# Control Systems

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Contents



***Abstract*—The objective of this manual is to introduce control system design at an elementary level.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/ketan/codes
```

## 1 PID Controller

### 1.1 Introduction

1.1.1. Tabulate the transfer functions of a PID controller and its variants.
**Solution:** See Table 1.1.1.

| Controller | Gain |
|---|---|
| PID | $K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$ |
| PD | $K_p(1 + T_d s)$ |
| PI | $K_p\left(1 + \frac{1}{T_i s}\right)$ |

TABLE 1.1.1

1.1.2. For a unity Feedback system

$$G(s) = \frac{K}{s(s+2)(s+4)(s+6)} \tag{1.1.2.1}$$

Design a PD Controller with $K_v = 2$ and Phase Margin 30°
**Solution:** The gain after cascading the PD Controller with G(s) is

$$G_c(s) = \frac{K_p(1 + T_d s)K}{s(s+2)(s+4)(s+6)} \tag{1.1.2.2}$$

Choosing $K_p = 1$ in ,

$$K_v = \lim_{s \to 0} sG_c(s) = 2 \tag{1.1.2.3}$$

$$\implies K = 96 \tag{1.1.2.4}$$

For Phase Margin 30°, at Gain Crossover Frequency $\omega$,

$$\tan^{-1}(T_d\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = -60° \tag{1.1.2.5}$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_d^2\omega^2 + 1}}{\omega\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{1.1.2.6}$$

By Hit and Trial, one of the best combinations is

$$\omega = 4 \tag{1.1.2.7}$$

$$T_d = 1.884 \tag{1.1.2.8}$$

We get a Phase Margin of 30.31°

1.1.3. Verify using a Python Plot
**Solution:** The following code plots Fig. 1.1.3

```
codes/ee18btech11021/EE18BTECH11021_3.
    py
```

1.1.4. Design a PI Controller with $K_v = \infty$ and Phase Margin 30°
**Solution:** From Table 1.1.1, the open loop gain in this case is

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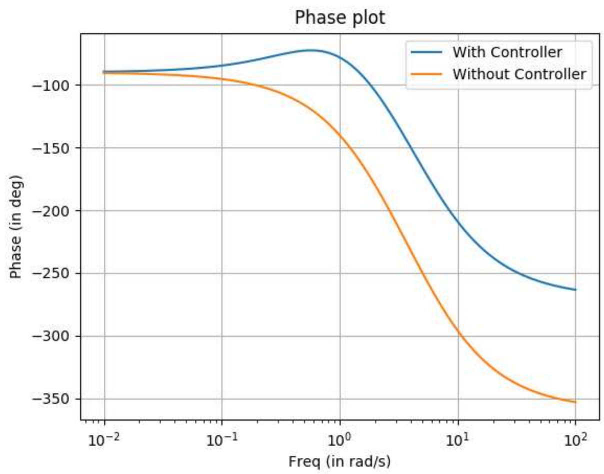


Fig. 1.1.3

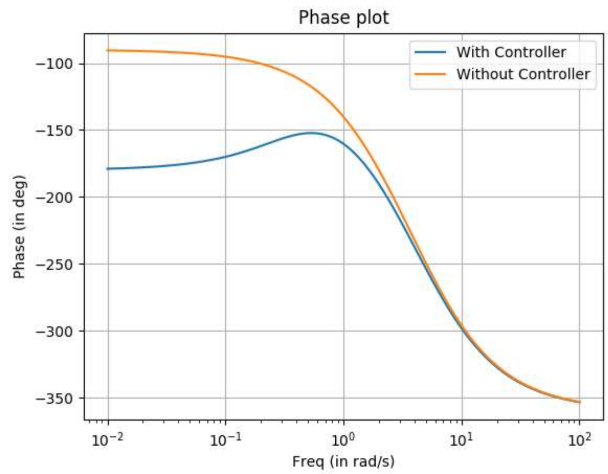


Fig. 1.1.5

$$G_1(s) = \frac{K_p\left(1 + \frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{1.1.4.1}$$

Choose $K_pK = 96$. Then

$$G_1(s) = \frac{96(T_i s + 1)}{T_i s^2(s+2)(s+4)(s+6)} \tag{1.1.4.2}$$

For Phase Margin 30°, at Gain Crossover Frequency $\omega$

$$\tan^{-1}(T_i\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \tag{1.1.4.3}$$

and

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{T_i^2\omega^2 + 1}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{1.1.4.4}$$

By Hit and Trial, one of the best combinations is

$$\omega = 0.75 \tag{1.1.4.5}$$
$$T_i = 2.713 \tag{1.1.4.6}$$

We get a Phase Margin of 25.53°

1.1.5. Verify using a Python Plot

**Solution:** The following code plots Fig. 1.1.5.

```
codes/ee18btech11021/EE18BTECH11021_4.
    py
```

1.1.6. Design a PID Controller with $K_v = \infty$ and Phase Margin 30°

**Solution:**

$$G_1(s) = \frac{K_p\left(1 + T_d s + \frac{1}{T_i s}\right)K}{s(s+2)(s+4)(s+6)} \tag{1.1.6.1}$$

Choose $K_pK = 96$. The open loop gain is

$$G_1(s) = \frac{96(T_iT_ds^2 + T_is + 1)}{T_is^2(s+2)(s+4)(s+6)} \tag{1.1.6.2}$$

For Phase Margin 30°, at Gain Crossover Frequency $\omega$,

$$\tan^{-1}\left(\frac{T_i\omega}{1 - TiT_dw^2}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{6}\right) = 30 \tag{1.1.6.3}$$

$$\left|G_1(\jmath\omega)\right| = \frac{96\sqrt{(1 - TiT_d\omega^2)^2 + T_i^2}}{T_i^2\omega^2\sqrt{(\omega^2+4)(\omega^2+16)(\omega^2+36)}} = 1 \tag{1.1.6.4}$$

By Hit and Trial, one of the best combinations is

$$\omega = 1 \tag{1.1.6.5}$$
$$T_i = 1.738 \tag{1.1.6.6}$$
$$T_d = 0.4 \tag{1.1.6.7}$$

We get a Phase Margin of 30°

1.1.7. Verify using a Python Plot

**Solution:** The following code plots Fig. 1.1.7

```
codes/ee18btech11021/EE18BTECH11021_5.py
```

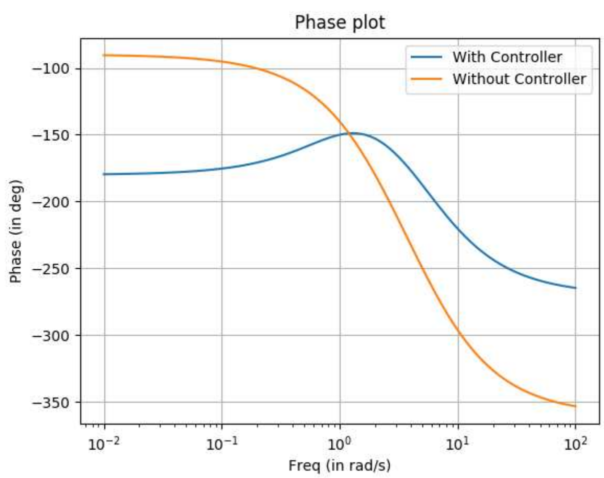


Fig. 1.1.7

## 2 Polar Plot

### 2.1 Introduction

2.1.1. Sketch the polar plot of

$$G(s) = \frac{1}{(s^2)(s+1)(s+2)}. \tag{2.1.1.1}$$

**Solution:** Substituting $s = \jmath\omega$ in (2.1.1.1),
Now the magnitude will be

$$r = |G(\jmath\omega)| = \frac{1}{(\omega^2)(\sqrt{1+\omega^2})(\sqrt{1+4\omega^2})} \tag{2.1.1.2}$$

$$\theta = \angle G(\jmath\omega) = -\tan^{-1}(0) - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{2.1.1.3}$$

$$= 180^\circ - \tan^{-1}(\omega) - \tan^{-1}(2\omega) \tag{2.1.1.4}$$

The polar plot is the $(r, \theta)$ plot for $\omega \in (0, \infty)$. The following python code generates the polar plot in Fig. 2.1.1

```
codes/ee18btech11028.py
```

The location of $(-1, 0)$ with respect to the polar plot provides information regarding the stability of the system.

- If $(-1, 0)$ is not enclosed, then it is stable.

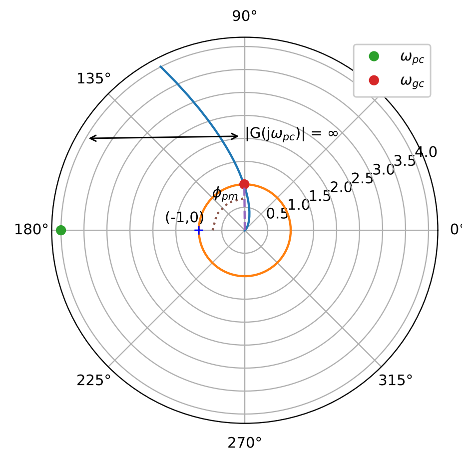


Fig. 2.1.1

- If $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 2.1.1, the point $(-1, 0)$ is enclosed by the polar plot, which implies system is not stable. The polar plot also provides info on the GM and PM, which can then be used for determining the stability of the system.

- If the $GM > 1 \cap PM > 0$, then the control system is **stable**.
- If the $GM = 1 \cap PM = 0$, then the control system is **marginally stable**.
- If the $GM < 1 \cup PM < 0$ , then the control system is **unstable**.

Therefore, our system is unstable.

### 2.2 Example

2.2.1. Sketch the Polar Plot of

$$G(s) = \frac{1}{s(1+s^2)} \tag{2.2.1.1}$$

**Solution:** From (2.2.1.1),

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1-\omega^2)} \tag{2.2.1.2}$$

$$\left|G(\jmath\omega)\right| = \frac{1}{\left|\omega(1-\omega^2)\right|} \tag{2.2.1.3}$$

$$\angle G(\jmath\omega) = \begin{cases} \frac{\pi}{2} & \omega > 1 \\ -\frac{\pi}{2} & 0 < \omega < 1 \end{cases} \tag{2.2.1.4}$$

The corresponding polar plot is generated in Fig. 2.2.1 using

```
codes/ee18btech11023.py
```

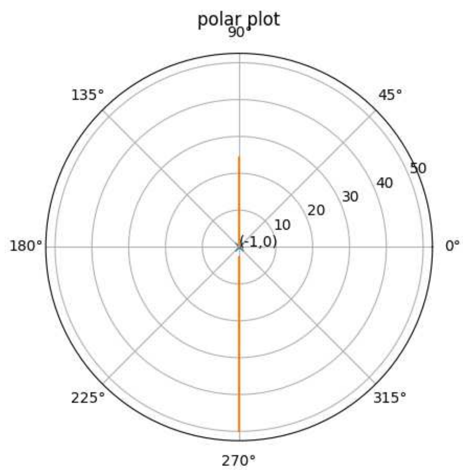


Fig. 2.2.1

In Fig. 2.2.1, (–1,0) is exactly on the polar plot. Hence, the system is marginally stable.

### *2.3 Example*

2.3.1. Sketch the Polar Plot for

$$G(s) = \frac{1}{(1+s)(1+2s)} \tag{2.3.1.1}$$

**Solution:** The following code generates Fig. 2.3.1

```
codes/ee18btech11012.py
```

The polar plot is to the right of $(-1, 0)$. Hence the closed loop system is stable.

### *2.4 Example*

2.4.1. Plot the polar plot of

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}. \tag{2.4.1.1}$$

**Solution:**
The following python code generates the polar plot in Fig. 2.4.1

```
codes/ee18btech11033.py
```

∵ (-1,0) is on the right side of the polar plot, the system is unstable.

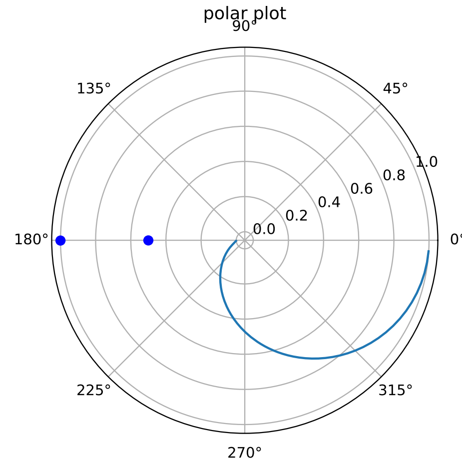


Fig. 2.3.1

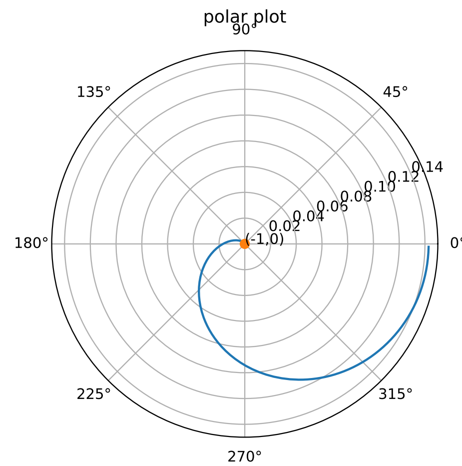


Fig. 2.4.1

### *2.5 Example*

2.1. Sketch the direct polar plot for a unity feedback system with open loop transfer function

$$G(s) = \frac{1}{s(1+s)^2} \tag{2.1.1}$$

**Solution:** The polar plot is obtained by plotting $(r, \phi)$

$$r = |H(\jmath\omega)||G(\jmath\omega)| \tag{2.1.2}$$

$$\phi = \angle H(\jmath\omega)G(\jmath\omega), 0 < \omega < \infty \tag{2.1.3}$$

The following code plots the polar plot in Fig. 2.1

```
codes/ee18btech11002/polarplot.py
```

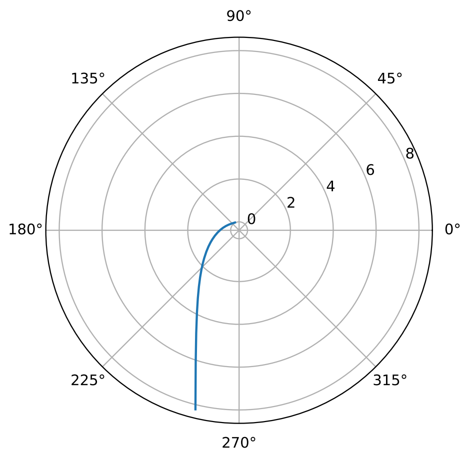


Fig. 2.1: Polar Plot

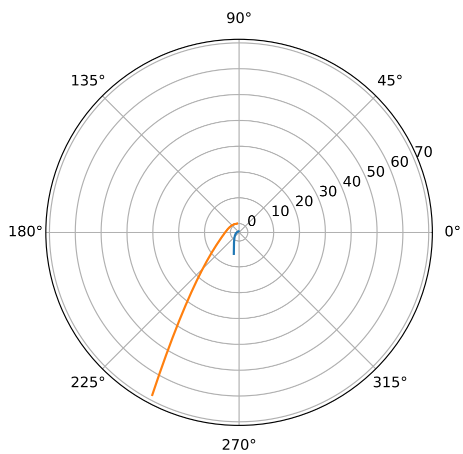


Fig. 2.2: Inverse Polar Plot

2.2. Sketch the inverse polar plot for (2.1.1)
**Solution:** The above code plots the polar plot in Fig. 2.2 by plotting $\left(\frac{1}{r}r, -\phi\right)$

## 2.6 *Example*

2.1. Plot the polar plot of

$$G(s) = \frac{100(s+5)}{s(s+3)(s^2+4)}. \tag{2.1.1}$$

**Solution:** The following python code generates the polar plot in Fig. 2.1

```
codes/ee18btech11042.py
```

Since (-1,0) is on the polar plot, the above system is marginally stable.

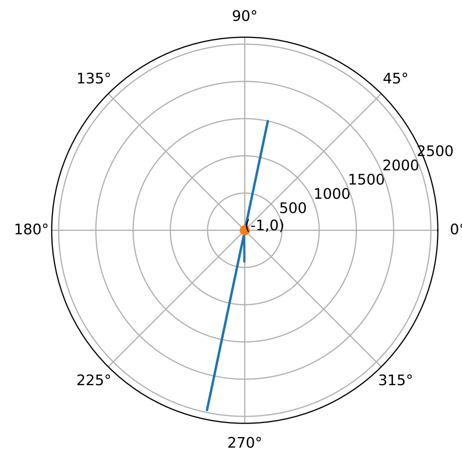


Fig. 2.1

## 2.7 *Example*

2.7.1. Sketch the Polar Plot of

$$G(s) = \frac{\left(1 + \frac{s}{29}\right)(1 + 0.0025s)}{(s^3)(1 + 0.005s)(1 + 0.001s)} \tag{2.7.1.1}$$

**Solution:** The following code generates the polar plot in Fig. 2.7.1

```
codes/ee18btech11029.py
```

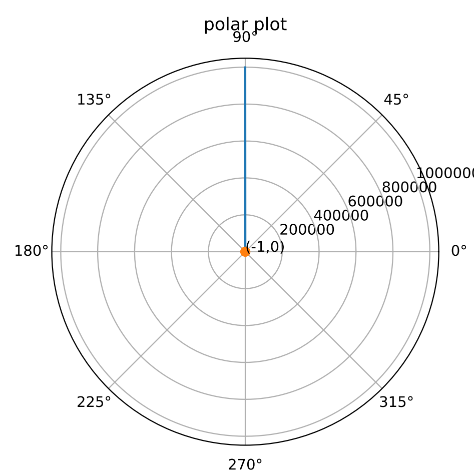


Fig. 2.7.1

- The polar plots use open loop transfer function to determine the stability and hence reference point is shifted to $(-1, 0)$
- If $(-1, 0)$ is left of the polar plot or $(-1, 0)$ is not enclosed, then it is stable

- If $(-1, 0)$ is on right side of the polar plot or $(-1, 0)$ is enclosed by polar plot then it is unstable.
- If $(-1, 0)$ is on the polar plot then it is marginally stable

In Fig. 2.7.1, $(-1, 0)$ is on the polar plot so the system is marginally stable.