Algebraic Topology

A Peek into the study of topological spaces

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§1. Preliminaries

§§1.1. Basics of sets and functions

Subsets of a set are denoted by $B = \{a \in A \mid \text{(conditions)}\}$, cardinality of a set by $\mid A \mid$ and cartesian product of 2 sets as $A \times B = \{(a,b) \mid a \in A, b \in B\}$. A function from A to B is denoted as $f : A \to B$ where A is domain of f and B is codomain. $f : a \mapsto b$ indicates that $f(a) = b$.
Range or Image of A under f defined as -

$$f(A) = \{b \in B \mid b = f(a), \text{for some a} \in A\}$$

Preimage or inverse image of C under f defined as -

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}$$

For each $\{b\} \in B$, the preimage of {b} under f is called fibers of f over b (can contain one or more elements). If $f : A \mapsto B$ and $g : B \mapsto C$ then the composite map $g \circ f : A \mapsto C$ is defined as $(g \circ f)(a) = g(f(a))$. A function f is injective if whenever $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$, surjective if for all $b \in B$, there is some $a \in A$ such that $f(a) = b$, and is bijective if it is both injective and surjective. A left inverse of $f : A \mapsto B$ is a function $g : B \mapsto A$ such that $g \circ f : A \mapsto A$ is the identity map, and a similar definiton for the right inverse. Important proofs considering $f : A \to B$

- f is injective if and only if f has left inverse
- f is surjective if and only if f has a right inverse
- f is a bijection iff there is a $g : B \mapsto A$ such that $f \circ g$ and $g \circ f$ are the identity maps on their respective domains (the map here is unique)
- if A and B have same cardinality, then f is bijective if and only if f is injective if and only if f is surjective

A permutation of set A is a bijection from A to itself. $f\mid_A$ is used to denote a restriction on domain A, if defined over a superset of A, and if $f\mid_A = g$, then f is an extention of g. For a non empty set A -

- A binary relation on set A is written as a $a \sim b$ if (a,b)$\in A \times A$
- A relation is an equivalence relation if it is reflexive, symmetric and transitive
- Equivalence class of some $a \in A$ is $\{x \in A \mid x \sim a\}$, elements of this set are equivalent to a and any one element of this class is called its representative
- A partition is any collection such that $A = \cup_i A_i$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ and in the indexing set.

Hence, for an equivalence relation on A, the set of all equivalence classes of A forms a partition of A and conversely, given a partition of A we can define an equivalence relation on A with the same equivalence classes as given.

§§1.2. Properties of Integers

There exists a minimal element in a non empty subset of $\mathbb{Z}^+$ (well-ordering property of $\mathbb{Z}$). If a, b $\in \mathbb{Z}$ with a $\neq 0$, a divides b if there is an element $c \in \mathbb{Z}$ such that $b = ac$ and is written by $a \mid b$. If

$a, b \in \mathbb{Z} - \{0\}$, there is a unique positive integer d (greatest common divisor of a and b) satisfying: (a) $d \mid a$ and $d \mid b$ and (b) if $e \mid a$ and $e \mid b$, then $e \mid d$. The gcd of a and b will be denoted by (a, b). If $(a, b) = 1$, we say that a and b are relatively prime.
If $a, b \in \mathbb{Z} - \{0\}$, there is a unique positive integer l, called the least common multiple of a and b (or l.c.m. of a and b), satisfying: (a) $a \mid l$ and $b \mid 1$ (sol is a common multiple of a and b), and (b) if $a \mid m$ and $b \mid m$, then $I \mid m$ (so I is the least such multiple). For all integers, $dl = ab$.
The Division Algorithm states that if $a, b \in \mathbb{Z} - 0$, then there exist unique $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r <\mid b \mid$ where q is the quotient and r the remainder.
The **Eucledian Algorithm** produces the greatest common divisor of two integers a and b by iterating the Division Algorithm, from $a = q_0 + b$, $b = q_1 r_0 + r_1$, till the last non zero remainder is found, which is (a,b).
The gcd of a and b is a $\mathbb{Z}$-linear combination of a and b. That is $(a, b) = ax + by$, where $x, y \in \mathbb{Z}$, and are not unique.
An element p of $\mathbb{Z}^+$ is called a prime if $p > 1$ and the only positive divisors of p are 1 and p. An integer $n > 1$ which is not prime is called composite. If p is a prime and $p \mid ab$, for some $a, b \in \mathbb{Z}$, then either $p \mid a$ or $p \mid b$. The Fundamental Theorem of Arithmetic says: if $n \in \mathbb{Z}$, $n > 1$, the n can be factored uniquely into the product of primes. The GCD of 2 numbers is the min of each prime exponent in both expressions, and LCM as the maximum.
The Euler φ function is defined as follows: for $n \in \mathbb{Z}^+$ let $\varphi(n)$ be the number of positive integers $a < n$ with a relatively prime to n, i.e. $(a, n) = 1$. For primes we have $\varphi(p^a) = p^a - p^{a-1}$ and under multiplication as $\varphi(ab) = \varphi(a)\varphi(b)$ if $(a, b) = 1$. Together with the formula above this gives a general formula for the values of $\varphi(n) : n = \prod p_i^{\alpha_i}$ then $\varphi(n) = \prod p_i^{\alpha_i - 1}(p_i - 1)$.

§§1.3. $\mathbb{Z}/n\mathbb{Z}$

We define a relation on $\mathbb{Z}$ as $a \sim b$ iff $n \mid (b - a)$. We can see that this relation is an equivalence relation. For any $k \in \mathbb{Z}$ we shall denote the equivalence class of a by $\overline{a}$, also called the congruence class or residue class of $a \pmod n$, consisting of the integers which differ from a by an integral multiple of n, ie $\overline{a} = \{a + kn \mid k \in \mathbb{Z}\}$. For $a \pmod n$ this has n distinguishable classes. The set of equivalence classes under the equivalence relation is denoted by $\mathbb{Z}/n\mathbb{Z}$. We define addition and multiplication of this to follow the modular arithmetic as $\overline{a + b} = \overline{a} + \overline{b}$ and $\overline{ab} = \overline{a} \cdot \overline{b}$. These operations are well defined and hence do not depend on the choice of representatives taken. An important subset of $\mathbb{Z}/n\mathbb{Z}$ consists of the residue classes which have a multiplicative inverse in $\mathbb{Z}/n\mathbb{Z}$.

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{\overline{a} \in \mathbb{Z}/n\mathbb{Z} \mid \text{ there exists } \overline{c} \in \mathbb{Z}/n\mathbb{Z} \text{ with } \overline{a} \cdot \overline{c} = \overline{1}\} \tag{1.1}$$

It is also the collection of residue classes whose representatives are relatively prime to n. If a is an integer relatively prime to n then the Euclidean Algorithm produces integers x and y satisfying $ax + ny = 1$, hence $ax \equiv 1 \pmod n$, so that $\overline{x}$ is the multiplicative inverse of $\overline{a}$.

§2. Groups - The basics

§§2.1. Basic Axioms

Definition 2.1. A **binary operation** $\star$ on a set G is a function $\star : G \times G \mapsto G$. For any $a, b \in G$ we shall write $a \star b$ for $\star(a, b)$.

This is associative if for all $a, b, c \in G$ $a \star (b \star c) = (a \star b) \star c$ and commutative if $a \star b = b \star a$ for all $a, b \in G$. Suppose that $\star$ is a binary operation on a set G and H is a subset of G. If the restriction of $\star$ to H is a binary operation on H, i.e., for all $a, b \in H, a \star b \in H$, then H is said to be closed under $\star$.

Definition 2.2. A **group** is an ordered pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G satisfying the following axioms:

- (a $\star$ b) $\star$ c = a $\star(b \star c)$, for all $a, b, c \in G$, i.e. is associative, and
- there exists an element $e \in G$, called an identity of G, such that for all $a \in G$ we have $a \star e = e \star a = a$, and
- for each $a \in G$ there is an element a^{-1} of G, called an inverse of a, such that $a \star a^{-1} = a^{-1} \star a = e$.

The group $(G, \star)$ is called **abelian** (or commutative) if $a \star b = b \star a$ for all $a, b \in G$.

For $n \in \mathbb{Z}^+$, $\mathbb{Z}/n\mathbb{Z}$ is an abelian group under the operation of addition of residue classes (with identity as $\overline{0}$ and inverse as $\overline{-a}$), and $n \in \mathbb{Z}^+$, $(\mathbb{Z}/n\mathbb{Z})^\times$ is an abelian group under multiplication of residue classes (with identity as $\overline{1}$, and inverse as defined above). If $(A, \star)$ and $(B, \diamond)$ are groups, we form $A \times B = \{(a, b) | a \in A, b \in B\}$ with component-wise operation as $(a_1, b_1)(a_2, b_2) = (a_1 \star a_2, b_1 \diamond b_2)$.

Theorem 2.3. If G is a group under $\star$, then -

- the identity of G is unique, and
- for each $a \in G, a^{-1}$ is unique, and
- $(a^{-1})^{-1} = a \quad \forall a \in G$, and
- $(a \star b)^{-1} = (b^{-1}) \star (a^{-1})$, and
- For any $a_1, a_2 \ldots a_n \in G$ the value of $a_1 \star a_2 \ldots \star a_n$ is independent of bracketing.

For ease of writing, we can ignore the operation $\star$ between two elements and simply write them as ab, with the group operation as $\cdot$, identity as 1, and a^{-1} as the inverse.
For a group G and $a, b \in G$, $ax = b$ and $ya = b$ have unique solutions as $a^{-1}b$ and ba^{-1} and similarly, the left and right cancellation laws hold.

Definition 2.4. For a group G and $x \in G$ define the order of x to be the smallest positive integer n such that $x^n = 1$, and denote this integer by $|x|$. If no positive power of x is the identity, the order of x is defined to be infinity. For a group however, order n implies that the cardinality of the group is n.

If $G = \{g_1, g_2, \ldots, g_n\}$ is a finite group with $g_1 = 1$, then multiplication/group table of G is the $n \times n$ matrix with g_{ij} as the i, j entry.

§§2.2. Dihedral Groups

For each $n \in \mathbb{Z}^+, n \geq 3$, let D_{2n} be the set of symmetries of a regular n-gon. Each symmetry s can be described uniquely by the corresponding permutation σ of $\{1, \ldots, n\}$ where if the symmetry s puts vertex i in the place where vertex j was originally, then σ is the permutation sending i to j. Now we see that we can make D_{2n} into a group by defining st for $s, t \in D_{2n}$ to be the symmetry obtained by first applying t then s to the n-gon. The binary operation on D_{2n} is associative since composition of functions is associative. The identity of D_{2n} is the identity symmetry, and the inverse of $s \in D_{2n}$ is the symmetry which reverses all motions of s.
We can show that $|D_{2n}| = 2n$, and is called the dihedral group of order $2n$ (Any adjacent pair of vertices can can end up in $n * 2$ positions, and once ordered pair determined, due to the rigidity of motion all other vertices are fixed). These are seen as the n rotations about the centre by $\frac{2\pi}{n}$ radians (labelled as r, with $|r| = n$), and n reflections about the lines of symmetry (labelled as s, with $|s| = 2$). The group can be shown to be represented as -

Definition 2.5. $D_{2n} = \{1, r, \ldots r^{n-1}, s, sr, \ldots, sr^{n-1}\}$, and having the property $r^i s = sr^{-i}$.

Definition 2.6. A subset S of elements of a group G with the property that every element of G can be written as a (finite) product of elements of S and their inverses is called a set of generators of G, writing it as $G = \langle S \rangle$.
Any equations in a general group G that the generators that is, from $S \cup \{1\}$ satisfy are called relations in G.

In general, if a group G is generated by a subset S and there is some collection of relations, say $R_1, R_2, \ldots, R_m$ such that any relation among the elements of S can be deduced from these, we call these generators and relations a presentation of G and write -

$$G = \langle s, R_1, R_2, \ldots, R_m \rangle$$

§§2.3. Symmetric groups

Theorem 2.7. For any non-empty Ω let S_Ω be the set of all bijections from Ω to itself, and is a group under function composition: $\circ$.

Proof. We can see this by noting that $\circ$ is a binary operation since if σ and τ are bijections, then so is the composition, associative is trivial, the identity is the identity map permutation, and the inverse satisfies $\sigma^{-1} \circ \sigma = 1$. This group is called the symmetric group on the set Ω. □

When Ω is the natural numbers till n, the symmetric group of degree n is called S_n. The order of S_n is $n!$.

We express the notation for writing elements of S_n with something known as cycle decomposition. The cycle $(a_1, a_2, \ldots, a_m)$ is the permutation which sends $a_i \mapsto a_{i+1}$ for $1 \leq i \leq m-1$ and sends $a_m \mapsto a_1$. We can generally group it into k cycles. For any $x \in \sigma$, find the immediate right neighbour, which is $\sigma(x)$, if there is no element to the right, cycle back to the start element. The product of all the cycles is called the cycle decomposition of σ. In general, we pick the smallest element to start a cycle which hasn't been picked yet, call it a. Read $\sigma(a)$, call it b, if it is a, it is the complete cycle. Similarly read off $\sigma(b)$. The length of a cycle is the number of integers which appear in it. Two cycles are called disjoint if they have no numbers in common.

We can also easily see that S_n is a non abelian group. Since disjoint cycles permute numbers which lie in disjoint sets it follows that disjoint cycles commute. The cycle decomposition of each permutation is the unique way of expressing a permutation as a product of disjoint cycles (up to rearranging its cycles and cyclically permuting the numbers within each cycle). We can also prove that the order of a permutation is the lcm of the lengths of the cycles in its cycle decomposition. The order of σ is n iff $\sigma^n(a_k) = a_k$ for all a_k in the cycle decomposition.

§§2.4. Matrix Groups

Definition 2.8. A field is a set F with binary operations $+$ and $\cdot$, on F such that $(F, +)$ is an abelian group (call its identity 0) and $(F - \{0\}, \cdot)$ is also an abelian group, and the following distributive law holds: $a \cdot (b + c) = a \cdot b + a \cdot c$, for all $a, b, c \in F$.

Examples include $\mathbb{Q}, \mathbb{R}, \mathbb{Z}/p\mathbb{Z}$, with p prime.

For each $n \in \mathbb{Z}^+$, let $GL_n(F) = \{A|$ A is a $n \times n$ matrix with entries from F and $\det(A) \neq 0\}$. Since matrix multiplication is associative, if $\det(A) \neq 0$ and $\det(B) \neq 0 \implies \det(AB) \neq 0$, $\det(A) \neq 0$ implies A^{-1} exists for each A such that $AA^{-1} = I$, the identity, making $GL_n(F)$ a group, called the general linear group of degree n .

§§2.5. Quarternion Group

$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$, over $\cdot$ as 1 as the identity element, -1 reversing the sign of any element, $i \cdot i = j \cdot j = k \cdot k = -1$, and ij and other elements similar to the cross-product (hence, is also non-abelian).

§§2.6. Homomorphisms and Isomorphisms

Definition 2.9. Let $(G, \star)$ and $(H, \diamond)$ be groups. A map $\varphi : G \to H$ such that $\varphi(x \star y) = \varphi(x) \diamond \varphi(y)$ for all $x, y \in$ G is called a homomorphism.

Intuitively, a map φ is a homomorphism if it respects the group structures of its domain and codomain.

Definition 2.10. A map $\varphi : G \mapsto H$ is called an isomorphism and G and H are said to be isomorphic ($G \cong H$) if -

- φ is a homomorphism, and
- φ is a bijection

Intuitively, G and H are the same group except that the elements and the operations may be written differently in G and H.

Theorem 2.11. If $\varphi : G \mapsto H$ is an isomorphism -

- $|G| = |H|$, and
- G is abelian if and only if H is abelian, and
- for all $x \in G$, $|x| = |\varphi(x)|$

Let $\mathcal{G}$ be a nonempty collection of groups. Here, the relation $\cong$ is an 1equivalence relation on $\mathcal{G}$ and the equivalence classes are called isomorphism classes. Up to isomorphism there are precisely two groups of order 6: S_3 and $\mathbb{Z}/6\mathbb{Z}$. If G and H are 2 finite groups with the generators $S = \{s_1, \ldots, s_m\}$ and $R = \{r_1, \ldots, r_m\}$ be the generators with all all relations satisfied by r_i also be satisfied by s_i. Then there is a unique homomorphism $\varphi : G \to H$ mapping $s_i \, to \, r_i$, and if the order of G is same as H, then φ is an isomorphism.

§§2.7. Group Actions

Definition 2.12. A group action of a group G on a set A is a map from $G \times A$ to A, as $g \cdot a$ for all $g \in G$ and $a \in A$, satisfying -

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$. Note here that on LHS $g_2 \cdot a$ is a member of A, but on the RHS $g_1 g_2$ is a member of G, and
- $1 \cdot a = a$ for all $a \in A$.

Theorem 2.13. Let G act on A. For each $g \in G$ define $\sigma_g : A \to A$ with the left action as $\sigma_g(a) = g \cdot a$. We now claim that ;

- for each fixed $g \in G$, σ_g is a permutation of A, since it is injective and maps A to A, hence being bijective and a permutation, and
- $\varphi : G \to S_A$ as $g \mapsto \sigma_g$ is a homomorphism by noting that $\varphi(g_1 g_2)(a) = \sigma_{g_1 g_2} = (g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a) = \sigma_{g_1}(\sigma_{g_1}(a)) = (\varphi(g_1) \circ \varphi(g_2))(a)$

The homomorphism φ is called the permutation representation associated to the given action. Thus actions of a group G on a set A and the homomorphisms from G into the symmetric group S_A are in bijective correspondence. If $ga = a\ \forall g \in G, a \in A$, it is called the trivial action, and the permutation representation is the trivial homomorphism which maps every element of G to the identity on S_A.

If G acts on A, and distinct elements of G induce district permutations of A, then the action is called faithful. A faithful action is therefore one in which the associated permutation representation is injective. The kernel of the action is $\{g \in G \mid gb = b \quad \forall b \in B\}$. For any group G and $A = G$, the left regular action of G on itself is defined as $g \cdot a = ga$. A group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity.

§3. Subgroups

Definition 3.1. For a group G, a subset H of G is a subgroup of G if H is nonempty and H is closed under products and inverses, that is for all $x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$. If H is a subgroup of G we shall write $H \leq G$. If $H \neq G$ then we write $H < G$.

Some examples include $\mathbb{Z} \leq \mathbb{Q}, \mathbb{Q} \leq \mathbb{R}$. $H = \{1\}$ is the trivial subgroup. The relation is a subgroup of is transitive: if H is a subgroup of a group G and K is a subgroup of H, then K is also a subgroup of G.

Proposition 3.2. A subset H of group G is a subgroup iff:

- $H \neq \emptyset$, and
- for all $x, y \in H$, $xy^{-1} \in H$

Furthermore, if H is finite, then it suffices to check that H is nonempty and closed under multiplication.

Proof. We can prove this by seeing that the if condition is by definition, and the only if condition by ensuring that $1(x = y)$, for every z in $H(x = 1, y = z)$, z^{-1} is in H, and for all u and v in H, $x = u, y = v^{-1}$, uv is in H. Hence H is a subgroup. If H is finite and closed under multiplcation, then $x^{n-1} = x^{-1}$ is an element of H, and hence is closed under inverses. □

If H and K be subgroups of G. Then $H \cup K$ is a subgroup if and only if either $H \subseteq K$ or $K \subseteq H$. However, $H \cap K$ is a subgroup of G, and so is an arbitrary intersection of subgroups of G.

§§3.1. Centralizers, Normalizers, Stabilizers and kernels

Definition 3.3. $C_G(A) = \{g \in G \mid gag^{-1} = a \quad \forall a \in A\}$ called the centralizer of A in G. It is also the set of elements of G which commute with every element of A.

It is a subgroup of G since the identity belongs to it, and if $x \in C_G(A)$, then x^{-1} also commutes with all elements of G, and finally if x and y commute with all elements of A, then so does xy. Hence $C_G(A) \leq G$.

Definition 3.4. Define $Z(G) = \{g \in G \mid gx = xg \quad \forall x \in G\}$, is the set of elements commuting with all elements of G. It is called the centre of G and $Z(G) = C_G(G)$, hence $Z(G) \leq G$.

Definition 3.5. Let $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. The normalizer of A in G is $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$, and that $C_G(A) \leq N_G(A)$, and $N_G(A) \leq G$.
For an abelian group $C_G(A) = N_G(A)$ for any subset A of G, and $Z(G) = G$

Definition 3.6. If G is a group acting on a set S, fix some element s of S, then the stabilizer of s in G is $G_s = \{g \in G \mid g \cdot s = s\}$. We can see by the axioms of group action that $G_s \leq G$.

Definition 3.7. Define the kernel of an action of G on S as $\{g \in G | \; g \cdot s = s \forall s \in S\}$, and we can see that the kernel is a subgroup of G.

Theorem 3.8. Lagranges Theorem : if G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$.

§§3.2. Cyclic groups and subgroups

Definition 3.9. A group H is cyclic if H can be generated by a single element, i.e. , there is some element x in H such that $H = \{x^n \mid n \in \mathbb{Z}\}$. We say that H is generated by x, and write $H = \langle x \rangle$ (also implies $H = \langle x^{-1} \rangle$.

For example, if $G = D_{2n}$, H is the subgroup of all rotations, then $H = \langle r \rangle$. If $H = \mathbb{Z}$, then $H = \langle 1 \rangle$. We state a bunch of propositions here, the proofs of which are trivial.

Proposition 3.10. If $H = \langle x \rangle$, then $|H| = |x|$, where if one side of this equality is infinite, so is the other.

Proposition 3.11. Let G be an arbitrary group, and x is in G. Let $m, n \in \mathbb{Z}$. If $x^n = x^m = 1$, then $x^d = 1$, where $d = (m, n)$. In particular, if $x^m = 1$ for some integer m, then $|x|$ divides m.

The proof just follows the Euclid Division Algorithm.

Theorem 3.12.

- if $n \in \mathbb{Z}^+$, and $\langle x \rangle$ and $\langle y \rangle$ are both cyclic groups of order n, then $x^k \mapsto y^k$ is well defined and an isomorphism, and
- if $\langle x \rangle$ is a infinite cyclic group, then $k \mapsto x^k$ is well defined and an isomorphism

where well defined follows from the above proposition, and the law of exponents make sure that the map is a homomorphism, and since the 2 groups have the same order and the map is surjective, bijectivity and hence the 2 groups being isomorphic follows.

Proposition 3.13. For a group G, $z \in G$ and $a \in \mathbb{Z} - \{0\}$,

- If $|x| = \infty$, then $|x^a| = \infty$, and
- If $|x| = n$, then $|x^a| = \frac{n}{(n,a)}$, and

- in particular to above, if a divides n, then $|x^a| = \frac{n}{a}$

In a similar manner as above, we state

Proposition 3.14. If $H = \langle x \rangle$,

- Assume $|x| = \infty$, then $H = \langle x^a \rangle$ iff $a = \pm 1$, and
- Assume $|x| = n$, then $H = \langle x^a \rangle$ iff $(a, n) = 1$. Hence, the number of generators of H is $\varphi(n)$

Hence, $\bar{a}$ generates $\mathbb{Z}/n\mathbb{Z}$ if and only if $(a, n) = 1$. Finally, putting it all together, we get -

Theorem 3.15. Let $H = \langle x \rangle$ be a cyclic group

- Every subgroup of H is cyclic, then either $K = \{1\}$ or $K = \langle x^d \rangle$, where d is the smallest positive integer such that $x^d \in K$, where $K \leq H$, and
- If $|H| = \infty$, then for any distinct nonnegative integers a and b, $\langle x^a \rangle \neq \langle x^b \rangle$ and $\langle x^m \rangle \neq \langle x^{|m|} \rangle$, and
- If $|H| = n$, then for each positive integer a dividing n there is a unique n subgroup of H of order a, which is the cyclic group $\langle x^d \rangle$, where $d = n/a$, and $\langle x^m \rangle \neq \langle x^{(n,m)} \rangle$

§§3.3. Subgroups generated by a subset of a group

Proposition 3.16. If $\mathcal{A}$ is any non empty collection of subgroups of G, then the intersection of all members of $\mathcal{A}$ is also a subgroup

Proof. Let $K = \bigcap_{H \in \mathcal{A}} H$, we see that since each H is a subgroup, 1 is in each H and hence in K, and if a, b are in K, then it is in each H, and hence so is ab^{-1} in each H and hence in K, and $K \leq G$. □

Definition 3.17. If A is any subset of G define $\langle A \rangle = \cap H$ such that $A \subseteq H, H \leq G$, called the subgroup of G generated by A.

We can see that A is the unique minimal element of $\mathcal{A}$. We define the closure of A as follows -

Definition 3.18. $\bar{A} = \{a_1^{\epsilon_1} \dots a_n^{\epsilon_n} \mid n \in \mathbb{Z},\ n \geq 0,\ a_i \in A,\ \epsilon_i = \pm 1\ \forall i\}$

and $\bar{A} = \{1\}$ if $A = \emptyset$.

Proposition 3.19. $\bar{A} = \langle A \rangle$

Proof. Since $\bar{A}$ is never empty, and we can see if a, b are in $\bar{A}$, ab^{-1} is again an element of $\bar{A}$, making it a subgroup. For each a in A, $a^1 = a$,and hence $A \subseteq \bar{A}$, and hence $\langle A \rangle \subseteq \bar{A}$. Since $\langle A \rangle$ is a subgroup, and hence closed under group operations and inverses, each element of $\bar{A}$ belongs in it. $\square$

If G is abelian, we can collect all the powers together, and if each a_i has a finite order d_i, then note that $|\langle A \rangle| \leq d_1 \dots d_k$. Non abelian nature complicates matters, and even finite order generators can result in a subgroup of infinite order.
A subgroup M of a group G is called a maximal subgroup if $M \neq G$ and the only subgroups of G which contain M are M and G. If H is a proper subgroup of a finite group G, then there is such an H. We can also prove that if $G = \langle x \rangle$ of order n, then H is maximal iff $H = \langle x^p \rangle$ for some prime p dividing n.
The lattice of subgroups of a given finite group G is constructed as follows: plot all subgroups of G starting at the bottom with 1 , ending at the top with G and, with subgroups of larger order positioned higher on the page than those of smaller order. We draw paths upwards between subgroups using the rule that there will be a line upward from A to B if $A \leq B$ and there are no subgroups properly between A and B.
This may not be easily (or at all) carried out for infinite groups. Isomorphic groups have the same lattices (converse is not true).

§4. Quotient Groups and Homomorphisms

§§4.1. Definitions

Definition 4.1. If $\varphi : G \to H$ is a homomorphism the kernel of it is the set $\{g \in G | \varphi(g) = 1_H\}$, and denoted by $\ker \varphi$.

Proposition 4.2. If $\varphi : G \to H$ is a homomorphism then

- $\varphi(1_G) = 1_H$, and
- $\varphi(g^{-1}) = (\varphi(g))^{-1}$, and
- $\varphi(g^n) = \varphi(g)^n$, and
- $\ker \varphi$ is a subgroup of G, and
- the image of G under φ $(im\varphi)$, is a subgroup of H .

The proof follows a similar argument using the definition of a homomorphism as we have done multiple times.

Definition 4.3. Let $\varphi : G \to H$ be a homomorphism with kernel K . The quotient group or factor group, G/K is the group whose elements are the fibers of φ with group operation : if X is the fiber above a and Y is the fiber above b then the product of X with Y is defined to be the fiber above the product ab. This is associative under multiplication since it is associative in H, the identity is the fiber over the identity of H, and the inverse of the fiber over a is the fiber over a^{-1}, making the set of fibers into a group.

Proposition 4.4. Let $\varphi : G \to H$ be a homomorphism with kernel K. Let $X \in G/K$ be the fiber above a, $X = \varphi^{-1}(a)$.

- For any $u \in X$, $X = \{uk \mid k \in K\}$, and
- For any $u \in X$, $X = \{ku \mid k \in K\}$

Proof. (1) and (2) have the same proof. Let $uK = \{uk | k \in K\}$. For any k in K, $\varphi(uk) = \varphi(u)\varphi(k) = a$, hence uk is in X, thus $uK \subseteq X$. Let any g in X, and put $k = u^{-1}g$, then $\varphi(k) = 1$, thus k is in K, making $g = uk$, and hence $X \subseteq uK$. □

Definition 4.5. For any $N \leq G$, and any $g \in G$, $gN = \{gn \mid n \in N\}$ is the left coset and $Ng = \{ng \mid n \in N\}$ is the right coset of N in G. Any element of a coset is called a representative for the coset.

Theorem 4.6. Let G be a group and let K be the kernel of some homomorphism from G to another group. Then the set whose elements are the left cosets of K in G with operation defined by $uK \circ vK = (uv)K$ forms a group, G/K . In particular, this operation is well defined, if u_1 and v_1 are the chosen representatives, then $uvK = u_1v_1K$. The same statement is true with right coset in place of left coset.

Proof. Let K is the kernel of some homomorphism $\varphi : G \to H$, and $X = \varphi^{-1}(a)$ and $Y = \varphi^{-1}(b)$. By definition of operation, if $Z = XY$ then $Z = \varphi^{-1}(ab)$. For any arbitrary representatives u, v of X and Y, we can see that uv is in Z. Hence Z is the left coset. We can show conversely every element z in Z is written as uv in the same way and well defined follows. The last statement follows from the definition. □

Example 4.7. For isomorphisms, $K = 1$ and G/K is isomorphic to G. If $H = 1$, then ker of homomorphism is G, and called the trivial homomorphism with G/G isomorphic to 1.

Proposition 4.8. If N is any subgroup of G, the set of left cosets form a partition of G. For all u, v in G, $uN = vN$ iff $v^{-1}u$ is in N, and $uN = vN$ iff u and v are representatives of the same coset.

Proof. Since N is a subgroup of G, 1 belongs to N, and $1 \cdot g = g$ belongs to gN, and any element of gN belongs to G by defintion, hence $G = \cup gN$ with g in G. If $uN \cap vN$ is not empty, and x is in the intersection, then $x = un = vm$, $u = vm_1$ for m, m_1, n in N. Hence any element ut of $uN = vm_1t$, which is in vN, and conversely we can prove the other inequality to show $vN = uN$. From this, if $u \cdot 1$ is in vN, this means that $v^{-1}u$ is in N, and equality of representatives follow. □

Proposition 4.9. Let G be a group and let N be a subgroup of G.

- The operation on the set of left cosets of N in G described by $uN \cdot vN = (uv)N$ is well defined if and only if $gng^{-1} \in N$ for all g in G and all n in N, and
- If well defined, the set of left cosets are a group, with identity 1N and inverse of gN is $g^{-1}N$

Proof. If well defined, then for any u, u_1 in uN and v, v_1 in vN using $u = 1, u_1 = n, v = v^{-1} = g^{-1}$, then $uvN = u_1v_1N$ implies $g^{-1}N = ng^{-1}N$, hence $ng^{-1} = g^{-1}n_1$, giving gng^{-1} in N. Conversely if gng^{-1} in N $u_1v_1 = unvm = uvv^{-1}nvm = uvn_1m = uvn_2$, hence uvN and u_1v_1N contain a common element and hence are equal. From above, verifying group axioms is easy. □

Definition 4.10. The set $gNg^{-1} = \{gng^{-1} \mid n \in N\}$ is the conjugate of N by g. g normalizes N if $gNg^{-1} = N$. A subgroup N of a group G is called normal if every element of G normalizes N. If N is a normal subgroup of G write $N \trianglelefteq G$.

Hence we conclude -

Theorem 4.11. If N is a subgroup of G, then the following are equivalent -

- $N \trianglelefteq G$, and
- $N_G(N) = G$, and
- $gN = Ng$ for all g in G, and
- the operation on left cosets makes it into a group, and
- $gNg^{-1} \subseteq N$ for all g in G

Proposition 4.12. A subgroup N of G is normal iff it is the kernel of some homomorphism.

Proof. If it is a kernel, we have already proved that it is normal. Let $N \trianglelefteq G$ let $H = G/N$ and define $\pi : G \to H$ as $\pi(g) = gN$, we can see from the above properties that it is a homomorphism. $\ker \pi = \{g \in G | gN = 1N\}$, thus g is in N, and hence N is the kernel of π. □

Definition 4.13. Let $N \trianglelefteq G$, the homomorphism $\pi : G \to G/N$, as $\pi(g) = gN$ is the natural projection. If $\bar{H} \leq G/N$, then the complete preimage of $\bar{H}$ in G is the preimage of it under the natural projection homomorphism.

If G is an abelian group, any subgroup N of G is normal. We can prove that quotient groups of cyclic groups are cyclic, with $|G/N| = |G|/|N|$.

§§4.2. Lagranges Theorem

Theorem 4.14. If G is a finite group and H is a subgroup of G, then the order of H divides the order of G and the number of left cosets of H in G is $\frac{|G|}{|H|}$.

Proof. If $|H| = n$, and the number of left cosets be k. Consider the map $H \to gH$ as $h \mapsto gh$, is clearly a surjection and by cancellation law, an injection, hence $|gH| = |H|$. Since G is partitioned into k disjoint subsets each of which has cardinality n, $|G| = kn$, thus completing the proof. □

Definition 4.15. If G is a group and $H \leq G$, the number of left cosets of H in G is called the index of H in G and denoted by $|G : H|$.

We see here 2 corollaries. If G is a finite group and x in G, then the order of x divides the order of G (by Lagrange Theorem). In particular $X^{|G|} = 1$ for all x in G. If G is a group of prime order p , then G is cyclic, hence $G \cong Z_p$.

Definition 4.16. Groups G in which the only normal subgroups are the trivial ones: 1 and G are called simple groups.

The full converse to Lagrange's Theorem is not true: namely, if G is a finite group and n divides $|G|$, then G need not have a subgroup of order n. However the following partial converses are true -

Theorem 4.17. *Cauchy Theorem* : If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p.

Theorem 4.18. *Sylow* : If G is a finite group of order $p^\alpha m$ where p is a prime and p doesnt divide m, the G has a subgroup of order p^α.

We postpone the proofs.

Definition 4.19. Let H and K be subgroups of a group and define $HK = \{hk \mid h \in H, k \in K\}$. This need not be a group.

Proposition 4.20. H and K are finite subgroups of a group then $|HK| = \frac{|H||K|}{|H \cap K|}$.

Proof. HK is the union of left cosets of K, and each coset of K has $|K|$ elements, and 2 cosets are the same iff $h_1 h_2^{-1}$ is in K, thus $h_1(H \cap K) = h_2(H \cap K)$, and the number of distinct cosets is hence $\frac{|H|}{|H \cap K|}$ by Lagranges theorem, each of which has $|K|$ number of elements, hence the formula. □

Theorem 4.21. If H and K are subgroups of a group, HK is a subgroup if and only if $HK = KH$.

Proof. If $HK = KH$, then identity is in both, and let a, b be in HK, then let $a = h_1 k_1$ and $b = h_2 k_2$, then using $HK = KH$ we can see that ab^{-1} is in HK, and hence is a subgroup. Conversly, if it is a subgroup, $K \leq HK$ and $H \leq HK$, thus $KH \leq HK$, and by taking any element and its inverse in HK the reverse inclusion follows. □

A corollary follows that if H and K are subgroups of G and $H \leq N_G(K)$, then HK is a subgroup of G. If $K \trianglelefteq G$, then $HK \leq G$ for any $H \leq G$. That is, HK is a subgroup if H normalizes K. In a finite group the number of left cosets of H in G equals the number of right cosets even though the left cosets are not right cosets in general (unless they are normal).

§§4.3. Isomorphism theorems

Theorem 4.22. The First Isomorphism Theorem: If $\varphi : G \to H$ is a homomorphism of groups, then $\ker \varphi \trianglelefteq G$ and $G/\ker \varphi \cong \varphi(G)$.

The corollary can be stated as - φ is injective iff $\ker \varphi = 1$ and $|G : \ker \varphi| = |\varphi(G)|$.

Theorem 4.23. The Second Isomorphism Theorem: If G is a group, A and B are subgroups, and let $A \leq N_G(B)$. Then AB is a subgroup, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

Proof. By the above corollary, AB is a subgroup. Since $B \leq N_G(B)$, thus $AB \leq N_G(B)$, B is normal in AB and hence AB/B is defined. $\varphi : A \to AB/B$ as $\varphi(a) = aB$. We see that this is a homomorphism, identity is the coset B, hence $\ker \varphi = A \cap B$. The result then follows from the first isomorphism theorem. □

Theorem 4.24. The Third Isomorphism Theorem: If G is a group and H and K are normal subgroups, with $H \leq K$, then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$.

Proof. The first part follows from the definition, and define $\varphi : G/H \to G/K$. We can see this is well defined since $gH \mapsto gK$, and that it is a surjective homomorphism. Noe that $\ker \varphi = K/H$, and the result follows from the first isomorphism theorem. □

Theorem 4.25. The Fourth Isomorphism Theorem: If G is a group and N is a normal subgroup of G, Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups $A/N = \bar{A}$. Then -

- $A \leq B$ iff $\bar{A} \leq \bar{B}$, and
- if $A \leq B$ then $|B : A| = |\bar{B} : \bar{A}|$, and
- $\overline{\langle A, B \rangle} = \langle \bar{A}, \bar{B} \rangle$, and
- $\overline{A \cap B} = \bar{A} \cap \bar{B}$, and
- $A \trianglelefteq G$ iff $\bar{A} \trianglelefteq \bar{G}$, and

§5. Group Actions

§§5.1. Permutation representations

If G is the group acting on set A, then for each g in G, $\sigma_g : A \to A$ is defined as $\sigma_g : a \mapsto g \cdot a$. The homomorphism defined as $\varphi : G \to S_A$ as $\varphi(g) = \sigma(g)$ is the permutation representation to the given action. The Kernel of the action : $\{g \in G \mid g \cdot a = a\}$ for all a in A. The stabilizer of a in G is : $\{g \in G \mid g \cdot a = a\}$ denoted by G_a. An action is faithful iff its kernel is the identity.

Proposition 5.1. For any group G and any nonempty set A there is a bijection between the actions of G on A and the homomorphisms of G into S_A .

This follows by noting that $g \cdot a = \varphi(g)(a)$. We shall say a given action of G on A affords or induces the associated permutation representation of G.

Proposition 5.2. If G acts on nonempty A, then the relation $a \sim b$ iff $a = g \cdot b$ is an equivalence relation. For each a in A, the number of elements in the equivalence class containing a is $|G : G_a|$

Proof. We can see that $a \sim a$ since it is connected by identity, it is symmetric since inverse of g connects b and a, and transitivity follows from the group action definition. Let $C_a = \{g \cdot a \mid g \in G\}$ is the equivalence class of a, define the map $g \cdot a \mapsto gG_a$. It is clearly a surjection, on g and $g \cdot a = h \cdot a$ iff $h^{-1}g$ is in G_a iff $gG_a = hG_a$ hence map is injective and thus a bijection. □

Definition 5.3. Let G be a group acting on the nonempty set A. The equivalence class $\{g \cdot a | g \in G\}$ is the orbit of G containing a. The action of G on A is called transitive if there is only one orbit, i.e., for any 2 elements a and b in A, $a = gb$ for some g in G.

Subgroups of symmetric groups are called permutation groups.

§§5.2. Groups acting on themselves by left multiplication

Here we consider $G = A$, i.e., $g \cdot a = ga$ for all g, a in G. For finite groups we label the elements as $g_1, \ldots, g_n$, with the permutation as $\sigma_g(i) = j$ iff $gg_i = g_j$. We can see that action of group on itself is transitive and faithful.

We consider a generalization of this action.

Definition 5.4. Let H be any subgroup of G and let A be the set of all left cosets of H in G. Define an action of G on A by $g \cdot aH = gaH$, for all g in G and aH in A. This clearly satisfies the axioms for group action. If H is the identity subgroup, then this specializes to the action described above, if it is of finite index m, then the permutation is described as $\sigma_g(i) = j$ iff $ga_iH = a_jH$.

Theorem 5.5. Let G be a group, let H be a subgroup of G and let G act by left multiplication on the set A of left cosets of H in G . Let π_H be the associated permutation representation afforded by this action. The G acts transitively on A, the stabilizer of H in A is H and the kernel of the action is $\cap_{x \in G} xHX^{-1}$ and it is the largest normal subgroup of G in H.

Proof. For any 2 cosets aH and bH, note that $g = ba^{-1}$ is the required element for transitivity. The stabilizer of H by definition is H itself ($\{g \in G | g \cdot 1H = 1H\}$. Note $\ker \pi_H = \{g \in G | gxH = xH\}$ for all x in G, thus $x^{-1}gx \in H$ and hence the relation follows. Note that $\ker \pi_h \trianglelefteq G$ and $\ker \pi_h \trianglelefteq H$, if any N is normal in H then $N = xNx^{-1} \leq xHx^{-1}$ thus N is a subgroup of the intersection. □

The corollary is the Cayley's theorem which we prove using $H = 1$, obtaining a homomorphism and an isomorphism since kernel is the identity.

Theorem 5.6. Every group is isomorphic to a subgroup of some symmetric group. If G is a group of order n, then G is isomorphic to a subgroup of S_n.

The permutation representation afforded by left multiplication on the elements of G (cosets of $H = 1$) is called the left regular representation of G. Another corollary is as follows -

Theorem 5.7. If G is a finite group of order n and p is the smallest prime dividing $|G|$, then any subgroup of index p is normal.

Let $H \leq G$ and $|G : H| = p$ and π_H be the permutation representation and $K = \ker \pi_H$, and let $|H : K| = k$ then $|G : K| = kp$ and G/K is isomorphic to a subgroup of S_p by the first isomorphism theorem, and by Lagrange's theorem, pk divides $p!$ and hence k divides $(p-1)!$. By minimality of p, thus $k = 1$, and hence $H = K \trianglelefteq G$.

§§5.3. Groups acting on themselves by conjugation

Here we again consider $G = A$ but acting by conjugation : $g \cdot a = gag^{-1}$ for all g, a in G. We can see that it satisfies the axioms for group action.

Definition 5.8. Two elements a and b of G are said to be conjugate in G if there is some g in G such that $b = gag^{-1}$ (i.e., if and only if they are in the same orbit of G acting on itself by conjugation). The orbits of G acting on itself by conjugation are called the conjugacy classes of G.

For abelian groups the conjugacy classes are just $\{a\}$. If $|G| > 1$, then $\{1\}$ is a conjugacy class, and one element subset is a conjugacy class iff a is in the centre of G. We now generalize to any subset S of G and the action as $g \cdot S = gSg^{-1}$ for any g in G, and S in the set of all subsets of G.

Definition 5.9. Two subsets S and T of G are said to be conjugate in G if there is some g in G such that $T = gSg^{-1}$.

Proposition 5.10. The number of conjugates of S is the index $|G : N_G(S)|$, and in particular the number of conjugates of s is the index $|G : C_G(s)|$.

Proof. The number of conjugates of S is the index $|G : G_s|$ where $G_S = \{g \in G | gSG^{-1} = S\} = N_G(S)$, from an above proposition, hence the first part follows, the second follows by noting that $N_G(\{s\}) = C_G(s)$ □

Theorem 5.11. The Class equation: If G is a finite group and $g_1, \dots g_r$ be representatives of the distinct conjugacy classes of G not contained in the center $Z(G)$ then $|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$.

Proof. This follows from the above proposition and noting that $\{x\}$ is a conjugacy class of size 1 iff x is in $Z(G)$. Since they contain just 1 element, hence the total number of elements of G follows as the sum of 1 centre order times and the number of conjugates for each representative for a conjugacy class. □

Theorem 5.12. If p is a prime and P is a group of order p^α for some $\alpha \geq 1$ then $Z(P) \neq 1$.

Proof. From definition we know that $C_P(g_i) \neq P$ hence p divides $|P : C_P(g_i)|$ hence by the class equation p divides $Z(P)$. □

A corollary follows that if $P = p^2$ for prime p then since $Z(P) \neq 1$ and hence $P/Z(P)$ is cyclic, making P abelian. If P has an element of order p^2 then it is cyclic and isomorphic to Z_{p^2} or $Z_p \times Z_p$.

Proposition 5.13. If σ, τ are elements of S_n and if σ has cycle decomposition $(a_1 a_2 ... a_{k_1})(b_1 b_2 ... b_{k_2})...$ then $\tau\sigma\tau^{-1}$ has cycle decomposition $(\tau(a_1)\tau(a_2) \dots \tau(a_{k1}))(\tau(b_1)\tau(b_2) \dots \tau(b_{k2})) \dots$.

Proof. This just follows from the fact that if $\sigma(i) = j$ then $(\tau\sigma\tau^{-1})\tau(i) = \tau(j)$. □

Definition 5.14. If $\sigma \in S_n$ is the product of disjoint cycles $n_1, \dots n_r$ with $n_1 \leq n_2 \dots \leq n_r$ then these r integers are called the cycle type of σ. A partition of n is a non decreasing sequence of integers whose sum is n.

Proposition 5.15. Two elements of S_n are conjugate iff they have the same cycle type. The number of conjugacy classes of S_n equals the number of partitions of n.

Proof. Conjugates have the same cycle type from above, and conversely we first order the cycles in increasing length and define τ mapping the i position in σ_1 to the i position in σ_2, and we note from the above prop that $\tau\sigma_1\tau^{-1} = \sigma_2$. □

Using these we obtain that if σ is a m cycle, then $|C_{S_n}(\sigma)| = m(n-m)!$, explicitly written as $C_{S_n}(\sigma) = \{\sigma^i\tau|0 \leq i \leq m-1, \tau \in S_{n-m}$, where S_{n-m} is the group fixing all integers in the m cycle.

Proposition 5.16. If $H \trianglelefteq G$ then for every conjugacy class $\mathcal{K}$ in G either $K \subseteq H$ or $K \cap H = \emptyset$.

Proof. This follows since if $x \in \mathcal{K} \cap H$, then $gxg^{-1} \in gHg^{-1}$ for all g. Since H is normal thus the relation follows. □

Definition 5.17. We now similarly define right group actions of G on A as a map from $A \times G$ to A that satisfies:

- $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$ for all a in A and g_1, g_2 in G, and
- $a \cdot 1 = a$ for all a in A

The conjugation action is written as $a^g = g^{-1}ag$ and we see this verifies the axioms for a group action. We see a left group action can be transformed to a right group action as $a \cdot g = g^{-1} \cdot a$, called corresponding group actions. Note that the relation conjugacy is the same for the left and right corresponding actions.

§6. The Fundamental Group

§§6.1. Homotopy of Paths

Definition 6.1. If f and f' are continuous maps of X to Y then f is homotopic to f' if there is a continuous $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$. F is called a homotopy between these two maps, and $f \simeq f'$. If f' is the constant map, then f is nullhomotopic.

Definition 6.2. Two paths f, f' are path homotopic if there is a continuous map $F : I \times I \to X$ such that $F(s, 0) = f(s), F(s, 1) = f'(s), F(0, t) = x_0$ and $F(1, t) = x_1$ for each s and t. F is a path homotopy between f, f' and we write $f \simeq_p f'$.

Lemma 6.3. $\simeq$ and $\simeq_p$ are equivalence relations, and define the path-homotopy equivalence class by $[f]$.

Proof. Reflexivity follows from $F(x, t) = f(x)$, Symmetry from $G(x, t) = F(x, 1-t)$, where $F(x, t)$ is the homotopy between f, f'. Transitivity follows from taking F to be the homotopy between f, f' and F' to be between f', f''. We then define $G(x, t) = F(x, 2t)$ for $t \in [0, 1/2]$ and $F(x, 2t-1)$ for $t \in [1/2, 1]$. G is well defined, and by pasting lemma on $X \times [0, 1/2], X \times [1/2, 1]$, continuity follows. Same proof holds for path homotopy. □

If f and g are homotopic, then $F(x, t) = (1-t)f(x) + tg(x)$ is called the straight line homotopy (this makes sense when we consider $Y \subset \mathbb{R}^n$). If Y is convex, then any 2 paths from the same endpoints are homotopic since F is contained in Y.

Definition 6.4. If f is a path in X from x_0 to x_1 and g from x_1 to x_2 then the product $f \star g$ as the path h given by $\begin{cases} h(s) = f(2s) & t \in [0, 1/2], \\ h(s) = g(2s-1) & t \in [1/2, 1]. \end{cases}$ Product operation induces a well defined operation as $[f] \star [g] = [f \star g]$. If F is a homotopy between f, f' and G between g, g' then H is defined as $\begin{cases} F(2s, t) & s \in [0, 1/2], \\ G(2s-1, t) & s \in [1/2, 1] \end{cases}$

Theorem 6.5. The operation $\star$ satisfies the following groupoid properties (defined only for those $[f], [g]$ such that $f(1) = g(0)$ -

- If $[f] \star ([g] \star [h])$ is defined, then so is $([f] \star [g]) \star [h]$ and are equal, and
- If e_x is the constant map at x, then $[f] \star [e_{x_1}] = [e_{x_0}] \star [f] = [f]$, and
- If $\bar{f}(s) := f(1-s)$, the reverse of f, then $[f] \star [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] \star [f] = [e_{x_1}]$

The proof for $(2), (3)$ directly follow from the definition, meanwhile for (1) we use the following provable theorem -

Theorem 6.6. Let f be a path in X., and $a_0, \ldots, a_n$ be numbers such that $0 = a_0 < a_1 < \ldots < a_n = 1$. Define $f_i : I \to X$ be the path equaling the positive linear map (straight line path joining $[a, b] \to [c, d]$) of I onto $[a_{i-1}, a_i]$ followed by f. Then $[f] = [f_1] \star \ldots \star [f_n]$.

§§6.2. The Fundamental Group

Definition 6.7. If X is a space and let x_0 is a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0 with the operation $*$ is called the fundamental group of X relative to the base point x_0, denoted by $\pi_1(X, x_0)$.

Restricted to this definition, the groupoid forms a group.

Definition 6.8. Let α be a path from x_0 to x_1, and define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the equation $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$ is well defined and a loop at x_1.

Theorem 6.9. The map $\hat{\alpha}$ is a group isomorphism.

Proof. Note that $\hat{\alpha}([f]) * \hat{\alpha}([g]) = [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] = \hat{\alpha}([f] * [g])$, proving homomorphism. Injectivity is immediate and for any $[h] \in \pi_1(X, x_1)$ note that $\hat{\beta}([h]) := [\alpha] * [f] * [\bar{\alpha}]$ works as an inverse. $\square$

Hence, if X is path connected, then for any 2 points x_0 and x_1 on X, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. Note that $\pi_1(X, x_0)$ gives us information of just the path component C containing x_0.

Definition 6.10. A space X is simply connected if it is path connected and $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence for every $x_0 \in X$. Hence $\pi_1(X, x_0) = 0$.

Lemma 6.11. In a simply connected space X any 2 paths having same initial and final points are path homotopic.

Proof. If α and β are 2 points from x_0 to x_1, then $[\alpha\bar{\beta}]$ is the one element group, or e_{x_0}, and hence $[\alpha] = [\beta]$. $\square$

Definition 6.12. If $h : (X, x_0) \to (Y, y_0)$ is a continuous map, then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ as $h_*([f]) = [h \circ f]$, and is called the homomorphism induced by h relative to the base point x_0.

It is well defined, and homomorphism follows from $(h \circ f) * (h \circ g) = h \circ (f * g)$.

Theorem 6.13. If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous maps, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is identity, then i_* is the identity homomorphism.

The corollary comes as follows : If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X and Y then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

§§6.3. Covering Spaces

Definition 6.14. Let $p : E \to B$ is a continuous surjective map. The open set U in B is said to be evenly covered by p if $p^{-1}(U)$ can be written as union of disjoint open sets V_α in E such that restriction of p to V_α is a homeomorphism of V_α onto U. This collection is called a partition of $p^{-1}(U)$ into slices. If every point b of B has such a neighbourhood that is evenly covered by p then it is called a covering map and E is the covering space of B.

The map $p : \mathbb{R} \to S^1$ is given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map. Note that if $p : E \to B$ is a covering map then it is a local homeomorphism of E with B. We also easily note that -

Theorem 6.15. Let $p : E \to B$ be a covering map. If B_0 is a subspace of B and if $E_0 = p^{-1}(B_0)$ then the map p_0 obtained by restricting p is a covering map.

and

Theorem 6.16. If $p : E \to B$ and $p' : E' \to B'$ are covering maps then $p \times p' : E \times E' \to B \times B'$ is a covering map.

§§6.4. The Fundamental Group of the Circle

Definition 6.17. Let $p : E \to B$ be a map. If f is a continuous mapping of X into B, a lifting of f is the map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

Lemma 6.18. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$. Any path $f : [0, 1] \to B$ beginning at b_0 has a unique lifting path $\tilde{f}$ beginning at e_0.

Proof. We cover B with open U which are evenly covered by p. We find subdivisions of the interval such that for each subdivision, its mapping lies in such a U. We define $\tilde{f}(0) = e_0$, and $\tilde{f}(s) = (p|V_0)^{-1} f(s)$, where $s \in [s_i, s_{i+1}]$ and V_0 is that slice of $p^{-1}(U)$ to which the interval mapping belongs to. $\tilde{f}$ is thus continuous on the interval and by the pasting lemma on the whole interval, and $p \circ \tilde{f} = f$ follows. Uniqueness follows from a similar argument. $\square$

Lemma 6.19. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$. Let the map $F : I \times I \to B$ is continuous and $F(0,0) = b_0$. Then there is a unique lifting of F to $\tilde{F} : I \times I \to E$ such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Proof. This follows the same argument as above, and use the previous lemma to extend the lifting function to the left edge and the bottom edge, from where we iteratively define onto rectangles (note that the map is already defined on the left and bottom edges of the rectangle) and then defining $\tilde{F}(x) = p_0^{-1}(F(x))$, which is continuous by the pasting lemma (where p_0 is the restriction of p to V_0, which is the slice containing the left and bottom edges). Uniqueness follows in the same way. If F is a homotopy, then it carries $0 \times I$ to b_0, and for $\tilde{F}$ this is the set $p^{-1}(b_0)$,which must be a one point set by connectedness and its discrete topology. Similarly for the right edge, and homotopy of $\tilde{F}$ follows. □

A corollary for this follows as -

Theorem 6.20. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$. Let f, g be paths in B from b_0 to b_1 and $\tilde{f}, \tilde{g}$ be their respective lifting to paths beginning at e_0 in E. If f, g are path homotopic, then $\tilde{f}, \tilde{g}$ end at the same point and are also path homotopic.

Definition 6.21. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$. Given $[f]$ in $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f in E beginning at e_0. Let $\phi([f])$ denote the endpoint $\tilde{f}(1)$, then it is a well defined set map, $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ and called the lifting correspondence derived from p.

Theorem 6.22. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$. If E is path connected, then the lifting correspondence is surjective and if it is simply connected then it is bijective.

Proof. If path connected, then given any $e_1 \in p^{-1}(b_0)$ there is a path $\tilde{f}$ between e_0, e_1, thus defining the loop $f = p \circ \tilde{f}$ at b_0. If E is simply connected, then taking any 2 elements $[f], [g]$ such that $\phi(f) = \phi(g)$, then they have the same start and end point, and from simply connectedness have a homotopy between them, say $\tilde{F}$, and thus $p \circ \tilde{F}$ is a homotopy between f, g, placing them in the same class. □

Theorem 6.23. The fundamental group of S^1 is isomorphic to the additive group of integers.

Proof. Let $p : \mathbb{R} \to S^1$ as $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map and $p(0) = b_0$. Then $p^{-1}(b_0)$ is $\mathbb{Z}$. Since $\mathbb{R}$ is simply connected the map $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijective. Given any $[f], [g]$ and their liftings $\tilde{f}, \tilde{g}$, let $\phi([f]) = n$ and $\phi([g]) = m$, and $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$ be another lifting of g as we can check from seeing that $g = p \circ \tilde{\tilde{g}}$. Thus $\tilde{f} * \tilde{\tilde{g}}$ is defined, and their endpoint is is $n + m$, which is by definition $\phi([f]) + \phi([g])$, making it a homomorphism. This group is hence infinite cyclic. □

We can also prove a generalization of this, which is -

Theorem 6.24. Let $p : E \to B$ is a covering map and $p(e_0) = b_0$.

- The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective, and
- Let $H = p_*(\pi_1(E, e_0))$, then the lifting correspondence ϕ induces an injective map $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$, which is bijective if E is path connected, and
- If f is a loop in B at b_0, then $[f] \in H$ iff f lifts to a loop in E based at e_0.

§§6.5. Retractions and Fixed points

Definition 6.25. $A \subset X$ is a retraction of X onto A if there is a continuous map $r : X \to A$ such that $r|A$ is the identity map.

Lemma 6.26. If A is a retract of X then the homomorphism of fundamental groups induced by the inclusion map $j : A \to X$ is injective.

Proof. If r is a retraction then note $r \circ j$ is the identity on A, and hence $r_* \circ j_*$ is the identity map on $\pi_1(A, a)$ making j_* injective. □

From the non triviality of the fundamental group of S^1 and triviality of the fundamental group of B^2 we have -

Theorem 6.27. There is no retraction from B^2 onto S^1.

Lemma 6.28. Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent -

- h is null homotopic
- h extends to a continuous map $k : B^2 \to X$
- h_* is the trivial homomorphism of fundamental groups

Proof. $1 \implies 2$ follows from considering $H : S^1 \times I \to X$ as a homotopy between h and constant. $\pi(x, t) = (1 - t)x$ which is continuous, closed and surjective, hence a quotient map, which then induces a continuous map $k : B^2 \to X$.
$2 \implies 3$ follows from $j : S^1 \to B^2$ to be the inclusion map and $h = k \circ j$ then j_* must be trivial since the fundamental group of B^2 is trivial and hence h_* is trivial.
$3 \implies 1$ follows from considering the loop $f = h \circ p_0$, where $p_0 : I \to S^1$ is the standard covering map, and F is the homotopy between f and constant. $p_o \times id$ is a quotient map, and hence F induces such an H between h and constant. □

A corollary follows from noting that the inclusion map $j : S^1 \to R^2 - 0$ is not null homotopic and the identity map $i : S^1 \to S^1$ is not nullhomotopic.

Definition 6.29. A vector field on B^2 is an ordered pair $(x, v(x))$ where $x \in B^2$ and $v(x)$ is a continuous map of $B^2 \to \mathbb{R}^2$.

Theorem 6.30. Given a non vanishing vector field on B^2, there exists a point of S^1 where the vector field points inward and another point where it points outward.

Proof. If $v(x)$ does not point inward at any point on S^1, consider its restriction on S^1 as w, since it extends to a coninous map $B^2 \to R^2 - 0$, by the previous theorem it is nullhomotopic. Not pointing inwards also implies that the homotopy of w with $j : S^1 \to R^2 - 0$, the inclusion map as $F(x, t) = tx + (1 - t)w$, implying j is also nullhomotopic, a contradiction. □

Similarly -

Theorem 6.31 (Brouwer Fixed Point Theorem)**.** If $f : B^2 \to B^2$ is continuous, then there exists a point $x \in B^2$ such that $f(x) = x$.

Proof. Assume the non vanishing field $v(x) = f(x) - x$, for it to point outward at any point x, $f(x) = x + ax$ for some positive a, which would lie outside the unit ball, this giving us a contradiction. □

A corollary of this follows as - Let A be a 3×3 matrix of positive real numbers, then A has a positive real eigenvalue. We do this showing that the octant is homemorphic to B^2 and since the entries of A are all positive, and also the coordinates are positive. Note that $x \mapsto T(x)/\|T(x)\|$ is a continuous map from $B^2 \to B^2$ has a fixed point x_0 then $T(x_0) = \|T(x_0)\|x_0$ thus it has a positive eigenvalue.
Similarly, deriving a homemorphism between $T = \{(x, y)|x \geq 0, y \geq 0, x + y \leq 1\}$ and B^2, and choosing vertices of T on each open set covering T, ϕ_i to be the partition of unity and $k(x) = \sum \phi_i(x)v_i$ and the exact same reasoning as above we show -

Theorem 6.32. There is an $\epsilon > 0$ such that for every open covering $\mathcal{A}$ of T by sets of diameter less than ϵ, some point of T belongs to atleast 3 elements of $\mathcal{A}$.

§§6.6. The Fundamental Theorem of Algebra

Theorem 6.33. A polynomial equation $x^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0 = 0$ of degree $n > 0$ with real or complex coefficients has atleast one real or complex root.

Proof. First consider $f : S^1 \to S^1$ as $f(z) = z^n$, and take p_0 to be the standard loop, its image under f_* is the loop wound n times, and hence is injective.

Let $g : S^1 \to \mathbb{R}^2 - 0$ as $g(z) = z^n$ and j is in the inclusion map, thus $g_* = j_* \circ f_*$ is also injective and hence not nullhomotopic.
Now given the monic polynomial as above, assume that $|a_{n-1}| + \ldots |a_0| < 1$, and define $k : B^2 \to \mathbb{R}^2 - 0$ to be the polynomial map, the restriction of which to S^1 is h. Since h extends, it is nullhomotopic, and note that $F : S^1 \times I \to \mathbb{R}^2 - 0$ defines the homotopy between h and g as $F(z,t) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0)$, thus making g nullhomotopic, contradicting the above.
For the general case substitute $x = cy$ for a large enough c for the above to be satisfied, with y_0 as the root, thus making $x_0 = cy_0$ as the original root. □

§§6.7. Borsuk Ulam Theorem

Theorem 6.34. If x is in S^n, a map $h : S^n \to S^m$ is said to be antipode preserving if $h(-x) = -h(x)$ for all x. If $h : S^1 \to S^1$ is a continuous and antipode preserving, then h is not nullhomotopic.

Proof. Let $b_0 = (0,1)$, note that using rotations, it suffices to prove the theorem assuming that $h(b_0) = b_0$. Let $q : S^1 \to S^1$ is the map $q(z) = z^2$, a quotient map, note that $q \circ h$ will induce a contiunous map such that $k \circ q = q \circ h$. q is also a covering map, and now take $\tilde{f}$ to be a path in S^1 from b_0 to $-b_0$ with $f = q \circ \tilde{f}$ as a non trivial element. Now, $k_*[f] = [k \circ q \circ \tilde{f}] = [q \circ h \circ \tilde{f}]$ is not trivial, and hence is injective, so is q_* from which it follows that h_* must be injective and hence not null homotopic. □

The corollary of this follows as : There is no continuous antipode preserving map $g : S^2 \to S^1$. If there was, then its restriction to the equator would be antipode preserving and hence not nullhomotopic, but an extension of the map of the equator to the sphere and hence B^2 would imply nullhomotopic.

Theorem 6.35 (Borsuk Ulam theorem for S^2)**.** Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point x of S^2 such that $f(x) = f(-x)$.

Proof. If $f(x) \neq f(-x)$ for all $x \in S^2$, then $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ is a continuous map $g : S^2 \to S^1$ such that it is antipode preserving, violating the corollary. □

Theorem 6.36 (The bisection theorem)**.** Given 2 bounded polygonal regions in $\mathbb{R}^2$, there is a line in $\mathbb{R}^2$ that bisects each of them.

Proof. Take a point $u \in S^2$ with the plane P as its normal vector. and $f_i(u)$ is the area A_i lying on the same side of P as u. Replacing $u \mapsto -u$ gives the other half space, so that $f_i(u) + f_i(-u) = \text{Area} A_i$. Now we take the map $F : S^2 \to \mathbb{R}^2$ as $F(u) = (f_1(u), f_2(u))$, and this gives us a u such that the area is bisected. □

§§6.8. Deformation Retracts and Homotopy Type

Lemma 6.37. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps which are homotopic and the imgae of the base point x_0 remains fixed at y_0 during the entire homotopy, then since for any loop f, $h \circ f$ and $k \circ f$ are homotopic and hence h_* and k_* are equal.

Theorem 6.38. The inclusion map $j : S^n \to \mathbb{R}^{n+1} - 0$ induces an isomorphism of fundamental groups

Proof. Let $b_0 = (1, 0, \ldots, 0)$ and $r : \mathbb{R}^{n+1} - 0 \to S^n$ is the map $r(x) = \frac{x}{\|x\|}$, then $r \circ j$ is the identity map, making $r_* \circ j_*$ the identity homomorphism of $\pi_1(S^n, b_0)$. Note $j \circ r$ is homotopic to the the identity map, with b_0 fixed, and by the previous lemma, $j_* \circ r_*$ is also the identity homomorphism, making j_* an isomorphism. □

Definition 6.39. If A is a subspace of X, then it is a deformation retract of x, if there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x, H(x, 1) \in A \forall x \in X$ and $H(a, t) = a \forall a \in A$, and H is called a deformation retraction of X onto A. Then $r : X \to A$ such that $r(x) = H(x, 1)$ is a retraction and H is a homotopy between I and $j \circ r$ (j is inclusion), thus resulting in j inducing an isomorphism of fundamental groups.

Definition 6.40. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps such that $g \circ f$ and $f \circ g$ are homotopic to the identity map on X and Y respectively, then f and g are called homotopy equivalences and each is said to be a homotopy inverse of the other. Homotopy equivalence is an equivalence relation, and 2 spaces which are homotopy equivalent have the same homotopy type.

Lemma 6.41. Let $h, k : X \to Y$ be homotopic continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$, then there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. If $H : X \times I \to Y$ is the homotopy between h and k, then $\alpha(t) = H(x_0, t)$.

Proof. Let $f_0, f_1 \in X \times I$ be loops such that $f_0(s) = (f(s), 0), f_1(s) = (f(s), 1)$ and $c \in X \times I$ with $c(t) = (x_0, t)$. Then $H \circ f_0 = h \circ f, H \circ f_1 = k \circ f$ and $H \circ c$ is the path α. Consider the following paths in $I \times I$ as $\beta_0(s) = (s, 0), \beta_1(s) = (s, 1), \gamma_0(t) = (0, t), \gamma_1(t) = (1, t)$, with $F \circ \beta_0 = f_0, F \circ \beta_1 = f_1, F \circ \gamma_0 = F \circ \gamma_1 = c$. Let G be the homotopy between $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ in $I \times I$, then $H \circ F \circ G$ is the homotopy between $(h \circ f) * \alpha$ and $\alpha * (k \circ f)$, and we are done. □

3 Corollaries follow as -

- Let $h, k : X \to Y$ be homotopic continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$, then if h_* is injective, surjective or trivial, then so is k_*, and
- Let $h : X \to Y$, if h is nullhomotopic, then h_* is the trivial homomorphism, and

- Let $f : X \to Y$ be continuous and $f(x_0) = y_0$. If f is a homotopic equivalence then $f_* : \pi_1(X, x_0) \to (Y, y_0)$ is an isomorphism.

§§6.9. The Fundamental Group of S^n

Theorem 6.42. Let $X = U \cup V$ where U, V are open sets of X. If $U \cap V$ is path connected, and $x_0 \in U \cap V$. If i, j are inclusion maps of U, V, then the images of i_*, j_* generate $\pi_1(X, x_0)$. This means that for any loop f at x_0 it is path homotopic to product of the form $(g_1 * (g_2 * (\ldots * g_n)))$, where each g_i is a loop based at x_0 lying either in U or V.

Proof. Choose by the Lebesgue number lemma, $b_0, \ldots b_m$ subdivison of I such that for each i, $f([b_{i-1}, b_i])$ is contained in U or V. Now, keep deleting b_i if $f(b_i) \notin U \cap V$, we would then be left with such division that $f(b_i) \in U \cap V$. Now define f_i to be the path in X which is the composite of the linear map from $[0, 1]$ to $[a_{i-1}, a_i]$ followed by f. Thus by a previous theorem $[f] = [f_1] * \ldots * [f_n]$. Now for each i choose $\alpha_i \in U \cap V$ (by path connectedness) as the path between x_0 and $f(a_i)$ and set $g_i = \alpha_{i-1} * f_i * \bar{\alpha}_i$. Thus we conclude that $[g_1] * \ldots * [g_n] = [f_1] * \ldots * [f_n]$. □

A corollary follows as : If $X = U \cup V$ with U, V open sets of X, with $U \cap V$ non empty and path connected. If U, V are simply connected then so is X.

Theorem 6.43. If $n \geq 2$, then n-sphere S^n is simply connected.

Proof. Let p and q be the north and south poles. Define $f : S^n - p \to \mathbb{R}^n$ as the stereographic projection, which is a homeomorphism. Similarly, $S^n - q$ is also homeomorphic to $\mathbb{R}^n$. Taking $U = S^n - p$ and $V = S^n - q$ and $X = S^n$ and noting that $S^n - p - q$ is path connected for $n \geq 2$, and are both simply connected being homeomorphic to $\mathbb{R}^n$, hence the result follows by the corollary. □

§§6.10. Fundamental Groups of some surfaces

A surface is basically a Hausdorff space with a countable basis, and every point has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^2$.

Theorem 6.44. $\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ are projection mappings, then define the homomorphism between the above 2 spaces as $\Phi([f]) = p_*([f]) \times q_*([f])$. Then map is surjective, since for any 2 loops g, h based in X, Y at x_0, y_0 the loop $f = g(s) \times h(s)$ is based in $X \times Y$ at $x_0 \times y_0$ with $\Phi([f]) = [g] \times [h]$. Also, the kernel of this homomorphism vanishes, since if $[p \circ f] \times [q \circ f] \simeq_p e_{x_0} \times e_{y_0}$, then $f \simeq_p e_{x0} \times e_{y0}$. □

The corollary follows as seeing that the fundamental group of the torus $T = S^1 \times S^1$ is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Definition 6.45. The projective plane P^2 is the quotient space obtained from S^2 by identifying each point $x \in S^2$ with the antipodal point $-x$.

Theorem 6.46. P^2 is a compact surface, and the quotient map $p : S^2 \to P^2$ is a covering map.

Proof. Note that p is an open map, since $p^{-1}(p(U)) = U \cup a(U)$, is also open (from the definition of a quotient map), where $a : S^2 \to S^2$ with $a(x) = -x$. Similarly, it is also a closed map. If we take such a U that no pair of equivalent antipodal points are in it, then $p : U \to p(U)$ is bijective, continuous and open, making it a homeomorphism. Similarly for $p(a(U))$, and hence we see that it is a covering map, with the disjoint open sets $U \cap a(U)$ homeomorphically mapped to the image. P^2 also hence has countable basis, and Hausdorff nature and P^2 being a surface follows in the same way. □

Thus, by a previous theorem (6.22) we see that there is a bijective map between $\pi_1(P^2, y)$ and $p^{-1}(y)$, the latter being a 2 element group, hence the former is a group of order 2 and isomorphic to $\mathbb{Z}/2$.

Theorem 6.47. The fundamental group of the figure eight is not abelian.

Proof. If $X = A \cap B$, which are the 2 circles at the common point x_0. Let E be the subspace of $\mathbb{R}^2$ consisting of the x, y axis and tiny circles tangent to the axis at each nonzero integer point. $p : E \to X$ maps the x axis and circles tangent to the y axis onto A homeomorphically, and maps y axis and circles tangent to the x axis onto B homeomorphically, and hence is a covering map. We take $\tilde{f} : I \to E$ along x axis to $(1, 0)$ and $\tilde{g}$ along y axis to $(0, 1)$. Thus $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$ are loops based at x_0 going along A and B. The lifting of $f * g$ goes along x axis and the circle tangent to it, whereas $g * f$ goes along y axis and circle tangent to it, thus are not abelian. □

We can see that the figure 8 X is a retract of the double torus $T\#T$, and hence the inclusion map between the fundamental groups is injective, so that $\pi_1(T\#T, x_0)$ is not abelian.

§7. Seperation Theorems in the Plane

§§7.1. Jordan Separation Theorem

Lemma 7.1. Let C be a compact subspace of S^2, with $b \in S^2 - C$, and h homeomorphism of $S^2 - b$ with $\mathbb{R}^2$. If U is a component of $S^2 - C$ and does not contain b, then $h(U)$ is a bounded component of $\mathbb{R}^2 - h(C)$, and if it contains b, then $h(U - b)$ is an unbounded component.

Proof. We see that if U is a component as above, then $U - b$ is connected, since if not, and A, B form a separation of $U - b$, then considering a small neighbourhood W around b, $W - b$ is homeomorphic to some open subset of $\mathbb{R}$, and hence is connected and contained in A. b cannot be a limit point of B and hence $A \cup \{b\}$ and B form a separation of U. Also if $\{U_\alpha\}$ are the components of $S^2 - C$, then $V_\alpha = h(U_\alpha - b)$ are components of $\mathbb{R}^2 - h(C)$. Now we extend h to $H : S^2 \to \mathbb{R}^2 \cup \infty$, with $H(b) = \infty$ and the last part of the lemma follows. □

Lemma 7.2. If $a, b \in S^2$, and A is a compact space, and let $f : A \to S^2 - a - b$ be a continuous map. If a, b are in the same component of $S^2 - f(A)$, then f is nullhomotopic.

Proof. We replace S^2 as $\mathbb{R}^2 \cup \infty$, and $a, b \mapsto 0, \infty$, then the lemma is equivalent to $g : A \to \mathbb{R}^2 - 0$ and 0 lying in the unbounded component, then g is nullhomotopic. We consider a large enough ball B containing $g(A)$, and p not in B. Then 0 and p are in the same unbounded component, and let $\alpha(t)$ be the path between 0 and p. Define $G : A \times I \to \mathbb{R}^2 - 0$ as $G(x, t) = g(x) - \alpha(t)$ (this is never 0 since α is not in B containing $g(A)$). Now we define $H : A \times I \to \mathbb{R}^2 - 0$ as $H(x, t) = tg(x) - p$ (non zero due to the same reason as above). Hence g is nullhomotopic. □

Definition 7.3. If X is connected, and $A \subset X$, then A separates X if $X - A$ is not connected. An arc A is a space homemorphic to $[0, 1]$. The endpoints of A are 2 points p, q of A such that $A - p$ and $A - q$ are connected, other points are called interior points. A simple closed curve is a space homeomorphic to S^1.

Theorem 7.4. The Jordan Separation Theorem : Let C be a simple closed curve in S^2, then C separates S^2.

Proof. Let $C = A_1 \cup A_2$ for 2 arcs, that intersect only in endpoints a, b, X denote $S^2 - a - b$, $U = S^2 - A_1$ and $V = S^2 - A_2$. Note that $X = U \cup V$ and $U \cap V = S^2 - C$, and by a previous theorem with $x_0 \in U \cap V$, the mapping of the inclusions $i : (U, x_0) \to (X, x_0)$ and $j : (V, x_0) \to (X, x_0)$ will induce $\pi_1(X, x_0)$. Take a loop $f : I \to U$ based at x_0, $p : I \to S^1$ is the standard loop and these induce continuous $h : S^1 \to U$ such that $h \circ p = f$. Note that $i \circ h : S^1 \to S^2 - a - b$, does not intersect connected A_1, which thus lies in the same component of $S^2 - h(S^1)$, and hence $i \circ h$ is nullhomotopic, leading to $(i \circ h)_*$ being the trivial homomorphism and using $(i \circ h)_*([p]) = [i \circ h \circ p] = [i \circ f] = i_*([f])$, the latter is trivial. Same proof goes for j_*, and thus $\pi_1(X, x_0)$ must be trivial, but X is homeomorphic to the punctured plane, and cannot have a trivial fundamental group. □

This generalizes to the following with the same proof -

Theorem 7.5. Let A_1 and A_2 be closed connected subsets of S^2 whose intersection consists of a and b only. Then $C = A_1 \cup A_2$ separates S^2.

§§7.2. Domain Invariance

Lemma 7.6 (Homotopy extension lemma)**.** Let X be a space such that $X \times I$ is normal. If A is a closed subspace of X and $f : A \to Y$ is a continuous map, with Y open, and if f is nullhomotopic, then it can be extended to a continuous map $g : X \to Y$ which is also nullhomotopic.

Proof. If $F : A \times I \to Y$ is a homotopy between f and a constant map, and we extend F to the space $X \times 1$ by setting it as the constant. Since F is a continuous map from $(A \times I) \cup (X \times 1)$, a closed subspace, by the Tietze extension theorem, it extends to $G : X \times I \to \mathbb{R}^n$. If we take $U = G^{-1}(Y)$ as an open subset, and I being compact, there is now an open set W containing A such that $W \times I \subset U$. With X being normal, there is a continuous function $\phi : X \times [0, 1]$ such that $\phi(x) = 0$ on $x \in A$ and $\phi(x) = 1$ on $x \in X - W$. Then $g(x) = G(x, \phi(x))$ carries X into Y, and for $x \in A$, $g(x) = f(x)$, with $H(x, t) = G(x, (1 - t)\phi(x) + t)$ being the homotopy between g and a constant map. □

Lemma 7.7 (Borsuk Lemma)**.** Let a, b be points of S^2. Let A be a compact space and let $f : A \to S^2 - a - b$ be a continuous injective map. If f is nullhomotopic, then a, b lie in the same complement of $S^2 - f(A)$.

Proof. The lemma is equivalent to: If A is a compact subspace of $\mathbb{R}^2 - 0$, and the inclusion $j : A \to \mathbb{R}^2 - 0$ is nullhomotopic, then 0 lies in the unbounded component of $\mathbb{R}^2 - A$. If C is the component containing 0, we assume C is bounded and let $\mathbb{R}^2 - A = C \cup D$. Define $h : \mathbb{R}^2 \to \mathbb{R}^2 - 0$ continuous equalling the identity outside C. If j is the inclusion map, and nullhomotopic, then by previous lemma, j is extended to k on $C \cup A$, which we then extend to a map $h\mathbb{R}^2 \to \mathbb{R}^2 - 0$ by setting it to identity on $D \cup A$, which is continuous by pasting lemma. If B is large enough ball containing $C \cup A$ (as we assume C is bounded), restricting h to B, we observe it is identity on the boundary, then we get a standard retraction of B into the boundary, which doesnt exist. □

Theorem 7.8 (Invariance of domain)**.** If U is an open subset of $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ is continuous and injective, then $f(U)$ is open and the inverse function f^{-1} is continuous.

Proof. we replace $\mathbb{R}^2$ by S^2. Then if B is any closed ball contained in U, and a, b are 2 points in $S^2 - f(B)$, with the identity map and its restriction to $h : B \to S^2 - a - b$ nullhomotopic, implies a, b are in the same component, thus $f(B)$ does not separate S^2. Now, we know that $C = f(Bd(B))$

is a simple closed curve, thus it separates S^2. We take V to be the set containing $f(IntB)$, and if we take a point a in V and not in $f(IntB)$, and b be a point of the other component, since $f(B)$ does not separate S^2, a, b lie in the same component of $S^2 - f(B)$, which is contained in $S^2 - f(BdB)$, a, b are in the same component of the latter, which gives a contradiction. Hence $f(IntB) = V$ and is open in S^2. Thus, for any ball B contained in U, $f(IntB)$ is open in S^2, making $f : U \to S^2$ an open map, from which $f(U)$ open and f^{-1} continuous follows. □

§§7.3. Jordan Curve theorem

Theorem 7.9. Let $X = U \cup V$ with both sets open, and $U \cap V$ can be written as the union of 2 disjoint open sets A and B. Take α to be the path in U from $a \in A$ to $b \in B$ and a path β in V from b to a. Let $f = \alpha * \beta$.

- $[f]$ generates an infinite cyclic subgroup of $\pi_1(X, a)$, and
- If $\pi_1(X, a)$ is itself infinite cyclic and generated by $[f]$, and
- If there is another path γ in U from a to a' in A, and a path δ in V from a' to a and $g = \gamma * \delta$, then the subgroups generated by $[f]$, $[g]$ intersect only in the identity.

Proof. We first construct E. We call Y to be the union of $U \times 2n$ and $V \times (2n+1)$ for all integer n, and E as the quotient space identifying $x \times (2n)$ and $x \times (2n-1)$ for $x \in A$ and $x \times 2n$ and $x \times (2n+1)$ for $x \in B$, and $\pi : Y \to E$ is the quotient map. The map $rho : Y \to X$ induces continuous and surjective map $p : E \to X$. We note that it is an open map by proving $\pi^{-1}\pi(W \times 2n)$ for any W open in U is open. Since the set $p^{-1}(U)$ is the union of disjoint, open and homeomorphically mapped sets $\pi(U \times 2n)$, hence p is a covering map. We then let $e_n = \pi(a \times 2n)$ and define liftings $\tilde{f}_n$ that begin at e_n and end at e_{n+1} and define $\tilde{\alpha}_n(s) = \pi(\alpha(s) \times 2n)$ and $\tilde{\beta}_n(s) = \pi(\beta(s) \times (2n+1))$ as the liftings of α, β, and $\tilde{f}_n = \tilde{\alpha}_n * \tilde{\beta}_n$. From this we directly note that since none of the $[f]^m$ are trivial, it generates an infinite cyclic group. We then define the lifting correspondence $\phi : \pi_1(X, a) \to p^{-1}(a)$ which is surjective and by a previous theorem, induces a injective map from the quotient group, the kernl being trivial else the quotient will be finite. Thus ϕ is bijective, and equals all of $\pi_1(X, a)$. For the last part we define a lifting as $\tilde{\gamma}(s) = \pi(\gamma(s) \times 0)$ and $\tilde{\delta(s)} = \pi(\delta(s) \times 1)$, hence $\tilde{g}$ is a loop in E, and thus can never equal an integer power of $[f]$. □

Theorem 7.10. Let D be an arc in S^2, then D does not separate S^2.

Proof. Since D is contractible, the identity map is nullhomotopic, and a, b are 2 points not in D, then the inclusion map is nullhomotopic, then by the Borsuk lemma implies a, b are in the same component. □

We can generalize this to prove that -

Theorem 7.11. Let D_1 and D_2 be closed sunsets of S^2 such that $S^2 - D_1 \cap D_2$ is simply connected. If neither D_1 nor D_2 separates S^2 then $D_1 \cup D_2$ does not separate S^2.

Theorem 7.12 (The Jordan Curve Theorem). If C is a simple closed curve in S^2, then C separates S^2 into 2 components W_1, W_2, each of which has C as the boundary.

Proof. Let $C = C_1 \cup C_2$, intersecting in p, q, with $X = S^2 - p - q$, $U = S^2 - C_1$ and $V = S^2 - C_2$. We assume that A_1, A_2 and the rest of the components union B are the components of $U \cap V$. We then let $a \in A_1$, $a' \in A_2$ and $b \in B$ and consider the loops as in the previous theorem as $f = \alpha * \beta$ and $g = \gamma * \delta$. Now taking $U \cap V$ to be the union of open $A_1 \cup A_2$ and B, and then A_1 and $A_2 \cup B$, we see $[f], [g]$ are both nontrivial, which is a contradiction. Now, finding a path α in $S^2 - C_2$ joining 2 points a, b of W_1, W_2, and must contain a point lying in C_1, with the reverse inclusion obvious, and hence C is the boundary for both. □

The proof above leads us to the generalization as -

Theorem 7.13. If C_1, C_2 are closed connected subsets of S^2 whose intersection consists of 2 points. If neither C_1 nor C_2 separate S^2, then $C_1 \cup C_2$ separates S^2 into 2 components.

§§7.4. Imbedding graphs (kinky) in a plane

Definition 7.14. A finite linear graph G is a Hausdorff space that is written as the union of finitely many arcs, each pair of which intersect at atmost 1 end point. The arcs are called edges and the endpoints are vertices. If G has an edge for every pair of distinct vertices then it is called a complete graph of n vertices, denoted G_n. A theta space is a Hausdorff space written as union of arcs A, B, C, each pair intersecting at their endpoints.

Theorem 7.15. If X is a theta space, and A, B, C are the arcs with union X, then X separates S^2 into 3 components, with boundaries $A \cup B, B \cup C, C \cup A$ respectively, and $A \cup B$ is the boundary of one of the components of $S^2 - A \cup B$.

Proof. If a, b be the endpoints of the arcs, then $A \cup B$ separates into U, U' and now consider $\bar{U} = U \cup A \cup B$ and C, the intersection of which are 2 points, and union separates into say V, W, then $S^2 - A \cup B \cup C$ is the union of 3 disjoint connected sets U, V, W and the boundary relation follows. □

Theorem 7.16. Let X be a utilities graph, where we have to join h_1, h_2, h_3 to e, g, w without intersecting, it cannot be imbedded in the plane.

Proof. Let $A = gh_1w, B = gh_2w, C = gh_3w$, note that the union of these arcs is a theta space. Thus, it is separated into 3 components, U, V, W with boundaries $A \cup B, B \cup C, C \cup A$. Then e must lie in one of these 3 components, it cannot be in U, since h_3 is not in that, and similarly for others. Hence we reach a contradiction. □

Theorem 7.17. If X is a subspace of S^2 that is a complete graph with 4 vertices, then it separates S^2 into 4 components, the boundaries of which are X_i, the union of those edges which do not have a_i as a vertex.

Proof. Set Y as the theta space by setting the arc $A = a_1a_2a_3, B = a_1a_3, C = a_1a_4a_3$, which separates S^2 into 3 components U, V, W and note that $a_2a_4 - a_2 - a_4$ being connected must lie in W, and note that a_2a_4 and $\bar{U} \cup \bar{V}$ intersect in a_2a_4, both are connected and do not separate, hence their union separates into W_1, W_2. Thus, 4 connected components are formed and the boundaries follow from the symmetry condition. □

Theorem 7.18. The complete graph on 5 vertices cannot be imbedded in the plane.

Proof. Let G be a 5 vertex graph, and X be the complete 4 vertex graph that doesn't contain one of the vertices in the former. Thus X separates into 4 components, and a_5 must be in one of them, thus the connected space $a_1a_5 \cup a_2a_5 \cup a_3a_5 \cup a_4a_5$ lies in the boundary of the component, a contradiction. □

§§7.5. Winding number of a simple closed curve

Definition 7.19. If $h : S^1 \to \mathbb{R}^2 - 0$ is a continuous map, then the induced homomorphism h_* carries the generator of $\pi_1(S_1, s_0)$ to some integral power of a generator of $\pi_1(\mathbb{R}^2 - 0, r_0)$, and the power is known as the winding number.

Lemma 7.20. If G is a subspace of S^2, a complete graph of 4 vertices and C be the subgraph $a_1a_2a_3a_4a_1$, a simple closed curve. If p, q are interior points f a_1a_3 and a_2a_4 respectively, then -

- p, q lie in different components of $S^2 - C$, and
- Inclusion $j : C \to S^2 - p - q$ induces an isomorphism of fundamental groups.

Proof. Note that $C \cup a_1a_3$ separates $S^2 - C$ into U, V, W, the latter having C as the boundary, and note that since $a_2a_4 - a_2 - a_4$ belongs in W, thus $q \in W$, and p is not in W, since it is in the theta space. The first result follows.
Let $X = S^2 - p - q$, and choose x, y interior to a_1a_2, a_3a4 respectively, with $\alpha = xa_1a_4y$ and $\beta = ya_3a_2x$, then $\alpha * \beta$ is a loop in C. Let $D_1 = pa_3a_2q, D_2 = qa_4a_1p$, with $U = S^2 - D_1$ and $V = S^2 - D_2$, then $X = U \cup V$ and $U \cap V = S^2 - D$, with α a path in U from x to y and β is a path in V from y to x, satisfying the hypothesis of Thm 7.9, leading to $\alpha * \beta$ being the generator of the fundamental group of X, making j_* surjective, and hence an isomorphism. □

Theorem 7.21. If C is a simple closed curve in S^2, and p, q lie in different of $S^2 - C$, then the inclusion $j : C \to S^2 - p - q$ induces an isomorphism of fundamental groups.

Proof. If a, b, c are distinct points of $\mathbb{R}^2$, if A is an arc with endpoints a, b and B is an arc with endpoints b, c then we can see that there is an arc in $A \cup B$ with endpoints a, c. We also note that if U is an open set of $\mathbb{R}^2$, any 2 points of U that can be connected by a path in U are the endpoints of an arc lying in U. Now take C, and assume that 0 lies in the bounded component. For the x axis, let a_1 be the largest point on negative x axis and a_3 be the smallest positive axis point, both in C. Let a be any point in the unbounded component. Now take paths $\alpha : I \to \mathbb{R}^2 - C_1$ and $\beta : I \to \mathbb{R}^2 - C_2$, where $C = C_1 \cup C_2$, from a to 0. Let α, β intersect C in a_2, a_4, and by the previous step, has an arc connecting a_2, a_4, and thus we have formed a subspace G which is a complete graph of 4 vertices, with C as a subgraph $a_1 a_2 a_3 a_4 a_1$. By the previous lemma, we get the isomorphism between some p, q in lying in the different components of $\mathbb{R}^2 - C$, and the result for any 2 such points in the components follows simply by translation arguments. □

§§7.6. Cauchy Integral Formula

Definition 7.22. If f is a loop in $\mathbb{R}^2$, and a is a point not in its image, then with $g(s) := \frac{f(s)-a}{\|f(s)-a\|}$, and hence g is a loop in S^1. If $p : \mathbb{R} \to S^1$ is the standard covering map, and $\tilde{g}$ is the lifting of g to S^1, then the difference $\tilde{g}(1) - \tilde{g}(0)$ is an integer, called the winding number of f wrt a, denoted by $n(f, a)$.

Definition 7.23. Let $F : I \times I \to X$ is a continuous map such that $F(0, t) = F(1, t)$, then for each t, $f_t(s)$ is a loop in X, and the map F is called a free homotopy, which is a homotopy between loops where the base point is allowed to move.

Theorem 7.24. If f is a loop in $\mathbb{R}^2 - a$ then -

- If $\bar{f}$ is the reverse of f, then $n(\bar{f}, a) = -n(f, a)$, which follows from replacing $s \mapsto 1 - s$, giving us the opposite sign in $\tilde{g}(1) - \tilde{g}(0)$, and
- If f is freely homotopic to f', then $n(f, a) = n(f', a)$, following from defining $G(s, t) = \frac{F(s,t)-a}{\|F(s,t)-a\|}$, a continuous map from the interval to the integers, and hence must be constant, and
- If a, b lie in the same component of $\mathbb{R}^2 - f(I)$, then $n(f, a) = n(f, b)$, which follows from defining a path α between a, b and the resulting $f(s) - \alpha(t)$ as the free homotopy.

Definition 7.25. Let f be a loop in X, which we call a simple loop provided $f(s) = f(s')$ iff $s = s'$ or they are $0, 1$. For such a simple loop, its image is a simple closed curve in X.

This, along with the theorems of the previous section, lead us to naturally see that -

Theorem 7.26. Let f be a simple loop in $\mathbb{R}^2$, If a lies in the unbounded component of $\mathbb{R}^2 - f(I)$, then $n(f, a) = 0$, but if it is in the bounded component then $n(f, a) = \pm 1$.

Lemma 7.27. If f is a piecewise differentiable loop in the complex plane, and a is a point not in the image of f, then

$$n(f,a) = \frac{1}{2\pi i}\int_f \frac{dz}{z-a}.$$

Proof. This just follows from defining $r(s) = \|f(s) - a\|$ and $\theta(s) = 2\pi\tilde{g}(s)$, which gives us $\int_f \frac{dz}{z-a} = \int_0^1 \frac{r'e^{i\theta}+ir\theta'e^{i\theta}}{re^{i\theta}}ds = i[\theta(1)-\theta(0)] = 2\pi i(\tilde{g}(1)-\tilde{g}(0))$. □

From this, the Cauchy integral formula follows -

Theorem 7.28. If C is a simple closed piecewise differentiable curve in the complex plane, and B is the bounded component of $\mathbb{R}^2 - C$. If $F(z)$ is analytic in an open set Ω containing B, C, then for each point $a \in B$,

$$F(a) = \pm\frac{1}{2\pi i}\int_C \frac{F(z)}{z-a}dz$$

(the $\pm$ follows from taking the appropriate orientation.

§8. The Seifert-van Kampen Theorem

Definition 8.1. We say subgroups $\{G_\alpha\}$ generate G if every element $x \in G$ can be written as a finite sum of elements of groups G_α. If the groups G_α generate G, we say that G is the sum of the groups G_α, with $G = \sum_{\alpha \in J} G_\alpha$. If the expression for each x is unique, i.e., only finitely many single J tuple such that $x = \sum x_\alpha$, then G is said to be a direct sum of these groups, with $G = \oplus_{\alpha \in J} G_\alpha$.

Lemma 8.2. Let G be an abelian group and $\{G_\alpha\}$ be a family of subgroups of G. If G is their direct sum, then given any abelian group H and any family of homomorphisms $h_\alpha : G_\alpha \to H$, then there exists a unique homomorphism $h : G \to H$ whose restriction to $\{G_\alpha\}$ is h_α for each α. The converse also holds.

Proof. We show the converse by letting $x = \sum x_\alpha = \sum y_\alpha$, and let $H = G_\beta$, with h_α as the trivial homomorphism for $\alpha \neq \beta$ identity otherwise. If an extension exists, then $h(x) = \sum h_\alpha(x_\alpha) = x_\beta = \sum h_\alpha(y_\alpha) = y_\beta$, hence G is the direct sum. If G is the direct sum, given h_α, if $x = \sum x_\alpha$, set $h(x) = \sum h_\alpha(x_\alpha)$ it is thus finite and unique, and the restriction on G_α obviously equals h_α on G_α. $\square$

The corollary follows as : If $G = G_1 \oplus G_2$, with G_1 as the direct sum of of H_α and G_2 is the direct sum of H_β, being disjoint, then G is the direct sum of the subgroups H_γ, and implies associativity of the direct sum.
Another corollary states that if $G = G_1 \oplus G_2$, then G/G_2 is isomorphic to G_1.

Definition 8.3. Let G_α be an indexed family of abelian groups, and G is an abelian group. and $i_\alpha : G_\alpha \to G$ is a monomorphism, such that G is the direct sum of $i_\alpha(G_\alpha)$, then G is the external direct sum of G_α relative to i_α.

Theorem 8.4. Given a family of abelian groups $\{G_\alpha\}$ there exists an abelian group G and a family of monomorphisms $i_\alpha : G_\alpha \to G$ such that G is the direct sum of $i_\alpha(G_\alpha)$.

Proof. Consider the cartesian product $\prod_{\alpha \in J} G_\alpha$ (an abelian group under coordinate wise addition) and G is the subgroup which consists of x_α such that $x_\alpha = 0_\alpha$ for all but finitely many α. We then define the monomorphism $i_\beta(x)$ as the tuple with x as the β coordinate, and identity otherwise. Since $x \in G$ has finitely many nonzero elements, it can be written uniquely as a finite sum of $i_\beta(G_\beta)$. $\square$

From a similar proof as previously in the section, we conclude -

Lemma 8.5. Let $\{G_\alpha\}$ be an indexed family of abelian groups, and G be an abelian group, with $i_\alpha : G_\alpha \to G$ be a family of homomorphisms, if each i_α is a monomorphism and G is the direct sum of the groups $i_\alpha(G_\alpha)$, then given any abelian group H and family of homomorphisms $h_\alpha : G_\alpha \to H$, then there exists a unique homomorphism $h : G \to H$ such that $h \circ i_\alpha = h_\alpha$. As beofre, the converse also holds.

Now, applying this lemma to get an identity map of G, we get the uniqueness of direct sums as-

Theorem 8.6. Let G_α be a family of abelian groups. Suppose G and G' are abelian groups and $i_\alpha : G_\alpha \to G$ and $i'_\alpha : G_\alpha \to G'$ are families of monomorphisms, such that G is the direct sum of $i_\alpha(G_\alpha)$ and G' is the direct sum of $i'_\alpha(G_\alpha)$. Then there is a unique isomorphism $\phi : G \to G'$ such that $\phi \circ i_\alpha = i'_\alpha$ for each α.

Definition 8.7. Let G be an abelian group and let $\{a_\alpha\}$ be an indexed family of elements of G and G_α be the subgroup of G generated by a_α. If G_α generate G, with each being infinite cyclic, and G is thier direct sum, then G is called a free abelian group having $\{a_\alpha\}$ as basis.

Lemma 8.8. Let G be an abelian group, with $\{a_\alpha\}$ be a family of elements of G that generates G. Then G is a free abelian group with this basis iff for any abelian group H and any family $\{y_\alpha\}$ of elements of H, there is a unique homomorphism h of G into H such that $h(a_\alpha) = y_\alpha$ for each α.

Proof. For the $\Leftarrow$, if we assume a_β to generate a finite cyclic group, then using $H = \mathbb{Z}$, there can be no homomorphism mapping a_β to 1 due to order, the direct sum follows from the first lemma. The $\Rightarrow$ holds by the same lemma. □

A corollary of this is the theorem -

Theorem 8.9. If G is a free abelian group with basis $\{a_1, \dots, a_n\}$, then n is uniquely determined by G, and the number of elements in a basis is called the rank of G.

§§8.1. Free Products of Groups

Definition 8.10. Let G be a group. If $\{G_\alpha\}$ is a family of subgroups of G, these groups generate G if every element $x \in G$ can be written as a finite product of elements of the groups G_α, i.e., $x = x_1 \cdot \dots \cdot x_n$ is called a word of length n, and represents x. One can obtain a reduced word, a word representing x of the form $y_1 \cdot \dots \cdot y_m$, where no group G_α contains both y_i, y_{i+1}, and where $y_i \neq 1$ for all i. The identity element is of length 0.

Definition 8.11. If G is a group and $\{G_\alpha\}$ be a family of subgroups generating G. Suppose $G_\alpha \cap G_\beta$ intersects in only identity for distinct α, β. We say that G is the free product of the groups G_α if for each $x \in G$, there is only one reduced word in the groups G_α that represents x, and we denote $G = \prod^*_{\alpha \in J} G_\alpha$.

We also note that for G to be the free product of these groups, the representation of 1 by the empty word must be unique.

Lemma 8.12. Let G be a group; let $\{G_\alpha\}$ be a family of subgroups of G. If G is the free product of these groups, then given any group H and and family of homomorphisms $h_\alpha : G_\alpha \to H$, there is a homomorphism $h : G \to H$ whose restriction equals h_α.

Proof. If h exists, then uniqueness just follows from $h(x) = \prod h_{\alpha_i}(x_i)$, where $x = (x_1 \dots x_n)$ is the representation of x. Now, given any word w, define $\phi(w) = h_{\alpha_1}(x_1)\dots h_{\alpha_n}(x_n)$, note that ϕ is well defined with ϕ of the empty word as 1 in H. From the reduction operations, we can see that if w' is also obtained from w, then $\phi(w') = \phi(w)$, thus if w is any word in the groups that represents x, then $h(x) = \phi(w)$. From this definition, we prove that h is a homomorphism, with $h(xy) = h(x)h(y)$. □

Definition 8.13. Let G_α be an indexed family of groups, G is a group, and $i_\alpha : G_\alpha \to G$ is a monomorphism, such that G is the free product of $i_\alpha(G_\alpha)$, then G is the external free product of G_α relative to i_α. G is of course not unique.

Theorem 8.14. Given a family G_α of groups, there exists G, a group and $i_\alpha : G_\alpha \to G$ is a family of monomorphisms, such that G is the free product of the groups $i_\alpha(G_\alpha)$.

Proof. We firstly assume G_α as disjoint, W the set of all reduced words in the elements of the groups G_α and $P(W)$ denote the set of all bijective functions in W. For each α and $x \in G_\alpha$, define $\pi_x : W \to W$ as $\pi_x(\phi) = (x)$, $\pi_x(w) = (x, x_1, \dots, x_n)$ if $\alpha_1 \neq \alpha$, $\pi_x(w) = (xx_1, \dots, x_n)$ if $\alpha_1 = \alpha$ and $x_1 \neq x^{-1}$ and $\pi_x(w) = (x_2, \dots, x_n)$ if $\alpha_1 = \alpha$ and $x_1 = x^{-1}$ and if $x = 1_\alpha$ then π_x as the identity map. Using the cases of the empty word, $\alpha \neq \alpha_1$, $\alpha = \alpha_1$ and $yx_1 \neq 1_\alpha$, and $\alpha = \alpha_1$ and $yx_1 = 1_\alpha$ we conclude that if $x, y \in G_\alpha$ and $z = xy$, then $\pi_z = \pi_x \circ \pi_y$. We also note that it is a bijective mapping by definition, and hence is an element of $P(W)$. We also note that if $i_\alpha(x) := \pi_x$, then it is a homomorphism and $x \neq 1\alpha \implies \pi_x(\phi) = (x)$ shows that π_x is not an identity map, and hence i_α is a monomorphism. Now taking G to be the subgroup of $P(W)$ generated by $G_\alpha; = i_\alpha(G_\alpha)$, noting that $G'_\alpha \cap G'_\beta$ consisnt of only the identity by applying it on ϕ, and similarly also proving that no non empty reduced word is the identity, we conclude that G is a free product of the groups G'_α. □

The lemmas for extension conditions of external free products and the uniqueness of free products follow from similar proofs as above -

Lemma 8.15. Let $\{G_\alpha\}$ be an indexed family of groups, and G be a group, with $i_\alpha : G_\alpha \to G$ be a family of homomorphisms, if each i_α is a monomorphism and G is the free product of the groups $i_\alpha(G_\alpha)$, then given any group H and family of homomorphisms $h_\alpha : G_\alpha \to H$, then there exists a unique homomorphism $h : G \to H$ such that $h \circ i_\alpha = h_\alpha$.

Theorem 8.16. Let G_α be a family of groups. Suppose G and G' are groups and $i_\alpha : G_\alpha \to G$ and $i'_\alpha : G_\alpha \to G'$ are families of monomorphisms, such that $i_\alpha(G_\alpha)$ and $i'_\alpha(G_\alpha)$ generate G and G'. If both G and G' have the previous extension property, then there is a unique isomorphism $\phi : G \to G'$ such that $\phi \circ i_\alpha = i'_\alpha$ for each α.

Using these 2 results, by a very similar proof done as for the above, we conclude the next 2 results as -

Lemma 8.17. Given a family G_α of groups, and a group G, a group and $i_\alpha : G_\alpha \to G$ as a family of homomorphisms, then if the extension condition holds, then each i_α is a monomorphism, and G is the free product of $i_\alpha(G_\alpha)$.

This leads to the corollary: If $G = G_1 * G_2$, where G_1 is the free product of H_α and G_2 is the free product of H_β, the index sets being disjoint, then G is the free product of the subgroups H_γ, over the union of the index sets, hence also implying associativity.

Theorem 8.18. Let $G = G_1 * G_2$, and N_i is a normal subgroup of G_i, and if N is the least normal subgroup of G containing N_1, N_2, then $G/N \cong (G_1/N_1) * (G_2/N_2)$.

Proof. Note that the composite of the inclusion and projection homomoprhisms $G_1 \to G_1 * G_2 \to (G_1 * G_2)/N$, carries N_1 to the identity, and hence induces the homomorphism $i_1 : G_1/N_1 \to (G_1 * G_2)/N$, and similarly $i_2 : G_2/N_2 \to (G_1 * G_2)/N$. Let H be any group, and $h_1 : G_1/N_1 \to H, h_2 : G_2/N_2 \to H$ be arbitrary homomorphisms, and consider the composite of $G_i \to G_i/N_i \to H$, which carries N_1, N_2 to identity , and thus induces a homomorphism $h : (G_1 * G_2)/N \to H$ satisfying $h_j = h \circ i_j$, thus making i_α monomorphisms, and hence $G_1 * G_2/N$ is the external free product relative to these monomorphisms. □

§§8.2. Free Groups

Definition 8.19. A family of elements a_α generate G if every element of G can be written as a product of powers of these elements. Suppose each a_α generates an infinite cyclic subgroup G_α of G. If G is the free product of these groups, then G is said to be a free group, and the family a_α is called a system of free generators for G.

Exact proofs as above lead us to conclude the uniqueness of the extension homomorphism and and the converse, along with the associativity of system of free generators.

Definition 8.20. If a_α is an arbitrary indexed family, and G_α denote set of symbols of the form a_α^n, and we make it into a group, and the external free products of G_α is called the free group on the elements a_α.

Definition 8.21. If G is a group, and $x, y \in G$ we denote the commutator of x, y as $[x, y] = xyx^{-1}y^{-1}$, and the subgroup G generated by set of all commutators in G is called the commutator subgroup of G and denoted $[G, G]$.

Lemma 8.22. Given G, the subgroup $[G, G]$ is normal and the quotient group $G/[G, G]$ is abelian. If $h : G \to H$ is any homomorphism from G to an abelian group H, then the kernel of h contains $[G, G]$, so h induces a homomorphism $k : G/[G, G] \to H$.

Proof. We show that any conjugate of a commutator is in $[G, G]$, by computing $g[x, y]g^{-1} = [gx, y] \cdot [y, g]$, which is in $[G, G]$. We see that for any arbitrary element $z = z_1 \ldots z_n \in [G, G]$, any conjugate can be written as $gzg^{-1} = gz_1g^{-1} \ldots gz_ng^{-1}$, and hence is in $[G, G]$, making it a normal subgroup. Noting that $a^{-1}b^{-1}ab[G, G] = [G, G]$, we get that $ab[G, G] = ba[G, G]$, thus making it abelian, and the last part follows from noting that h carries $[G, G]$ to the identity, since H is abelian. □

From the previous lemma it follows that -

Theorem 8.23. If G is a free group with free generators a_α, then $G/[G, G]$ is a free abelian group with basis $[a_\alpha]$, where $[a_\alpha]$ denotes the coset of a_α in $G/[G, G]$.

§§8.3. The actual Theorem

The modern version is stated as -

Theorem 8.24. If $X = U \cup V$, U, V open in X, $U, V, U \cap V$ are path connected, and $x_0 \in U \cap V$. If H is a group and $\phi_1 : \pi_1(U, x_0) \to H$ and $\phi_2 : \pi_2(V, x_0) \to H$ be homomorphisms and with the inclusions induced as shown below. If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there is a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, "\Phi"] & H \\
& \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Proof. Uniqueness follows from noticing that Φ is completely determined by ϕ_1 and ϕ_2. We first define ρ such that for each loop f based at x_0, $\rho(f) = \phi_1([f]_U)$ if f lies in U, and $\rho(f) = \phi_2([f]_V)$ if in V. Note it is well defined, and it satisfies $[f]_{U/V} = [g]_{U/V} \implies \rho(f) = \rho(g)$ and if both f, g

lie in U/V, then $\rho(f * g) = \rho(f) \cdot \rho(g)$.
We extend ρ to σ, with each path f in U or V, and each $x \in X$ choose a path α_x from x_0 to x, as constant if in $U \cap V$, or lying either in U, V Thus define a loop $L(f) = \alpha_x * f * \bar{\alpha}_y$ and define $\sigma(f) = \rho(L(f))$. We can check that it restricts to ρ for loops, and satisfies the above 2 conditions.
Now we extend σ to τ for any arbitrary path f in X. Given a f, we choose a subdivision $s_0 < \ldots < s_n$ such that each subinterval lies in U, V, and use $[f] = [f_1] * \ldots * [f_n]$. Note that it is well defined by homotopy, and it is an extension of σ. We show that if $[f] = [g]$, then $\tau(f) = \tau(g)$. If f, g are paths and F is a homotopy between them, then with the subdivisions as above, and defining $\beta_i = F(s_i, t)$ and noting that $f_i * \beta_i \sim_p \beta_{i-1} * g_i$ and it follows from the relation on $\sigma(f_i) = \sigma(\beta_{i-1}) \cdot \sigma(g_i) \cdot \sigma(\beta_i)^{-1}$. Now substituting this, we get $\tau(f) = \sigma(f_1) \cdot \ldots \cdot \sigma(f_n) = \tau(g)$.
Joinging 2 paths as before with subdivisons, and using the above, we get $\tau(f * g) = \tau(f) \cdot \tau(g)$.
Now define $\Phi([f]) = \tau(f)$, being a well defined homomorphism, and we finally get $\Phi \circ j_1 = \phi_1$ and the other relation follows similarly. □

The classical theorem is stated as -

Theorem 8.25. If the hypotheses of previous theorem are satisfied, and $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is a homomorphism extending j_1, j_2, the inclusion homomorphisms. Then j is surjective, and its kernel is the least normal subgroup N of the free product that contains all elements represented by words of the form $(i_1(g)^{-1}, i_2(g))$, for $g \in \pi_1(U \cap V, x_0)$.

Proof. Since $\pi_1(X, x_0)$ is generated by j_1, j_2, j is surjective. Since ker$j$ is normal, and $ji_1(g) = i_*(g) = ji_2(g)$, thus $i_1(g)^{-1}i_2(g)$ is in the kernel of j for each $g \in \pi_1(U \cap V, x_0)$, thus j induces an epimorphism $k : \pi_1(U, x_0) * \pi_1(V, x_0)/N \to \pi_1(X, x_0)$, and let H denote the former group. We see that with the hypotheses of the previous satisfied, there is a homomorphism $\Phi : \pi_1(X, x_0) \to H$, and if we taken any generator $g \in \pi_1(U, x_0)$, we have $\Phi \circ k(gN) = \Phi(j_i(g)) = gN$, and hence Φ is the left inverse of k, making k injective and hence $N = kerj$. □

We see as a corollary that if the hypotheses are satisfied and $U \cap V$ is simply connected, then the map $k : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is an isomorphism. However, if V is simply connected, then there is an isomorphism $k : \pi_1(U, x_0)/N \to \pi_1(X, x_0)$, where N is the least normal subgroup of $pi_1(U, x_0)$ containing the image of $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(X, x_0)$.

§§8.4. Fundamental Group of Wedge of Circle

Definition 8.26. If $X = S_1 \cup \ldots \cup S_n$ is a Hausdorff space, each of which is homeomorphic to the unit circle, and there is a point p such that $S_i \cap S_j = \{p\}$ for $i \neq j$, then X is called the wedge of the circles $s_1, \ldots, S_n$.

Theorem 8.27. If X is the wedge of the circles $s_1, \ldots, S_n$, and p is the common point, then $pi_1(X, p)$ is a free group, and f_i is the generator of $\pi_1(S_i, p)$, then the loops $f_1, \ldots, f_n$ are a system of free generators for $\pi_1(X, p)$.

Proof. Let q_i be a point in S_i different from p, and $W_i = S_i - q_i$ and set $U = S_1 \cup W_2 \ldots \cup W_n$ and $V = W_1 \cup S_2 \ldots \cup S_n$. Note that $U \cap V = W_1 \cup \ldots \cup W_n$, since each W_i is homeomorphic to the interval, having p as the deformation retract, and there is a deformation retraction of $U \cap V$ onto p. It follows that $U \cap V$ is simply connected and hence from the previous section corollary $\pi_1(X, p)$ is a free product of $\pi_1(U, p)$ and $\pi_1(V, p)$. Similarly, S_1 is a deformation retract of U and $S_2 \cup \ldots \cup S_n$ is of V. Thus, f_1 is a generator of the former, and by induction hypothesis, $pi_1(V, p)$ is a free group with $f_2, \ldots, f_n$ as generators, and the theorem then follows. □

Definition 8.28. Let X be a space as a union of X_α, the topology of X is said to be coherent with the subspaces X_α provided a subset C of X is closed in X if $C \cap X_\alpha$ is closed in X_α for each α. Let X be a space that is the union of the subspaces S_α, each of which is homeomorphic to the unit circle. If there is a point p of X as above, and the topology of X is coherent with the subspaces S_α, then X is called the wedge of the circles $S\alpha$.

Lemma 8.29. Let X be the wedge of S_α, then it is normal and any compact subspace of X is contained in union of finitely many circles S_α.

Proof. If A, B are disjoint closed sets in X, and if B does not include $\{p\}$, choose disjoint open subsets U_α, V_α containing $p \cup A \cap S_\alpha$ and $B \cap S_\alpha$, define $U = \cup U_\alpha, V = \cup V_\alpha$ are disjoint open sets, and hence X is normal. If C is compact, choose a point $x_\alpha \in C \cap (S_\alpha - p)$, this set $D = \{x_\alpha\}$ is closed in X, contained in C, and hence D must be finite. □

Theorem 8.30. Let X be the wedge of S_α, and p is the common point, then $\pi_1(X, p)$ is free group, and f_α is a loop in S_α as the generator of $\pi_1(S_\alpha, p)$ then the loops f_α represent a system of free generators for $\pi_1(X, p)$.

Proof. Let i_α be the homomorphism induced by inclusion and let G_α be the image of i_α. If f is any loop in X based at p, then the image set of f is compact, and lies in finite union of subspaces S_α, and $[f]$ is a product of elements of finite G_α, thus G_α generate $\pi_1(X, p)$ and i_α is a monomorphism from the previous theorem. Similarly, a reduced word cannot map to $\pi_1(X, p)$ according to the same. □

Lemma 8.31. Given an index set J, there exists a space X such that is a wedge of S_α for $\alpha \in J$.

Proof. Give J the discrete topology, define $E = S^1 \times J$, choose $b_0 \in S^1$, and X is the quotient space obtained from E by collapsing $b_0 \times J$ to p, and $\pi : E \to X$ is the quotient map, $S_\alpha = \pi(S^1 \times \alpha)$. Note that if C is closed in $S^1 \times \alpha$ then $\pi^{-1}(\pi(C))$ is either C or $C \cup b_0 \times J$, hence $\pi(X)$ is closed in X, thus S_α is closed in X, and hence π maps $S^1 \times \alpha$ homeomorphically to S_α. Let $D \subset X$ and $D \cap S_\alpha$ is closed for each α, note that $\pi^{-1}(D) \cap (S^1 \times \alpha) = \pi_\alpha^{-1}(D \cap S_\alpha)$, thus $\pi^{-1}(D)$ is closed in $S^1 \times \alpha$, which is closed in $S^1 \times J$, and hence D is closed in X by quotient topology. □

§§8.5. A useful Theorem

Theorem 8.32. Let X be Hausdorff, A a closed path connected subspace, a continuous map $h : B^2 \to X$ mapping bijectively $Int(B^2) \mapsto X - A$, and $S^1 \mapsto A$. If $p \in S^1$ and $a = h(p)$, let $k : (S^1, p) \to (A, a)$ is obtained by restricting h, then the homomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ induced by inclusion is surjective and its kernel is the least normal subgroup of $\pi_1(A, a)$ containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(A, a)$.

Proof. Let $x_0 = h(0)$, $U = X - x_0$ and $C = h(B^2)$. If $\pi : B \to C^2$ is the map by restricting h, then note that $\pi \times id$ is a closed map, thus a quotient map. Similarly, its restriction $\pi' : (B^2 - 0_\times I \to (C - x_0) \times I$ is also a quotient map. The deformation retract of $B^2 - 0$ onto S^1 induces a deformation retraction of $C - x_0$ onto $\pi(S^1)$, and we extend this to all of U by fixing it, thus A is a deformation retract of U.

Thus our theorem reduces to - Let f be a loop whose class generates $\pi_1(S^1, p)$. Then the inclusion of U into X induces an epimorphism $\pi_1(U, a) \to \pi_1(X, a)$, whose kernel is the least normal subgroup containing $[g] = [h \circ f]$.

Consider a point b not in A, a point of $U - A$, and the homomorphism $\pi_1(U, b) \to \pi_1(X, b)$ induced by the inclusion. Let $X = U \cup V$, where $V = X - A$, note that U is path connected, V is simply connected, and $U \cap V = V - x_0$ is path connected and homeomorphic to $Int(B^2) - 0$, and hence the FG is infite cyclic. By the previous section corollary, the homomorphism $\pi_(U, b) \to \pi_1(X, b)$ is surjective, and the kernel is the LNS containing the image of the infinite cyclic group $\pi_1(U \cap V, b)$.

Let q be a point such that $h(q) = b$, and f_0 be the loop based at q, generating the FG of this space, then $g_0 = h \circ f_0$ represents the generator of the FG of $U \cap V$. We then obtain the same result with a base point a, by defining γ between q and p, and $\delta = h \circ \gamma$, and noting that $\delta([g_0]) = [g]$. $\square$

§§8.6. Fundamental groups of the Torus and Dunce Cap

Theorem 8.33. The fundamental group of the torus has a presentation consisting of two generators α, β and a single relation $\alpha\beta\alpha^{-1}\beta^{-1}$.

Proof. Let $X = S^1 \times S^1$ be the torus, $h : I^2 \to X$ obtained by restricting the covering map as before. Let $p = (0, 0)$, $a = h(p)$ and $A = h(BdI^2)$, then the hypothesis of the previous theorem is satisfied, and since A is the wedge of 2 circles, so the fundamental group of A is free. Define the path along x axis on I^2 to be a_0, the one along y axis to be b_0, and the paths $\alpha = h \circ a_0$ and similarly $\beta = h \circ b_0$ are the generators of $\pi_1(A, a)$. Similarly consider a_1, b_1 parallel to a_0, b_0, define $f = a_0 * b_1 * \bar{a_1} * \bar{b_0}$, it is a generator of $\pi_1(BdI^2, p)$, thus by the previous theorem $\pi_1(X, a)$ is the quotient of the free group on the free gen $[\alpha], [\beta]$ with the least normal subgroup as $g = h \circ f = [\alpha][\beta][\alpha^{-1}][\beta^{-1}]$. $\square$

The corollary using a theorem from the subsection of free groups shows us that the fundamental group of the torus is a free abelian group of rank 2.

Definition 8.34. If n is a positive integer greater than 1, and r be the rotation of S^1 through $2\pi/n$, the quotient space X formed from B^2 by identifying $x \in S^1$ with points $r(x), \ldots, r^{n-1}(x)$, called the n-fold dunce cap. If $\pi : B^2 \to X$ is the quotient map, we can see that it is a closed map, and hence X is compact.

Lemma 8.35. If $\pi : E \to X$ is a closed quotient map, and E is normal, then X too is normal.

Proof. If E is normal, let A, B are disjoint closed sets of X, then $\pi^{-1}(A), \pi^{-1}(B)$ are disjoint closed sets of E, and hence have U, V disjoint open sets containing them. Now, we take closed sets $C = E - U$ and $D = E - V$ as closed sets, and by definition of a closed map, $\pi(C), \pi(D)$ are closed in X. Note that $U_0 = X - \pi(C)$ and $V_0 = X - \pi(D)$ are open sets containing A, B and are disjoint, and are the required sets, making X normal. □

Note that the 2 fold dunce cap is homeomorphic to the projective plane P^2!

Theorem 8.36. The fundamental group of the n-fold dunce cap is a cyclic group of order n.

Proof. Let $h : B^2 \to X$ be the quotient map, $A = h(S^1)$, $p = (1, 0)$ and $a = h(p)$. Note that the arc from p to $r(p)$ is mapped by h to A, making the latter homeomorphic to a circle, with infinite cyclic fundamental group. If $\gamma(t) = (\cos(2\pi t/n), \sin(2\pi t/n))$ is this path, then $\alpha = h \circ \gamma$ is the generator of $\pi_1(A, a)$, and $f = \gamma * (r \circ \gamma) * \ldots * (r^{n-1} \circ \gamma)$ generates $\pi_1(S^1, p)$, and $h \circ f = \alpha * \alpha \ldots * \alpha$, a n fold product, thus the result follows. □

Welcome to the end! Tis been a wild ride!