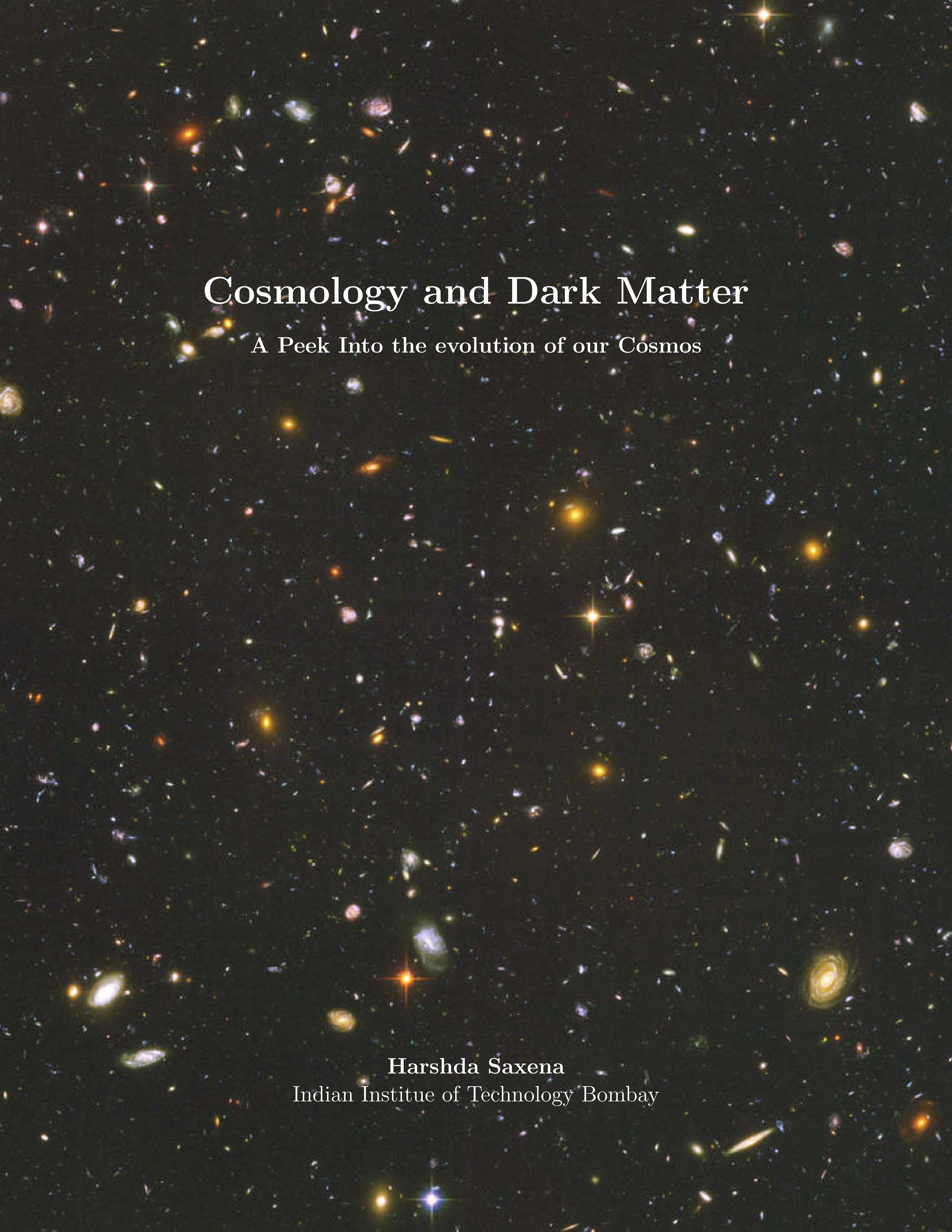

Cosmology and Dark Matter

A Peek Into the evolution of our Cosmos

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Contents

1 Introduction

This is a short mid-term report on Cosmology and Dark Matter under the SoS initiative of the MnP club, under my mentor Riya Singh, and in the year 2020. In the first half of the summer, I have completed-

- An Introduction to Modern Cosmology by Andrew Liddle
- Introduction to Tensor Calculus by Kees Dullemond and Kasper Peeters (first 5 chapters)
- Physics- Volume 1 by Resnick, Halliday and Krane (Chapter 20- The Special Theory of Relativity)

In this report, I would briefly summarize the important results, formulae and theory covered by me in the past month, and would encourage the reader to look for small proofs that I would have missed, due to the time/page limitations, in the books stated above. Let us now get right into the discussions of different Cosmological Models, initially from a Newtonian approach, after which we shall discuss some Tensor Calculus and Special Theory of Relativty, the pre-requisites of General Theory of Relativity, and finally into Modern Cosmology (to be completed in the next half).

2 A Brief Introduction to Our Cosmos

Cosmology is a branch of astronomy concerned with the studies of the origin and evolution of the universe, from the Big Bang, to today, and on into the future. It consists of 2 parts- Cosmogony (The origin of the universe) and Cosmography (mapping the features of the universe). In this part, we shall give a small overview of the sections to come, the details of which will be discussed in the next sections.

The cornerstone of Modern Cosmology is the belief that the place occupied by us in the universe is in no way special, and is known as the Cosmological Principle (it is also called as the Copernican Principle, but this is in fact a generalization, since he believed that the Sun was the centre). This then implies, that the Universe is globally homogeneous at the scale of the largest structures in the universe. Hence 2 properties of the Universe at the global scale include -

- Homogeneity - The Universe “looks”same at each point.
- Isotropy - The Universe “looks” the same in all directions.

2.1 Structures in the Universe

Since the structures we talk about are in astronomical terms, it makes sense to use a different scale of units, $\mathcal{M}_\odot$as the Mass of the Sun and 1 parsec (3.26 light years) as the distance unit. The smallest structure we will talk about are stars, a collection of which (and much more!) make a Galaxy, with an approximate distance of 1 Mpc between galaxies. The Sun, along with 400 billion other stars, reside in the Milky Way galaxy. A distance of about 100 Mpc shows a cluster of galaxies, which further group to form Superclusters, which are joined by filaments/walls of galaxies (the Milky way resides in the Virgo Supercluster), and lastly there are Cosmic voids, which contain very few galaxies, and hence are mostly “empty”.

These structures, can be "seen", and hence the light from them resides in the visible region of the EM spectrum, but other wavebands of light are equally important to obtain data. Microwaves are used for observing the Cosmic Microwave Background Radiation (CMBR) which tells us about the curvature of the universe and the density of matter in the universe, Radio waves for observing distant galaxies, Infrared for closer galaxies due to less scattering, and X rays for clusters of galaxies.

The universe is thought to be made up of mostly 4 kinds of particles -

- **Baryons :** A general term for particles made up of 3 quarks (quarks are considered to be one of the elementary particles in the Standard Model). These are mostly non-relativistic particles (rest-mass energy much more than their kinetic energy) which consist of protons and neutrons, and to the annoyance of particle physicists, electrons too. These make the atoms which make up majority of the structures of the universe.
- **Radiation :** These are relativistic particles, having rest mass 0, and are called photons. These interact with baryons through Thompson Scattering (at low energies) and Compton Scattering (at high energies).
- **Neutrinos :** These are highly relativistic particles, have a very low mass due to which they can be assumed as massless, and are very weakly interacting particles. They are mostly created in radioactive processes.
- **Dark Matter and Dark Energy :** These are proposed to make up 95 % of the matter-energy content of the universe, but their actual composition remains a mystery.

2.2 The Expanding Universe - Hubble's Law

In the early 1910s, it was observed that galaxies seemed to be moving away from us, and as a result, the light coming from them was red shifted (the Doppler effect of light waves). Redshift (z) is defined as -

$$z = \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}} \tag{1}$$

And after applying special relativity, the redshift comes out in terms of the relative velocity (v) as -

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{2}$$

which reduces to $z = v/c$ on applying the approximation of $v << c$. Edwin Hubble (the pioneer of modern cosmology), in 1929 upon observing that the (recession) velocity of these galaxies was directly proportional to their distance from us (r), gave us the Hubble's Law -

$$\vec{v} = H_0 \vec{r} \tag{3}$$

where H_0 is the proportionality constant, known as Hubble's constant ($H_0 = 100h$ km/sMpc, where h ranges from $(0.64, 0.8)$)

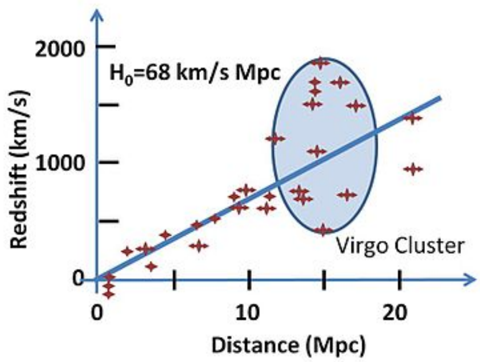

Figure 1: Fit of redshift velocities to Hubble's Law [1].

This relation does not mean that there is a "center" of the universe, from where $\vec{r}$ is measured, but it is measured from the observer as the origin, which works due to the linearity of the relation. Another important point to note is that this does not hold for nearby galaxies, due to the peculiar velocity of galaxies. The fact that the universe is expanding, also lead to the idea of The Big Bang Theory being developed.

3 Differential Equations involved in Cosmology

3.1 The Friedmann Equation

The Friedmann equation describes the expansion of the Universe, and is therefore one of the most important equations in cosmology. Consider a uniformly expanding medium with mass density ρ. According to the cosmological principle, we can consider any point to be the centre, hence consider a small volume containing mass m at a distance $\vec{r}$. According to Newtons laws, this "particle" has a gravitational potential energy and a kinetic energy, and the total energy (U) can be given by -

$$U = \frac{m\dot{r}^2}{2} - \frac{4\pi G\rho m r^2}{3} \tag{4}$$

We know change into a different coordinate system, the **comoving coordinate system**, which, as the name implies, is carried along with the expansion. Objects are assumed to be fixed in this frame (no peculiar velocities).

$$\vec{r} = a(t)\vec{x} \tag{5}$$

Here $\vec{r}$ is the real distance and $\vec{x}$ is the comoving distance and $a(t)$ is called the scale factor of the universe, where we have used isotropy in assuming that it is a function of time only. Substituting (5) in (4), using the fact that $\dot{x}$ is 0, and rearranging, we get the **The Friedmann Equation** -

$$(\frac{\dot{a}}{a})^2 = \frac{8G\pi\rho}{3} - \frac{kc^2}{a^2} \tag{6}$$

where we have made the substitution $kc^2 = -2U/mx^2$. Here k is independant of x and tells us about the curvature of the universe. There are 3 possible geometries to the universe, relating to the values of k, as summarized below. Observations indicate that the universe is very close to spatial flatness, and hence we will be setting $k = 0$ in many places.

curvature	geometry	angles of triangle	circumference of circle	type of Universe
$k > 0$	spherical	$> 180^o$	$c < 2\pi r$	Closed
$k = 0$	flat	180^o	$c = 2\pi r$	Flat
$k < 0$	hyperbolic	$< 180^o$	$c > 2\pi r$	Open

Figure 2: A summary of different geometries.

3.2 The Fluid Equation

The fluid equation describe the change of ρ with time by including a pressure term. According to the first law of thermodynamics $\frac{\mathrm{d}E}{\mathrm{d}t} + p\frac{\mathrm{d}V}{\mathrm{d}t} = T\frac{\mathrm{d}S}{\mathrm{d}t}$, using comoving coordinates, using $E = mc^2$, and assuming reversible expansion ($\frac{\mathrm{d}S}{\mathrm{d}t} = 0$) and putting it all together we get -

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + \frac{p}{c^2}) = 0 \tag{7}$$

We should not think of this pressure as helping the expansion along, but see its effect through the work done during expansion. Upon differentiating the Friedmann equation, and substituting in the fluid equation, we come across the **Acceleration Equation** -

$$\frac{\ddot{a}}{a} = \frac{-4G\pi}{3}(\rho + \frac{3p}{c^2}) \tag{8}$$

Note the independence from k in the equation. From here on we will also be using the natural units, setting $c = 1$. Substituting the comoving coordinates in Hubble's law, we also get $H = \frac{\dot{a}}{a}$, and define the current value of the Hubble's constant as H_0.

3.3 Simple Cosmological Models

In order to discover how the Universe might evolve, we need some idea of what is in it. In a cosmological context, this is done by specifying the relationship between the mass density ρ and the pressure p. This relationship is known as the equation of state. We will consider simplified models here, either purely radiation dominated or matter dominated.

3.3.1 Matter-Dominated universe

This refers to non-relativistic matter dominating the universe, so that the pressure applied becomes 0. Solving the fluid and acceleration equations we get the following relations -

$$\rho(a) = \frac{\rho_0}{a^3} \tag{9}$$

$$a(t) = (\frac{t}{t_0})^{2/3} \tag{10}$$

$$H = \frac{2}{3t} \tag{11}$$

implying that the rate of expansion decreases with time. As we shall see, ρ_{matter} falls of more slowly with $a(t)$, compared to a radiation dominated universe, and thus a matter dominated universe is more stable.

3.3.2 Radiation-Dominated universe

This refers to non-relativistic matter dominating the universe, and using $p = \frac{\rho c^2}{3}$ (according to radiation theory). A similar analysis yields -

$$\rho(a) = \frac{\rho_0}{a^4} \tag{12}$$

$$a(t) = (\frac{t}{t_0})^{1/2} \tag{13}$$

$$H = \frac{1}{2t} \tag{14}$$

implying that the universe expands much more slowly and $\rho_{radiation}$ falls off faster, hence leading to matter eventually dominating later.

3.4 Density Parameter

The density parameter is a very useful way of specifying the density of the Universe. **Critical density (ρ_c)** is defined as the density at which the universe is flat and is given by -

$$\rho_c = \frac{3H^2}{8G\pi}, \tag{15}$$

which yields $\rho_c(t_0) = 1.88h^2 \times 10^-26kg/m^3$. $\rho_c(t_0)$ is a very small number, but it is very close to the actual density of our universe. Density Parameter (Ω) is defined as -

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)} \tag{16}$$

and substituting this in (6) gives us,

$$\Omega(t) - 1 = \frac{k}{a^2H^2} \tag{17}$$

implying that if $\Omega = 1$, it pertains to a flat universe, but even a small deviation from the critical density will make the geometry of the universe closed or open, that is, a flat universe is in unstable equilibrium.

3.5 Cosmological Constant

3.5.1 Deceleration Parameter

The deceleration parameter is a way of quantifying the change of rate of the Hubble parameter. We define the deceleration parameter q_0 as

$$q_0 = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}(t_0)^2} \tag{18}$$

The larger the value of q_0, the more rapid the deceleration. Considering a matter-dominated universe ($p = 0$), and substituting the Friedmann and Acceleration equations, we get $q_0 = \frac{\Omega_0}{2}$, and this greatly simplifies finding out the values of Ω_0, since the deceleration is easier to measure than the density. To widespread surprise, the observations indicate that the Universe appears to be accelerating at present, $q_0 < 0$. None of the cosmological models that we have discussed so far are capable of satisfying this condition, as can be seen directly from the acceleration equation. This introduces the need for a cosmological constant.

3.5.2 Cosmological constant as a solution

When formulating general relativity, Einstein believed that the Universe was static, but found that his theory of general relativity did not permit it. In order to arrange a static Universe, he proposed a change to the equations, something he would later famously call his "greatest blunder". That was the introduction of a cosmological constant. The cosmological constant Λ appears in the Friedmann equation as an extra term, giving -

$$H^2 = \frac{8G\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{19}$$

The original idea was for Λ to balance ρ and k, for H to be 0, but this is unstable to small perturbations. However, this was a boon as realized later, since from the new acceleration equation

$$\frac{\ddot{a}}{a} = \frac{-4G\pi}{3}(\rho + \frac{3p}{c^2}) + \frac{\Lambda}{3} \tag{20}$$

A positive cosmological constant gives a positive contribution to a, and so acts effectively as a repulsive force, which can explain the accelerating universe observations. The introduction of Λ can be thought of as a cosmic fluid with energy density ρ_Λ, pressure p_Λ and critical density Ω_Λ given by -

$$\Omega + \Omega_\Lambda = 1 + \frac{k}{a^2H^2} \tag{21}$$

$$\rho_\Lambda = \frac{\Lambda}{8G\pi} \tag{22}$$

$$p_\Lambda = -\rho_\Lambda c^2 \tag{23}$$

The negative effective pressure in the above equations is thought to drive the expansion. The deceleration parameter then becomes -

$$q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda \tag{24}$$

which can provide for an accelerating universe. The cosmological constant also provides a useful way to parametrize possible cosmological models.

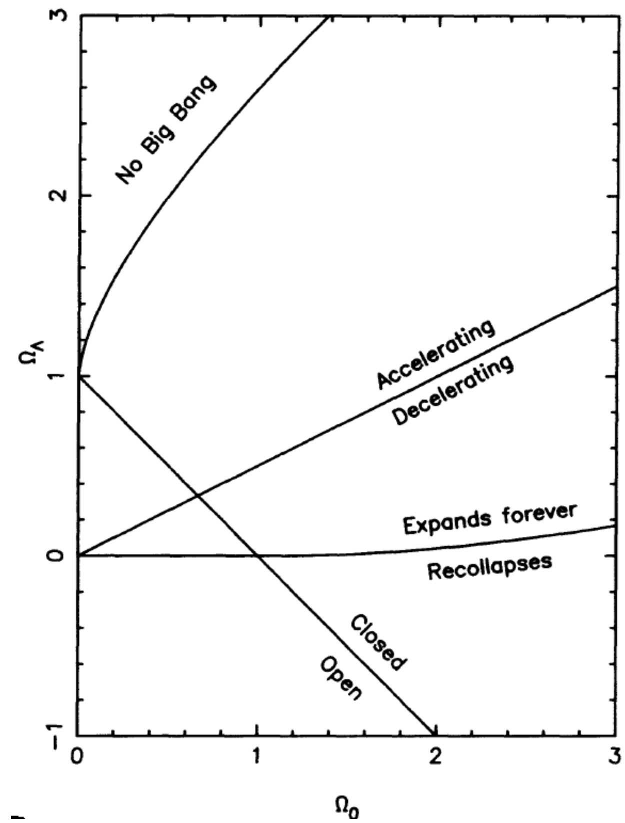

Figure 3: Different models for the Universe can be identified by their location in the plane showing the densities of matter and Λ. This figure indicates the main properties in different regions, with the labels indicating the behaviour on each side of the dividing lines.

3.6 Age of the Universe

One of the quantities that we can predict from a cosmological model, from the solution $a(t)$ for the expansion, is the age of the Universe t_o. By Hubble's Law, the easiest prediction we can make about the age is H_0^-1 which comes to around $9.77h^-1 \times 10^9$ years, and is known and **Hubble time**. This is a relatively good estimate, as a first guess. In a flat, purely matter dominated universe, we get $H_0 = \frac{2}{3t_0}$, which comes preposterously low as an estimate. In an open universe, since it will be lesser matter dominated, it would have taken more time for the universe to slow to its current rates, hence estimates increase. However, the cosmological constant comes once again to the rescue and we find that retaining flat geometry and Λ as +ve, we get better estimates of the age, currently pegged at 13.8 billion years by putting the observed values of $\Omega_0 = 0.3$ and $h = 0.72$ below, obtained by integration of (19).

$$H_0 t_0 = \frac{2}{3\sqrt{1-\Omega_0}} \ln\left(\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right) \tag{25}$$

4 Dark Matter

The need for dark matter as a form of "invisible" matter providing the needed gravitational attraction has pretty old origins, but it still remains a mystery to scientists. In this section we will first explore the proofs for the existence of a dark form of matter, and later discuss possible candidates for it.

4.1 Existence of Dark Matter?

4.1.1 Visible Matter density

From the crude estimates that a typical galaxy weighs about $10^{11} M_0$ and that galaxies are typically about a megaparsec apart, we know that the Universe cannot be a long way from the critical density. From stellar theory, we can estimate the amount of density in stars, and that comes out to be a small fraction of critical density-

$$\Omega_{stars} = \frac{\rho_{stars}}{\rho_c} \approx 0.005 \to 0.01 \tag{26}$$

Even on considering the gases which also contribute to the mass density, the theory of nucleosynthesis (formation of elements) predicts that the total baryonic density can be -

$$0.016 < \Omega_B h^2 < 0.024 \tag{27}$$

still a small fraction of ρ_c.

4.1.2 Galaxy Rotation curves

The general argument is to look at motions of various kinds of astronomical objects, and assess whether the visible material is sufficient to provide the inferred gravitational force. If it is not, the excess gravitational attraction must be due to extra, invisible, material. A galaxy rotation curve shows the velocity of of matter rotating in a spiral disk, as a function of distance from the centre. According to Newtons Laws, this should follow the relation -

$$v = \sqrt{\frac{GM(R)}{R}} \tag{28}$$

and at large distances, enclosing almost the whole mass, should fall off as $\propto \frac{1}{\sqrt{R}}$, but is found to be almost constant.

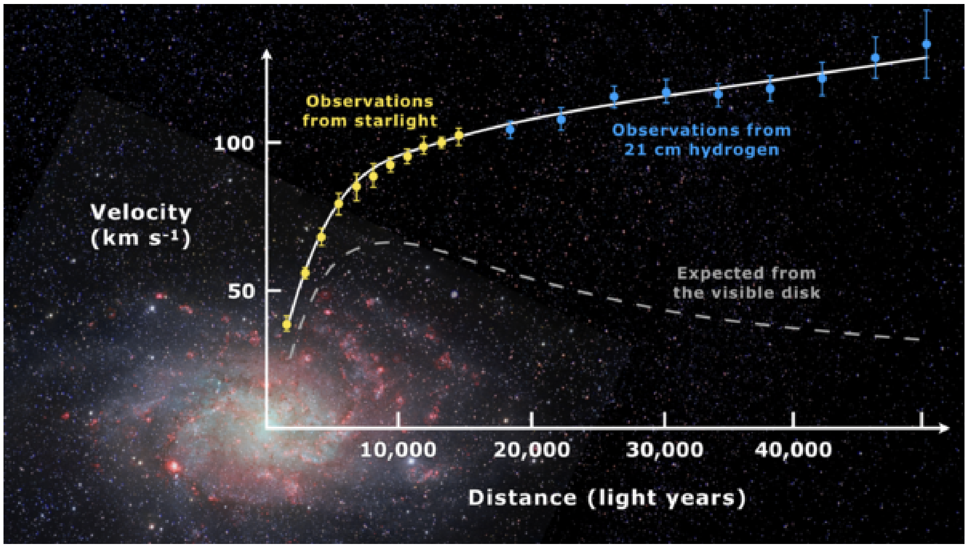

Figure 4: Rotation curve of spiral galaxy Messier 33 (yellow and blue points with error bars), and a predicted one from distribution of the visible matter (gray line). The discrepancy between the two curves can be accounted for by adding a dark matter halo surrounding the galaxy. [2].

This is an example of dark matter which can provide for the extra gravitational force, and is assumed to be in a halo around the galaxy. Further analysis shows that

$$\Omega_{darkmatter} \approx 0.1 \tag{29}$$

making up for a large amount of dark matter in the universe.

4.1.3 Galaxy Clusters

Galaxy clusters are the largest collection of objects (gravitationally), and have 2 parts - the galaxies that make them up and hot gas between these galaxies. The existence of this gas was found in the X-Ray region, and are believed to be very hot remnants from galaxy formation. There is on an average, 5-10 times more gas than stars in these clusters. These high temperature gases exert very high pressure, but are confined in these clusters by gravitational attraction. However, there simply isn't enough mass to keep them confined, and analysis shows that around 10 times the baryon density given by nucleosynthesis, is required to provide for the needed attraction.

4.1.4 Other evidences

Further dynamical evidence for the existence of dark matter comes from the motions of galaxies relative to one another (i.e. the deviations from the cosmological principle). Galaxies possess relative motions, the peculiar velocities mentioned earlier, which allows one to estimate their mass under the assumption that their gravitational interaction is responsible for the motions, which are often termed bulk flows. Once again, the amount of mass predicted from observing the bulk flows is much more than then mass from visible matter.

Structures in the universe originated form irregularities that grew out of gravitational attraction. Baryonic matter cannot account for the amount of gravitational attraction needed to form the structures observed, and hence is a strong proof of the existence of dark matter. Structure formation, bulk flows and the Ω_0 vs Ω_Λ plots estimate -

$$\Omega_0 \approx 0.3 \tag{30}$$
$$\Omega_\Lambda \approx 0.7 \tag{31}$$

4.2 Candidates for Dark Matter

The prediction of non-baryonic dark matter is one of the boldest and most striking in all of cosmology, and if ultimately verified, for example by direct detection of dark matter particles, will be amongst cosmology's most notable successes. Dark matter can be briefly summarized as Hot Dark Matter (HDM) which contains of relativistic particles, and Cold Dark Matter (CDM) consisting of non-relativistic particles.

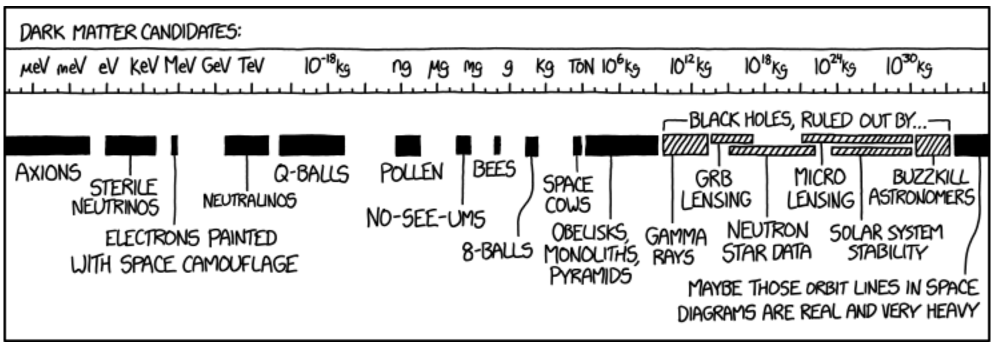

Figure 5: LOL.[3]

- **Neutrinos -** In the Standard Model of particle interactions the neutrino is a massless particle, and is present in the Universe in great abundance, being about as numerous as photons. If the Standard Model is extended to permit the neutrinos to have a small mass (a few tens of electron-volts), this would not affect their number density but they would have enough density to imply a closed Universe! They (muon and tau neutrinos) would be a candidate for HDM, but HDM cannot explain structure formation, so CDM is also required. If we consider heavier mass neutrinos, they could be a candidate for CDM, but this is unlikely.

- **Light Supersymmetric Particles (LSPs) -** Particle physicists regard supersymmetry as the most solidly-founded extension to standard particle theory, and it has the nice property of associating a new companion particle to each of the particles we already know about. In the simplest scenarios, the LSP is stable and is an excellent CDM candidate. Depending on the model the particle in question might be called the photino, or gravitino, or neutralino. They are also sometimes known as WIMPs — Weakly Interacting Massive Particles.

- **Axions -** The axion is a hypothetical elementary particle postulated to resolve the strong CP problem in quantum chromodynamics. If axions exist and have low mass within a specific range, they are of interest as a possible component of cold dark matter.

- **Primordial Black Holes -** A population of primordial black holes, meaning black holes formed early in the Universe's history rather than at a star's final death throes, would act like cold dark matter. However if they are made of baryons they must form before nucleosynthesis to avoid the nucleosynthesis bound of equation (27)

- **MAssive Compact Halo Objects (MACHOs) -** MACHOs have been detected by gravitational lensing of stars in the Large Magellanic Cloud (LMC). The idea was to monitor LMC stars, which lie outside the galactic halo. If there are invisible massive objects in the halo, and they happen to pass extremely close to our line of sight to the LMC star, then their gravitational field can bend and focus light from the star, temporarily brightening it, and this phenomenon is called microlensing. This was observed in the 1990s, when the light curve (plot of brightness vs time) of a star peaked aperiodically.

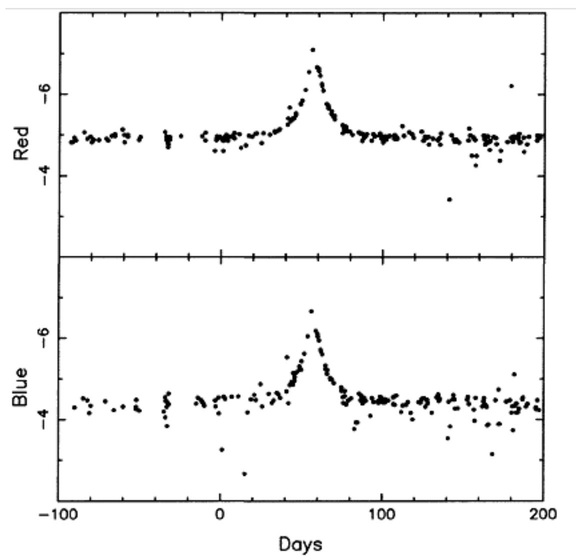

Figure 6: Light curves for a star in the LMC, obtained by the MACHO collaboration. The x-axis is in days with an arbitrary origin, while the y-axis shows the brightness of the star in red light and in blue light .

The best hope for dark matter detection is that the particles also interact through the Weak force, and not just the gravitational force. Any DM signal should show annual modifications, and hence can be filtered out from noise.

5 CMB and Neutrinos

5.1 The Form of CMB

The crucial observation which swayed the Big Bang/Steady State Universe debate in favour of the former was the detection of the Cosmic Microwave Background Radiation reported in 1965. This radiation bathes the Earth from all directions, and is now known to accurately take on the form of a black-body with temperature $T_0 \approx 2.72$K. This yields the energy density of the radiation as -

$$\Omega_{radiation} = 2.47 \times 10^{-5} h^{-2} \tag{32}$$

However, we know that -

$$\rho_{radiation} \propto \frac{1}{a^4} \tag{33}$$

$$\implies T_{radiation} \propto \frac{1}{a} \tag{34}$$

which shows that the CMB cools as the universe expands. The energy distribution, given by Planck's formula as below -

$$\epsilon(f)\,\mathrm{d}f = \frac{8\pi h}{c^3}\frac{f^3\,\mathrm{d}f}{e^{\frac{hf}{kT}}-1} \tag{35}$$

and since $\frac{f}{T}$, remains constant, the form of blackbody is preserved, and shows a similar distribution at a lower T and lower number density.The number density of photons to that of baryons is give by -

$$1.7\times 10^9 = \frac{n_\gamma}{n_B} \tag{36}$$

5.2 The formation of CMB

The radiation emitted by constituent particles of the plasma in the early universe couldn't travel very far, before they hit an electron and ricocheted, still trapped within the plasma . As space expanded and cooled further, around the 3000K (we shall see how we got this number) mark, around 38,0000 years after the big bang, neutral atoms could finally form. The light emitted by the plasma just before it neutralized, could now freely stream through the universe and this process is called decoupling, the light finally redshifting to form the CMB we see today.

Since decoupling happened when the Universe was only about one thousandth of its present size, and the photons have been travelling uninterrupted since then, they come from a considerable distance away. Those we see originate on the surface of a very large sphere centred on our location, called the surface of last scattering, with a radius of about $6000h^{-1}Mpc$. Recombination refers to the epoch where electrons joined the nuclei to create atoms, whereas decoupling refers to the epoch after which the photons will not scatter again. If recombination were instantaneous and complete, the two would coincide, but in practice each process takes some time and decoupling follows recombination. We can get a close estimate of the temperature at which this happened, using the Saha equation, derived assuming only hydrogen is present, thermal and chemical equilibrium and $\chi = \frac{n_p}{n_B}$ where n_p is the number of free protons. The abundance can be shown as -

$$\frac{1-\chi}{\chi} \simeq 3.8\frac{n_B}{n_\gamma}(\frac{k_BT}{m_ec^2})^{\frac{3}{2}}e^{\frac{13.6eV}{k_BT}} \tag{37}$$

putting $\chi_{rec} = 0.1$, we get $T_{rec} \approx 3600K$, close to the actual value of 3000K.

5.3 Anisotropies in the CMB

The fundamental measurement in microwave background studies is the temperature of the microwave background seen in a given direction on the sky, $T(\theta,\phi)$. Usually the mean temperature T is subtracted and a dimensionless temperature anisotropy is defined.

$$\frac{\Delta T}{T}(\theta,\phi) = \frac{T(\theta,\phi)-\bar{T}}{\bar{T}} \tag{38}$$

The next step is to carry out an expansion in spherical harmonics -

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_m^l(\theta,\phi) \tag{39}$$

where a_{lm} tells about the size of irregularities, and we study the radiation angular power spectrum ($C_l =< |a_{lm}| >^2$), the rotational invariance making sure than C_l is independant of m, and gives aniostropies on scale of $\frac{180}{l}$. The $l = 1$ gives the dipole due to relative motion of earth wrt the CMB, and peaks onward $l = 2$ are studied and plotted in the $\frac{(l)(l+1)C_l}{2\pi}$ vs l graph - For $l <= 15$,

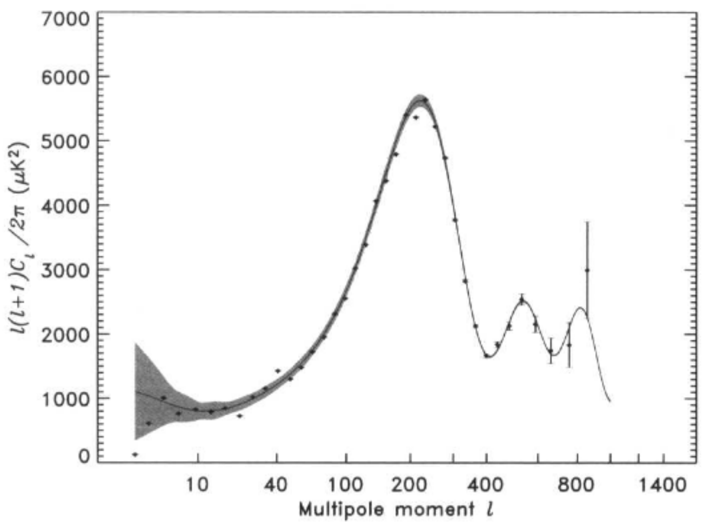

Figure 7: The radiation angular power spectrum as measured by the WMAP satellite, shown as the black dots. The solid line shows a theoretical prediction from their best-fit cosmological model, which fits the data extremely well.

it is fairly constant and called Sachs-Wolfe plateau, at $l \approx 200$, we begin to see the effects of the oscillations in the cosmic fluid at the time of decoupling. The location of the first peak, gives the geometry of the universe and the plot also shows that $\Omega_0 + \Omega_\Lambda = 1.02 \pm 0.02$, placing the universe very close to spatial flatness.

5.4 Neutrinos

Studies of neutrinos coming from the Sun, those interacting in the Earth's atmosphere, and those created on Earth via nuclear interactions, have all shown evidence that neutrinos possess the ability to change their type as they travel. This phenomenon is known as neutrino oscillations and can be understood in particle physics models, but only those where the neutrino rest-mass is nonzero. The reason we expect there to be Cosmic Neutrino Background (CNB) is because the high density of early universe would ensure their thermal equilibrium with photons and other particles, with interactions like -

$$p + e^- \Longleftrightarrow n + \nu_e \tag{40}$$

$$\gamma + \gamma \Longleftrightarrow \nu_\mu + \bar{\nu_\mu} \tag{41}$$

The fermionic properties of neutrinos make their number density around 7/8 times that of photons making up the CMB, and since neutrinos interact much more weakly than photons, they decouple before, whereas electron-positron annihilation's boost the photons temperature by a factor of $\sqrt[3]{\frac{11}{4}}$.

Putting this together, and assuming neutrinos to be relativistic, we get -

$$\Omega_\nu = 3 \times \frac{7}{8} \times (\frac{11}{4})^{\frac{4}{3}} \Omega_{rad} \tag{42}$$
$$= 1.68 \times 10^{-5} h^2$$

$$T_\nu = \sqrt[3]{\frac{11}{4}} T = 1.95K \tag{43}$$

Assuming light mass for neutrinos ($m_\nu c^2 << 1MeV$), we get a slightly different Ω_ν, which can make light neutrinos one of the good HDM candidates. Heavy neutrinos, also provide a good CDM candidate. Existence of CNB is also crucial for nucleosynthesis, the correct matter-radiation epoch and structure formation. However due to our experimental limits, we have not been able to detect it yet.

6 The Early Universe

The different eras of the universe are summarized in the table below -

Time	Temperature	What's going on?
$t < 10^{-10}$ s	$T > 10^{15}$ K	Open to speculation!
10^{-10} s $< t < 10^{-4}$ s	10^{15} K $> T > 10^{12}$ K	Free electrons, quarks, photons, neutrinos; everything is strongly interacting with everything else.
10^{-4} s $< t < 1$ s	10^{12} K $> T > 10^{10}$ K	Free electrons, protons, neutrons, photons, neutrinos; everything is strongly interacting with everything else.
1 s $< t < 10^{12}$ s	10^{10} K $> T > 10000$ K	Protons and neutrons have joined to form atomic nuclei, and so we have free electrons, atomic nuclei, photons, neutrinos; everything is strongly interacting with everything else except the neutrinos, whose interactions are now too weak. The Universe is still radiation dominated.
10^{12} s $< t < 10^{13}$ s	10000 K $> T > 3000$ K	As before, except that now the Universe is matter dominated.
10^{13} s $< t < t_0$	3000 K $> T > 3$ K	Atoms have now formed from the nuclei and the electrons. The photons are no longer interacting with them, and are cooling to form what we will see as the microwave background.

Figure 8: Different stages of the Universe's evolution (taking $\Omega_0 = 0.3$ and $h = 0.72$). Some numbers are approximate.

6.1 Inflation

Inflation is not a replacement for the Hot Big Bang theory, but rather an extra add-on idea which is supposed to apply during some very early stage of the Universe's expansion.

6.1.1 Need for Inflation

From equation 17, and substituting the values of aH, we get that the $|\Omega_{tot} - 1|$ is an increasing function of time, leading to the universe getting more and more curved as time goes by. However, for the universe to be as flat as observed today, the early universe, at the time of electro-weak symmetry breaking would have to satify $|\Omega_{tot} - 1| < 10^{-30}$, which is extraordinarily close to ρ_c.

The distance which light could have travelled during the lifetime of the Universe gives rise to a region known as the observable Universe. One of the most important properties of the microwave background is that it is very nearly isotropic. Being at the same temperature is the characteristic of thermal equilibrium, and so this observation is naturally explained if different regions of the sky have been able to interact and move towards thermal equilibrium. However due to the finite speed of light and age of the universe, these far-separated regions of the universe had no interaction between them to establish thermal equilibrium. Similarly, the anisotropies observed in the CMB have to be pre-existing, and cannot be created afterwards.

Another inconsistency was the existence of relic particles, particularly Magnetic Monopoles, which were extremely massive (10^{16} GeV) and should have been highly abundant, but they have not been observed yet, and should have dominated over radiation much before the equality epoch.

6.1.2 Inflation as a solution

Inflation came to the rescue, and was defined as a period where the universe was accelerating, $\ddot{a}(t) > 0$. From the acceleration equations, this implies negative pressure, $p < -\frac{\rho c^2}{3}$. Applying the Friedmann equations, and assuming the cosmological constant to be the dominating factor, we get

$$a(t) = e^{(\frac{\Lambda}{3})t} \tag{44}$$

a much more dramatic expansion, which is supposed to happen around 10^{-34} sec. Applying this condition to the flatness problem, we see that during a perfect exponential growth of the universe, we get -

$$|\Omega_{tot} - 1| \propto e^{-\sqrt{\frac{4\Lambda}{3}}t} \tag{45}$$

and hence it drives Ω_{tot} towards 1. Calculating the amount of inflation to achieve the spatial flatness as seen today, the required value is that $a(t)$ increases by a factor of 10^{27} by the end of inflation, which happen very quickly, since the increase is exponential.

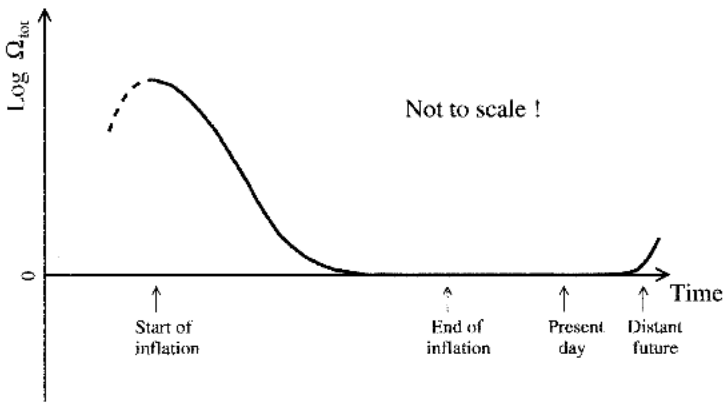

Figure 9: Possible evolution of the density parameter Ω_{tot}- There might or might not be a period before inflation, indicated by the dashed line..

Inflation greatly increases the size of a region of the Universe, while keeping its characteristic scale, the Hubble scale, fixed. This means that a small patch of the Universe, small enough to achieve thermalization before inflation, can expand to be much larger than the size of our presently observable Universe. Then the microwaves coming from opposite sides of the sky really are at the same temperature because they were once in equilibrium, hereby solving the horizon problem. The dramatic expansion of the inflationary era dilutes away any unfortunate relic particles, because their density is reduced by the expansion more quickly than the cosmological constant. Provided enough expansion occurs, this dilution can easily make sure that the particles are not observable today.

6.2 Baryogenesis

Baryogenesis is the hypothetical theory that took place during the early universe, producing baryon asymmetry, that is, the imbalance of matter and antimatter in the early universe. The following Sakharov conditions are needed to generate the observed asymmetry -

- **Baryon number violation -** Reactions in the Standard Model preserve baryon number, new interactions are being explored in GUT's to explain this.
- **C and CP violation -** C and CP violation refers to two symmetries typically obeyed by particle interactions — that interaction rates are unchanged if one switches the charge (C) or parity (P) of the particles, or both (CP). CP violation is observed in interactions of particles called neutral K mesons, though at a very low level and without the presence of baryon number violation. Such violation, at a much larger level, would be needed in any interactions able to generate the baryon number.
- **Departure from Thermal Equilibrium -** This preferentially chooses one direction for reactions to lead into, hence amplifying any original asymmetry present. With reactions like

$$\gamma + \gamma \longleftrightarrow p + \bar{p} \tag{46}$$

  if by some mechanism, an additional proton was created for every such billion pairs, after sufficient cooling and annihilation of matter-antimatter pairs, that one proton would be enough to create the observed baryonic density. This mechanism currently remains unknown.

6.3 Nucleosynthesis

The formation of nuclei, which took place at about 1 second after the big bang by a process called nucleosynthesis. The origin of heavy elements are indeed the cores of stars, but lighter elements like D, He-3, Li, He-4 could not have been created, and their abundances have to come from the primordial gas. At high temperatures, protons and neutrons would be in equilibrium, and after sufficient cooling, bind to form nuclei. The relative ratios of their abundances are given by-

$$\frac{N_n}{N_p} = (\frac{m_n}{m_p})^{\frac{3}{2}} e^{-\frac{(m_n - m_p)c^2}{k_B T}} \tag{47}$$

Until $k_B T >> (m_n - m_p)c^2$, the $\frac{N_n}{N_p}$ is almost 1. The reactions converting neutrons to protons and vice versa are -

$$n + \nu_e \longleftrightarrow p + e^- \tag{48}$$

$$n + e^+ \longleftrightarrow p + \bar{\nu_e} \tag{49}$$

These reactions proceed in both directions till $k_B T \approx 0.8MeV$, after which $\frac{N_n}{N_p} = \frac{1}{5}$, and will only change by decay of free neutrons. Some of the reactions which make these elements are -

$$p + n \rightarrow D \tag{50}$$

$$D + p \rightarrow He^3 \tag{51}$$

$$D + D \rightarrow He^4 \tag{52}$$

The delay before He^4 appearing, leads to the decay of neutrons, and reduces the ratio to $\frac{N_n}{N_p} = \frac{1}{8}$. In the Early universe, the major abundances were of H(not enough neutrons to bind to protons) and He^4 (the most stable light nucleus). A detailed analysis gives the amounts as - H $\approx$ 77%, He^4 $\approx$ 23%, D$\approx 10^{-4}$%, He^3 $\approx 10^{-5}$% and Li^7 $\approx 10^{-10}$% by mass. This allows us to fix the constraints of $\Omega_B h^2$ within a very small range to agree with observations.

6.4 Observational Cosmology

Observational cosmology considers how objects with given properties, such as luminosity and size, will appear to us. In particular, it is concerned with the dependence of that appearance on the cosmological model.

6.4.1 Light Propagation and Redshift

We derive our results using the Robertson-Walker metric for spacetime given by -

$$ds^2 = -c^2 dt^2 + a(t)^2 (\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2) \tag{53}$$

Light propagation obeys $ds = 0$, using symmetry of space, and finding the time difference between emission and reception, we get the relation between emitted and received wavelengths as -

$$\frac{\lambda_r}{\lambda_e} = \frac{a(t_r)}{a(t_e)} \tag{54}$$

$$\frac{a(t_0)}{a(t_e)} = 1 + z \tag{55}$$

Using this metric, we can also calculate how far light could have travelled during the lifetime of the universe, assuming a matter dominated flat universe with no cosmological constant -

$$\int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} = \int_0^{t_0} \frac{cdt}{a(t)} \implies \int_0^{r_0} dr = ct_0^{\frac{2}{3}} \int_0^{t_0} \frac{dt}{t^{\frac{2}{3}}} \implies r_0 = 3ct_o \tag{56}$$

giving the radius of the observable universe as approximately 46.5 billion light years. However, the different evolution's of scale factors leads to different conclusions, but we must keep in mind that the universe was opaque until the time of formation of CMB, and hence the result is pretty accurate. Also, note that the observable radius is not ct_0, since the universe is expanding.

6.4.2 Luminosity Distance

The luminosity distance is a way of expressing the amount of light received from a distant object. The luminosity distance is the distance that the object appears to have, assuming the inverse square law for the reduction of light intensity with distance holds. It is the ratio of the Luminosity and radiation flux density of an object, which in turn depends upon the redshift and is given by -

$$d_{lum} = a_0 r_0 (1+z) \tag{57}$$

where $a_0 r_0$ is the physical distance to the object. The geometry of the universe also affects this, with hyperbolic geometry enhancing and spherical opposing this effect. The luminosity distance depends on the cosmological model we have under discussion, and hence can be used to tell us which cosmological model describes our Universe. In particular, we can plot the luminosity distance against redshift for different cosmologies, and observed data only fit in with a universe of $\Omega_0 \approx 0.3$ and $\Omega_\Lambda \approx 0.7$.

6.4.3 Angular Diameter Distance

The angular diameter distance is a measure of how large objects appear to be. As with the luminosity distance, it is defined as the distance that an object of known physical extent appears to be at, under the assumption of Euclidean geometry. The angular diameter distance and angular size we perceive of an object with physical size l is given by -

$$d_{diam} = \frac{a_0 r_0}{1+z} \tag{58}$$

$$d\theta = \frac{l(1+z)}{a_0 r_0} \tag{59}$$

A key application of the angular diameter distance is in the study of features in the cosmic microwave background radiation.

7 Special Theory Of Relativity

7.1 Problems in Classical Mechanics

Higher precision and ability to achieve higher speeds in modern times has shown many inconsistencies in our idea of absolute space and time. The biggest of these is the pion - a meson that has a lifetime of about 26.0 ns at rest, appears to last longer at higher speeds when created in labs and hence, also travels much different distances in relative frames. Also, since classical mechanics allows particles to travel faster than light, it also violates causality, when observed by specific frames. Another inconsistency was about our classical views of energy, and how certain particle interactions tend to "violate" that. Arguably the most important, which lead Einstein to search for an alternative theory, was that a transformation to a different inertial frame, when applied to Maxwell's theory of Electromagnetism, do not remain invariant. This lead him to put down the postulates of STR to lead to a consistent theory, which can be summarized in the few words - " Motion affects Measurement".

7.2 Postulates of STR

The postulates, or general statements which provide for a basis of the theory of STR are as follows

- *The Principle of Relativity* - The laws of physics are the same in all inertial frames
- *The Principle of constancy of c* - The speed of light in free space has the same value c in all inertial reference frames

These 2 postulates lead to another consequence - it is impossible to accelerate a particle to a speed greater than c, no matter how much Kinetic energy we give it.

7.3 Lorentz Transformation

There are multiple ways to derive the time dilation and length contraction equations in STR, however we will focus on the methods of Lorentz Transformations. The transformation equations in general have - an observer S in an inertial frame, another observer S' in another inertial frame moving relative to S, and a common event observed. Without loss of generality, assuming the relative motion of frames is in the xx' direction, with velocity $\vec{u}$. Galilean transformations gives the relationships as -

$$x' = x - ut \tag{60}$$
$$y' = y \tag{61}$$
$$z' = z \tag{62}$$
$$t' = t \tag{63}$$

The relativistic corrections to these, which can derived from the postulates given above by assuming space-time homogeneity and symmetry are -

$$x' = \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(x - ut) \tag{64}$$
$$y' = y \tag{65}$$
$$z' = z \tag{66}$$
$$t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma(t - \frac{ux}{c^2}) \tag{67}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ is known as the Lorentz factor. These equations also reduce to the Galilean equations in the limit $u << c$ or $c \to \infty$. From the above equations, we can see the effects of time dilation (the clock of an observer in relative motion seems to run slowly when timed by the clocks in the frame of reference which is "at rest") and length contraction (the distance between 2 points measured simultaneously, by an observer in relative motion is observed to be lesser than the one "at rest", but only along the axis of relative motion). We can also derive the relation between velocities as -

$$v_x{}' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - u\Delta t)}{\gamma(\Delta t - \frac{u\Delta x}{c^2})} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \tag{68}$$

$$v_y{}' = \frac{v_y}{\gamma(1 - \frac{uv_x}{c^2})} \tag{69}$$

$$v_z{}' = \frac{v_z}{\gamma(1 - \frac{uv_x}{c^2})} \tag{70}$$

We can also put $c = v_x$ and check that $v_x{}' = c$, always, irrespective of u.

7.4 Energy and Momentum in STR

Applying the classical definition of momentum to frames in relative motion, we see that momentum is not conserved in such inertial frames. Einstein gave a new definition of momentum, so that it would be conserved in all inertial frames, and would reduce to the classical definition at lower speeds. The relativistic formula for the momentum of a particle moving with velocity is -

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{71}$$

Since energy of a particle is defined from Work-Energy theorem, we get the total KE of a particle as -

$$KE = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2 \tag{72}$$

which, once again reduces to the classical formula $KE = \frac{mv^2}{2}$ at low speeds. We call the mc^2 term as rest energy, or the total energy of a particle as measured from a frame in which it is at rest. Only the total energy, the combination of rest energy and kinetic energy is conserved in all inertial frames. A little manipulation of the above equation yield the total energy (E) as -

$$E^2 = p^2c^2 + m^2c^4 \tag{73}$$

8 Brief Introduction to Tensors

Let us take a manifold (=space) with dimension n. We will denote the components of a vector with the numbers $v_1 \dots v_n$. If one modifies the vector basis, in which the components are expressed, then these components will change, too. Such a transformation can be written using matrices. We shall employ a convention where we shall omit the summation sign, assuming the index over which the summation is performed as the repeated one -

$$\begin{aligned} v_\mu{}' &= \sum_{\nu=1}^{n} A_{\mu\nu} v_\nu \quad (\forall \mu \in |1 \le \mu \le n) \\ \implies v_\mu{}' &= A_{\mu\nu} v_\nu \end{aligned} \tag{74}$$

The transformation of basis vectors and components of vectors, can be written as follows -

$$e_\mu{}' = \Lambda_{\mu\nu} e_\nu \tag{75}$$

$$v'_\mu = (\Lambda_{\mu\nu}{}^{-1})^T v_\nu \tag{76}$$

The normal vectors are called ‘contravariant vectors’, because they transform contrary to the basis vector columns. A ‘gradient’, which we so far have always regarded as a true vector, will from now on be called a ‘covariant vector’ or ‘covector’: it transforms in the same way as the basis vector columns. Applying chain rule, we get the gradient (w) transformation as -

$$w_\mu{}' = (\frac{\partial x_\nu}{\partial x_\mu{}'}) w_\nu \tag{77}$$

Taking A as the vector components matrix, this turns out to be -

$$w' = (A^{-1})^T w \tag{78}$$

To distinguish vectors from covectors we will denote vectors with an arrow ($\vec{v}$) and covectors with a tilde ($\widetilde{w}$). To make further distinction between contravariant and covariant vectors we will put the contravariant indices as superscript and the covariant indices with subscripts.
To see an example of how tensors arise, we shall consider inner products. Inner products between a vector and covector are invariant under all transformations. We can introduce a new inner product between two contravariant vectors which also has this invariance property. To do this we introduce a covector w_μ and define the inner product between x^μ and y^ν with respect to this covector in the following way, which is invariant -

$$s = w_\nu w_\mu x^\nu y^\mu = \begin{pmatrix} x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} w_1 \cdot w_1 & w_1 \cdot w_2 & w_1 \cdot w_3 \\ w_2 \cdot w_1 & w_2 \cdot w_2 & w_2 \cdot w_3 \\ w_3 \cdot w_1 & w_3 \cdot w_2 & w_3 \cdot w_3 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \tag{79}$$

the two appearances of the covector w are combined into one object: some kind of product of w with itself, we call this object g. So, we define the inner product with respect to the object g as -

$$s = g_{\mu\nu} x^\mu y^\nu \tag{80}$$

We do no longer regard g as built out of two covectors, but regard it as a matrix-like set of numbers on itself. This curious object, which looks like a matrix, but does not transform as one, is an example of a tensor. There are tensors of rank 3, as opposed to tensors of rank 0 (scalars), rank 1 (vectors and covectors) and rank 2 (matrices and the other kind of tensors).
A formal definition of tensors is as follows - An (N,M)-tensor at a given point in space can be described by a set of numbers with N + M indices which transforms, upon coordinate transformation given by the matrix A, in the following way -

$$t'^{\alpha_1 \ldots \alpha_N}{}_{\beta_1 \ldots \beta_M} = A^{\alpha_1}{}_{\mu_1} \ldots A^{\alpha_N}{}_{\mu_N} A^{\nu_1}{}_{\beta_1} \ldots A^{\nu_M}{}_{\beta_M} t^{\mu_1 \ldots \mu_N}{}_{\nu_1 \ldots \nu_M} \tag{81}$$

A tensor t is called symmetric in the indices μ and ν if the components are equal upon exchange of the index-values and called anti-symmetric if the components are equal but opposite upon exchange of the index-values. We can also create a ’dual basis’, by constructing a set of ‘unit covectors’ that serve as a basis for the covectors. They transform as -

$$\widetilde{e}'^\alpha = A^\alpha{}_\beta \widetilde{e}^\beta \tag{82}$$

Basis-covector-columns transform as vectors, in contrast to basis-vector-columns, which transform as covectors. We can then express these tensors in terms of 'basis tensors' as $t = t^{\mu\nu}{}_{\rho}\vec{e}_{\mu} \otimes \vec{e}_{\nu} \otimes \widetilde{e}^{\rho}$, where $\otimes$ is tensor outer product. Tensors can be regarded as an object, a function or an operator. To define distance from A to B, we divide the distance into small lengths ds, and then integrate to find distance along this path.

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{83}$$

where information of length/distance is stored in g and is called the metric of the manifold. This tensor is also symmetric, and this symmetry is preserved in any transformation. We can also transform between covariant and contravariant vectors by using metric tensors. Tensor calculus is a very important part in understanding GR.

9 Modified PoA

- Till 15 May - A first course in General Relativity by Bernard Schutz, to gain a basic overview on GR.
- 15 May to 5 June - Modern Cosmology by Scott Dodelson, to gain a deeper insight into the application of GR into cosmology
- 10 June - End-report submission

References

[1] *Hubble's law* https://en.wikipedia.org/wiki/Hubble's_law

[2] *Galaxy rotation curve* https://en.wikipedia.org/wiki/Galaxy_rotation_curve

[3] https://xkcd.com/2035/