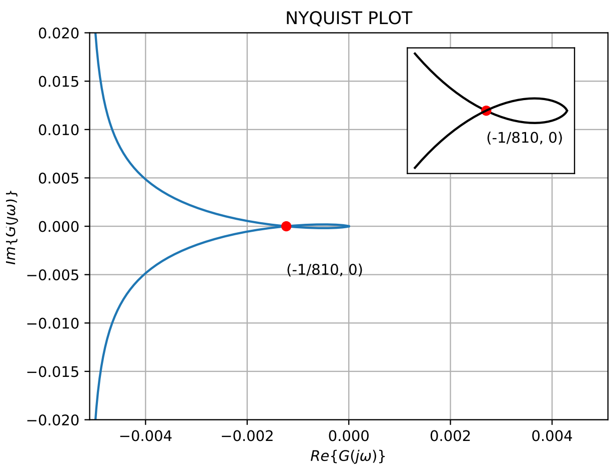


Fig. 2.2.2: Nyquist plot for $G_1(s)H(s)$

half circle will enclose the right hand side of end points of the Nyquist plot. So for (2.2.1.9),

$$\implies \frac{-1}{K} < \frac{-1}{810} \implies K < 810 \qquad (2.2.1.10)$$

And also,

$$K > 0 \qquad (2.2.1.11)$$

$$\implies 0 < K < 810 \qquad (2.2.1.12)$$

The following python code generates Fig. 2.2.2

codes/ee18btech11028_1.py

2.2.2. Using Nyquist criterion, find out the range of K for which the closed loop system will be stable.

$$G(s) = \frac{K}{(s+1)(s+3)}$$

$$H(s) = \frac{1}{(s+5)(s+7)} \qquad (2.2.2.1)$$

The system flow can be described by Fig. 2.2.3 From (2.2.2.1),

$$L(s) = G(s)H(s)$$

$$= \frac{K}{(s+1)(s+3)(s+5)(s+7)} \qquad (2.2.2.2)$$

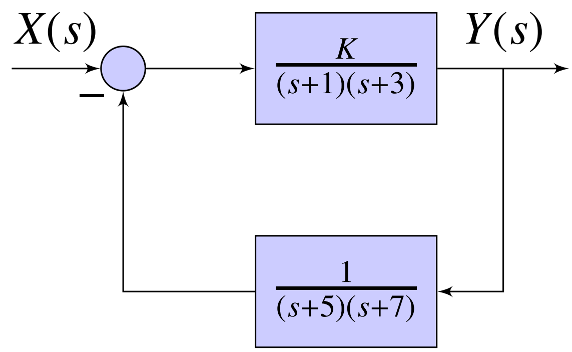


Fig. 2.2.3

$$L(j\omega) = G(\jmath\omega)H(\jmath\omega)$$

$$= \frac{K}{(j\omega+1)(j\omega+3)(j\omega+5)(j\omega+7)} \qquad (2.2.2.3)$$

The Nyquist plot is a graph of $\mathrm{Re}\{L(jw)\}$ vs $\mathrm{Im}\{L(j\omega)\}$. Let's take K=1 and draw the nyquist plot.

The following python code generates the Nyquist plot.

/codes/es17btech11015.py

The Fig. 2.2.4 shows the Nyquist plot for K = 1

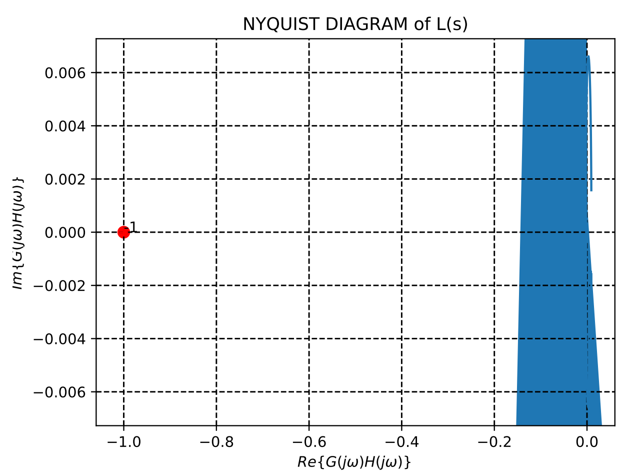


Fig. 2.2.4

Nyquist criterion-For the stable system :

$$Z = P + N = 0, \qquad (2.2.2.4)$$

where,
Z = Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane

P = Poles of $G(s)H(s)$ in right half of s plane

N = No. of encirclements of $G(s)H(s)$ about -1 in the Nyquist plot

Since from the equation (2.2.2.2), P = 0

So, for Z to be equal to 0 ,we have to choose the range of K such that N is equal to 0. From the figure 2.2.4, we can observe that the plot is not cutting the x-axis. If we consider the Nyquist plot with K term even then the plot won't cut the x-axis.
So, N = 0 irrespective of K.
Therefore, the system is stable for

$$-\infty < K < \infty \tag{2.2.2.5}$$

2.3 M and N Circles

2.3.1. What are Constant M and N circles and how can we determine closed loop frequency response using M and N circles?
Solution: M circles are called constant magnitude Loci and N circles are called as constant phase angle Loci.These are helpful in determining the closed-loop frequency response of unity negative feedback systems.

Constant-Magnitude Loci(Mcircle): Let $G(\jmath\omega)$ be complex quantity it can be written as

$$G(\jmath\omega) = X + \jmath Y \tag{2.3.1.1}$$

where X,Y are real quantities. Let M be magnitude of closed loop transfer function.

$$M = \left|\frac{X + \jmath Y}{1 + X + \jmath Y}\right| \tag{2.3.1.2}$$

$$M^2 = \frac{X^2 + Y^2}{(1 + X)^2 + Y^2} \tag{2.3.1.3}$$

Hence,

$$X^2\left(1 - M^2\right) - 2M^2X - M^2 + \left(1 - M^2\right)Y^2 = 0 \tag{2.3.1.4}$$

If $M = 1$, then from Equation (2.3.6.4), we obtain $X = \frac{-1}{2}$ This is the equation of a straight line parallel to the Y axis and passing through the point $\left(\frac{-1}{2}, 0\right)$.

If $M \neq 1$ Equation (2.3.1.4) can be written as

$$X^2 + \frac{2M^2}{M^2 - 1}X + \frac{M^2}{M^2 - 1} + Y^2 = 0 \tag{2.3.1.5}$$

Simplifying,

$$\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{\left(M^2 - 1\right)^2} \tag{2.3.1.6}$$

Equation (2.3.6.6) is the equation of a circle with center $\left(-\frac{M^2}{M^2-1}, 0\right)$ and radius $\left|\frac{M}{M^2-1}\right|$
Thus the intersection of Nquist plot with M circle at a frequency(ω) results as the magnitude of closed loop transfer function as M at frequency (ω)
Constant-Phase-Angle Loci (N Circles): Finding Phase angle α from (2.3.6.3) we get,

$$\alpha = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right) \tag{2.3.1.7}$$

$$\text{Let } tan\alpha = N \tag{2.3.1.8}$$

$$N = tan\left(\tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right)\right) \tag{2.3.1.9}$$

Simplifying,

$$N = \frac{Y}{X^2 + X + Y^2} \tag{2.3.1.10}$$

Further Simplifying..

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \tag{2.3.1.11}$$

Equation (2.3.6.11) is the equation of a circle with center at $\left(\frac{-1}{2}, \frac{1}{2N}\right)$ and radius $\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}$
Thus the intersection of Nquist plot with N circle at a frequency(ω) results as the phase of closed loop transfer function as $tan^{-1}(N)$ at frequency (ω)

2.3.2. For unity Feedback system given below,obtain closed loop frequency response using constant M and N circles.

$$G(s) = \frac{10}{s(s + 1)(s + 2)} \tag{2.3.2.1}$$

Solution: The following code plots Fig. 2.3.4

```
codes/ee18btech11017_code1.py
```

2.3.3. Find the intersection of M and N circles with

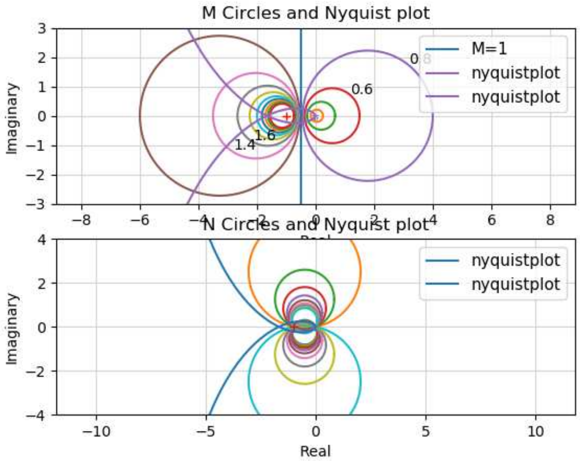


Fig. 2.3.1

Nyquist plot at different frequencies.
Solution: The following code finds intersection of M and N circles with Nyquist plot at different frequencies

```
codes/ee18btech11020/
    ee18btech11020_code2.py
```

The points M and frequencies are listed in Table 2.3.3

M in dB	**M**	ω
13.08	4.51	1.8
7.640	2.41	1.98
5.153	1.81	2.07
-0.81	0.91	2.37
-10.17	0.31	3.17
-40	0.01	8.47

TABLE 2.3.1

The points M and frequencies are listed in Table2.3.4
The constant N locus for given value of α is not the entire circle but only an arc.This is beacuse tangent of angle remains same if $+180°$ or $-180°$ is added to the angle.

2.3.4. Plot Magnitude and Phase plot from the values obtained above.
Solution: The following code plots Fig. 2.3.2

```
codes/ee18btech11020/
    ee18btech11020_code3.py
```

α	**N**	ω
113.49	-2.3	6.684
111.03	-2.6	7.67
105.94	-3.5	10.011
101.30	-5	14.09
102.80	-4.4	24.77
77.73	4.6	34.13

TABLE 2.3.2

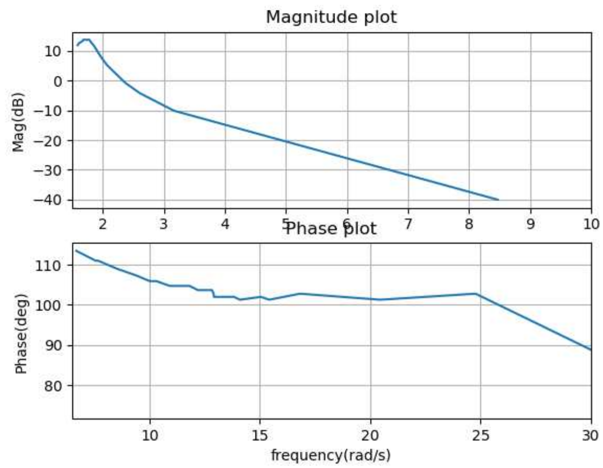


Fig. 2.3.2

2.3.5. Compare the above plot with bode plot of closed loop transfer function.
Solution: The following code plots Fig. 2.3.3

```
codes/ee18btech11020_code4.py
```

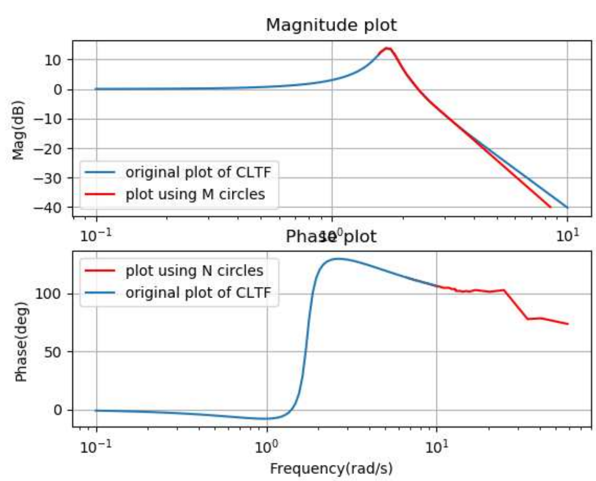


Fig. 2.3.3

2.3.6. What are Constant M and N circles and how can we determine closed loop frequency response using M and N circles?

Solution: M circles are called constant magnitude Loci and N circles are called as constant phase angle Loci.These are helpful in determining the closed-loop frequency response of unity negative feedback systems.

Constant-Magnitude Loci(Mcircle): Let $G(\jmath\omega)$ be complex quantity it can be written as

$$G(\jmath\omega) = X + \jmath Y \tag{2.3.6.1}$$

where X,Y are real quantities. Let M be magnitude of closed loop transfer function.

$$M = \left|\frac{X + \jmath Y}{1 + X + \jmath Y}\right| \tag{2.3.6.2}$$

$$M^2 = \frac{X^2 + Y^2}{(1+X)^2 + Y^2} \tag{2.3.6.3}$$

Hence,

$$X^2\left(1 - M^2\right) - 2M^2X - M^2 + \left(1 - M^2\right)Y^2 = 0 \tag{2.3.6.4}$$

If $M = 1$, then from Equation (2.3.6.4), we obtain $X = \frac{-1}{2}$ This is the equation of a straight line parallel to the Y axis and passing through the point $\left(\frac{-1}{2}, 0\right)$.

If $M \neq 1$ Equation (2.3.6.4) can be written as

$$X^2 + \frac{2M^2}{M^2 - 1}X + \frac{M^2}{M^2 - 1} + Y^2 = 0 \tag{2.3.6.5}$$

Simplifying,

$$\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{(M^2 - 1)^2} \tag{2.3.6.6}$$

Equation (2.3.6.6) is the equation of a circle with center $\left(-\frac{M^2}{M^2-1}, 0\right)$ and radius $\left|\frac{M}{M^2-1}\right|$

Thus the intersection of Nquist plot with M circle at a frequency(ω) results as the magnitude of closed loop transfer function as M at frequency (ω)

Constant-Phase-Angle Loci (N Circles): Finding Phase angle α from (2.3.6.3) we get,

$$\alpha = \tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right) \tag{2.3.6.7}$$

$$\text{Let } tan\alpha = N \tag{2.3.6.8}$$

$$N = tan\left(\tan^{-1}\left(\frac{Y}{X}\right) - \tan^{-1}\left(\frac{Y}{1 + X}\right)\right) \tag{2.3.6.9}$$

Simplifying,

$$N = \frac{Y}{X^2 + X + Y^2} \tag{2.3.6.10}$$

Further Simplifying..

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \tag{2.3.6.11}$$

Equation (2.3.6.11) is the equation of a circle with center at $\left(\frac{-1}{2}, \frac{1}{2N}\right)$ and radius $\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}$

Thus the intersection of Nquist plot with N circle at a frequency(ω) results as the phase of closed loop transfer function as $tan^{-1}(N)$ at frequency (ω)

For unity Feedback system given below,obtain closed loop frequency response using constant M and N circles.

$$G(s) = \frac{50(s + 3)}{s(s + 2)(s + 4)} \tag{2.3.6.12}$$

Solution: The following code plots Fig. 2.3.4

```
codes/ee18btech11017_code1.py
```

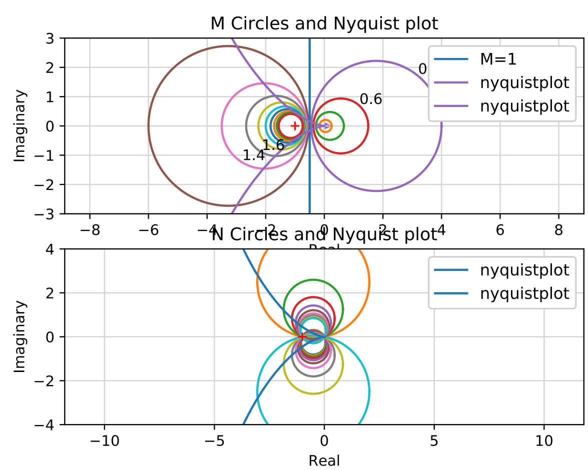


Fig. 2.3.4

The following code finds intersection of M and N circles with Nyquist plot at different frequencies

```
codes/ee18btech11017_code2.py
```

The points M and frequencies are listed in Table 2.3.3

M in dB	M	ω
5.15	1.81	5.08
7.64	2.41	6.34
6.48	2.11	7.58
-0.81	0.91	9.86
-10.17	0.31	14.33
-40	0.01	57.91

TABLE 2.3.3

The points M and frequencies are listed in Table2.3.4

α	N	ω
-38.65	-0.80	5.42
-66.50	-2.30	6.40
-78.60	-5.00	6.72
-101.50	4.90	7.32
-158.19	0.4	11.68
-174.28	0.1	46.01

TABLE 2.3.4

The constant N locus for given value of α is not the entire circle but only an arc.This is beacuse tangent of angle remains same if $+180°$ or $-180°$ is added to the angle.
The following code plots Fig. 2.3.5

```
codes/ee18btech11017_code3.py
```

The following code plots Fig. 2.3.6

```
codes/ee18btech11017_code4.py
```

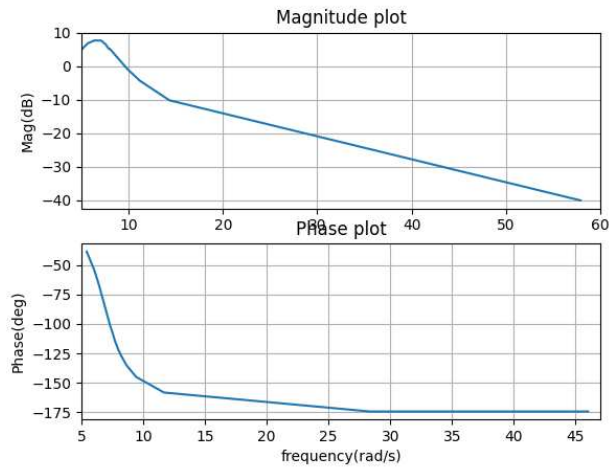


Fig. 2.3.5

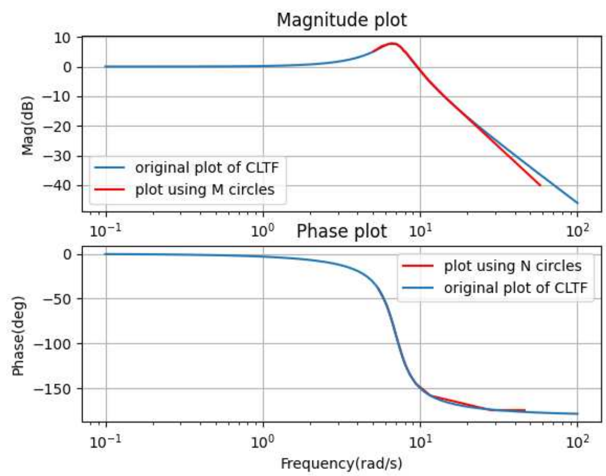


Fig. 2.3.6

2.4 *Nichol's Chart*

2.1. Nichols chart is the plot of gain and phase that is the magnitude(in dB) on the vertical axis and phase(in deg) on the horizontal axis for the given transfer function. It is called the gain phase plot. These plots are used for the stability analysis of the system.
There are four cases for finding the stability (r is magnitude in dB and ϕ is phase in deg)
Case 1 : System with no unstable pole.
For the stable function T(s) whose Nichols plot intersects dB line at least one time.The system is stable if and only if one of the following holds
1) the steady gain $| ko |$ is less than 0 dB
2) The Nichols plot T(jω) intersects the line segment C := $[(\phi,r) : r = 0$ dB , $-180° < \phi < 180°]$for the function with the positive steady gain ko and $| ko |$ larger than dB.
Case 2: System with one unstable pole.
The system is stable if and only if the half part of the Nichols plot crosses the line segment C

:= [(ϕ,r) : r = 0 dB , $-180°$ <ϕ <180°], the steady gain ko is negative and | ko | is larger than 0 dB.
Case 3 : System with 2k unstable poles.
The feedback system is stable if and only if the plot intersects the line segment C := [(ϕ,r) : r = 0 dB , $-180°$ <ϕ <180°], the steady gain ko is positive and | ko | is larger than 0 dB
Case 4 : System with 2k+1 unstable poles.
The stability analysis of the system with 2k+1 unstable poles is equivalent to that of the shifting Nichols plot of the function with one unstable pole.
Consider the closed loop transfer function

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{2.1.1}$$

The system flow is shown below

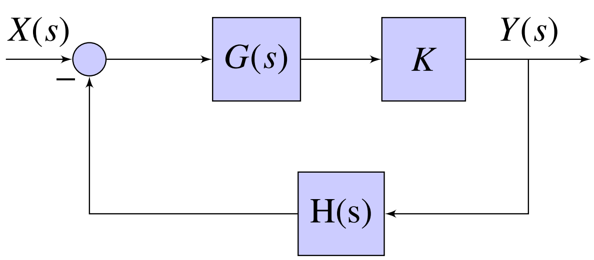


Fig. 2.4.1

Also, for the percentage overshoot(PO) we need the characteristic equation of the function so as to determine the damping ratio. The general characteristic equations is

$$s^2 + 2\zeta\omega s + \omega^2 \tag{2.1.2}$$

where ζ is the damping ratio and ω is the natural frequency.
Since all the systems above are third order or fourth order systems we need to decompose them into first order and second order to find out the PO.

$$PO = \exp\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} * 100 \tag{2.1.3}$$

2.5 Stability

2.2. Using Nichol's chart , find out whether each of the system below are stable or not

$$G_1(s) = \frac{50}{s(s+3)(s+6)} \tag{2.2.1}$$
$$H_1(s) = 1 \tag{2.2.2}$$

$$G_2(s) = \frac{9}{s^2(s+3)} \tag{2.2.3}$$
$$H_2(s) = (s+4) \tag{2.2.4}$$

$$G_3(s) = \frac{20}{s(s+1)} \tag{2.2.5}$$
$$H_3(s) = \frac{s+3}{s+4} \tag{2.2.6}$$

$$G_4(s) = \frac{100(s+5)}{s(s^2+4)(s+3)} \tag{2.2.7}$$
$$H_4(s) = 1 \tag{2.2.8}$$

For the above systems also estimate the percentage overshoot that can be expected when a step input is given to the system.

2.3. From (2.2.1) and (2.2.2),

$$G_1(s)H_1(s) = \frac{50}{s(s+3)(s+6)} \tag{2.3.1}$$

Solution: From (2.1.1),

$$T_1(s) = \frac{50}{s^3 + 9s^2 + 18s + 50} \tag{2.3.2}$$

codes/es17btech11009_1.py

The above code gives the following plot for the closed loop system as shown in Fig 2.5.1
The given system has no unstable poles so according to case 1, we could conclude that the plot satisfies the stability condition therefore, the system is stable.

2.4. From (2.3.2), There are no zeroes and poles are shown in Table 2.5.1

Poles	Zeros
$p_1 = -7.4$	
$p_2 = -0.75 + 2.47j$	
$p_3 = -0.75 - 2.47j$	

TABLE 2.5.1

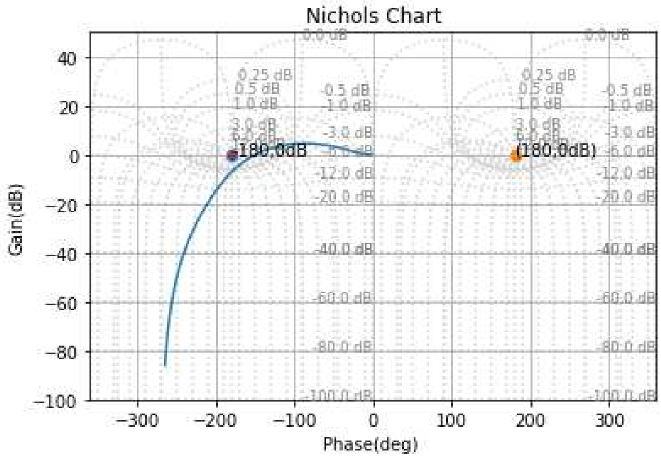


Fig. 2.5.1

Since we have two conjugate poles, The approximated transfer function is

$$T(s) = \frac{K_1}{(s - p_2)(s - p_3)} \tag{2.4.1}$$

$$T(s) = \frac{K_1}{s^2 + 1.5s + 6.66} \tag{2.4.2}$$

The characteristic equation of (2.4.2) is,

$$s^2 + 1.5s + 6.66 = 0 \tag{2.4.3}$$

From (2.1.2) and (2.1.3),
$\zeta = 0.29$ and $\omega = 2.58$
Percentage overshoot $= 38.4\%$

2.5. The following code generates step response of the function (2.3.2) as shown in Fig 2.5.2

```
codes/es17btech11009_11.py
```

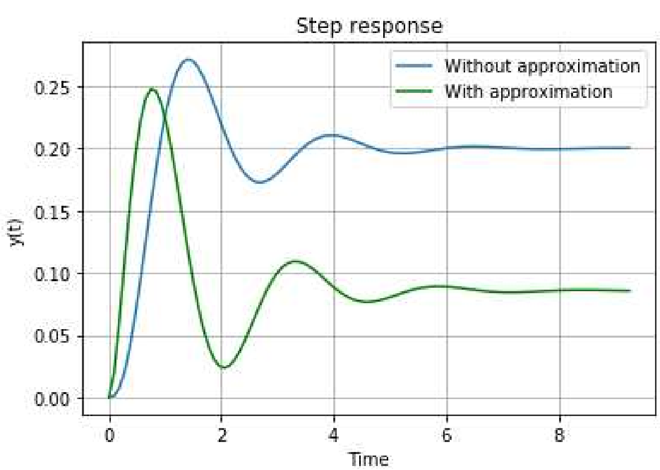


Fig. 2.5.2

2.6. From (2.2.3) and (2.2.4),

$$G_2(s)H_2(s) = \frac{9(s+4)}{s^2(s+3)} \tag{2.6.1}$$

Solution: From (2.1.1),

$$T_2(s) = \frac{9}{s^3 + 3s^2 + 9s + 36} \tag{2.6.2}$$

```
codes/es17btech11009_2.py
```

The above code gives the following plot for the closed loop system as shown in Fig 2.5.3 The given system has two unstable poles so

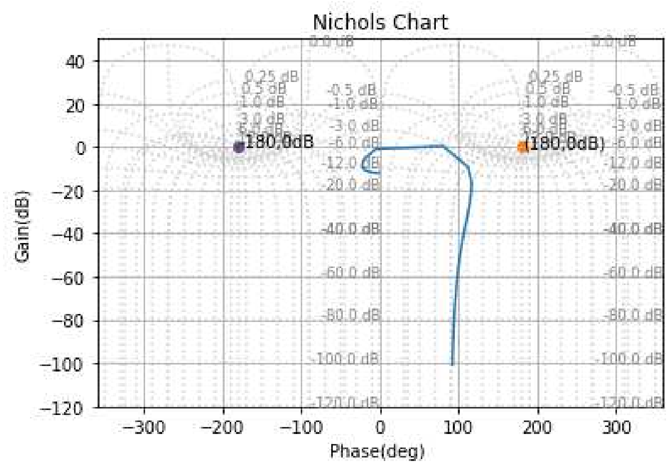


Fig. 2.5.3

according to case 3, we could conclude that the system in unstable.

2.7. From (2.6.2), there are no zeroes and poles are shown in Table 2.5.2

Poles	Zeros
$p_1 = -3.43$	
$p_2 = 0.21 + 3.2j$	
$p_3 = 0.21 - 3.2j$	

TABLE 2.5.2

Since we have two conjugate poles, The approximated transfer function is

$$T(s) = \frac{K_1}{(s - p_2)(s - p_3)} \tag{2.7.1}$$

$$T(s) = \frac{K_1}{s^2 + 0.42s + 10.28} \tag{2.7.2}$$

The characteristic equation of (2.7.2) is,

$$s^2 + 0.42s + 10.28 = 0 \tag{2.7.3}$$

From (2.1.2) and (2.1.3),
$\zeta = 0.065$ and $\omega = 3.2$
Percentage overshoot = 81.8%

2.8. The following code generates step response of the function (2.6.2) as shown in Fig 2.5.4

```
codes/es17btech11009_21.py
```

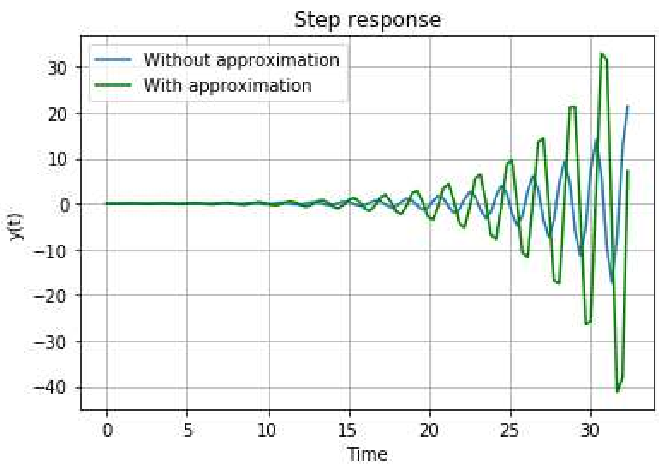


Fig. 2.5.4

2.9. From (2.2.5) and (2.2.6),

$$G_3(s)H_3(s) = \frac{20(s+3)}{s(s+1)(s+4)} \tag{2.9.1}$$

Solution: From (2.1.1),

$$T_3(s) = \frac{20(s+4)}{s^3 + 5s^2 + 24s + 60} \tag{2.9.2}$$

```
codes/es17btech11009_3.py
```

The above code gives the following plot of the closed loop system as shown in Fig 2.5.5 The given system has no unstable poles so according to case 1, we could conclude that the plot satisfies the stability condition.The system is stable.

2.10. From (2.9.2), The zeros and the poles are shown in Table 2.5.3
Since we have two conjugate poles, The approximated transfer function is

$$T(s) = \frac{K_1}{(s-p_2)(s-p_3)} \tag{2.10.1}$$

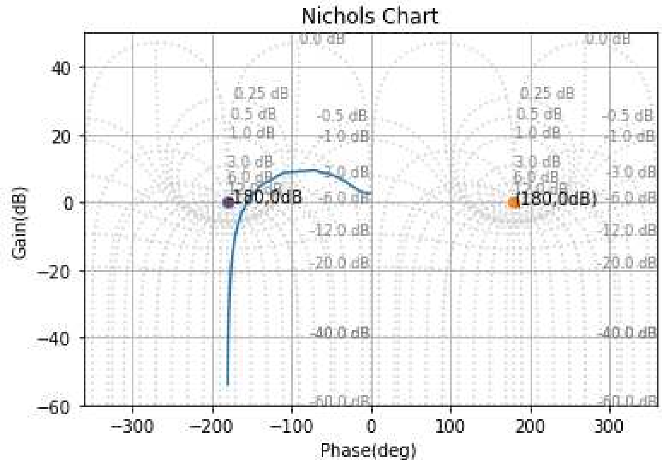


Fig. 2.5.5

Poles	Zeros
$p_1 = -3.27$	$z_1 = -4$
$p_2 = -0.86 + 4.19j$	
$p_3 = -0.86 - 4.19j$	

TABLE 2.5.3

$$T(s) = \frac{K_1}{s^2 + 1.72s + 18.28} \tag{2.10.2}$$

The characteristic equation of (2.10.2) is,

$$s^2 + 1.72s + 18.28 = 0 \tag{2.10.3}$$

From (2.1.2) and (2.1.3),
$\zeta = 0.201$ and $\omega = 4.27$
Percentage overshoot = 52.2%

2.11. The following code generates step response of the function (2.9.2) as shown in Fig 2.5.6

```
codes/es17btech11009_31.py
```

2.12. From (2.2.7) and (2.2.8),

$$G_4(s)H_4(s) = \frac{100(s+5)}{s(s^2+4)(s+3)} \tag{2.12.1}$$

Solution: From (2.1.1),

$$T_4(s) = \frac{100(s+5)}{s^4 + 3s^3 + 4s^2 + 112s + 500} \tag{2.12.2}$$

```
codes/es17btech11009_4.py
```

The above code gives the following plot for the closed loop system as shown in Fig 2.5.7. The given system has 2 unstable poles so according to case 3, we could conclude that the system

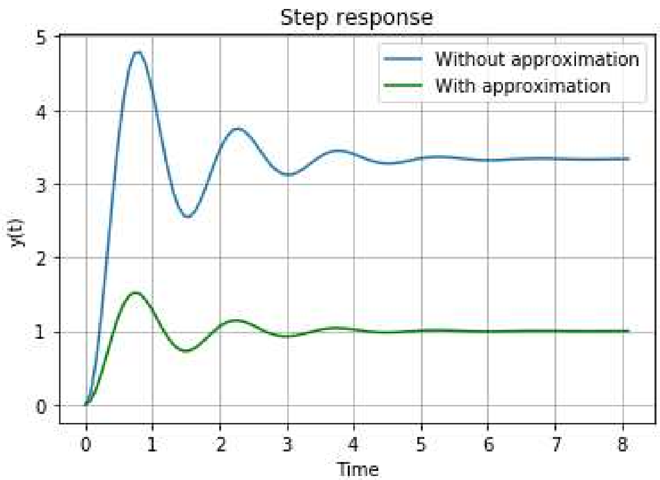


Fig. 2.5.6

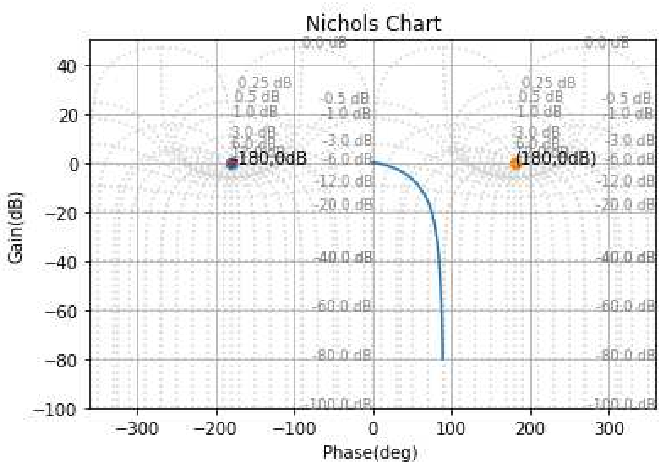


Fig. 2.5.7

is unstable.

2.13. From (2.12.2), The zeroes and poles are shown in Table 2.5.4

Poles	Zeros
$p_1 = 2.5 + 4.4j$	$z_1 = -5$
$p_2 = 2.5 - 4.4j$	
$p_3 = -4.04 + 1.6j$	
$p_3 = -4.04 - 1.6$	

TABLE 2.5.4

We have four conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_3)(s - p_4)} \tag{2.13.1}$$

$$T(s) = \frac{K_1}{s^2 + 5.08s + 25.89} \tag{2.13.2}$$

The characteristic equation of (2.13.2) is,

$$s^2 + 1.72s + 18.28 = 0 \tag{2.13.3}$$

From (2.1.2) and (2.1.3),
$\zeta = 0.5$ and $\omega = 5.08$
Percentage overshoot = 16.36%

2.14. The following code generates step response of the function (2.12.2) as shown in Fig 2.5.8

```
codes/es17btech11009_41.py
```

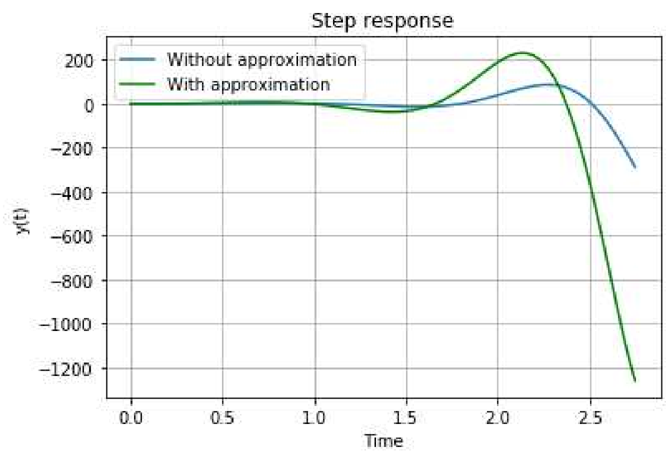


Fig. 2.5.8

2.6 Range of K

2.15. Using Nichol's chart, find out the range of K for which the closed loop systems will be stable

$$G_5(s) = \frac{K}{s(s+6)} \tag{2.15.1}$$

$$H_5(s) = \frac{1}{s+9} \tag{2.15.2}$$

$$G_6(s) = \frac{K(s+2)(s+4)}{s^2 - 3s + 10} \tag{2.15.3}$$

$$H_6(s) = \frac{1}{s} \tag{2.15.4}$$

$$G_7(s) = \frac{K}{(s+1)(s+3)} \tag{2.15.5}$$

$$H_7(s) = \frac{s+5}{s+7} \tag{2.15.6}$$

For the above systems also estimate the percentage overshoot that can be expected when a step input is given to the system.

2.16. From (2.15.1) and (2.15.2),

$$G_5(s)H_5(s) = \frac{K}{s(s+6)(s+9)} \tag{2.16.1}$$

Solution: From (2.1.1),

$$T_5(s) = \frac{K(s+9)}{s^3 + 15s^2 + 54s + K} \tag{2.16.2}$$

codes/es17btech11009_5.py

The above code gives the following plot for k = 2 as shown in Fig 2.6.1

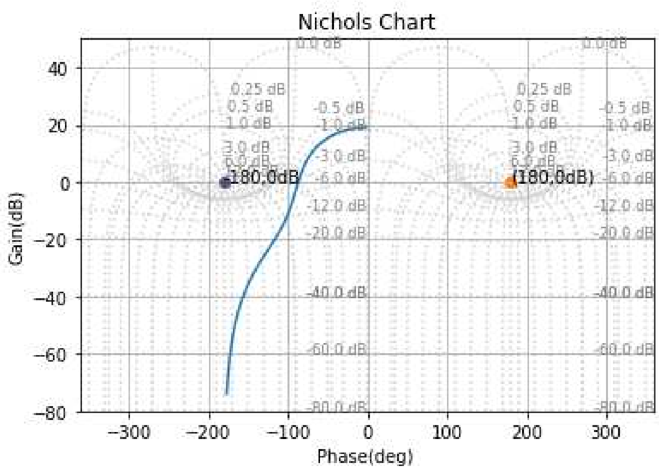


Fig. 2.6.1

By increasing k, we could see that the plot shifts in upward direction, so the range of k for which the system is stable is 0 <K <810.

2.17. Let k= 500
From (2.12.2), The zeroes and poles are shown in Table 2.6.1

Poles	Zeros
$p_1 = -13.72$	$z_1 = -9$
$p_2 = -0.63 + 6j$	
$p_3 = -0.63 - 6j$	

TABLE 2.6.1

Since we have two conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s-p_2)(s-p_3)} \tag{2.17.1}$$

$$T(s) = \frac{K_1}{s^2 + 1.26s + 36.39} \tag{2.17.2}$$

The characteristic equation of (2.17.2) is,

$$s^2 + 1.26s + 36.39 = 0 \tag{2.17.3}$$

From (2.1.2) and (2.1.3),
$\zeta = 0.1$ and ω =6.03
Percentage overshoot = 72.9%

2.18. The following code generates step response of the function (2.16.2) as shown in Fig 2.6.2

codes/es17btech11009_51.py

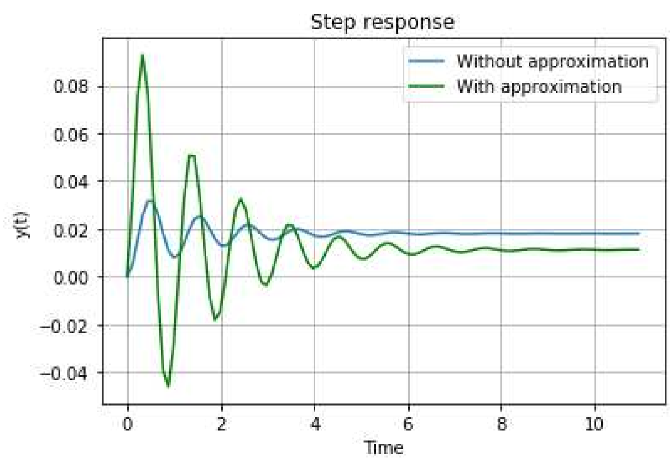


Fig. 2.6.2

2.19. From (2.15.3) and (2.15.4),

$$G_6(s)H_6(s) = \frac{K(s+2)(s+4)}{s(s^2 - 3s + 10)} \tag{2.19.1}$$

Solution: From (2.1.1)

$$T_6(s) = \frac{K\left(s^3 + 6s^2 + 8s\right)}{(s^3 - 3s^2 + 10s) + (s^2 + 6s + 8)K} \tag{2.19.2}$$

codes/es17btech11009_6.py

The above code gives the following plot for k = 5 as shown in Fig 2.6.3
By increasing k, we could see that the plot shifts in upward direction, so the range of k for which the system is stable is 3.9 <K <∞.

2.20. Let k= 5
From (2.19.2), The zeros and poles are shown in Table 2.6.2

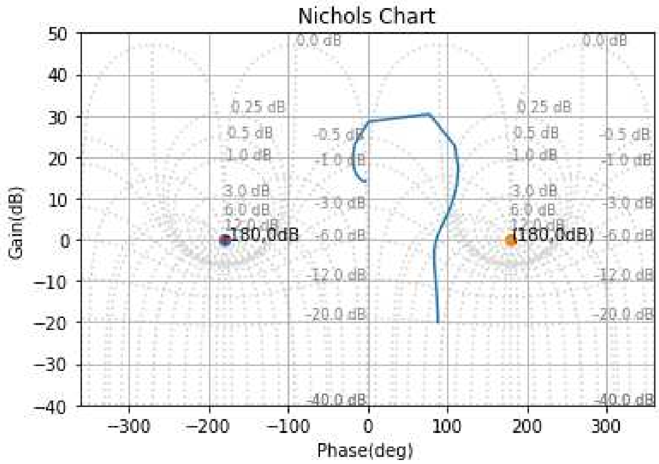


Fig. 2.6.3

Poles	Zeros
$p_1 = -1.02$	$z_1 = -28.6$
$p_2 = -0.48 + 6.22j$	$z_2 = -1.39$
$p_3 = -0.48 - 6.22j$	$z_2 = 0$

TABLE 2.6.2

Since we have two conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_2)(s - p_3)} \tag{2.20.1}$$

$$T(s) = \frac{K_1}{s^2 + 0.96s + 38.9} \tag{2.20.2}$$

The characteristic equation of (2.20.2) is,

$$s^2 + 0.96s + 38.9 = 0 \tag{2.20.3}$$

From (2.1.2) and (2.1.3),
ζ =0.07 and $\omega = 6.23$
Percentage overshoot = 81%

2.21. The following code generates step response of the function (2.19.2) as shown in Fig 2.6.4

```
codes/es17btech11009_61.py
```

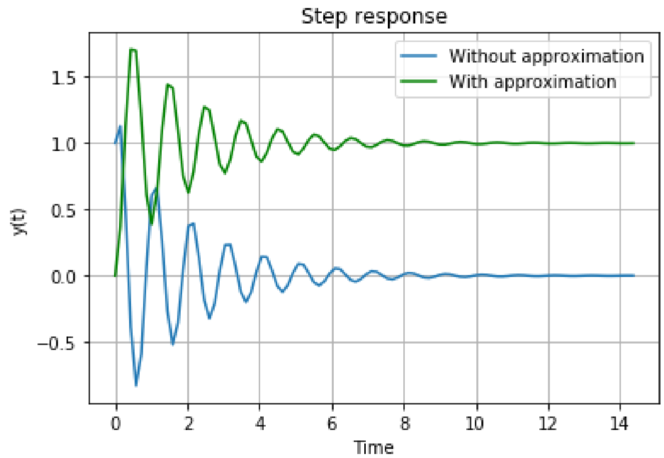


Fig. 2.6.4

2.22. From (2.15.5) and (2.15.6),

$$G_7(s) H_7(s) = \frac{K}{(s+1)(s+3)(s+5)(s+7)} \tag{2.22.1}$$

Solution: From (2.1.1),

$$T_7(s) = \frac{K\left(s^2 + 12s + 35\right)}{(s^4 + 16s^3 + 86s^2 + 176s + 105 + K)} \tag{2.22.2}$$

```
codes/es17btech11009_7.py
```

The above code gives the following plot for k = 10 as shown in Fig 2.6.5

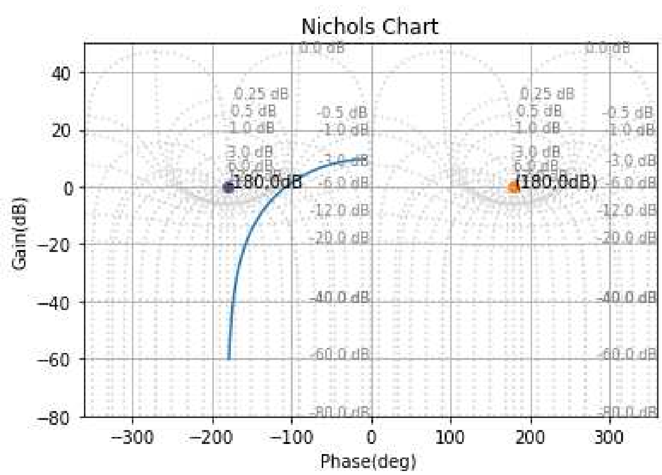


Fig. 2.6.5

By varying k, we could see that the the range of k for which the system is stable is 3 <K <∞.

2.23. Let k= 1000
From (2.22.2), The zeros and poles are shown in Table 2.6.3
Since we have four conjugate poles, Consider

Poles	Zeros
$p_1 = -8.28 + 3.65$	$z_1 = -7$
$p_2 = -8.28 - 3.65j$	$z_2 = -5$
$p_3 = 0.28 + 3.65j$	
$p_3 = 0.28 - 3.65j$	

TABLE 2.6.3

the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_3)(s - p_4)} \tag{2.23.1}$$

$$T(s) = \frac{K_1}{s^2 + 0.56s + 13.39} \tag{2.23.2}$$

The characteristic equation of (2.23.2) is,

$$s^2 + 0.56s + 13.39 = 0 \tag{2.23.3}$$

From (2.1.2) and (2.1.3),
ζ =0.076 and $\omega = 3.65$
Percentage overshoot = 79.4%

2.24. The following code generates step response of the function (2.22.2) as shown in Fig 2.6.6

```
codes/es17btech11009_71.py
```

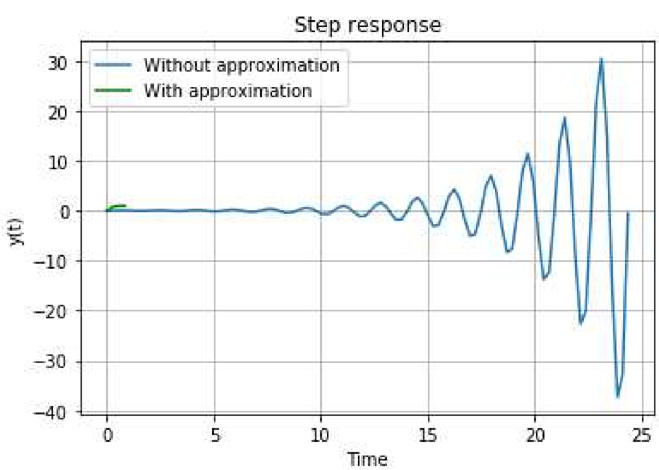


Fig. 2.6.6

2.7 *Closed Loop Frequency Response*

2.25. For unity feedback (negative) systems given below, obtain closed loop frequency response using constant M and N circles.

$$G_8(s) = \frac{10}{s(s+1)(s+2)} \tag{2.25.1}$$

$$G_9(s) = \frac{1000}{(s+3)(s+4)(s+5)(s+6)} \tag{2.25.2}$$

$$G_0(s) = \frac{50(s+3)}{s(s+2)(s+4)} \tag{2.25.3}$$

For the above systems also estimate the percentage overshoot that can be expected when a step input is given to the system.

Constant M and N Circles

Let $G(j\omega)$ be complex quantity it can be written as

$$G(j\omega) = x + jy \tag{2.25.4}$$

where x,y are real quantities.
M circles are called constant magnitude Loci and N circles are called as constant phase angle Loci.These are helpful in determining the closed-loop frequency response of unity negative feedback systems.

Mcircle(Constant-Magnitude Loci):

Let M be magnitude of closed loop transfer function. From (2.25.4)

$$M = \left|\frac{x + jY}{1 + x + jy}\right| \tag{2.25.5}$$

$$M^2 = \frac{x^2 + y^2}{(1+x)^2 + y^2} \tag{2.25.6}$$

Hence,

$$X^2\left(1 - M^2\right) - 2M^2X - M^2 + \left(1 - M^2\right)Y^2 = 0 \tag{2.25.7}$$

If $M = 1$, then from Equation (2.25.6), we obtain $x = \frac{-1}{2}$ This is the equation of a straight line parallel to the Y axis and passing through the point $\left(\frac{-1}{2}, 0\right)$.
If $M \neq 1$ Equation (2.25.7) can be written as

$$x^2 + \frac{2M^2}{M^2 - 1}x + \frac{M^2}{M^2 - 1} + y^2 = 0 \tag{2.25.8}$$

Simplifying,

$$\left(x + \frac{M^2}{M^2 - 1}\right)^2 + y^2 = \frac{M^2}{(M^2 - 1)^2} \tag{2.25.9}$$

Equation (2.25.9) is the equation of a circle with center $\left(-\frac{M^2}{M^2-1}, 0\right)$ and radius $\left|\frac{M}{M^2-1}\right|$
Thus the intersection of Nquist plot with M

circle at a frequency(ω) results as the magnitude of closed loop transfer function as M at frequency (ω) **N Circles(Constant-Phase-Angle Loci):** Finding Phase angle α from (2.25.6) we get,

$$\alpha = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right) \tag{2.25.10}$$

$$\text{Let } tan\alpha = N \tag{2.25.11}$$

$$N = tan\left(\tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{1+x}\right)\right) \tag{2.25.12}$$

Simplifying,

$$N = \frac{y}{x^2 + x + y^2} \tag{2.25.13}$$

Further Simplifying..

$$\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \tag{2.25.14}$$

Equation (2.25.14) is the equation of a circle with center at $\left(\frac{-1}{2}, \frac{1}{2N}\right)$ and radius $\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}$ Thus the intersection of Nquist plot with N circle at a frequency(ω) results as the phase of closed loop transfer function as $tan^{-1}(N)$ at frequency (ω)

2.26. From (2.25.1),

$$G_8(s) = \frac{10}{s(s+1)(s+2)} \tag{2.26.1}$$

Solution: From (2.1.1),

$$T_8(s) = \frac{10}{s^3 + 3s^2 + 2s + 10} \tag{2.26.2}$$

The following code gives the nichol plot of (2.26.2) shown in Fig 2.7.1

```
codes/es17btech11009_8.py
```

The M and N circles of T($j\omega$) in the gain phase plane are transformed into M and N contours in rectangular co-ordinates. A point on the constant M loci in T($j\omega$) plane is transferred to gain phase plane by drawing the vector directed from the origin of T($j\omega$) plane to a particular point on M circle and then measuring the length in dB and angle in degree.

2.27. The following code plots M and N contours in rectangle co-ordinates look like as shown in Fig. 2.7.2.

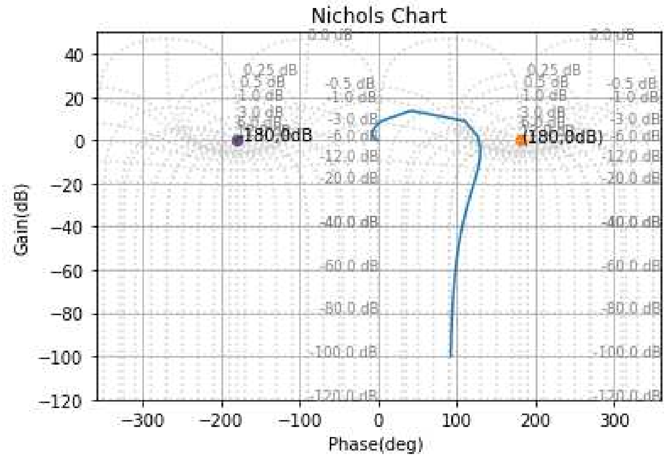


Fig. 2.7.1

```
codes/es17btech11009_8_code1.py
```

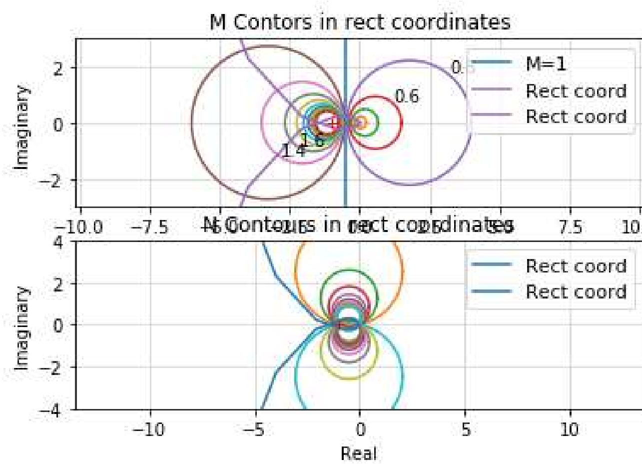


Fig. 2.7.2

2.28. To find the intersection of M and N contours in the rectangle co-ordinates at different frequencies.
Solution: The following code finds intersection of M and N contours with rectangular co-ordinates at different frequencies

```
codes/es17btech11009_8_code2.py
```

The points M and frequencies are listed in Table 2.7.1

2.29. The points N and frequencies are listed in Table2.7.2 The constant N locus for given value of α is not the entire circle but only an arc.This is beacuse tangent of angle remains same if $+180°$ or $-180°$ is added to the angle.

2.30. From (2.26.2),

M in dB	M	ω
13.64	4.81	1.68
11.84	3.91	1.85
7.64	2.41	1.98
5.15	1.81	2.07
-4.29	0.61	2.61
-40	0.01	8.71

TABLE 2.7.1

α	N	ω
-78.69	-5	10.351
-77.10	-4.4	11.789
-75.2	-3.8	14.027
-66.5	-2.3	27.06
-63.4	-2.0	6.115
5.7	0.1	27.066

TABLE 2.7.2

The zeroes and poles are shown in Table 2.7.3

Poles	Zeros
$p_1 = -3.3$	
$p_2 = 0.15 + 1.73j$	
$p_3 = 0.15 - 1.73j$	

TABLE 2.7.3

Since we have two conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_1)(s - p_2)} \tag{2.30.1}$$

$$T(s) = \frac{K_1}{s^2 + 0.3s + 3} \tag{2.30.2}$$

The characteristic equation of (2.30.2) is,

$$s^2 + 0.3s + 3 = 0 \tag{2.30.3}$$

From (2.1.2) and (2.1.3),
ζ =0.0086 and ω = 1.732
Percentage overshoot = 75.5%

2.31. The following code generates step response of the function (2.26.2) as shown in Fig 2.7.3

```
codes/es17btech11009_81.py
```

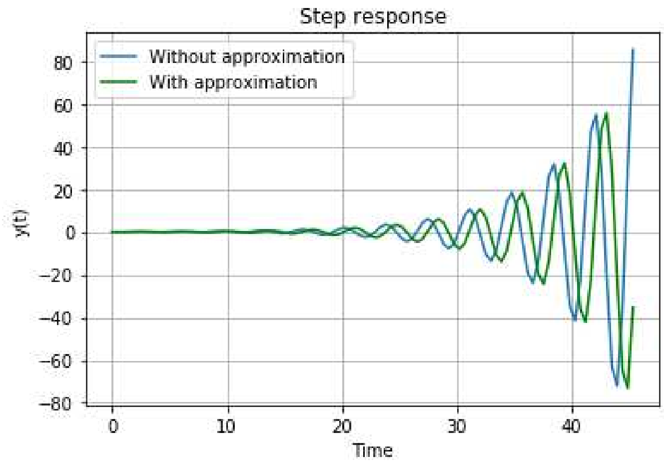


Fig. 2.7.3

2.32. From (2.25.2),

$$G_9(s) = \frac{1000}{(s+3)(s+4)(s+5)(s+6)} \tag{2.32.1}$$

Solution: From (2.1.1),

$$T_9(s) = \frac{1000}{s^4 + 18s^3 + 119s^2 + 342s + 1360} \tag{2.32.2}$$

The following code gives the nichols plot as shown in Fig 2.7.4

```
codes/es17btech11009_9.py
```

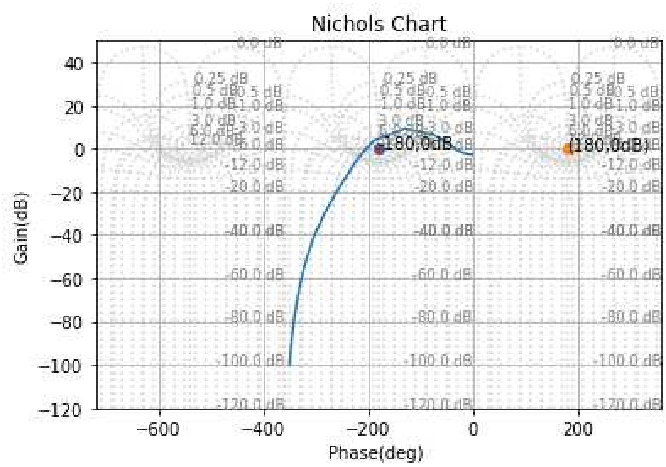


Fig. 2.7.4

2.33. The following code plots M and N contours in rectangle co-ordinates which look like as shown in Fig. 2.7.5.

```
codes/es17btech11009_9_code1.py
```

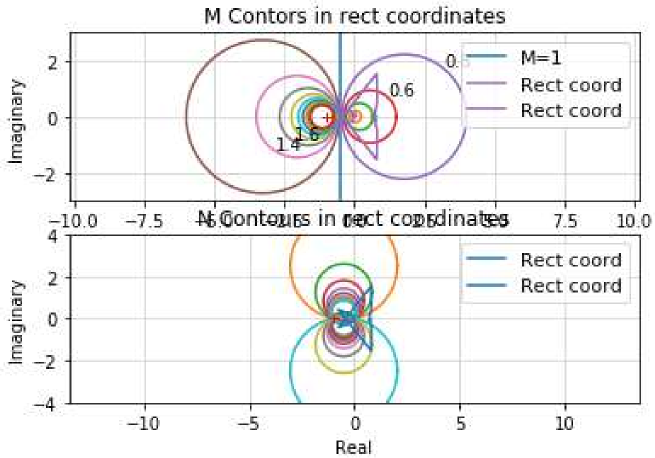


Fig. 2.7.5

2.34. The following code finds intersection of M and N contours with rectangular co-ordinates at different frequencies

```
codes/es17btech11009_9_code2.py
```

The points M and frequencies are listed in Table 2.7.4

M in dB	M	ω
9.57	3.01	4.67
5.15	1.81	5.01
-4.29	0.61	5.83
-10.17	0.31	6.7
-40	0.01	15.14

TABLE 2.7.4

2.35. The points N and frequencies are listed in Table2.7.5

α	N	ω
-78.69	-5.0	7.76
-77.1	-4.4	7.553
-74.05	-3.5	6.944
5.71	0.1	36.128
34.99	0.7	22.57

TABLE 2.7.5

2.36. From (2.22.2),
The zeroes and poles are shown in Table 2.7.6

Poles	Zeros
$p_1 = -8.55 + 3.89j$	
$p_2 = -8.55 - 3.89j$	
$p_3 = -0.44 + 3.89j$	
$p_3 = -0.44 - 3.89j$	

TABLE 2.7.6

We have four conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_3)(s - p_4)} \tag{2.36.1}$$

$$T(s) = \frac{K_1}{s^2 + 0.56s + 13.39} \tag{2.36.2}$$

The characteristic equation of (2.36.2) is,

$$s^2 + 0.56s + 13.39 = 0 \tag{2.36.3}$$

From (2.1.2) and (2.1.3),
ζ =0.11 and $\omega = 3.91$
Percentage overshoot = 71%

2.37. The following code generates step response of the function (2.32.2) as shown in Fig 2.7.6

```
codes/es17btech11009_91.py
```

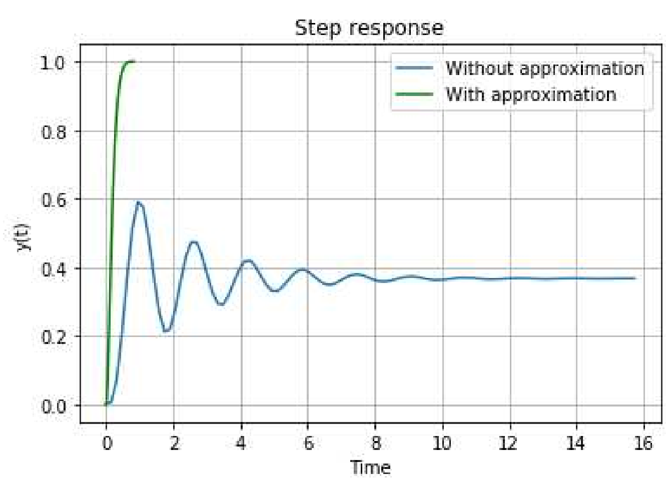


Fig. 2.7.6

2.38. From (2.25.3),

$$G_0(s) = \frac{50(s + 3)}{s(s + 2)(s + 4)} \tag{2.38.1}$$

Solution: From (2.1.1),

$$T_0(s) = \frac{50(s+3)}{s^3 + 6s^2 + 58s + 150} \tag{2.38.2}$$

The following code gives the nichols plot as shown in Fig 2.7.7

```
codes/es17btech11009_10.py
```

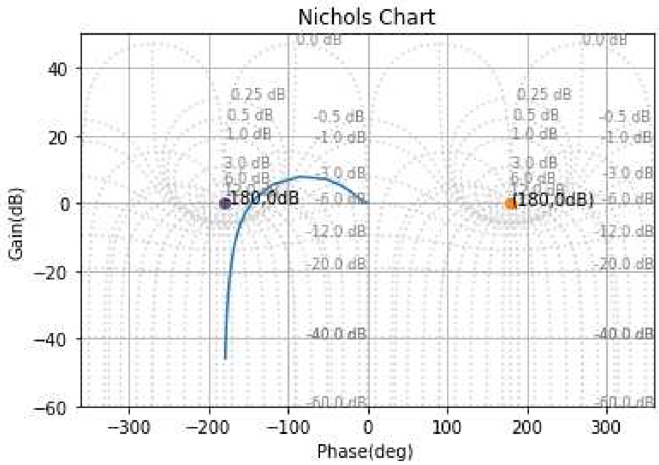


Fig. 2.7.7

2.39. The following code plots M and N contours in rectangle co-ordinates which look like as shown in Fig. 2.7.8.

```
codes/es17btech11009_10_code1.py
```

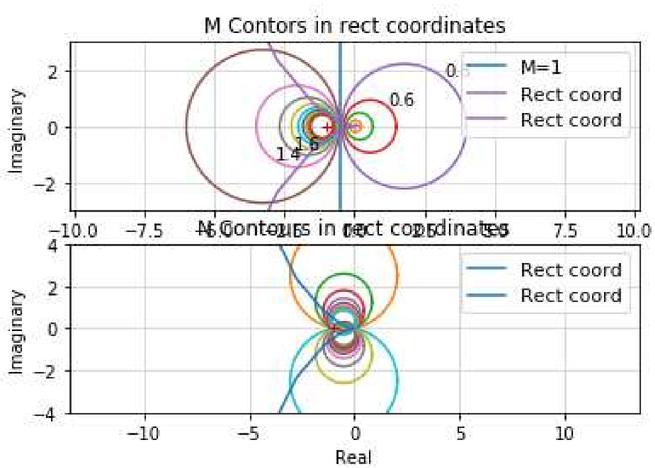


Fig. 2.7.8

2.40. The following code finds intersection of M and N contours with rectangular co-ordinates at different frequencies

```
codes/es17btech11009_10_code2.py
```

The points M and frequencies are listed in Table 2.7.7

M in dB	**M**	ω
7.64	2.41	6.34
6.48	2.11	7.58
-0.81	0.91	9.86
-10.17	0.31	14.33
-40	0.01	57.91

TABLE 2.7.7

2.41. The points N and frequencies are listed in Table2.7.8

α	**N**	ω
-66.50	-2.30	6.40
-78.60	-5.00	6.72
-101.50	4.90	7.32
-158.19	0.4	11.68
-174.28	0.1	46.01

TABLE 2.7.8

2.42. From (2.38.2), The zeroes and poles are shown in Table 2.7.9

Poles	**Zeros**
$p_1 = -3$	$z_1 = -3$
$p_2 = -1.46 + 6.84j$	
$p_3 = -1.46 - 6.84j$	

TABLE 2.7.9

Since we have two conjugate poles, Consider the approximated transfer function

$$T(s) = \frac{K_1}{(s - p_2)(s - p_3)} \tag{2.42.1}$$

$$T(s) = \frac{K_1}{s^2 + 2.92s + 48.91} \tag{2.42.2}$$

The characteristic equation of (2.42.2) is,

$$s^2 + 2.92s + 48.91 = 0 \tag{2.42.3}$$

From (2.1.2) and (2.1.3),
ζ =0.2 and $\omega = 6.99$
Percentage overshoot = 52.7%

2.43. The following code generates step response of the function (2.38.2) as shown in Fig 2.7.9

codes/es17btech11009_101.py

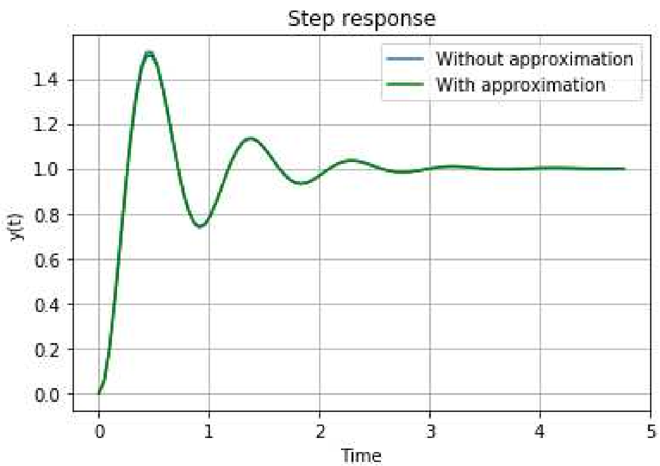


Fig. 2.7.9

2.8 Nyquist and Routh-Hurwitz

2.8.1. Find the range of k such that given characteristic equation

$$s(s^3 + 2s^2 + s + 1) + k(s^2 + s + 1) = 0 \quad (2.8.1.1)$$

is stable.
Solution: The General form of characteristic equation :

$$1 + G(s)H(s) = 0 \quad (2.8.1.2)$$

2.8.2. We draw nyquist plot for open loop transfer function, which is G(s)H(s). When system is marginally stable nyquist plot ((G(s)H(s))) passes through (-1,0),then 1+G(s)H(s) nyquist plot passes through (0,0). We will make real and imaginary part of characteristic equation to 0.So,that we get k for system to be marginally stable. For the system to be stable , the range of k becomes any value greater than k mimimum.

$$Realpart = \omega^4 - \omega^2(k+1) + k = 0 \quad (2.8.2.1)$$

$$Imaginarypart = -\omega^3 + \omega(w+1) = 0 \quad (2.8.2.2)$$

By equating real and imaginary to 0 .We get,

$$k = 0 \quad (2.8.2.3)$$

So,we got minimum value of k is 0 for system to be stable. Then the range of k is

$$0 < k < \infty \quad (2.8.2.4)$$

2.8.3. For a nyquist plot,no.of clock wise encirclement's around the point(-1,0) for a open loop transfer function gives the total no. right hand side zeros plus total no.of right hand side poles, which gives us a idea about stability of system.

2.8.4. Nyquist plot for different values of k.

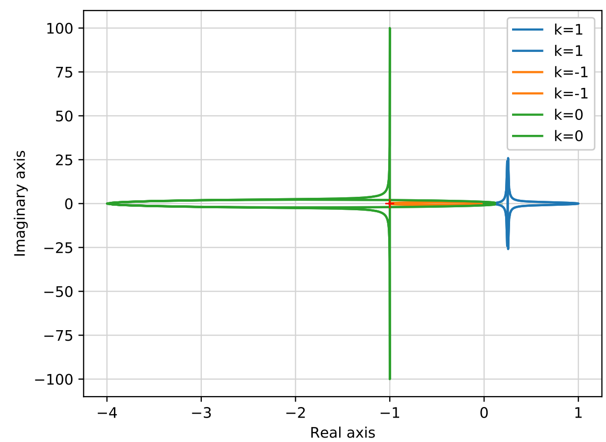


Fig. 2.8.1

In plot, we can see at k=0 ,the plot passes through (-1,0) and at k = 1 no,of encirclements about (-1,0) is 0 which implies system is stable as no.of right hand side zeros(positive values) are 0 and also verifies our above result of k range.
Code for Nyquist plot

codes/ee18btech11042_1.py

2.8.5. Verify using Routh hurwitz criterion
Solution: Routh -hurwitz criterion says system is marginally when no.of sign changes is 0 in matrix and any row of matrix is completely 0. From this , we get minimum value of k for system to be stable

$$s^4 + 2s^3 + s^2(k+1) + s(k+1) + k = 0 \quad (2.8.5.1)$$

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & k+1 & k \\ 2 & k+1 & 0 \\ \frac{k+1}{2} & k & 0 \\ \frac{(k-1)^2}{2} & 0 & 0 \\ k & 0 & 0 \end{vmatrix}$$

2.8.6. For the system to be stable,all values of that matrix should be greater than or equal to 0.So,minimum value of k is,

$$k = 0 \tag{2.8.6.1}$$

The range of k system to be stable

$$0 < k < \infty \tag{2.8.6.2}$$

2.8.7. Verify it using following routh -hurwitz code.

```
codes/ee18btech11042_2.py
```

2.9 Nyquist and Routh-Hurwitz

2.9.1. In the block diagram Fig.2.9.1

$$G(s) = \frac{K}{(s+4)(s+5)} \tag{2.9.1.1}$$

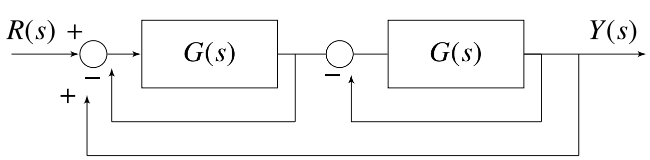


Fig. 2.9.1

2.9.2. Find the range of K for stability by Nyquist criterion
Solution:

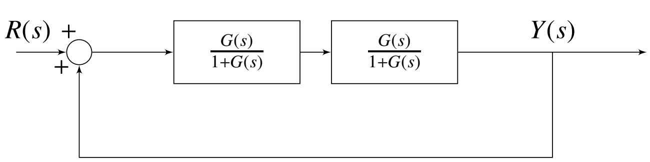


Fig. 2.9.2

The open loop transfer function from Fig.2.9.2

$$T(s) = \left(\frac{\frac{K}{(s+4)(s+5)}}{1 + \frac{K}{(s+4)(s+5)}}\right)^2 \tag{2.9.2.1}$$

$$T(\jmath\omega) = \left(\frac{\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}{1 + \frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}\right)^2 \tag{2.9.2.2}$$

- Since it is connected in positive feedback the transfer function cuts at $(1, \jmath 0)$

$$\implies \text{Re}\{T(\jmath\omega)\} = 1 \tag{2.9.2.3}$$
$$\implies \text{Im}\{T(\jmath\omega)\} = 0 \tag{2.9.2.4}$$

$$\left(\frac{\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}{1 + \frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}\right)^2 = 1 + \jmath 0 \tag{2.9.2.5}$$

$$(\jmath\omega + 4)(\jmath\omega + 5) + 2K = 0 \tag{2.9.2.6}$$

$$-\omega^2 + 9\jmath\omega + 20 + 2K = 0 \tag{2.9.2.7}$$

From (2.9.2.4)

$$20 + 2K = 0 \tag{2.9.2.8}$$
$$\implies K = -10 \tag{2.9.2.9}$$

The minimum value of stability for the system to be stable is

$$K_{min} > -10 \tag{2.9.2.10}$$

The range of K for which the system is stable is

$$-10 < K < \infty \tag{2.9.2.11}$$

2.9.3. From the table.2.9.1, Stability criterion for K is N+P=Z
2.9.4. Verify the Nyquist plots by

```
codes/ee18btech11029_1.py
```

2.9.5. Verify the result using Routh-Hurwitz criterion

K	P	N	Z	Description
-10	0	0	0	System is marginally stable
-9	0	0	0	System is stable
-11	0	1	1	System is unstable

TABLE 2.9.1

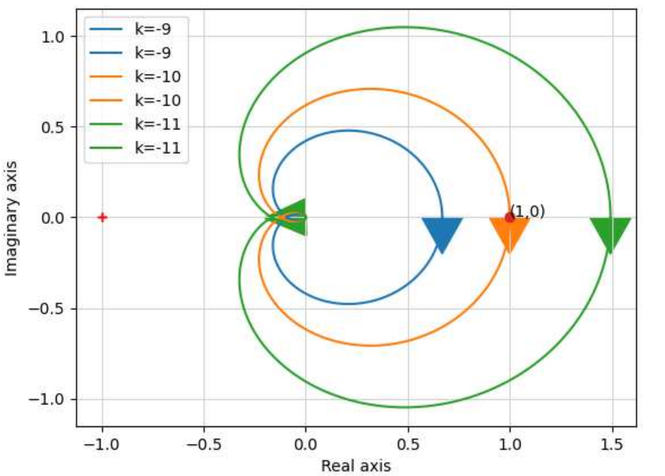


Fig. 2.9.3: Nyquist Plot

Solution: The characteristic equation is

$$1 - T(s) = 0 \tag{2.9.5.1}$$

$$1 - \left(\frac{\frac{K}{(s+4)(s+5)}}{1 + \frac{K}{(s+4)(s+5)}}\right)^2 = 0 \tag{2.9.5.2}$$

$$1 + 2\left(\frac{K}{(s+4)(s+5)}\right) = 0 \tag{2.9.5.3}$$

$$s^2 + 9s + 20 + 2K = 0 \tag{2.9.5.4}$$

$$\begin{vmatrix} s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 20+2K \\ 9 & 0 \\ 20+2K & 0 \end{vmatrix} \tag{2.9.5.5}$$

For a system to be stable it should not have any sign changes

$$20 + 2K > 0 \tag{2.9.5.6}$$

This is valid for all positive values of K but the minimum value of K is

$$K > -10 \tag{2.9.5.7}$$

So the range of K for stability is

$$-10 < K < \infty \tag{2.9.5.8}$$

2.9.6. Verify the result by

```
codes/ee18btech11029_2.py
```

2.10 Nyquist

Consider the system shown in Fig. 2.10.1 below. Sketch the nyquist plot of the system when

1) $G_c(s) = 1$
2) $G_c(s) = 1 + \frac{1}{s}$

and determine the maximum value of K for stability. Take

$$G(s) = \frac{K}{s(1+s)(1+4s)} \tag{2.10.1}$$

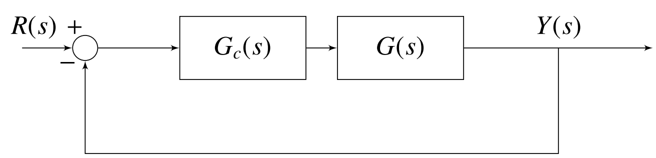


Fig. 2.10.1

Solution: For $G_c(s) = 1$,

The open loop transfer function is

$$G_c(s)G(s) = \frac{K}{s(1+s)(1+4s)} \tag{2.10.2}$$

$$G_c(\jmath\omega)G(\jmath\omega) = \frac{K}{\jmath\omega(1+\jmath\omega)(1+4\jmath\omega)} \tag{2.10.3}$$

$$= \frac{K}{\jmath\omega(1-4\omega^2+5\jmath\omega)} \tag{2.10.4}$$

$$= \frac{K\left(-5\omega - \jmath\left(1-4\omega^2\right)\right)}{\omega\left(\left(1-4\omega^2\right)^2 + 25\omega^2\right)} \tag{2.10.5}$$

The maximum K for stability is where the nyquist plot of open loop transfer function cuts the coordinate $(-1, \jmath 0)$

$$\implies \text{Re}\{G(\jmath\omega)G_c(\jmath\omega)\} = -1 \tag{2.10.6}$$

$$\implies \text{Im}\{G(\jmath\omega)G_c(\jmath\omega)\} = 0 \tag{2.10.7}$$

$$\implies \mathrm{Re}\left\{G(\jmath\omega)\,G_c(\jmath\omega)\right\} = \frac{-5K\omega}{\omega\left((1-4\omega^2)^2+25\omega^2\right)} \tag{2.10.8}$$

$$\implies \mathrm{Im}\left\{G(\jmath\omega)\,G_c(\jmath\omega)\right\} = \frac{-K\left(1-4\omega^2\right)}{\omega\left((1-4\omega^2)^2+25\omega^2\right)} \tag{2.10.9}$$

From (2.10.9) and (2.10.7)

$$1-4\omega^2 = 0 \implies \omega = \frac{1}{2} \tag{2.10.10}$$

From (2.10.8),(2.10.6) and substituting $\omega = \frac{1}{2}$

$$\frac{-5K\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{25}{4}\right)} = -1 \implies K = \frac{5}{4} = 1.25 \tag{2.10.11}$$

For $K < 0$ the system with negative feedback is unstable the range of K is

$$0 < K < \frac{5}{4} \tag{2.10.12}$$

Sketching the Nyquist plot for $G(s)G_c(s)$ in Fig. 2.10.2 The following code gives the nyquist plot

```
codes/ee18btech11034/ee18btech11034_1.py
```

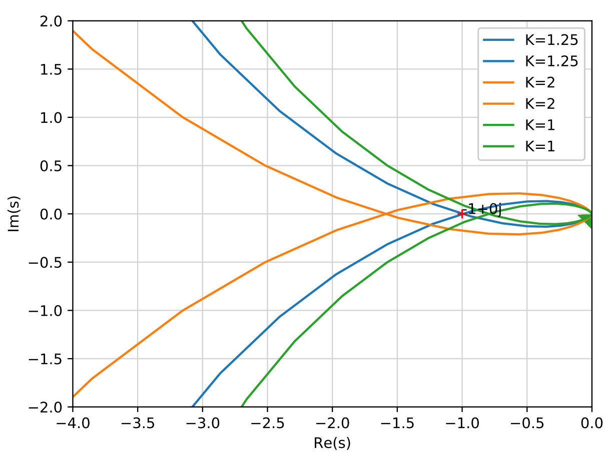


Fig. 2.10.2

Stability Criterian for K

$$N + P = Z \tag{2.10.13}$$

K	P	N	Z	Description
1.25	0	0	0	System is marginally stable
2	0	1	1	System is unstable
1	0	0	0	System is stable

TABLE 2.10.1

From the Fig.2.10.2

$$K_{max} = \frac{5}{4} \tag{2.10.14}$$

Solution: For $G_c(s) = \frac{1+s}{s}$,
the open loop transfer function is

$$G_c(s)\,G(s) = \frac{K(s+1)}{s^2(1+s)(1+4s)} \tag{2.10.15}$$

$$G_c(s)\,G(s) = \frac{K}{s^2(1+4s)} \tag{2.10.16}$$

$$G_c(\jmath\omega)\,G(\jmath\omega) = \frac{K}{(\jmath\omega)^2(1+4\jmath\omega)} \tag{2.10.17}$$

$$= \frac{\frac{-K}{\omega^2}(1-4\jmath\omega)}{1+16\omega^2} \tag{2.10.18}$$

From (2.10.7)

$$\implies \mathrm{Im}\left\{G(\jmath\omega)\,G_c(\jmath\omega)\right\} = \frac{4K}{\omega(1+16\omega^2)} = 0 \tag{2.10.19}$$

This is possible when

$$K = 0 \tag{2.10.20}$$

The system is unstable for both

$$K < 0 \tag{2.10.21}$$

$$K > 0 \tag{2.10.22}$$

Sketching the Nyquist plot for $G(s)G_c(s)$ in Fig. 2.10.3 The following code gives the nyquist plot

```
codes/ee18btech11034/ee18btech11034_2.py
```

From (2.10.13)

From (2.10.20) K_{max} must be 0 which is not

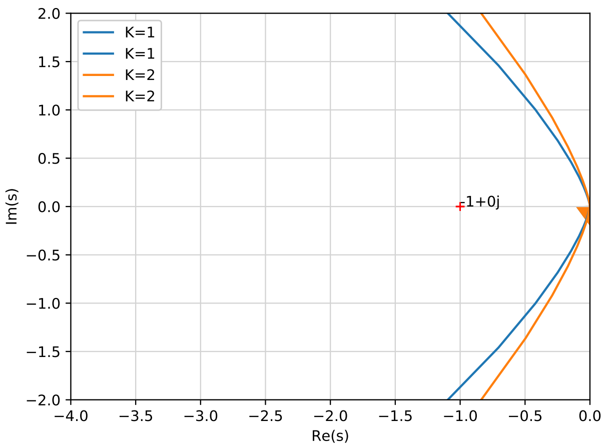


Fig. 2.10.3

K	P	N	Z	**Description**
1	0	1	1	System is unstable
2	0	1	1	System is unstable

TABLE 2.10.2

possible. Hence the system is unstable for all real K

2.11 Nyquist

Sketch the Nyquist plot for a closed loop system having open-loop transfer function

$$G(s)H(s) = \frac{2e^{-s\tau}}{s(1+s)(1+0.5s)} \tag{2.11.1}$$

Determine the maximum value of τ for the system to be stable.

Solution: From (2.11.1),

$$\implies \text{Re}\{G(\jmath\omega)H(\jmath\omega)\} =$$

$$-4\left[\frac{3\omega^2\cos(\omega\tau) - \left(\omega^3 - 2\omega\right)\sin(\omega\tau)}{(3\omega^2)^2 + (\omega^3 - 2\omega)^2}\right] \tag{2.11.2}$$

$$\implies \text{Im}\{G(\jmath\omega)H(\jmath\omega)\} =$$

$$4\left[\frac{\left(\omega^3 - 2\omega\right)\cos(\omega\tau) + 3\omega^2\sin(\omega\tau)}{(3\omega^2)^2 + (\omega^3 - 2\omega)^2}\right] \tag{2.11.3}$$

Determining the stability of closed loop transfer function using Nyquist stability Criterion.

$$Z = P + N \tag{2.11.4}$$

Poles of open loop transfer function are on left half of s-plane.Therefore,$P = 0$
To ensure that the system is stable N should be 0
For maximum value of τ for stability ,the nyquist plot cuts the real axis at -1+j0.

$$G(s)H(s) = -1 + j0 \tag{2.11.5}$$

$$\text{Im}\{G(\jmath\omega)H(\jmath\omega)\} = 0 \tag{2.11.6}$$

$$\text{Re}\{G(\jmath\omega)H(\jmath\omega)\} = -1 \tag{2.11.7}$$

From (2.11.3) and (2.11.6)

$$\implies \tan(\omega\tau) = \frac{-\left(\omega^3 - 2\omega\right)}{3\omega^2} \tag{2.11.8}$$

From (2.11.2) and (2.11.7) and substituting $\tan(\omega\tau) = \frac{-\left(\omega^3-2\omega\right)}{3\omega^2}$

$$\implies \omega^6 + 5\omega^4 + 4\omega^2 - 16 = 0 \tag{2.11.9}$$

Solving (2.11.9) graphically.

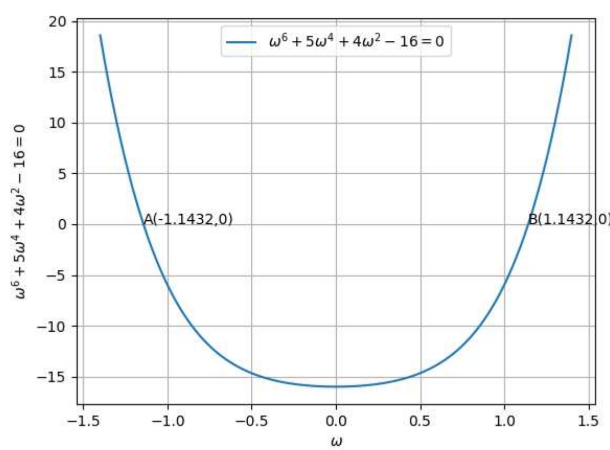


Fig. 2.11.1

Python code for the above plot is

codes/ee18btech11035_1.py

ω = 1.1432,-1.1432 (As, ω is positive)
Therefore,ω = 1.1432

Substituting ω in (2.11.8)

$$\tan(1.1432\tau) = 0.2021 \tag{2.11.10}$$
$$\tau = 0.1744 \tag{2.11.11}$$

The following python code generates the Nyquist plot.

```
codes/ee18btech11035_2.py
```

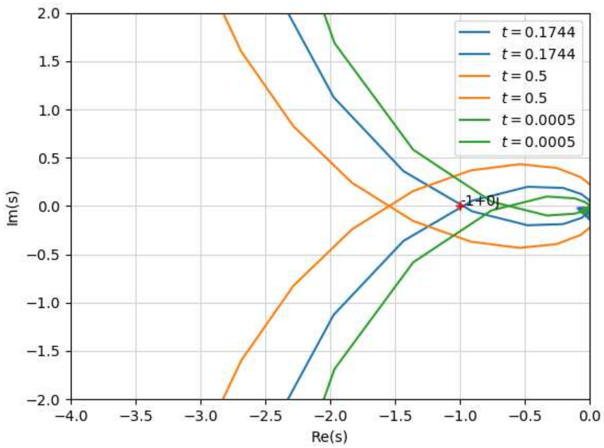


Fig. 2.11.2: Nyquist plot for variable τ

From the above figure (2.11.2) $\tau \leq 0.1744$ for a stable system.

τ	**P**	**N**	**Z**	**Description**
0.1744	0	1	1	System is Marginally stable
0.5	0	0	0	System is unstable
0.0005	0	0	0	System is stable

TABLE 2.11.1

Therefore, $\tau_{max} = 0.1744$

.

3 Design in Frequency Domain

3.1

3.2 Lead Compensator

3.1. For a unity feedback system shown in 3.2.1, $\frac{10}{s(s+1)}$. Design a lead compensator such that the phase margin of the system is 45° and appropriate steady state error is less than or equal to $\frac{1}{15}$ units of the final output value. Further the gain crossover frequency of the system must be less than 7.5rad/sec.

3.2. For the control system shown in 3.2.1 write the steady state output for step input.
Solution:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{3.2.1}$$
$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{sR(s)G(s)}{1+G(s)} \tag{3.2.2}$$
$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{G(s)}{1+G(s)} \tag{3.2.3}$$

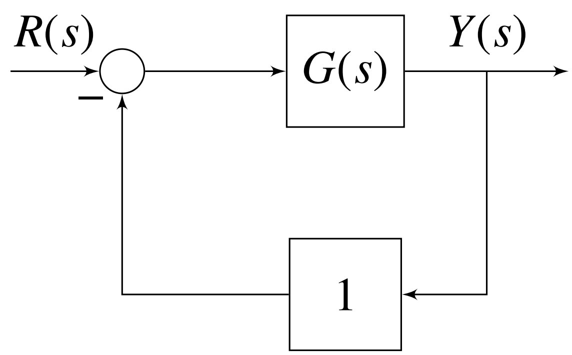


Fig. 3.2.1

3.3. What do you mean by steady state error and write the expression for steady state error for control system shown in 3.2.1 considering step input.
Solution: Steady-state error is the difference between the input and the output for a prescribed test input as time tends to infinity.

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)} \tag{3.3.1}$$

3.4. Write the general expression for the transfer function of a phase lead compensator.
Solution:

$$G_c(s) = K_{comp} \frac{(1+\alpha T s)}{(1+T s)} \tag{3.4.1}$$

3.2.2 shows the compensated control system.

3.5. Calculate the steady state output value and steady state error for the control system shown in 3.2.1, where G(s) = $\frac{10}{s(s+1)}$. Consider the input to be unit step.

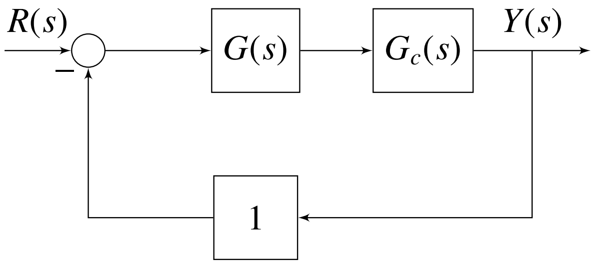


Fig. 3.2.2

Solution:
Steady state value:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{10}{10 + s(s+1)} \tag{3.5.1}$$

$$\lim_{t\to\infty} y(t) = 1 \tag{3.5.2}$$

Steady state error:

$$e_{ss} = \lim_{s\to 0} \frac{s(s+1)}{10 + s(s+1)} \tag{3.5.3}$$

$$e_{ss} = 0 \tag{3.5.4}$$

3.6. Choose a value of K_{comp} which satisfies the steady state error condition.
Solution: As the steady state error for unit step response is always zero, any value of K_{comp} satisfies the steady state error condition in compensated system. For simplicity let us choose $K_{comp} = 1$.

3.7. Calculate phase margin and gain cross over frequency of open-loop transfer function G(S).
Solution:
Gain cross over frequency:

$$G(j\omega) = \frac{10}{j\omega(j\omega + 1)} \tag{3.7.1}$$

$$|G(j\omega)| = \frac{10}{\sqrt{\omega^4 + \omega^2}} \tag{3.7.2}$$

$$\frac{10}{\sqrt{\omega^4 + \omega^2}} = 1 \tag{3.7.3}$$

$$\omega_{gc} = 3.084 \tag{3.7.4}$$

Phase Margin:

$$\phi = -90^\circ - tan^{-1}(\omega) \tag{3.7.5}$$

$$pm = 180^\circ - 90^\circ - tan^{-1}(\omega_{gc}) \tag{3.7.6}$$

$$pm = 17.966^\circ \tag{3.7.7}$$

$$\tag{3.7.8}$$

3.8. Write the expression for maximum phase of a lead compensator and the frequency where it occurs
Solution:

$$\phi_{max} = sin^{-1}(\frac{\alpha - 1}{\alpha + 1}) \tag{3.8.1}$$

$$\omega_m = \frac{1}{T\alpha} \tag{3.8.2}$$

3.9. Calculate the value of ϕ_{max} required to meet desired phase margin.
Solution:

$$\phi_{max} = 45^\circ - pm + 15^\circ \tag{3.9.1}$$

$$\phi_{max} = 45^\circ - 17.966^\circ + 15^\circ \tag{3.9.2}$$

$$\phi_{max} = 42.034^\circ \tag{3.9.3}$$

Here the extra 15^o has been added to compensate for the shift in ω_{gc}.

3.10. Using (3.8.1) calculate the value of α
Solution:

$$sin(42.034^\circ) = \frac{\alpha - 1}{\alpha + 1} \tag{3.10.1}$$

$$0.669 = \frac{\alpha - 1}{\alpha + 1} \tag{3.10.2}$$

$$0.331\alpha = 1.669 \tag{3.10.3}$$

$$\alpha = 5.04 \tag{3.10.4}$$

3.11. Choose appropriate value for ω_m.
Solution:

- For maximum increase in phase margin we have to ensure that ϕ_{max} occurs at frequency close to ω_{gc} of G(s).

$$\omega_m = 3.084 rad/sec \tag{3.11.1}$$

- We know that ω_{gc} gets shifted slightly when we cascade a compensator to original transfer function, to compensate for the shift we have already added an extra 15° to ϕ_{max}.

3.12. Using (3.8.2) calculate the value of T.
Solution:

$$T = \frac{1}{\omega_m \alpha} \tag{3.12.1}$$

$$T = \frac{1}{15.54} \tag{3.12.2}$$

$$T = 0.064 \tag{3.12.3}$$

3.13. Write the final expression of the Lead compensator designed.

Solution:

$$G_c(s) = \frac{(1 + 0.322s)}{(1 + 0.064s)} \tag{3.13.1}$$

- Zero at s = -3.084
- Pole at s = -15.54

3.14. Verify using a python plot.
Solution:

```
codes/ee18btech11044_2.py
```

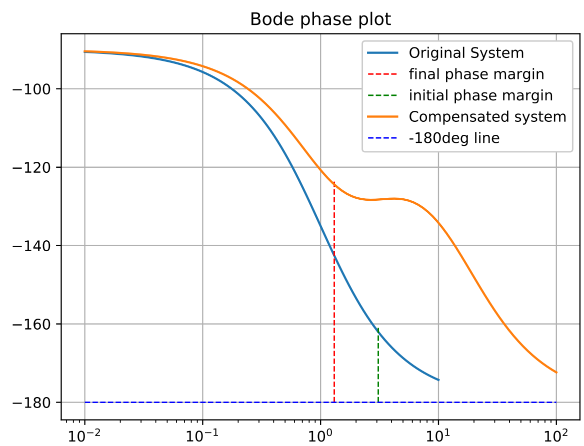


Fig. 3.2.3

3.3 Lag Compensator

3.3.1. Given the unity feedback system, with

$$G(S) = \frac{K(s + 10)(s + 11)}{s(s + 3)(s + 6)(s + 9)} \tag{3.3.1.1}$$

Use frequency response method to design a lag compensator to yield Kv = 1000 and peak overshoot of 15% .Use second order approximation.
Solution: Fig. 3.3.1 models the equivalent of compensated closed loop system.

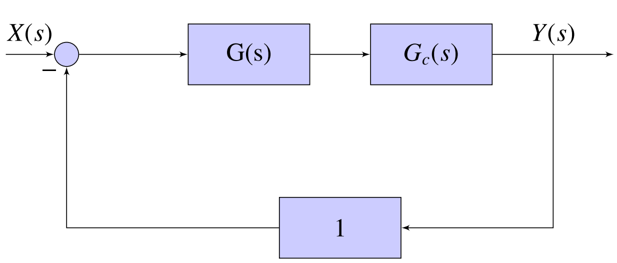


Fig. 3.3.1

Velocity constant

$$K_v = \lim_{t \to 0} sG(s) \tag{3.3.1.2}$$

$$\lim_{t \to 0} s \frac{K(s + 10)(s + 11)}{s(s + 3)(s + 6)(s + 9)} = 1000 \tag{3.3.1.3}$$

$$\implies K = 1473 \tag{3.3.1.4}$$

Bode plot of G(s) for the value of k

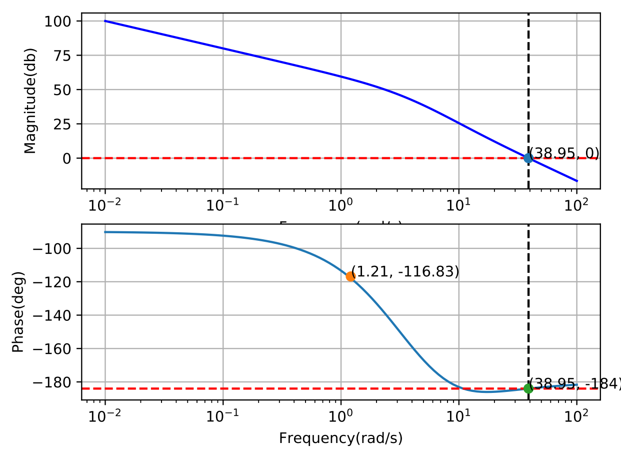


Fig. 3.3.2

The following code verifies the result.

```
codes/ee18btech11030/ee18btech11030.py
```

Relation between %OS and Damping ratio

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{(\pi)^2 + (\ln(\%OS/100))^2}} \tag{3.3.1.5}$$

$$\implies \zeta = 0.517 \tag{3.3.1.6}$$

Phase Margin for a Damping ratio is given by Eq (3.3.1.7)

$$\phi_m = 90^\circ - \arctan(\frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta} \tag{3.3.1.7}$$

$$\implies \phi_m = 53.17^\circ \tag{3.3.1.8}$$

From Fig : 3.3.2 of uncompensated system with K = 1473

$$\phi_m = -4^\circ,\ \omega_{PM} = 38.95 rad/sec \tag{3.3.1.9}$$

$$\phi_{ml} \text{ lies in } (5,12) \tag{3.3.1.10}$$

$$\phi_{total} = 53.17^\circ - (-4^\circ) + \phi ml \tag{3.3.1.11}$$

$$\phi_{total} = 63.17^\circ \tag{3.3.1.12}$$

Note : Adding 6 degrees phase angle to compensate the phase angle contribution of the lag compensator.
From Figure 3.3.2

$$\phi_{total} = 63.17^\circ \text{ at, } \omega_{PM} \quad = 1.21 rad/sec \tag{3.3.1.13}$$

At this Phase Margin frequency, the magnitude plot must go through 0 dB. But The magnitude of uncompensated system at 1.21 rad/sec is 57.55 dB = 754.2

3.3.2. Designing Lag Compensator Gc(s)
Solution: General lag compensator

$$G_c(s) = \left(\frac{s + \frac{1}{T}}{s + \frac{1}{T\alpha}} \right) \quad \alpha > 1 \tag{3.3.2.1}$$

- First draw the high-frequency asymptote at -57.55 dB.So that magnitude at 1.21 rad/sec becomes 0 dB.
- Arbitrarily select the higher break frequency to be about one decade below the phase-margin frequency, or 0.121 rad/sec.
- Starting at the intersection of this frequency with the lag compensator's high-frequency asymptote, draw a -20 dB/decade line until 0 dB is reached.That intersection gives the lower break frequency.
- The lower break frequency is found to be 0.0001604 rad/sec.
- The compensator must have a dc gain of unity to retain the value of Kv that we have already designed by setting K = 1473.
- Gain in the lag compensator = 0.001326

$$Gain = \frac{0.0001604}{0.121} = 0.001326 \tag{3.3.2.2}$$

Hence the lag compensator transfer function is

$$G_c(s) = \frac{0.001326(s + 0.121)}{s + 0.0001604} \tag{3.3.2.3}$$

3.3.3. Verifying Lag Compensator using Plots
Solution: Magnitude and Phase plot
The following code

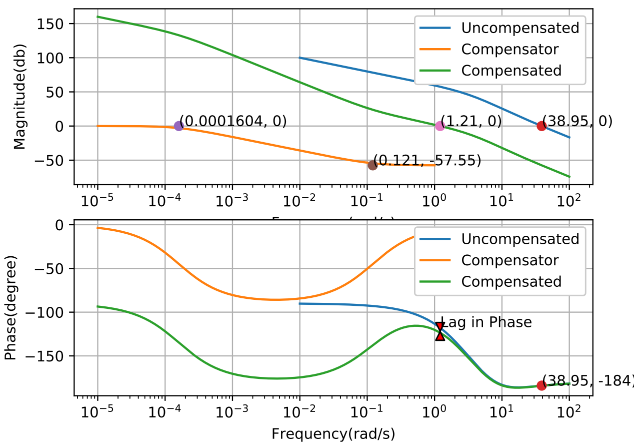


Fig. 3.3.3

codes/ee18btech11030/ee18btech11030_1.py

Specification	Propsed	Actual
OS%	15%	15.16%
K_v	1000	1000.47

TABLE 3.3.1: Comparing the Proposed and Actual results

3.3.4. Verifying in time domain
Solution: Time response for a unit step function

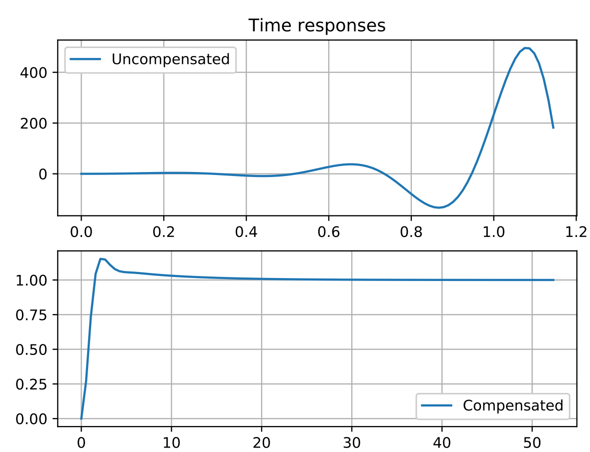


Fig. 3.3.4

The following code

codes/ee18btech11030/ee18btech11030_2.py

3.4 Lead Compensator

3.4.1. For a unity feedback system shown in Fig. 1

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{3.4.1.1}$$

Design a lead compensator to yield a $K_v = 2$ and a phase margin of 30°.

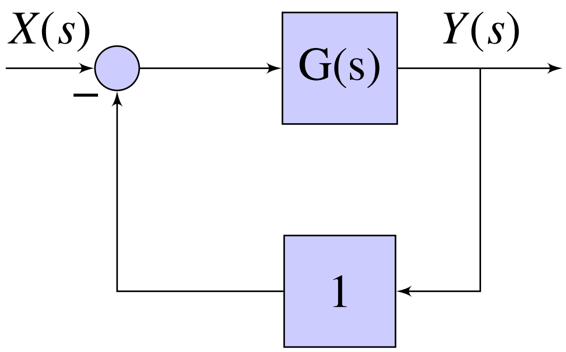


Fig. 3.4.1

Solution: For unity feedback we have Velocity error constant (K_v)

$$K_v = \lim_{s\to 0} sG(s) \tag{3.4.1.2}$$

$$\lim_{s\to 0}\left(\frac{K}{(2+s)(4+s)(6+s)}\right) = 2 \tag{3.4.1.3}$$

$$\implies K = 96 \tag{3.4.1.4}$$

Check the phase margin and gain crossover frequency by running the following code

```
codes/ee18btech11036_1.py
```

- The Phase margin: 19.76°
- Gain Crossover Frequency:1.469 rad/sec

The Bode plot of system is as shown,
Therefor amount of phase to be added: 30-19.76=10.24
The circuit of lead compensator is given by
Transfer function:

$$C(s) = \beta\left(\frac{1 + j\tau\omega}{1 + j\beta\tau\omega}\right) \tag{3.4.1.5}$$

$$\beta = \left(\frac{R_2}{R_1 + R_2}\right) \tag{3.4.1.6}$$

$$\tau = R_1 C \tag{3.4.1.7}$$

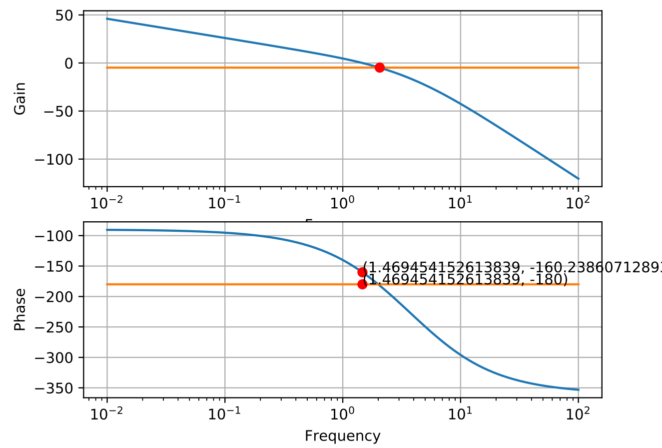


Fig. 3.4.2

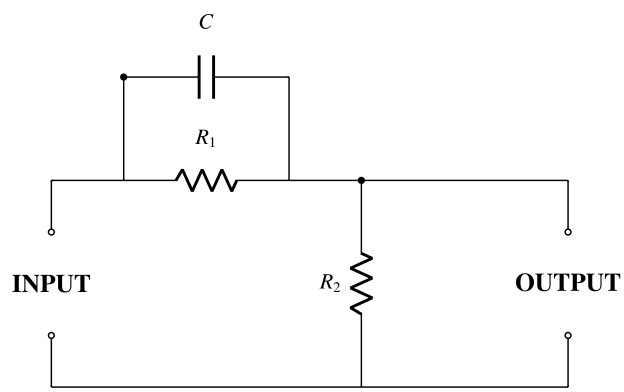


Fig. 3.4.3

Find the values of β and τ
Solution: The maximum phase lead compensated by a lead compensator is given by

$$\phi = \sin^{-1}\frac{1-\beta}{1+\beta} \tag{3.4.1.8}$$

at

$$\omega = \frac{1}{\sqrt{\beta}\tau} \tag{3.4.1.9}$$

Now we know that from Gain crossover frequency

$$\omega = 1.469 rad/sec \tag{3.4.1.10}$$

and the phase margin to be added:

$$\phi = 10.24° \tag{3.4.1.11}$$

But to compensate for the added magnitude of lead compensator, a correction factor of $10° - 20°$ is added.Hence

$$\phi = 30.24° \implies \beta = 0.33 \tag{3.4.1.12}$$

From the bode plot ω is chosen at which gain of original system is

$$-20\log\left(1/\sqrt{\beta}\right) = -4.81 \tag{3.4.1.13}$$

Find the plot using the following code

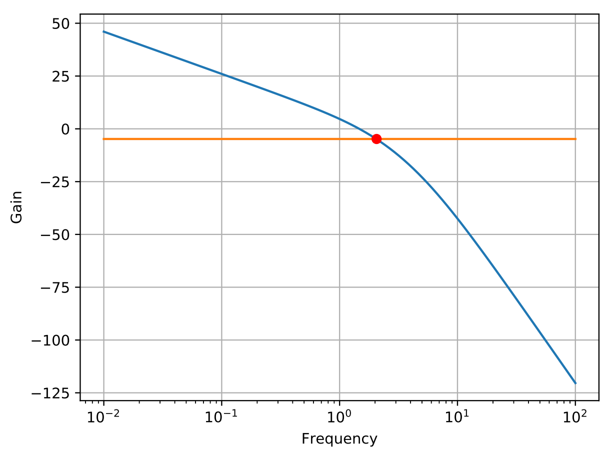


Fig. 3.4.4

```
codes/ee18btech11036_4.py
```

From plot ω=2.009 rad/sec
Solving equations 3.4.1.8 and 3.4.1.9:

$$\tau = 0.828 \tag{3.4.1.14}$$
$$\beta = 0.33 \tag{3.4.1.15}$$
$$\tag{3.4.1.16}$$

New Transfer Function:

$$G(s) = \frac{96\,(1+0.828s)}{(s)\,(2+s)\,(4+s)\,(6+s)\,(1+0.273s)} \tag{3.4.1.17}$$

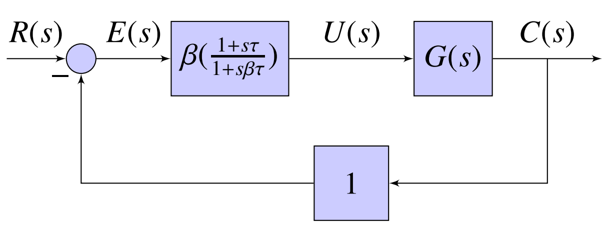


Fig. 3.4.5

Verify your results from the following code:

```
codes/ee18btech11036_2.py
```

- The Phase margin: 29.269°
- The Gain Crossover Frequency: 2.02 rad/sec

The Bode plot is as shown,

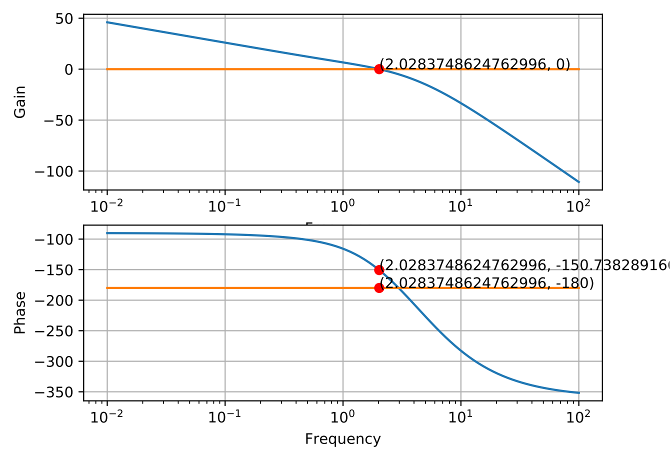


Fig. 3.4.6

3.4.2. For a unity feedback system

$$G(s) = \frac{K}{(s)(s+2)(s+4)(s+6)} \tag{3.4.2.1}$$

Design a lag compensator to yield a $K_v = 2$ and Phase Margin of 30° **Solution:** Fig.3.4.7 models the equivalent of compensated closed loop system.

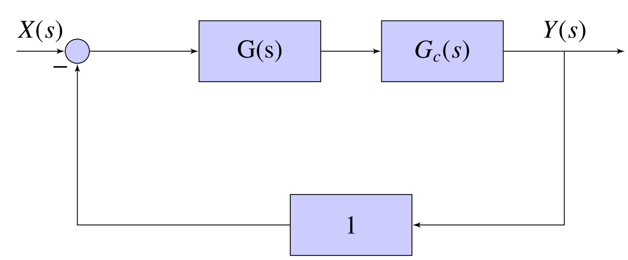


Fig. 3.4.7

Static Velocity Error Constant K_v is the steady-state error of a system for a unit-ramp input i.e.,

$$K_v = \lim_{s\to 0} sG(s)G_c(s) \tag{3.4.2.2}$$

Therefore,

$$K_v = \lim_{s \to 0} s \frac{K}{s(s+2)(s+4)(s+6)} \frac{Ts+1}{\beta Ts+1}$$
$$\implies 2 = \frac{K}{(0+2)(0+4)(0+6)} \frac{T(0)+1}{\beta T(0)+1}$$
$$\therefore K = 96 \quad (3.4.2.3)$$

$$G(s) = \frac{96}{s(s+2)(s+4)(s+6)} \quad (3.4.2.4)$$

Substituting $s = \jmath\omega$ in (3.4.2.4),

$$G(\jmath\omega) = \frac{96}{(j\omega)(j\omega+2)(j\omega+4)(j\omega+6)} \quad (3.4.2.5)$$

$$|G(\jmath\omega)| = \frac{|96|}{\omega\sqrt{4+\omega^2}\sqrt{16+\omega^2}\sqrt{36+\omega^2}} \quad (3.4.2.6)$$

$$\angle G(\jmath\omega) = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) \quad (3.4.2.7)$$

The standard Transfer equation of Lag Compensator and its Phase and Gain

$$G_c(s) = \frac{Ts+1}{\beta Ts+1} \quad (3.4.2.8)$$
$$|G_c(s)| = \frac{1}{\beta} \frac{1+\left(\frac{\omega}{T}\right)^2}{1+\left(\frac{\omega}{\beta T}\right)^2} \quad (3.4.2.9)$$
$$\angle G_c(s) = \tan^{-1}(\omega T) - \tan^{-1}(\omega\beta T) \quad (3.4.2.10)$$

Where $\beta > 1$.
It can be approximated that for $\omega > \frac{1}{T}$

$$|G_c(s)| = \frac{1}{\beta} \quad (3.4.2.11)$$

and Phase to be very small($< 12^\circ$).
The Phase Margin(PM) of the Transfer function G(s)
From (3.4.2.6) and (3.4.2.6)
At Gain Crossover,

$$|G(s)| = 1 \quad (3.4.2.12)$$
$$\implies \omega_{gc} = 1.47 rad/sec \quad (3.4.2.13)$$
$$\implies \angle G\left(\jmath\omega_{gc}\right) = -160.26^\circ \quad (3.4.2.14)$$
$$PM = 180^\circ + \angle G\left(\jmath\omega_{gc}\right) \quad (3.4.2.15)$$
$$\implies PM = 19.74^\circ \quad (3.4.2.16)$$

The following are the Bode plots of uncompensated system

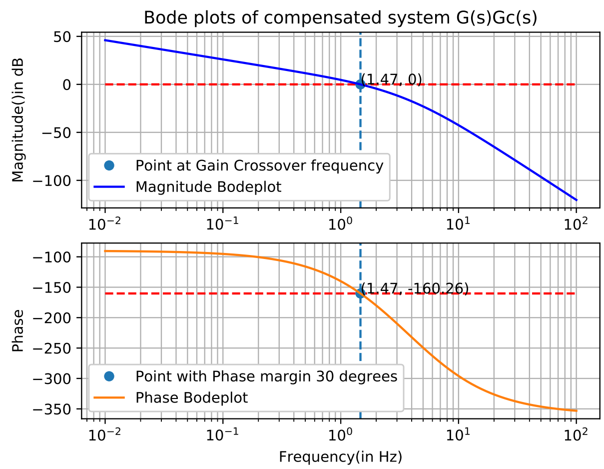


Fig. 3.4.8

The code for Bode plots of uncompensated system

codes/ee18btech11046_1.py

The lag companstor form is given in (3.4.2.8), Let

$$G'(s) = G(s)G_c(s) \quad (3.4.2.17)$$

The $PM = 30^\circ$ when $\angle G'(\jmath\omega) = -150^\circ$
Since the addition of compensator reduces Gain of system, thereby reducing Gain Crossover frequency which increases Phase Margin(PM) of system.
Since, Compansator also has small negative phase(say ϵ),let $\epsilon = 5^\circ$.i.e, $\angle G_c(s) = 5$

$$\angle G'(s) = \angle G(s) + \angle G_c(s) \quad (3.4.2.18)$$
$$\implies -150^\circ = \angle G(s) - 5^\circ \quad (3.4.2.19)$$
$$\implies \angle G(s) = -145^\circ \quad (3.4.2.20)$$

The value of ω where $\angle G(s) = -145^\circ$ is

$$\angle G(s) = -145^\circ \quad (3.4.2.21)$$
$$\implies \omega_{req} = 1.10953 rad/sec \quad (3.4.2.22)$$

The value $\frac{1}{T}$ is exactly 2 octaves below ω_{req}

obtained in (3.4.2.22)

$$\frac{1}{T} = \frac{\omega_{req}}{4} \tag{3.4.2.23}$$

$$\implies T = 3.605 \tag{3.4.2.24}$$

Now we should take β such that Gain Crossover frequency occurs at ω_{req} i.e., to make $\left|G^{'}(\jmath\omega)\right| = 1$
From (3.4.2.11),

$$\left|G^{'}\left(\jmath\omega_{gc}\right)\right| = \left|G\left(\jmath\omega_{gc}\right)\right|\left|G_c\left(\jmath\omega_{gc}\right)\right| = 1 \tag{3.4.2.25}$$

$$\implies 1.4936 \times \frac{1}{\beta} = 1 \tag{3.4.2.26}$$

$$\implies \beta = 1.4936 \tag{3.4.2.27}$$

Substituting values of T and β obtained from(3.4.2.24) and(3.4.2.27) in (3.4.2.8) The required Compensator Transfer is

$$G_c(s) = \frac{3.605s + 1}{5.384s + 1} \tag{3.4.2.28}$$

The following are the Bode plots of compensated system

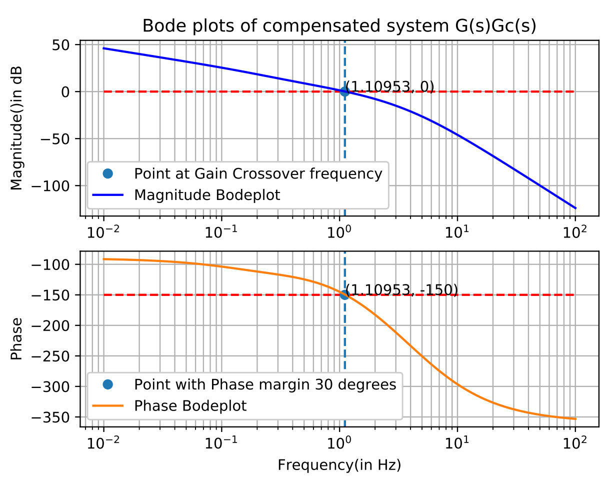


Fig. 3.4.9

The code for Bode plots of compensated system

codes/ee18btech11046_2.py

3.4.3. For a unity feedback system

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{3.4.3.1}$$

Design a lag lead compensator to yield a K_v = 2 and a phase margin of 30°. First we will design a lead compensator and for that whole system we will design a lag compensator which will finally be the lag lead compensator of the orignal transfer function.

Solution: For unity feedback we have Velocity error constant (K_v)

$$K_v = \lim_{s\to 0} sG(s) \tag{3.4.3.2}$$

$$\lim_{s\to 0}\left(\frac{K}{(2+s)(4+s)(6+s)}\right) = 2 \tag{3.4.3.3}$$

$$\implies K = 96 \tag{3.4.3.4}$$

Check the phase margin and gain crossover frequency by running the following code

codes/es17btech11019_1.py

- The Phase margin: 19.76°
- Gain Crossover Frequency:1.469 rad/sec

The plot of system is as shown,

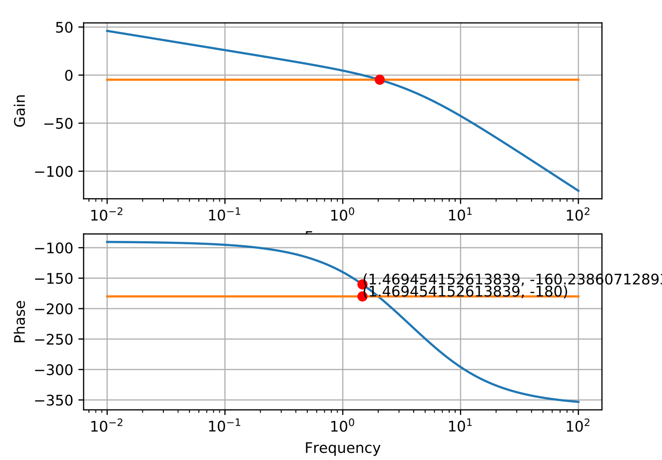


Fig. 3.4.10

Therefor amount of phase to be added: 30-19.76=10.24
Transfer function:

$$C(s) = \beta\left(\frac{1 + j\tau\omega}{1 + j\beta\tau\omega}\right) \tag{3.4.3.5}$$

Find the values of β and τ

Solution: The maximum phase lead compensated by a lead compensator is given by

$$\phi = \sin^{-1}\frac{1-\beta}{1+\beta} \tag{3.4.3.6}$$

at

$$\omega = \frac{1}{\sqrt{\beta}\tau} \tag{3.4.3.7}$$

Now we know that from Gain crossover frequency

$$\omega = 1.469 rad/sec \tag{3.4.3.8}$$

and the phase margin to be added:

$$\phi = 10.24^\circ \tag{3.4.3.9}$$

But to compensate for the added magnitude of lead compensator, a correction factor of $10^\circ - 20^\circ$ is added.Hence

$$\phi = 30.24^\circ \implies \beta = 0.33 \tag{3.4.3.10}$$

From the bode plot ω is chosen at which gain of original system is

$$-20\log\left(1/\sqrt{\beta}\right) = -4.81 \tag{3.4.3.11}$$

Find the plot using the following code

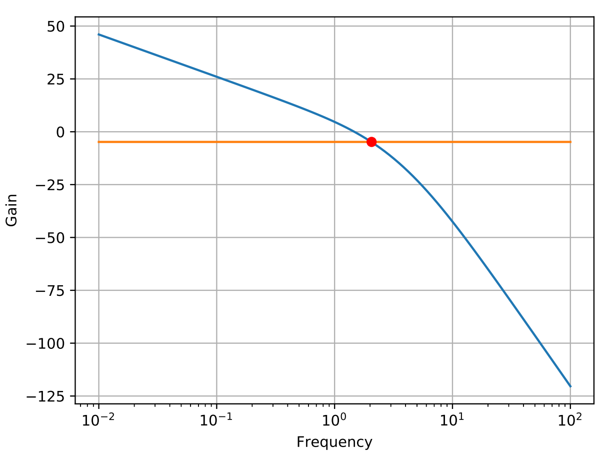


Fig. 3.4.11

codes/es17btech11019_2.py

From plot ω=2.009 rad/sec

Solving equations 3.4.3.6 and 3.4.3.7

$$\tau = 0.828 \tag{3.4.3.12}$$
$$\beta = 0.33 \tag{3.4.3.13}$$
$$\tag{3.4.3.14}$$

New Transfer Function:

$$G(s) = \frac{96\,(1+0.828s)}{(s)\,(2+s)\,(4+s)\,(6+s)\,(1+0.273s)} \tag{3.4.3.15}$$

Verify your results from the following code:

codes/es17btech11019_3.py

- The Phase margin: 29.269°
- The Gain Crossover Frequency: 2.02 rad/sec

The plot is as shown,

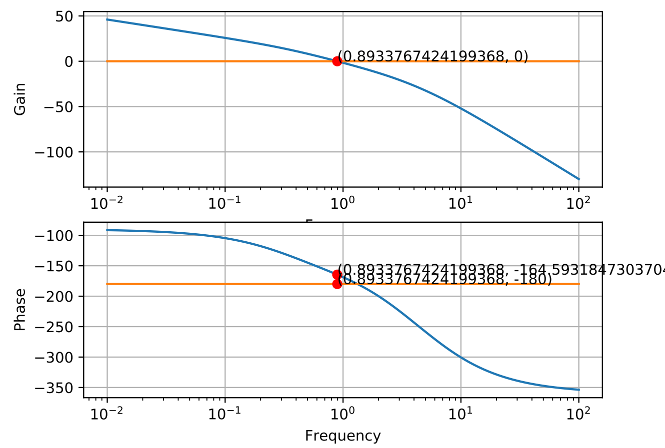


Fig. 3.4.12

Now for lag compensator of this whole lead compensated part Transfer function:

$$C'(s) = \left(\frac{1 + j\tau\omega}{1 + j\alpha\tau\omega}\right) \tag{3.4.3.16}$$

Find the values of α

Solution:

$$\alpha = \frac{1}{\beta} \tag{3.4.3.17}$$

Solving equations 3.4.3.17

$$\alpha = 3.03 \tag{3.4.3.18}$$

New Transfer Function:

$$G(s) = \frac{96\,(1+0.828s)\,(1+0.828s)}{(s)\,(2+s)\,(4+s)\,(6+s)\,(1+0.273s)\,(1+2.50884s)} \tag{3.4.3.19}$$

Final plot is,
Find the plot using the following code

```
codes/es17btech11019_4.py
```

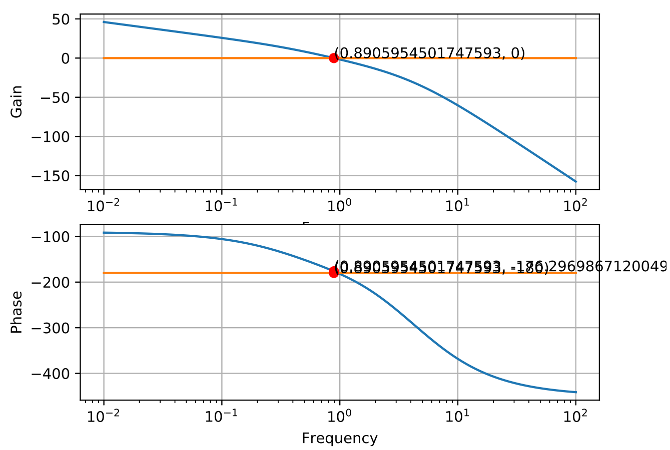


Fig. 3.4.13

3.5 Compensator for Overshoot

3.5.1. Given the unity feedback system of Fig. 3.5.1 , with

$$G(s) = \frac{K}{s(s+5)(s+20)} \tag{3.5.1.1}$$

The uncompensated system has about 55% peak overshoot and a peak time if 0.5 seconds when $K_v = 10$. Use frequency response technique to design a lead compensator to reduce the percent overshoot to 10% , while keeping the peak time and steady state error about the same or less. Consider second order approximations.

3.5.2. **Solution:**

$$K_v = \lim_{s\to 0} sG(s) = 10 \tag{3.5.2.1}$$

$$\implies K = 1000 \tag{3.5.2.2}$$

The bode plot for G(s)is as follows :

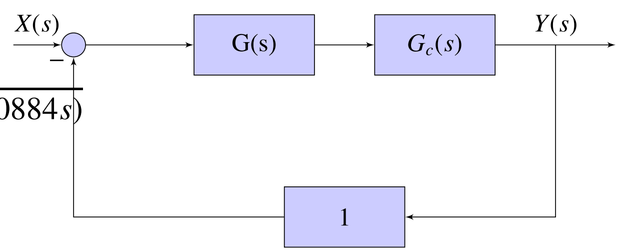


Fig. 3.5.1

$$G(s) = \frac{1000}{s(s+5)(s+20)} \tag{3.5.2.3}$$

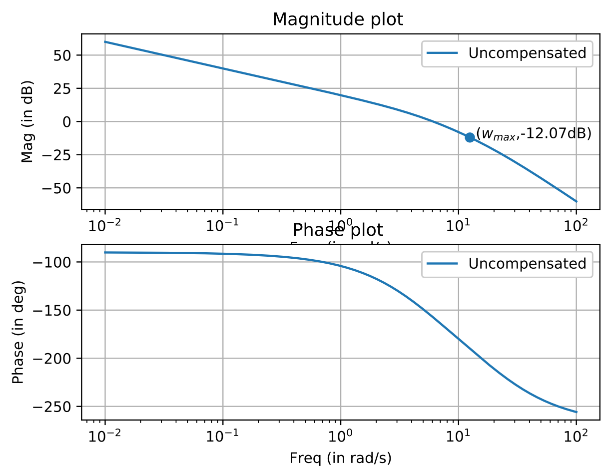


Fig. 3.5.2: G(s) Bode Plot

$$\zeta = \frac{-\ln\left(\frac{OS\%}{100}\right)}{\sqrt{\pi^2 + \left(\ln\left(\frac{OS\%}{100}\right)\right)^2}} \tag{3.5.2.4}$$

$$PhaseMargin = \phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}\right) \tag{3.5.2.5}$$

The following code computes the above quantities.

```
codes/ee18btech11026/ee18btech11026_1.py
```

The required additional phase contribution by the compensator will be:

$$\phi_{max} = 58.9 - 21.16 + correction factor \tag{3.5.2.6}$$

Specifications	Actual	Expected
OS%	55%	10%
ζ	0.186	0.591
ϕ_m	21.16°	58.59°
T_p	0.5	<= 0.5
K_v	10	<= 10

TABLE 3.5.1: Table of Specifications

$$CorrectionFactor = 25^\circ \tag{3.5.2.7}$$

$$\phi_{max} = 62^\circ \tag{3.5.2.8}$$

Note : Since we know that the lead network will also increase the phase-margin frequency, we add a correction factor to compensate for the lower uncompensated system's phase angle.Choosing the correction factor is a trail and error procedure so as to reach our expected specifications.

The gain compensator's T.F will be of the form:

$$G_c(s) = \frac{1}{\beta}\left(\frac{s+\frac{1}{T}}{s+\frac{1}{T\beta}}\right) \tag{3.5.2.9}$$

This form of T.F does not influence the steady state error.

Important Relations to find T and β:

$$\phi_{max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} \tag{3.5.2.10}$$

The Compensator's magnitude at the phase margin frequency ω_{max}

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}} \tag{3.5.2.11}$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} \tag{3.5.2.12}$$

Using the above formulae :

$$\beta = 0.062 \tag{3.5.2.13}$$

$$|G_c(j\omega_{max})| = 12.07dB \tag{3.5.2.14}$$

If we select ω_{max} to be the new phase-margin frequency, the uncompensated system's magnitude at this frequency must be -12.07 dB to yield a 0 dB crossover at ω_{max} for the compensated system.

From the bode plot of the un-compensated system, find ω_{max} where the magnitude is -12.07 dB. This becomes our new phase-margin frequency.

$$\omega_{max} = 12.5rad/sec \tag{3.5.2.15}$$

$$T = 0.321 \tag{3.5.2.16}$$

The Compensator's T.F is as follows :

$$G_c(s) = 16.13\left(\frac{s+3.115}{s+50.25}\right) \tag{3.5.2.17}$$

The open loop T.F for the compensated system is :

$$G(s).G_c(s) = 16130\left(\frac{(s+3.115)}{s(s+50.25)(s+5)(s+20)}\right) \tag{3.5.2.18}$$

3.5.3. **Verification :** We could observe the affect of the lead-phase compensator from the phase plots.

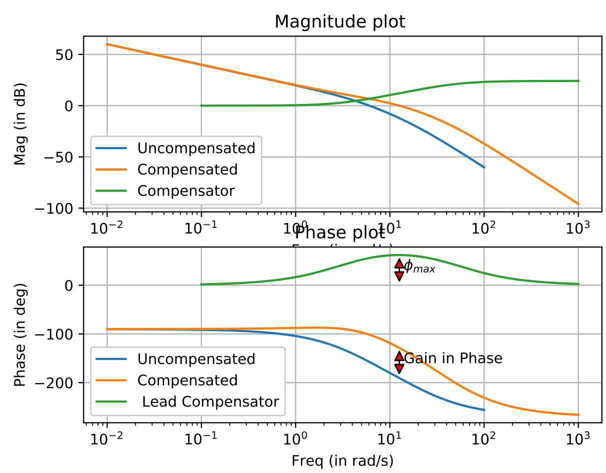


Fig. 3.5.3: Combined Bode Plots

The time responses for a unit step input in a unity feedback system with and without a compensator are as follows :

These plots are generated using the below code:

```
codes/ee18btech11026/ee18btech11026_2.py
```

3.5.4. **Result :** The below is the summary for the designed lead-compensator

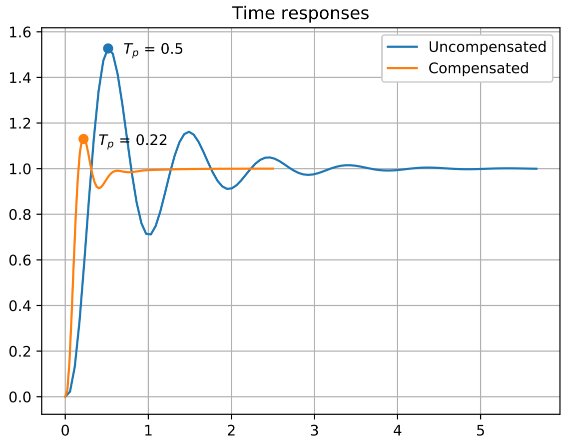


Fig. 3.5.4: Time response for a unit step input

Specifications	Expected	Proposed
OS%	10%	11%
T_p	<= 0.5	0.22
K_v	<= 10	10

TABLE 3.5.2: Comparing the desired and obtained results

3.6 Lead Compensator

3.6.1. For a control system in unity feedback with a transfer function

$$G(s) = \frac{10K}{s(s+1)(s+5)} \tag{3.6.1.1}$$

Design a lead compensator with a 60° phase margin and an appropriate error constant of 5

Solution: Before adding a compensator, we first find a value of gain for an error constant of 5. As the system has one pole at the origin, the appropriate error constant would be the velocity constant K_v. For an error constant of 5, we solve the following equation:

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{10K}{(s+1)(s+5)} = 2K \tag{3.6.1.2}$$

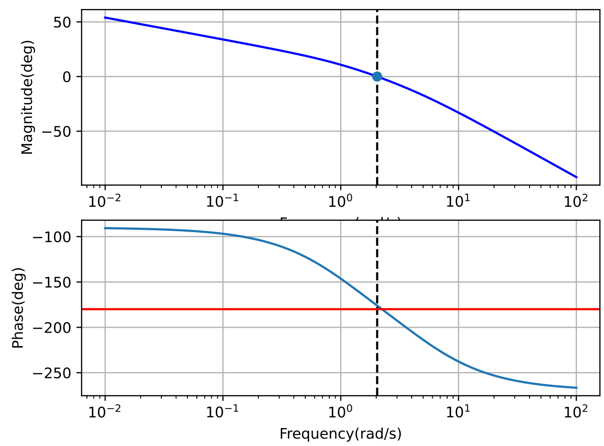
$$\implies K = \frac{K_v}{2} = 2.5 \tag{3.6.1.3}$$

So, the new transfer function becomes

$$G(s) = \frac{25}{s(s+1)(s+5)} \tag{3.6.1.4}$$

The gain crossover frequency ω_{gc} and phase margin ϕ_M is calculated from the plots using the following code

```
codes/ee18btech11051/
    ee18btech11051_code1.py
```



This can also be calculated from the equations

$$\omega\sqrt{\omega^2+1}\sqrt{\omega^2+25} = 25 \tag{3.6.1.5}$$

$$\phi_M = -90^\circ - \tan^{-1}\omega_{gc} - \tan^{-1}(\frac{\omega_{gc}}{10}) \tag{3.6.1.6}$$

Solving which, we get $\phi_M = 3.96^\circ, \omega_{gc} = 2.03$ The following code computes the margins and frequencies:

```
codes/ee18btech11051/
    ee18btech11051_code2.py
```

The maximum phase of the compensator is given by-

$$\phi_M = 60^\circ - phasemargin + anglecorrection \tag{3.6.1.7}$$

$$\phi_M = 56^\circ + anglecorrection \tag{3.6.1.8}$$

Here, the angle correction is added to compensate the early zero added due to the lead compensator. The highest phase margin is achieved when ϕ_M is close to 90°. To achieve it, we take angle correction = 33°. The phase lead compensator will have a transfer function of the form:

$$G_C(s) = \left(\frac{1+\alpha Ts}{1+Ts}\right), \alpha > 1 \tag{3.6.1.9}$$

Note that this transfer function doesn't change the error constant. Now we solve for α using the equation

$$\alpha = \frac{1+\sin\phi_M}{1-\sin\phi_M} = \frac{1+\sin 89^{\circ}}{1-\sin 89^{\circ}} = 13,130.56 \tag{3.6.1.10}$$

Now to get the phase gain, we set the frequency for maximum phase of compensator ω_m to the previous gain crossover frequency. And using it, the value of T is calculated as follows

$$\omega_m = 2.03 rad/sec \tag{3.6.1.11}$$

$$T = \frac{1}{\omega_m \alpha} = 0.0000375 \tag{3.6.1.12}$$

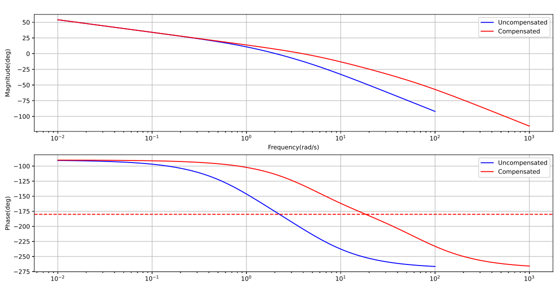
The Compensator will have the transfer function as follows :

$$G_c(s) = \left(\frac{1+0.5s}{1+0.0000375s}\right) \tag{3.6.1.13}$$

The open loop T.F for the compensated system is :

$$G(s)G_c(s) = 25\left(\frac{(1+0.5s)}{s(s+1)(s+5)(1+0.0000375s)}\right) \tag{3.6.1.14}$$

To observe the changes, we plot the compensated system



Using the code previously used to calculate the phase margin, the phase margin of the compensated system turns out to not go beyond 46°. So, we use a two stage lead compensator to get the required phase margin. The new compensator can be thought of as the square of the previous compensator

$$G_C^2(s) = \left(\frac{1+\alpha Ts}{1+Ts}\right)^2, \alpha > 1 \tag{3.6.1.15}$$

The values of α and T can still be calculated as done previously, but for half of the required increase in phase, as the compensator gives double of the configured phase. So, for

$$\phi_M = \frac{(56^{\circ}+26^{\circ})}{2} = 41^{\circ} \tag{3.6.1.16}$$

$$\alpha = \frac{1+\sin(38^{\circ})}{1-\sin 38^{\circ}} = 5 \tag{3.6.1.17}$$

$$\omega_m = 2.6 \tag{3.6.1.18}$$

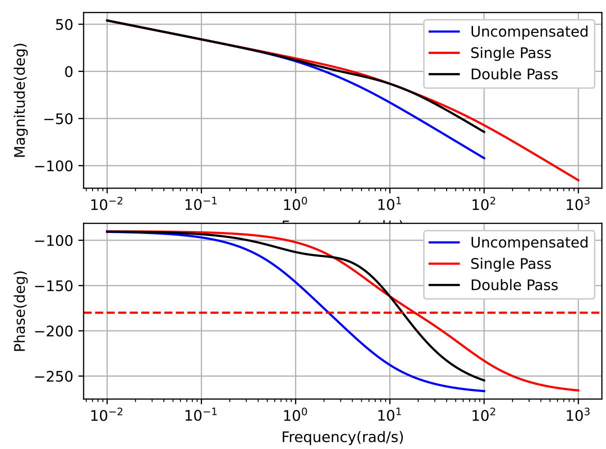
$$T = \frac{1}{\omega_m \alpha} = 0.0769 \tag{3.6.1.19}$$

Now, the new compensator and the compensated transfer functions are

$$G_c^2(s) = \frac{(1+0.38s)^2}{(1+0.0769s)^2} \tag{3.6.1.20}$$

$$G(s)G_c^2(s) = 25\left(\frac{(1+0.38s)^2}{s(s+1)(s+5)(1+0.0769s)^2}\right) \tag{3.6.1.21}$$

Verification : Now we plot the newly compensated transfer function.



Parameter	Expected	Final
Phase Margin	60°	59.68°
K_v	5	5
ω_{gc}	>= 2.03	3.01

TABLE 3.6.1: Design Parameters

The design reqirements and the achieved parameters are listed in the table 3.6.1.

As it can be seen from the final readings in the table 3.6.1, the desired transfer function is achieved.

3.6.2. An aircraft roll control system can be represented by a block diagram shown in Fig. 3.6.1 with G(s) in feedback system, whose error K_v = 5. Determine K

$$G(s) = \frac{10K}{s(s+1)(s+5)} \tag{3.6.2.1}$$

The block diagram is given by Fig.3.6.1

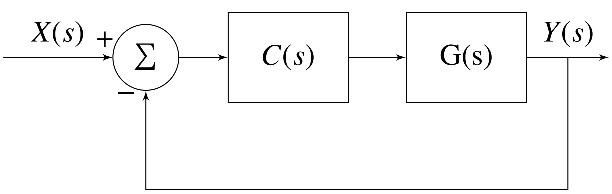


Fig. 3.6.1

For unity feedback we have Velocity error constant (K_v)

$$K_v = \lim_{s\to 0} sG(s) \tag{3.6.2.2}$$

$$\lim_{s\to 0}\left(\frac{10K}{(s+1)(s+5)}\right) = 5 \tag{3.6.2.3}$$

$$\implies K = 2.5 \tag{3.6.2.4}$$

It's Phase Margin = 3.94°
and Gain Crossover Frequency = 2.03 rad/s
Refer Fig. 3.6.2 for plot G(s).

```
codes/es17btech11002_1_new.py
```

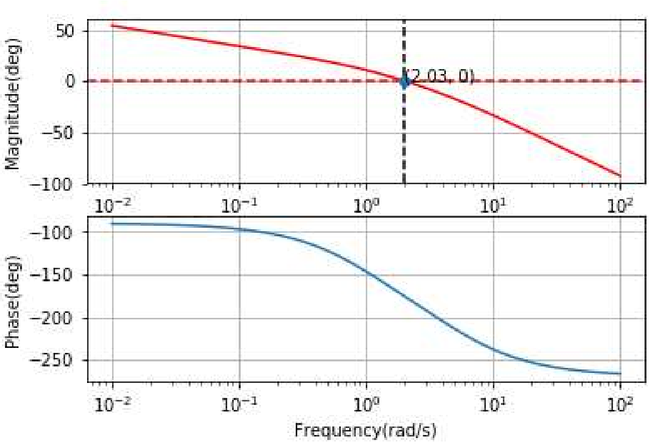


Fig. 3.6.2

Compensator required phase angle (ϕ_m) and Phase Margin Frequency $\left(\omega_{pm}\right)$,

$$\phi_m = -(180^\circ + \theta) + PM + 5 = 65^\circ \tag{3.6.2.5}$$

$$\omega_{pm} = 1.25 rad/s. \tag{3.6.2.6}$$

Attenuation factor ($\alpha\beta$) is given by

$$\alpha = 0.5 \tag{3.6.2.7}$$

$$\beta = 20 \tag{3.6.2.8}$$

Lead and Lag Compensator Design Parameter is given in TABLE 3.6.2 And Compensator

Zeros/Poles	**Parameter**	**Value**
z_{lead}	$\omega_{pm}\sqrt{\alpha}$	0.279
p_{lead}	$\frac{z_{lead}}{\alpha}$	5.590
z_{lag}	$0.1\omega_{pm}$	0.125
p_{lag}	$\frac{z_{lag}}{\beta}$	0.00625

TABLE 3.6.2: Zeroes and Poles

obtained has transfer function

$$G_c(s) = \frac{(s+0.279)(s+0.125)}{(s+5.590)(s+0.00625)} \tag{3.6.2.9}$$

Refer Fig3.6.3 for plot $G(s)G_c(s)$.

```
codes/es17btech11002_2_new.py
```

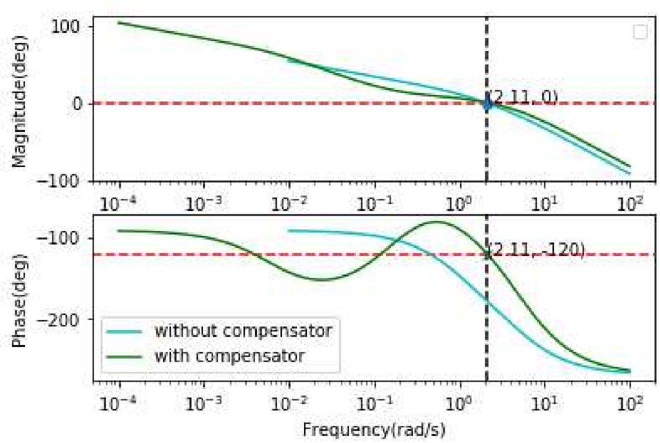


Fig. 3.6.3

NOTE : The idea of using a lead-lag network is to provide the attenuation of a phase-lag network and the lead-phase angle of a phase-lead network. This points should be noted while designing a controller, and parameters to be changed accordingly to get exact results.

3.6.3. For a unity feedback system given below refer, 3.6.4, with

$$G(s) = \frac{K}{s(s+5)(s+11)} \tag{3.6.3.1}$$

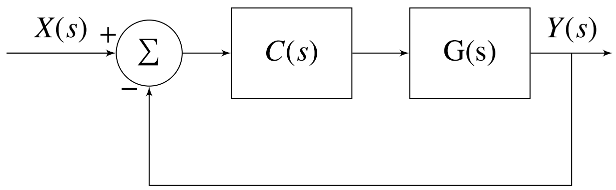


Fig. 3.6.4

Find the value of the gain, K, of the uncompensated system operating at 30% overshoot.

Solution:
The damping ratio ζ is given by:

$$\zeta = -\frac{\ln \frac{\%OS}{100}}{\sqrt{\pi^2 + \ln \frac{\%OS}{100}^2}} \tag{3.6.3.2}$$

Therefore, solving the above equation with %OS = 30, we get,

$$\zeta = 0.358 \tag{3.6.3.3}$$

Further, we need to find the point on the root locus which crosses the 0.358 damping ratio line.
Let this point be $-\sigma_d + j\omega_d$, where σ_d is the exponential damping frequency and ω_d is the damped frequency of oscillation.
And the relation between σ_d and ω_d is given by,

$$\omega_d = \sigma \tan(\arccos\zeta)) \tag{3.6.3.4}$$

where,

$$\sigma_d = \zeta\omega_n \tag{3.6.3.5}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{3.6.3.6}$$

ω_n is the natural frequency.

Now, by solving the below equation and equating the real and imaginary parts to zero,

$$1 + G(-\sigma_d + j\omega_d) = 0 \tag{3.6.3.7}$$

We get,

$$\sigma_d = 1.464 \tag{3.6.3.8}$$

$$\omega_d = 3.818 \tag{3.6.3.9}$$

$$Gain, K = 218.6 \tag{3.6.3.10}$$

The peak time, T_p is given by,

$$T_p = \frac{\pi}{\omega_d} \tag{3.6.3.11}$$

And, settling time is given by,

$$T_s = \frac{4}{\sigma_d} \tag{3.6.3.12}$$

So, we get,

$$T_p = 0.823 \tag{3.6.3.13}$$

$$T_s = 2.732 \tag{3.6.3.14}$$

$$K_v = \lim_{s\to 0} sG(s) \tag{3.6.3.15}$$

$$\lim_{s\to 0}\left(\frac{K}{(s+5)(s+11)}\right) = \frac{218.6}{(5)(11)} \tag{3.6.3.16}$$

$$\implies K_v = 3.975 \tag{3.6.3.17}$$

Design a lag-lead compensator to:

a) Decrease the peak time by a factor of 2
b) Decrease the percent overshoot by a factor of 2
c) Improve the steady state error by a factor of 30

Solution:
Lead Design: Using the required specifications, we can calculate the damping ratio and the natural frequency, Using eq. 3.6.3.2, we get,

$$\zeta = 0.517 \tag{3.6.3.18}$$

And,

$$\omega_d = \frac{\pi}{T_p} = \omega_n\sqrt{1-\zeta^2} = 7.634. \tag{3.6.3.19}$$

Hence, $\omega_n = 8.919$.
Thus, the desired pole is located at,

$$-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -4.61 + j7.634 \tag{3.6.3.20}$$

Let us now assume a lead compensator zero at -5. The summation of the system's original poles and lead compensator zero to the design point is -171.2°. Thus, the compensator pole must contribute 171.2° - 180° = -8.8°.

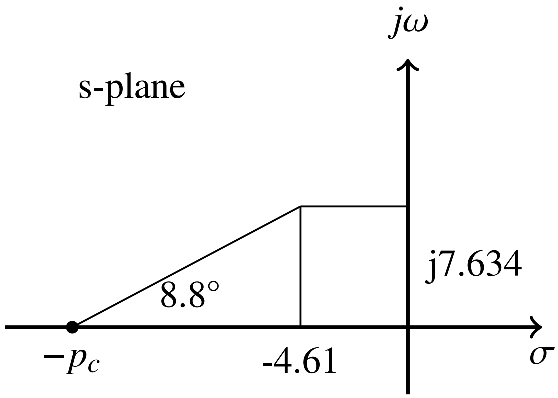


Fig. 3.6.5

Refer figure 3.6.5 for clarification.

$$\tan(8.8°) = \frac{7.634}{p_c - 4.61} \tag{3.6.3.21}$$

Hence, $p_c = 53.92$.
Therefore, the compensated open-loop transfer function is,

$$\frac{K}{s(s+11)(s+53.92)} \tag{3.6.3.22}$$

Evaluating the gain for this function at the desired pole, we get $K = 4430$.

Lag Design:
The lead compensated $K_v = 7.469$.
We need an improvement over the lead compensated system of,

$$\frac{(30)(3.975)}{7.469} = \frac{119.25}{7.469} \tag{3.6.3.23}$$

$$\implies = 15.97 \tag{3.6.3.24}$$

$$K_v = \lim_{s\to 0} sG(s) \tag{3.6.3.25}$$

Choose p_c (compensator pole) = 0.001, we get z_c (compensator zero) = 0.001597. Thus, the compensator is given by,

$$G_{lag}(s) = \frac{s+0.01597}{s+0.001} \tag{3.6.3.26}$$

So, the final compensated open loop transfer function is,

$$C(s)G(s) = \frac{4430(s+0.01597)}{s(s+11)(s+53.92)(s+0.001)} \tag{3.6.3.27}$$

Plot the graph after adding a lag-lead compensator.
Solution:
See Fig. 3.6.6 generated by

codes/ep18btech11016/ep18btech11016.py

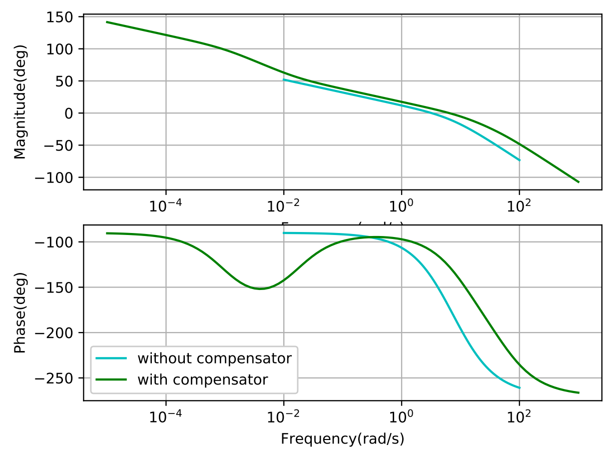


Fig. 3.6.6

3.7 *Lag Lead Compensator*

3.1. For the unity feedback system shown in Fig. 3.7.1 , with

$$G(s) = \frac{K}{s(s+1)(s+4)} \tag{3.1.1}$$

Design a lag-lead compensator to yield a $K_v = 12$ as well as peak overshoot of 12% and peak time of less than or equal to 2 seconds.

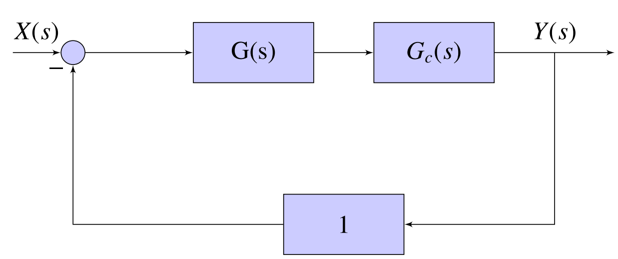


Fig. 3.7.1

3.2. **Solution:**

$$K_v = \lim_{s \to 0} sG(s) = 12 \tag{3.2.1}$$

$$\implies K = 48 \tag{3.2.2}$$

The bode plot for G(s)is as follows :

$$G(s) = \frac{48}{s(s+1)(s+4)} \tag{3.2.3}$$

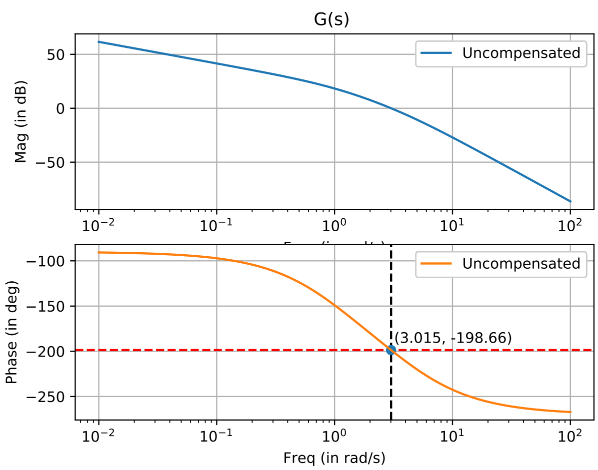


Fig. 3.7.2: G(s) Bode Plot

Damping Ratio ζ

$$\zeta = \frac{-\ln\left(\frac{OS\%}{100}\right)}{\sqrt{\pi^2 + \left(\ln\left(\frac{OS\%}{100}\right)\right)^2}} \tag{3.2.4}$$

Phase Margin ϕ_M

$$\phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right) \tag{3.2.5}$$

closed-loop Bandwidth ω_{bw}

$$\omega_{bw} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{3.2.6}$$

The following code computes the above quantities.

```
codes/ee18btech11049/ee18btech11049_1.py
```

$$\zeta = 0.557 \tag{3.2.7}$$

$$\phi_M = 56.13° \tag{3.2.8}$$

$$\omega_{bw} = 2.27 \text{ rad/sec} \tag{3.2.9}$$

The required phase margin to yield a 12% OS is 56.13°
Let us select ω =1.83 rad/s as the new phase-margin frequency.
At this frequency, the uncompensated phase is -176 and would require, if we add a -6 contribution from the lag compensator, a +56 contribution from the lead compensator.

$$G_{Lead}(s)\,G_{Lag}(s) = \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\gamma T_2}}\right) \tag{3.2.10}$$

Choose the lag compensator 1-decade below, so that its will have minimal effect at the new phase-margin frequency.

$$\phi_{max,lead} = 56 = \sin^{-1}\frac{1-\beta}{1+\beta} \tag{3.2.11}$$

$$\beta = 0.092 \tag{3.2.12}$$

$$\gamma = 10.86 \text{ since } \gamma = \frac{1}{\beta} \tag{3.2.13}$$

Thus with $\gamma = 10.86$

$$G_{Lag}(s) = \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\gamma T_2}}\right) = \frac{s + 0.183}{s + 0.0168} \tag{3.2.14}$$

$$\omega_{max} = \frac{1}{T_1\sqrt{\beta}} \tag{3.2.15}$$

Using Values of $\omega_{max} = 1.83$ and $\beta = 0.094$

$$G_{Lag}(s) = \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right) = \frac{s + 0.55}{s + 5.69} \tag{3.2.16}$$

The lag-lead Compensated System's open loop transfer function is

$$G_{total}(s) = \frac{48\,(s + 0.183)\,(s + 0.55)}{s\,(s+1)\,(s+4)\,(s + 0.0168)\,(s + 5.69)} \tag{3.2.17}$$

3.3. **Verification :** We could observe the affect of the lag-lead compensator from these phase

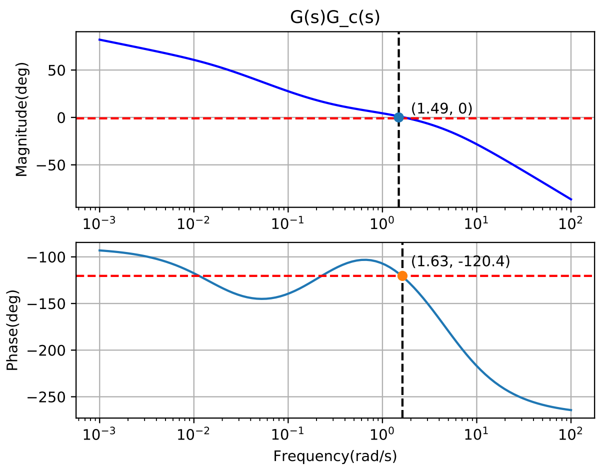


Fig. 3.7.3: G(s) Compensated Bode Plot

plots.

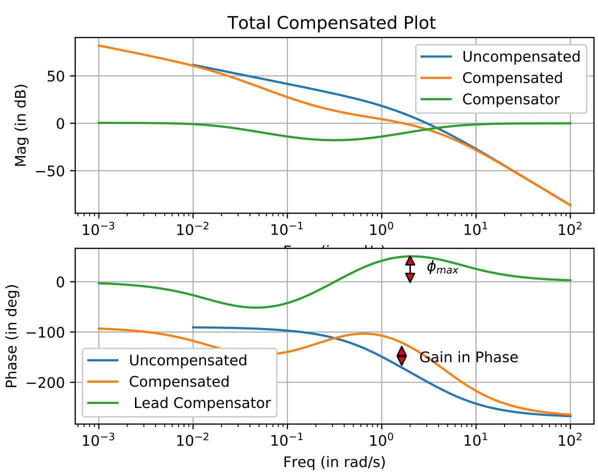


Fig. 3.7.4: Combined Bode Plots

These plots are generated using the below code:

codes/ee18btech11049/ee18btech11049_2.py

3.4. **Result :** The below is the summary for the designed lead-lead compensator.

Specifications	Expected	Proposed
OS%	12%	10.2%
T_p	<= 2	1.61
ϕ_M	56	59.6
ω_{bw}	>= 2.27	3
ω_{gc}	-	1.63
K_v	12	12

TABLE 3.7.1: Comparing the desired and obtained results

3.8 *Lag Lead Compensator*

3.8.1. Using the frequency response method, design a lag-lead compensator for the unity feedback system given

$$G(S) = \frac{K(s+7)}{s(s+5)(s+15)} \tag{3.8.1.1}$$

The following specifications must be met: Peak overshoot = 15%, settling time = 0.1 second and velocity error constant = 1000 Use second order approximation.
Solution: Figure: **??** models the equivalent of compensated closed loop system. Velocity error constant

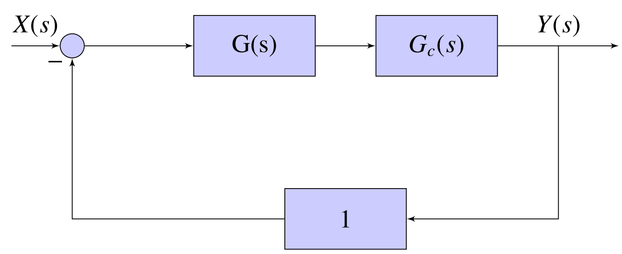


Fig. 3.8.1

$$K_v = \lim_{s\to 0} sG(s) \tag{3.8.1.2}$$

$$\lim_{s\to 0} s\frac{K(s+7)}{s(s+5)(s+15)} = 1000 \tag{3.8.1.3}$$

$$\implies K = 10714 \tag{3.8.1.4}$$

Bode plot of G(s) for the value of K The following code verifies the result.

codes/ee18btech11012/ee18btech11012_1.py

Relation between %OS and Damping ratio

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{(\pi)^2 + (\ln(\%OS/100))^2}} \tag{3.8.1.5}$$

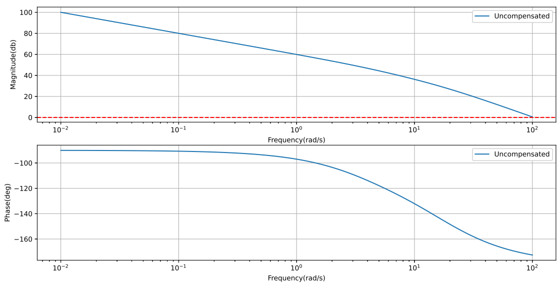


Fig. 3.8.2

$$\implies \zeta = 0.517 \tag{3.8.1.6}$$

Phase Margin for a Damping ratio is given by

$$\phi_m = 90^\circ - \arctan(\frac{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}{2\zeta} \tag{3.8.1.7}$$

$$\implies \phi_m = 53.17^\circ \tag{3.8.1.8}$$

For an additional 5°for lag compensation,Phase margin is

$$\phi_m = 53.17^\circ + 5^\circ = 58.17^\circ \tag{3.8.1.9}$$

Note : Adding 5°phase angle to compensate the phase angle contribution of the lag compensator. Bandwidth frequency is given by

$$\omega_{BW} = \omega_n(\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2})} \tag{3.8.1.10}$$

where

$$\omega_n = \frac{4}{T_s\zeta} \tag{3.8.1.11}$$

Given settling time = 0.1 sec then

$$\omega_n = 77.37 rad/sec \tag{3.8.1.12}$$

then

$$\omega_{BW} = 96.91 rad/sec \tag{3.8.1.13}$$

3.8.2. Designing Lag-Lead Compensator Gc(s)
Solution: General lag-lead compensator

$$G_c(s) = \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\gamma T_2}}\right) \tag{3.8.2.1}$$

- Choose the new phase-margin frequency

$$\omega_{Pm} = 0.8\omega_{BW} \quad = 77.53 rad/sec \tag{3.8.2.2}$$

- At this phase-margin frequency,Phase angle is -170.52°.
- Then the conribution required from the lead is

$$\phi_{max} = 58.17 - (180 - 170.52) = 48.69^\circ. \tag{3.8.2.3}$$

- Now Using the relation

$$\phi_{max} = \sin^{-1}(\frac{1-\beta}{1+\beta}) \tag{3.8.2.4}$$

then we get

$$\beta = 0.142 \tag{3.8.2.5}$$

- Lag Compensator Design:The Compensator must have a dc gain of unity to retain the value of Kv that we have already designed by setting K = 10714.

$$z_{clag} = \frac{\omega_{Pm}}{10} = \frac{77.53}{10} = 7.753 \tag{3.8.2.6}$$

$$p_{clag} = z_{clag} * \beta = 1.102 \tag{3.8.2.7}$$

Gain in the lag compensator is

$$K_{clag} = \frac{p_{clag}}{z_{clag}} = 0.1421 \tag{3.8.2.8}$$

- Hence the lag compensator transfer function is

$$G_{clag}(s) = \frac{0.1421(s+7.753)}{s+1.102} \tag{3.8.2.9}$$

- Lead Compensator Design:DC gain for this must be unity.
Relations to find T and β: The Compensator's magnitude at the phase margin frequency ω_{max}

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}} \tag{3.8.2.10}$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} \tag{3.8.2.11}$$

So,To find transfer function

$$z_{lead} = \frac{1}{T_2} = \omega_{Pm} * \sqrt{\beta} = 29.92 \tag{3.8.2.12}$$

$$p_{lead} = \frac{z_{lead}}{\beta} = 205.74, K_{lead} = \frac{p_{lead}}{z_{lead}} = 7.04 \tag{3.8.2.13}$$

- Thus lead compensator transfer function is

$$G_{lead} = \frac{7.04(s + 29.22)}{s + 205.74} \tag{3.8.2.14}$$

- So the overall compensator tranfer function is

$$G_c(s) = G_{clag}(s)G_{lead}(s) \tag{3.8.2.15}$$

$$G_c(s) = \frac{1.000384(s + 7.753)(s + 29.23)}{(s + 1.102)(s + 205.7)} \tag{3.8.2.16}$$

3.8.3. Verifying Lag-lead Compensator using Plots
Solution: Magnitude and Phase plot The following code

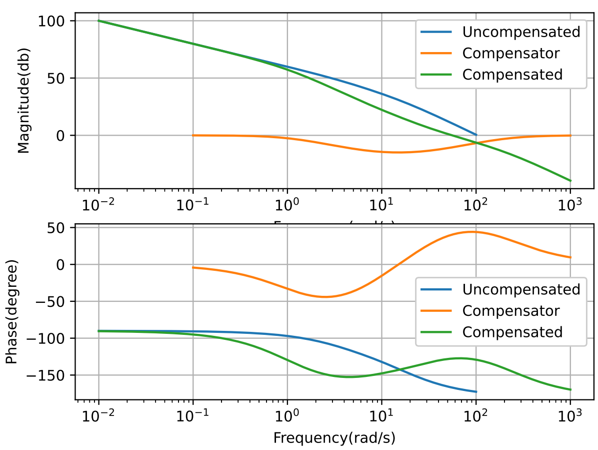


Fig. 3.8.3

```
codes/ee18btech11012/ee18btech11012_2.py
```

3.8.4. Verifying in time domain
Solution: Time response for a unit step function The following code can be verified

```
codes/ee18btech11012/ee18btech11012_3.py
```

3.8.5. Verifying the designed lag-lead compensator
Solution:

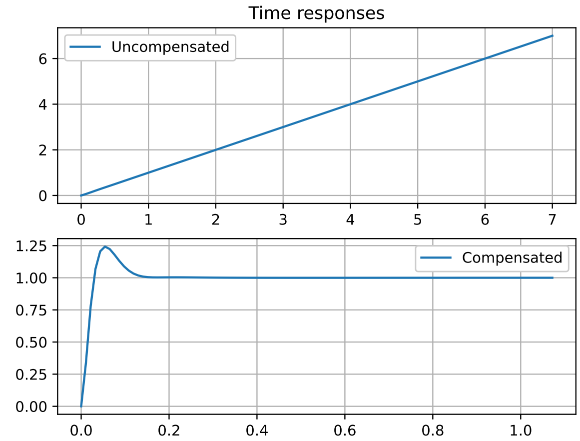


Fig. 3.8.4

Parameter Specification	**Proposed**	**Actual**
Phase Margin	53.17°	53.3994°
K_v	1000	1023.67
Phase Margin frequency	77.53	55.5874

TABLE 3.8.1: Comparing the desired and obtained results

3.9

3.10

3.11

4 PID Controller Design

4.1 PD

4.1. For a unity feedback system shown in Fig. 4.1.1

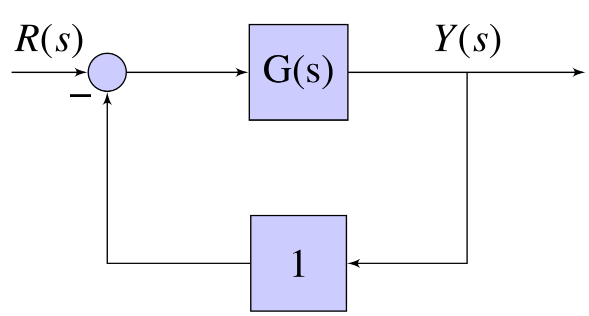


Fig. 4.1.1

$$G(s) = \frac{K}{s(s + 1)} \tag{4.1.1}$$

Design a PD controller such that the phase margin is 45°and appropriate steady state error is less than or equal to $\frac{1}{15}$ units of the final output value. Further the gain crossover frequency of the system must be less than 7.5 rad/s.
Solution: Using TABLE 4.2.1 The gain after cascading the PD controller with G(s) is

$$G_c(s) = \frac{K_P(1+T_d s)K}{s(s+1)} \tag{4.1.2}$$

Type	Poles	Input	Steady State Error
Type 0	0	Step	$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)}$
Type 1	1	Ramp	$e_{ss} = \frac{1}{\lim_{s\to 0} sG(s)}$
Type 2	2	Parabolic	$e_{ss} = \frac{1}{1+\lim_{s\to 0} s^2G(s)}$

TABLE 4.1.1: System Types and Poles at Origin

Using TABLE 4.1.1, (4.1.2) is Type 1 system.

$$e_{ss} = \frac{1}{\lim_{s\to 0} sG_c(s)} \tag{4.1.3}$$

$$e_{ss} \le \frac{1}{15} \lim_{s\to 0} sG_c(s) \tag{4.1.4}$$

$$\implies K_P K \ge \sqrt{15} \tag{4.1.5}$$

For Phase Margin 45°, at gain crossover frequency ω,

$$\tan^{-1}(T_d\omega) - \tan^{-1}(\omega) = -45^\circ \tag{4.1.6}$$

$$\left|G_c(\jmath\omega)\right| = \frac{\sqrt{15}\sqrt{T_d^2\omega^2+1}}{\omega\sqrt{\omega^2+1}} = 1 \tag{4.1.7}$$

By Hit and Trial, one of the best combinations is

$$\omega = 2.893 \tag{4.1.8}$$

$$T_d = -0.71 \tag{4.1.9}$$

Parameters	Required	Obtained
ω	≤ 7.5	2.893
Phase Margin	45°	45°
T_d	Not Given	-0.71

TABLE 4.1.2

4.2. Verify using a Python Plot
Solution: The following code plots Fig. 4.1.2

```
codes/ee17btech11031_pd_ke.py
```

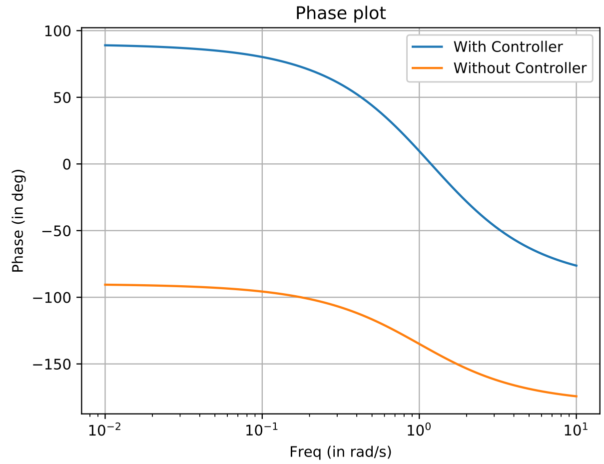


Fig. 4.1.2

4.2 PID

4.2.1. Tabulate the transfer functions of a PID controller and its variants.
Solution: See Table 4.2.1.

Controller	Gain
PID	$K_p\left(1+T_d s+\frac{1}{T_i s}\right)$
PD	$K_p(1+T_d s)$
PI	$K_p\left(1+\frac{1}{T_i s}\right)$

TABLE 4.2.1

4.2.2. For a unity Feedback system

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{4.2.2.1}$$

Design a PD Controller with $K_v = 2$ and Phase Margin 30°
Solution: The gain after cascading the PD Controller with G(s) is

$$G_c(s) = \frac{K_p(1+T_d s)K}{s(s+2)(s+4)(s+6)} \tag{4.2.2.2}$$

Choosing $K_p = 1$ in ,

$$K_v = \lim_{s\to 0} sG_c(s) = 2 \tag{4.2.2.3}$$

$$\implies K = 96 \tag{4.2.2.4}$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\tan^{-1}(T_d\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = -60^\circ \quad (4.2.2.5)$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_d^2\omega^2+1}}{\omega\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \quad (4.2.2.6)$$

By Hit and Trial, one of the best combinations is

$$\omega = 4 \quad (4.2.2.7)$$

$$T_d = 1.884 \quad (4.2.2.8)$$

We get a Phase Margin of 30.31°

4.2.3. Verify using a Python Plot
Solution: The following code plots Fig. 4.2.1

```
codes/ee18btech11021/EE18BTECH11021_3.py
```

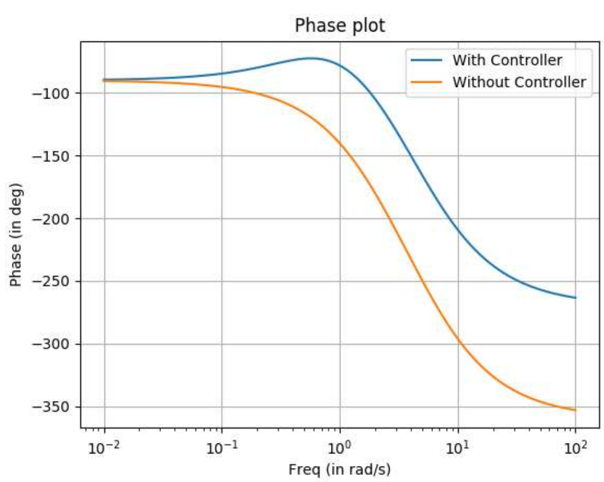


Fig. 4.2.1

4.2.4. Design a PI Controller with $K_v = \infty$ and Phase Margin 30°
Solution: From Table 4.2.1, the open loop gain in this case is

$$G_1(s) = \frac{K_p\left(1+\frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \quad (4.2.4.1)$$

Choose $K_pK = 96$. Then

$$G_1(s) = \frac{96(T_i s+1)}{T_i s^2(s+2)(s+4)(s+6)} \quad (4.2.4.2)$$

For Phase Margin 30°, at Gain Crossover Frequency ω

$$\tan^{-1}(T_i\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \quad (4.2.4.3)$$

and

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_i^2\omega^2+1}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \quad (4.2.4.4)$$

By Hit and Trial, one of the best combinations is

$$\omega = 0.75 \quad (4.2.4.5)$$

$$T_i = 2.713 \quad (4.2.4.6)$$

We get a Phase Margin of 25.53°

4.2.5. Verify using a Python Plot
Solution: The following code plots Fig. 4.2.2.

```
codes/ee18btech11021/EE18BTECH11021_4.py
```

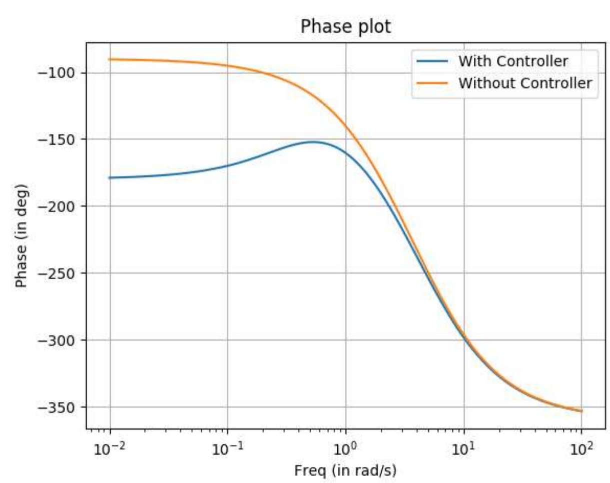


Fig. 4.2.2

4.2.6. Design a PID Controller with $K_v = \infty$ and Phase Margin 30°
Solution:

$$G_1(s) = \frac{K_p\left(1 + T_d s + \frac{1}{T_i s}\right) K}{s(s+2)(s+4)(s+6)} \quad (4.2.6.1)$$

Choose $K_p K = 96$. The open loop gain is

$$G_1(s) = \frac{96(T_i T_d s^2 + T_i s + 1)}{T_i s^2 (s+2)(s+4)(s+6)} \quad (4.2.6.2)$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\tan^{-1}\left(\frac{T_i \omega}{1 - TiT_d w^2}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \quad (4.2.6.3)$$

$$\left|G_1(j\omega)\right| = \frac{96\sqrt{(1 - TiT_d\omega^2)^2 + T_i^2}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \quad (4.2.6.4)$$

By Hit and Trial, one of the best combinations is

$$\omega = 1 \quad (4.2.6.5)$$

$$T_i = 1.738 \quad (4.2.6.6)$$

$$T_d = 0.4 \quad (4.2.6.7)$$

We get a Phase Margin of 30°

4.2.7. Verify using a Python Plot

Solution: The following code plots Fig. 4.2.3

```
codes/ee18btech11021/EE18BTECH11021_5.py
```

4.3 Position Control System

4.3.1. A position control system is to be designed such that maximum peak overshoot is less than 25 %. Further, appropriate error constant should be 50. For the motor to be used, load and torque speed curve is shown below, where, J_1 = 2 kg-m2 , J_2 = 18 kg-m2 , f_1 = 2 N-m-s/rad, f_2 = 36 N-ms/rad. (Although obvious, consider position as the controlled variable and armature voltage as the manipulated variable.).

(i) Design a lead compensator for the system.

(ii) Design a lag compensator for the system.

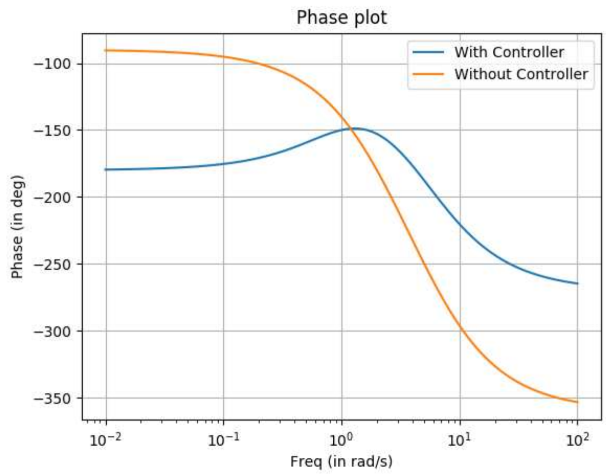


Fig. 4.2.3

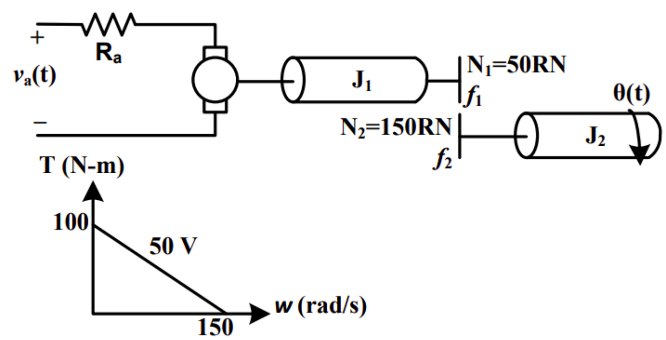


Fig. 4.3.1

Solution: Solving the system shown in 4.3.1, From speed-torque curve of DC Motor in figure. Let $\mathbf{T_m}$ be the torque exerted by DC Motor. 4.3.1

$$\mathbf{T_m} = \frac{\mathbf{K_T}}{R_a} V_a - \frac{\mathbf{K_T.K_v^T}}{R_a}\boldsymbol{\omega_1} \quad (4.3.1.1)$$

$$\mathbf{T_m} = 2V_a - \frac{2}{3}\boldsymbol{\omega_1} \quad (4.3.1.2)$$

Change in torque across ends = torque applied on load + viscous friction. On J_1 at one end torque T_m is applied and at the other end T_1 exists.

$$\mathbf{T_m} = \mathbf{T_1} + J_1\ddot{\boldsymbol{\theta}}_1 + f_1\dot{\boldsymbol{\theta}}_1 \quad (4.3.1.3)$$

Variable	Description
$\mathbf{T_m}$	Torque applied by motor on left of J_1
$\mathbf{T_1}$	Torque existing on right of J_1
$\mathbf{T_2}$	Torque existing on left of J_2
$\boldsymbol{\theta_1}$	Position vector of J_1
$\boldsymbol{\theta_2}$	Position vector of J_2
V_a	Armature voltage of DC Motor
$\boldsymbol{\omega_1}$	Angular velocity of J_1
J_1	Moment of Intertia of first load
J_2	Moment of Intertia of second load
f_1	Viscous friction on J_1
f_2	Viscous friction on J_2

TABLE 4.3.1: List of Variables

Similarly for J_2

$$\mathbf{T_2} = J_2\ddot{\boldsymbol{\theta}}_2 + f_2\dot{\boldsymbol{\theta}}_2 \tag{4.3.1.4}$$

$$\mathbf{T_2} = \frac{N_2}{N_1}\mathbf{T_1} \text{ (Gear Train Formula)} \tag{4.3.1.5}$$

$$\boldsymbol{\theta_2} = \frac{N_1}{N_2}\boldsymbol{\theta_1} \text{ (Gear Train Formula)} \tag{4.3.1.6}$$

Vector/Matrix	Dimension
$\boldsymbol{\theta_1}\&\boldsymbol{\theta_2}$	1X3
$\boldsymbol{\theta}$	2X3
$\mathbf{T_m}$	1X3
$\mathbf{K}$	2X3
$\mathbf{K_v}$	1X3
$\mathbf{K_T}$	1X3

TABLE 4.3.2: Vectors and Matrices

Converting to State Space model

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\theta_1} \\ \boldsymbol{\theta_2} \end{pmatrix} \tag{4.3.1.7}$$

$$3\mathbf{T_m} = \begin{pmatrix} 3J_1 & J_2 \end{pmatrix}\ddot{\boldsymbol{\theta}} + \begin{pmatrix} 3f_1 & f_2 \end{pmatrix}\dot{\boldsymbol{\theta}} \tag{4.3.1.8}$$

$$\mathbf{T_m} = 2V_a - \begin{pmatrix} \frac{2}{3} & 0 \end{pmatrix}\dot{\boldsymbol{\theta}} \tag{4.3.1.9}$$

$$\boldsymbol{\theta} = \begin{pmatrix} N_2 & 0 \\ 0 & N_1 \end{pmatrix} K \tag{4.3.1.10}$$

This is the state space model obtained

$$\ddot{\boldsymbol{\theta}} = \begin{pmatrix} \frac{-13}{6} & 0 \\ 0 & \frac{-5}{3} \end{pmatrix}\dot{\boldsymbol{\theta}} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \end{pmatrix} V_a \tag{4.3.1.11}$$

$$\dot{\boldsymbol{\theta}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\dot{\boldsymbol{\theta}} \tag{4.3.1.12}$$

On solving the above State Space Model

$$V_a(s) = (6s + 10)(s\theta_2(s)) \tag{4.3.1.13}$$

$$G(s) = K\frac{\theta(s)}{V_a(s)} = \frac{K}{2s(3s+5)} \tag{4.3.1.14}$$

From Error Constant K = 500

$$G(s) = \frac{\theta(s)}{V_a(s)} = 250\frac{1}{s(3s+5)} \tag{4.3.1.15}$$

$$\zeta = 0.0695 \tag{4.3.1.16}$$

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 81.6\% \tag{4.3.1.17}$$

$$\phi_M = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}\right) \tag{4.3.1.18}$$

$$\phi_{max} = 39.5^\circ - 7.35^\circ + \text{ correction factor} \tag{4.3.1.19}$$

$$\phi_{max} = 57^\circ \tag{4.3.1.20}$$

Specifications	Actual	Expected
OS%	81.6%	25%
ζ	0.0695	0.403
ϕ_m	7.35°	39.5°

TABLE 4.3.3: Table of Specifications

Designing a lead compensator

$$G_c(s) = \frac{1}{a}\frac{s + \frac{1}{T}}{s + \frac{1}{aT}}(a < 1) \tag{4.3.1.21}$$

$$\sin\phi_{max} = \frac{a-1}{a+1} \tag{4.3.1.22}$$

$$a = 0.1 \tag{4.3.1.23}$$

$$|G(j\omega_c)| = \frac{1}{\sqrt{a}} = 10dB \quad (4.3.1.24)$$

$$\omega_c = 5^\circ \text{ (Refer figure 4.3.3)} \quad (4.3.1.25)$$

$$T = \frac{1}{\omega_c\sqrt{a}} = 0.632 \quad (4.3.1.26)$$

$$G_c(s) = 10\frac{s+1.6}{s+16} \quad (4.3.1.27)$$

$$G(s)G_c(s) = 2500\frac{s+1.6}{s(3s+5)(s+16)} \quad (4.3.1.28)$$

Designing a lag compensator

$$G_c(s) = \frac{1}{b}\frac{s+\frac{1}{T}}{s+\frac{1}{bT}}(b>1) \quad (4.3.1.29)$$

$$\phi_{max} = 39.5^\circ - 7.35^\circ + correction factor \quad (4.3.1.30)$$

$$\phi_{max} = 45^\circ \quad (4.3.1.31)$$

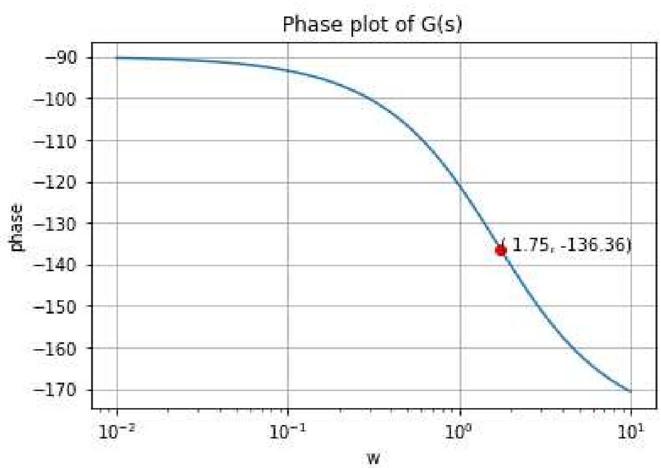


Fig. 4.3.2: Phase plot of G(s)

ω_c = Frequency at which phase of bode plot of G(s) is -180 +ϕ_{max} i.e. -135 °
$\omega_c = 1.75 rad/sec$ as in Figure 4.3.2
We place the zero at

$$\omega = 0.2\omega_c = 0.35 rad/sec \quad (4.3.1.32)$$

$$\implies T = 2.85 \quad (4.3.1.33)$$

codes/ee18btech11001/ee18btech11001_1.py

The magnitude of $G(j\omega)$ at the new gain crossover frequency $\omega_c = 1.75 rad/sec$ is 26 dB

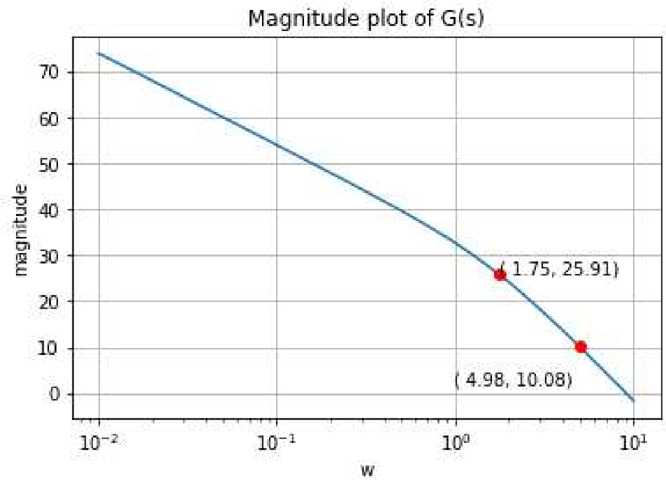


Fig. 4.3.3: Magnitude plot of G(s)

as in figure 4.3.3.In order to have ω_c as the new gain crossover frequency, the lag compensator must give an attenuation of -26db at ω_c

$$-20\log b = -26dB \quad (4.3.1.34)$$

$$b = 19.95 \approx 20 \quad (4.3.1.35)$$

$$G_c(s) = 0.05\frac{s+0.35}{s+0.0175} \quad (4.3.1.36)$$

$$G(s)G_c(s) = 12.5\frac{s+0.35}{s(3s+5)(s+0.0175)} \quad (4.3.1.37)$$

Performance Evaluation of compensators
The following code plots the performance curves

codes/ee18btech11001/ee18btech11001_2.py

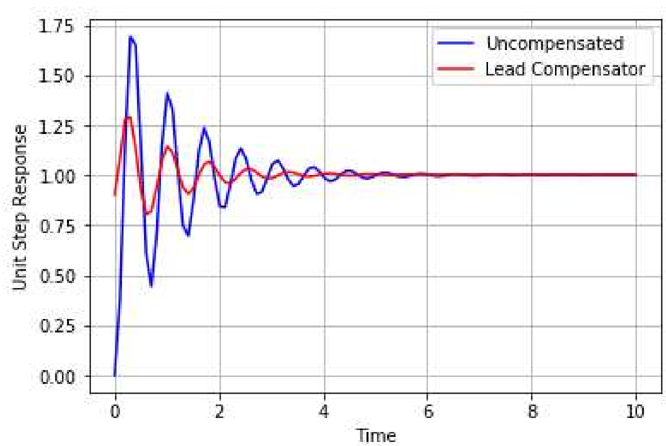


Fig. 4.3.4: Performance of Lead Compensator

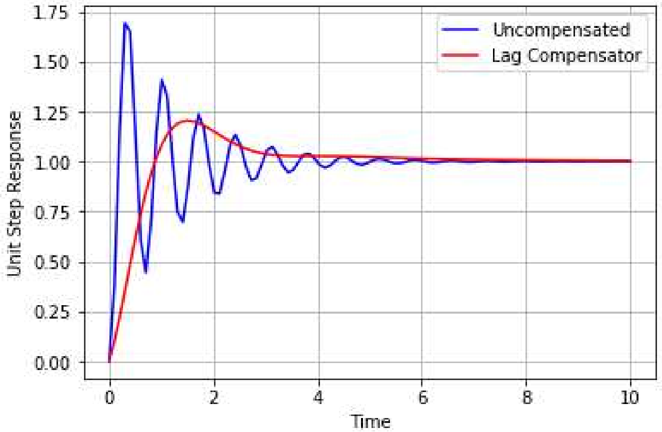


Fig. 4.3.5: Performance of Lag Compensator

Compensator	**Actual OS%**	**Expected OS%**
Lead Compensator	26%	25%
Lag Compensator	25%	24%

TABLE 4.3.4: Performance comparison

Figures 4.3.4 and 4.3.5 show the reduced overshoot & settling time for unit step input.

This verifies that the designed lead and lag compensators work as per specifications.