From Transversality to the Jordan-Brouwer Separation Theorem and its Consequences

Harshda Saxena

Mentor : Prof. Sugata Mondal
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Preliminaries

We will be dealing with only manifolds as subsets of Euclidean space, thereby avoiding charts and atlases (works because of Whitney's Embedding Theorem!)

Smooth maps

- A mapping f of an open set $U \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ is called *smooth* if it has continuous partial derivatives of all orders.
- Maps on arbitrary subsets are smooth if they can be locally extended to a smooth map on open sets.
- A smooth map $f : X \to Y$ of subsets of two Euclidean spaces is a *diffeomorphism* if it is one to one and onto, and if the inverse map $f^{-1} : Y \to X$ is also smooth.

Manifold

X is a *k-dimensional manifold* if it is locally diffeomorphic to $\mathbb{R}^k$.

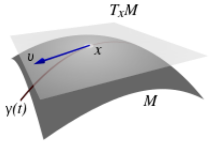
A diffeomorphism $\phi : U(\subset \mathbb{R}^k) \to V$ is called a *parametrization* of the neighborhood V.
The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a *coordinate system* on V.

Immersing Ourselves in Derivative Maps

Tangent Spaces

Let $\phi : U \to X$ be a local parametrization around x, where $\phi(0) = x$ and define the *tangent space* of X at x to be the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^n$. It is denoted by $T_x(X)$.

We can prove that $T_x(X)$ is well defined, and with the same dimension k of X.



Immersion

If dim $X \leq$ dim Y, and $df_x : T_x(X) \to T_y(Y)$ is injective, then f is called an *immersion* at x. The *cannonical immersion* is an inclusion map. For an immersion f, it is locally equivalent to the canonical immersion near x.

An immersion which is injective and proper is an *embedding*, and maps X diffemomorphically onto a submanifold of Y.

Immersing wasn't enough, let's submerse!

Definition

Let $\dim X \geq \dim Y$, and $f : X \to Y$ such that $df_x : T_x(X) \to T_y(Y)$ is surjective, then f is called a *submersion* at x. The *canonical submersion* is the standard projection, and similarly, a submersion f is locally equivalent to the canonical submersion near x.

Regular Values

For smooth map $f : X \to Y$, a point $y \in Y$ is called a *regular value* if $df_x : T_x(X) \to T_y(Y)$ is surjective for every point x such that $f(x) = y$, meanwhile x is a *regular point*.

Pre-Image Theorem

If y is a regular value of $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of X, with $\dim f^{-1}(y) = \dim X - \dim Y$. Also note that if Z is the preimage of a regular value y in Y under smooth $f : X \to Y$, then the kernel of the derivative $df_x : T_x(X) \to T_y(Y)$ at any point $x \in Z$ is the tangent space $T_x(Z)$ to Z at x.

Regular values have beautiful properties, which we shall see soon.

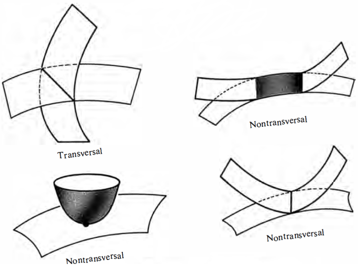
Some much needed pictures

Maps transversal to a submanifold of Y

A smooth map $f : X \to Y$ is transversal to submanifold $Z \subset Y$ ($f \pitchfork Z$) if at every preimage point of Z, $\operatorname{im}(df_x) + T_y(Z) = T_y(Y)$. Also, $f^{-1}(Z)$ is a submanifold of X. The codimension of $f^{-1}(Z)$ in X is same as the codimension of Z in Y.

Submanifolds transversal to each other

2 submanifolds X and Z of Y are said to be transversal ($X \pitchfork Z$), iff for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$. The intersection of 2 transversal submanifolds of Y is another submanifold and $\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z$.



Let's set some boundaries!

Definitions

Let the upper half space in $\mathbb{R}^k$ be $\mathbb{H}^k$. We call a subset X of $\mathbb{R}^n$ a k-dimensional *manifold-with-boundary* if every point of X possesses a neighbourhood diffeomorphic to an open set in $\mathbb{H}^k$. If X is a k-dimensional manifold with a boundary, then ∂X is a $k-1$ dimensional manifold without a boundary.

A useful theorem

Let f be a smooth map from manifold X with boundary to boundaryless manifold Y. Suppose that both f and ∂f are transversal with respect to a boundaryless submanifold Z in Y. Then preimage $f^{-1}(Z)$ is a manifold with the boundary $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$, and the codimension of $f^{-1}(Z)$ in X equals codimension of Z in Y.

The Classification of One Manifolds

Every compact, connected $1D$ manifold with a boundary is diffeomorphic to $[0, 1]$ or S^1.

Transversality Homotopy Theorem

For any smooth map $f : X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.

Everything's black or white - Intersections in Binary

Mod 2 intersection of a map with a submanifold

If X is a compact manifold and $f : X \to Y$ is a smooth map transversal to closed Z in Y and $\dim X + \dim Z = \dim Y$, then by codimension summing, $f^{-1}(Z)$ is a closed 0 dimensional submanifold and hence a finite set. The mod 2 intersection number $I_2(f, Z)$ of the map f with Z is the number of points in $f^{-1}(Z)$ modulo 2. For arbitrary smooth g, select homotopic map f transversal to Z and define $I_2(f, Z) = I_2(g, Z)$.

Mod 2 intersection of 2 submanifolds

If X is a compact submanifold of Y and Z is a closed submanifold of complementary dimension, define the mod 2 intersection number of X with Z by $I_2(X, Z) = I_2(i, Z)$ where $i : X \hookrightarrow Y$ is the inclusion. If $X \mathrel{\overline{\pitchfork}} Z$, then $I_2(X, Z) = \#\{X \cap Z\}$ mod 2.

Boundary Theorem

If X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension.

Mod 2 degree of a function

If $f : X \to Y$ is a smooth map of compact X into connected Y such that $\dim X = \dim Y$, then $I_2(f, \{y\})$ is the same for all y in Y, and called the mod 2 degree of f denoted as $\deg_2(f)$.

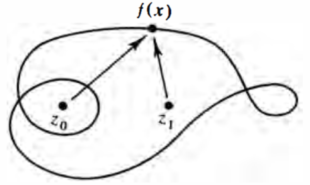
A much needed unwind

A hypersurface in a manifold is a submanifold of codimension one.

Definition

Let X be a compact connected hypersurface and $f : X \to \mathbb{R}^n$ a smooth map. Take any point z in $\mathbb{R}^n$ not in $f(X)$. Define $u(x) = \frac{f(x)-z}{|f(x)-z|}$ as a map from X to S^{n-1}. Define the mod 2 winding number of f around z to be $W_2(f, z) = deg_2(u)$.

We use $W_2(X, z)$ for the winding number of the inclusion map of X around z.



Thus, the winding number modulo 2 counts the number of times a function map winds around a specific point not in the image of the map.

Obvious? Apparently not!

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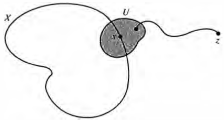
The Jordan Brower Seperation Theorem

The complement of the compact connected hypersurface X in $\mathbb{R}^n$ consists of 2 connected open sets,
the "outside" $D_0 = \{z : W_2(X, z) = 0\}$
and "inside" $D_1 = \{z : W_2(X, z) = 1\}$
where $z \in \mathbb{R}^n - X$. Further, $\bar{D}_1$ is a compact manifold with boundary with $\partial \bar{D}_1 = X$.

Onto a proof

Claim 1

Let x is any point of X and U is any neighbourhood of x in $\mathbb{R}^n$ and z is any point of $\mathbb{R}^n - X$, then there is a point of U that can be joined to z by a curve not intersecting X.



Proof

Fix z in $\mathbb{R}^n - X$, and let S be the set of points in X such that the above is true. We aim to show that S is both open and closed, thus making it the entirety of X by connectedness (it is non empty, see the straight line joining the closest point to z in X).

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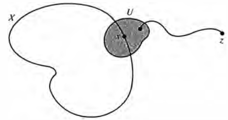

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Let s_i be a sequence of points in S, which converge to some s, and let $s \notin S$. Let U be any nbd of s, and we must have some s_i in U, consider U_i nbd of s_i. WLOG we can assume $U_i \subset U$, and must have a point z_i. This point is also in U, and we have a contradiction. Hence S is closed.

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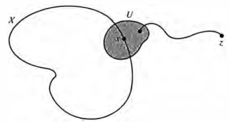

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Let s_i be a sequence of points in S, which converge to some s, and let $s \notin S$. Let U be any nbd of s, and we must have some s_i in U, consider U_i nbd of s_i. WLOG we can assume $U_i \subset U$, and must have a point z_i. This point is also in U, and we have a contradiction. Hence S is closed.
Let s be any point in S, then there is a nbd U around s (from the Local Immersion theorem) such that $U \cap X$ is a cannonical immersion of the form $(u_1, \ldots, u_{n-1}, 0)$. For this U let $y = (v_1, \ldots, v_n)$ ($v_n \neq 0$, let it be positive) be the point connecting to z w/o intersecting X. Take any $s_1 \neq s$, with $s_1 \in U \cap X$, take any nbd V of s_1 and note that $V \cap U \neq \emptyset$. Thus, there exists some $v' \in V \cap U$ such that v_n' is positive, thus there is a curve joining v_n' to v_n and hence to z. Since $U \cap X \subset S$ is open in X, s is an interior point, and hence S is open.

Some connections?

Claim 2

$\mathbb{R}^n - X$ has atmost 2 connected components.

Proof

Similarly, using the local immersion theorem, there is a nbd U of s such that $U \cap X$ is a cannonical immersion of the form $(u_1, \ldots, u_{n-1}, 0)$. Take any ball B_ϵ around s, and note that $B_\epsilon - X \cap B_\epsilon$ has 2 components. Take z_0, z_1 in these 2 components, and by the previous claim, any arbit point $y \in \mathbb{R}^n - X$ can be connected to some point in U, which can be connected to either z_0 or z_1 by a path not intersecting X.

Claim 3

If z_0 and z_1 belong to the same connected component of $\mathbb{R}^n - X$, then $W_2(X, z_0) = W_2(X, z_1)$

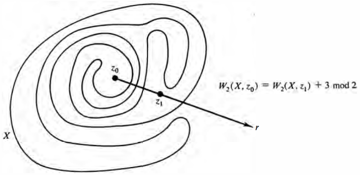
Proof

If z_0, z_1 belong to the same connected component of $\mathbb{R}^n - X$, define the path (from above) between the 2 points as $\gamma : [0, 1] \to \mathbb{R}^n - X$, with $\gamma(0) = z_0$ and $\gamma(1) = z_1$, and define $U(x, t) = \frac{x - \gamma(t)}{|x - \gamma(t)|}$, a homotopy between u_0 and u_1, hence $\deg_2(u_0) = \deg_2(u_1)$, and the relation on W_2 follows.

Dont get wound up just yet!

Looking at rays from a point in the complement of X

Given a point z and a direction vector $\vec{v} \in S^{n-1}$, consider the ray r emanating from z in the direction of $\vec{v}$, that is $r = \{z + t\vec{v} : t \geq 0\}$. Let $g : \mathbb{R}^n - \{z\}$ such that $g(y) = \frac{y-z}{|y-z|}$, and note that $u : X \to S^{n-1}$ is just the composite of g and i. Hence $g \circ i \pitchfork \{\vec{v}\} \iff i \pitchfork r$, and hence we see the ray is transversal to X iff $\vec{v}$ is a regular value of the map $u : X \to S^{n-1}$. As a consequence of Sards *almost every ray from z intersects X transversally*.



Relations between the winding numbers of 2 points

Suppose that r is a ray emanating from z_0 that intersects X transversally in a nonempty (necessarily finite) set. Suppose that z_1 is any other point on r (but not on X), and ℓ be the number of times r intersects X between z_0 and z_1. Hence $\vec{v}$ is a regular value for both u_0 and u_1 from above. From definition, we now conclude that $W_2(X, z_0) = \#\{u_0^{-1}(\vec{v})\}$ mod $2 = (\#\{u_1^{-1}(\vec{v})\} + \ell) \mod 2 = W_2(X, z_1) + \ell \mod 2$.

Almost done!

Combining everything from above, we note that if r is a ray emanating from z_0 and is transveral to X (can always be chosen), and there exist 1 point of intersection of r and X between z_0 and z_1, thus $W_2(X, z_0) \neq W_2(X, z_1)$, and from above, belong to the disconnected components $D_0 = \{z : W_2(X, z) = 0\}$ and $D_1 = \{z : W_2(X, z) = 1\}$ which form a partition on $\mathbb{R}^n - X$.

The "Outside"

Since X is compact, it fits in some ball B. Pick any $z \notin \mathbb{R}^n - B$, and a hyperplane H containing z disjoint from B. Then the half of S^{n-1} corresponding to the half-space determined by H that doesn't contain B is not hit by u at all; so $\deg_2 u = 0$.

$\overline{D_1}$ is a compact manifold and $\partial D_1 = X$

We see that $\overline{D_1}$ is a manifold (compactness is obvious from above) since the interior is locally diffeomorophic to $\mathbb{R}^n$, and points on both X and $\overline{D_1}$, from above X divides any neighbourhood of such points into 2 connected components, each of which can be seen to be diffeomorphic to $\mathbb{H}^n$, hence it is a manifold with boundary, the points of boundary are those of X from the above argument.

A berserk application

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Manifolds

Transversality

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Borsuk Ulam theorem

If $f : S^k \to \mathbb{R}^{k-1}$ is a smooth map whose image does not contain the origin and if f satisfies $f(-x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$.

Proof

We use induction on k, and prove for base case of $k = 1$. We can show that if this theorem is true, then it is equivalent to the following: if $f : S^k \to S^k$ carries antipodal to antipodal points, then $\deg_2(f) = 1$. We can prove for S^1 as assuming such an f WLOG such that $f(\cos t, \sin t) = (\cos g(t), \sin g(t))$. Then the antipodal condition gives $g(s + \pi) = g(s) + \pi q$, where q is odd. We then note $\deg_2(f) = q \mod 2 = 1$.

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Now, let S^{k-1} be the equator, embedded in S^k, and the restriction of f to the equator as g. Use Sard to select a unit vector $\vec{a}$ which is a regular value for both $\frac{g}{|g|} : S^{k-1} \to S^k$ and $\frac{f}{|f|} : S^k \to S^k$. Regularity for g simply means that g never intersects the line ℓ along $\vec{a}$, and regularity for f leads to $f \overline{\pitchfork} \ell$.

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By definition $W_2(f, 0) = \# \left(\frac{f}{|f|}\right)^{-1}(\vec{a}) \mod 2$. Note that $+\vec{a}$ and $-\vec{a}$ are hit same amount of times, and if f_+ is the restriction of f to the upper hemisphere, thus $\# f_+^{-1}(\ell)$.

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If $f : S^k \to \mathbb{R}^{k-1}$ is a smooth map whose image does not contain the origin and if f satisfies $f(-x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$.

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Let V be the orthogonal complement of ℓ, and $\pi : \mathbb{R}^{k+1} \to V$ be the projection. With g being anti-symmetric, so is the projection, and $\pi \circ g : S^{k-1} \to V$, and it can't be 0, since g doesn't intersect ℓ by hypothesis, thus by inductive hypothesis $W_2(\pi \circ g, 0) = 1$.

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Since $f_+ \overline{\pitchfork} l$, thus $\pi \circ f_+$ is transversal to $\{0\}$. Therefore,
$1 = W_2(\pi \circ g, 0) = \# (\pi \circ f_+)^{-1}(0) \mod 2 = \# f_+^{-1}(\ell) \mod 2 = W_2(f, 0)$.

Thank You, and a meme

THANK YOU!

You can find a detailed set of my notes of Differential Topology by Gullemin and Pollack here