Control Systems

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CONTENTS

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***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Signal Flow Graph

1.1 Mason's Gain Formula

1.1.1. The Block diagram of a system is illustrated in the figure shown, where $X(s)$ is the input and $Y(s)$ is the output. Draw the equivalent signal flow graph.
Solution: The signal flow graph of the block diagram in Fig. 1.1.1.1 is available in Fig. 1.1.1.2

1.1.2. Draw all the forward paths in Fig. 1.1.1.2 and compute the respective gains.
Solution: The forward paths are available in

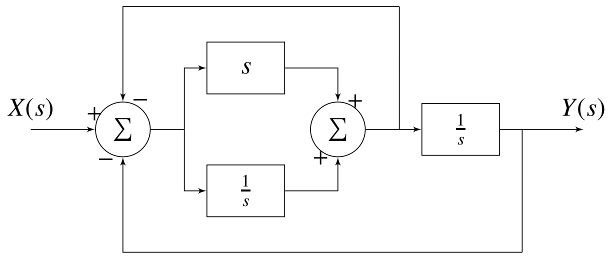


Fig. 1.1.1.1: Block Diagram

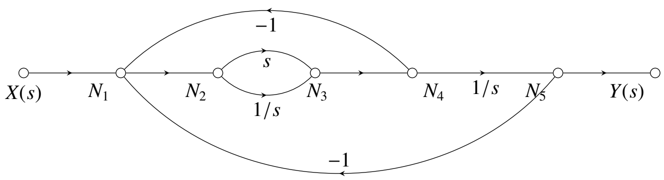


Fig. 1.1.1.2: Signal Flow Graph

Figs. 1.1.2.3 and 1.1.2.4. The respective gains are

$$P_1 = s\left(\frac{1}{s}\right) = 1 \tag{1.1.2.1}$$

$$P_2 = (1/s)(1/s) = 1/s^2 \tag{1.1.2.2}$$

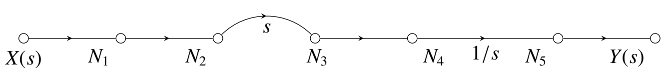


Fig. 1.1.2.3: P_1

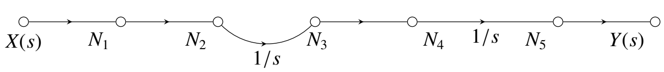


Fig. 1.1.2.4: P_2

1.1.3. Draw all the loops in Fig. 1.1.1.2 and calculate the respective gains.
Solution: The loops are available in Figs. 1.1.3.5-1.1.3.8 and the corresponding gains are

$$L_1 = (-1)(s) = -s \tag{1.1.3.1}$$

$$L_2 = s\left(\frac{1}{s}\right)(-1) = -1 \tag{1.1.3.2}$$

$$L_3 = \left(\frac{1}{s}\right)(-1) = -\frac{1}{s} \tag{1.1.3.3}$$

$$L_4 = \left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(-1) = -\frac{1}{s^2} \tag{1.1.3.4}$$

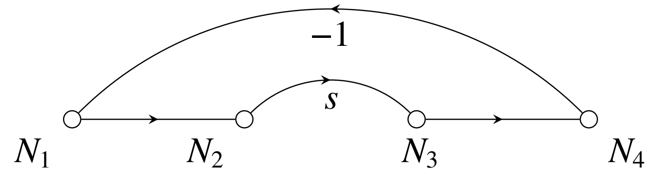


Fig. 1.1.3.5: L_1

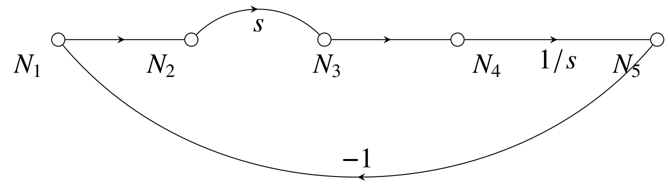


Fig. 1.1.3.6: L_2

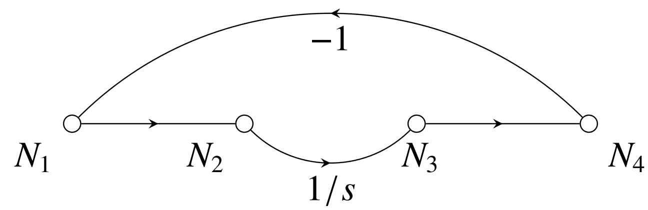


Fig. 1.1.3.7: L_3

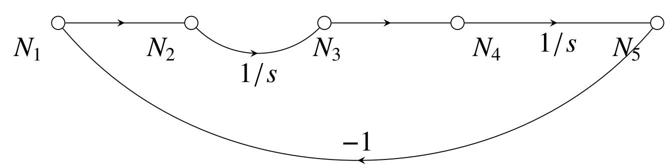


Fig. 1.1.3.8: L_4

1.1.4. State Mason's Gain formula and explain the parameters through a table.
Solution: According to Mason's Gain Formula,

$$T = \frac{Y(s)}{X(s)} \tag{1.1.4.1}$$

$$= \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \tag{1.1.4.2}$$

where the parameters are described in Table 1.1.4

1.1.5. List the parameters in Table 1.1.4 for Fig. 1.1.1.2.
Solution: The parameters are available in Table 1.1.5

Variable	Description
P_i	ith forward path
L_j	jth loop
Δ	$1 - \sum_{L_i} + \sum_{L_i \cap L_j = \phi} L_i L_j - \sum_{L_i \cap L_j \cap L_k = \phi} L_i L_j L_k + \dots$
Δ_i	$1 - \sum_{L_k \cap P_i = \phi} L_k + \sum_{L_k \cap L_j \cap P_i = \phi} L_k L_j - \dots$

TABLE 1.1.4

Path	Value	Parameter	Value	Remarks
P_1	1	Δ_1	1	All loops intersect with P_1
P_2	$\frac{1}{s^2}$	Δ_2	1	All loops intersect with P_2
L_1	$-s$	Δ	$1 - \sum_i L_i$	All loops intersect
L_2	-1			
L_3	-$\frac{1}{s}$			
L_4	-$\frac{1}{s^2}$			

TABLE 1.1.5

1.1.6. Find the transfer function using Mason's Gain Formula.
Solution: From (1.1.4.2) and 1.1.5,

$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{1.1.6.1}$$

$$= \frac{1 + \frac{1}{s^2}}{1 - (-s - 1 - \frac{1}{s} - \frac{1}{s^2})} \tag{1.1.6.2}$$

$$= \frac{s^2 + 1}{s^3 + 2s^2 + s + 1} \tag{1.1.6.3}$$

after simplification.

1.2 Matrix Formula

1.2.1. Write the transition equations in Fig. 1.1.1.2.
Solution: The equations are

$$N_1 = X(s) - N_4 - N_5 \tag{1.2.1.1}$$

$$N_2 = N_1 \tag{1.2.1.2}$$

$$N_3 = N_2(s + 1/s) \tag{1.2.1.3}$$

$$N_4 = N_3 \tag{1.2.1.4}$$

$$Y(s) = N_5 = N_4/s \tag{1.2.1.5}$$

1.2.2. Obtain the state transition matrix from (1.2.1.5)
Solution: The state transition matrix is

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & s + 1/s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/s & 0 \end{pmatrix} \tag{1.2.2.1}$$

1.2.3. State the equivalent matrix form of Mason's gain formula
Solution: Let

$$\mathbf{U} = (\mathbf{I} - \mathbf{T})^{-1} \tag{1.2.3.1}$$

The gain from node m to node n of the graph is U_{nm}

1.2.4. Find the transfer function for the sytem in Fig. 1.1.1.1
Solution: From (1.2.2.1) and (1.2.3.1),

$$\mathbf{I} - \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -s - 1/s & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1/s & 1 \end{pmatrix} \tag{1.2.4.1}$$

$$\implies U_{40} = \frac{\begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -s - 1/s & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1/s \end{vmatrix}}{\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -s - 1/s & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1/s & 1 \end{pmatrix}}, \tag{1.2.4.2}$$

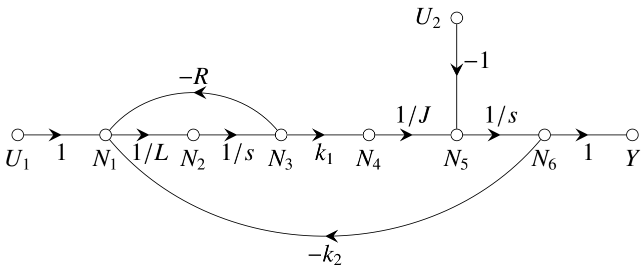
using the cofactor expansion and Problem 1.2.3. The gain is obtained as (1.1.6.3) after expanding the determinants and simplifying.

1.2.5. Write a program to compute the gain using the matrix method.
Solution: The following code computes the transfer function

```
codes/ee18btech11007/MasonsGain.py
```

1.3 Example

1.1. In a system whose signal flow graph is shown in the figure, $U_1(s)$ and $U_2(s)$ are inputs. Find the transfer function $\frac{Y(s)}{U_1(s)}$.



Solution: The desired transfer function is given by

$$\left.\frac{Y(s)}{U_1(s)}\right|_{U_2(s)=0} \tag{1.1.1}$$

The corresponding transition equations are

$$N_1 = U_1 - RN_3 - k_2N_6 \tag{1.1.2}$$
$$N_2 = \frac{N_1}{L} \tag{1.1.3}$$
$$N_3 = \frac{N_2}{s} \tag{1.1.4}$$
$$N_4 = k_1N_3 \tag{1.1.5}$$
$$N_5 = \frac{N_4}{J} \tag{1.1.6}$$
$$N_6 = \frac{N_5}{s} \tag{1.1.7}$$

and the state transition matrix is

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & -R & 0 & 0 & -k_2 \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{s} & 0 \end{pmatrix} \tag{1.1.8}$$

Defining

$$\mathbf{U} = (\mathbf{I} - \mathbf{T})^{-1} \tag{1.1.9}$$

$$= \begin{pmatrix} 1 & 0 & R & 0 & 0 & k_2 \\ \frac{-1}{L} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{s} & 1 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{J} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{s} & 1 \end{pmatrix}^{-1} \tag{1.1.10}$$

the gain of the system is given by

$$U_{50} = \frac{Y(s)}{U_1(s)} = \frac{k_1}{s^2LJ + sRJ + k_1k_2} \tag{1.1.11}$$

computed by

```
codes/ee18btech11041.py
```

2 Bode Plot

2.1 Introduction

2.1. For an LTI system, the Bode plot for its gain defined as

$$G(s) = 20\log|H(s)| \tag{2.1.1}$$

is as illustrated in the Fig. 2.1. Express $G(f)$ in terms of f.

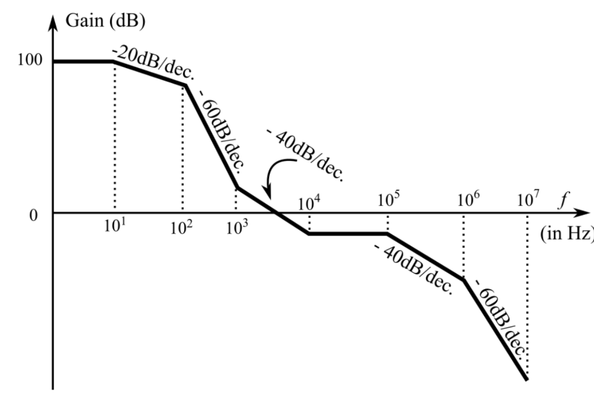


Fig. 2.1

Solution:

$$G(f) = \begin{cases} 100 & 0 < f < 10^1 \\ 120 - 20\log(f) & 10 < f < 10^2 \\ 200 - 60\log(f) & 10^2 < f < 10^3 \\ 140 - 40\log(f) & 10^3 < f < 10^4 \\ -20 & 10^4 < f < 10^5 \\ 180 - 40\log(f) & 10^5 < f < 10^6 \\ 300 - 60\log(f) & 10^6 < f < 10^7 \end{cases} \tag{2.1.2}$$

2.2. Express the slope of $G(f)$ in terms of f.
Solution: The desired slope is

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} \tag{2.2.1}$$

$$\nabla G(f) = \begin{cases} 0 & 0 < f < 10^1 \\ -20 & 10 < f < 10^2 \\ -60 & 10^2 < f < 10^3 \\ -40 & 10^3 < f < 10^4 \\ 0 & 10^4 < f < 10^5 \\ -40 & 10^5 < f < 10^6 \\ -60 & 10^6 < f < 10^7 \end{cases} \quad (2.2.2)$$

2.3. Express the change of slope of $G(f)$ in terms of f.
Solution:
$\Delta(\nabla G(f))$ = Change of slope G(f) at f

$$\Delta(\nabla G(f)) = \begin{cases} -20 & f = 10^1 \\ -40 & f = 10^2 \\ +20 & f = 10^3 \\ +40 & f = 10^4 \\ -40 & f = 10^5 \\ -20 & f = 10^6 \end{cases} \quad (2.3.1)$$

2.4. Tabulate the poles and zeros of $H(s)$ using (2.3.1).
Solution: Table 2.4 provides the details.

f **(Hz)**	$\Delta(\nabla G(f))$	**Pole**	**Zero**
10^1	-20	1	0
10^2	-40	2	0
10^3	20	0	1
10^4	40	0	2
10^5	-40	2	0
10^6	-20	1	0
Total		**6**	**3**

TABLE 2.4

2.5. Obtain the transfer function of $H(s)$.
Solution: From Table 2.4,

$$H(s) = \frac{K(s + j2\pi 10^3)(s + j2\pi 10^4)^2}{(s + j2\pi 10^1)(s + j2\pi 10^2)^2(s + j2\pi 10^5)^2(s + j2\pi 10^6)} \quad (2.5.1)$$

2.6. Justify the above results.
Solution: Let us consider a generalized transfer gain

$$H(s) = k\frac{(s - z_1)(s - z_2)...(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)....(s - p_{n-1})(s - p_n)} \quad (2.6.1)$$

The gain

$$\begin{aligned} G(f) &= 20\log|H(s)| \\ &= 20\log|k| + 20\log|s - z_1| \\ &+ 20\log|s - z_2| + \cdots + 20\log|s - z_m| \\ &- 20\log|s - p_1| - 20\log|s - p_2| \\ &- \cdots - 20\log|s - z_n| \end{aligned} \quad (2.6.2)$$

Substituting $s = \jmath\omega$, for real z_1

$$20\log|s - z_1| = 20\log\left|\sqrt{\omega^2 + z_1^2}\right| \quad (2.6.3)$$

$$= \begin{cases} 20\log|z_1|, & \omega \ll z_1 \\ 20\log|\omega|, & \omega \gg z_1 \end{cases} \quad (2.6.4)$$

Taking the derivative,

$$\frac{d\,(20\log|s - z_1|)}{d\,(\log|\omega|)} = \begin{cases} 0, & \omega \ll z_1 \\ 20, & \omega \gg z_1 \end{cases} \quad (2.6.5)$$

Thus, when a zero is encountered, the gradient of $H(\jmath\omega)$ jumps by +20 in the log scale. When a pole is encountered, the gradient falls by -20. Note that this is a very loose justification, but works well in practice.

2.7. Obtain the Bode plot and the slope plot for $H(s)$ and verify with Fig. 2.1
Solution: Bode Plot of obtained Transfer Function is Fig. **??**, obtained from (2.5.1), is a close reconstruction of Fig. **??**.

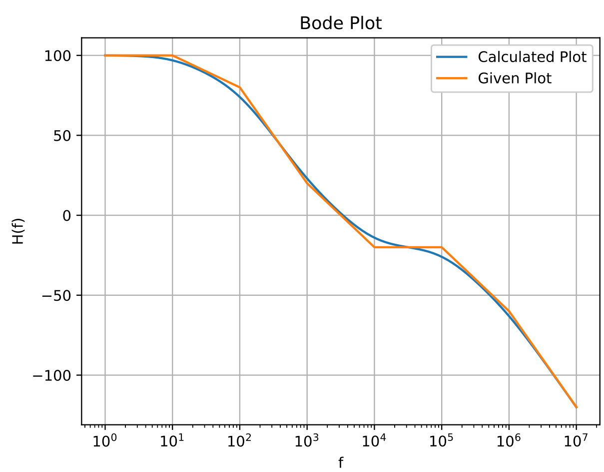


Fig. 2.7

2.2 *Example*

2.2.1. The asymptotic Bode magnitude plot of minimum phase transfer function G(s) is show in

Fig. 2.2.1 . Express $20\log\left|G(\jmath\omega)\right|$ as a function

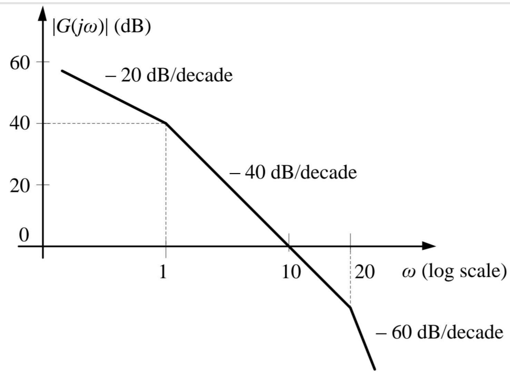


Fig. 2.2.1

of ω using Fig. 2.2.1.
Solution: The desired expression (in dB) is

$$|G(j\omega)| = \begin{cases} 60 - 20(\log(\omega) - \log(0.1)) & 0.1 < \omega < 1 \\ 80 - 40(\log(\omega) - \log(0.1)) & 1 < \omega < 20 \\ 126.02 - 60(\log(\omega) - \log(0.1)) & 20 < \omega \end{cases} \tag{2.2.1.1}$$

2.2.2. Express the slope of $20\log\left|G(\jmath\omega)\right|$ as a function of ω.
Solution: The desired slope is

$$\nabla 20\log|G(j\omega)| = \begin{cases} -20 & \omega < 1 \\ -40 & 1 < \omega < 20 \\ -60 & 20 < \omega \end{cases} \tag{2.2.2.1}$$

2.2.3. Express the change of slope of $20\log\left|G(\jmath\omega)\right|$ as a function of ω.
Solution:

$$\Delta(\nabla 20\log|G(j\omega)|) = \begin{cases} -20 & \omega = 0 \\ -20 & \omega = 1 \\ -20 & \omega = 20 \end{cases} \tag{2.2.3.1}$$

2.2.4. Find the poles and zeros of $G(s)$.
Solution: From (2.2.3.1), the poles are located at 0,1,20. There are no zeros.

2.2.5. Find $G(s)$
Solution:

$$G(s) = \frac{k}{s(1+s)(20+s)} \tag{2.2.5.1}$$

2.2.6. Obtain the Bode plot of $G(s)$ through a python code and compare with the line plot of the expression that you obtained in Problem 2.2.1
Solution: Fig. 2.2.6 shows the Bode plot of the transfer function obtained. The **Line plot** is the approximation of the **caluculated bode plot**.

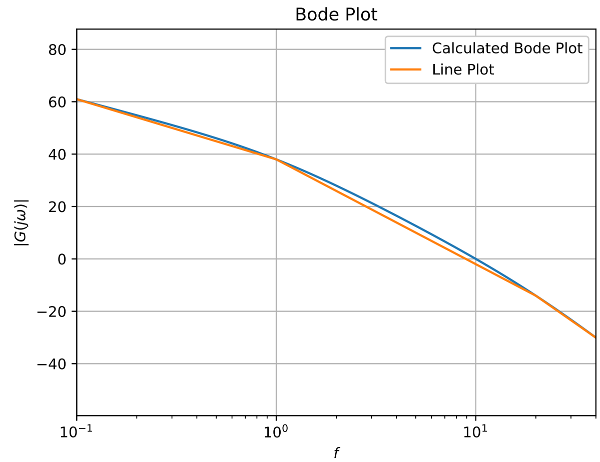


Fig. 2.2.6

2.2.7. Verify if at very high frequency ($\omega \to \infty$), the phase angle $\angle G(j\omega) = -3\pi/2$.
Solution: Phase ϕ is the sum of all the phases corresponding to each pole and zero.

$$\Rightarrow G(j\omega) = \frac{k}{j\omega(1+j\omega)(20+j\omega)} \tag{2.2.7.1}$$

$$\implies \phi = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{20}\right) \tag{2.2.7.2}$$

$$= -90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{20}\right) \tag{2.2.7.3}$$

$$\implies \lim_{\omega\to\infty} \phi = -3\pi/2 \tag{2.2.7.4}$$

2.3 *Phase*

2.3.1. The asymptotic Bode phase plot of

$$G(s) = \frac{k}{(s+0.1)(s+10)(s+p_1)} \tag{2.3.1.1}$$

with k and p_1 both positive, is shown in Fig. 2.3.1. Express it as a piecewise linear function of $\log(\omega)$.

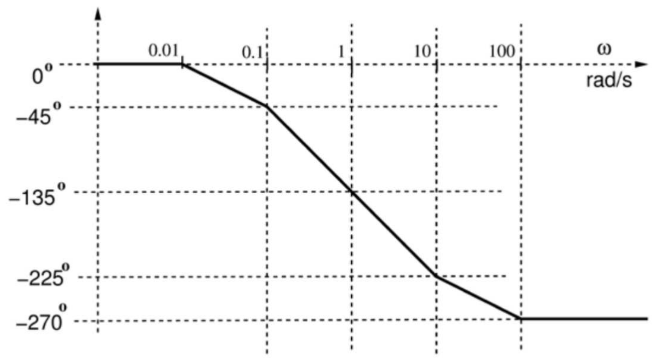


Fig. 2.3.1

Solution: The desired expression is

$$\phi(\omega) = \begin{cases} 0 & 0 < \omega < 0.01 \\ -90 - 45\log(\omega) & 0.01 < \omega < 0.1 \\ -135 - 90\log(\omega) & 0.1 < \omega < 10 \\ -180 - 45\log(\omega) & 10 < \omega < 100 \\ -90 & 100 < \omega \end{cases} \tag{2.3.1.2}$$

2.3.2. Find p_1.
Solution: Let

$$G_1(s) = \frac{1}{(s+p_1)} \tag{2.3.2.1}$$

The equivalent Bode phase is

$$\phi_1(\omega) = \angle G_1(\jmath\omega)$$
$$= \begin{cases} 0 & 0 < \omega < \frac{p_1}{10} \\ -45 \times \left(\log\left(\frac{10\omega}{p_1}\right)\right) & \frac{p_1}{10} < \omega < 10p_1 \\ -90 & 10p_1 < \omega \end{cases} \tag{2.3.2.2}$$

Similarly, let

$$G_2(s) = \frac{k}{(s+0.1)(s+10)}. \tag{2.3.2.3}$$

The equivalent Bode phase is

$$\phi_2(\omega) = \angle G_2(\jmath\omega)$$
$$= \begin{cases} 0 & 0 < \omega < 0.01 \\ -90 - 45\log(\omega) & 0.01 < \omega < 100 \\ -180 & 100 < \omega \end{cases}. \tag{2.3.2.4}$$

Hence, from (2.3.2.2) and (2.3.2.4),

$$\phi(\omega) = \phi_1(\omega) + \phi_2(\omega) \tag{2.3.2.5}$$

From (2.3.1.2) and (2.3.2.4)

$$\phi(\omega) = \phi_2(\omega) \quad 0 < \omega < 0.1 \tag{2.3.2.6}$$
$$\implies \phi_1(\omega) = 0 \quad 0 < \omega < 0.1 \tag{2.3.2.7}$$

This is obvious from Figs. 2.3.2 and Fig. 2.3.1. Thus,

$$\frac{p_1}{10} = 0.1 \implies p_1 = 1 \tag{2.3.2.8}$$

The following code generates Fig. 2.3.2

```
codes/ee18btech11037.py
```

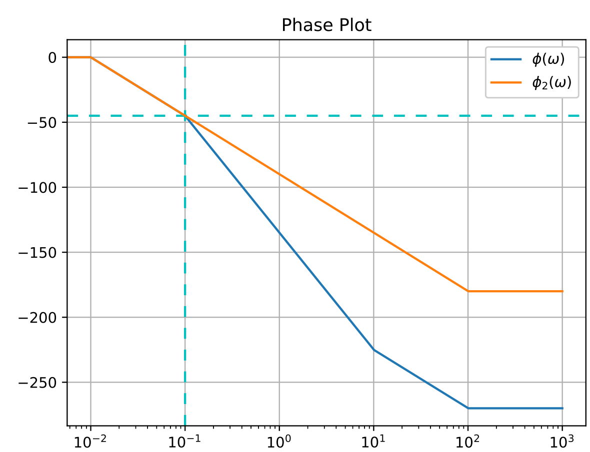


Fig. 2.3.2

2.3.3. Find the value of p_1 using phase of the transfer function.
Solution:

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{p_1}\right) \tag{2.3.3.1}$$

From the plot,

$$-45^\circ = -\tan^{-1}\left(\frac{0.1}{0.1}\right) - \tan^{-1}\left(\frac{0.1}{10}\right) - \tan^{-1}\left(\frac{0.1}{p_1}\right) \tag{2.3.3.2}$$

p_1 is approximately 1, i.e, for p_1 in 0.95 to 1.05 the ϕ is approximately equals to -45°.

3 Second order System

3.1 Damping

3.1.1. List the different kinds of damping for a second order system defined by

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2} \tag{3.1.1.1}$$

where ω is the natural frequency and ζ is the damping factor.
Solution: The details are available in Table 3.1.1

Damping Ratio	Damping Type
$\zeta > 1$	Overdamped
$\zeta = 1$	Critically Damped
$0 < \zeta < 1$	Underdamped
$\zeta = 0$	Undamped

TABLE 3.1.1

3.1.2. Classify the following second-order systems according to damping.
a) $H(s) = \frac{15}{s^2+5s+15}$
b) $H(s) = \frac{25}{s^2+10s+25}$
c) $H(s) = \frac{35}{s^2+18s+35}$
Solution: For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{3.1.2.1}$$

$$\omega^2 = 25, 2\zeta\omega = 10 \tag{3.1.2.2}$$

$$\implies \omega = 1, \zeta = 1 \tag{3.1.2.3}$$

and the system is critically damped. Similarly, the damping factors for other systems in Problem 3.1.2 are calculated and listed in Table 3.1.2

H(s)	ω	ζ	Damping Type
$\frac{35}{s^2+18s+35}$	$\sqrt{35}$	$\sqrt{\frac{81}{35}} > 1$	Overdamped
$\frac{25}{s^2+10s+25}$	5	1	Critically Damped
$\frac{15}{s^2+5s+15}$	$\sqrt{15}$	$\sqrt{\frac{5}{12}} < 1$	Underdamped

TABLE 3.1.2

3.1.3. Find the step response of each $H(s)$ in Table 3.1.2.
Solution:

a) For

$$H(s) = \frac{15}{s^2 + 5s + 15}, \tag{3.1.3.1}$$

the step response is

$$y(t) = 25te^{-5t}u(t) \tag{3.1.3.2}$$

b) For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{3.1.3.3}$$

the step response is

$$y(t) = \frac{30}{\sqrt{35}}e^{\frac{-5t}{2}} \sin\left(\frac{\sqrt{35}}{2}t\right)u(t) \tag{3.1.3.4}$$

c) For

$$H(s) = \frac{35}{s^2 + 18s + 35}, \tag{3.1.3.5}$$

the step response is

$$y(t) = \frac{35}{2\sqrt{46}}\left[e^{(-9+\sqrt{46})t} - e^{(-9-\sqrt{46})t}\right]u(t) \tag{3.1.3.6}$$

3.1.4. Illustrate the effect of damping by plotting the step responses in (3.1.3.2)-(3.1.3.6)
Solution: The following code

```
codes/ee18btech11012.py
```

plots the desired graphs in Fig. 3.1.4.

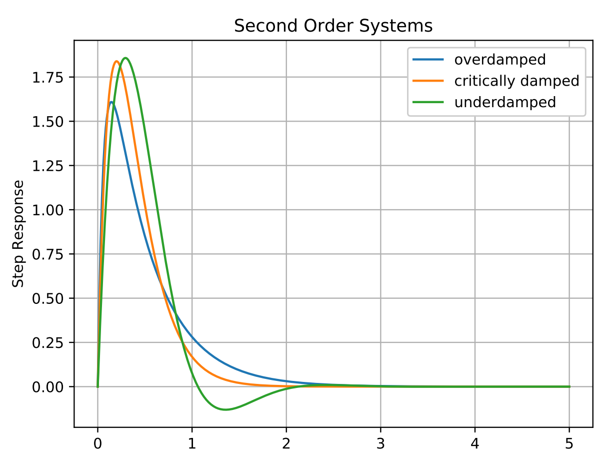


Fig. 3.1.4

3.2 Example

3.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{3.1.1}$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{3.1.2}$$

are at

$$s = -1 \tag{3.1.3}$$

i.e., the left half of s-plane. Hence the system is stable.

3.2. Find and sketch the step response $c(t)$ of the system.
Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1 + 2s + s^2}\right] \tag{3.2.1}$$

$$= \frac{1}{s(1+s)^2} \tag{3.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \tag{3.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \tag{3.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{3.2.5}$$

The following code plots $c(t)$ in Fig. 3.2

```
codes/ee18btech11002/plot.py
```

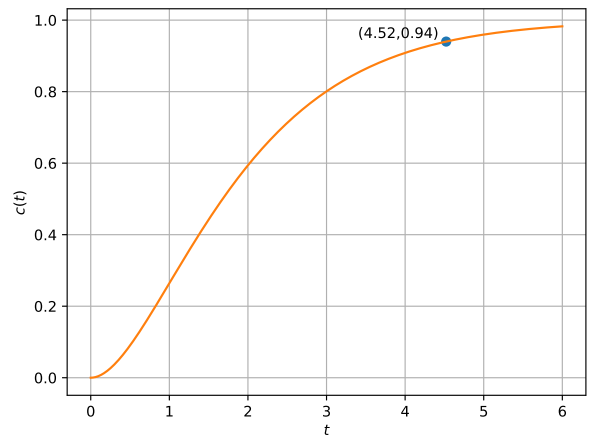


Fig. 3.2

3.3. Find the steady state response of the system using the final value theorem. Verify using 3.2.5
Solution: To know the steady response value of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{3.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1 + s + s^2}\right) = \frac{1}{1 + 0 + 0} = 1 \tag{3.3.2}$$

Using 3.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{3.3.3}$$

$$= (1 - 0 - 0) = 1 \tag{3.3.4}$$

3.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{3.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{3.4.2}$$

3.3 Settling Time

3.3.1. Find the closed loop transfer function for the system in Fig. given that

$$G(s) = \frac{1}{s^2 + 2s} \tag{3.3.1.1}$$

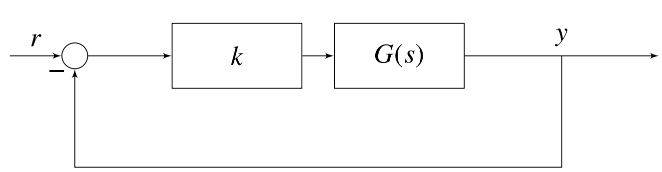


Fig. 3.3.1

Solution: The closed loop transfer function is

$$H(s) = \frac{kG(s)}{1 + kG(s)} \tag{3.3.1.2}$$

$$= \frac{k}{s^2 + 2s + k} \tag{3.3.1.3}$$

after substituting from (3.3.1.1).

3.3.2. Find the step response of the system.
Solution: From (3.3.1.3), the step response is

$$Y(s) = \frac{k}{s^2 + 2s + k}\frac{1}{s} \tag{3.3.2.1}$$

$$\implies y(t) = \Big[1 + \frac{k}{\left(2\sqrt{1-k}\right)\left(-1 + \sqrt{1-k}\right)} e^{\left(-1+\sqrt{1-k}\right)t} + \frac{k}{\left(2\sqrt{1-k}\right)\left(1 + \sqrt{1-k}\right)} e^{\left(-1-\sqrt{1-k}\right)t}\Big] u(t) \quad k \neq 1 \tag{3.3.2.2}$$

and

$$y(t) = \left(1 - e^{-t} - te^{-t}\right) u(t) \quad k = 1 \tag{3.3.2.3}$$

3.3.3. Find the steady state step response of the system usng the final value theorem.
Solution: From (3.3.2.1),

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{3.3.3.1}$$

$$= 1 \tag{3.3.3.2}$$

3.3.4. Find the step response for an overdamped system.
Solution: For an overdamped system, $k < 1$,

$$\implies y(t) = \left[1 - e^{-t}\left\{\frac{\sinh\left(\sqrt{1-k}\right)t}{\sqrt{1-k}} + \cosh\left(\sqrt{1-k}\right)t\right\}\right] u(t) \tag{3.3.4.1}$$

3.3.5. Find the step response for an underdamped system.
Solution: In this case, $k > 1$.

$$\implies y(t) = \left[1 - e^{-t}\left\{\frac{\sin\left(\sqrt{k-1}\right)t}{\sqrt{k-1}} + \cos\left(\sqrt{k-1}\right)t\right\}\right] u(t) \tag{3.3.5.1}$$

3.3.6. Find the step response for a critically damped system.
Solution: For $k = 1$,

$$y(t) = \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{3.3.6.1}$$

3.3.7. The settling time t_s is defined as the first instant where

$$|y(t_s) - y_s s| \leq 0.02 \tag{3.3.7.1}$$

where $y_s s$ is the steady state value of $y(t)$. Find k for which the settling time is minimum.

4 Routh Hurwitz Criterion

4.1 Routh Array

4.1.1. Generate the Routh array for the polynomial,

$$f(s) = s^7 + s^6 + 7s^5 + 14s^4 + 31s^3 + 73s^2 + 25s + 200 \tag{4.1.1.1}$$

Solution:

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \end{vmatrix} \tag{4.1.1.2}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \end{vmatrix} \tag{4.1.1.3}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 0 & 0 & 0 & \end{vmatrix} \tag{4.1.1.4}$$

When such a case is encountered, we take the derivative of the expression formed the the coefficients above it i.e derivative of $8s^4+48s^2+200$.

$$\frac{d}{dx}(8s^4 + 48s^2 + 200) = 32s^3 + 96s$$

The coefficients of obtained expression are placed in the table.

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \end{vmatrix} \tag{4.1.1.5}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \end{vmatrix} \tag{4.1.1.6}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \end{vmatrix} \tag{4.1.1.7}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \\ 200 & & & \end{vmatrix} \tag{4.1.1.8}$$

So, the above one is the Routh-Hurwitz Table.

4.1.2. Find the number of roots of the polynomial in the right half of the s-plane.
Solution: The number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column. From 4.1.1.8, the polynomial in (4.1.1.1) has 4 roots lie on right-side of Imaginary Axis.

4.1.3. Write a Python code for generating each stage of the Routh Table.
Solution: The following code

```
codes/ee18btech11014/ee18btech11014.py
```

generates the various stages.

4.1.4. Find the roots of the polynomial in in (4.1.1.1) and verify that 4 roots are in the right half s-plane.
Solution: The following code generates the necessary roots.

```
codes/ee18btech11014/Roots.py
```

4.2 *Marginal Stability*

4.2.1. Consider a unity feedback system as shown in Fig. 4.2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{4.2.1.1}$$

where k greater than 0. Find its closed loop transfer function.

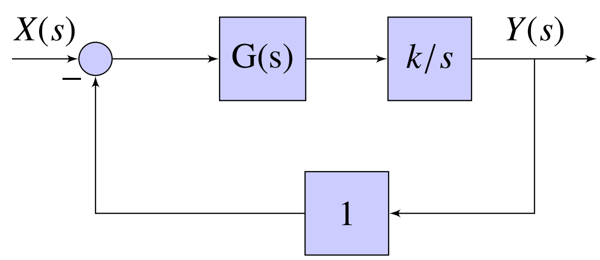


Fig. 4.2.1

Solution: $\because H(s) = 1$ in Fig. 4.2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{4.2.1.2}$$

$$\implies T(s) = \frac{k}{s^3 + 3s^2 + 2s} \tag{4.2.1.3}$$

4.2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{4.2.2.1}$$

$$\implies 1 + \left[\frac{k}{s^3 + 3s^2 + 2s}\right] = 0 \tag{4.2.2.2}$$

$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{4.2.2.3}$$

4.2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on jω axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into

an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0 \quad (4.2.3.1)$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix} \begin{vmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{vmatrix} \quad (4.2.3.2)$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \quad (4.2.3.3)$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \quad (4.2.3.4)$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \quad (4.2.3.5)$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \quad (4.2.3.6)$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 4.2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \quad (4.2.3.7)$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \quad (4.2.3.8)$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \quad (4.2.3.9)$$

$$\implies k = 6 \quad \because k > 0 \quad (4.2.3.10)$$

4.2.4. Repeat the above using the determinant method.

Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 \cdots & & \\ 0 & a_1 & a_3 \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \quad (4.2.4.1)$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \quad (4.2.4.2)$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \quad (4.2.4.3)$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \quad (4.2.4.4)$$

$$\cdots \quad (4.2.4.5)$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (4.2.2.1),

$$D_1 = 1 \neq 0 \quad (4.2.4.6)$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \quad (4.2.4.7)$$

4.2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.

Solution: The following code verifies the stability using the tabular method

```
codes/ee18btech11005_2.py
```

and the following one verifies using the determinant method.

```
codes/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infinity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.
- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the

right side of s-plane.That means it has sign changes.

4.3 Stability

4.3.1. The characteristic equation of linear time invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.1)$$

Find the condition for the system to be BIBO stable using the Routh Array.
solution

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.2)$$

The Routh hurwitz criterion:-

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 3 & k \\ 3 & 1 & 0 \\ \frac{8}{3} & k & 0 \\ \frac{\frac{8}{3}-3k}{\frac{8}{3}} & 0 & 0 \\ k & 0 & 0 \end{vmatrix} \quad (4.3.1.3)$$

From the above array, the given system is stable if

$$k > 0$$
$$\frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0 \quad (4.3.1.4)$$
$$\implies 0 < k < \frac{8}{9} \quad (4.3.1.5)$$

4.3.2. Modify the Python code in Problem 4.2.5 to verify your solution by choosing two different values of k.
Solution: The following code

```
codes/ee18btech11008.py
```

provides the necessary soution for $k = 0.5, 3$.

- $k = 0.5 < \frac{8}{9}$ has no sign changes in first column of its routh array.So the system is stable.
- $k = 3 > \frac{8}{9}$ has 2 sign changes in first column of its routh array.So the system is unstable.

4.4 Example

4.4.1. Consider a standard control system with negative feedback and the transfer functions

$$G(s) = \frac{1}{(s+1)(s+2)} \quad (4.4.1.1)$$

and

$$H(s) = \frac{s+\alpha}{s} \quad (4.4.1.2)$$

Find α so that the closed loop system has poles on the imaginary axis. **Solution:** The Characteristic equation is

$$1 + G(s)H(s) = 0 \quad (4.4.1.3)$$

$$\implies 1 + \left[\frac{1}{(s+1)(s+2)}\right]\left[\frac{s+\alpha}{s}\right] = 0 \quad (4.4.1.4)$$

$$\implies s^3 + 3s^2 + 3s + \alpha = 0 \quad (4.4.1.5)$$

Constructing the routh array for (4.4.1.5),

$$\begin{vmatrix} s^3 \\ s^2 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 3 & \alpha \end{vmatrix} \quad (4.4.1.6)$$

$$\implies \begin{vmatrix} s^3 \\ s^2 \\ s^1 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 3 & \alpha \\ \frac{9-\alpha}{3} & 0 \end{vmatrix} \quad (4.4.1.7)$$

$$\implies \begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 3 & \alpha \\ \frac{9-\alpha}{3} & 0 \\ \alpha & 0 \end{vmatrix} \quad (4.4.1.8)$$

For poles on the imaginary axis any one of the rows should be zero. Thus,

$$\frac{9-\alpha}{3} = 0 \implies \alpha = 9 \quad (4.4.1.9)$$

Substituting $\alpha = 9$ in (4.4.1.5),

$$s^3 + 3s^2 + 3s + 9 = 0 \quad (4.4.1.10)$$
$$\implies s = -3, \pm j\sqrt{3} \quad (4.4.1.11)$$

verifying that poles lie on the imaginary axis.

4.5 Example

4.5.1. A closed loop system has the characteristic equation given by

$$s^3 + Ks^2 + (K+2)s + 3 = 0 \quad (4.5.1.1)$$

Determine the condition for K for which the system is stable.

Solution: The Routh array for (4.5.1.1) is

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & K+2 & 0 \\ K & 3 & 0 \\ \frac{K^2+2K-3}{K} & 0 & 0 \\ 3 & 0 & 0 \end{vmatrix} \tag{4.5.1.2}$$

For the system to be stable, there should be no sign changes in the first column of the Routh array. Thus,

$$\{K > 0\} \cap \left\{ \frac{K^2 + 2K - 3}{K} > 0 \right\} \tag{4.5.1.3}$$

$$\implies K > 1 \tag{4.5.1.4}$$

The following program computes the routh-array and stabilty for different values of K.

```
codes/ee18btech11039.py
```

5 State-Space Model

5.1 Controllability and Observability

5.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
Solution: The model is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{5.1.1}$$

with parameters listed in Table 5.1.

Variable	Size	Description
$\mathbf{u}$	$p \times 1$	input(control) vector
$\mathbf{y}$	$q \times 1$	output vector
$\mathbf{x}$	$n \times 1$	state vector
$\mathbf{A}$	$n \times n$	state or system matrix
$\mathbf{B}$	$n \times p$	input matrix
$\mathbf{C}$	$q \times n$	output matrix
$\mathbf{D}$	$q \times p$	feedthrough matrix

TABLE 5.1

5.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5.2.1}$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \tag{5.2.2}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{5.2.3}$$

$$+ (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{5.2.4}$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{5.2.5}$$

Substituting from (**??**) in the above,

$$\mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\mathbf{I})\mathbf{U}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{5.2.6}$$

5.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I} \tag{5.3.1}$$

5.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.4.1}$$

$$D = 0 \tag{5.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.3}$$

find $\mathbf{A}$ and $\mathbf{C}$ such that the state-space realization is in *controllable canonical form.*
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{5.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{5.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{5.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{5.4.7}$$

so the above equation can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.4.8}$$

Letting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.4.9}$$

$$\mathbf{X}_1 = \begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} \tag{5.4.10}$$

$$\mathbf{X} = \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix}, \tag{5.4.11}$$

$$\mathbf{X}_1(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{5.4.12}$$

$$Y = \mathbf{C}\mathbf{X}_1(s) \tag{5.4.13}$$

where

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{5.4.14}$$

5.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form*.
Solution: Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}, \tag{5.5.1}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.5.2}$$

$$\implies U(s) = Y(s)(s^3 + 3s^2 + 2s + 1) \tag{5.5.3}$$

$$\text{or, } Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.4}$$

Let

$$X_1(s) = Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.5}$$

$$X_2(s) = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{5.5.6}$$

$$X_3(s) = s^{-1}(U(s) - Y(s)) \tag{5.5.7}$$

$$\begin{aligned} sX_1(s) &= -3Y(s) + X_2(s) \\ \implies sX_2(s) &= -2Y(s) + X_3(s) \\ sX_3(s) &= U(s) - Y(s) \end{aligned} \tag{5.5.8}$$

Substituting $Y = X_1(s)$ the above,

$$sX_1(s) = -3X_1(s) + X_2(s) \tag{5.5.9}$$

$$sX_2(s) = -2X_1(s) + X_3(s) \tag{5.5.10}$$

$$sX_3(s) = U(s) - X_1(s) \tag{5.5.11}$$

which can be expressed as

$$\begin{pmatrix} sX_1(s) \\ sX_2(s) \\ sX_3(s) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.5.12}$$

$$\text{or, } \begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \\ Y(s) &= \mathbf{B}\mathbf{X}(s) \end{aligned} \tag{5.5.13}$$

where

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.5.14}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{5.5.15}$$

5.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

5.7. Theoretically, show that eigenvaues of **A** are the poles of $H(s)$.
Solution: As we know that the characteristic equation is det(sI-A)

$$s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.7.1}$$

$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{5.7.2}$$

$$\implies \left| s\mathbf{I} - \mathbf{A} \right| = s(s^2 + 3s + 2) + 1(1) \tag{5.7.3}$$

$$= s^3 + 3s^2 + 2s + 1 \tag{5.7.4}$$

which is the denominator of $H(s)$ in (5.4.1)

5.2 Second Order System

5.2.1. Consider a state-variable model of a system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} r \tag{5.2.1.1}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.2.1.2}$$

where y is the output, and r is the input.

5.2.2. List the various state matrices in (5.2.1.1)

5.2.3. Find the the system transfer function $H(s)$.
Solution: From (5.1.1) and , (5.3.1), the transfer function for the state space model is

$$H(s) = C(sI - A)^{-1}B + D \tag{5.2.3.1}$$

$$= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + 2\beta & 1 \\ -\alpha & s \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{s(s + 2\beta) + \alpha} \tag{5.2.3.2}$$

$$= \frac{b_1(s + 2\beta) + b_2}{s^2 + 2s\beta + \alpha} \tag{5.2.3.3}$$

$$\implies H(s) = \frac{b_1 s}{s^2 + 2s\beta + \alpha} + \frac{2b_1\beta + b_2}{s^2 + 2s\beta + \alpha} \tag{5.2.3.4}$$

5.2.4. Find the Damping ratio ζ and the Undamped natural frequency ω_n of the system.
Solution: Generally for a second order system the transfer function is given by 3.1.1.1

$$H(s) = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \tag{5.2.4.1}$$

Comparing the denominator of the above with (5.2.3.4),

$$2\zeta\omega_n = 2\beta, \tag{5.2.4.2}$$

$$\omega_n^2 = \alpha \tag{5.2.4.3}$$

$$\implies \zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha} \tag{5.2.4.4}$$

5.2.5. Using Table 3.1.1, explain how the damping conditions depend upon α and β.

5.3 Example

5.3.1. The state equation and the output equation of a control system are

$$\dot{\mathbf{x}} = \begin{pmatrix} -4 & -1.5 \\ 4 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} \mathbf{u} \tag{5.3.1.1}$$

$$\mathbf{y} = \begin{pmatrix} 1.5 & 0.625 \end{pmatrix} \mathbf{x} \tag{5.3.1.2}$$

Find the transfer function of the system
Solution: The system matrices are

$$\mathbf{A} = \begin{pmatrix} -4 & -1.5 \\ 4 & 0 \end{pmatrix} \tag{5.3.1.3}$$

$$\mathbf{B} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{5.3.1.4}$$

$$\mathbf{C} = \begin{pmatrix} 1.5 & 0.625 \end{pmatrix} \tag{5.3.1.5}$$

and the transfer function is

$$T(s) = \mathbf{C}\left[(sI - \mathbf{A})^{-1}\right]\mathbf{B} + \mathbf{D} \tag{5.3.1.6}$$

$$= \left(\frac{6s+10}{(s^2+4s+6)}\right) \tag{5.3.1.7}$$

5.3.2. Verify your answer using a python code
Solution: The following python code gives the desired answer

```
codes/ee18btech11023.py
```

5.4 Example

5.4.1. Consider the state space realization :

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & -9 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ 45 \end{pmatrix} u(t) \tag{5.4.1.1}$$

with initial conditions

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.4.1.2}$$

where $u(t)$ is the step function. Find

$$\lim_{t\to\infty} \|\mathbf{x}(t)\| \tag{5.4.1.3}$$

Solution: From (5.2.4),

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{5.4.1.4}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & -9 \end{pmatrix} \tag{5.4.1.5}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 45 \end{pmatrix} \tag{5.4.1.6}$$

$$U(s) = \frac{1}{s} \tag{5.4.1.7}$$

The following code

```
codes/ee18btech11026.py
```

yields

$$\mathbf{X}(s) = \begin{pmatrix} 0 \\ \frac{45}{s(s+9)} \end{pmatrix} \tag{5.4.1.8}$$

Using the final value theorem,

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{s\to 0} \mathbf{X}(s) \tag{5.4.1.9}$$

$$= \begin{pmatrix} 0 \\ 5 \end{pmatrix} \tag{5.4.1.10}$$

$$\implies \lim_{t\to\infty} \|\mathbf{x}(t)\| = 5 \tag{5.4.1.11}$$

5.5 Example

5.5.1. A second-order LTI system is described by the following state equations

$$\frac{\partial x_1(t)}{\partial t} - x_2(t) = 0 \tag{5.5.1.1}$$

$$\frac{\partial x_2(t)}{\partial t} + 2x_1(t) + 3x_2(t) = r(t) \tag{5.5.1.2}$$

$$c(t) = x_1(t). \tag{5.5.1.3}$$

where $x_1(t)$ and $x_2(t)$ are the two state variables and r(t) denotes the input. The output is $c(t)$. Express this in terms of the state space model.
Solution: From (5.1.1), (5.5.1.1)-(5.5.1.3) can be expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{5.5.1.4}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{5.5.1.5}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \tag{5.5.1.6}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.5.1.7}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{5.5.1.8}$$

$$\mathbf{D} = 0 \tag{5.5.1.9}$$

5.5.2. Find the system transfer function $H(s)$.
Solution: From (5.3.1),

$$H(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I} \tag{5.5.2.1}$$

$$= \frac{1}{s^2 + 3s + 2} \tag{5.5.2.2}$$

using the code in

```
codes/ee18btech11031/ee18btech11031.py
```

5.5.3. Identify the damping type.
Solution: From (3.1.1.1) and

$$\omega = \sqrt{2}, \zeta = \frac{3}{2\sqrt{2}} > 1 \tag{5.5.3.1}$$

Fom Table 3.1.1, the sytem is overdamped.

5.5.4. Find and plot the unit step response for the system.
Solution:

$$Y(s) = U(s)H(s) \tag{5.5.4.1}$$

$$= \frac{1}{s(s+1)(s+2)} \tag{5.5.4.2}$$

$$\implies y(t) = \left(\frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}\right)u(t) \tag{5.5.4.3}$$

The following code plots the step response in Fig. 5.5.4 using the code in

```
codes/ee18btech11031/ee18btech11031_2.py
```

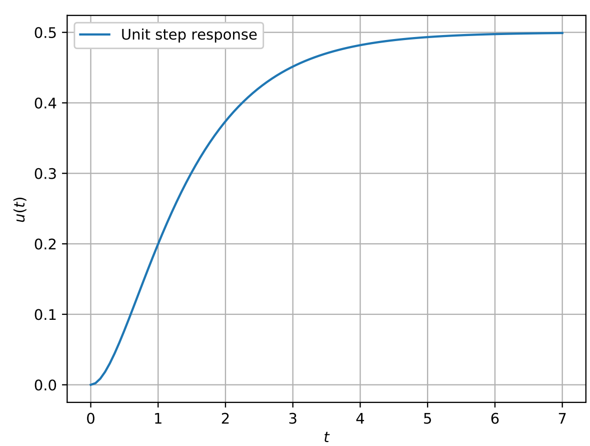


Fig. 5.5.4

5.6 Example

5.6.1. Consider the system described by the following state space representation

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbf{u} \tag{5.6.1.1}$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} \tag{5.6.1.2}$$

If u(t) is a unit step input and

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.6.1.3}$$

Find the output $y(1)$.
Solution: From 5.6.1.4

$$\mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D})\mathbf{U}(s) + \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) \quad (5.6.1.4)$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \quad (5.6.1.5)$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.6.1.6)$$
$$\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (5.6.1.7)$$
$$\mathbf{D} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad (5.6.1.8)$$
$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (5.6.1.9)$$

The following code computes

```
codes/ee18btech11047.py
```

$$Y(s) = \frac{s^2+2s+1}{s^3+2s^2} \quad (5.6.1.10)$$
$$\implies y(t) = \left(\frac{1}{4}e^{-2t} + \frac{3}{4} + \frac{1}{2}t\right)u(t) \quad (5.6.1.11)$$
$$\text{or, } y(1) = \frac{1}{4}\left(5+e^{-2}\right) \quad (5.6.1.12)$$

5.7 Example

5.7.1. Find the transfer function of the following system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}u(t) \quad (5.7.1.1)$$

$$\mathbf{y} = \begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{x} \quad (5.7.1.2)$$

Solution: By comparing the above equations to (5.1.1),

$$\mathbf{D} = 0 \quad (5.7.1.3)$$
$$\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (5.7.1.4)$$
$$\mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5.7.1.5)$$
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \quad (5.7.1.6)$$

From (5.3.1)

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{DI} \quad (5.7.1.7)$$
$$= \frac{s+4}{s^2-s-4} \quad (5.7.1.8)$$

where the following code computes the transfer function.

```
codes/ee18btech11040.py
```

6 Nyquist Plot

6.1 Introduction

6.1.1. The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{\pi e^{-0.25s}}{s} \quad (6.1.1.1)$$

6.1.2. Find $\mathrm{Re}\{G(\jmath\omega)\}$ and $\mathrm{Im}\{G(\jmath\omega)\}$.
Solution: From (6.1.1.1),

$$G(j\omega) = \frac{\pi}{\omega}(-\sin 0.25\omega - j\cos 0.25\omega) \quad (6.1.2.1)$$
$$\implies \mathrm{Re}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-\sin 0.25\omega) \quad (6.1.2.2)$$
$$\mathrm{Im}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-j\cos 0.25\omega) \quad (6.1.2.3)$$

6.1.3. Sketch the Nyquist plot.
Solution: The Nyquist plot is a graph of $\mathrm{Re}\{G(\jmath\omega)\}$ vs $\mathrm{Im}\{G(\jmath\omega)\}$. The following python code generates the Nyquist plot in Fig. 6.1.3

```
codes/ee18btech11007/ee18btech11007.py
```

6.1.4. Find the point at which the Nyquist plot of G(s) passes through the negative real axis
Solution: Nyquist plot cuts the negative real axis at ω for which

$$\angle G(\jmath\omega) = -\pi \quad (6.1.4.1)$$

From (6.1.1.1),

$$G(\jmath\omega) = \frac{\pi e^{-\frac{\jmath\omega}{4}}}{\jmath\omega} = \frac{\pi e^{-\jmath\left(\frac{\omega}{4}+\frac{\pi}{2}\right)}}{\omega} \quad (6.1.4.2)$$
$$\implies \angle G(\jmath\omega) = -\left(\frac{\omega}{4}+\frac{\pi}{2}\right) \quad (6.1.4.3)$$

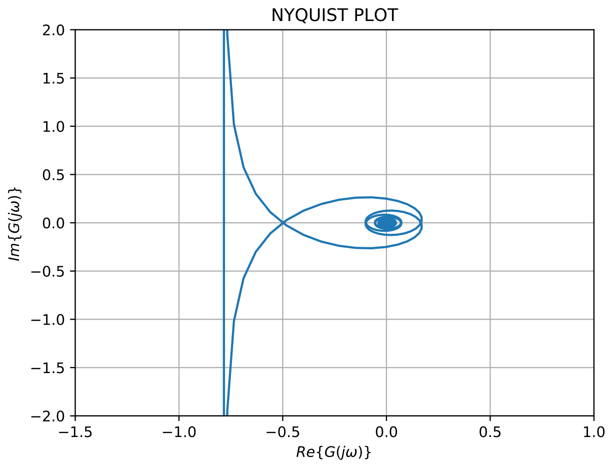


Fig. 6.1.3

From (6.1.4.3) and (6.1.4.1),

$$\frac{\omega}{4} + \frac{\pi}{2} = \pi \tag{6.1.4.4}$$

$$\implies \omega = 2\pi \tag{6.1.4.5}$$

Also, from (6.1.1.1),

$$\left|G(\jmath\omega)\right| = \frac{\pi}{|\omega|} \tag{6.1.4.6}$$

$$\implies \left|G(\jmath 2\pi)\right| = \frac{1}{2} \tag{6.1.4.7}$$

6.1.5. Use the Nyquist Stability criterion to determine if the system in (6.1.4.3) is stable.

Variable	Value	Description
Z	0	Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane
P	0	Poles of $G(s)H(s)$ in right half of s plane
N	0	No of clockwise encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot

TABLE 6.1.5

Solution: Consider Table 6.1.5. According to the Nyquist stability criterion,

a) If the open-loop transfer function $G(s)$ has a zero pole of multiplicity l, then the Nyquist plot has a discontinuity at ω =0. During further analysis it should be assumed that the phasor travels l times clock-wise along a semicircle of infinite radius. After applying this rule, the zero poles should be neglected, i.e. if there are no other unstable poles, then the open-loop transfer function $G(s)$ should be considered stable.
b) If the open-loop transfer function $G(s)$ is stable, then the closed-loop system is unstable for any encirclement of the point -1. If the open-loop transfer function $G(s)$ is unstable, then there must be one counter clock-wise encirclement of -1 for each pole of $G(s)$ in the right-half of the complex plane.
c) The number of surplus encirclements (N + P greater than 0) is exactly the number of unstable poles of the closed-loop system.
d) However, if the graph happens to pass through the point $-1+\jmath 0$, then deciding upon even the marginal stability of the system becomes difficult and the only conclusion that can be drawn from the graph is that there exist zeros on the $\jmath\omega$ axis.

From (6.1.1.1), $G(s)$ is stable since it has a single pole at $s = 0$. Further, from Fig. 6.1.3, the Nyquist plot doesnot encircle $s = -1$. From Theorem 6.1.5b, we may conclude that the system is stable.

6.2 Example

6.2.1. Find the number of encirclements around the point -1+j0 in the complex plane by the Nyquist plot of

$$G(s) = \frac{1-s}{4+2s}. \tag{6.2.1.1}$$

Solution: The following code generates the Nyquist plot in Fig. 6.2.1. Is is easy to verify that the number of encirclements of -1+j0 is zero.

7 Compensators

7.1 Phase Lead

7.1.1. Consider a control system with

$$G(s) = \frac{1}{s(3s+1)} \tag{7.1.1.1}$$

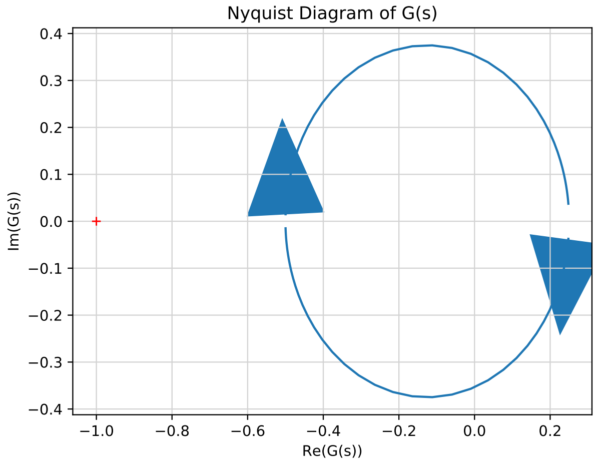


Fig. 6.2.1

Find its phase margin.
Solution:

$$G(\jmath\omega) = \frac{1}{(j\omega)(3j\omega+1)} \tag{7.1.1.2}$$

$$\implies \left|G(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} \tag{7.1.1.3}$$

$$\implies \angle G(\jmath\omega) = -\tan^{-1}(3\omega) - 90^\circ \tag{7.1.1.4}$$

At Gain Crossover,

$$\left|G(\jmath\omega)\right| = 1 \tag{7.1.1.5}$$

$$\implies \frac{1}{\omega\left(\sqrt{9\omega^2+1}\right)} = 1 \tag{7.1.1.6}$$

$$\implies \omega_{gc} = 0.531 \tag{7.1.1.7}$$

$$\implies \angle G(\jmath\omega) = -147.88^\circ \tag{7.1.1.8}$$

$$\implies PM = 32.12^\circ \tag{7.1.1.9}$$

7.1.2. The minimum acceptable PM for a control system is 45°. Design a suitable *lead compensator* for (7.1.1.1).
Solution: Let the desired compensator be

$$D(s) = \frac{3(s+\frac{1}{3T})}{(s+\frac{1}{T})} \tag{7.1.2.1}$$

Choosing T = 1,

$$D(s) = \frac{3(s+\frac{1}{3})}{(s+1)} \tag{7.1.2.2}$$

By cascading the Compensator and the Open Loop Transfer Function,

$$G_1(s) = D(s)G(s) \tag{7.1.2.3}$$

$$= \frac{1}{s(3s+1)}\frac{3(s+\frac{1}{3})}{(s+1)} \tag{7.1.2.4}$$

$$\implies G_1(s) = \frac{1}{s(s+1)} \tag{7.1.2.5}$$

$$\implies G_1(\jmath\omega) = \frac{1}{(j\omega)(j\omega+1)} \tag{7.1.2.6}$$

$$\implies \left|G_1(\jmath\omega)\right| = \frac{1}{\omega\left(\sqrt{\omega^2+1}\right)} \tag{7.1.2.7}$$

$$\angle G_1(\jmath\omega) = -\tan^{-1}(\omega) - 90^\circ \tag{7.1.2.8}$$

At Gain Crossover,

$$\left|G_1(\jmath\omega)\right| = 1 \tag{7.1.2.9}$$

$$\implies \frac{1}{\omega\left(\sqrt{\omega^2+1}\right)} = 1 \tag{7.1.2.10}$$

$$\implies \omega_{gc} = 0.786 \tag{7.1.2.11}$$

$$\implies \angle G(\jmath\omega) = -128.167^\circ \tag{7.1.2.12}$$

$$\implies PM = 51.83^\circ \tag{7.1.2.13}$$

7.1.3. Verify the above improvement in Phase Margin with the help of a Python Code
Solution: The following code generates Fig. 7.1.3

```
codes/ee18btech11021.py
```

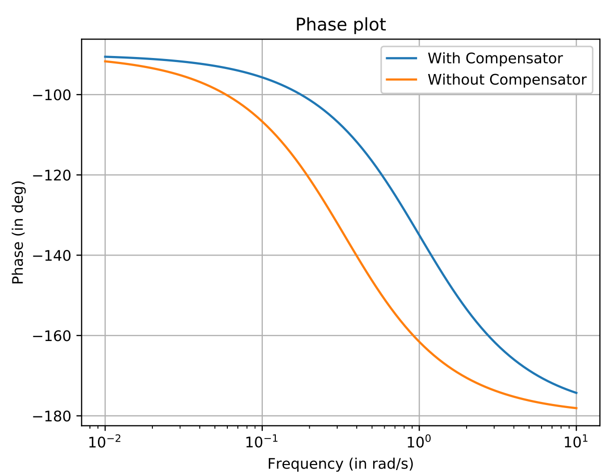


Fig. 7.1.3

7.2 Lag Lead

7.1. Write the general expression for the transfer function of a Phase lag-lead compensator. **So-**

lution: The Transfer Function of the Phase lag-lead compensator is

$$H(s) = \frac{(1+\alpha sT_1)(1+\beta sT_2)}{(1+T_1 s)(1+T_2 s)} \tag{7.1.1}$$

α and β are generally chosen in such a way that

$$\alpha\beta = 1 \tag{7.1.2}$$

7.2. Write the expression for phase introduced by phase lag-lead compensator. **Solution:**

$$\phi = \tan^{-1}\left(\frac{\omega T_1}{\beta}\right) + \tan^{-1}(\beta\omega T_2) - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) \tag{7.2.1}$$

7.3. Consider the Transfer function of a phase lag-lead compensator

$$C(s) = \frac{(1+\frac{s}{0.1})(1+\frac{s}{100})}{(1+\frac{s}{1})(1+\frac{s}{10})} \tag{7.3.1}$$

Find the frequency range in which the phase (lead) introduced by the compensator reaches the maximum
Solution: From (7.1.1) and (7.3.1),

$$\alpha = 10, T_1 = 1, \tag{7.3.2}$$
$$\beta = 0.1, T_2 = 0.1. \tag{7.3.3}$$

Thus,

$$\angle C(\jmath\omega) = \underbrace{\tan^{-1}(10\omega) - \tan^{-1}(\omega)}_{lead} + \underbrace{\tan^{-1}(0.01\omega) - \tan^{-1}(0.1\omega)}_{lag} \tag{7.3.4}$$

As we are trying to find the range of frequencies in which phase lead introduced by the compensator is maximum, the phase introduced by the lag part of compensator will be close to zero. Hence, the problem can be expressed as

$$\min_{\omega} \tan^{-1}(10\omega) - \tan^{-1}(\omega) \tag{7.3.5}$$

which can be obtained by

$$\frac{d}{d\omega}\left[\tan^{-1}(10\omega) - \tan^{-1}(\omega)\right] = 0 \tag{7.3.6}$$
$$\implies \frac{10}{1+100\omega^2} - \frac{1}{1+\omega^2} = 0 \tag{7.3.7}$$
$$\text{or, } \omega = \frac{1}{\sqrt{10}} \tag{7.3.8}$$

7.4. Verify your result using a python plot.
Solution: The following code

```
codes/ee18btech11044.py
```

generates Fig. 7.4.

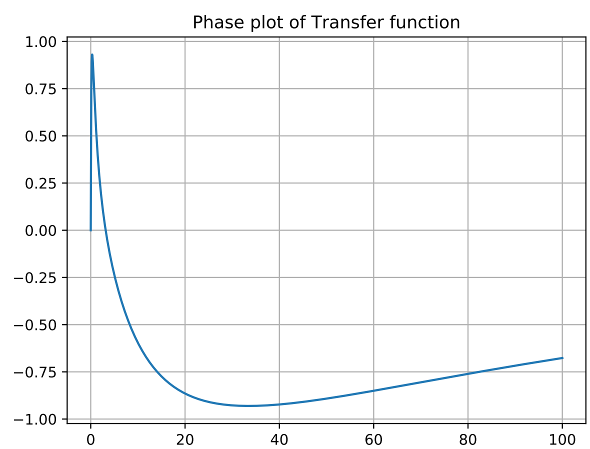


Fig. 7.4

7.3 Example

7.1. Consider the system in Section 7.1.

$$G(s) = \frac{1}{s(3s+1)} \tag{7.1.1}$$

Find the step response for the system in Fig. 7.3.

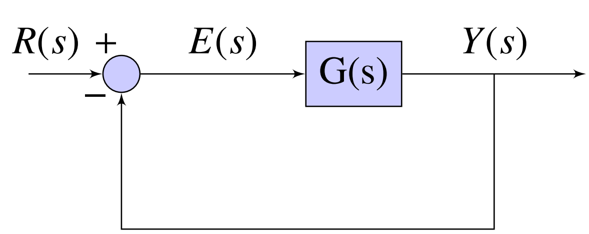


Fig. 7.1

Solution: The step response is

$$Y(s) = \frac{G(s)}{1+G(s)}\frac{1}{s} \tag{7.1.2}$$
$$= \frac{1}{(s)(3s^2+s+1)} \tag{7.1.3}$$
$$= \frac{-3s-1}{3s^2+s+1} + \frac{1}{s} \tag{7.1.4}$$

$$\implies y(t) = \left[1 - e^{-\frac{t}{6}}\left\{\cos\left(\frac{\sqrt{11}t}{6}\right) - \frac{1}{\sqrt{11}}\sin\left(\frac{\sqrt{11}t}{6}\right)\right\}\right]u(t) \tag{7.1.5}$$

7.2. Find the step response for the system shown in Fig. 7.2 that is obtained by adding the lead compensator to the system in Fig. 7.1

$$D(s) = \frac{3(s+\frac{1}{3})}{(s+1)} \tag{7.2.1}$$

Solution: The step response is given by

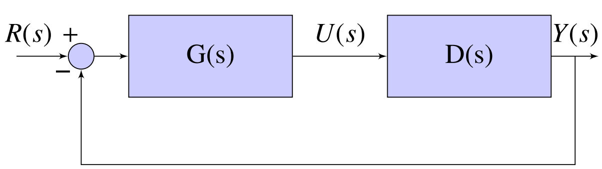


Fig. 7.2

$$Y_1(s) = \frac{G_1(s)}{1+G_1(s)}\frac{1}{s} \tag{7.2.2}$$
$$= \frac{1}{s(s^2+s+1)} \tag{7.2.3}$$

where

$$G_1(s) = G(s)D(s) = \frac{1}{s(s+1)} \tag{7.2.4}$$

$$\implies y(t) = \left[1 - e^{\frac{-t}{2}}\left\{\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right\}\right]u(t) \tag{7.2.5}$$

7.3. Show that the lead compensator reduces the settling time.
Solution: The following code generates Fig. 7.3 which is self-explanatory.

```
codes/ee18btech11027.py
```

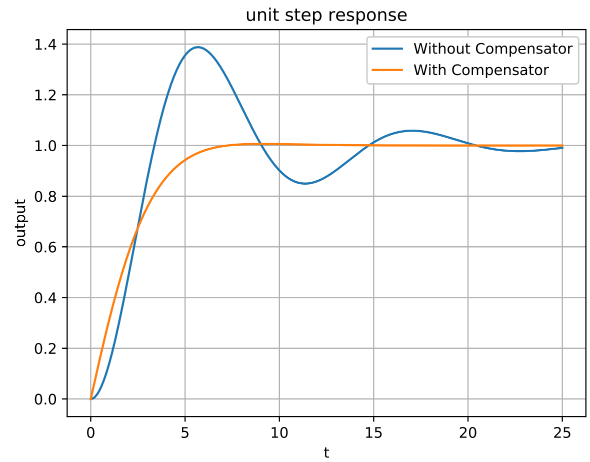


Fig. 7.3

8 Gain Margin

8.1 Introduction

8.1.1. The open loop transfer function of a feedback control system is

$$G(s) = \frac{1}{s(1+2s)(1+s)} \tag{8.1.1.1}$$

Find the magnitude and phase of $\left|G(\jmath\omega)\right|$.
Solution:

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1+2\jmath\omega)(1+\jmath\omega)} \tag{8.1.1.2}$$
$$= \frac{1}{\jmath\omega(1+3\jmath\omega-2\omega^2)} \tag{8.1.1.3}$$
$$= \frac{1}{\jmath\omega-3\omega^2-2\jmath\omega^3} \tag{8.1.1.4}$$
$$= \frac{1}{-3\omega^2+\jmath\omega(1-2\omega^2)} \tag{8.1.1.5}$$

$$\implies \angle G(\jmath\omega) = -tan^{-1}\left(\frac{\omega(1-2\omega^2)}{-3\omega^2}\right) \tag{8.1.1.6}$$

8.1.2. The frequency at which the phase of open-loop transfer function reaches -180° or +180° depending upon the range of tan inverse function) is defined to be the phase crossover frequency. Find the phase crossover frequency for (8.1.1.1). **Solution:** From (8.1.1.6), at $\omega = \omega_{pc}$

$$\omega(1-2\omega^2) = 0 \tag{8.1.2.1}$$

$$\implies \omega_{pc} = \frac{1}{\sqrt{2}} \tag{8.1.2.2}$$

8.1.3. The gain Margin is given by,

$$GM = -20\log_{10}\left|G\left(\jmath\omega_{pc}\right)\right| = 20\log_{10} k_g \tag{8.1.3.1}$$

where

$$k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} \tag{8.1.3.2}$$

Find the GM for (8.1.1.6).
Solution:

$$\left|G\left(\jmath\omega_{pc}\right)\right| = \frac{1}{\left(\frac{3}{2}\right)} \implies k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} = \frac{3}{2}3.5dB \tag{8.1.3.3}$$

The greater the Gain Margin (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.

8.1.4. Obtain the GM from the Bode plot.
Solution: The following code

```
codes/ee18btech11016.py
```

plots the amplitude and phase of (8.1.1.1) in Fig. 8.1.4. From Fig. 8.1.4,

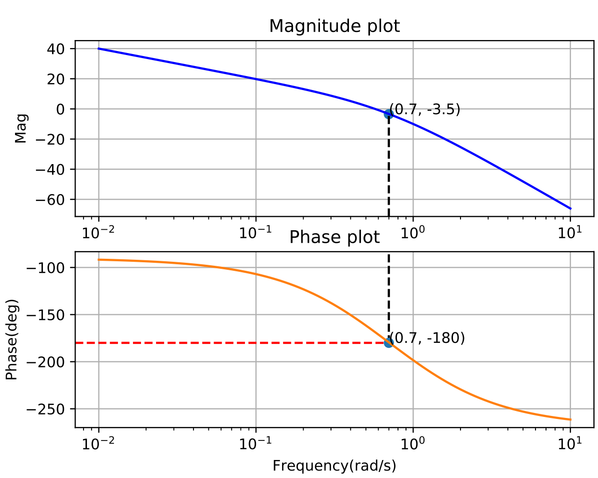


Fig. 8.1.4

$$20\log_{10}\left|G\left(\jmath\omega_{pc}\right)\right| = -3.5dB, \quad \omega_{pc} = -180° \tag{8.1.4.1}$$

$$\implies GM = +3.5dB. \tag{8.1.4.2}$$

8.1.5. A positive GM indicates closed loop stability with unity feedback. Verify this for (8.1.1.1).
Solution: The characteristic equation is

$$1 + G(s) = 0 \implies 2s^3 + 3s^2 + s + 1 = 0 \tag{8.1.5.1}$$

Constructing the routh array

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \end{vmatrix} \tag{8.1.5.2}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{8.1.5.3}$$

There are no sign changes in the first column of the routh array. $\therefore$ the system is stable.

8.1.6. Instead of unity feedback, consider a system with

$$H(s) = \frac{1}{s+1} \tag{8.1.6.1}$$

Find the magnitude and phase of $\left|G(\jmath\omega)H(\jmath\omega)\right|$
Solution:

$$\because G(s)H(s) = \frac{1}{s(1+2s)(s+1)^2}, \tag{8.1.6.2}$$

$$G(\jmath\omega)H(\jmath\omega) = \frac{1}{(2\omega^4 - 4\omega^2) + j(\omega - 5\omega^3)} \tag{8.1.6.3}$$

$$\implies \angle G(\jmath\omega)H(\jmath\omega) = -tan^{-1}\left(\frac{\omega - 5\omega^3}{2\omega^4 - 4\omega^2}\right) \tag{8.1.6.4}$$

8.1.7. Compute the open loop gain margin for this system.
Solution: For ω=ω_{pc}

$$\text{Im}\{G(\jmath\omega)H(\jmath\omega)\} = 0. \tag{8.1.7.1}$$

$$\implies \omega(1-5\omega^2) = 0 \tag{8.1.7.2}$$

$$\implies \omega_{pc} = \frac{1}{\sqrt{5}} \tag{8.1.7.3}$$

Hence,

$$GM = -20\log_{10}|G\left(\jmath\omega_{pc}\right)H\left(\jmath\omega_{pc}\right)| = 20\log_{10}k_g \tag{8.1.7.4}$$

where

$$k_g = \frac{1}{|G\left(\jmath\omega_{pc}\right)H\left(\jmath\omega_{pc}\right)|} \tag{8.1.7.5}$$

$$= \frac{18}{25} = 2.853dB. \tag{8.1.7.6}$$

Hence $GM < 0$ and the system is unstable.

8.1.8. Obtain the GM from the Bode plot.
Solution: The following code

```
codes/ee18btech11016_2.py
```

plots the amplitude and phase of (8.1.1.1) in Fig. 8.1.8.

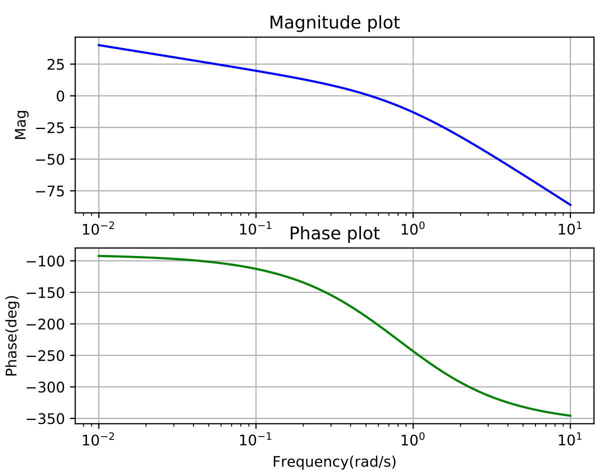


Fig. 8.1.8

8.1.9. Show that the closed loop transfer function

$$T(s) = \frac{1}{1 + (s(1+2s)(1+s)^2)} \tag{8.1.9.1}$$

is unstable using the Routh Hurwitz criterion.
Solution: The characteristics equation is

$$2s^4 + 5s^3 + 4s^3 + s + 1 = 0 \tag{8.1.9.2}$$

Constructing routh array for the above,

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 2 & 4 & 1 \\ 5 & 1 & 0 \\ (18/5) & 1 & 0 \end{vmatrix} \tag{8.1.9.3}$$

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (18/5) & 1 & 0 \\ (-7/18) & 0 & 0 \end{vmatrix} \tag{8.1.9.4}$$

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (18/5) & 1 & 0 \\ (-7/18) & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{8.1.9.5}$$

There are 2 sign changes in the first column of the routh array. So, 2 poles lie on right half of s-plane. Therefore,the system is unstable.

8.2 Example

8.2.1. Fig. 8.2.1.1 shows the Bode magnitude and phase plots of

$$G(s) = \frac{n_0}{s^3 + d_2s^2 + d_1s + d} \tag{8.2.1.1}$$

Find $\left|G\left(\jmath\omega_{pc}\right)\right|$.

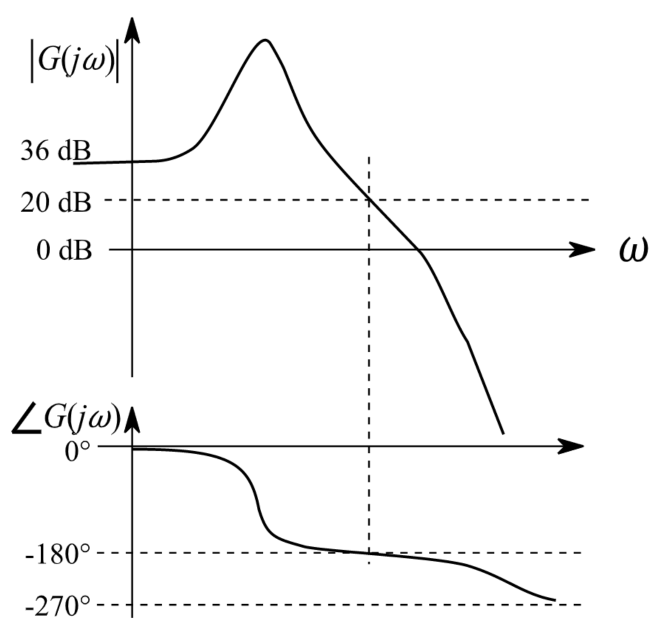


Fig. 8.2.1.1

Solution: From Fig. 8.2.1.1,

$$\angle G\left(\jmath\omega_{pc}\right) = 180^\circ \tag{8.2.1.2}$$

$$\implies 20\log\left|G\left(\jmath\omega_{pc}\right)\right| = 20 \tag{8.2.1.3}$$

8.2.2. Consider the negative unity feedback configuration with gain k in the feed forward path as shown in Fig. 8.2.2.1. Find the condition for the closed loop system to be stable.

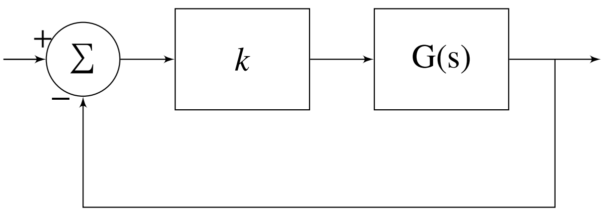


Fig. 8.2.2.1

Solution: The open loop gain for the system in Fig. 8.2.2.1 is

$$G_1(s) = kG(s) \tag{8.2.2.1}$$

$$\implies \angle G_1\left(\jmath\omega_{pc}\right) = \angle G\left(\jmath\omega_{pc}\right) \tag{8.2.2.2}$$

$$= 180^\circ \quad \text{and} \tag{8.2.2.3}$$

$$20\log\left|G_1\left(\jmath\omega_{pc}\right)\right| = 20\log|k| + 20\log\left|G\left(\jmath\omega_{pc}\right)\right| \tag{8.2.2.4}$$

$$= 20\left(1+\log|k|\right) \tag{8.2.2.5}$$

from (8.2.1.3). From (8.1.3.1) and (8.2.2.5), the GM of $G_1(s)$ is

$$-20\log\left|G_1\left(\jmath\omega_{pc}\right)\right| = -20\left(1+\log|k|\right) \tag{8.2.2.6}$$

For stability, $GM > 0$

$$\implies -20\left(1+\log|k|\right) > 0 \tag{8.2.2.7}$$

$$\implies |k| < 0.1 \tag{8.2.2.8}$$

8.3 Example

8.3.1. Consider the unity feedback control system in Fig. 8.3.2.1. Find the value of K such that $PM = 30^\circ$.

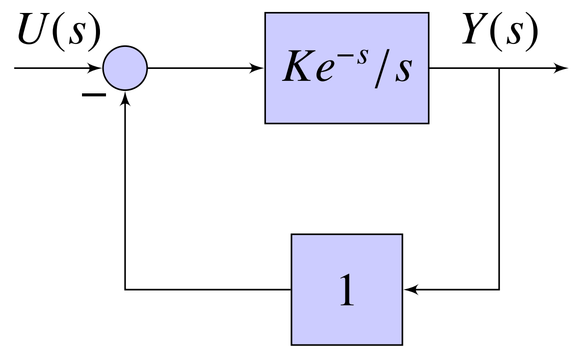


Fig. 8.3.1.1

Solution: From Fig. 8.3.2.1,

$$G(s) = \frac{Ke^{-s}}{s} \tag{8.3.1.1}$$

$$H(s) = 1 \tag{8.3.1.2}$$

$$\implies \tag{8.3.1.3}$$

$$\left|G\left(\jmath\omega_{gc}\right)H\left(\jmath\omega_{gc}\right)\right| = 1 \tag{8.3.1.4}$$

$$\implies \omega_{gc} = K \tag{8.3.1.5}$$

Thus,

$$\angle G\left(\jmath\omega_{gc}\right)H\left(\jmath\omega_{gc}\right) = \angle\frac{Ke^{-jK}}{jK} \tag{8.3.1.6}$$

$$= -90^\circ - K\left(\frac{180}{\pi}\right) \tag{8.3.1.7}$$

$$\implies PM = 180^\circ - 90^\circ - K\left(\frac{180}{\pi}\right) \tag{8.3.1.8}$$

$$= 30^\circ \tag{8.3.1.9}$$

$$\implies K = \frac{\pi}{3} \tag{8.3.1.10}$$

8.3.2. Verify your result by plotting the gain and phase plots of $G\left(j\left(\omega\right)\right)$

Solution: The following code plots Fig. 8.3.2.1

```
codes/ee18btech11038.py
```

9 Phase Margin

9.1 Intoduction

9.1. The open loop transfer function of a system is

$$G(s) = \frac{2}{(s+1)(s+2)} \tag{9.1.1}$$

Find its magnitude and phase response.

Solution: Substituting $s = \jmath\omega$ in (9.1.1),

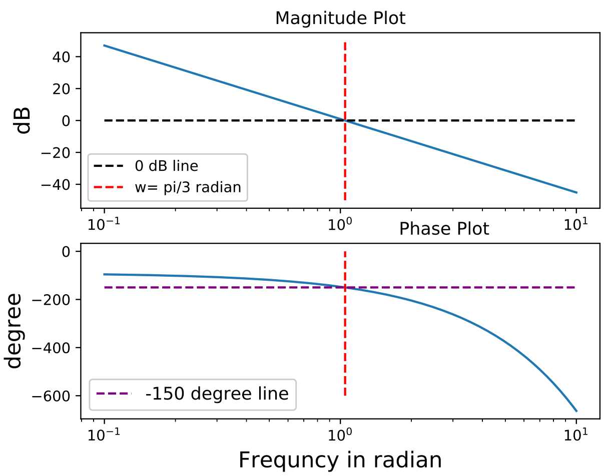


Fig. 8.3.2.1

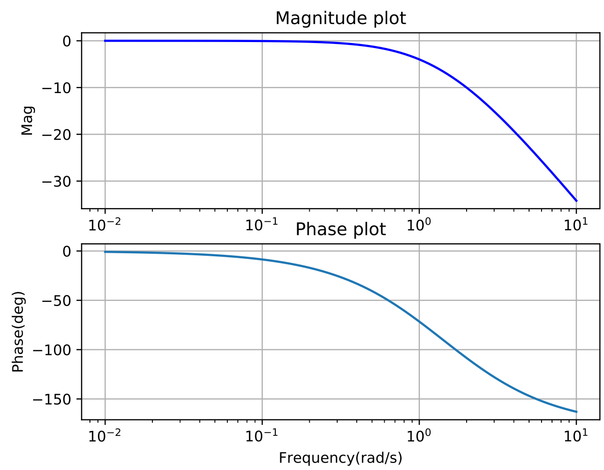


Fig. 9.4.1

$$G(\jmath\omega) = \frac{1}{(j\omega+1)(j\omega+2)} \tag{9.1.2}$$

$$\implies \left|G(\jmath\omega)\right| = \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} \tag{9.1.3}$$

$$\angle G(\jmath\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) \tag{9.1.4}$$

9.2. Find ω for which the gain of (9.1.1) first becomes 1.
Solution: From (9.1.3)

$$\left|G(\jmath\omega)\right| = 1 \tag{9.2.1}$$

$$\implies \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} = 1 \tag{9.2.2}$$

$$\implies \omega_{gc} = 0 \tag{9.2.3}$$

which is the desired frequency.

9.3. Find $\angle G(\jmath\omega_{gc}) + 180°$. This is known as the *phase margin*(PM)
Solution: From (9.1.4),

$$\angle G(\jmath\omega) = 0° \implies PM = 180° \tag{9.3.1}$$

9.4. Verify your result by plotting the gain and phase plots of $G(\jmath\omega)$.
Solution: The following code plots Fig. 9.4.1

```
codes/ee18btech11017.py
```

The Phase plot is as shown,

9.5. A positive phase margin for the open loop system indicates a stable closed loop system. (9.3.1) indicates that $G(s)$ with unity feedback is stable. Show that the roots of $1+G(s)$ lie in the left half plane proving closed loop stability.
Solution: Let the closed loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)} \tag{9.5.1}$$

Then

$$1 + G(s) = 0 \tag{9.5.2}$$

$$\implies s^2 + 3s + 4 = 0 \tag{9.5.3}$$

$$\text{or } s = -1.5 + 1.3j, -1.5 - 1.3j \tag{9.5.4}$$

Since the roots are in the left half plane, the system is stable.

9.6. Instead of unity feedback, consider a system with

$$H(s) = \frac{50}{s+1} \tag{9.6.1}$$

Compute the open loop phase margin for this system.
Solution:

$$\because G(s)H(s) = \frac{100}{(s+1)^2(s+2)}, \tag{9.6.2}$$

the magnitude and phase are

$$\left|G(\jmath\omega)H(\jmath\omega)\right| = \frac{10^2}{\sqrt{(\omega^2+1)^2}\sqrt{\omega^2+4}} \tag{9.6.3}$$

$$\angle G(\jmath\omega)H(\jmath\omega) = -\tan^{-1}\frac{\omega}{2} - 2\tan^{-1}(\omega) \tag{9.6.4}$$

The gain crossover frequency is given by

$$\frac{10^2}{\sqrt{\omega_{gc}^2+4}\sqrt{(\omega_{gc}^2+1)^2}} = 1 \tag{9.6.5}$$

$$\tag{9.6.6}$$

$$\omega_{gc}^6 + 6\omega_{gc}^4 + 9\omega_{gc}^2 - 9996 = 0 \tag{9.6.7}$$

$$\implies \omega_{gc} = 4.42 \tag{9.6.8}$$

From (9.6.4) and (9.6.8), the phase margin is

$$PM = 180^\circ - 2\tan^{-1}(\omega_{gc}) - \tan^{-1}\left(\frac{\omega_{gc}}{2}\right) \tag{9.6.9}$$

$$\implies P.M = -40.15^\circ \tag{9.6.10}$$

9.7. Verify your result through the magnitude and phase plot.
Solution: The following code plots Fig. 9.7.1

```
codes/ee18btech11017_2.py
```

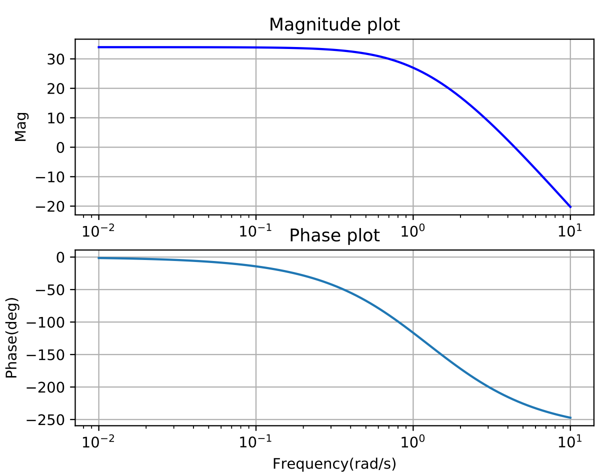


Fig. 9.7.1

9.8. Since the PM in (9.6.10) is negative, the closed loop system is unstable . Verify this using the Routh-Hurwitz criterion.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{9.8.1}$$

$$\implies s^3 + 4s^2 + 5s + 102 = 0 \tag{9.8.2}$$

Constructing the routh array for (9.8.2),

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \end{vmatrix} \tag{9.8.3}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \\ 102 & 0 & 0 \end{vmatrix} \tag{9.8.4}$$

$\because$ there are two sign changes in the first column of the routh array, two poles lie on right half of s-plane. Therefore,the system is unstable.

9.2 *Example*

9.1. For the feedback system in Fig. 9.1.1,

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}. \tag{9.1.1}$$

Find $k > 0$ for which the gain margin of system is exactly 0 dB and phase margin of system is exactly 0 degree.

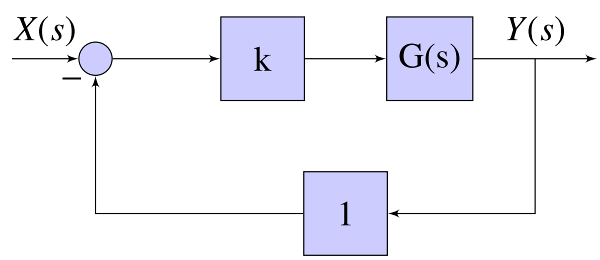


Fig. 9.1.1

Solution: From the given information, system can be destabilized with a marginal increase in the gain. Hence the system is marginally stable. The characteric equation for (9.1.1) is

$$s^3 + 6s^2 + 11s^1 + (6+k) \tag{9.1.2}$$

and the corresponding Routh array is

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 11 \\ 6 & (6+k) \\ \frac{66-(6+k)}{6} & 0 \\ (6+k) & 0 \end{vmatrix} \tag{9.1.3}$$

For the system to be marginally stable,

$$\frac{66-(6+K)}{6} > 0 \implies k = 60 \qquad (9.1.4)$$

The following code

```
codes/ee18btech11036.py
```

verifies that the system is marginally stable for $k = 60$.

10 Oscillator

10.1 Introduction

10.1.1. A unity feedback control system is characterised by the open-loop transfer function

$$G(S) = \frac{2(s+1)}{s^3 + ks^2 + 2s + 1} \qquad (10.1.1.1)$$

Find the value of the k for which the system oscillates at 2 rad/s. Verify your result through a program.

Solution: Fig. 10.1.1.1 models the equivalent closed loop system.

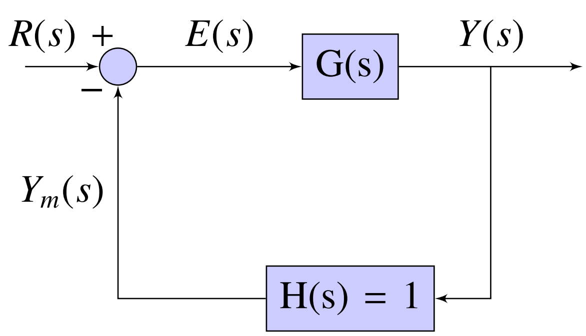


Fig. 10.1.1.1

The characteristic equation is

$$1 + G(s) = 0 \qquad (10.1.1.2)$$

$$\implies 1 + \frac{2(s+1)}{s^3 + ks^2 + 2s + 1} = 0 \qquad (10.1.1.3)$$

$$\text{or, } s^3 + ks^2 + 4s + 3 = 0 \qquad (10.1.1.4)$$

Constructing the routh array for (10.1.1.4)

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ k & 3 \\ \frac{3-4k}{k} & 0 \\ 3 & 0 \end{vmatrix} \qquad (10.1.1.5)$$

For the system to oscillate, poles should lie on the imaginary axis.

$$\implies \frac{3-4k}{k} = 0, \text{ or, } k = \frac{3}{4} \qquad (10.1.1.6)$$

Substituting in (10.1.1.4),

$$s^3 + \frac{3}{4}s^2 + 4s + 3 = 0 \qquad (10.1.1.7)$$

$$\implies s = \frac{-3}{4}, \pm 2\jmath \qquad (10.1.1.8)$$

The following code verifies the result.

```
codes/ee18btech11030/ee18btech11030.py
```

10.1.2. Sketch the impulse response of the closed loop system.

Solution: The closed loop response

$$G_m(s) = \frac{G(s)}{1+G(s)} = \frac{2(s+1)}{s^3 + \frac{3}{4}s^2 + 4s + 3} \qquad (10.1.2.1)$$

$$= \frac{8}{73(s+\frac{3}{4})} + \frac{-8s+152}{73(s^2+4)} \qquad (10.1.2.2)$$

$$\implies g_m(t) = \frac{8}{73}e^{-\frac{3t}{4}}u(t) - \left(\frac{8}{73}\right)\sin(2t) + \left(\frac{152}{73}\right)\cos(2t) \qquad (10.1.2.3)$$

The following code

```
codes/ee18btech11030/ee18btech11030_1.py
```

plots Fig. 10.1.2.1 . This shows that system oscillates at 2 rad/sec.

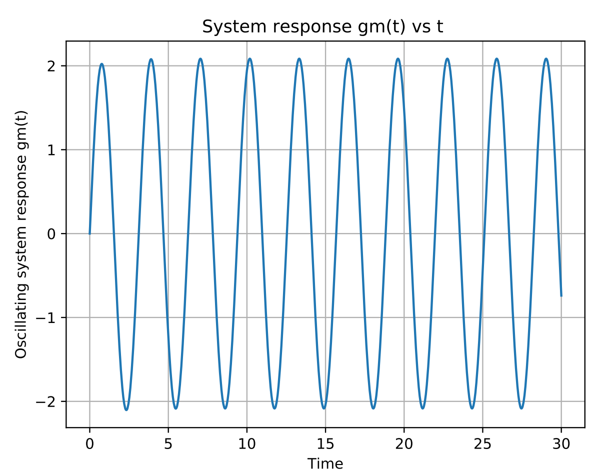


Fig. 10.1.2.1

10.2 Example

10.2.1. Fig. 10.2.1.1 shows a Hartley oscillator.

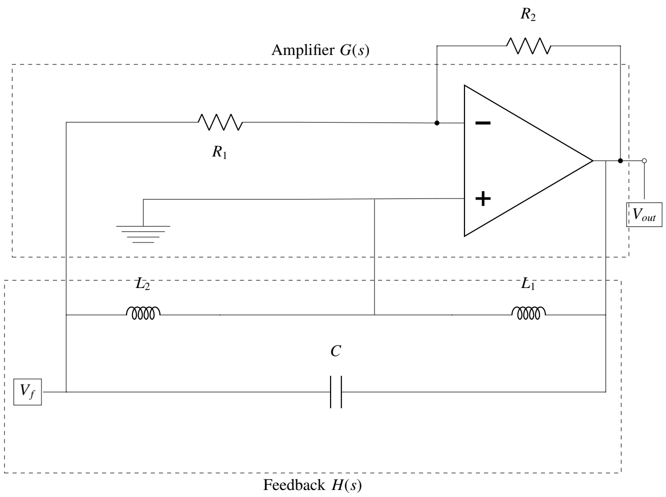


Fig. 10.2.1.1: Hartley oscillator

10.2.2. Draw the equivalent block diagram of the oscillator in Fig. 10.2.1.1.
Solution: See Fig. .

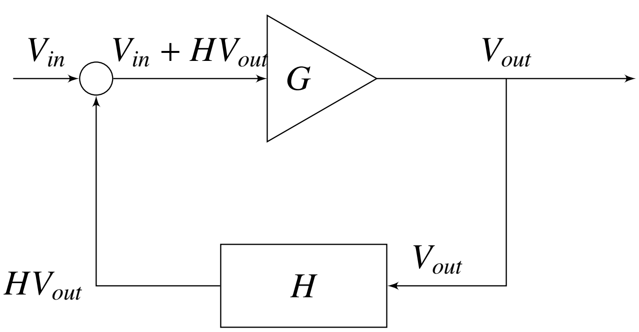


Fig. 10.2.2.1: block diagram for oscillator

10.2.3. Find G and H.
Solution: From the figure 10.2.3.1

W.K.T, no current flows in the opamp terminals.
and,(in S-domain)

$$A(V_1 - V_2) = V_{out} \tag{10.2.3.1}$$
$$v_2 = 0 \tag{10.2.3.2}$$
$$V_1 = V_{out} + iR_2 \tag{10.2.3.3}$$

where,
A is the gain through the amplifier,

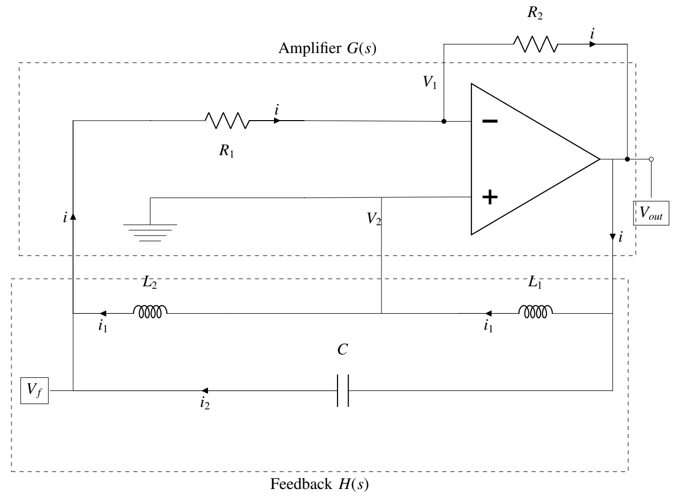


Fig. 10.2.3.1: Amplifier written in equivalent circuit form

Assuming everything at 0 initially.

$$V_{out} - i_1 S L_1 = 0 \tag{10.2.3.4}$$
$$i_1(S L_1 + S L_2) = i_2(\frac{1}{S C}) \tag{10.2.3.5}$$

On solving

$$i_1 = \frac{V_{out}}{S L_1} \tag{10.2.3.6}$$
$$i_2 = ((S L_1 + S L_2)S C)(\frac{V_{out}}{S L_1}) \tag{10.2.3.7}$$
$$\tag{10.2.3.8}$$

Now,

$$i = i_1 + i_2 \tag{10.2.3.9}$$
$$i = ((S L_1 + S L_2)S C + 1)\frac{V_{out}}{S L_1} \tag{10.2.3.10}$$
$$\tag{10.2.3.11}$$

Finding V_1

$$V_1 = V_{out} + iR_2 \tag{10.2.3.12}$$
$$= V_{out}(1 + R_2\frac{((S L_1 + S L_2)S C + 1)}{S L_1}) \tag{10.2.3.13}$$
$$\tag{10.2.3.14}$$

Gain, G(s) is given by,

$$G = kA \quad (10.2.3.15)$$

$$\therefore A = \frac{V_{out}}{V_1} \quad (10.2.3.16)$$

$$= \frac{1}{1 + R_2 \frac{((SL_1+SL_2)SC+1)}{SL_1}} \quad (10.2.3.17)$$

where, k is some real number.

$$H(s) = \frac{V_{out}}{V_f} = \frac{i_1 SL_1}{i_1 SL_2} = \frac{L_1}{L_2} \quad (10.2.3.18)$$

10.2.4. Show that the gain of the oscillator is

$$Gain = \frac{V_{out}}{V_{in}} = \frac{G}{1 - GH} \quad (10.2.4.1)$$

Solution: From figure 10.2.2.1 Oscillators gain can be given as follows:

$$G(V_{in} + HV_{out}) = V_{out} \quad (10.2.4.2)$$

$$G(V_{in} = (1 - GH)V_{out} \quad (10.2.4.3)$$

$$\frac{V_{out}}{V_{in}} = \frac{G}{1 - GH} \quad (10.2.4.4)$$

resulting in (10.2.4.1).

10.2.5. State the condition for sustained oscillations. Justify.
Solution: Condition for sustained oscillation is given by

$$GH = 1 \quad (10.2.5.1)$$

Along with, total phase gain o the circuit should be 0 or 2π
Justification: as, when $GH = 1$, gain becomes infinity, and theoretically we can get output, without actually providing input
Total phase gain should be so, as we want our signal to be in phase after every loop traversal.

10.2.6. Find the frequency of oscillation using the condition that $GH = 1$.
Solution: Now,we know that $GH = 1$ for sustained oscillations, putting the the above terms in the equation
on solving,
putting that in and equating $GH = 1$ we get,

$$1 = (\frac{L_1}{L_2})\frac{k}{1 + R_2 \frac{((SL_1+SL_2)SC+1)}{SL_1}} \quad (10.2.6.1)$$

As we need, to find frequency, put S =jw

$$1 = (\frac{L_1}{L_2})\frac{k}{1 + R_2 \frac{((j\omega L_1+j\omega L_2)j\omega C+1)}{j\omega L_1}} \quad (10.2.6.2)$$

$$1 = (\frac{L_1}{L_2})\frac{k}{1 - jR_2 \frac{(-(\omega L_1+\omega L_2)\omega C+1)}{\omega L_1}} \quad (10.2.6.3)$$

$$(10.2.6.4)$$

To satisfy the above equation, equating imaginary term to Zero.

$$\omega L_1 + \omega L_2 = \frac{1}{\omega C} \quad (10.2.6.5)$$

$$\omega = \frac{1}{\sqrt{(L_1 + L_2)(C)}} \quad (10.2.6.6)$$

$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)(C)}} \quad (10.2.6.7)$$

Therefor, G for sustained oscillations can be given by,

$$G = \frac{1}{H} = \frac{L_2}{L_1} \quad (10.2.6.8)$$

10.2.7. For Hartley oscillator frequency generated can be given as

$$f = \frac{1}{2\pi\sqrt{(L_1 + L_2)C}} \quad (10.2.7.1)$$

We know that for an opamp gain is given by:

$$G = \frac{R_2}{R_1} \quad (10.2.7.2)$$

Here,

$$G(S) = \frac{R_2}{R_1} = \frac{L_2}{L_1} \quad (10.2.7.3)$$

referring to 10.2.6.8
And,

$$H(s) = \frac{V_o}{V_f} = \frac{L_1}{L_2} \quad (10.2.7.4)$$

referring to 10.2.3.18

10.2.8. **Simulation:**
Taking the following values,and applying in 10.2.7.1

Component	Value
R_1	10KΩ
R_2	100KΩ
R_3	∼
L_1	$1\mu H$
L_2	$1\mu H$
C	120 pF

We get f = 103 MHz
Feedback factor for Hartley given by:

$$H = \frac{L_1}{L_2} = 1 \tag{10.2.8.1}$$

W.K.T, $GH = 1$, for sustained oscillation
∴ Minimum amplification Gain,G = 1
(GH ¡=1 for a stable system)

11 Root Locus

11.1 Introduction

11.1.1. A feedback control system has the characteristic equation

$$s^2 + 6Ks + 2s + 5 = 0, \quad K > 0. \tag{11.1.1.1}$$

Find the open loop gain.
Solution: can be expressed as

$$1 + \frac{6ks}{s^2 + 2s + 5} = 0 \tag{11.1.1.2}$$
$$\implies 1 + KG(s) = 0 \tag{11.1.1.3}$$
$$\implies G(s) = \frac{6s}{s^2 + 2s + 5} \tag{11.1.1.4}$$

11.1.2. Find the poles and zeros of $G(s)$
Solution: The poles and zeros are at

$$\begin{aligned} p_1, p_2 &= -1 \pm 2\jmath \\ z &= 0 \end{aligned} \tag{11.1.2.1}$$

11.1.3. Find the number of root locus branches.
Solution: For $G(s)$, let

- P- No. of finite poles
- Z- No. of finite zeros

The number of root locus branches

$$N = \begin{cases} P & P > Z, \\ Z & P < Z. \end{cases} \tag{11.1.3.1}$$

The root locus branches start at the open loop poles and end at open loop zeros. From (11.1.2.1),

$$P = 2, Z = 1 \implies N = 2 \tag{11.1.3.2}$$

One branch originates at each pole and ends at zero and the other branch starts from other pole and goes to infinity.

11.1.4. Find the centroid and the angle of asymptotes.
Solution: Some of the root locus branches approach infinity when $P \neq Z$. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as centroid. The ordinate of the centroid is given by

$$\frac{\sum_{i=1}^{P} p_i - \sum_{j=1}^{Z} z_j}{P - Z} = \frac{-2 - 0}{2 - 1} \tag{11.1.4.1}$$
$$= -2 \tag{11.1.4.2}$$

after substituting from (11.1.2.1). Thus, the centroid is at $(-2, 0)$. The formula for angle of asymptotes θ is

$$\theta = \frac{(2q + 1)\pi}{P - Z}, \quad q = 0, 1, \dots P - Z - 1 \tag{11.1.4.3}$$
$$= \pi \tag{11.1.4.4}$$

One branch meets the real axis at a breakaway point then goes to zero at origin and other goes to infinity following the asymptote($y = 0$).

11.1.5. Find the intersection points of root locus branches with the imaginary axis.
Solution: Using the Routh array method,

11.1.6. Find the angle of departure with respect to the pole p_k.
Solution: The angle of departure with respect to the kth (complex) pole is defined as

$$\phi_k^d = \begin{cases} \pi - \phi_k, & \mathrm{Im}\,(p_k) \neq 0 \\ 0 & \mathrm{Im}\,(p_k) = 0 \end{cases} \tag{11.1.6.1}$$

where,

$$\phi_k = \sum_{i=1}^{P} \underline{/p_i - p_k} - \sum_{j}^{Z} \underline{/p_k - z_j} \tag{11.1.6.2}$$

For $p_1 = -1 + 2\jmath$,

$$\phi_1 = \underline{/-1 + 2\jmath - (-1 - 2\jmath)} - \underline{/-1 + 2i - 0} \tag{11.1.6.3}$$

$$\implies = \frac{\pi}{2} + \left(\pi - \tan^{-1} 2\right) = \frac{3\pi}{2} - \tan^{-1} 2 \tag{11.1.6.4}$$

11.1.7. Find the angle of arrival with respect to the zero z_k

Solution: The angle of arrival with respect to the kth (complex) zero is defined as

$$\phi_k^a = \begin{cases} \pi - \phi_k, & \text{Im}\,(z_k) \neq 0 \\ 0 & \text{Im}\,(z_k) = 0 \end{cases} \tag{11.1.7.1}$$

where,

$$\phi_k = \sum_{i=1}^{P} \underline{/z_k - p_i} - \sum_{j}^{Z} \underline{/z_k - z_j} \tag{11.1.7.2}$$

Since there are no complex zeros there is no angle of arrival.

11.1.8. Find the breakaway point for the root locus.

Solution: From (11.1.1) and (11.1.1.4),

$$K = \frac{-\left(s^2 + 2s + 5\right)}{6s}, \tag{11.1.8.1}$$

$$\frac{dK}{ds} = 0 \implies \left[1 - \frac{5}{s^2}\right] = 0 \tag{11.1.8.2}$$

$$\implies s = -\sqrt{5} \tag{11.1.8.3}$$

which is the breakaway point that lies only in the left half plane. Some information on the breakaway point is available below.

- While varying K, the point where the Root Locus enters the real axis is called th breakaway point.
- It is the point on a real axis segment of the root locus between two real poles where the two real closed-loop poles meet and diverge to become complex conjugates.
- As the root locus is symmetric about the real axis there will be two roots at the breakaway point.

11.1.9. Plot the Root Locus.

Solution: The following code generates the desired plot in Fig. 11.1.9.1.

```
codes/ee18btech11046.py
```

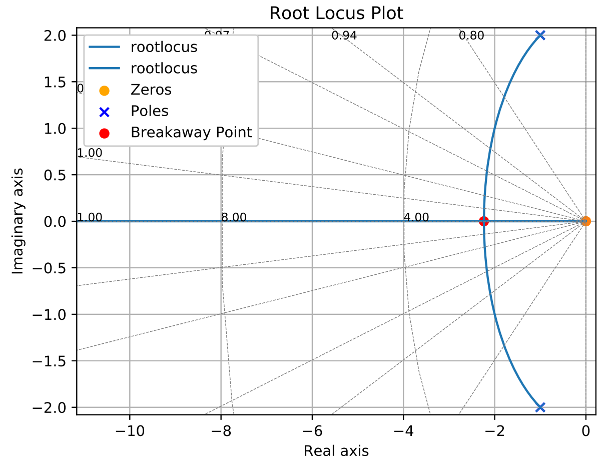


Fig. 11.1.9.1

11.2 Example

11.1. A unity negative feedback system has the open loop transfer function

$$G(s) = \frac{K}{s(s+1)(s+3)} \tag{11.1.1}$$

Find the value of the gain K (>0) at which the root locus crosses the imaginary axis.

Solution: From , the characteristic function of the control system is

$$s^3 + 4s^2 + 3s + K = 0 \tag{11.1.2}$$

If all elements of any row of the Routh array table are zero, then the root locus branch intersects the imaginary axis. The Routh array is

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ 4 & K \\ (12-K)/4 & 0 \\ K & \end{vmatrix} \tag{11.1.3}$$

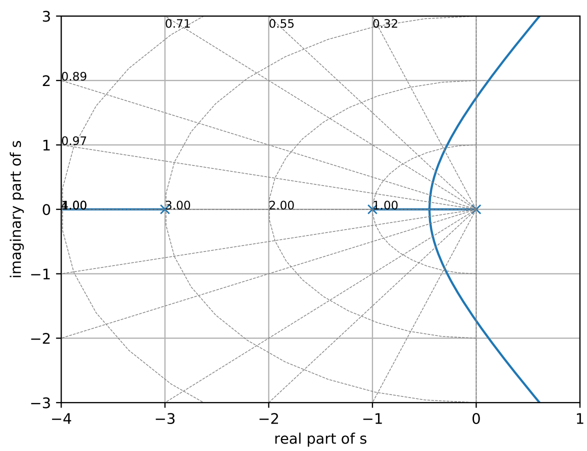
$$\implies \frac{12-K}{4} = 0 \tag{11.1.4}$$

$$\text{or, } K = 12. \tag{11.1.5}$$

The following code plots the root locus in Fig. 11.2.1

```
codes/ee18btech11050.py
```

11.2. Root Locus plot



0.71
0.55
0.32
0.89
0.97
4.00
3.00
2.00
1.00
3
2
1
0
−1
−2
−3
−4
−3
−2
−1
0
1
imaginary part of s
real part of s

Fig. 11.2.1