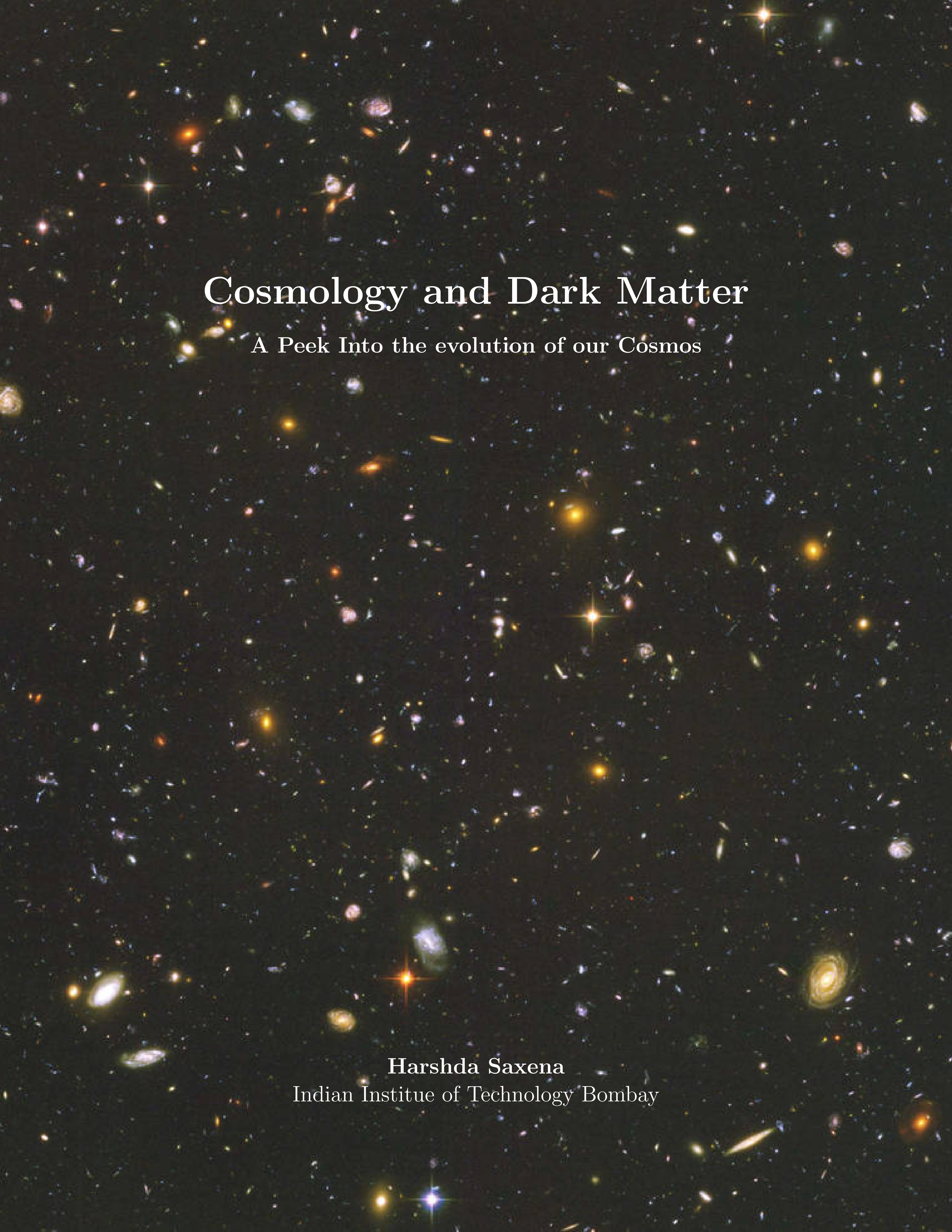

Cosmology and Dark Matter

A Peek Into the evolution of our Cosmos

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Contents

1 Introduction

This is a brief report on Cosmology and Dark Matter, the topic of my SoS project under the MnP club, and my mentor Riya Singh, in the year 2020. During the course of the summer, I have completed-

- An Introduction to Modern Cosmology by Andrew Liddle
- Introduction to Tensor Calculus by Kees Dullemond and Kasper Peeters (first 5 chapters)
- Physics- Volume 1 by Resnick, Halliday and Krane (Chapter 20- The Special Theory of Relativity)
- A First Course in General relativity by Bernard Schutz
- Modern Cosmology by Scott Dodelson (first 6 chapters)

In this report, I would briefly summarize the important results, formulae and theory covered by me in the past 2 months, and would encourage the reader to look for small proofs that I would have missed, due to the time/page limitations, in the books stated above. Let us now get right into the discussions of different Cosmological Models, initially building up on prerequisites like Tensor Calculus, Special Theory of Relativity and finally General Theory of Relativity, and finally applying these to the study of Cosmology, first from the Newtonian approach, and then from the modern relativistic approach.

2 An Introduction to Tensors

Let us take a manifold (=space) with dimension n. We will denote the components of a vector $\vec{v}$ with the numbers $v_1 \dots v_n$. If one modifies the vector basis, in which the components are expressed, then these components will change, too. Such a transformation can be written using matrices. We shall employ a convention where we shall omit the summation sign, assuming the index over which the summation is performed as the repeated one -

$$v_\mu{}' = \sum_{\nu=1}^{n} A_{\mu\nu} v_\nu \quad (\forall \mu \in |1 \leq \mu \leq n) \tag{1}$$

$$\implies v_\mu{}' = A_{\mu\nu} v_\nu \tag{2}$$

The transformation of basis vectors and components of vectors, can be written as follows -

$$e_\mu{}' = \Lambda_{\mu\nu} e_\nu \tag{3}$$

$$v'_\mu = (\Lambda_{\mu\nu}{}^{-1})^T v_\nu \tag{4}$$

The normal vectors are called 'contravariant vectors', because they transform contrary to the basis vector columns. A 'gradient', which we so far have always regarded as a true vector, will from now on be called a 'covariant vector' or 'covector': it transforms in the same way as the basis vector columns. Applying chain rule, we get the gradient (w) transformation as -

$$w_\mu{}' = \left(\frac{\partial x_\nu}{\partial x_\mu{}'}\right) w_\nu \tag{5}$$

Taking A as the vector components matrix, this turns out to be -

$$w' = (A^{-1})^T w \tag{6}$$

To distinguish vectors from covectors we will denote vectors with an arrow ($\vec{v}$) and covectors with a tilde ($\widetilde{w}$). To make further distinction between contravariant and covariant vectors we will put the contravariant indices as superscript and the covariant indices with subscripts.
To see an example of how tensors arise, we shall consider inner products. Inner products between a vector and covector are invariant under all transformations. We can introduce a new inner product between two contravariant vectors which also has this invariance property. To do this we introduce a covector w_μ and define the inner product between x^μ and y^ν with respect to this covector in the following way, which is invariant -

$$s = w_\nu w_\mu x^\nu y^\mu = \begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} w_1 \cdot w_1 & w_1 \cdot w_2 & w_1 \cdot w_3 \\ w_2 \cdot w_1 & w_2 \cdot w_2 & w_2 \cdot w_3 \\ w_3 \cdot w_1 & w_3 \cdot w_2 & w_3 \cdot w_3 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \tag{7}$$

the two appearances of the covector w are combined into one object: some kind of product of w with itself, we call this object g. So, we define the inner product with respect to the object g as -

$$s = g_{\mu\nu} x^\mu y^\nu \tag{8}$$

We do no longer regard g as built out of two covectors, but regard it as a matrix-like set of numbers on itself. This curious object, which looks like a matrix, but does not transform as one, is an example of a tensor. There are tensors of rank 3, as opposed to tensors of rank 0 (scalars), rank 1 (vectors and covectors) and rank 2 (matrices and the other kind of tensors).
A formal definition of tensors is as follows - An (N,M)-tensor at a given point in space can be described by a set of numbers with N + M indices which transforms, upon coordinate transformation given by the matrix A, in the following way -

$$t'^{\alpha_1 \ldots \alpha_N} \quad {}_{\beta_1 \ldots \beta_M} = A^{\alpha_1}{}_{\mu_1} \ldots A^{\alpha_N}{}_{\mu_N} A^{\nu_1}{}_{\beta_1} \ldots A^{\nu_M}{}_{\beta_M} t^{\mu_1 \ldots \mu_N} \quad {}_{\nu_1 \ldots \nu_M} \tag{9}$$

A tensor t is called symmetric in the indices μ and ν if the components are equal upon exchange of the index-values and called anti-symmetric if the components are equal but opposite upon exchange of the index-values. We can also create a 'dual basis', by constructing a set of 'unit covectors' that serve as a basis for the covectors. They transform as -

$$\widetilde{e}'^{\alpha} = A^{\alpha}{}_{\beta} \widetilde{e}^{\beta} \tag{10}$$

Basis-covector-columns transform as vectors, in contrast to basis-vector-columns, which transform as covectors. We can then express these tensors in terms of 'basis tensors' as $t = t^{\mu\nu}{}_{\rho} \vec{e}_\mu \otimes \vec{e}_\nu \otimes \widetilde{e}^\rho$, where $\otimes$ is tensor outer product. Tensors can be regarded as an object, a function or an operator.
To define distance from A to B, we divide the distance into small lengths ds, and then integrate to find distance along this path.

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$
$$\mathbf{g}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$$

where information of length/distance is stored in g and is called the metric of the manifold. This tensor is also symmetric, and this symmetry is preserved in any transformation. We can also transform between covariant and contravariant vectors by using metric tensors. A one-form $\widetilde{p}$ is defined to have the same magnitude as its associated vector $\vec{p}$ and is defined as $\widetilde{p}^2 = \eta_{\alpha\beta} p_\alpha p_\beta$. We will regard normal's to a surface as one-forms, rather than vectors to make sure the metric doesn't enter the definition.

3 Special Theory Of Relativity

3.1 Problems in Classical Mechanics

Higher precision and ability to achieve higher speeds in modern times has shown many inconsistencies in our idea of absolute space and time. The biggest of these is the pion - a meson that has a lifetime of about 26.0 ns at rest, appears to last longer at higher speeds when created in labs and hence, also travels much different distances in relative frames. Also, since classical mechanics allows particles to travel faster than light, it also violates causality, when observed by specific frames. Another inconsistency was about our classical views of energy, and how certain particle interactions tend to "violate" that. Arguably the most important, which lead Einstein to search for an alternative theory, was that a transformation to a different inertial frame, when applied to Maxwell's theory of Electromagnetism, do not remain invariant. This lead him to put down the postulates of STR to lead to a consistent theory, which can be summarized in the few words - " Motion affects Measurement".

3.2 Postulates of STR

The postulates, or general statements which provide for a basis of the theory of STR are as follows

- *The Principle of Relativity* - The laws of physics are the same in all inertial frames
- *The Principle of constancy of c* - The speed of light in free space has the same value c in all inertial reference frames

These 2 postulates lead to another consequence - it is impossible to accelerate a particle to a speed greater than c, no matter how much Kinetic energy we give it.
A coordinate system in STR is called inertial if the distance between points is independent of time, the clocks that sit at every point ticking off the time coordinate t are synchronized and all run at the same rate and that the geometry of space at any constant time t is Euclidean.

3.3 Spacetime Diagrams

A spacetime diagram is a continuous set of points, or a manifold, and points to the close relation between space and time. The 4 axes are coordinates in the order (t,x,y,z). We will consider natural units here. A single point in this space is a point of fixed x and fixed t, and is called an event. A line in the space gives a relation x = x(t), and so can represent the position of a particle at different times. This is called the particle's world line. Its slope is related to its velocity as $slope = \frac{dt}{dx} = \frac{1}{v}$, and a photon always travels in the $slope = 1$ line on this diagram (in natural units) and a curve represents an accelerated world line. Consider an observer O and another observer $\bar{O}$ moving with velocity $\vec{v}$ wrt O in the x direction. The spacetime axes in the diagrams are given in the figure.

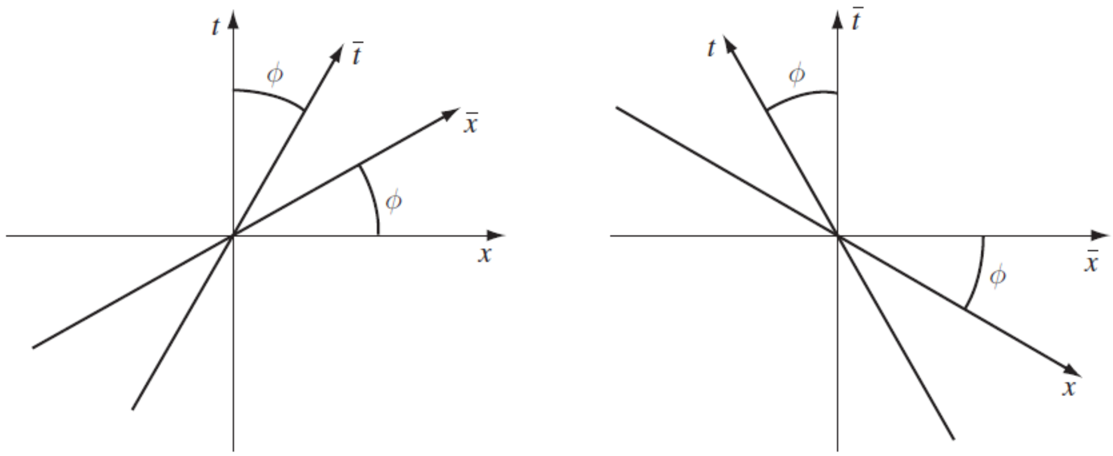

Figure 1: Left - Axes of $\bar{O}$ in O's spacetime diagram and vice versa for right.

We shall define the interval between any two events that are separated by coordinate increments $(\Delta t, \Delta x, \Delta y, \Delta z)$ to be

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{11}$$

We can also prove the fundamental theorem that the interval between any two events is the same when calculated by any inertial observer:

$$\Delta s^2 = \Delta \bar{s}^2 \tag{12}$$

If the value of this interval (Δs^2) is positive, the 2 events are called spacelike separated, timelike separated if negative and null separated if 0. Equipped with this, we can go on to further derive the Lorentz transformations between 2 frames.

3.4 Lorentz Transformation

There are multiple ways to derive the time dilation and length contraction equations in STR, however we will focus on the methods of Lorentz Transformations. The transformation equations in general have - an observer S in an inertial frame, another observer S' in another inertial frame moving relative to S, and a common event observed. Without loss of generality, assuming the relative motion of frames is in the xx' direction, with velocity $\vec{u}$. Galilean transformations gives the relationships as -

$$x' = x - ut \tag{13}$$
$$y' = y \tag{14}$$
$$z' = z \tag{15}$$
$$t' = t \tag{16}$$

The relativistic corrections to these, which can derived from the postulates given above by assuming space-time homogeneity and symmetry are -

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(x - ut) \tag{17}$$

$$y' = y \tag{18}$$

$$z' = z \tag{19}$$

$$t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(t - \frac{ux}{c^2}) \tag{20}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ is known as the Lorentz factor. These equations also reduce to the Galilean equations in the limit $u << c$ or $c \to \infty$. From the above equations, we can see the effects of time dilation (the clock of an observer in relative motion seems to run slowly when timed by the clocks in the frame of reference which is "at rest") and length contraction (the distance between 2 points measured simultaneously, by an observer in relative motion is observed to be lesser than the one "at rest", but only along the axis of relative motion). We can also derive the relation between velocities as -

$$v_x{}' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - u\Delta t)}{\gamma(\Delta t - \frac{u\Delta x}{c^2})} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{21}$$

$$v_y{}' = \frac{v_y}{\gamma(1 - \frac{uv_x}{c^2})} \tag{22}$$

$$v_z{}' = \frac{v_z}{\gamma(1 - \frac{uv_x}{c^2})} \tag{23}$$

We can also put $c = v_x$ and check that $v_x{}' = c$, always, irrespective of u. We define vectors as $\Delta\vec{x} \xrightarrow{O} \Delta x^\alpha$. Using the same rules we used in tensor caculus, Lorentz transformations of the vectors and basis vectors can be represented as -

$$\Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} \Delta x^{\beta} \tag{24}$$

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}_{\alpha} \vec{e}_\beta \tag{25}$$

where the transformation matrix for the previous case is -

$$\Lambda^{\bar{\beta}}_{\alpha} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{26}$$

3.5 Energy and Momentum in STR

Applying the classical definition of momentum to frames in relative motion, we see that momentum is not conserved in such inertial frames. Einstein gave a new definition of momentum, so that it would be conserved in all inertial frames, and would reduce to the classical definition at lower speeds. The relativistic formula for the momentum of a particle moving with velocity $\vec{v}$ is -

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{27}$$

Since energy of a particle is defined from Work-Energy theorem, we get the total KE of a particle as -

$$KE = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 \tag{28}$$

which, once again reduces to the classical formula $KE = \frac{mv^2}{2}$ at low speeds. We call the mc^2 term as rest energy, or the total energy of a particle as measured from a frame in which it is at rest. Only the total energy, the combination of rest energy and kinetic energy is conserved in all inertial frames. A little manipulation of the above equation yield the total energy (E) as -

$$E^2 = p^2c^2 + m^2c^4 \tag{29}$$

An accelerated particle has no inertial frame in which it is always at rest. However, there is an inertial frame which momentarily has the same velocity as the particle, but which a moment later is of course no longer comoving with it. This frame is the momentarily comoving reference frame (MCRF), and is an important concept. The *Four-Momentum* of a particle is defined as $\vec{p} = m\vec{U}$, where m is the rest mass with components defined as -

$$\vec{p} \xrightarrow{O} (E, p^1, p^2, p^3) \tag{30}$$

and p^0 is the Energy of the particle. $\vec{U} = \vec{e}_0$ in the MCRF of the particle, and is called the Four-Velocity. The four velocity is given by -

$$\vec{U} = \frac{d\vec{x}}{d\tau} \tag{31}$$

where $d\tau^2 = -ds^2$ is called the *proper time* of the particle, and measures the time according to its own clock. We can also similarly define the acceleration four vector as $\vec{a} = \frac{d\vec{U}}{d\tau}$. We hence define the relation between the momentum and energy as measured by an observer in the frame invariant way as -

$$-\vec{p} \cdot \vec{U}_{obvs} = \bar{E} \tag{32}$$

Any particle whose four-momentum is null must have rest mass zero, and conversely. The only known zero rest-mass particle is the photon. However keep in mind that we cannot assign a rest frame to a photon.

3.6 Tensors in STR

The transformation matrix can be represented as -

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}} = \Lambda^{\beta}_{\bar{\alpha}} \tag{33}$$

Nomenclature to indicate derivatives is given by $\phi_{,x}$, to represent derivative of a scalar ϕ wrt x. Differentiation of a function produces a tensor of one higher (covariant) rank. Consider a (1,1) tensor T whose components are functions of position. We can write T as $T = T^{\alpha}_{\beta}\widetilde{w}^{\beta} \otimes \vec{e}_{\alpha}$. We define the ordinary derivative of this function along the world line as -

$$\frac{dT}{d\tau} = T^{\alpha}_{\beta,\gamma}U^{\gamma} \tag{34}$$

$$\nabla T = T^{\alpha}_{\beta,\gamma}\widetilde{w}^{\beta} \otimes \widetilde{w}^{\gamma} \otimes \vec{e}_{\alpha} \tag{35}$$

and the notation to show derivative along a vector/world line as - $\frac{d\mathbf{T}}{d\tau} = \nabla_{\vec{U}}\mathbf{T}$. The above equations make use of the fact that basis one forms were constant everywhere, which is not the case for GR. In many interesting situations in astrophysical GR, the source of the gravitational field can be taken to be a perfect fluid as a first approximation. In general, a 'fluid' is a special kind of continuum. A continuum is a collection of particles so numerous that the dynamics of individual particles cannot be followed, leaving only a description of the collection in terms of 'average' or 'bulk' quantities. Let us define n as the number density in the MCRF of the fluid, m as the mass of each particle, and the Number Flux 4 vector as $\vec{N} = n\vec{U}$. Thus, in any frame, the time component of $\vec{N}$ is the number density and the spatial components are the fluxes across surfaces of the various coordinates. In a frame O with velocities (v_x, v_y, v_z), we get the following relations-

$$\begin{aligned} \vec{N} &\xrightarrow{O} (n\gamma, v^i n\gamma) \\ \vec{N} \cdot \vec{N} &= -n^2 \end{aligned} \tag{36}$$

On similar lines, the Stress-Energy tensor can be defined in terms of its components as -

$$\begin{aligned} \mathbf{T}(\widetilde{dx}^{\alpha}, \widetilde{dx}^{\beta}) &= T^{\alpha\beta} \\ \mathbf{T} = \vec{p} \otimes \vec{N} &= mn\vec{U} \otimes \vec{U} = \rho\vec{U} \otimes \vec{U} \end{aligned} \tag{37}$$

where we can think of $T^{\alpha\beta}$ as flux of α momentum across a surface of constant x^{β}. We can prove that T is a symmetric tensor. We can also derive the mathematical relations leading to the laws such as The First and Second Laws of Thermodynamics as -

$$d\rho - \frac{(\rho + p)dn}{n} = nTdS \tag{38}$$

and a general conservation law, including the law of conservation of energy van be written as -

$$T^{\alpha}{}_{,} = 0 \tag{39}$$

and the conservation of particles (assuming no nuclear reactions or others which change the total number of particles) as -

$$N^{\alpha}_{,\alpha} = (nU^{\alpha})_{,\alpha} = 0 \tag{40}$$

We will mostly deal with **Perfect Fluids** in our study. A perfect fluid in relativity is defined as a fluid that has no viscosity and no heat conduction in the MCRF. It is a generalization of the 'ideal gas' of ordinary thermodynamics. We can show the stress energy tensor of a perfect fluid to be -

$$\mathbf{T} = (\rho + p)\vec{U} \otimes \vec{U} + pg^{-1} \tag{41}$$

These are of great importance in GR, since Einstein modified the classical understanding to show that T, the stress energy tensor is the source of the gravitational field, which shows that even components such as pressure and momentum contribute to gravitation, and not just the energy density.

4 General Theory of relativity

Experiments have successfully shown that a Lorentz frame at"rest" on the Earths surface is not inertial, and have come to the conclusion that gravity does in fact affect light. Freely falling frames however, can be shown to be inertial since all freely falling particles maintain uniform velocity in this frame. What we now call the Einstein equivalence principle says that we can discover how all the other forces of nature behave in a gravitational field by postulating that the differential equations that describe the laws of physics have the same local form in a freely falling inertial frame as they do in SR, i.e. when there are no gravitational fields. We keep in mind that our arguments are only valid locally, since the gravitational field of Earth is not uniform and it is hence impossible to create a global inertial frame, and non-uniformity's in the field are regarded as tidal forces. Any gravitational field can be regarded as uniform over a small enough region of space and time, and so we can always set up local inertial frames. They are analogous to the MCRFs of fluids.

4.1 Christoffel Symbols

Consider the derivative of a vector field $\vec{V}$ wrt x^β, which by our knowledge of chain rule can be written as -

$$\frac{\partial \vec{V}}{\partial x^\beta} = V^\alpha_{;\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta} \tag{42}$$

The final term in this equation is itself a vector, it can be written as a linear combination of the basis vectors; we introduce the Christoffel symbols to denote the coefficients in this combination -

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta}\vec{e}_\mu \tag{43}$$

It needs three indices: α gives the basis vector being differentiated; β gives the coordinate with respect to which it is being differentiated; and μ denotes the component of the resulting derivative vector. We then see that the derivative equation becomes -

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \tag{44}$$

and is called the covariant derivative, and denoted as $\partial(\vec{V})^\alpha_\beta = (\partial_\beta \vec{V})^\alpha = V^\alpha_{;\beta}$. The divergence of a field can be represented as $V^\alpha_{;\alpha}$. We can extend the same definition to one forms and see that they transform as -

$$\partial(\widetilde{p})_{\alpha\beta} = p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^\mu_{\alpha\beta} \tag{45}$$

We also realize that the Christoffel symbols are symmetric in the covariant indices. Some index gymnastics gives us the relation between the metric tensor and the symbols as -

$$\Gamma^\gamma_{\beta\mu} = \frac{g^{\alpha\mu}}{2}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{46}$$

4.2 Curved Manifolds

A manifold is any set that can be continuously parametrized. The number of independent parameters is the dimension of the manifold, and the parameters themselves are the coordinates of the manifold. We shall consider diffrentiable manifolds, that is manifolds which are continously diffrentiable. A differentiable manifold on which a symmetric (0,2) tensor field **g** has been singled out to act as the metric at each point, and which is positive definite, is called a Riemannian manifold. The metric of course gives a way to define lengths of curves with the help of λ as a parameter we define the length as -

$$l = \int_{\lambda_0}^{\lambda_1} |g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}|^{\frac{1}{2}} d\lambda \tag{47}$$

Since these manifolds are locally flat (a Minkowski frame locally), Christoffel symbols vanish and covariant derivatives become equal to the partial derivatives. We only need remember that in any coordinates the square root of the negative of the determinant of $g_{\alpha\beta}$ is the thing to multiply by dx^4 to get the true, or proper, volume element.
It is important to distinguish two different kinds of curvature: intrinsic and extrinsic. Consider, for example, a cylinder. Since a cylinder is round in one direction, we think of it as curved. This is its extrinsic curvature: the curvature it has in relation to the flat three-dimensional space it is part of. On the other hand, a cylinder can be made by rolling a flat piece of paper without tearing or crumpling it, so the intrinsic geometry is that of the original paper: it is flat. The intrinsic geometry of an n-dimensional manifold considers only the relationships between its points on paths that remain in the manifold (for the cylinder, in the two-dimensional surface).

4.3 Parallel Transport and Geodesics

An intrinsic geometry can be visualized by parallel transporting a vector. Consider a flat space and a closed curve drawn on it, and draw a vector tangent to a point on the curve. Now at at each point a vector is drawn parallel to the one at the previous point. This construction is carried around the loop and the vector finally drawn at the initial point is, of course, parallel to the original one. On a curved surface, like a sphere, a completely different thing happens, where the final vector is rotated relative to the original one, an effect of the spheres intrinsic curvature, and this construction is called parallel transporting a vector. This result has radical implications: on a curved manifold it simply isn't possible to define globally parallel vector fields.We can still define local parallelism, for instance how to move a vector from one point to another, keeping it parallel and of the same length. Suppose a vector field $\vec{V}$ is defined on the sphere, and we examine how it changes along a curve, and if $\vec{U} = \frac{d\vec{x}}{d\lambda}$ is the tangent along the curve, then the frame invariant definition of parallel transport of $\vec{V}$ along $\vec{U}$ can be written as -

$$\frac{d\vec{V}}{d\lambda} = U^{\beta} V^{\alpha}_{;\beta} = \nabla_{\vec{U}} \vec{V} = 0 \tag{48}$$

In a curved space, we can also draw lines that are 'as nearly straight as possible' by demanding parallel-transport of the tangent vector. These are called geodesics, and can be written as (with λ as the parameter of the curve) -

$$\nabla_{\vec{U}} \vec{U} = 0 \tag{49}$$

$$\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) + \Gamma^{\alpha}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0 \tag{50}$$

4.4 The Curvature Tensor

We can now write a mathematical description of the intrinsic curvature of a manifold by taking the example of the parallel-transport of a vector around a closed loop, and take it as our definition of curvature. We find the change in the vector which is parallel transported, and hence define the Riemann curvature tensor **R** as the (1,3) tensor which when supplied with the basis vectors of the loop and the parallel transport vector give the components of change in the vector, and is denoted by $R^{\alpha}{}_{\beta\mu\nu}$ characterizes the curvature in a tensorial way, and is given by -

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\sigma\mu} \Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\nu} \Gamma^{\sigma}{}_{\beta\mu} \tag{51}$$

$$[\nabla_{\alpha}, \nabla_{\beta}] V^{\mu} = R^{\mu}{}_{\nu\alpha\beta}\, v^{\nu} \tag{52}$$

Lowering the first index, we can see that R_{α} is antisymmetric on the first pair and on the second pair of indices, and symmetric on exchange of the two pairs, and it being 0 implies a flat manifold. In a curved space, parallel lines when extended do not remain parallel. This can now be formulated mathematically in terms of the Riemann tensor. We define a 'connecting vector' ξ which 'reaches' from one geodesic to another, connecting points at equal intervals in λ. Geodesics in flat space maintain their separation; those in curved spaces don't.

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^{\alpha} = R^{\alpha}_{\mu\nu\beta}\vec{V}^{\mu}\vec{V}^{\nu}\xi^{\beta} \tag{53}$$

This is called the equation of geodesic deviation and shows mathematically that the tidal forces of a gravitational field (which cause trajectories of neighboring particles to diverge) can be represented by curvature of a spacetime in which particles follow geodesics. We can find the Bianchi identities by playing around with the definition of R, which leads us to -

$$R_{\alpha\beta\mu\nu;\lambda} \quad + R_{\alpha\beta\lambda\mu;\nu} \quad + R_{\alpha\beta\nu\lambda;\mu} \quad = 0 \tag{54}$$

The Ricci Tensors $R_{\alpha\beta}$ is the contraction of $R^{\mu}{}_{\alpha\mu\beta}$ on the first and third indices. Other contractions would in principle also be possible: on the first and second, the first and fourth, etc. But because R is antisymmetric, all these contractions either vanish identically or reduce to $\pm R_{\alpha\beta}$. Therefore the Ricci tensor is essentially the only contraction of the Riemann tensor. The Ricci scalar is further defined as - $R = g^{\mu\nu}R_{\mu\nu}$. We can further define the Einstein Tensor as -

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = G^{\beta\alpha} \tag{55}$$
$$G^{\alpha\beta}_{;\beta} = 0$$

A few identifications and postulates -

- Spacetime (the set of all events) is a four-dimensional manifold with a metric.
- The metric of spacetime can be put in the Lorentz form $\eta_{\alpha\beta}$at any particular event by an appropriate choice of coordinates.
- Weak Equivalence Principle: Freely falling particles move on timelike geodesics of spacetime. This generalizes to the Einstein Equivalence Principle: Any local physical experiment not involving gravity will have the same result if performed in a freely falling inertial frame as if it were performed in the flat spacetime of special relativity.

For weak gravitational fields (where, in Newtonian language, the gravitational potential energy of a particle is much less than its rest-mass energy) the ordinary Newtonian potential ϕ completely determines the metric, which has the form -

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2 + dy^2 + dz^2) \tag{56}$$

Finding the equations of motion for a particle following the geodesic, we get the generalized equation as -

$$m\frac{dp_{\beta}}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}p^{\nu}p\alpha \tag{57}$$

We therefore have the following important result: if all the components of g are independent of x_{β} for some fixed index , then p_{β} is a constant along any particle's trajectory. The time component generalizes the Law of Conservation of Energy as $\frac{dp^0}{d\tau} = -m\frac{d\phi}{d\tau}$, whereas the spatial components to first approximation generalize the Newtons Laws of Motion as - $\frac{dp^i}{d\tau} = -m\phi_{,j}\delta^{ij}$. For weak fields, in which we are working now, we get $-p_0 = m + m\phi + \frac{p^2}{2m}$ (conserved energy) if the field is static (independent of time). If the metric is further axially symmetric, then we get $p_\phi = mr^2\omega$ (angular momentum conserved).

4.5 Field Equations of GR

The field equations are the heart and soul of GR, and are the relativistic counterparts of the Newtonian equation $\nabla^2\phi = 4\pi G\rho$ and are given by (in geometrized units) -

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = 8\pi T^{\alpha\beta} \tag{58}$$

the addition of Λ was done later, and is called the Cosmological Constant. Since the absence of gravity leaves spacetime flat, a weak gravitational field is one in which spacetime is 'nearly' flat. This is defined as a manifold on which coordinates exist in which the metric has components $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $h_{\alpha\beta} << 1$ everywhere. We see that, under a background Lorentz transformation, $h_{\alpha\beta}$ transforms as if it were a tensor in SR all by itself! It is, of course, not a tensor, but just a piece of $g_{\alpha\beta}$. But this restricted transformation property leads to a convenient fiction: we can think of a slightly curved spacetime as a flat spacetime with a 'tensor' h defined on it. We can also have gauge transformations, which changes coordinates in the way $h_{\alpha\beta} \longrightarrow h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, which gives us a nice description of the curvature tensor in terms of h only and hence we can get the weak-field Einstein equations as -

$$\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu} \tag{59}$$

where $\Box$ is the D'Alembertian operator. Comparing this to Newtonian equations, we get $\bar{h}^{00} = -4\phi$. We also define the relativistic mass, as the total mass of the object (different from the Newtonian concept of mass) with the far field potential after solving the above equations as -

$$(\phi)_{\text{relativistic far field}} = -\frac{M}{r} \tag{60}$$

Newtonian gravity is produced exclusively by the curvature of time in spacetime. Spatial curvature comes in only at the level of post-Newtonian corrections, which we shall see in the next section.

5 A few Applications of GTR

5.1 Gravitational Waves

Gravitational waves can be generated from rapidly moving sources, with a changing quadruple moment. The 3-d wave equation $(-\frac{\partial^2}{\partial t^2} + \nabla^2)\bar{h}^{\alpha\beta} = 0$, and has the complex solution as $\bar{h}^{\alpha\beta} = A^{\alpha\beta}exp(ik_\alpha x^\alpha)$, where the four vector k is null to satisfy the wave equation, and can be written as - $\vec{k} \longrightarrow (w, k)$, and this is called a plane wave and by theorems of Fourier analysis all solutions are superpositions of this basic solution. Changing the gauge, we can use the traceless-transverse gauge which imposes 2 further conditions - $A^\alpha{}_\alpha = 0$ and $A_{\alpha\beta}U^\beta = 0$. In background Minkowski spacetime and orienting the axis so that the wave travels in the +ve dirn, we get the Matrix of A as -

$$(A^{TT}_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{61}$$

A free particle following the geodesic equations can be shown to always be at a constant coordinate position as the wave passes by, the proper distance (as opposed to the coordinate distance) does change with time as $\Delta l \approx [1 + \frac{h^{TT}_{xx}}{2}]l$. Gravitational waves create a bigger distance change if the original distance is bigger. This is the reason that modern gravitational wave detectors, are designed and built on huge scales, measuring changes in separations over many kilometers (for ground-based detectors like LIGO) or millions of kilometers (in space like LISA). Two particles initially separated in the x direction by a distance ϵ have a separation vector $\vec{\xi}$ whose components proper lengths obey when a wave passes through -

$$\frac{\partial^2 \xi^x}{\partial t^2} = \frac{\epsilon}{2}\frac{\partial^2 h^{TT}_{xx}}{\partial t^2} \tag{62}$$

$$\frac{\partial^2 \xi^y}{\partial t^2} = \frac{\epsilon}{2}\frac{\partial^2 h^{TT}_{xy}}{\partial t^2} \tag{63}$$

$$\frac{\partial^2 \xi^i}{\partial t^2} = -R^i{}_{0j0}\,\xi^j \tag{64}$$

Consider a ring of particles initially at rest in the x y plane, as in. Suppose a wave has $h^{TT}_{xx} \neq 0, h^{TT}_{xy} = 0$ (the + polarization) Then the particles will be moved (in terms of proper distance relative to the one in the center) in the way shown in (b), as the wave oscillates. If, instead, the wave had $h^{TT}_{xy} \neq 0, h^{TT}_{xx} = 0$ (the x polarization) then the picture would distort as in (c). Since h^{TT}_{xy} and h^{TT}_{xx} are independent, (b) and (c) provide a pictorial representation for two different linear polarizations. Notice that the two states are simply rotated 45° relative to one another. This contrasts with the two polarization states of an electromagnetic wave, which are 90° to each other.

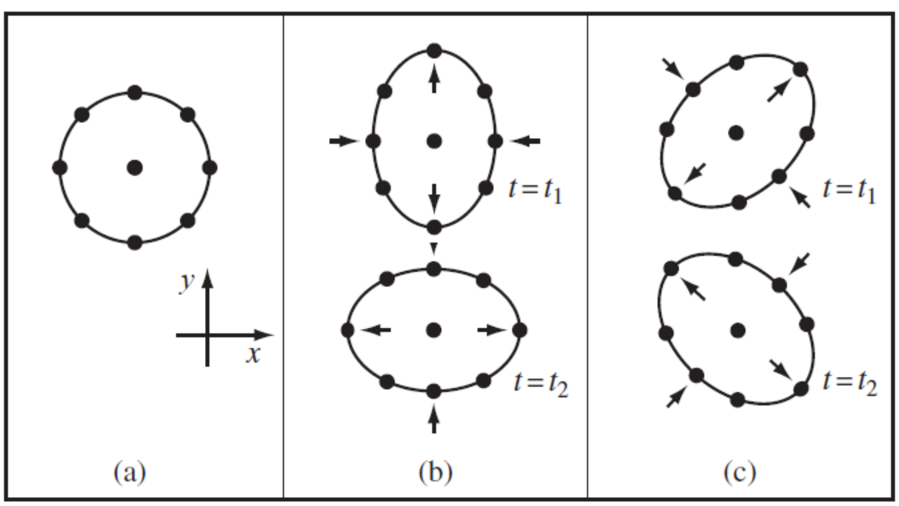

Figure 2: (a) A circle of free particles before a wave traveling in the z direction reaches them. (b) Distortions of the circle produced by a wave with the '+' polarization. The two pictures represent the same wave at phases separated by 180○. Particles are positioned according to their proper distances from one another. (c) As (b) for the '×' polarization.

If the waves have short wavelength, then they basically follow a null geodesic, and they parallel-transport their polarization tensor. This is exactly the same as for electromagnetic waves, so that photons and gravitational waves leaving the same source at the same time will continue to travel together through the universe, provided they move through vacuum. If nearby null geodesics converge, then gravitational and electromagnetic waves traveling on them will be focused and become stronger. This is called gravitational lensing.

To discuss the slow motion wave generation, we solve the Field Equations, assuming the time dependant part of T to be sinusoidal (astrophysical sources are roughly periodic: pulsating stars, pulsars, binary systems), we assume the source is much much smaller than the distance consideration, we get the exact solution as a retarded integral -

$$\bar{h}_{\mu\nu}(t, x_i) = 4 \int \frac{T_{\mu\nu}(t - R, y^i)}{r} d^3y \tag{65}$$

where R is $|x^i - y^i|$, which generalizes to $\bar{h}_{\mu\nu}(t, x_i) = -\frac{2}{r} I_{jk,00}(t - r)$, where I is the referred to as the quadruple moment tensor of the mass distribution: $I^{lm} = \int T^{00} x^l x^m d^3x$. These are majorly generated by four groups of sources: binary systems - mostly inspiral events ending in the merger of the two objects which itself might produce a burst of radiation, spinning neutron stars (only if they aren't symmetrical about the rotation axis), gravitational collapse - mostly supernova explosions, and the waves due to the Big Bang which are very weak but might be detected upon increasing sensitivity.

5.2 Spherical Solutions for Stars

We define a static spacetime to be one in which we can find a time coordinate t with two properties: (i) all metric components are independent of t, and (ii) the geometry is unchanged by time reversal, tßt. We then get the metric for a static, spherically symmetric spacetime as -

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega \tag{66}$$

demanding asymptotically flat spacetime. These have physical interpretation attached to them, the Λ term changing the proper radial distance between particles, and the Φ term relating to the redshift, or the energy measured by a local inertial observer. The stress–energy tensor involves both p and ρ, but these may be related by an equation of state. Using this and the conversation laws of the Stress Energy tensor, we get the major 3 equations governing the stars as -

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \tag{67}$$

$$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r(r - 2m(r)} \tag{68}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho \tag{69}$$

inside the star. Solving these outside the star, we get the Schwarzschild metric as -

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2 \tag{70}$$

A more general treatment, establishes Birkhoff 's theorem, that the Schwarzschild solution, is the only spherically symmetric, asymptotically flat solution to Einstein's vacuum field equations, even if

we drop our initial assumptions that the metric is static. This means that even a radially pulsating or collapsing star will have a static exterior metric of constant mass M. Dividing the 1st and 2nd equation we obtain an equation known as Tolman–Oppenheimer–Volkov (T–O–V) equation. Inside the star, however, assuming an equation of state as ρ constant, we get the Schwarzschild solution. Buchdahl (1981) found a solution for the equation of state for relativistic stars as $\rho = 12(p_* p)^{\frac{1}{2}} - 5p$, p_* being an arbitary constant, and using causal relation of the speed of sound inside the star to be less than 1, we can get exact solutions which are too extensive to write down here.
We see that there are no uniform-density stars with radii smaller than (9/4)M, because to support them in a static configuration requires pressures larger than infinite! This is in fact true of any stellar model, and is known as Buchdahl's theorem (Buchdahl 1959). The figure below shows a brief overview of the Stellar life cycle, which is highly dependant on the mass of the star, its rotation, its magnetic field and its composition.

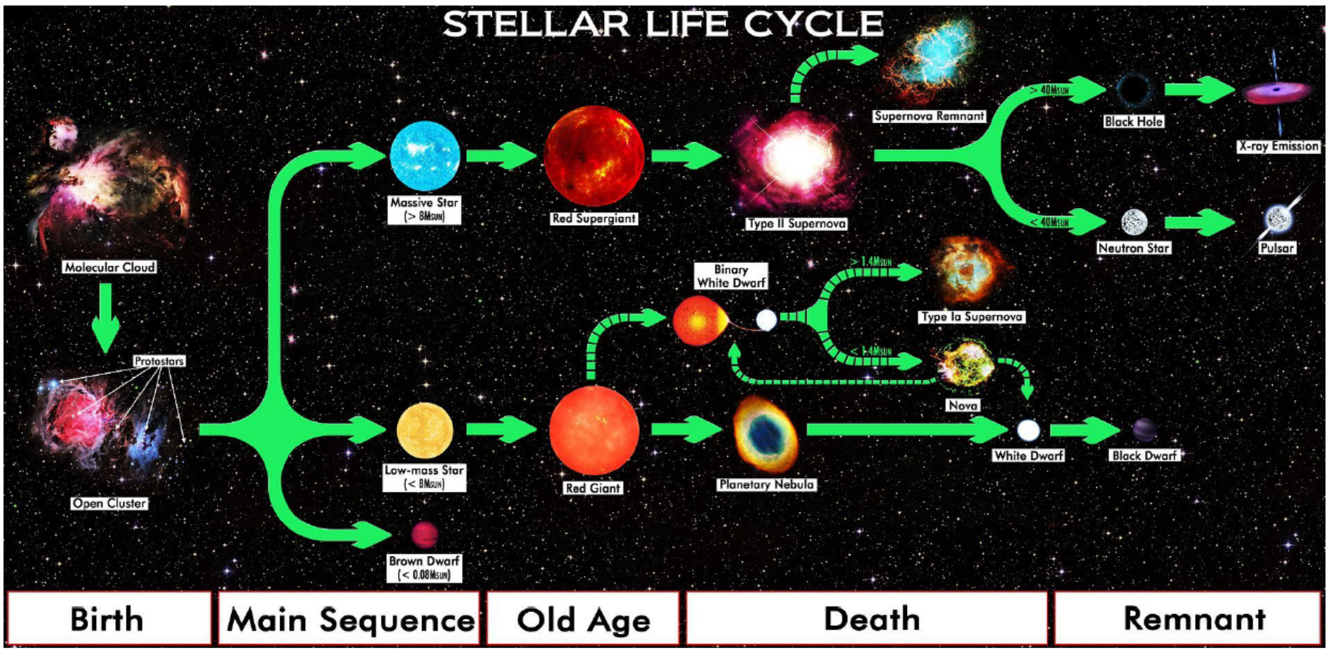

Figure 3: A brief overview of Stellar Evolution.[4]

The quantum mechanical pressure of a cold electron gas, is $p \approx \frac{\rho}{3}$ and the maximum pressure comes from these in White Dwarfs, whereas the maximum mass is contributed from cold nonrelativistic gas of nuclei.

5.3 Perihelion Shift

To solve for the trajectories of particles and photons we solve the momentum equation of the particle and photon, which gives us the equations of orbits in terms of energy E and angular momentum L -

$$\textbf{particle} - \left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right) \tag{71}$$

$$\textbf{photon} - \left(\frac{dr}{d\lambda}\right)^2 = (E)^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2}\right) \tag{72}$$

using this we define the effective potentials and the right terms, so the right hand side becomes $E^2 - V(r)^2$. A circular orbit (r = const) is possible only at a minimum or maximum of V^2, and we can calculate the derivative of V and equate it to 0, which gives us 3M radius for photons, and 2 radii for particles, and a stable circular orbit at radius r has angular momentum $(\frac{L}{m})^2 = \frac{Mr}{1-\frac{3M}{r}}$. We can get the angular velocity and hence period of the particle for a circular orbit, and these come out the same as Newtonian predictions. A slightly noncircular orbit will oscillate in and out about a central radius r. In Newtonian gravity the orbit is a perfect ellipse, which means, among other things, that it is closed: after a fixed amount of time it returns to the same point (same r and ϕ). In GR, this does not happen. However, when the effects of relativity are small and the orbit is nearly circular, the relativistic orbit must be almost closed: it must look like an ellipse which slowly rotates about the center. One way to describe this is to look at the perihelion of the orbit, the point of closest approach to the star. The perihelion will rotate around the star in some manner, and observerations can hope to measure this. We get $\frac{d\phi}{d\tau} = \frac{L}{mr^2}$, from the definition of L, squaring this and dividing to obtain an expression in terms of $\frac{dr}{d\phi}$, and solving we get the equation as $\frac{1}{r} - \frac{Mm^2}{L^2} = y_0 + Acos(k\phi + b)$, the orbit returns to the same r when $k\phi$ goes through 2π therefore the change in ϕ from one perihelion to the next is -

$$\Delta\phi = 2\pi(1 - \frac{6M^2m^2}{L^2})^{\frac{1}{2}} \tag{73}$$

The perihelion advance, then, from one orbit to the next is then $6\pi\frac{M^2m^2}{L^2}$ radians per orbit, it has been measured for Mercury to be 43"/century, not explainable by Newtonian gravity, and the demonstration that Einstein's theory predicts exactly that amount was the first evidence in favor of the theory.

5.4 Gravitational Lensing

In this section we treat the analogous effect for photons, their deflection from straight-line motion as they pass through a gravitational field. Historically, this was the first general-relativistic effect to have been predicted before it was observed, and its confirmation in the eclipse of 1919 (see McCrae 1979) made Einstein an international celebrity. Similar to the treatment of particles, we calculate the $\frac{d\phi}{dr}$ in terms of the impact parameter ($b = \frac{L}{E}$) -

$$\frac{d\phi}{dr} = \frac{1}{r^2}[\frac{1}{b^2} - \frac{1}{r^2}(1 - \frac{2M}{r})]^{\frac{-1}{2}} \tag{74}$$

including the $\frac{1}{r^3}$ term, we calculate the correction (removing this term leads to Newton's solution of straight line orbit), assuming $\frac{M}{r} << 1$, we get the solutions as -

$$\phi = \phi_0 + \frac{2M}{b} + arcsin(by) - 2M(\frac{1}{b^2} - y^2)^{\frac{1}{2}} \tag{75}$$

where we define $y = \frac{1}{r}(1 - \frac{M}{r})$, and using our approximations, we get the net deflection as -

$$\Delta\phi = \frac{4M}{b} \tag{76}$$

It may of course happen that photons from the same star will travel trajectories that pass on opposite sides of the deflecting star and intersect each other after deflection. As astronomers have built larger and more powerful telescopes, able to see much greater distances into the universe, they have revealed a sky filled with lensed images. Of particular importance is lensing by clusters of galaxies. There are so many galaxies in the universe that, beyond any given relatively distant cluster, there is a high probability that there will be another group of galaxies located in just the right position to be lensed into multiple images in our telescope: the probability of 'being in exactly the right spot' has become reasonably large. What is more, the masses of galaxy clusters are huge, so the deflections are much bigger than the sizes of the more distant galaxy images, so separating them is not a problem. In fact, we often see multiple images of the same object, created by the irregularities of the lensing mass distribution.

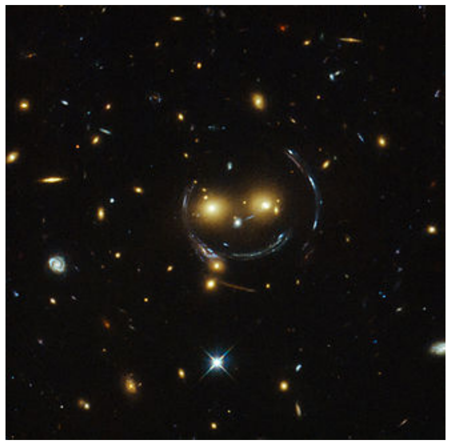

Figure 4: Smiley" or "Cheshire Cat" image of galaxy cluster (SDSS J1038+4849) and gravitational lensing (an "Einstein ring") discovered by an international team of scientists, imaged with HST.[5]

Even more important than the creation of separate images can be the brightening of single images by the focusing of light from them, and brightening can be used to determine the amount of mass in the intervening cluster, which was instrumental in the study of dark matter, as we will see below.

5.5 Blackholes

As long ago as the late 1700's, the British physicist John Michell and the French mathematician and physicist Pierre Laplace speculated (independently) on the possibility that stars might exist whose escape velocity was larger than the speed of light. The star would then be dark, invisible. For a spherical star, this is a simple computation. By conservation of energy, a particle launched from the surface of a star with mass M and radius R will just barely escape if its gravitational potential energy balances its kinetic energy which gives the radius if the blackhole $R = \frac{2GM}{c^2}$, remarkably, as we shall see this is exactly the modern formula for the radius of a black hole in general relativity. For Michell and Laplace the star was dark because light could not escape to infinity. The star was still there, shining light. The light would still leave the surface, but gravity would eventually pull it back, like a ball thrown upwards. In relativity, the light never leaves the 'surface' of a black hole; and this surface is itself not the edge of a massive body but just empty space, left behind by the inexorable collapse of the material that formed the hole.

It is clear that something funny goes wrong with the line element in Schwarzschild metric, at r = 2M, but what is not clear is whether the problem is with the geometry or just with the coordinates. If we calculate the proper time it takes fir an object to fall to $r = 2M$ radially, we get a finite time - $\Delta\tau = \frac{4M}{3}[\frac{R}{2M} - 1]^{\frac{3}{2}}$. However if we calculate the coordinate time, we see the integral diverges, thus from an outside observer, a particle takes infinite time to reach the horizon. Inside $r < 2M$ we see from the metric that the timelike coordinate has become spacelike, and vice versa. Everything inside $r = 2M$ is trapped and, moreover, doomed to encounter the singularity at $r = 0$, since it is the future of every timelike and null world line. Once a particle crosses the surface, it cannot be seen by an external observer, since to be seen means to send out a photon which reaches the external observer. This surface is therefore called a horizon, and is known as the Schwarzschild horizon. We then move into the Kruskal–Szekeres coordinates, in which the coordinates act more "well behaved", and the metric is given by -

$$ds^2 = -\frac{32M^3}{r}e^{\frac{r}{2M}}(dv^2 - du^2) + r^2 d\Omega^2 \tag{77}$$

where $u = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} cosh(\frac{t}{4M})$ and $v = (\frac{r}{2M} - 1)^{\frac{1}{2}} e^{\frac{r}{4M}} sinh(\frac{t}{4M})$ for $r < 2M$ and exchange outside the horizon, showing the singularity at $r = 0$. The coordinates u and v are not particularly good for describing the geometry far from the star, the coordinates t and r are best there; indeed, they were constructed in order to be well behaved there. But if we are interested in the horizon, then we use u and v. The central, even astonishing property of the Schwarzschild horizon is that anything that crosses it can not get back outside it. The definition of a general horizon (called an event horizon) focuses on this property. An event horizon is the boundary in spacetime between events that can communicate with distant observers and events that cannot. While the detailed structure of an event horizon is not easy to compute, some important general properties of horizons are understood, and they underpin the confidence with which astronomers now employ black holes in models of complex astrophysical phenomena.

- It is believed that any horizon will eventually become stationary, provided that it is not constantly disturbed by outside effects like accretion. So an isolated black hole should become stationary. The principal result is that a stationary vacuum black hole is characterized by just two numbers: its total mass M and total angular momentum J.

- If the black hole is not in vacuum, its structure may be more general. It may carry an electric charge Q and, in principle, a magnetic monopole moment F, although magnetic monopoles are not found in Nature.

- If gravitational collapse is nearly spherical, then all nonspherical parts of the mass distribution – quadrupole moment, octupole moment – except possibly for angular momentum, are radiated away in gravitational waves, and a stationary black hole of the Kerr type, a rotating blackhole, stable to perturbations is left behind.

- An important general result concerning nonstationary horizons is the area theorem of Hawking : in any dynamical process involving black holes, the total area of all the horizons cannot decrease in time, owing to the entropy relation with area as - $S = \frac{k_B A}{4h}$.

- Inside the Schwarzschild and Kerr horizons there are curvature singularities where the curvature, and hence the tidal gravitational force, becomes infinite. Many physicists expect this shortcoming to be remedied by a quantum theory of gravity.

The Kerr black hole is axially symmetric but not spherically symmetric (that is rotationally symmetric about one axis only, which is the angular-momentum axis), and is characterized by two parameters, M and J, the metric of which is given by the heinous equation -

$$ds^2 = -\frac{\Delta - a^2sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mrsin^2\theta}{\rho^2}dtd\phi + \frac{(r^2+a^2)^2 - a^2\Delta sin^2\theta}{\rho^2}sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{78}$$

$$\Delta = r^2 - 2Mr + a^2 \tag{79}$$

$$\rho^2 = r^2 + a^2cos^2\theta \tag{80}$$

The presence of $g_{t\phi} \neq 0$ affects particles trajectories. We can show that a particle with no initial L from infinity is 'dragged' just by the influence of gravity so that it acquires an angular velocity in the same sense as that of the source of the metric. This effect is often called the dragging of inertial frames. We shall see that the surface where $g_{tt} = 0$ lies outside the horizon; it is called the ergosphere. It is sometimes also called the 'static limit', since inside it no particle can remain at fixed r, θ, ϕ ,the dragging of orbits has become so strong that even a photon cannot move in the direction opposite the rotation, and is given by $r_{\text{ergosphere}} = M + (M^2 - a^2cos^2\theta)^{\frac{1}{2}}$. In the Kerr solution, the horizon occurs at $\Delta = 0$, and is given by $r_{\text{horizon}} = M + (M^2 - a^2)^{\frac{1}{2}}$, with the Area of horizon as $A = 4\pi(r^2 + a^2)$, we can similarly caluclate the particle trajectories given the Kerr metric and define potentials as we did before. We have also observed blackholes of stellar mass formed from the death-throes of stars, supermassive blackholes and the centre of galaxies, intermediate blackholes from the collapse of Population III stars, and dynamical blackholes. Further advanced treatment using quantum mechanics shows that blackholes behave like blackbodies with temperatures $T = \frac{\hbar}{8\pi kM}$, which gives us insight into phenomenon like Hawking radiation, which is due to quantum fluctuations at the horizon of the blackhole. With the basics of GR in hand, we can now move onto the study of Cosmology.

6 A Brief Introduction to Our Cosmos

Cosmology is a branch of astronomy concerned with the studies of the origin and evolution of the universe, from the Big Bang, to today, and on into the future. It consists of 2 parts- Cosmogony (The origin of the universe) and Cosmography (mapping the features of the universe). In this part, we shall give a small overview of the sections to come, the details of which will be discussed in the next sections.

The cornerstone of Modern Cosmology is the belief that the place occupied by us in the universe is in no way special, and is known as the Cosmological Principle (it is also called as the Copernican Principle, but this is in fact a generalization, since he believed that the Sun was the centre). This then implies, that the Universe is globally homogeneous at the scale of the largest structures in the universe. Hence 2 properties of the Universe at the global scale include -

- Homogeneity - The Universe "looks"same at each point.
- Isotropy - The Universe "looks" the same in all directions.

6.1 Structures in the Universe

Since the structures we talk about are in astronomical terms, it makes sense to use a different scale of units, $\mathcal{M}_\odot$as the Mass of the Sun and 1 parsec (3.26 light years) as the distance unit. The smallest structure we will talk about are stars, a collection of which (and much more!) make a Galaxy, with an approximate distance of 1 Mpc between galaxies. The Sun, along with 400 billion other stars, reside in the Milky Way galaxy. A distance of about 100 Mpc shows a cluster of galaxies, which further group to form Superclusters, which are joined by filaments/walls of galaxies (the Milky way resides in the Virgo Supercluster), and lastly there are Cosmic voids, which contain very few galaxies, and hence are mostly "empty".

These structures, can be "seen", and hence the light from them resides in the visible region of the EM spectrum, but other wavebands of light are equally important to obtain data. Microwaves are used for observing the Cosmic Microwave Background Radiation (CMBR) which tells us about the curvature of the universe and the density of matter in the universe, Radio waves for observing distant galaxies, Infrared for closer galaxies due to less scattering, and X rays for clusters of galaxies.

The universe is thought to be made up of mostly 4 kinds of particles -

- **Baryons :** A general term for particles made up of 3 quarks (quarks are considered to be one of the elementary particles in the Standard Model). These are mostly non-relativistic particles (rest-mass energy much more than their kinetic energy) which consist of protons and neutrons, and to the annoyance of particle physicists, electrons too. These make the atoms which make up majority of the structures of the universe.

- **Radiation :** These are relativistic particles, having rest mass 0, and are called photons. These interact with baryons through Thompson Scattering (at low energies) and Compton Scattering (at high energies).

- **Neutrinos :** These are highly relativistic particles, have a very low mass due to which they can be assumed as massless, and are very weakly interacting particles. They are mostly created in radioactive processes.

- **Dark Matter and Dark Energy :** These are proposed to make up 95 % of the matter-energy content of the universe, but their actual composition remains a mystery.

6.2 The Expanding Universe - Hubble's Law

In the early 1910s, it was observed that galaxies seemed to be moving away from us, and as a result, the light coming from them was red shifted (the Doppler effect of light waves). Redshift (z) is defined as -

$$z = \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}} \tag{81}$$

And after applying special relativity, the redshift comes out in terms of the relative velocity (v) as -

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{82}$$

which reduces to $z = v/c$ on applying the approximation of $v << c$. Edwin Hubble (the pioneer of modern cosmology), in 1929 upon observing that the (recession) velocity of these galaxies was directly proportional to their distance from us (r), gave us the Hubble's Law -

$$\vec{v} = H_0 \vec{r} \tag{83}$$

where H_0 is the proportionality constant, known as Hubble's constant ($H_0 = 100h$ km/sMpc, where h ranges from $(0.64, 0.8)$)

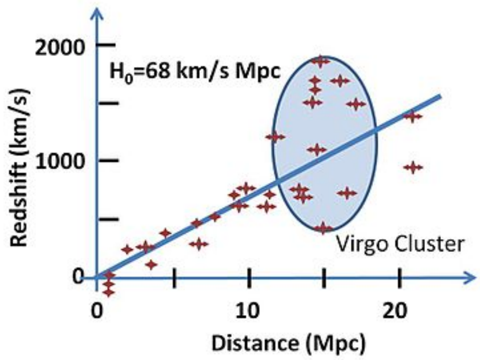

Figure 5: Fit of redshift velocities to Hubble's Law [1].

This relation does not mean that there is a "center" of the universe, from where $\vec{r}$ is measured, but it is measured from the observer as the origin, which works due to the linearity of the relation. Another important point to note is that this does not hold for nearby galaxies, due to the peculiar velocity of galaxies. The fact that the universe is expanding, also lead to the idea of The Big Bang Theory being developed.

7 Differential Equations involved in Cosmology

7.1 The Cosmological Metric

The metric tensor that represents a cosmological model must incorporate the observed homogeneity and isotropy. Thus, if at one moment, t_0, the hypersurface of constant time has the line element $dl^2 = h_{ij}(t_0)dx^i dx^j$, this also guarentees that h must be independant of the spatial coordinates to preserve isotropy and we can extend this to the whole spacetime as - $ds^2 = -dt^2 + R^2(t)h_{ij}dx^i dx^j$. A necessary condition for homogeneity is certainly that the Ricci scalar curvature of the three-dimensional metric, R, must have the same value at every point: every scalar must be independent of position at a fixed time. Enforcing this condition we get the Robertson-Walker metric as -

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \tag{84}$$

Notice that we can, without loss of generality, scale the coordinate r in such a way as to make k take one of the three values +1, 0, 1. Putting the different values of k, we see it is representative of the curvature of the universe, which we will see more about in the next section. We can see this relation if we put $k = 0$, we get the metric of flat Euclidean space. Putting $k = 1$, we can paramterize space using sinusoidal functions to get the time hypersurface as $dl^2 = a^2[d\chi^2 + sin^2\chi(d\theta^2 + sin^2\phi^2)]$ which is the metric of a three-sphere of radius $a(t_0)$, i.e. of the set of points in four-dimensional Euclidean space that are all at a distance a from the origin. This model is called the closed, or spherical Robertson–Walker metric. Putting k = -1, we parametrize it with the hyperbolic sin function as - $dl^2 = a^2[d\chi^2 + sinh^2\chi(d\theta^2 + sin^2\phi^2)]$. This is called the hyperbolic, or open, Robertson–Walker model. For flat FRW metric, we can calculate the Christoffel symbols as ${\Gamma^0}_{0\alpha} = 0$, ${\Gamma^0}_{ij} = \delta_{ij}\dot{a}a$ and ${\Gamma^i}_{0j} = \delta_{ij}\frac{\dot{a}}{a}$. The Ricci Tensor can be calculated as - $R_{00} = -3\frac{\ddot{a}}{a}$ and $R_{ij} = \delta_{ij}(2\dot{a}^2 + a\ddot{a})$, which gives us the Ricci Scalar as - $R = 6(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2)$, which is isotropic and homogeneous at each point in space, as needed.

7.2 The Friedmann Equation

The Friedmann equation describes the expansion of the Universe, and is therefore one of the most important equations in cosmology. Consider a uniformly expanding medium with mass density ρ. According to the cosmological principle, we can consider any point to be the centre, hence consider a small volume containing mass m at a distance $\vec{r}$. According to Newtons laws, this "particle" has a gravitational potential energy and a kinetic energy, and the total energy (U) can be given by -

$$U = \frac{m\dot{r}^2}{2} - \frac{4\pi G\rho m r^2}{3} \tag{85}$$

We know change into a different coordinate system, the **comoving coordinate system**, which, as the name implies, is carried along with the expansion. Objects are assumed to be fixed in this frame (no peculiar velocities).

$$\vec{r} = a(t)\vec{x} \tag{86}$$

Here $\vec{r}$ is the real distance and $\vec{x}$ is the comoving distance and $a(t)$ is called the scale factor of the universe, where we have used isotropy in assuming that it is a function of time only. Substituting the above equations, using the fact that $\dot{x}$ is 0, and rearranging, we get the **The Friedmann Equation** -

$$(\frac{\dot{a}}{a})^2 = \frac{8G\pi\rho}{3} - \frac{kc^2}{a^2} \tag{87}$$

where we have made the substitution $kc^2 = -2U/mx^2$. Here k is independent of x and tells us about the curvature of the universe, as we had shown previously. There are 3 possible geometries to the universe, relating to the values of k, as summarized below. Observations indicate that the universe is very close to spatial flatness, and hence we will be setting $k = 0$ in many places. This is the Newtonian analog of the Friedmann Equation.

curvature	geometry	angles of triangle	circumference of circle	type of Universe
$k > 0$	spherical	$> 180^o$	$c < 2\pi r$	Closed
$k = 0$	flat	180^o	$c = 2\pi r$	Flat
$k < 0$	hyperbolic	$< 180^o$	$c > 2\pi r$	Open

Figure 6: A summary of different geometries.

Writing down the Einstein Field equations for such a universe, we can immediately notice that G_{tt} and G_{rr} are the only non trivial values, related by the Bianchi identies, hence only one component of the Einstein tensor is independent and is given by $G_{tt} = 3(\frac{\dot{a}}{a})^2 + \frac{3k}{a^2}$ and when substituted into $G_{tt} = 8\pi T_{tt}$ gives us the Friedmann equation, exactly the same as when derived by Newtons Laws (surprise surprise!).

7.3 The Fluid Equation

The fluid equation describe the change of ρ with time by including a pressure term. According to the first law of thermodynamics $\frac{\mathrm{d}E}{\mathrm{d}t} + p\frac{\mathrm{d}V}{\mathrm{d}t} = T\frac{\mathrm{d}S}{\mathrm{d}t}$, using comoving coordinates, using $E = mc^2$, and assuming reversible expansion ($\frac{\mathrm{d}S}{\mathrm{d}t} = 0$) and putting it all together we get -

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + \frac{p}{c^2}) = 0 \tag{88}$$

We should not think of this pressure as helping the expansion along, but see its effect through the work done during expansion. Upon differentiating the Friedmann equation, and substituting in the fluid equation, we come across the **Acceleration Equation** -

$$\frac{\ddot{a}}{a} = \frac{-4G\pi}{3}(\rho + \frac{3p}{c^2}) \tag{89}$$

Note the independence from k in the equation. From here on we will also be using the natural units, setting $c = 1$. Substituting the comoving coordinates in Hubble's law, we also get $H = \frac{\dot{a}}{a}$, and define the current value of the Hubble's constant as H_0.
Looking at the spatial part of Einstein's equations, where there acceleration $\ddot{a}$ explicitly appears. We can derive this from the Christoffel symbols, which implies the following simple 'equation of motion' for the scale factor, and gives us the Acceleration Equation. The acceleration is produced not only by energy density but by also the pressure exerted, and gives rise to what we call as the active gravitational mass - $(\rho + 3p)$.

7.4 Simple Cosmological Models

In order to discover how the Universe might evolve, we need some idea of what is in it. In a cosmological context, this is done by specifying the relationship between the mass density ρ and the pressure p. This relationship is known as the equation of state. We will consider simplified models here, either purely radiation dominated or matter dominated. Through most of the early universe, reactions proceeded rapidly enough to keep particles in equilibrium, different species sharing a common temperature. We will often want to express the energy density and pressure in terms of this temperature. For this reason, and many others which will emerge over the next few chapters, it is convenient to introduce the occupation number, or distribution function of a species. The energy density can be represented as $\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x}, \vec{p}) E(p)$, where i labels different species, g_i is the degeneracy of the species and f is the distribution functions. The pressure can be similarly expressed as an integral over the distribution function, $P_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x}, \vec{p}) \frac{p^2}{3E(p)}$.

7.4.1 Matter-Dominated universe

This refers to non-relativistic matter dominating the universe, so that the pressure applied becomes 0. Solving the fluid and acceleration equations we get the following relations -

$$\rho(a) = \frac{\rho_0}{a^3} \tag{90}$$

$$a(t) = (\frac{t}{t_0})^{2/3} \tag{91}$$

$$H = \frac{2}{3t} \tag{92}$$

implying that the rate of expansion decreases with time. As we shall see, ρ_{matter} falls of more slowly with $a(t)$, compared to a radiation dominated universe, and thus a matter dominated universe is more stable. We can also get these equations from the equation of motion for matter $T^{\mu\nu}{}_{;\nu} = 0$, because of isotropy, the spatial components of this equation must vanish identically. Only the time component $= 0$ is nontrivial, which gives $\frac{d}{dt}(\rho a^3) = 0$ for matter-dominated era, implying the density being inversely proportional to the volume.

7.4.2 Radiation-Dominated universe

This refers to non-relativistic matter dominating the universe, and using $p = \frac{\rho c^2}{3}$ (according to radiation theory). A similar analysis yields -

$$\rho(a) = \frac{\rho_0}{a^4} \tag{93}$$

$$a(t) = (\frac{t}{t_0})^{1/2} \tag{94}$$

$$H = \frac{1}{2t} \tag{95}$$

implying that the universe expands much more slowly and $\rho_{radiation}$ falls off faster, hence leading to matter eventually dominating later. Similar to the previous treatment of the stress-energy tensor, and using radiation theory $\frac{d}{dt}(\rho a^3) = \frac{-\rho}{3}(\frac{d}{dt}(a^3)$ for radiation-dominated era. We could also have a look at the geodesic equations the temporal part of which gives us the equation as $\frac{dE}{dt} + \frac{\dot{a}}{a}E = 0$, which shows us mathematically, that Energy of radiation goes down as $\frac{1}{a}$.

7.5 Density Parameter

The density parameter is a very useful way of specifying the density of the Universe. **Critical density (ρ_c)** is defined as the density at which the universe is flat and is given by -

$$\rho_c = \frac{3H^2}{8G\pi}, \tag{96}$$

which yields $\rho_c(t_0) = 1.88h^2 \times 10^-26kg/m^3$. $\rho_c(t_0)$ is a very small number, but it is very close to the actual density of our universe. Density Parameter (Ω) is defined as -

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \tag{97}$$

and substituting this in (6) gives us,

$$\Omega(t) - 1 = \frac{k}{a^2H^2} \tag{98}$$

implying that if $\Omega = 1$, it pertains to a flat universe, but even a small deviation from the critical density will make the geometry of the universe closed or open, that is, a flat universe is in unstable equilibrium.

7.6 Cosmological Constant

7.6.1 Deceleration Parameter

The deceleration parameter is a way of quantifying the change of rate of the Hubble parameter. We define the deceleration parameter q_0 as

$$q_0 = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2} \tag{99}$$

The larger the value of q_0, the more rapid the deceleration. Considering a matter-dominated universe ($p = 0$), and substituting the Friedmann and Acceleration equations, we get $q_0 = \frac{\Omega_0}{2}$, and this greatly simplifies finding out the values of Ω_0, since the deceleration is easier to measure than the density. To widespread surprise, the observations indicate that the Universe appears to be accelerating at present, $q_0 < 0$. None of the cosmological models that we have discussed so far are capable of satisfying this condition, as can be seen directly from the acceleration equation. This introduces the need for a cosmological constant.

7.6.2 Cosmological constant as a solution

When formulating general relativity, Einstein believed that the Universe was static, but found that his theory of general relativity did not permit it. In order to arrange a static Universe, he proposed a change to the equations, something he would later famously call his "greatest blunder". That was the introduction of a cosmological constant. He introduced the Cosmological Constant in his Field Equations, changing them to -

$$G + \Lambda g = 8\pi T \tag{100}$$

using this and deriving the Friedmann equation in the same way we did in the same way we did with noticing the temporal component of the field equations, Λ appears in the Friedmann equation as an extra term, giving -

$$H^2 = \frac{8G\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{101}$$

The original idea was for Λ to balance ρ and k, for H to be 0, but this is unstable to small perturbations. However, this was a boon as realized later, since from the new acceleration equation

$$\frac{\ddot{a}}{a} = \frac{-4G\pi}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda}{3} \tag{102}$$

A positive cosmological constant gives a positive contribution to a, and so acts effectively as a repulsive force, which can explain the accelerating universe observations. The introduction of Λ can be thought of as a cosmic fluid with energy density ρ_Λ, pressure p_Λ and critical density Ω_Λ given by -

$$\Omega + \Omega_\Lambda = 1 + \frac{k}{a^2H^2} \tag{103}$$

$$\rho_\Lambda = \frac{\Lambda}{8G\pi} \tag{104}$$

$$p_\Lambda = -\rho_\Lambda c^2 \tag{105}$$

The negative effective pressure in the above equations is thought to drive the expansion. The deceleration parameter then becomes -

$$q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \tag{106}$$

which can provide for an accelerating universe. The cosmological constant also provides a useful way to parametrize possible cosmological models.

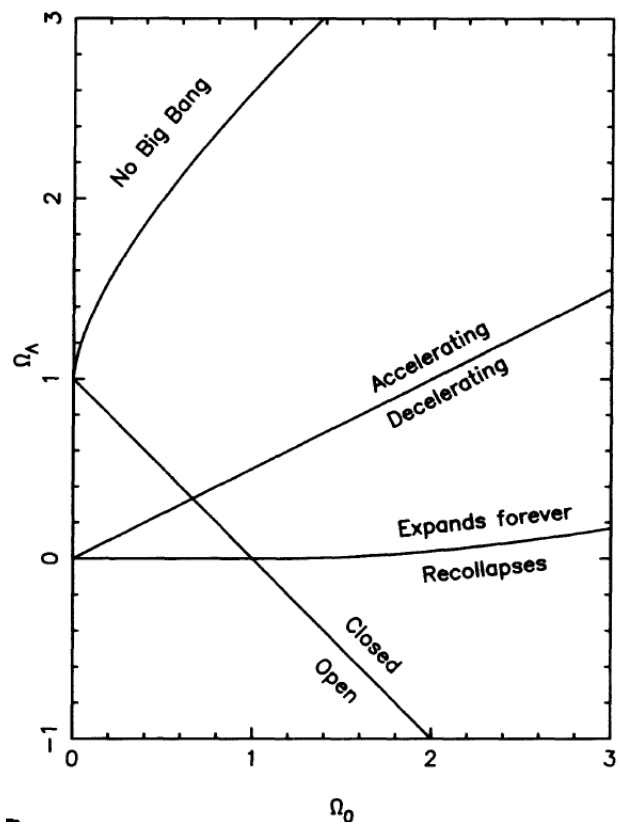

Figure 7: Different models for the Universe can be identified by their location in the plane showing the densities of matter and Λ. This figure indicates the main properties in different regions, with the labels indicating the behaviour on each side of the dividing lines.

7.7 Age of the Universe

One of the quantities that we can predict from a cosmological model, from the solution $a(t)$ for the expansion, is the age of the Universe t_o. By Hubble's Law, the easiest prediction we can make about the age is $H_0^{-}1$ which comes to around $9.77h^{-}1 \times 10^9$ years, and is known and **Hubble time**. This is a relatively good estimate, as a first guess. In a flat, purely matter dominated universe, we get $H_0 = \frac{2}{3t_0}$, which comes preposterously low as an estimate. In an open universe, since it will be lesser matter dominated, it would have taken more time for the universe to slow to its current rates, hence estimates increase. However, the cosmological constant comes once again to the rescue and we find that retaining flat geometry and Λ as +ve, we get better estimates of the age, currently pegged at 13.8 billion years by putting the observed values of $\Omega_0 = 0.3$ and $h = 0.72$

below, obtained by integration of (19).

$$H_0 t_0 = \frac{2}{3\sqrt{1-\Omega_0}} \ln\left(\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right) \tag{107}$$

8 Dark Matter

The need for dark matter as a form of "invisible" matter providing the needed gravitational attraction has pretty old origins, but it still remains a mystery to scientists. In this section we will first explore the proofs for the existence of a dark form of matter, and later discuss possible candidates for it.

8.1 Existence of Dark Matter?

8.1.1 Visible Matter density

From the crude estimates that a typical galaxy weighs about $10^{11} M_0$ and that galaxies are typically about a megaparsec apart, we know that the Universe cannot be a long way from the critical density. From stellar theory, we can estimate the amount of density in stars, and that comes out to be a small fraction of critical density-

$$\Omega_{stars} = \frac{\rho_{stars}}{\rho_c} \approx 0.005 \rightarrow 0.01 \tag{108}$$

Even on considering the gases which also contribute to the mass density, the theory of nucleosynthesis (formation of elements) predicts that the total baryonic density can be -

$$0.016 < \Omega_B h^2 < 0.024 \tag{109}$$

still a small fraction of ρ_c.

8.1.2 Galaxy Rotation curves

The general argument is to look at motions of various kinds of astronomical objects, and assess whether the visible material is sufficient to provide the inferred gravitational force. If it is not, the excess gravitational attraction must be due to extra, invisible, material. A galaxy rotation curve shows the velocity of of matter rotating in a spiral disk, as a function of distance from the centre. According to Newtons Laws, this should follow the relation -

$$v = \sqrt{\frac{GM(R)}{R}} \tag{110}$$

and at large distances, enclosing almost the whole mass, should fall off as $\propto \frac{1}{\sqrt{R}}$, but is found to be almost constant.

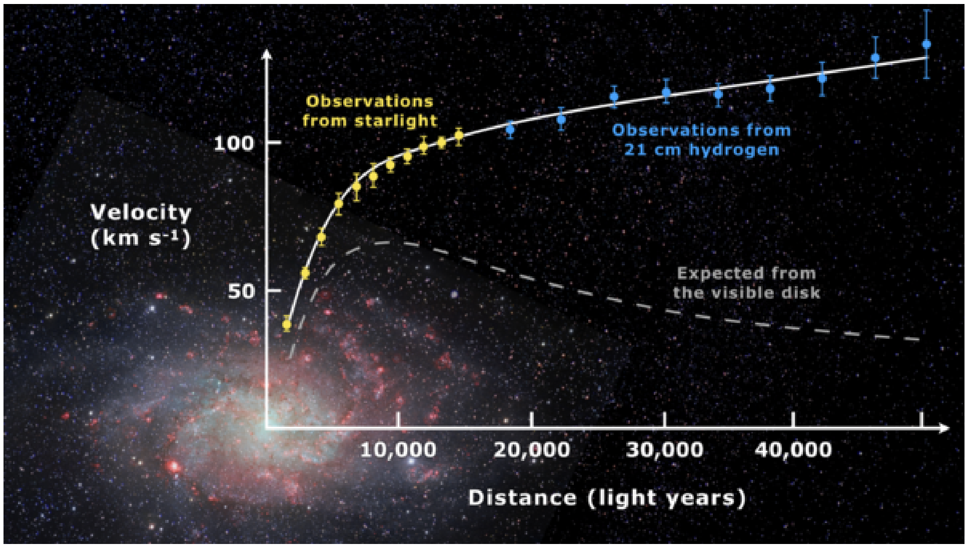

Figure 8: Rotation curve of spiral galaxy Messier 33 (yellow and blue points with error bars), and a predicted one from distribution of the visible matter (gray line). The discrepancy between the two curves can be accounted for by adding a dark matter halo surrounding the galaxy. [2].

This is an example of dark matter which can provide for the extra gravitational force, and is assumed to be in a halo around the galaxy. Further analysis shows that

$$\Omega_{darkmatter} \approx 0.1 \tag{111}$$

making up for a large amount of dark matter in the universe.

8.1.3 Galaxy Clusters

Galaxy clusters are the largest collection of objects (gravitationally), and have 2 parts - the galaxies that make them up and hot gas between these galaxies. The existence of this gas was found in the X-Ray region, and are believed to be very hot remnants from galaxy formation. There is on an average, 5-10 times more gas than stars in these clusters. These high temperature gases exert very high pressure, but are confined in these clusters by gravitational attraction. However, there simply isn't enough mass to keep them confined, and analysis shows that around 10 times the baryon density given by nucleosynthesis, is required to provide for the needed attraction.

8.1.4 Other evidences

Further dynamical evidence for the existence of dark matter comes from the motions of galaxies relative to one another (i.e. the deviations from the cosmological principle). Galaxies possess relative motions, the peculiar velocities mentioned earlier, which allows one to estimate their mass under the assumption that their gravitational interaction is responsible for the motions, which are often termed bulk flows. Once again, the amount of mass predicted from observing the bulk flows is much more than then mass from visible matter.

Structures in the universe originated form irregularities that grew out of gravitational attraction. Baryonic matter cannot account for the amount of gravitational attraction needed to form the structures observed, and hence is a strong proof of the existence of dark matter. Structure formation, bulk flows and the Ω_0 vs Ω_Λ plots estimate -

$$\Omega_0 \approx 0.3 \tag{112}$$

$$\Omega_\Lambda \approx 0.7 \tag{113}$$

8.2 Candidates for Dark Matter

The prediction of non-baryonic dark matter is one of the boldest and most striking in all of cosmology, and if ultimately verified, for example by direct detection of dark matter particles, will be amongst cosmology's most notable successes. Dark matter can be briefly summarized as Hot Dark Matter (HDM) which contains of relativistic particles, and Cold Dark Matter (CDM) consisting of non-relativistic particles.

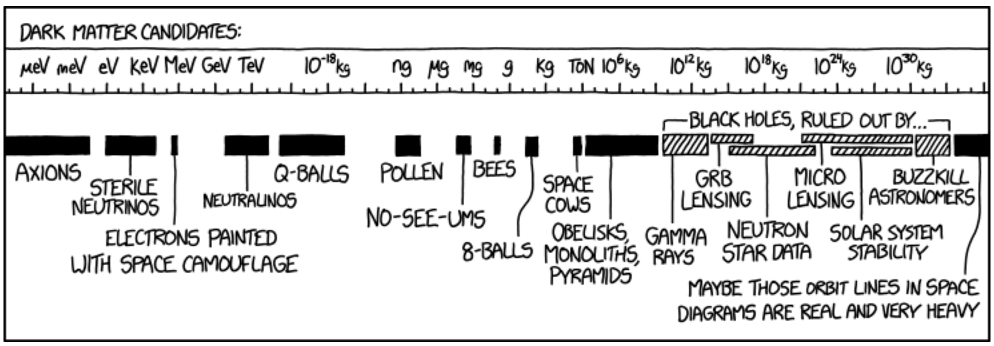

Figure 9: LOL.[3]

- **Neutrinos -** In the Standard Model of particle interactions the neutrino is a massless particle, and is present in the Universe in great abundance, being about as numerous as photons. If the Standard Model is extended to permit the neutrinos to have a small mass (a few tens of electron-volts), this would not affect their number density but they would have enough density to imply a closed Universe! They (muon and tau neutrinos) would be a candidate for HDM, but HDM cannot explain structure formation, so CDM is also required. If we consider heavier mass neutrinos, they could be a candidate for CDM, but this is unlikely.

- **Light Supersymmetric Particles (LSPs) -** Particle physicists regard supersymmetry as the most solidly-founded extension to standard particle theory, and it has the nice property of associating a new companion particle to each of the particles we already know about. In the simplest scenarios, the LSP is stable and is an excellent CDM candidate. Depending on the model the particle in question might be called the photino, or gravitino, or neutralino. They are also sometimes known as WIMPs — Weakly Interacting Massive Particles.

- **Axions -** The axion is a hypothetical elementary particle postulated to resolve the strong CP problem in quantum chromodynamics. If axions exist and have low mass within a specific range, they are of interest as a possible component of cold dark matter.

- **Primordial Black Holes -** A population of primordial black holes, meaning black holes formed early in the Universe's history rather than at a star's final death throes, would act like cold dark matter. However if they are made of baryons they must form before nucleosynthesis to avoid the nucleosynthesis bound of equation (27)

- **MAssive Compact Halo Objects (MACHOs) -** MACHOs have been detected by gravitational lensing of stars in the Large Magellanic Cloud (LMC). The idea was to monitor LMC stars, which lie outside the galactic halo. If there are invisible massive objects in the halo, and they happen to pass extremely close to our line of sight to the LMC star, then their gravitational field can bend and focus light from the star, temporarily brightening it, and this phenomenon is called microlensing. This was observed in the 1990s, when the light curve (plot of brightness vs time) of a star peaked aperiodically.

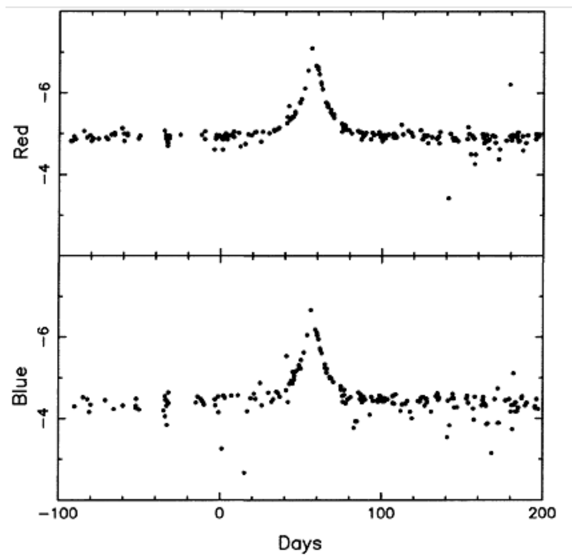

Figure 10: Light curves for a star in the LMC, obtained by the MACHO collaboration. The x-axis is in days with an arbitrary origin, while the y-axis shows the brightness of the star in red light and in blue light .

The best hope for dark matter detection is that the particles also interact through the Weak force, and not just the gravitational force. Any DM signal should show annual modifications, and hence can be filtered out from noise.

9 CMB and Neutrinos

9.1 The Form of CMB

The crucial observation which swayed the Big Bang/Steady State Universe debate in favour of the former was the detection of the Cosmic Microwave Background Radiation reported in 1965. This radiation bathes the Earth from all directions, and is now known to accurately take on the form of a black-body with temperature $T_0 \approx 2.72$K. This yields the energy density of the radiation as -

$$\Omega_{radiation} = 2.47 \times 10^{-5} h^{-2} \tag{114}$$

However, we know that -

$$\rho_{radiation} \propto \frac{1}{a^4} \tag{115}$$

$$\implies T_{radiation} \propto \frac{1}{a} \tag{116}$$

which shows that the CMB cools as the universe expands. The energy distribution, given by Planck's formula as below -

$$\epsilon(f)\,\mathrm{d}f = \frac{8\pi h}{c^3}\frac{f^3\,\mathrm{d}f}{e^{\frac{hf}{kT}}-1} \tag{117}$$

and since $\frac{f}{T}$, remains constant, the form of blackbody is preserved, and shows a similar distribution at a lower T and lower number density.The number density of photons to that of baryons is give by -

$$1.7\times 10^9 = \frac{n_\gamma}{n_B} \tag{118}$$

The energy density associated with this radiation is given by -

$$p_\gamma = 2\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{\frac{p}{T}}-1} \tag{119}$$

2 for the bosonic spin states, evaluating this using the Riemann Zeta function we get $p_\gamma = \frac{\pi^2 T^4}{15}$, confirming what we already know.

9.2 The formation of CMB

The radiation emitted by constituent particles of the plasma in the early universe couldn't travel very far, before they hit an electron and ricocheted, still trapped within the plasma . As space expanded and cooled further, around the 3000K (we shall see how we got this number) mark, around 38,0000 years after the big bang, neutral atoms could finally form. The light emitted by the plasma just before it neutralized, could now freely stream through the universe and this process is called decoupling, the light finally redshifting to form the CMB we see today.

Since decoupling happened when the Universe was only about one thousandth of its present size, and the photons have been travelling uninterrupted since then, they come from a considerable distance away. Those we see originate on the surface of a very large sphere centred on our location, called the surface of last scattering, with a radius of about $6000h^{-1}Mpc$. Recombination refers to the epoch where electrons joined the nuclei to create atoms, whereas decoupling refers to the epoch after which the photons will not scatter again. If recombination were instantaneous and complete, the two would coincide, but in practice each process takes some time and decoupling follows recombination. We can get a close estimate of the temperature at which this happened, using the Saha equation, derived assuming only hydrogen is present, thermal and chemical equilibrium and $\chi = \frac{n_p}{n_B}$ where n_p is the number of free protons. The abundance can be shown as -

$$\frac{1-\chi}{\chi} \simeq 3.8\frac{n_B}{n_\gamma}\left(\frac{k_BT}{m_ec^2}\right)^{\frac{3}{2}} e^{\frac{13.6eV}{k_BT}} \tag{120}$$

putting $\chi_{rec} = 0.1$, we get $T_{rec} \approx 3600K$, close to the actual value of 3000K.

9.3 Anisotropies in the CMB

The fundamental measurement in microwave background studies is the temperature of the microwave background seen in a given direction on the sky, $T(\theta,\phi)$. Usually the mean temperature T is subtracted and a dimensionless temperature anisotropy is defined.

$$\frac{\Delta T}{T}(\theta,\phi) = \frac{T(\theta,\phi)-\bar{T}}{\bar{T}} \tag{121}$$

The next step is to carry out an expansion in spherical harmonics -

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_m^l(\theta,\phi) \tag{122}$$

where a_{lm} tells about the size of irregularities, and we study the radiation angular power spectrum ($C_l =< |a_{lm}|>^2$), the rotational invariance making sure than C_l is independant of m, and gives aniostropies on scale of $\frac{180}{l}$. The $l = 1$ gives the dipole due to relative motion of earth wrt the CMB, and peaks onward $l = 2$ are studied and plotted in the $\frac{(l)(l+1)C_l}{2\pi}$ vs l graph - For $l <= 15$,

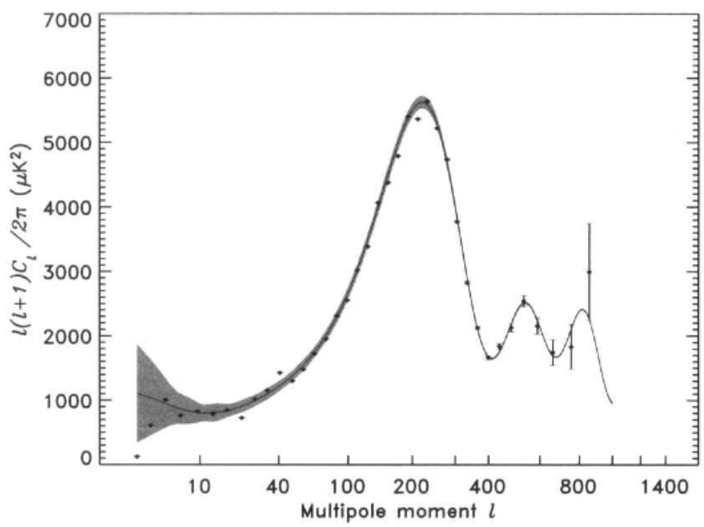

Figure 11: The radiation angular power spectrum as measured by the WMAP satellite, shown as the black dots. The solid line shows a theoretical prediction from their best-fit cosmological model, which fits the data extremely well.

it is fairly constant and called Sachs-Wolfe plateau, at $l \approx 200$, we begin to see the effects of the oscillations in the cosmic fluid at the time of decoupling. The location of the first peak, gives the geometry of the universe and the plot also shows that $\Omega_0 + \Omega_\Lambda = 1.02 \pm 0.02$, placing the universe very close to spatial flatness.

9.4 Neutrinos

Studies of neutrinos coming from the Sun, those interacting in the Earth's atmosphere, and those created on Earth via nuclear interactions, have all shown evidence that neutrinos possess the ability to change their type as they travel. This phenomenon is known as neutrino oscillations and can be understood in particle physics models, but only those where the neutrino rest-mass is nonzero. The reason we expect there to be Cosmic Neutrino Background (CNB) is because the high density of early universe would ensure their thermal equilibrium with photons and other particles, with interactions like -

$$p + e^- \Longleftrightarrow n + \nu_e \tag{123}$$

$$\gamma + \gamma \Longleftrightarrow \nu_\mu + \bar{\nu_\mu} \tag{124}$$

The fermionic properties of neutrinos make their number density around 7/8 times that of photons making up the CMB, and since neutrinos interact much more weakly than photons, they decouple before, whereas electron-positron annihilation's boost the photons temperature by a factor of $\sqrt[3]{\frac{11}{4}}$. Putting this together, and assuming neutrinos to be relativistic, we get -

$$\Omega_\nu = 3 \times \frac{7}{8} \times (\frac{11}{4})^{\frac{4}{3}} \Omega_{rad} \tag{125}$$
$$= 1.68 \times 10^{-5} h^2$$

$$T_\nu = \sqrt[3]{\frac{11}{4}} T = 1.95K \tag{126}$$

Assuming light mass for neutrinos ($m_\nu c^2 << 1MeV$), we get a slightly different Ω_ν, which can make light neutrinos one of the good HDM candidates. Heavy neutrinos, also provide a good CDM candidate. Existence of CNB is also crucial for nucleosynthesis, the correct matter-radiation epoch and structure formation. However due to our experimental limits, we have not been able to detect it yet. The energy density of a massive neutrino is given by -

$$\rho_v = 2 \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_v^2}}{e^{\frac{p}{T_v}} + 1} \tag{127}$$

10 The Early Universe

The Boltzmann equation formalizes the statement that the rate of change in the abundance of a given particle is the difference between the rates for producing and eliminating that species. Suppose that we are interested in the number density n_1 of species 1. For simplicity, let's suppose that the only process affecting the abundance of this species is 1+2 $\iff$ 3+4, the Boltzmann equation for this system in an expanding universe is -

$$a^{-3} \frac{d(n_1 a^3)}{dt} = \int \frac{d^3p_1}{2(2\pi)^3 E_1} \int \frac{d^3p_2}{2(2\pi)^3 E_2} \int \frac{d^3p_3}{2(2\pi)^3 E_3} \int \frac{d^3p_4}{2(2\pi)^3 E_4} \tag{128}$$
$$(2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4) \delta^3(E_1 + E_2 - E_3 - E_4) |M|^2 [f_3 f_4 - f_1 f_2] \tag{129}$$

where f is the occupation number and the delta terms account for momentum and energy conservation. Defining the thermally averaged cross-section as $< \sigma v >$ and $n_i^{(0)}$ as the species-dependent equilibrium number density converts the right hand side of the eqution to $n_1^{(0)} n_2^{(0)} < \sigma v > (\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}})$ and the equation to a ODE. We see at large rates, or at early times, the plasma was effectively in equilibrium called Nuclear Statistical Equilibrium (NSE). We can use these equations with a few more approximations to find densities of protons, neutrons and elements during Nucleosynthesis. The different eras of the universe are summarized in the table below -

Time	Temperature	What's going on?
$t < 10^{-10}$ s	$T > 10^{15}$ K	Open to speculation!
10^{-10} s $< t < 10^{-4}$ s	10^{15} K $> T > 10^{12}$ K	Free electrons, quarks, photons, neutrinos; everything is strongly interacting with everything else.
10^{-4} s $< t < 1$ s	10^{12} K $> T > 10^{10}$ K	Free electrons, protons, neutrons, photons, neutrinos; everything is strongly interacting with everything else.
1 s $< t < 10^{12}$ s	10^{10} K $> T > 10000$ K	Protons and neutrons have joined to form atomic nuclei, and so we have free electrons, atomic nuclei, photons, neutrinos; everything is strongly interacting with everything else except the neutrinos, whose interactions are now too weak. The Universe is still radiation dominated.
10^{12} s $< t < 10^{13}$ s	10000 K $> T > 3000$ K	As before, except that now the Universe is matter dominated.
10^{13} s $< t < t_0$	3000 K $> T > 3$ K	Atoms have now formed from the nuclei and the electrons. The photons are no longer interacting with them, and are cooling to form what we will see as the microwave background.

Figure 12: Different stages of the Universe's evolution (taking $\Omega_0 = 0.3$ and $h = 0.72$). Some numbers are approximate.

10.1 Inflation

Inflation is not a replacement for the Hot Big Bang theory, but rather an extra add-on idea which is supposed to apply during some very early stage of the Universe's expansion.

10.1.1 Need for Inflation

From equation 17, and substituting the values of aH, we get that the $|\Omega_{tot} - 1|$ is an increasing function of time, leading to the universe getting more and more curved as time goes by. However, for the universe to be as flat as observed today, the early universe, at the time of electro-weak symmetry breaking would have to satify $|\Omega_{tot} - 1| < 10^{-30}$, which is extraordinarily close to ρ_c.

The distance which light could have travelled during the lifetime of the Universe gives rise to a region known as the observable Universe. One of the most important properties of the microwave background is that it is very nearly isotropic. Being at the same temperature is the characteristic of thermal equilibrium, and so this observation is naturally explained if different regions of the sky have been able to interact and move towards thermal equilibrium. However due to the finite speed of light and age of the universe, these far-separated regions of the universe had no interaction between them to establish thermal equilibrium. Similarly, the anisotropies observed in the CMB have to be pre-existing, and cannot be created afterwards.

Another inconsistency was the existence of relic particles, particularly Magnetic Monopoles, which were extremely massive (10^{16} GeV) and should have been highly abundant, but they have not been observed yet, and should have dominated over radiation much before the equality epoch.

10.1.2 Inflation as a solution

Inflation came to the rescue, and was defined as a period where the universe was accelerating, $\ddot{a}(t) > 0$. From the acceleration equations, this implies negative pressure, $p < -\frac{\rho c^2}{3}$. Applying the Friedmann equations, and assuming the cosmological constant to be the dominating factor, we get

$$a(t) = e^{(\frac{\Lambda}{3})t} \tag{130}$$

a much more dramatic expansion, which is supposed to happen around 10^{-34} sec. Applying this condition to the flatness problem, we see that during a perfect exponential growth of the universe, we get -

$$|\Omega_{tot} - 1| \propto e^{-\sqrt{\frac{4\Lambda}{3}}t} \tag{131}$$

and hence it drives Ω_{tot} towards 1. Calculating the amount of inflation to achieve the spatial flatness as seen today, the required value is that $a(t)$ increases by a factor of 10^{27} by the end of inflation, which happen very quickly, since the increase is exponential.

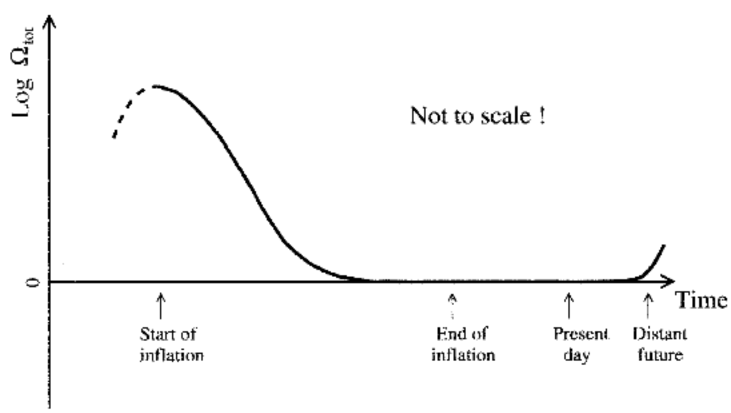

Figure 13: Possible evolution of the density parameter Ω_{tot}- There might or might not be a period before inflation, indicated by the dashed line..

Inflation greatly increases the size of a region of the Universe, while keeping its characteristic scale, the Hubble scale, fixed. This means that a small patch of the Universe, small enough to achieve thermalization before inflation, can expand to be much larger than the size of our presently observable Universe. Then the microwaves coming from opposite sides of the sky really are at the same temperature because they were once in equilibrium, hereby solving the horizon problem. The dramatic expansion of the inflationary era dilutes away any unfortunate relic particles, because their density is reduced by the expansion more quickly than the cosmological constant. Provided enough expansion occurs, this dilution can easily make sure that the particles are not observable today.

10.2 Baryogenesis

Baryogenesis is the hypothetical theory that took place during the early universe, producing baryon asymmetry, that is, the imbalance of matter and antimatter in the early universe. The following Sakharov conditions are needed to generate the observed asymmetry -

- **Baryon number violation -** Reactions in the Standard Model preserve baryon number, new interactions are being explored in GUT's to explain this.
- **C and CP violation -** C and CP violation refers to two symmetries typically obeyed by particle interactions — that interaction rates are unchanged if one switches the charge (C) or parity (P) of the particles, or both (CP). CP violation is observed in interactions of particles called neutral K mesons, though at a very low level and without the presence of baryon number violation. Such violation, at a much larger level, would be needed in any interactions able to generate the baryon number.
- **Departure from Thermal Equilibrium -** This preferentially chooses one direction for reactions to lead into, hence amplifying any original asymmetry present. With reactions like

$$\gamma + \gamma \longleftrightarrow p + \bar{p} \tag{132}$$

  if by some mechanism, an additional proton was created for every such billion pairs, after sufficient cooling and annihilation of matter-antimatter pairs, that one proton would be enough to create the observed baryonic density. This mechanism currently remains unknown.

10.3 Nucleosynthesis

The formation of nuclei, which took place at about 1 second after the big bang by a process called nucleosynthesis. The origin of heavy elements are indeed the cores of stars, but lighter elements like D, He-3, Li, He-4 could not have been created, and their abundances have to come from the primordial gas. At high temperatures, protons and neutrons would be in equilibrium, and after sufficient cooling, bind to form nuclei. The relative ratios of their abundances are given by-

$$\frac{N_n}{N_p} = (\frac{m_n}{m_p})^{\frac{3}{2}} e^{-\frac{(m_n - m_p)c^2}{k_B T}} \tag{133}$$

Until $k_B T >> (m_n - m_p)c^2$, the $\frac{N_n}{N_p}$ is almost 1. The reactions converting neutrons to protons and vice versa are -

$$n + \nu_e \longleftrightarrow p + e^- \tag{134}$$

$$n + e^+ \longleftrightarrow p + \bar{\nu_e} \tag{135}$$

These reactions proceed in both directions till $k_B T \approx 0.8 MeV$, after which $\frac{N_n}{N_p} = \frac{1}{5}$, and will only change by decay of free neutrons. Some of the reactions which make these elements are -

$$p + n \rightarrow D \tag{136}$$

$$D + p \rightarrow He^3 \tag{137}$$

$$D + D \rightarrow He^4 \tag{138}$$

The delay before He^4 appearing, leads to the decay of neutrons, and reduces the ratio to $\frac{N_n}{N_p} = \frac{1}{8}$. In the Early universe, the major abundances were of H(not enough neutrons to bind to protons) and He^4 (the most stable light nucleus). A detailed analysis gives the amounts as - H $\approx$ 77%, He^4 $\approx$ 23%, D$\approx$10^{-4}%, He^3 $\approx$10^{-5}% and Li^7 $\approx$10^{-10}% by mass. This allows us to fix the constraints of $\Omega_B h^2$ within a very small range to agree with observations.

10.4 Perturbations to the Flat FRW metric

Whereas the smooth universe is characterized by a single function, a(t), which depends only on time and not on space, the perturbed universe requires two more functions, Φ and Ψ, both of which depend on both space and time. The metric then becomes - $g_{00} = -1 - 2\Psi$ and $g_{ii} = a^2(1 + 2\Phi)$ where the perturbations are very small and we only go to first order. We can then calculate the momentum of photons as $P^0 = p(1 - \Psi)$, which shows that in an overdense region photons lose energy and redshift as they move out. Moving further, we can use the Boltzmann equation to describes the change in the photon momentum as it moves through a perturbed FRW universe as -

$$-\frac{dp}{pdt} = H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p^i}}{a}\frac{\partial \Psi}{\partial x^i} \tag{139}$$

The first term accounts for the loss of momentum due to the Hubble expansion, the second term says that a photon in a deepening gravitational well loses energy and a photon traveling into a well gains energy because it is being pulled toward the center. Solving the Boltzmann equation at zero order in this universe we come to the conclusion $T \propto \frac{1}{a}$, reassuring us that we are on the right path. The terms in the first order equation account for "free streaming," which translates into anisotropics on increasingly small scales as the universe evolves and account for the effect of gravity. Studying Compton scattering for the study of the CMB in this universe and to a certain degree of approximation we see that, when Compton scattering is very efficient, only the monopole perturbation survives; all other moments are washed out. Intuitively, strong scattering means that the mean free path of a photon is very small. These nearby electrons most likely had a temperature very similar to the point of observation. Therefore, photons from all directions have the same temperature. This is the characteristic signature of a monopole distribution: the temperature on the sky is uniform. Further analysis on giving electrons a bulk velocity shows that it also carries a dipole moment. In a similar manner we can solve the Boltzmann equation for baryons and CDM, and gain further insight on the density calculations that we described above. We can also solve the Einstein Field equations in this universe having scalar perturbations, using the calculations above for the stress-energy tensor, we find equations governing the evolution of the two functions which describe scalar metric perturbations. Moving a step further we can solve for vector and tensor perturbations too, which will give us the wave equations which we saw for gravitational waves namely - $\ddot{h_\alpha} + 2\frac{\dot{a}}{a}\dot{h_\alpha} + k^2 h_\alpha$, where α denotes the 2 polarizations of the waves. The further study of these perturbations is quite advanced, which we will not cover here. Now that we have a good enough backing of the theory, finally we go into the observational parts of studying the universe, which were just as useful for developing our understanding of the universe.

10.5 Observational Cosmology

Observational cosmology considers how objects with given properties, such as luminosity and size, will appear to us. In particular, it is concerned with the dependence of that appearance on the cosmological model.

10.5.1 Light Propagation and Redshift

We derive our results using the Robertson-Walker metric for spacetime given by -

$$ds^2 = -c^2dt^2 + a(t)^2\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2\right) \tag{140}$$

Light propagation obeys $ds = 0$, using symmetry of space, and finding the time difference between emission and reception, we get the relation between emitted and received wavelengths as -

$$\frac{\lambda_r}{\lambda_e} = \frac{a(t_r)}{a(t_e)} \tag{141}$$

$$\frac{a(t_0)}{a(t_e)} = 1 + z \tag{142}$$

Using this metric, we can also calculate how far light could have travelled during the lifetime of the universe, assuming a matter dominated flat universe with no cosmological constant -

$$\int_0^{r_0} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^{t_0} \frac{cdt}{a(t)} \implies \int_0^{r_0} dr = ct_0^{\frac{2}{3}} \int_0^{t_0} \frac{dt}{t^{\frac{2}{3}}} \implies r_0 = 3ct_o \tag{143}$$

giving the radius of the observable universe as approximately 46.5 billion light years. However, the different evolution's of scale factors leads to different conclusions, but we must keep in mind that the universe was opaque until the time of formation of CMB, and hence the result is pretty accurate. Also, note that the observable radius is not ct_0, since the universe is expanding.

10.5.2 Luminosity Distance

The luminosity distance is a way of expressing the amount of light received from a distant object. The luminosity distance is the distance that the object appears to have, assuming the inverse square law for the reduction of light intensity with distance holds. It is the ratio of the Luminosity and radiation flux density of an object, which in turn depends upon the redshift and is given by -

$$d_{lum} = a_0 r_0 (1 + z) \tag{144}$$

where $a_0 r_0$ is the physical distance to the object. The geometry of the universe also affects this, with hyperbolic geometry enhancing and spherical opposing this effect. The luminosity distance depends on the cosmological model we have under discussion, and hence can be used to tell us which cosmological model describes our Universe. In particular, we can plot the luminosity distance against redshift for different cosmologies, and observed data only fit in with a universe of $\Omega_0 \approx 0.3$ and $\Omega_\Lambda \approx 0.7$.

10.5.3 Angular Diameter Distance

The angular diameter distance is a measure of how large objects appear to be. As with the luminosity distance, it is defined as the distance that an object of known physical extent appears to be at, under the assumption of Euclidean geometry. The angular diameter distance and angular size we perceive of an object with physical size l is given by -

$$d_{diam} = \frac{a_0 r_0}{1 + z} \tag{145}$$

$$d\theta = \frac{l(1 + z)}{a_0 r_0} \tag{146}$$

A key application of the angular diameter distance is in the study of features in the cosmic microwave background radiation.

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