

# Control Systems

G V V Sharma\*

## CONTENTS

1	Circuit Design	1
2	Assignment	5

**Abstract**—This manual is an introduction to control systems based on GATE problems. Links to sample Python codes are available in the text.

Download python codes using

svn co <https://github.com/gadepall/school/trunk/control/codes>

## 1 CIRCUIT DESIGN

1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume 'A' as 'G' and 'β' as 'H')

**Solution:** Let  $G(f)$  be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20 \log(f) & 10^5 < f < 10^6 \\ 320 - 40 \log(f) & 10^6 < f < 10^7 \\ 460 - 60 \log(f) & 10^7 < f \end{cases} \quad (1.1.1)$$

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \quad (1.1.2)$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade and When a zero is encountered the slope**

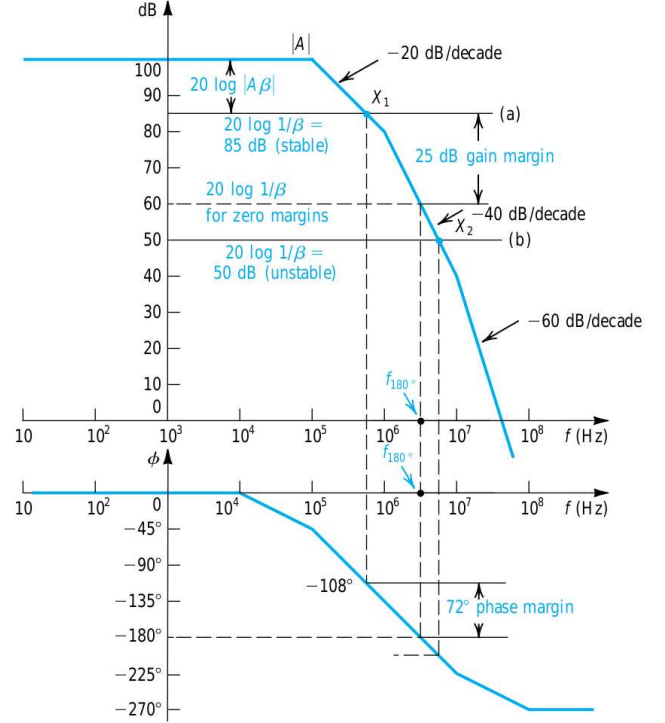


Fig. 1.1: Magnitude and Phase Bode Plot

**always increases by 20 dB/decade.** So, by observing Fig. 1.1 it can be concluded that we are having Poles at  $f = 10^5 \text{ Hz}$ ,  $10^6 \text{ Hz}$ ,  $10^7 \text{ Hz}$  and No Zeros.

So, the Open-Loop Transfer Function  $G(f)$  is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (1.1.3)$$

1.2. Calculate the Phase of Open-Loop Transfer Function.

**Solution:**

$$\phi(f) = - \left[ \tan^{-1} \left( \frac{f}{10^5} \right) + \tan^{-1} \left( \frac{f}{10^6} \right) + \tan^{-1} \left( \frac{f}{10^7} \right) \right] \quad (1.2.1)$$

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

1.3. Find the PM from Fig. 1.1, given that the feedback gain  $H(f)$  is constant and given by

$$20 \log \left( \frac{1}{H(f)} \right) = 85 \text{ dB} \quad (1.3.1)$$

$$\text{or, } H(f) = 5.623 \times 10^{-5}. \quad (1.3.2)$$

**Solution:** From the figure,

$$20 \log |G(f_1)| = 85 \text{ dB} \quad (1.3.3)$$

$$\Rightarrow 20 \log |G(f_1)| = 20 \log \left( \frac{1}{H(f_1)} \right) \quad (1.3.4)$$

$$\text{or, } |G(f_1)H(f_1)| = 1 \quad (1.3.5)$$

and

$$f_1 = 0.493 \text{ MHz}, \quad (1.3.6)$$

from (1.4.4) and (1.1.3). Also,

$$\therefore \angle H(f) = 0, \forall f \quad (1.3.7)$$

$$\angle G(f_1)H(f_1) = \angle G(f_1) = -108^\circ \quad (1.3.8)$$

$$\Rightarrow PM = 180^\circ - 108^\circ = 72^\circ \quad (1.3.9)$$

using (1.3.6) in (1.2.1).

1.4. Find the GM.

**Solution:** The crossover frequency  $f_\pi$  is defined as

$$\angle G(f_\pi)H(f_\pi) = 180^\circ \quad (1.4.1)$$

$$\Rightarrow \angle G(f_\pi) = 180^\circ \quad (1.4.2)$$

$$\Rightarrow f_\pi = 3.34 \text{ MHz} \quad (1.4.3)$$

by solving (1.2.1). From Fig. 1.1,

$$20 \log |G(f_\pi)| = 60 \text{ dB} \quad (1.4.4)$$

$$\Rightarrow 20 \log |G(f_\pi)| - 20 \log \left( \frac{1}{H(f_\pi)} \right) = (60 - 85) \text{ dB} \quad (1.4.5)$$

$$\Rightarrow GM = |20 \log |G(f_\pi)H(f_\pi)|| = 25 \text{ dB} \quad (1.4.6)$$

1.5. Break the Transfer Function  $G(f)$  into Simple Blocks and Create a Block Diagram for  $G(f)$ .

**Solution:**

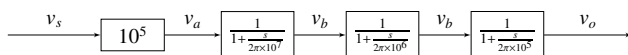


Fig. 1.5

1.6. Find the Gain of RC-Circuit in Fig. 1.6 and identify the pole location.

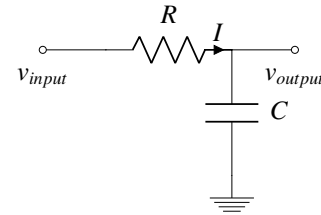


Fig. 1.6

**Solution:**

$$v_o = v_i \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \quad (1.6.1)$$

$$\Rightarrow \frac{v_o}{v_i} = \frac{1}{1 + sCR} \quad (1.6.2)$$

Thus, there is a pole at

$$s = -\frac{1}{RC} \quad (1.6.3)$$

1.7. Find the Gain of Operational Amplifier. The circuit diagram of Equivalent Circuit is 1.7.

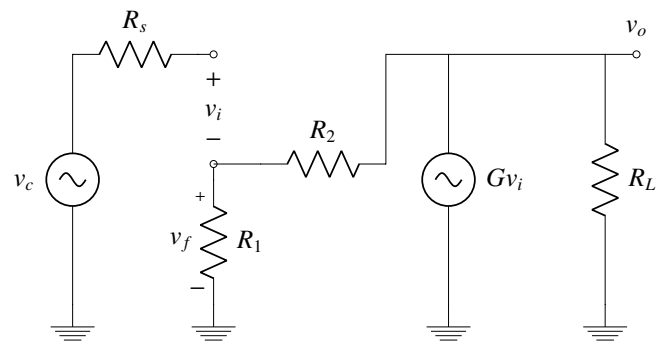


Fig. 1.7

**Solution:**

Applying KVL and KCL,

$$v_o = Gv_i \quad (1.7.1)$$

As no current flows through  $R_s$ ,

$$v_i = v_c - v_f \quad (1.7.2)$$

$$v_f = \frac{R_1}{R_1 + R_2} v_o \quad (1.7.3)$$

$$v_i = \frac{v_o}{G} \quad (1.7.4)$$

$$\frac{v_o}{G} = v_c - \frac{R_1}{R_1 + R_2} v_o \quad (1.7.5)$$

$$\frac{v_o}{v_c} = \frac{G}{1 + G \frac{R_1}{R_1 + R_2}} \quad (1.7.6)$$

So, Gain of the Circuit is  $\frac{G}{1 + G \frac{R_1}{R_1 + R_2}}$

1.8. Design a Circuit Model that follows the Transfer Function  $G(f)$

**Solution:**

Our Design for Modelling the Transfer Function is based on Poles of RC-Circuit and Gain of Operational Amplifier.

So, the Circuit Diagram is,

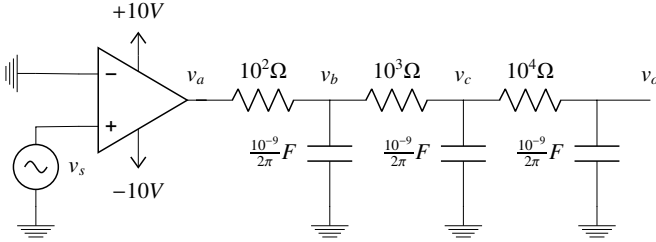


Fig. 1.8

Assuming, Open-Loop Gain of Operational Amplifier is  $10^5$  and also assuming Operational Amplifier doesn't have any Poles.

Equivalent Circuit of the circuit is

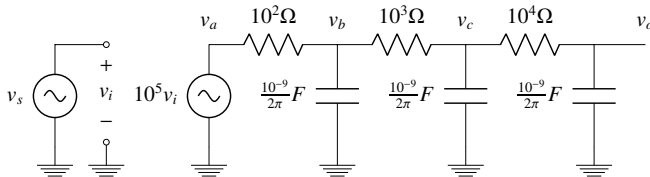


Fig. 1.8

The cascade of RC Circuits are used to introduce poles in the circuit and Op-Amp are used to achieve the Gain required.

At the Operational Amplifier,

$$v_i = v_s \quad (1.8.1)$$

$$v_a = 10^5 v_i \quad (1.8.2)$$

$$v_a = 10^5 v_s \quad (1.8.3)$$

At the first RC-Circuit,

$$2\pi RC = 10^{-7} \quad (1.8.4)$$

$$v_b = \frac{v_a}{1 + j\frac{f}{10^7}} \quad (1.8.5)$$

$$v_b = \frac{10^5 v_i}{1 + j\frac{f}{10^7}} \quad (1.8.6)$$

At the second RC-Circuit,

$$2\pi RC = 10^{-6} \quad (1.8.7)$$

$$v_c = \frac{v_b}{1 + j\frac{f}{10^6}} \quad (1.8.8)$$

$$v_c = \frac{10^5 v_i}{(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (1.8.9)$$

At the third RC-Circuit,

$$2\pi RC = 10^{-5} \quad (1.8.10)$$

$$v_o = \frac{v_c}{1 + j\frac{f}{10^5}} \quad (1.8.11)$$

$$v_o = \frac{10^5 v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (1.8.12)$$

The RC Circuits introduces poles at  $f = 10^7 \text{ Hz}$ ,  $10^6 \text{ Hz}$ ,  $10^5 \text{ Hz}$  respectively from left to right and Op-Amp introduced a Gain =  $10^5$ . So, the value of  $v_o$  is

$$v_o = \frac{10^5 v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (1.8.13)$$

So, Open-Loop Gain is

$$G = \frac{10^5}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (1.8.14)$$

1.9. Design a Circuit Model that follows the Feedback Transfer Function  $H(f)$

**Solution:**

On Bode Plot is  $H$  is independent of frequency. So,  $H$  should not involve any Reactive Elements. So,  $H$  is a combination of Resistors or

a Voltage Divider.

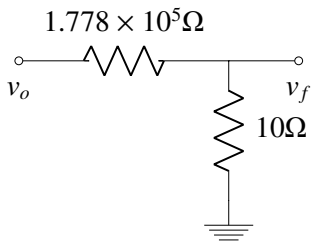


Fig. 1.9

$$v_f = \frac{10}{10 + 1.778 \times 10^5} \times v_o \quad (1.9.1)$$

$$v_f \approx 5.623 \times 10^{-5} v_o \quad (1.9.2)$$

$$\frac{v_f}{v_o} \approx 5.623 \times 10^{-5} \quad (1.9.3)$$

$$H(f) = 5.623 \times 10^{-5} \quad (1.9.4)$$

1.10. Draw the Magnitude and Phase Bode Plots of  $G(f)$

**Solution:** Magnitude Plot is 1.10

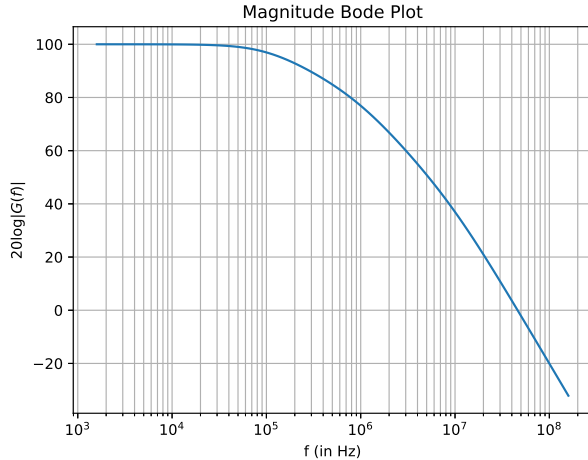


Fig. 1.10: Magnitude Bode Plot

Phase Plot is 1.10

Python Code for Magnitude and Phase Bode Plots is at

codes/ee18btech11014/Bode\_Plot.py

1.11. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits. Also draw its Equivalent Circuit

**Solution:**

The Closed-Loop Circuit is

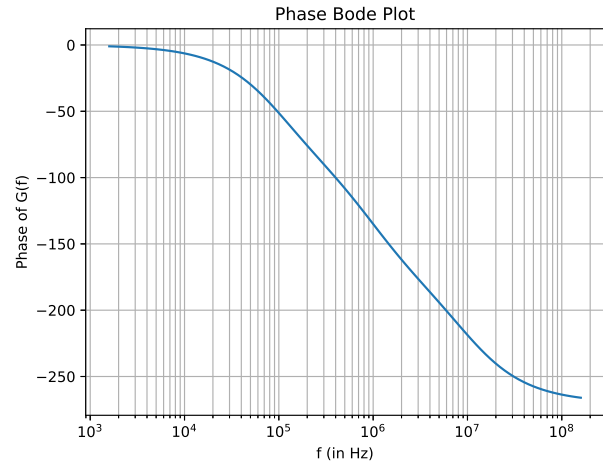


Fig. 1.10: Phase Bode Plot

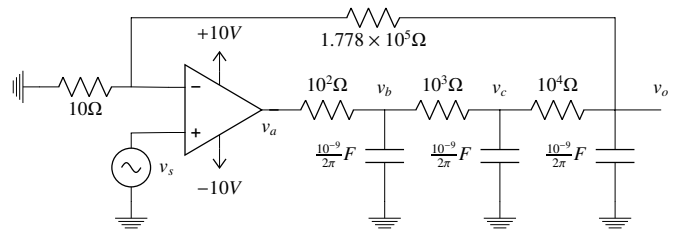


Fig. 1.11

The Equivalent Circuit of Closed-Loop Circuit is

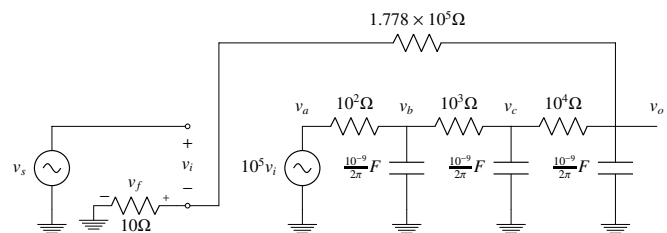


Fig. 1.11

From the Equivalent Circuit Diagram,

$$G = \frac{v_o}{v_i} = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (1.11.1)$$

$$H = \frac{v_f}{v_o} = 5.623 \times 10^{-5} \quad (1.11.2)$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \quad (1.11.3)$$

$$\frac{v_o}{G} = v_s - H v_o \quad (1.11.4)$$

$$\frac{v_o}{v_s} = \frac{G}{1 + GH} \quad (1.11.5)$$

So, the Closed-Loop Gain,

$$T = \frac{v_o}{v_s} = \frac{10^5}{5.623 + \left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (1.11.6)$$

## 2 ASSIGNMENT

- 2.1. Find the frequencies for which phase margins are  $90^\circ$  and  $45^\circ$  respectively?

**Solution:**  $\because \angle H(f) = 1$ , Let Phase Margin be  $\alpha = 90^\circ$ . Then,

$$\angle G(f_{90})H(f_{90}) = \angle G(f_{90}) = 90^\circ - 180^\circ \quad (2.1.1)$$

$$= -90^\circ \quad (2.1.2)$$

$$\Rightarrow |G(f_{90})H(f_{90})| = 1 \quad (2.1.3)$$

So, by the definition of Phase-Margin, at  $\phi = -90^\circ$ ,  $|GH| = 1$ . The value of  $\phi = -90^\circ$  between poles  $f = 10^5 \text{ Hz}$ ,  $10^6 \text{ Hz}$ . Assuming the Poles are farther apart,

$$\tan^{-1}\left(\frac{f}{10^7}\right) \approx 0 \quad (2.1.4)$$

where  $10^5 < f < 10^6$

So,

$$-\tan^{-1}\left(f/10^5\right) - \tan^{-1}\left(f/10^6\right) = -90 \quad (2.1.5)$$

$$\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) = 90 \quad (2.1.6)$$

$$\tan^{-1}\left(f/10^5\right) = 90 - \tan^{-1}\left(f/10^6\right) \quad (2.1.7)$$

$$\tan^{-1}\left(f/10^5\right) = \cot^{-1}\left(f/10^6\right) \quad (2.1.8)$$

$$\tan^{-1}\left(f/10^5\right) = \tan^{-1}\left(10^6/f\right) \quad (2.1.9)$$

$$f^2 = 10^{11} \quad (2.1.10)$$

$$f = 3.162 \times 10^5 \quad (2.1.11)$$

So, the approximate value of  $f$  at which Phase Margin is  $90^\circ$  is  $f = 3.162 \times 10^5 \text{ Hz}$ .

Similarly let Phase Margin be  $\alpha = 45^\circ$ . Then,

$$\alpha = \phi - (-180^\circ) \quad (2.1.12)$$

$$\phi = -180^\circ + \alpha \quad (2.1.13)$$

$$\phi = -135^\circ \quad (2.1.14)$$

So, by the definition of Phase-Margin, at  $\phi = -135^\circ$ ,  $|GH| = 1$ . The value of  $\phi = -135^\circ$  approximately at poles  $f = 10^6 \text{ Hz}$ .

So, the approximate value of  $f$  at which Phase Margin is  $45^\circ$  is  $f = 10^6$ .

- 2.2. Find the minimum values of Closed-Loop Voltage Gain for which phase margins are  $90^\circ$  and  $45^\circ$  respectively

**Solution:**

For  $\alpha = 90^\circ$ ,

$$f = 3.162 \times 10^5 \quad (2.2.1)$$

By substituting  $f$  in Open-Loop Gain  $G(f)$  (assuming poles are far part),

$$G(f) = 200 - 20 \log(3.162 \times 10^5) \quad (2.2.2)$$

$$G(f) = 90 \text{ dB} \quad (2.2.3)$$

$$G = 3.1625 \times 10^4 \quad (2.2.4)$$

At that  $f = 3.162 \times 10^5$ ,

$$H = \frac{1}{G} \quad (2.2.5)$$

$$H = 3.162 \times 10^{-5} \quad (2.2.6)$$

The minimum value of Closed-Loop Gain occurs at  $|GH| \gg 1$  and the value of Closed-Loop Gain is  $T = \frac{1}{H}$

$$T = \frac{1}{H} = 3.1625 \times 10^4 \quad (2.2.7)$$

**So, The minimum value of Closed-Loop Gain with Phase Margin equal to  $\alpha = 90^\circ$  is  $T_{min} = 3.1625 \times 10^4$ .**

For  $\alpha = 45^\circ$ ,

$$f = 10^6 \quad (2.2.8)$$

By substituting  $f$  in Open-Loop Gain  $G(f)$

(assuming poles are far part),

$$G(f) = 200 - 20\log(10^6) \quad (2.2.9)$$

$$G(f) = 80\text{dB} \quad (2.2.10)$$

$$G = 10^4 \quad (2.2.11)$$

At that  $f = 10^6$ ,

$$H = \frac{1}{G} \quad (2.2.12)$$

$$H = 10^{-4} \quad (2.2.13)$$

The minimum value of Closed-Loop Gain occurs at  $|GH| \gg 1$  and the value of Closed-Loop Gain is  $T = \frac{1}{H}$

$$T = \frac{1}{H} = 10^4 \quad (2.2.14)$$

**So, The minimum value of Closed-Loop Gain with Phase Margin equal to  $\alpha = 45^\circ$  is  $T_{min} = 10^4$ .**

2.3. Design a Feedback circuit for Phase Margin  $\alpha = 45^\circ$ .

**Solution:**

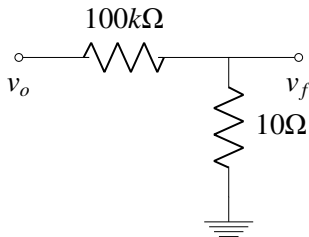


Fig. 2.3

$$v_f = \frac{10}{10 + 10^5} \times v_o \quad (2.3.1)$$

$$v_f \approx 10^{-4} v_o \quad (2.3.2)$$

$$\frac{v_f}{v_o} \approx 10^{-4} \quad (2.3.3)$$

$$H(f) = 10^{-4} \quad (2.3.4)$$

2.4. Design a Feedback circuit for Phase Margin  $\alpha = 90^\circ$ .

**Solution:**

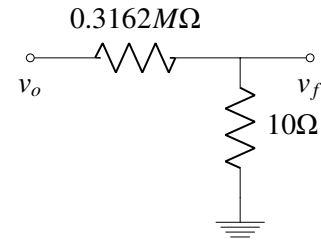


Fig. 2.4

$$v_f = \frac{10}{10 + 3.162 \times 10^5} \times v_o \quad (2.4.1)$$

$$v_f \approx 3.162 \times 10^{-5} v_o \quad (2.4.2)$$

$$\frac{v_f}{v_o} \approx 3.162 \times 10^{-5} \quad (2.4.3)$$

$$H(f) = 3.162 \times 10^{-5} \quad (2.4.4)$$

2.5. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits for phase Margin  $\alpha = 45^\circ$ . Also draw its Equivalent Circuit

**Solution:**

The Closed-Loop Circuit is

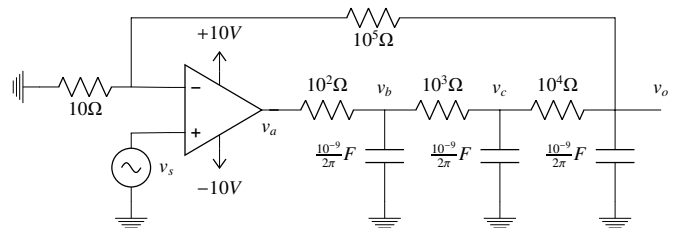


Fig. 2.5

The Equivalent Circuit of Closed-Loop Circuit is

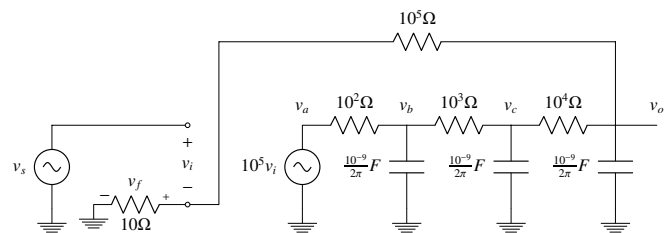


Fig. 2.5

From the Equivalent Circuit Diagram,

$$G = \frac{v_o}{v_i} = \frac{10^5}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (2.5.1)$$

$$H = \frac{v_f}{v_o} = 10^{-4} \quad (2.5.2)$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \quad (2.5.3)$$

$$\frac{v_o}{G} = v_s - Hv_o \quad (2.5.4)$$

$$\frac{v_o}{v_s} = \frac{G}{1 + GH} \quad (2.5.5)$$

So, the Closed-Loop Gain,

$$T = \frac{v_o}{v_s} = \frac{10^5}{10 + (1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (2.5.6)$$

2.6. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits for phase Margin  $\alpha = 90^\circ$ . Also draw its Equivalent Circuit

**Solution:**

The Closed-Loop Circuit is

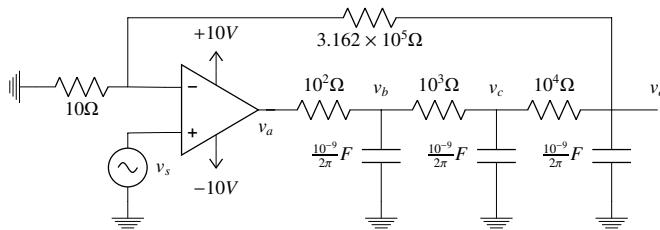


Fig. 2.6

The Equivalent Circuit of Closed-Loop Circuit is

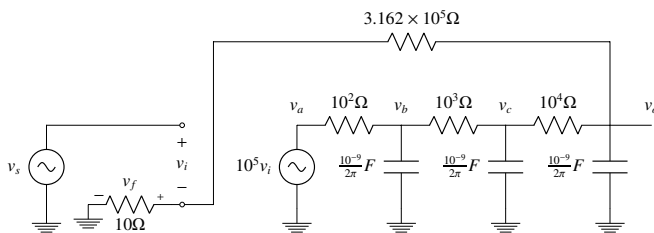


Fig. 2.6