Control Systems

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Contents

***Abstract*—The objective of this manual is to introduce control system design at an elementary level.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/ketan/codes
```

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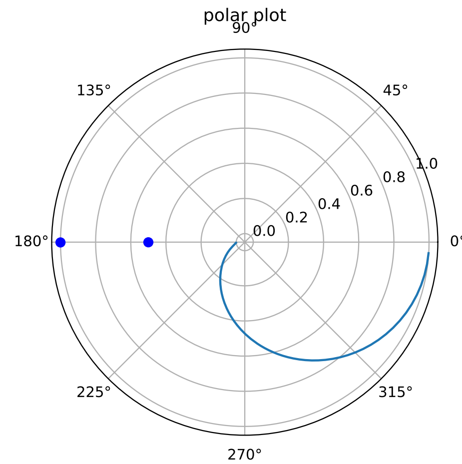

Fig. 1.1.1

1 Frequency Response Analysis

1.1 Polar Plot

1.1.1. Sketch the Polar Plot for

$$G(s) = \frac{1}{(1+s)(1+2s)} \tag{1.1.1}$$

Solution: The following code generates Fig. 1.1.1

```
codes/ee18btech11012.py
```

The polar plot is to the right of $(-1, 0)$. Hence the closed loop system is stable.

1.1.2. Sketch the polar plot of

$$G(s) = \frac{1}{(s^2)(s+1)(s+2)}. \tag{1.1.2}$$

Solution: Substituting $s = \jmath\omega$ in (1.1.2), Now the magnitude will be

$$r = |G(\jmath\omega)| = \frac{1}{(\omega^2)(\sqrt{1+\omega^2})(\sqrt{1+4\omega^2})} \tag{1.1.3}$$

$$\theta = \angle G(\jmath\omega) = -\tan^{-1}(0) - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.4}$$

$$= 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{1.1.5}$$

The polar plot is the (r, θ) plot for $\omega \in (0, \infty)$. The following python code generates the polar plot in Fig. 1.1.2

```
codes/ee18btech11028.py
```

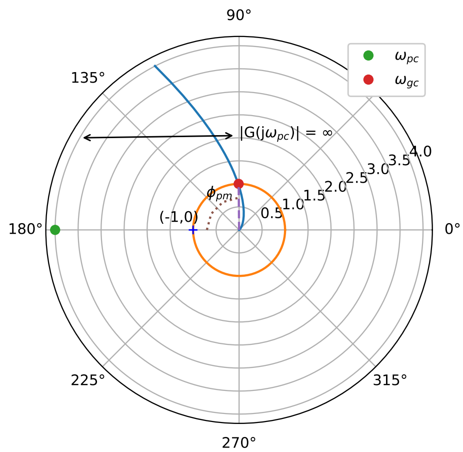

Fig. 1.1.2

The location of $(-1, 0)$ with respect to the polar plot provides information regarding the stability of the system.

- If $(-1, 0)$ is not enclosed, then it is stable.
- If $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 1.1.2, the point $(-1, 0)$ is enclosed by the polar plot, which implies system is not stable. The polar plot also provides info on the GM and PM, which can then be used for determining the stability of the system.

- If the $GM > 1 \cap PM > 0$, then the control system is **stable**.
- If the $GM = 1 \cap PM = 0$, then the control system is **marginally stable**.
- If the $GM < 1 \cup PM < 0$, then the control system is **unstable**.

Therefore, our system is unstable $\because GM < 1 \cap PM < 0$.

1.1.3. Sketch the Polar Plot of

$$G(s) = \frac{1}{s(1 + s^2)} \tag{1.1.6}$$

Solution: From (1.1.6),

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1 - \omega^2)} \tag{1.1.7}$$

$$\left|G(\jmath\omega)\right| = \frac{1}{\left|\omega(1 - \omega^2)\right|} \tag{1.1.8}$$

$$\angle G(\jmath\omega) = \begin{cases} \frac{\pi}{2} & \omega > 1 \\ -\frac{\pi}{2} & 0 < \omega < 1 \end{cases} \tag{1.1.9}$$

The corresponding polar plot is generated in Fig. 1.1.3 using

```
codes/ee18btech11023.py
```

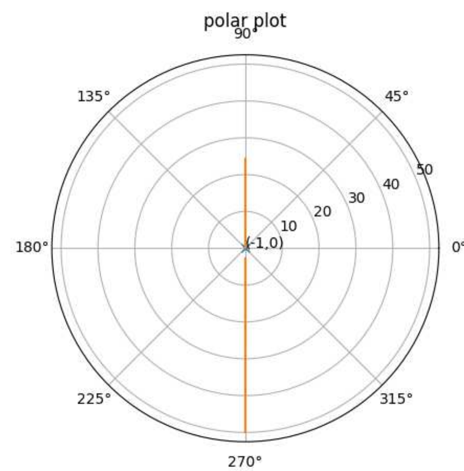

Fig. 1.1.3

In Fig. 1.1.3, (–1,0) is exactly on the polar plot. Hence, the system is marginally stable.

1.1.4. Sketch the Polar Plot of

$$G(s) = \frac{\left(1 + \frac{s}{29}\right)(1 + 0.0025s)}{(s^3)(1 + 0.005s)(1 + 0.001s)} \tag{1.1.10}$$

Solution: The following code generates the polar plot in Fig. 2.6.1

```
codes/ee18btech11029.py
```

- The polar plots use open loop transfer function to determine the stability and hence reference point is shifted to $(-1, 0)$
- If $(-1, 0)$ is left of the polar plot or $(-1, 0)$ is not enclosed, then it is stable
- If $(-1, 0)$ is on right side of the polar plot or $(-1, 0)$ is enclosed by polar plot then it is unstable.

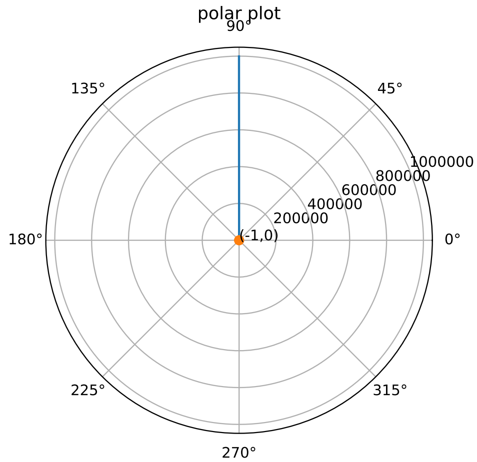

Fig. 1.1.4

- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 2.6.1, $(-1, 0)$ is on the polar plot so the system is marginally stable.

1.1.5. Plot the polar plot of

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}. \qquad (1.1.11)$$

Solution:

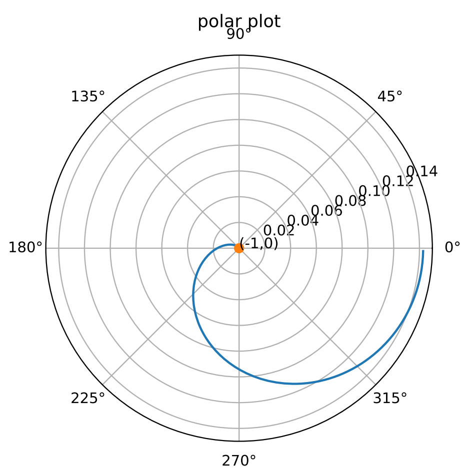

Fig. 1.1.5

The following python code generates the polar plot in Fig. 1.1.5

```
codes/ee18btech11033.py
```

∵ (-1,0) is on the right side of the polar plot, the system is unstable.

1.1.6. Plot the polar plot of

$$G(s) = \frac{100(s+5)}{s(s+3)(s^2+4)}. \qquad (1.1.12)$$

Solution: The following python code generates the polar plot in Fig. 1.1.6

```
codes/ee18btech11042.py
```

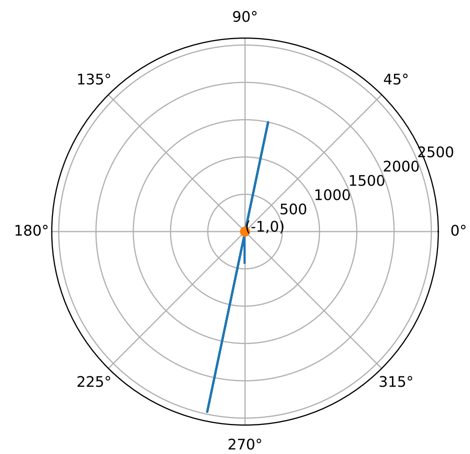

Fig. 1.1.6

Since (-1,0) is on the polar plot, the above system is marginally stable.

1.2 Direct and Inverse Polar Plot

Sketch the direct polar plot for a unity feedback system with open loop transfer function

$$G(s) = \frac{1}{s(1+s)^2} \qquad (1.2.1)$$

Solution: The polar plot is obtained by plotting (r, ϕ)

$$r = |H(\jmath\omega)||G(\jmath\omega)| \qquad (1.2.2)$$

$$\phi = \angle H(\jmath\omega)G(\jmath\omega), 0 < \omega < \infty \qquad (1.2.3)$$

The following code plots the polar plot in Fig. 1.2.1

```
codes/ee18btech11002/polarplot.py
```

Sketch the inverse polar plot for (1.2.1)

Solution: The above code plots the polar plot in Fig. 1.2.2 by plotting $\left(\frac{1}{r}r, -\phi\right)$

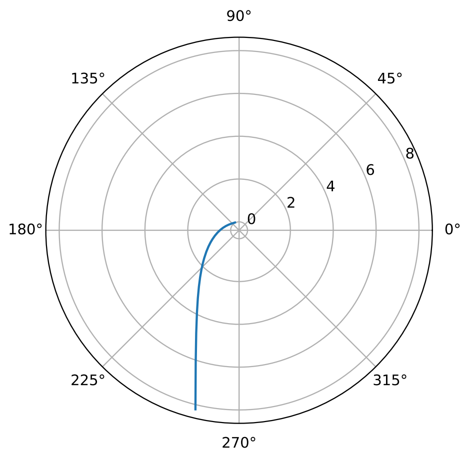

Fig. 1.2.1: Polar Plot

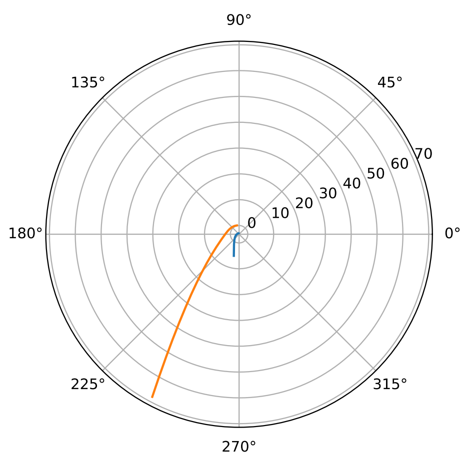

Fig. 1.2.2: Inverse Polar Plot

1.3 Bode Plot

1.3.1. Sketch the Bode Magnitude and Phase plot for the following system. Also compute the gain margin and the phase margin.

$$G(s) = \frac{10}{s(1+0.5s)(1+.01s)} \tag{1.3.1}$$

Solution: The Bode magnitude and phase plot are available in Fig. 1.3.1 and generated by

```
codes/ee18btech11048.py
```

The pole-zero locations are available in Table 1.3.1.

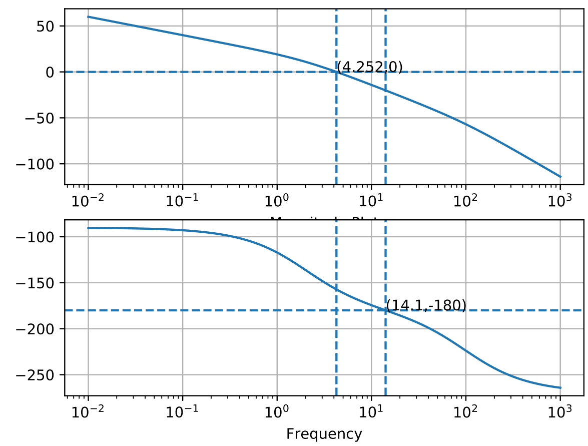

Fig. 1.3.1: Graphs

Zeros	Poles
-	0
	-2
	-100

TABLE 1.3.1: Zeros and Poles

The *Gain* and *Phase* of (1.3.2) are

$$\left|G(\jmath\omega)\right| = \frac{100}{\omega\sqrt{(0.5\omega)^2+1}\sqrt{(0.01\omega)^2+1}} \tag{1.3.2}$$

$$\angle G(\jmath\omega) = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{100}\right) \tag{1.3.3}$$

Hence,

$$\left|G\left(\jmath\omega_{gc}\right)\right| = 1 \tag{1.3.4}$$

$$\implies \omega_{gc} = 4.25 \tag{1.3.5}$$

$$\angle G\left(\jmath\omega_{gc}\right) = -157.2 \tag{1.3.6}$$

$$\implies PM = 22.8 \tag{1.3.7}$$

Similarly,

$$\angle G\left(\jmath\omega_{pc}\right) = -180^\circ \tag{1.3.8}$$

$$\implies \omega_{pc} = 14.1 \tag{1.3.9}$$

$$\implies -\left|G\left(\jmath\omega_{pc}\right)\right| = -20.2dB \tag{1.3.10}$$

$$\implies GM = 20.2dB \tag{1.3.11}$$

1.3.2. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} \tag{1.3.12}$$

Also compute gain margin and phase margin .
Solution: From (1.3.12), we have

$$G(\jmath\omega) = \frac{75(1+0.2\jmath\omega)}{\jmath\omega\left((\jmath\omega)^2+16\jmath\omega+100\right)} \tag{1.3.13}$$

poles = 0 , -8-6j , -8+6j
zeros = -5
Gain and phase plots are shown in Fig. 1.3.2

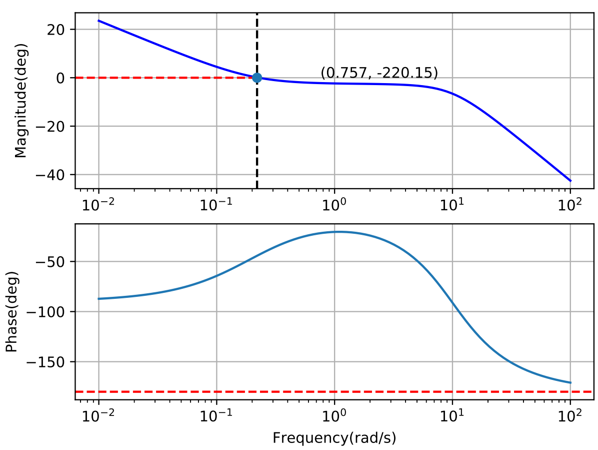

Fig. 1.3.2: a

The following code plots Fig. 1.3.2

```
codes/ee18btech11049.py
```

Solving

$$\left|G(\jmath\omega)\right| = \frac{75\sqrt{\omega^2+25}}{\omega\sqrt{(\omega+6)^2+64}\sqrt{(\omega-6)^2+64}} = 1, \tag{1.3.14}$$

or from Fig. 1.3.2, the gain crossover frequency

$$\implies \omega_{gc} = 0.757 \tag{1.3.15}$$

$$\angle G\left(\jmath\omega_{gc}\right) = -88.3 \tag{1.3.16}$$

$$\implies PM = 91.7 \tag{1.3.17}$$

Solution: From Fig. 1.3.2 ,we can say that phase never crosses $-180°$. So , the gain margin is *infinite*. Which means we can add any gain, and the equivalent closed loop system never becomes unstable.

1.3.3. Plot the Bode magnitude and phase plots for the following system

$$G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)} \tag{1.3.18}$$

Also compute gain margin and phase margin .
Solution: Substituting $s = \jmath\omega$ in (3.7.1.1) and assuming K =1,

$$G(\jmath\omega) = \frac{(\jmath\omega)^2}{(1+0.2\jmath\omega)(1+0.02\jmath\omega)} \tag{1.3.19}$$

The corner frequencies are

$$\omega_{c1} = 1/0.2 = 5 \tag{1.3.20}$$

$$\omega_{c2} = 1/0.02 = 50 \tag{1.3.21}$$

$$20\log\left|G(\jmath\omega)\right| = 20\log\left|(\jmath\omega)^2\right| - 20\log\left|(1+0.2\jmath\omega)\right| - 20\log\left|(1+0.02\jmath\omega)\right| \tag{1.3.22}$$

The various values of G($\jmath\omega$) are available in Table 3.7.1, in the increasing order of their corner frequencies also slope contributed by each term and the change in slope at the corner frequency. The pase

TERM	Corner Freq	Slope	Slope change
$(\jmath\omega)^2$	$--$	+40	$--$
$\frac{1}{1+\jmath 0.2}$	$\omega_{c1} = \frac{1}{0.2}$	-20	40-20=20
$\frac{1}{1+\jmath 0.02}$	$\omega_{c2} = \frac{1}{0.02}$	-20	20-20=0

TABLE 1.3.2: Magnitude

$$\phi = \angle G(\jmath\omega) = 180° - tan^{-1}(0.2\omega) - tan^{-1}(0.02\omega) \tag{1.3.23}$$

The phase angle of G($\jmath\omega$) are calculated for various value of ω in Table 1.3.3. The magnitude and phase plot are generated in Fig. 1.3.3 using the following python code

```
codes/es17btech11002.py
```

$\because$ the gain crossover frequency is 2 and the corresponding gain At $\omega = 2$ is 13dB,

$$20\log K = -13db \tag{1.3.24}$$

$$\implies K = 0.65 \tag{1.3.25}$$

ω	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(0.02\omega)$	$\phi = \angle G(j\omega)$
0.5	5.7	0.6	174
1	11.3	1.1	168
2	21.8	2.3	156
5	45	5.7	130
10	63.4	11.3	106
50	84.3	45	50

TABLE 1.3.3: Phase

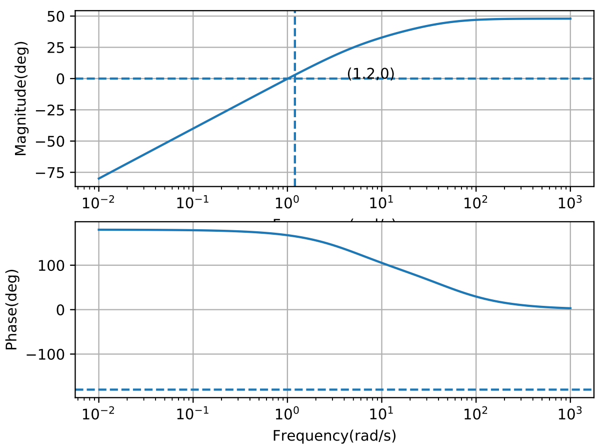

Fig. 1.3.3: Graphs

Solving (1.3.19) or from Fig. 1.3.3, the gain crossover frequency,

$$\omega_{gc} = 1.2 \tag{1.3.26}$$

$$\implies PM = 344.8 \tag{1.3.27}$$

From Fig. 1.3.3 ,we can say that phase never crosses $-180°$. So, the gain margin is *infinite*. Which means we can add any gain, and the equivalent closed loop system never goes unstable.

1.3.4. Sketch the bode magnitude and phase plots for the closed loop (negative feedback) system given by:

$$G(s) = \frac{100(s+2)(s+4)}{s^2 - 3s + 10} \tag{1.3.28}$$

$$H(s) = \frac{1}{s} \tag{1.3.29}$$

Solution: The system can be represented as:

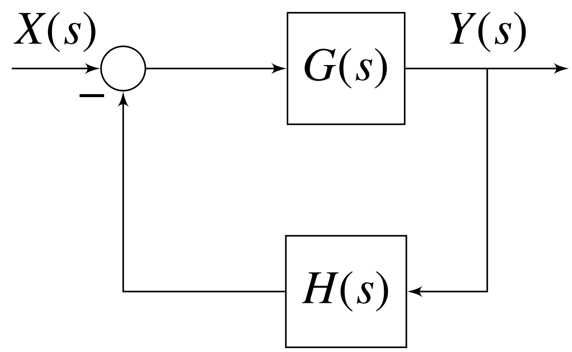

Fig. 1.3.4: Block diagram for the system

The closed loop transfer function of the system is given by:

$$G_m(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s).H(s)} \tag{1.3.30}$$

$$= \frac{100s(s+2)(s+4)}{s^3 + 97s^2 + 610s + 800} \tag{1.3.31}$$

Evaluate at $s = \jmath\omega$:

$$G_m(\jmath\omega) = \frac{100\jmath\omega(\jmath\omega+2)(\jmath\omega+4)}{(\jmath\omega)^3 + 97(\jmath\omega)^2 + 610(\jmath\omega) + 800} \tag{1.3.32}$$

$$= \frac{-600\omega^2 + \jmath\left(800\omega - 100\omega^3\right)}{800 - 97\omega^2 + \jmath(610\omega - \omega^3)} \tag{1.3.33}$$

From (1.3.33):

$$\left|G_m(\jmath\omega)\right| = \frac{\sqrt{(600\omega^2)^2 + (800\omega - 100\omega^3)^2}}{\sqrt{(800 - 97\omega^2)^2 + (610\omega - \omega^3)^2}} \tag{1.3.34}$$

$$\underline{/G_m}(\jmath\omega) = \tan^{-1}\left(\frac{\omega^2 - 8}{6\omega}\right) - \tan^{-1}\left(\frac{610\omega - \omega^3}{800 - 97\omega^2}\right) \tag{1.3.35}$$

The following code plots the bode magnitude and phase plots in Fig. 1.3.5:

```
codes/ee18btech11045/
    ee18btech11045_bode1.py
```

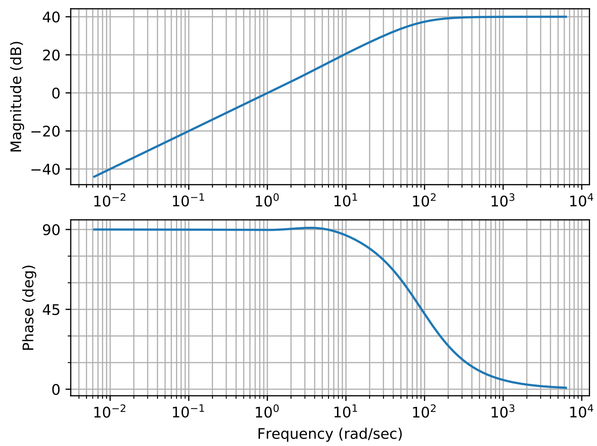

Fig. 1.3.5: Bode plot for $G_m(\jmath\omega)$

$$G(\jmath\omega)H(\jmath\omega) = \left(\frac{100(\jmath\omega+2)(\jmath\omega+4)}{(\jmath\omega)^2 - 3\jmath\omega + 10}\right)\left(\frac{1}{\jmath\omega}\right) \tag{1.3.36}$$

$$= \frac{100\left(-\omega^2 + 8 + \jmath 6\omega\right)}{3\omega^2 + \jmath\left(10\omega - \omega^3\right)} \tag{1.3.37}$$

Using (1.3.37)

$$\left|G(\jmath\omega)H(\jmath\omega)\right| = \frac{100\sqrt{(8-\omega^2)^2 + (6\omega)^2}}{\sqrt{(3\omega^2)^2 + (10\omega - \omega^3)^2}} \tag{1.3.38}$$

$$\underline{/G}(\jmath\omega)H(\jmath\omega) = \tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right) - \tan^{-1}\left(\frac{10-\omega^2}{3\omega}\right) \tag{1.3.39}$$

At the phase crossover frequency ω_{pc}:

$$\left|\underline{/G}(\jmath\omega)H(\jmath\omega)\right| = 180 \tag{1.3.40}$$

$$\implies \tan^{-1}\left(\frac{6\omega_{pc}}{8-\omega_{pc}^2}\right) - \tan^{-1}\left(\frac{10-\omega_{pc}^2}{3\omega_{pc}}\right) = 180 \tag{1.3.41}$$

Solving the above equation:

$$\frac{6\omega_{pc}}{8-\omega_{pc}^2} = \frac{10-\omega_{pc}^2}{3\omega_{pc}} \tag{1.3.42}$$

$$\implies \omega_{pc} = 5.8 rad/sec \tag{1.3.43}$$

$$\left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_{pc}} = 28.1 dB \tag{1.3.44}$$

Gain Margin GM :

$$GM = 0 - \left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_{pc}} dB \tag{1.3.45}$$

$$= -28.1 dB \tag{1.3.46}$$

At the gain crossover frequency ω_{gc}:

$$\left|G(\jmath\omega)H(\jmath\omega)\right|_{\omega=\omega_{gc}} = 1 \tag{1.3.47}$$

From (1.3.38),

$$10^4\left(\left(8-\omega^2\right)^2 + (6\omega)^2\right) = 9\omega^4 + \left(10\omega - \omega^3\right)^2 \tag{1.3.48}$$

$$\implies \omega_{gc} = 100.15 rad/sec \tag{1.3.49}$$

Substitute ω_{gc} in (1.3.39):

$$\underline{/G}(\jmath\omega)H(\jmath\omega)_{\omega=\omega_{gc}} = 265^\circ \tag{1.3.50}$$

Phase Margin PM:

$$PM = 180^\circ - \underline{/G}(\jmath\omega)H(\jmath\omega)_{\omega=\omega_{gc}} \tag{1.3.51}$$

$$= 180^\circ - 265^\circ = -85^\circ \tag{1.3.52}$$

The following code is used to verify the gain and phase margins:

```
codes/ee18btech11045/
    ee18btech11045_bode2.py
```

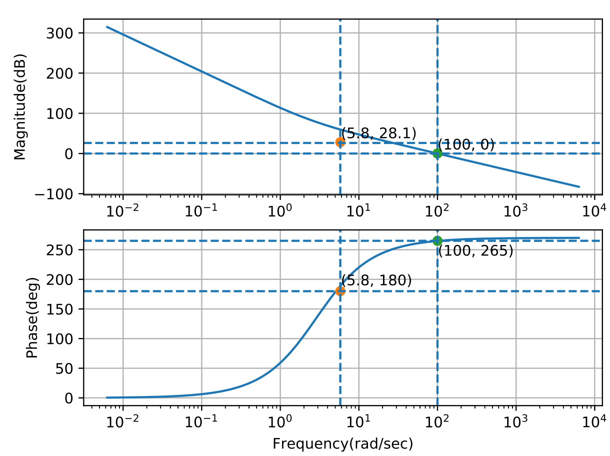

Fig. 1.3.6: Bode plot for $G(\jmath\omega)H(\jmath\omega)$

As both the Gain Margin (GM) and Phase

Margin (PM) are found to be negative, the system is unstable.

2 Stability in Frequency Domain

2.1 Nyquist Criterion

2.1.1. Using Nyquist criterion find the range of K for which closed loop system is stable.

$$G(s) = \frac{K}{s(s+6)} \tag{2.1.1.1}$$

$$H(s) = \frac{1}{s+9} \tag{2.1.1.2}$$

Solution: The system flow can be described as,

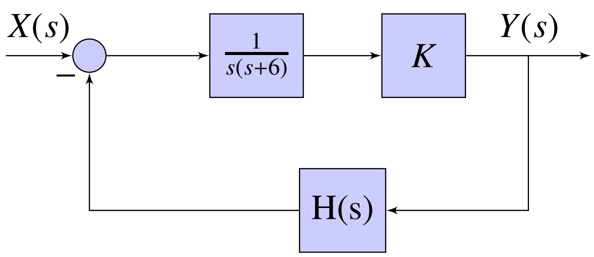

Fig. 2.1.1

$$G_1(s) = \frac{1}{s(s+6)}. \tag{2.1.1.3}$$

For Nyquist plot,

$$\text{Im}\left\{G_1(\jmath\omega)H(\jmath\omega)\right\} = \frac{-(54-\omega^2)}{(\omega)(\omega^2+56)(\omega^2+81)} \tag{2.1.1.4}$$

$$\text{Re}\left\{G_1(\jmath\omega)H(\jmath\omega)\right\} = \frac{-15\omega}{(\omega)(\omega^2+56)(\omega^2+81)} \tag{2.1.1.5}$$

From (2.1.1.4) and (2.1.1.5)
Nyquist Stability Criterion:

$$N = Z - P \tag{2.1.1.6}$$

where Z is # unstable poles of closed loop transfer function, P is # unstable poles of open loop transfer function and N is # clockwise encirclement of $(-1/K, 0)$.
For stable system,

$$Z = 0 \tag{2.1.1.7}$$

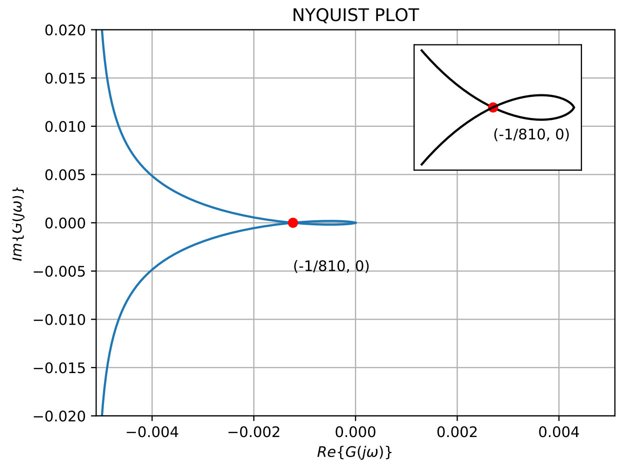

Fig. 2.1.2: Nyquist plot for $G_1(s)H(s)$

From (2.1.1.2) and (2.1.1.3),

$$P = 0 \tag{2.1.1.8}$$

$$\implies N = 0 \tag{2.1.1.9}$$

Since, there is a zero at origin, an infinite radius half circle will enclose the right hand side of end points of the Nyquist plot. So for (2.1.1.9),

$$\implies \frac{-1}{K} < \frac{-1}{810} \implies K < 810 \tag{2.1.1.10}$$

And also,

$$K > 0 \tag{2.1.1.11}$$

$$\implies 0 < K < 810 \tag{2.1.1.12}$$

The following python code generates Fig. 2.1.2

```
codes/ee18btech11028_1.py
```

2.1.2. Using Nyquist criterion, find out whether the system below is stable or not

$$G(s) = \frac{41}{s^2(s+3)} \tag{2.1.2.1}$$

$$H(s) = (s+4) \tag{2.1.2.2}$$

Solution: According to the Nyquist criteria the number of unstable closed-loop poles (Z) is equal to the number of unstable open-loop poles (P) plus the number of clockwise encirclements (N) of the point (-1,j0) of the Nyquist plot of G(s)H(s), i.e

$$Z = N + P \tag{2.1.2.3}$$

Open loop transfer function :

$$G(s)H(s) = \frac{41(s+4)}{s^2(s+3)} \tag{2.1.2.4}$$

Closed loop transfer function:

$$T(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{41}{s^3+3s^2+41s+164} \tag{2.1.2.5}$$

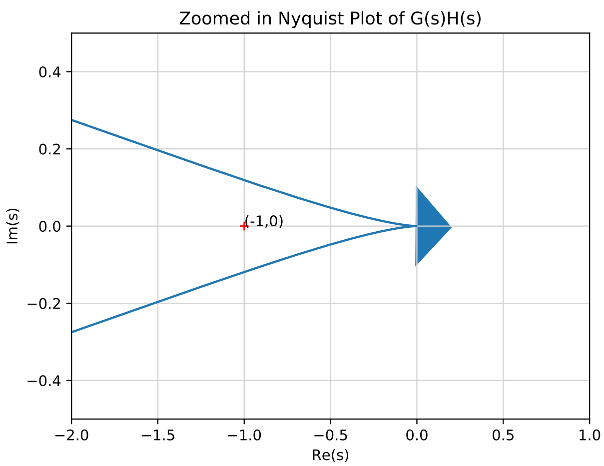

Fig. 2.1.3

In Fig.2.1.3 it can be seen that there is a clockwise encirclement around (-1+0j). As the open loop transfer function has zero pole of multiplicity 2, therefore it should be assumed that the phasor travels 2 times clock-wise along a semicircle of infinite radius.

N=2, P=0

$$\implies Z = 2 \tag{2.1.2.6}$$

Therefore, The system T(s) is unstable as it has two poles on the right side of the s plane. The following code generates the nyquist plot

```
codes/ee18btech11041.py
```

2.1.3. Using Nyquist criterion, find out the range of K for which the closed loop system will be stable.

$$G(s) = \frac{K(s+2)(s+4)}{s^2-3s+10}; H(s) = \frac{1}{s} \tag{2.1.3.1}$$

Solution: The system flow can be described by Fig. 2.1.4 From (2.1.3.1),

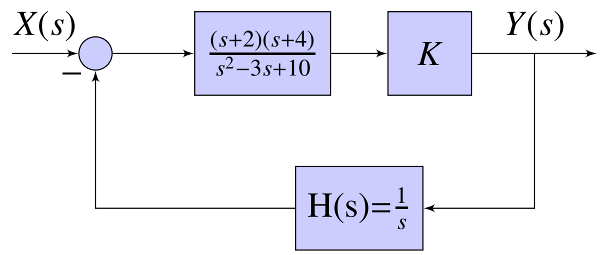

Fig. 2.1.4

$$G(s)H(s) = \frac{K(s+2)(s+4)}{s(s^2-3s+10)} \tag{2.1.3.2}$$

$$G(\jmath\omega)H(\jmath\omega) = \frac{K(j\omega+2)(j\omega+4)}{j\omega((10-\omega^2)-3j\omega)} \tag{2.1.3.3}$$

$$\text{Re}\left\{G(\jmath\omega)H(\jmath\omega)\right\} = \frac{K(84\omega^2-9\omega^4)}{\omega^6-11\omega^4+100\omega^2} \tag{2.1.3.4}$$

$$\text{Im}\left\{G(\jmath\omega)H(\jmath\omega)\right\} = \frac{K(-\omega^5+36\omega^3-80\omega)}{\omega^6-11\omega^4+100\omega^2} \tag{2.1.3.5}$$

The Nyquist plot is a graph of Re $\{G(\jmath\omega)H(\jmath\omega)\}$ vs Im $\{G(\jmath\omega)H(\jmath\omega)\}$. Let's take K =1 and draw the nyquist plot .

The following python code generates the Nyquist plot in Fig. 2.1.5

```
codes/ee18btech11016.py
```

Note that this nyquist plot is plotted when K=1.

Nyquist criterion-For the stable system :

$$Z = P + N = 0, \tag{2.1.3.6}$$

Since from the equation (2.1.3.2), P = 2 as the number of poles on right hand side of s-plane is equal to 2 .So, for Z to be equal to 0 ,we have to choose the range of K such that N should be equal to -2.

If we consider the Nyquist plot with K term i.e. of equation (2.1.3.2) , then it will cut x-axis at x = -0.254K , x = 0 and at x =

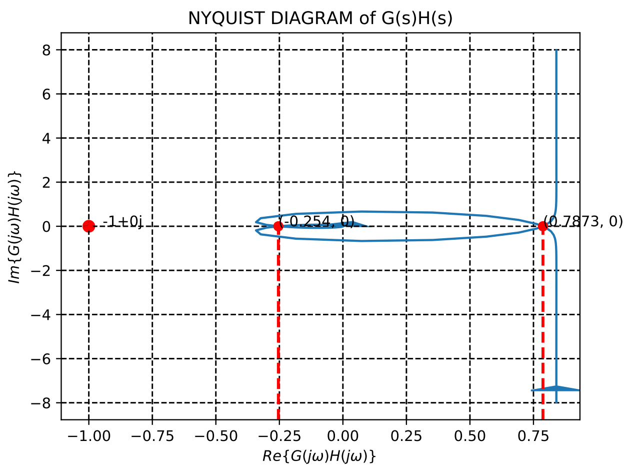

Fig. 2.1.5

Variable	Description
Z	Poles of $\frac{G(s)}{1+G(s)H(s)}$in right half of s plane
N	No of encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot
P	Poles of $G(s)H(s)$ in right half of s plane

TABLE 2.1.1

0.7873K (as we have nyquist graph at K=1, now we just need to multiply the intersected coordinates on x-axis by K).

So, we have to make sure that $(-1+\ j0)$ should be included in between x = -0.254K to x = 0, because then only N = -2 (as the no. of encirclements are 2 in anticlockwise direction in this case so N=-2)

$$-0.254K < -1 < 0 \tag{2.1.3.7}$$

So,

$$K > \frac{1}{0.254} \tag{2.1.3.8}$$

i.e.

$$K > 3.937 \tag{2.1.3.9}$$

Hence K > 3.937 ensures that the system is stable as no. of poles on the right hand side of s-plane (in this case) is 0.

2.2

2.3

2.4

2.5

2.6 Nyquist and Routh-Hurwitz

2.6.1. In the block diagram Fig.2.6.1

$$G(s) = \frac{K}{(s+4)(s+5)} \tag{2.6.1.1}$$

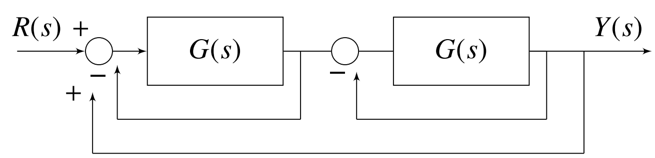

Fig. 2.6.1

2.6.2. Find the range of K for stability by Nyquist criterion
Solution:

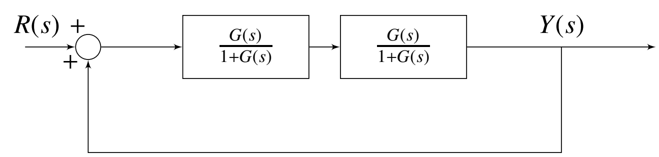

Fig. 2.6.2

The open loop transfer function from Fig.2.6.2

$$T(s) = \left(\frac{\frac{K}{(s+4)(s+5)}}{1+\frac{K}{(s+4)(s+5)}}\right)^2 \tag{2.6.2.1}$$

$$T(\jmath\omega) = \left(\frac{\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}{1+\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}\right)^2 \tag{2.6.2.2}$$

- Since it is connected in positive feedback the transfer function cuts at $(1, \jmath 0)$

$$\implies \text{Re}\{T(\jmath\omega)\} = 1 \tag{2.6.2.3}$$

$$\implies \text{Im}\{T(\jmath\omega)\} = 0 \tag{2.6.2.4}$$

$$\left(\frac{\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}{1+\frac{K}{(\jmath\omega+4)(\jmath\omega+5)}}\right)^2 = 1 + \jmath 0 \tag{2.6.2.5}$$

$$(\jmath\omega + 4)(\jmath\omega + 5) + 2K = 0 \tag{2.6.2.6}$$

$$-\omega^2 + 9\jmath\omega + 20 + 2K = 0 \tag{2.6.2.7}$$

From (2.6.2.4)

$$20 + 2K = 0 \tag{2.6.2.8}$$

$$\implies K = -10 \tag{2.6.2.9}$$

The minimum value of stability for the system to be stable is

$$K_{min} > -10 \tag{2.6.2.10}$$

The range of K for which the system is stable is

$$-10 < K < \infty \tag{2.6.2.11}$$

2.6.3. From the table.2.6.1, Stability criterion for K is N+P=Z

K	P	N	Z	Description
-10	0	0	0	System is marginally stable
-9	0	0	0	System is stable
-11	0	1	1	System is unstable

TABLE 2.6.1

2.6.4. Verify the Nyquist plots by

codes/ee18btech11029_1.py

2.6.5. Verify the result using Routh-Hurwitz criterion
Solution: The characteristic equation is

$$1 - T(s) = 0 \tag{2.6.5.1}$$

$$1 - \left(\frac{\frac{K}{(s+4)(s+5)}}{1 + \frac{K}{(s+4)(s+5)}}\right)^2 = 0 \tag{2.6.5.2}$$

$$1 + 2\left(\frac{K}{(s+4)(s+5)}\right) = 0 \tag{2.6.5.3}$$

$$s^2 + 9s + 20 + 2K = 0 \tag{2.6.5.4}$$

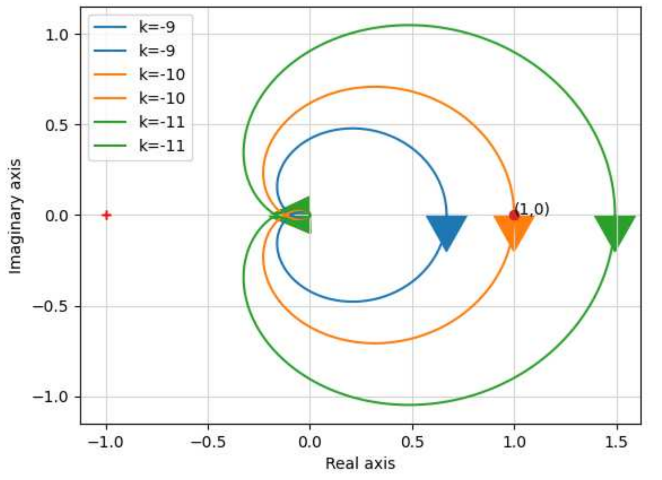

Fig. 2.6.3: Nyquist Plot

$$\begin{vmatrix} s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 20+2K \\ 9 & 0 \\ 20+2K & 0 \end{vmatrix} \tag{2.6.5.5}$$

For a system to be stable it should not have any sign changes

$$20 + 2K > 0 \tag{2.6.5.6}$$

This is valid for all positive values of K but the minimum value of K is

$$K > -10 \tag{2.6.5.7}$$

So the range of K for stability is

$$-10 < K < \infty \tag{2.6.5.8}$$

2.6.6. Verify the result by

codes/ee18btech11029_2.py

2.7 Nyquist

Consider the system shown in Fig. 2.7.1 below. Sketch the nyquist plot of the system when

1) $G_c(s) = 1$
2) $G_c(s) = 1 + \frac{1}{s}$

and determine the maximum value of K for stability. Take

$$G(s) = \frac{K}{s(1+s)(1+4s)} \tag{2.7.1}$$

Solution: For $G_c(s) = 1$,

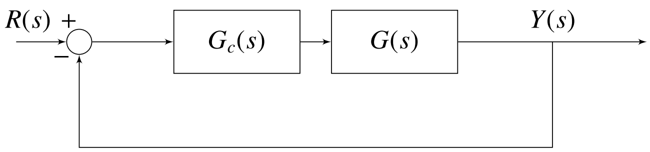

Fig. 2.7.1

The open loop transfer function is

$$G_c(s)\,G(s) = \frac{K}{s(1+s)(1+4s)} \tag{2.7.2}$$

$$G_c(\jmath\omega)\,G(\jmath\omega) = \frac{K}{\jmath\omega(1+\jmath\omega)(1+4\jmath\omega)} \tag{2.7.3}$$

$$= \frac{K}{\jmath\omega\left(1-4\omega^2+5\jmath\omega\right)} \tag{2.7.4}$$

$$= \frac{K\left(-5\omega-\jmath\left(1-4\omega^2\right)\right)}{\omega\left(\left(1-4\omega^2\right)^2+25\omega^2\right)} \tag{2.7.5}$$

The maximum K for stability is where the nyquist plot of open loop transfer function cuts the coordinate $(-1, \jmath 0)$

$$\implies \operatorname{Re}\{G(\jmath\omega)\,G_c(\jmath\omega)\} = -1 \tag{2.7.6}$$

$$\implies \operatorname{Im}\{G(\jmath\omega)\,G_c(\jmath\omega)\} = 0 \tag{2.7.7}$$

$$\implies \operatorname{Re}\{G(\jmath\omega)\,G_c(\jmath\omega)\} = \frac{-5K\omega}{\omega\left(\left(1-4\omega^2\right)^2+25\omega^2\right)} \tag{2.7.8}$$

$$\implies \operatorname{Im}\{G(\jmath\omega)\,G_c(\jmath\omega)\} = \frac{-K\left(1-4\omega^2\right)}{\omega\left(\left(1-4\omega^2\right)^2+25\omega^2\right)} \tag{2.7.9}$$

From (2.7.9) and (2.7.7)

$$1-4\omega^2 = 0 \implies \omega = \frac{1}{2} \tag{2.7.10}$$

From (2.7.8),(2.7.6) and substituting $\omega = \frac{1}{2}$

$$\frac{-5K\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{25}{4}\right)} = -1 \implies K = \frac{5}{4} = 1.25 \tag{2.7.11}$$

For $K < 0$ the system with negative feedback is unstable the range of K is

$$0 < K < \frac{5}{4} \tag{2.7.12}$$

Sketching the Nyquist plot for $G(s)G_c(s)$ in Fig. 2.7.2 The following code gives the nyquist plot

```
codes/ee18btech11034/ee18btech11034_1.py
```

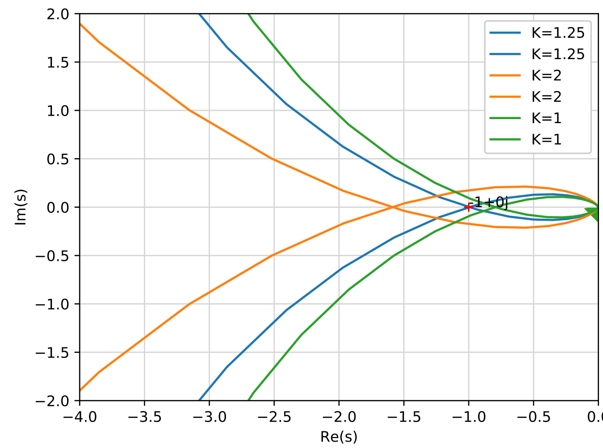

Fig. 2.7.2

Stability Criterian for K

$$N + P = Z \tag{2.7.13}$$

K	P	N	Z	Description
1.25	0	0	0	System is marginally stable
2	0	1	1	System is unstable
1	0	0	0	System is stable

TABLE 2.7.1

From the Fig.2.7.2

$$K_{max} = \frac{5}{4} \tag{2.7.14}$$

Solution: For $G_c(s) = \frac{1+s}{s}$,
the open loop transfer function is

$$G_c(s)\,G(s) = \frac{K(s+1)}{s^2(1+s)(1+4s)} \tag{2.7.15}$$

$$G_c(s)\,G(s) = \frac{K}{s^2(1+4s)} \tag{2.7.16}$$

$$G_c(\jmath\omega)G(\jmath\omega) = \frac{K}{(\jmath\omega)^2(1+4\jmath\omega)} \tag{2.7.17}$$

$$= \frac{\frac{-K}{\omega^2}(1-4\jmath\omega)}{1+16\omega^2} \tag{2.7.18}$$

From (2.7.7)

$$\implies \text{Im}\{G(\jmath\omega)G_c(\jmath\omega)\} = \frac{4K}{\omega(1+16\omega^2)} = 0 \tag{2.7.19}$$

This is possible when

$$K = 0 \tag{2.7.20}$$

The system is unstable for both

$$K < 0 \tag{2.7.21}$$

$$K > 0 \tag{2.7.22}$$

Sketching the Nyquist plot for $G(s)G_c(s)$ in Fig. 2.7.3 The following code gives the nyquist plot

```
codes/ee18btech11034/ee18btech11034_2.py
```

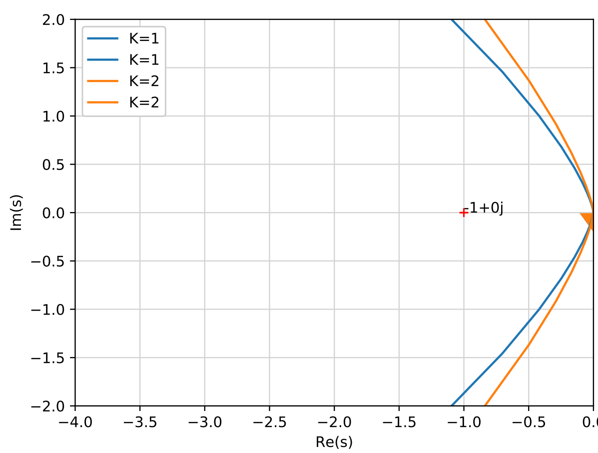

Fig. 2.7.3

From (2.7.13)

K	P	N	Z	Description
1	0	1	1	System is unstable
2	0	1	1	System is unstable

TABLE 2.7.2

From (2.7.20) K_{max} must be 0 which is not possible. Hence the system is unstable for all real K

2.8 *Nyquist*

Sketch the Nyquist plot for a closed loop system having open-loop transfer function

$$G(s)H(s) = \frac{2e^{-s\tau}}{s(1+s)(1+0.5s)} \tag{2.8.1}$$

Determine the maximum value of τ for the system to be stable.

Solution: From (2.8.1),

$$\implies \text{Re}\{G(\jmath\omega)H(\jmath\omega)\} = -4\left[\frac{3\omega^2\cos(\omega\tau) - \left(\omega^3 - 2\omega\right)\sin(\omega\tau)}{(3\omega^2)^2 + (\omega^3 - 2\omega)^2}\right] \tag{2.8.2}$$

$$\implies \text{Im}\{G(\jmath\omega)H(\jmath\omega)\} = 4\left[\frac{\left(\omega^3 - 2\omega\right)\cos(\omega\tau) + 3\omega^2\sin(\omega\tau)}{(3\omega^2)^2 + (\omega^3 - 2\omega)^2}\right] \tag{2.8.3}$$

Determining the stability of closed loop transfer function using Nyquist stability Criterion.

$$Z = P + N \tag{2.8.4}$$

Poles of open loop transfer function are on left half of s-plane.Therefore,$P = 0$
To ensure that the system is stable N should be 0
For maximum value of τ for stability ,the nyquist plot cuts the real axis at -1+j0.

$$G(s)H(s) = -1 + j0 \tag{2.8.5}$$

$$\text{Im}\{G(\jmath\omega)H(\jmath\omega)\} = 0 \tag{2.8.6}$$

$$\text{Re}\{G(\jmath\omega)H(\jmath\omega)\} = -1 \tag{2.8.7}$$

From (2.8.3) and (2.8.6)

$$\implies \tan(\omega\tau) = \frac{-\left(\omega^3 - 2\omega\right)}{3\omega^2} \tag{2.8.8}$$

From (2.8.2) and (2.8.7) and substituting $\tan(\omega\tau) = \frac{-(\omega^3-2\omega)}{3\omega^2}$

$$\implies \omega^6 + 5\omega^4 + 4\omega^2 - 16 = 0 \tag{2.8.9}$$

Solving (2.8.9) graphically.

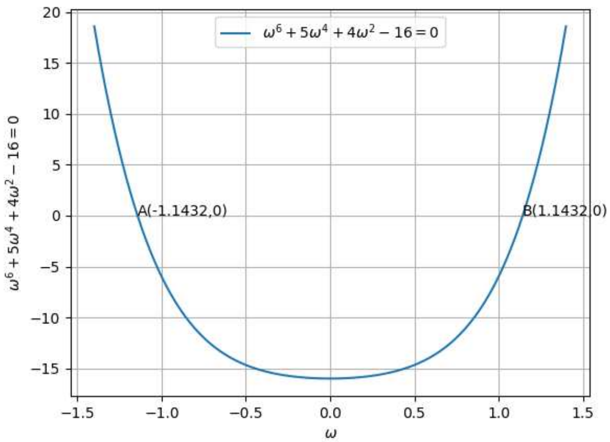

Fig. 2.8.1

Python code for the above plot is

```
codes/ee18btech11035_1.py
```

ω = 1.1432,-1.1432 (As, ω is positive)
Therefore,ω = 1.1432

Substituting ω in (2.8.8)

$$\tan(1.1432\tau) = 0.2021 \tag{2.8.10}$$
$$\tau = 0.1744 \tag{2.8.11}$$

The following python code generates the Nyquist plot.

```
codes/ee18btech11035_2.py
```

From the above figure (2.8.2) $\tau \leq 0.1744$ for a stable system.

τ	P	N	Z	Description
0.1744	0	1	1	System is Marginally stable
0.5	0	0	0	System is unstable
0.0005	0	0	0	System is stable

TABLE 2.8.1

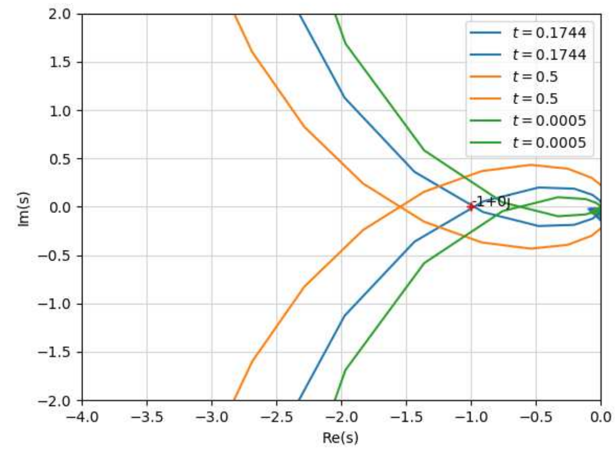

Fig. 2.8.2: Nyquist plot for variable τ

Therefore,τ_{max} = 0.1744

.

3 Design in Frequency Domain

3.1

3.2

3.3

3.4 Lead Compensator

3.4.1. For a unity feedback system shown in Fig. 1

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{3.4.1.1}$$

Design a lead compensator to yield a $K_v = 2$ and a phase margin of 30°.

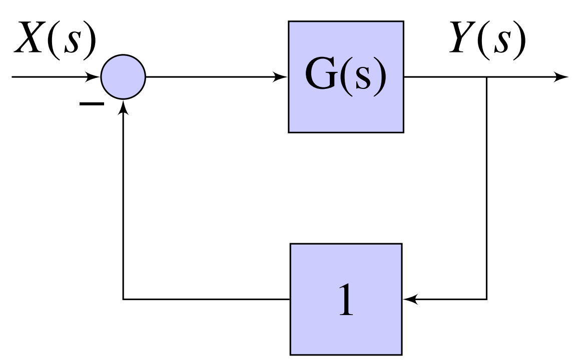

Fig. 3.4.1

Solution: For unity feedback we have Velocity error constant (K_v)

$$K_v = \lim_{s \to 0} sG(s) \tag{3.4.1.2}$$

$$\lim_{s\to 0}\left(\frac{K}{(2+s)(4+s)(6+s)}\right) = 2 \tag{3.4.1.3}$$
$$\implies K = 96 \tag{3.4.1.4}$$

Check the phase margin and gain crossover frequency by running the following code

```
codes/ee18btech11036_1.py
```

- The Phase margin: 19.76°
- Gain Crossover Frequency:1.469 rad/sec

The Bode plot of system is as shown,

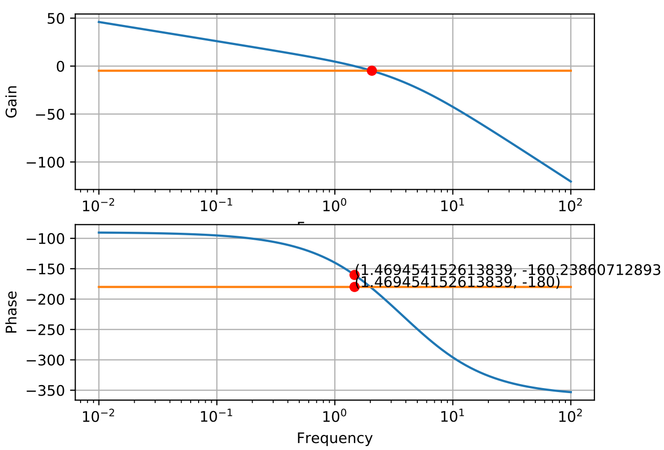

Fig. 3.4.2

Therefor amount of phase to be added: 30-19.76=10.24
The circuit of lead compensator is given by

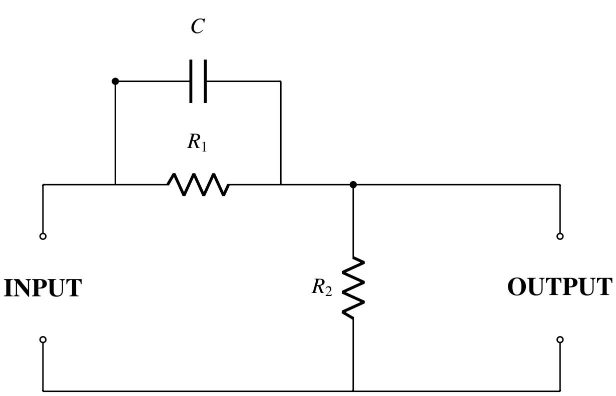

Fig. 3.4.3

Transfer function:

$$C(s) = \beta\left(\frac{1+j\tau\omega}{1+j\beta\tau\omega}\right) \tag{3.4.1.5}$$

$$\beta = \left(\frac{R_2}{R_1+R_2}\right) \tag{3.4.1.6}$$
$$\tau = R_1 C \tag{3.4.1.7}$$

Find the values of β and τ

Solution: The maximum phase lead compensated by a lead compensator is given by

$$\phi = \sin^{-1}\frac{1-\beta}{1+\beta} \tag{3.4.1.8}$$

at

$$\omega = \frac{1}{\sqrt{\beta}\tau} \tag{3.4.1.9}$$

Now we know that from Gain crossover frequency

$$\omega = 1.469 rad/sec \tag{3.4.1.10}$$

and the phase margin to be added:

$$\phi = 10.24° \tag{3.4.1.11}$$

But to compensate for the added magnitude of lead compensator, a correction factor of 10° – 20° is added.Hence

$$\phi = 30.24° \implies \beta = 0.33 \tag{3.4.1.12}$$

From the bode plot ω is chosen at which gain of original system is

$$-20\log\left(1/\sqrt{\beta}\right) = -4.81 \tag{3.4.1.13}$$

Find the plot using the following code

```
codes/ee18btech11036_4.py
```

From plot ω=2.009 rad/sec
Solving equations 3.4.1.8 and 3.4.1.9:

$$\tau = 0.828 \tag{3.4.1.14}$$
$$\beta = 0.33 \tag{3.4.1.15}$$
$$\tag{3.4.1.16}$$

New Transfer Function:

$$G(s) = \frac{96(1+0.828s)}{(s)(2+s)(4+s)(6+s)(1+0.273s)} \tag{3.4.1.17}$$

Verify your results from the following code:

```
codes/ee18btech11036_2.py
```

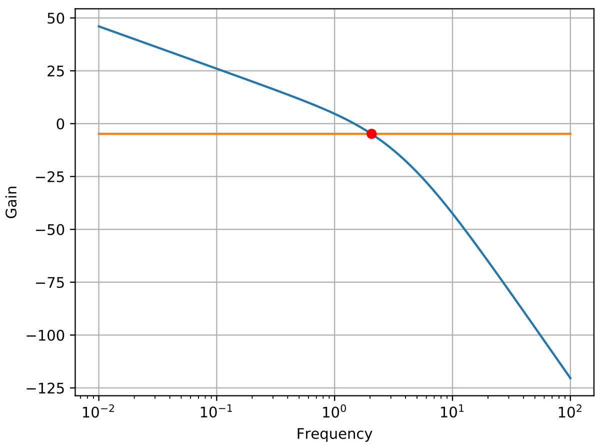

Fig. 3.4.4

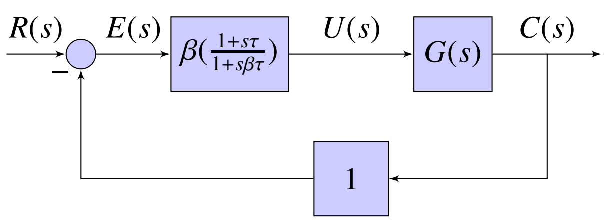

Fig. 3.4.5

- The Phase margin: 29.269°
- The Gain Crossover Frequency: 2.02 rad/sec

The Bode plot is as shown,

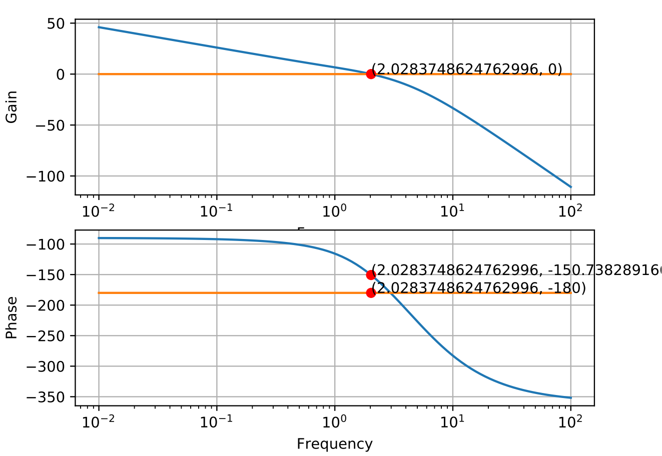

Fig. 3.4.6

3.5

3.6

3.7 Lead Compensator

3.7.1. An aircraft roll control system can be represented by a block diagram shown in Fig. 3.7.1 with G(s) in feedback system, whose error K_v = 5. Determine K

$$G(s) = \frac{10K}{s(s+1)(s+5)} \tag{3.7.1.1}$$

The block diagram is given by Fig.3.7.1

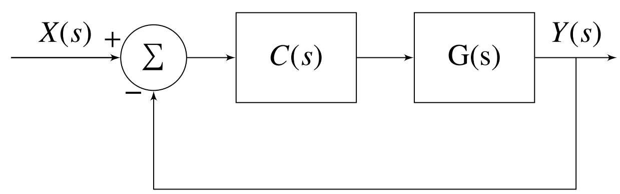

Fig. 3.7.1

For unity feedback we have Velocity error constant (K_v)

$$K_v = \lim_{s\to 0} sG(s) \tag{3.7.1.2}$$

$$\lim_{s\to 0}\left(\frac{10K}{(s+1)(s+5)}\right) = 5 \tag{3.7.1.3}$$

$$\implies K = 2.5 \tag{3.7.1.4}$$

It's Phase Margin = 3.94°
and Gain Crossover Frequency = 2.03 rad/s
Refer Fig. 3.7.2 for plot G(s).

codes/es17btech11002_1_new.py

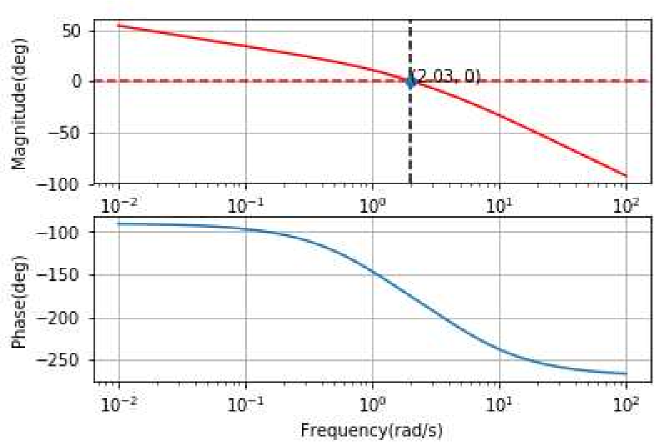

Fig. 3.7.2

Compensator required phase angle (ϕ_m) and Phase Margin Frequency $\left(\omega_{pm}\right)$,

$$\phi_m = -(180^\circ + \theta) + PM + 5 = 65^\circ \quad (3.7.1.5)$$
$$\omega_{pm} = 1.25 rad/s. \quad (3.7.1.6)$$

Attenuation factor ($\alpha\beta$) is given by

$$\alpha = 0.5 \quad (3.7.1.7)$$
$$\beta = 20 \quad (3.7.1.8)$$

Lead and Lag Compensator Design Parameter is given in TABLE 3.7.1 And Compensator

Zeros/Poles	Parameter	Value
z_{lead}	$\omega_{pm}\sqrt{\alpha}$	0.279
p_{lead}	$\frac{z_{lead}}{\alpha}$	5.590
z_{lag}	$0.1\omega_{pm}$	0.125
p_{lag}	$\frac{z_{lag}}{\beta}$	0.00625

TABLE 3.7.1: Zeroes and Poles

obtained has transfer function

$$G_c(s) = \frac{(s+0.279)(s+0.125)}{(s+5.590)(s+0.00625)} \quad (3.7.1.9)$$

Refer Fig3.7.3 for plot $G(s)G_c(s)$.

```
codes/es17btech11002_2_new.py
```

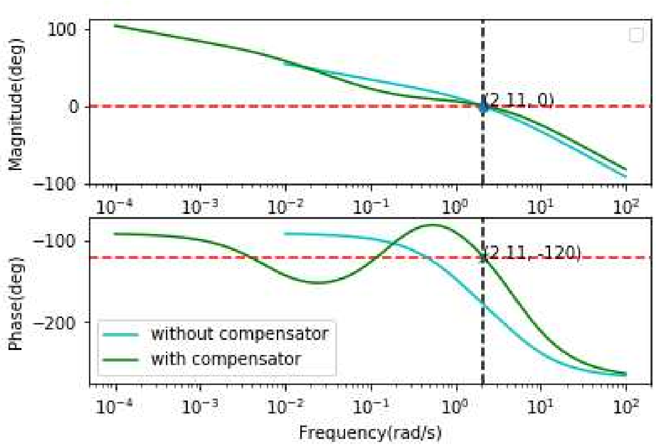

Fig. 3.7.3

NOTE : The idea of using a lead-lag network is to provide the attenuation of a phase-lag network and the lead-phase angle of a phase-lead network. This points should be noted while designing a controller, and parameters to be changed accordingly to get exact results.

3.8

3.9

3.10

4 PID Controller Design

4.1 Introduction

4.1.1. Tabulate the transfer functions of a PID controller and its variants.
Solution: See Table 4.1.1.

Controller	Gain
PID	$K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$
PD	$K_p(1 + T_d s)$
PI	$K_p\left(1 + \frac{1}{T_i s}\right)$

TABLE 4.1.1

4.1.2. For a unity Feedback system

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \quad (4.1.2.1)$$

Design a PD Controller with $K_v = 2$ and Phase Margin 30°
Solution: The gain after cascading the PD Controller with G(s) is

$$G_c(s) = \frac{K_p(1 + T_d s)K}{s(s+2)(s+4)(s+6)} \quad (4.1.2.2)$$

Choosing $K_p = 1$ in ,

$$K_v = \lim_{s \to 0} sG_c(s) = 2 \quad (4.1.2.3)$$
$$\implies K = 96 \quad (4.1.2.4)$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\tan^{-1}(T_d\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = -60^\circ \quad (4.1.2.5)$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_d^2\omega^2 + 1}}{\omega\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \quad (4.1.2.6)$$

By Hit and Trial, one of the best combinations is

$$\omega = 4 \quad (4.1.2.7)$$

$$T_d = 1.884 \tag{4.1.2.8}$$

We get a Phase Margin of 30.31°

4.1.3. Verify using a Python Plot

Solution: The following code plots Fig. 4.1.1

```
codes/ee18btech11021/EE18BTECH11021_3.py
```

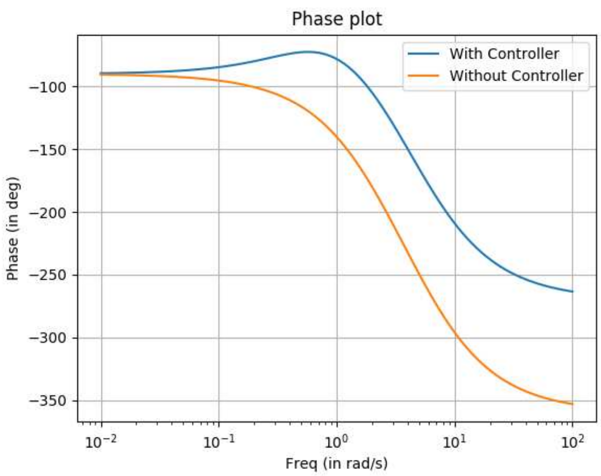

Fig. 4.1.1

4.1.4. Design a PI Controller with $K_v = \infty$ and Phase Margin 30°

Solution: From Table 4.1.1, the open loop gain in this case is

$$G_1(s) = \frac{K_p\left(1 + \frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{4.1.4.1}$$

Choose $K_pK = 96$. Then

$$G_1(s) = \frac{96(T_i s + 1)}{T_i s^2(s+2)(s+4)(s+6)} \tag{4.1.4.2}$$

For Phase Margin 30°, at Gain Crossover Frequency ω

$$\tan^{-1}(T_i\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \tag{4.1.4.3}$$

and

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_i^2\omega^2 + 1}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{4.1.4.4}$$

By Hit and Trial, one of the best combinations is

$$\omega = 0.75 \tag{4.1.4.5}$$

$$T_i = 2.713 \tag{4.1.4.6}$$

We get a Phase Margin of 25.53°

4.1.5. Verify using a Python Plot

Solution: The following code plots Fig. 4.1.2.

```
codes/ee18btech11021/EE18BTECH11021_4.py
```

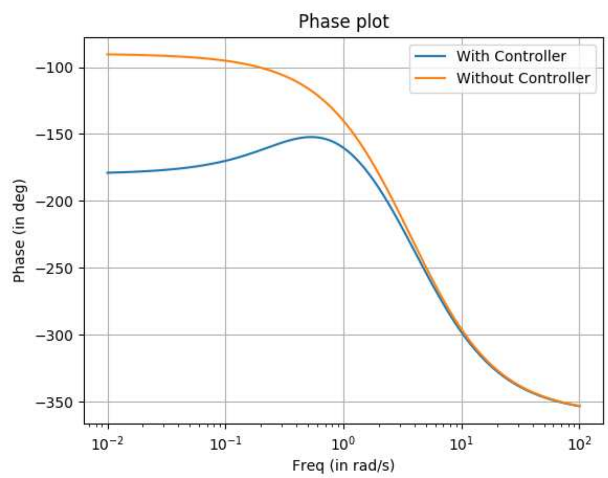

Fig. 4.1.2

4.1.6. Design a PID Controller with $K_v = \infty$ and Phase Margin 30°

Solution:

$$G_1(s) = \frac{K_p\left(1 + T_d s + \frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{4.1.6.1}$$

Choose $K_pK = 96$. The open loop gain is

$$G_1(s) = \frac{96(T_iT_d s^2 + T_i s + 1)}{T_i s^2(s+2)(s+4)(s+6)} \tag{4.1.6.2}$$

For Phase Margin 30°, at Gain Crossover Frequency ω,

$$\tan^{-1}\left(\frac{T_i\omega}{1 - TiT_d w^2}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \tag{4.1.6.3}$$

$$\left|G_1\left(\jmath\omega\right)\right| = \frac{96\sqrt{(1 - TiT_d\omega^2)^2 + T_i^2}}{T_i^2\omega^2\sqrt{(\omega^2 + 4)(\omega^2 + 16)(\omega^2 + 36)}} = 1 \tag{4.1.6.4}$$

By Hit and Trial, one of the best combinations is

$$\omega = 1 \tag{4.1.6.5}$$
$$T_i = 1.738 \tag{4.1.6.6}$$
$$T_d = 0.4 \tag{4.1.6.7}$$

We get a Phase Margin of 30°

4.1.7. Verify using a Python Plot

Solution: The following code plots Fig. 4.1.3

```
codes/ee18btech11021/EE18BTECH11021_5.py
```

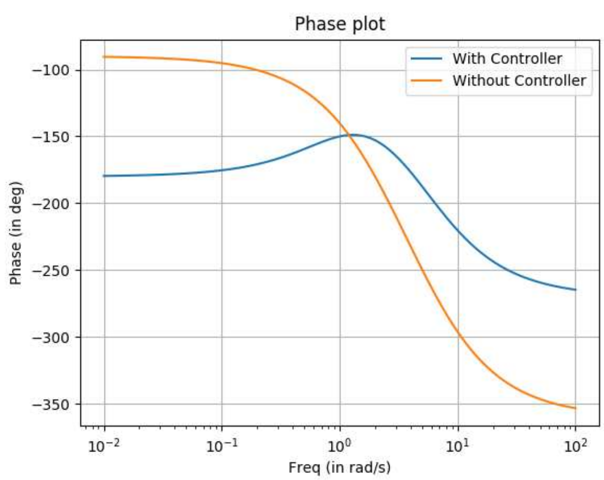

Fig. 4.1.3