Modifying the Poisson Equation in KFT

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Contents

1 Modifying Gravity and the need for screening

Some modified gravity theories aim to get rid of the need of dark energy, in lieu of a large distance modification of GR. Thus, there needs to be significant modification at large scales to explain late-time acceleration. However, the MG theory must tend to GR on the smallest of scales due to the stringent constraints set up by the solar system scales. Adding a scalar degree of freedom changes the strength of gravity on sub-horizon scales, thereby changing the growth rate of structure formation. However, recovering GR on small scales invites the need for an additional screening mechanism to screen the scalar interaction at solar system scales.

This is predominantly achieved through nonlinear screening mechanisms introducing a characteristic scale where modified gravity transitions to GR. There are multiple mechanisms to implement the screenings in scalar-tensor gravity theories:

- Screening at large field values such as in chameleon or syymetron models: operates in regions where the Newtonian gravitational potential exceeds a given threshold, $|\psi_N| \geqslant \Lambda_T$ (sec. 3.1 of [3])
- Screening with first derivatives such as in k-mouflage: operates when the local gravitational acceleration passes a given threshold value, $|\nabla\psi_N| \geqslant \Lambda_T^2$ (Sec 3.2 of [3])
- Screening with second derivatives such as in the Vainshtein mechanism: operates when curvature or local densities become large, $\left|\nabla^2\psi_N\right| \geqslant \Lambda_T^3$ (Sec 3.3 of [3])
- Linear suppression effects such as the Yukawa suppression or linear shielding mechanisms: operates when separations cross the scale set by the linearised mass or sound speed of the field (section 3.4 of [3])

We shall look into some of these screenings, and their prediction for nonlinear structure formation using the KFT formalism.

2 Evolution Equations with an interaction term, and the first and second order expressions

This section illustrates an intuition for why screening mechanisms involve a G_{eff} which is k-dependent, and the troubles that come with including the general interaction terms in the KFT formalism.
Following [2], if we consider perturbations around the FLRW metric in the usual conformal Newtonian gauge as:

$$ds^2 = -(1+2\psi)dt^2 + a^2(1+2\phi)\delta_{ij}dx^i dx^j$$

and using the quasistatic approximation of neglecting the time derivatives of the perturbed quantities compared to the spatial derivatives. The modification of gravity due to the scalar mode can be described by BD gravity, and the modified equations become:

$$\begin{aligned}
\phi + \psi &= -\varphi \\
\frac{\Delta\psi}{a^2} &= 4\pi G\rho_m\delta - \frac{\Delta\varphi}{2a^2} \\
(3+2\omega_{BD})\frac{\Delta\varphi}{a^2} &= 8\pi G\rho_m\delta - I(\varphi)
\end{aligned}$$

with I encoding the non-linear interaction terms, and ω_{BD} is the Brans-Dicke scalar which can be a function of time. The interaction term is included to get rid of the constraints between solar system measurements and large-scale measurements which are contradictory, and can be expanded as:

$$I(\varphi) = M_1(k)\varphi + \frac{1}{2}\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta_D(k-k_1-k_2)M_2(k_1,k_2)\varphi(k_1)\varphi(k_2) + O(3)$$

For the chameleon mechanism, the BD scalar acquires a mass, and mediates a force which decays exponentially above a length scale of the inverse of mass, thereby hiding the scalar interaction above a certain length and recovering GR. In this case, M_1 determines the mass term in the cosmological background and M_i, $(i > 1)$ describe the change of the mass term due to the change of the energy density. In the DGP braneworld model, there is a large $M_2(k)$ term dominating over the linear term, the perturbations tend to GR. Thus, the Poisson equation becomes:

$$-\frac{k^2}{a^2}\psi = \frac{1}{2}\kappa^2\rho_m\big[1 + \frac{k^2/a^2}{3\Pi(k)}\big]\delta(k) + \frac{1}{2}\frac{k^2}{a^2}S(k)$$

with

$$\Pi(k) = \frac{1}{3}\big((3+2\omega_{BD})\frac{k^2}{a^2} + M_1\big)$$

and $S(k)$ is the source term obtained perturbatively as

$$S(k) = -\frac{1}{6\Pi(k)}(\frac{\kappa^2\rho_m}{3})^2\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta_D(k-k_1-k_2)M_2(k_1,k_2)\frac{\delta(k_1)\delta(k_2)}{\Pi(k_1)\Pi(k_2)} + O(3)$$

Using these, we can, for different M_i represent different MG models, especially $f(R)$ gravity and nDGP models.

If we include just the $M_1(k)$ term, we can see that the Poisson equation becomes

$$-k^2\psi = 4\pi G\rho_m a^2\delta(k)\big[1 + \frac{k^2/a^2}{(3+2\omega_{BD})k^2/a^2 + M_1(k)}\big]$$

This leads to $G_{eff}(k,a)$, and a effective G dependent on scale, with the growth equations becoming:

$$\ddot{D}_+ + 2H\dot{D}_+ - 4\pi G(k,a)\rho_m a^2 D_+ = 0$$

Performing a change in coordinates, it leads to the following:

$$D_+'' + (\frac{3}{a} + \frac{E'}{E})D_+' - \frac{3\Omega_m G(k,a)}{2a^2 G_N}D_+ = 0$$

$$D_+'' + (\frac{3}{a} + \frac{E'}{E})D_+' - \frac{3\Omega_m}{2a^2}(1 + \frac{k^2/a^2}{(3+2\omega_{BD}k^2/a^2 + M_1(k)})D_+ = 0$$

This thus leads to the growth factor being dependent on k as well. This leads to changes in the interaction term which we must take into account, something that will be covered in later sections. It leads to an additional moments with the damped initial power spectrum, and additional filter functions.

However, the moment we significantly alter the Poisson equation, that is, if we include the source term $S(k)$ as defined above, things become complicated, since the "linear" growth equation becomes non-linear in D_+. Explicitly, if we add the second order in the source term to our Poisson equation (with $M_1 = 0$ for simplicity, which occurs in DGP models), it becomes:

$$-\frac{k^2}{a^2}\psi = 4\pi G\rho_m[1+\frac{1}{3+2\omega_{BD}}]\delta - \frac{k^2}{12a^2\Pi(k)}(\frac{8\pi G\rho_m}{3})^2\int\frac{d^3k_1}{(2\pi)^3}\frac{M_2(k_1,k-k_1)\delta(k_1)\delta(k-k_1)}{\Pi(k_1)\Pi(k-k_1)}$$

In some cases, M_2 becomes separable, such as that of the nDGP model in which

$$M_2(k_1,k-k_1) = \frac{2r_c^2}{a^4}(k_1^2(k-k_1)^2 - (k_1\cdot(k-k_1))^2)$$

and

$$\Pi(k,\tau) = \beta(\tau)\frac{k^2}{a^2}$$

with $\beta(\tau) = 1 - 2Hr_c(1+\frac{\dot{H}}{3H^2})$. In this case, if one of the parameters is along the z axis, we

$$M_2(k_1,k-k_1) = \frac{2r_c^2}{a^4}(k_1^2(k-k_1)^2)(1-cos^2\theta)$$

, and replacing it by its average (4/3) for simplicity, we get the source term as:

$$\begin{aligned} S(k) &= \frac{8r_c^2}{3(1+2/3\omega_{BD})^2}\int\frac{d^3k_1}{(2\pi)^3}\delta(k_1)\delta(k-k_1) \\ &= \frac{8r_c^2}{3(1+2/3\omega_{BD})^2}\delta(k)\star\delta(k) \end{aligned}$$

Putting this simplified equation in the modified Poisson equation we get the "linear" growth equation as (with $G_{eff} = G_N(1+\frac{1}{3+2\omega_{BD}})$):

$$D_+'' + (\frac{3}{a}+\frac{E'}{E})D_+' = \frac{3\Omega_m G_{eff}}{2a^2}D_+ - \frac{2\Omega_m^2 H^2 r_c^2\delta_0}{3a^2(1+2/3\omega_{BD})^2}D_+^2$$

Since this is pretty unwieldy to solve, we can try to linearize it, and then take the partial derivatives of this equation wrt E and G, as done in the taylor expansion function. Unfortunately, due to the D_+^2 term, the Taylor expansion changes the homogenous part, so the ansatz $C(a,x)D_+(a)$ no longer works. Completing this is left for future work, and we now go into seeing how our lives become easier with a scale dependent G predicted by numerical simulations in the paper [3].

3 Including screening effects in a scale dependent G_eff

Motivated by the variety of screening mechanisms available to scalar-tensor theories, [3] introduces a parametrisation of the modified gravitational forces that can act on spherical top-hat overdensities. They propose that a modification of gravity on the spherical collapse of a top hat can be parametrized as:

$$\Delta\psi = \kappa^2\frac{a^2}{2}\rho_m(1+\frac{\Delta G_{eff}}{G})\delta$$

with $\kappa^2 = 8\pi G$ and the field equations as $G_{\mu\nu} = \kappa^2(T_{\mu\nu}+T_{\mu\nu}^{eff})$, with the new effective stress-energy tensor encapsulating the extra terms in the field equation.

Viable modified gravity theories introducing deviations from GR at large scales need to recover Einstein gravity in high-density regions in order to comply with the tight constraints inferred from Solar-System tests, thus requiring $\frac{\Delta G_{eff}}{G} \to 0$. This is predominantly achieved through nonlinear screening mechanisms introducing a characteristic scale where modified gravity transitions to GR. [3] propose a parametrisation of the different effects of screening or linear suppression mechanisms on the modified spherical collapse through $\frac{\Delta G_{eff}}{G}$ as:

$$\frac{G_{eff}}{G} = A + \sum_{i}^{N_0} B_i \prod_{j}^{N_i} b_{ij} \left(\frac{r}{r_{0ij}}\right)^{a_{ij}} \left(\left[1 + \left(\frac{r_{0ij}}{r}\right)^{a_{ij}}\right]^{\frac{1}{b_{ij}}} - 1\right)$$

This basically defines a combination of interpolations between regimes of different radial dependence. The parameter A describes the modification of the gravitational coupling in the fully screened limit, which is 1 if we want to recover GR. For one summand ($N_0 = 1$) and factor ($N_1 = 1$), B is the effective enhancement in the fully unscreened limit, r_0 is the screening scale, and a determines the radial dependence of the screened solution along with b that defines an interpolation rate between the screened and unscreened limits. The product takes into account multiple screening or suppression effects (with number of factors N_i for each summand).

3.1 Explicit model 1 : Chameleon model

[3] considers a chameleon model with a constant Brans-Dicke parameter $\omega > -3/2$, which has the following form of the scalar field equation:

$$\Delta\phi = -\frac{1}{3 + 2\omega}\left[\kappa^2 \rho_m - R_0 \left(\frac{1-\phi}{1-\phi_0}\right)^{\alpha-1}\right]$$

They proceed to parametrize it in the form above. They obtain $A = 1$, and since 2 suppression effects need to be described, $N_0 = 1, N_1 = 2$. They also obtain $B = \frac{1}{3+2\omega}$. For $\omega = 0$, as in f(R) gravity, they find the following set of parameters that describe the interpolation: $(a_1, b_1, r_{01}) = (8.34, 6.21, 0.520)$ and $(a_2, b_2, r_{02}) = (-5.82, 1.52, 28.1)$.
The symmetron mechanism can be treated in a similar way.

3.2 Explicit model 2 : k-mouflage mechanism

This mechanism can operate in scalar-tensor theories with non-canonical kinetic contributions.In this case, there is only one suppression mechanism. The effective modification they obtain is

$$\frac{\Delta G_{eff}}{G} = \frac{2\beta^2}{\kappa_\chi(r)} = \frac{\beta\kappa}{GM/r^2}\sqrt{-2X}$$

where β is the coupling strength of the model, $\chi = X/\mathcal{M}^4 = -\partial^\mu\phi\partial_\mu\phi/2\mathcal{M}^4$, $\mathcal{M}$ is a model parameter characterising the suppression scale, $\kappa(\chi) = -1 + \chi + \kappa_0\chi^2, \kappa_0 < 0$. They obtain $B = -\sqrt{2}C_1/3$, where $C_1 = -3\sqrt{2}\beta^2$. Further, there is the relation of $a(b-1)/b = 4/3$, and we can calculate b from the relations $r_0 = (2(b/3)^3 C_2^2)^{1/4}$, where $C_2 = \frac{3\beta\kappa M}{2\pi\mathcal{M}^2}\sqrt{-3\kappa_0}$, with M as the is the mass enclosed within the radius r and the relation $\frac{\Delta G_{eff}}{G} = (2^{1/b} - 1)b$ valid for $r = r_0$.

3.3 Explicit model 3 : Vainshtein mechanism

The Vainshtein mechanism [3] operates, for instance, in DGP braneworld gravity. This is described by the equation of motion:

$$\nabla^2\phi + \frac{r_c^2}{3\beta}\left[(\nabla^2\phi)^2 - (\nabla_i\nabla^j\phi)(\nabla^i\nabla^j\phi)\right] = \frac{\kappa^2}{3\beta}\rho_m\delta$$

with r_c being a crossover-scale characterising the impact of the propagation of the graviton into the 4D brane universe, and $\beta = 1 + 2\sigma H r_c(1 + \frac{H'}{3H})$, with $\sigma = \pm 1$. Positive σ represents the normal branch whereas the negative sign is obtained in the self-accelerating branch. The self-accelerating branch suffers from a ghost instability, and hence we will not use it here. It is parametrized in the way above using $A = 1, N_0 = 1, N_1 = 1, B = 1/(3\beta), a = 3, b = 2, r_0 = r_v$, where r_v is the Vainsthein radius.

3.4 Explicit model 4 : Yukawa suppression

Similar to Model 1, they consider a scalar-tensor theory in Brans-Dicke representation with a constant Brans-Dicke parameter ω such that the quasistatic scalar field equation becomes:

$$\nabla^2\delta\phi - m^2\delta\phi + \frac{\kappa^2}{3+2\omega}\delta\rho_m = 0$$

It can be parametrized as above with $A = 1, N_0 = 1, N_1 = 1,\ B = \frac{1}{3+2\omega}$ and solving for a, b with the equations $a(b-1)/b = -2$, $r_0 = \frac{1}{m\sqrt{2b/3}}$ and $\frac{\Delta G_{eff}}{G} = (2^{1/b} - 1)b$ valid for $r = r_0$.

4 KFT for the case of screening in nDGP

Since life is certainly easier in the Fourier space, we simply write the parameterized gravitational coupling as:

$$\frac{G_{eff}(a,k)}{G} = A + \sum_i^{N_0} B_i \prod_j^{N_i} b_{ij}\left(\frac{k_{0ij}}{k}\right)^{a_{ij}}\left(\left[1 + \left(\frac{k}{k_{0ij}}\right)^{a_{ij}}\right]^{\frac{1}{b_{ij}}} - 1\right)$$

with k_0 characterizing an effective screening Fourier wave number. The general parameterization which follows in Section 6 can be used, where the screening scale $y_h/y_0 \to k_0/k$, so that we can parameterize the screening wave number as well, a similar approach as followed by [1].

We now apply the KFT formalism to the nDGP model, and proceed to generalize it in Section 6. The Poisson equation now becomes:

$$\begin{aligned}
-k^2\Phi &= 4\pi G_{eff}(k,a)a^2\delta\rho \\
G_{eff} &= G_N\left(1 + \frac{2}{3\beta}\left(\frac{k_*}{k}\right)^3\left(\left[1 + \left(\frac{k}{k_*}\right)^3\right]^{1/2} - 1\right)\right)
\end{aligned}$$

Some limits to note instantly are:

- For large scales, ie $k << k_*$, $G_{eff} \to G_{eff}^{lin} = G_N\left(1 + \frac{1}{3\beta}\right)$

- For small scales, solar system scales and smaller, ie $k >> k_*$, $G_{eff} \to G_N \left(1 + \frac{2}{3\beta}(\frac{k_*}{k})^{3/2}\right) \to G_N$, as it should to satisfy stringent solar system tests of GR.

Deriving the linear growth equations in the same way as before, using the quasistatic approximation for the metric perturbations we get:

$$\nabla^2 \varphi = \frac{3aG_{eff}(k,a)}{2m^2}\delta = A_\varphi \delta$$

$$D''_+ + (\frac{3}{a} + \frac{E'}{E})D'_+ - \frac{3\Omega_m G_{eff}(k,a)}{2a^2 G_N} D_+ = 0$$

Similarly, using the above we can get $\dot{m} = mD_+ A_\varphi$, with the difference of A_φ being scale dependent. Following the KFT formalism, the effective force to the free motion of particles with the effective gravitational potential $\phi = \varphi + \frac{\dot{m}}{m}\psi = \varphi + A_\varphi D_+ \psi$, with the Poisson equation becoming

$$\nabla^2 \phi = A_\varphi(k,a)(\delta - \delta^{lin}) = A_\varphi(k,a)\delta^{nl}$$

Thus, we now have 2 non-linearity scales, one characterized by k_*, the scale below which the force is screened, and the one characterized by the density perturbations becoming O(1), called k_0. Now, we calculate the particle-particle potential v as in KFT after going into Fourier space, and using the Yukawa approximation:

$$\begin{aligned}\tilde{v} &= \frac{-1}{nk^2} A_\varphi(k,a)\left[1 - \sqrt{\frac{P_\delta^{lin}}{P_\delta}}\right] \\ &= \frac{-1}{nk^2} A_\varphi^{GR} \frac{G_{eff}}{G_N} \frac{k^2}{k_0^2 + k^2} \\ &= -\frac{A_\varphi^{GR} G_{eff}}{nG_N(k_0^2 + k^2)}\end{aligned}$$

For the computation of the non-linear power spectrum with KFT, we evaluate the following term:

$$\begin{aligned}\vec{k} \cdot (\nabla \tilde{v} \star P_\delta)(k) &= \vec{k} \cdot \int_{k'} i(k-k')\left(\frac{-A_\varphi^{GR} G_{eff}(k-k',a)}{nG_N}\right)\left(\frac{1}{k_0^2 + (k-k')^2}\right)\bar{P}_\delta(k') \\ &= \frac{-iA_\varphi^{GR}}{n} \int_{k'} \frac{k\cdot(k-k')}{k_0^2 + (k-k')^2}\left(1 + \frac{2}{3\beta}\left(\frac{k_*}{k-k'}\right)^3\left[\left(1 + \left(\frac{k-k'}{k_*}\right)^3\right)^{1/2} - 1\right]\right)\bar{P}_\delta(k')\end{aligned}$$

Now, we substitute $y = \frac{k'}{k}$, $y_0 = \frac{k_0}{k}$, $y_* = \frac{k_*}{k}$ and the cosine μ of the polar angle between k and k',and get the following:

$$\vec{k}\cdot(\nabla\tilde{v}\star P_\delta)(k) = \frac{-iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\int y^2 dy d\mu\left(\frac{1-y\mu}{1+y_0^2+y^2-2y\mu}\right)$$
$$\times\left(1+\frac{2}{3\beta}\left(\frac{y_*^3}{(1+y^2-2y\mu)^{3/2}}\right)\left[\left(1+\left(\frac{(1+y^2-2y\mu)^{3/2}}{y_*^3}\right)\right)^{1/2}-1\right]\right)\bar{P}_\delta(ky)$$
$$= \frac{-iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\int y^2 dy d\mu\left(\frac{1-y\mu}{1+y_0^2+y^2-2y\mu}\right)\bar{P}_\delta(ky)$$
$$-\frac{iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\frac{2}{3\beta}\int y^2 dy d\mu\left(\frac{1-y\mu}{1+y_0^2+y^2-2y\mu}\right)\left(\frac{y_*^3}{(1+y^2-2y\mu)^{3/2}}\right)$$
$$\times\left[\left(1+\left(\frac{(1+y^2-2y\mu)^{3/2}}{y_*^3}\right)\right)^{1/2}-1\right]\bar{P}_\delta(ky)$$

Thus,

$$\vec{k}\cdot(\nabla\tilde{v}\star P_\delta)(k) = \frac{-iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\int y^2 dy\bar{P}_\delta(ky)J(y,y_0) - \frac{iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\frac{2y_*^3}{3\beta}\int y^2 dy\bar{P}_\delta(ky)J'(y,y_0,y_*)$$

where

$$J(y,y_0) = \int_{-1}^{1} d\mu\left(\frac{1-y\mu}{1+y_0^2+y^2-2y\mu}\right)$$
$$J'(y,y_0,y_*) = \int_{-1}^{1} d\mu\left(\frac{1-y\mu}{1+y_0^2+y^2-2y\mu}\right)\left(\frac{1}{(1+y^2-2y\mu)^{3/2}}\right)\left[\left(1+\left(\frac{(1+y^2-2y\mu)^{3/2}}{y_*^3}\right)\right)^{1/2}-1\right]$$

One immediate consistency check follows in the regime of $y_*\to 0$, in this regime for all scales of interest, we shouldn't expect any new terms in the mean-field interaction term. We can check that for $y_*\to 0$, $J'(y,y_0,y_*)\propto\frac{1}{y_*^{3/2}}$, and the additional term in $\vec{k}\cdot(\nabla\tilde{v}\star P_\delta)(k)$ has a y_*^3 multiplied, thus the additional term goes as $y_*^{3/2}\to 0$, thus we have satisfied our consistency check.

We can also check that the new integral $J'(y,y_0,y_*)$ is indeed well-behaved by evaluating in Matehmatica:

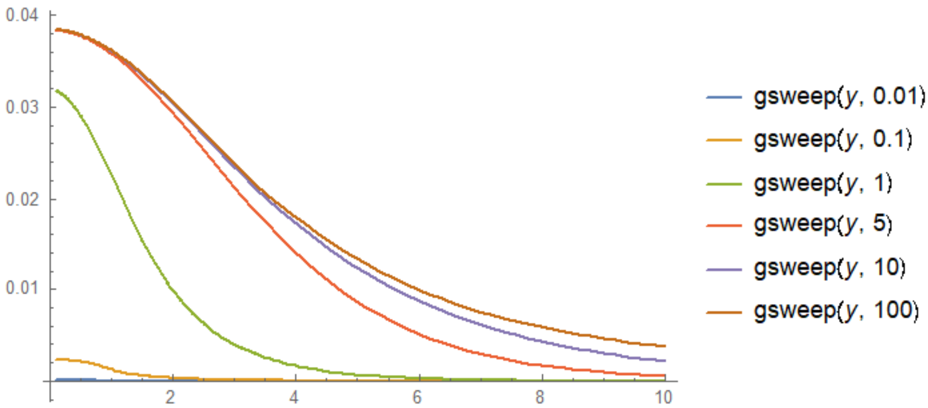

Figure 1: The $y_*^3 J'$ integral as a function of y evaluated for different k_* screening wavenumbers, the lower k_* (the second parameter in gsweep), the closer to 0 is the integrand, the lower the contribution in the interaction term

5 Coding it up

First, from the above, we need to calculate the interaction term. We will calculate this using the Hamiltonian propagator, since it is easier to implement:

$$\begin{aligned}
\langle S_I \rangle &= -2ink \cdot \int_0^t dt' g_H(t,t')(\nabla\tilde{v} \star \bar{P}_\delta)(k) \\
&= 2\int_0^t dt' g_H(t,t')\left[A_\varphi^{GR}\frac{k^3}{(2\pi)^2}\int y^2 dy \bar{P}_\delta(ky)J(y,y_0) + A_\varphi^{GR}\frac{k^3}{(2\pi)^2}\frac{2y_*^3}{3\beta}\int y^2 dy \bar{P}_\delta(ky)J'(y,y_0,y_*)\right] \\
&= 3\int_0^t dt' g_H(t,t')\frac{a}{m}D_+^2\left[\sigma^2_{J,y_0} + \frac{2y_*^3}{3\beta}\sigma^2_{J',y_0,y_*}\right]
\end{aligned}$$

where

$$\begin{aligned}
\sigma^2_{J',y_0,y_*} &= \frac{k^3}{(2\pi)^2}\int_0^\infty y^2 J'(y,y_0,y_*)\bar{P}_\delta^{(i)} \\
\sigma^2_{J,y_0} &= \frac{k^3}{(2\pi)^2}\int_0^\infty y^2 J(y,y_0)\bar{P}_\delta^{(i)}
\end{aligned}$$

Note here that there is no G_{eff} multiplied in the interaction term, since A_φ^{GR} just has a G_N term, and the effect of the G_{eff} has already been accounted for in additional interaction term which was something we calculated in the convolution part.

In the code, the only changes we have made are to include the k-dependence in a gScreening.cpp file, and to calculate this additional interaction term of $J'(y,y_0,y_*)$ in the mean field calculation of the forceterm.cpp file. The relevant code can be found here.

The G_{eff} now is dependent on the Hubble parameter, hence the gScreening class needs a cosmological model as input. It also needs to incorporate a k-dependence, which is included by:

```
double gScreening::operator () (double a, double k)
{
  if (std::isnan(k) == false)
    this->k_internal = k;
  double r_c = 2.0/70.0; /* Using the scale of H_0rc = 2 */
  double beta = 1.0+2.0*r_c*(cos_model->hubbleFunction(a)/cos_model->hubbleFunction
    (0.001))*(1.0+((cos_model->dexpansionFunction_da(a)/cos_model->
    dexpansionFunction_da(0.001))*70.0*(a/0.001))/(3.0*(cos_model->hubbleFunction(a)/
    cos_model->hubbleFunction(0.001))));
  double k_star = 10.0; /* till now, free parameter*/
  double Geff_Gn = 1.0 + (2.0/(3.0*beta))*pow (k_star/this->k_internal,3.0)*(pow(1.0
    + pow(this->k_internal/k_star,3.0),0.5)-1.0);

  return Geff_Gn;
}
```

with `k_internal` being set to 1.0.

For the forceTerm class, we take the additional value of `k_star` as input. A point to note here is that though the linear growth equation becomes scale dependent due to its dependence on G_{eff} in theory, since we simply use D_+ to set the time coordinate, it does not make physical sense to keep a time coordinate as scale dependent. Thus, we simply make a choice of k to set the time coordinate, that being the GR limit, and hence D_+ remains unchanged. Thus, the only relevant changes are to include the additional factors of $J', \sigma^2_{J',y_0,y_*}$ as:

```
double forceTerm::J_prime (double y, double y0, double y_star) /*J_prime definition*/
{
  astro::integrator integrate_J_prime ([&] (double x)
      { return
          (1-y*x)/(1+y0*y0 + y*y -2*x*y)/pow(1+y*y-2*x*y,1.5) *(pow(1+ pow(1+y*y-2*y*x,1.5)/pow(y_star,3),0.5)-1); },1.0e-2);
  /*std::cout<<integrate_J_prime (-1.0,1.0);*/
  return integrate_J_prime (-1.0,1.0);
}
```

and the change in the interaction term:

```
double forceTerm::interaction_term_hamilton (double k, double a)
{
  astro::integrator integrate
    ([&] (double x)
    { double r_c = 2.0/cosmological_model->hubbleFunction(1);
      double beta = 1.0+2.0*r_c*(cosmological_model->hubbleFunction(a)/cosmological_model->hubbleFunction(a_min))*(1.0+((cosmological_model->dexpansionFunction_da(a)/cosmological_model->dexpansionFunction_da(a_min))*cosmological_model->hubbleFunction(1)*(a/a_min))/(3.0*(cosmological_model->hubbleFunction(a)/cosmological_model->hubbleFunction(a_min)))) ;
      return
        1./a_min*propagator_h->g_qp(a,x)*
        pow(Dplus(x,k),2.)*(sigma_J_sq_table(k,x)+ 2*sigma_J_prime_sq_table(k,x)*pow(k_star/k,3)/(3*beta))/pow(x/a_min,2.)/E(x); },1.0e-2);
  return 3.0*integrate (a_min, a);
}
```

Note that here, since we will be changing the interaction term, the GR non-linear spectra will also change, which is undesirable, since we need the unchanged GR spectra to compare. We then need to include a boolean parameter, which switches between the 2 interaction terms to calculate the non-linear spectra differently for the GR and MG case. We first define the normal interaction term:

```
double forceTerm::interaction_term_normal (double k, double a) /*3*int g_H *G_eff* 1/(a^2 D_plus' E)* D_plus^2*sigma_J^2 D_plus' gets cancelled from t->a variable change in intg*/
{ astro::integrator integrate
    ([&] (double x)
    { //astro::cosmologicalParameters * p = cos_model->getParameters ();
      //screeningParameters * q = reinterpret_cast<screeningParameters*> (p->par);
      return
        1./a_min*propagator_h->g_qp(a,x)*
        pow(Dplus(x,k),2.)*(sigma_J_sq_table(k,x))/pow(x/a_min,2.)/E(x); },1.0e-2);
  /*std::cout<<integrate (a_min, a);*/
  return 3.0*integrate (a_min, a);
}
```

and switch the operator according to the parameter provided:

```
double forceTerm::operator () (double k, double a)
{
    if (fabs (a-a_save) > 1.0e-3)
    {
      k_scale = get_Yukawa_scale (1.0);
      init_tables();
    }
    a_save = a;
```

```

  switch(proptype){
    case (0):{
      return interaction_term_hamilton (k, a);
      break;}
    case(1):{
      return interaction_term_normal (k, a);
      break;}
    default:{
      throw std::runtime_error ("Error: Unknown propagator type");
      break;}
  }

}
```

as well as the non-linear spectra:

```
  for (unsigned int i = 0; i < data.n; i++)
  {
    data.x[i] = astro::x_linear (i, data.n, 0.05, 5.0); /*f_v*/

    switch(proptype){
      case (0):{
      data.y[i] = 1.0-exp (-0.5*interaction_term_hamilton (data.x[i], a));
      break;}
      case(1):{
      data.y[i] = 1.0-exp (-0.5*interaction_term_normal (data.x[i], a));
      break;}
      default:{
        throw std::runtime_error ("Error: Unknown propagator type");
        break;}

      }
  }
```

and pass on an additional parameter to the forceTerm class which declares which interaction term in being used, and make the change in main.cpp which passes the relevant values for the LCDM and MG models.

We then run this code, with 2 free parameters, those being `k_star` and `r_c`.
Upon changing `k_star` over 3 magnitudes, we observe the following:

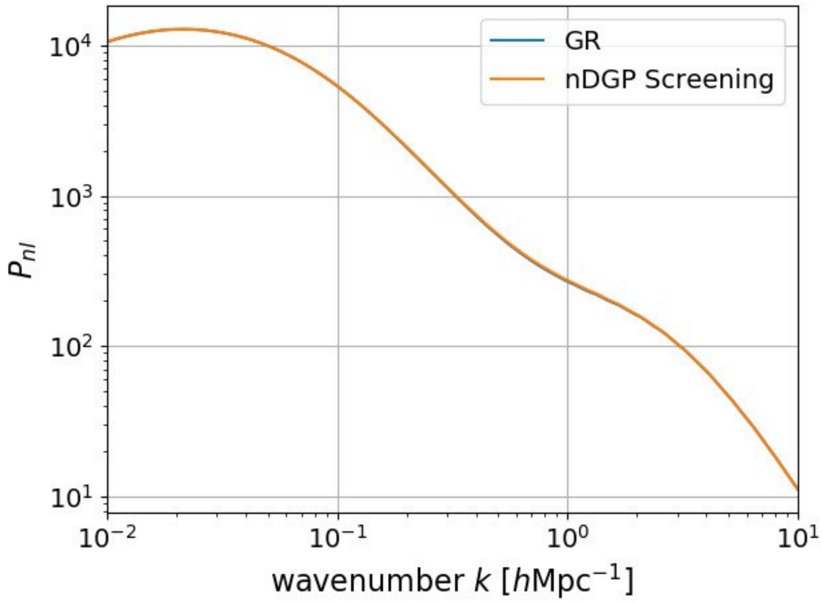

Figure 2: The nonlinear spectrum as obtained for the value of $k_* = 0.1$.

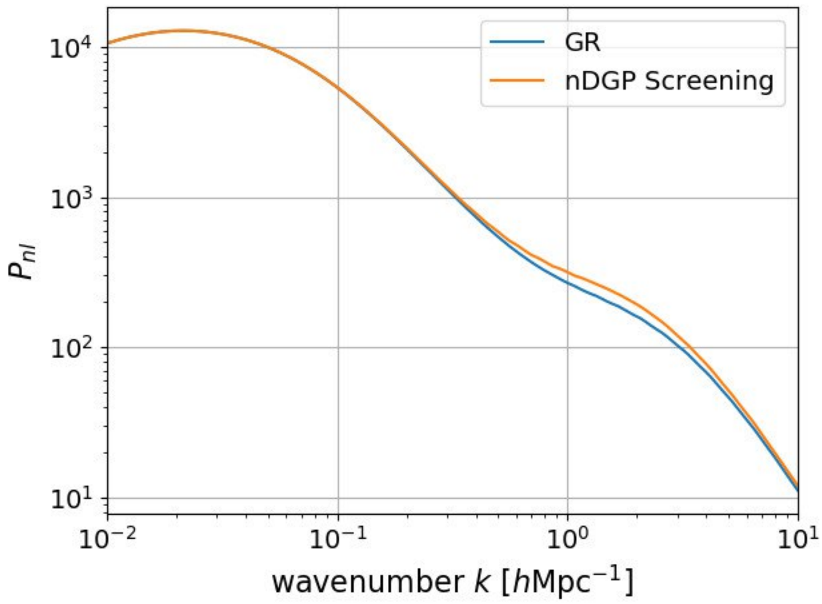

Figure 3: The nonlinear spectrum as obtained for the value of $k_* = 1$.

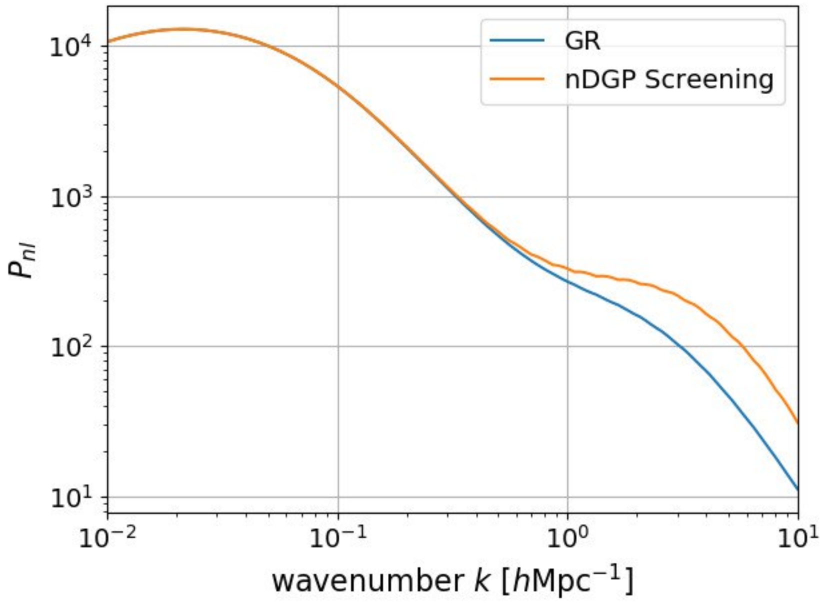

Figure 4: The nonlinear spectrum as obtained for the value of $k_* = 10$.

According to the plots, the screening changes the power spectrum on small scales and enhances the power spectrum. There is an agreement at large-scales between the GR and MG models due to the normalization of the power spectrum. Another thing to note is that for low values of k_*, there is a large agreement with GR, as required. For larger k_*, there is higher deviation from the GR in the intermediate scales, around 400% for $k_* \approx 10$. This will be the regime in which the Taylor expansion approach of deviations from GR will not apply.

To get an idea of the actual value of k_*, we note in Section 6 the general parametrization of the screening wavenumber. An order of magnitude calculation can be followed through for the nDGP case:

$$
\begin{aligned}
y_0 \approx \frac{1}{k_*} &= p_4 a^{p_5} \\
k_* &= \frac{1}{p_4 a^{p_5}} \\
&= \frac{1}{p_4 (\frac{p_1 p_3}{p_1 - 1})^{p_5}} \\
&= \frac{1}{p_4 \cdot 3^{p_5}} \\
&= \frac{3 \cdot (3\beta)^{2/3}}{2\Omega_m^{1/3} (H_0 r_c)^{2/3}}
\end{aligned}
$$

Here, we approximate β as:

$$\begin{aligned}\beta &= 1 + 2Hr_c\left(1 + \frac{\dot{H}}{3H^2}\right)\\ &\leqslant 1 + 2H_0 r_c\\ &\leqslant 5\end{aligned}$$

And thus we get for the screening wavenumber:

$$\begin{aligned}k_* &\leqslant \frac{3 \cdot (15)^{2/3}}{2(0.3)^{1/3}(2)^{2/3}}\\ &\leqslant 8.5\end{aligned}$$

Thus, depending on corrections to β, the value of k_* is approximately O(10), in which region there is high deviation from GR.

The paper [1] also suggests 2 possibilities of r_c: $H_0 r_c = 2.0$ and $H_0 r_c = 0.5$, resulting in a negligible change of the plot, implying that changes within r_c may not be very significant.

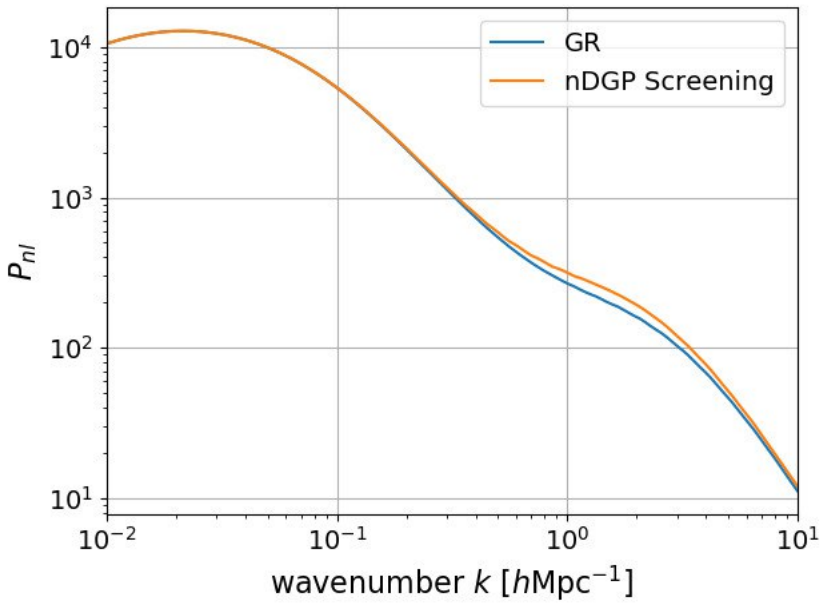

Figure 5: The nonlinear spectrum as obtained for the value of $k_* = 1$ and $r_c = 0.5/H_0$.

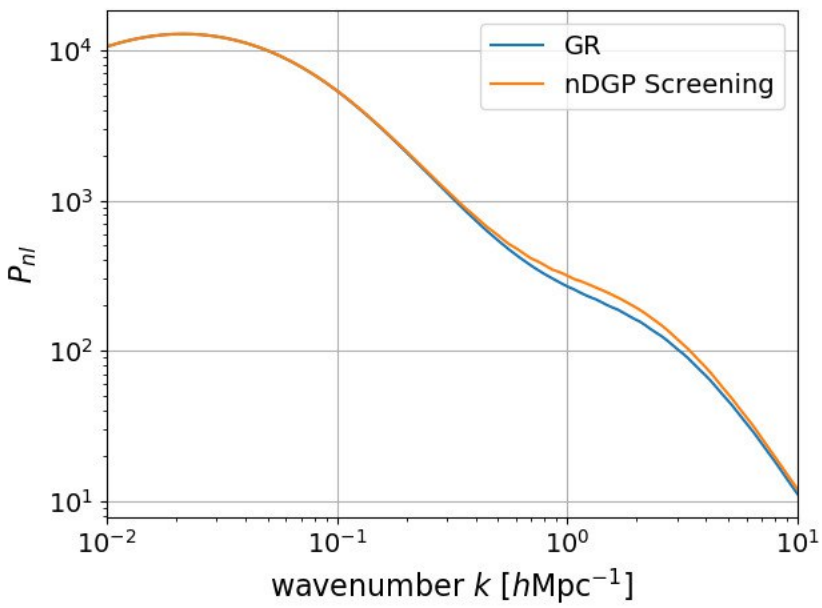

Figure 6: The nonlinear spectrum as obtained for the value of $k_* = 1$ and $r_c = 2.0/H_0$.

6 The general case of screening and its interaction term

This section builds on the explicit example section of Section 3. [3] construct a parametrisation for the nonlinear modified structure formation described by the spherical collapse model, and describe the effective gravitational coupling considering a single element, $N_0 = N_1 = 1$.

They propose 8 parameters to describe the general model:

- $A = p_0$
- $b = p_1$ as the interpolation rate parameter, the same b as that which appears in the parametrization in Section 3
- $B = p_2$, the maximal unscreened modification, the same B as that which appears in the parametrization in Section 3
- In the screened or suppressed limit, p_3 is defined as $\frac{\Delta G_{eff}}{G}|_{scr} \propto r^{p_3}$
- The rest of the quantities are used to parametrize the screening scale r_0, which we treat as a free parameter in our code, however, for completeness, they are represented as:

$$y_0 = r_0/(ar_{th}) = p_4 a^{p_5} (2GH_0 M_{vir})^{p_6} \left(\frac{y_{env}}{y_h}\right)^{p_7}$$

where r_{th} is the comoving top hat radius at an initial scale factor $a_i << 1$

- A point to note however, is that if p_7 is non-zero, we need to solve a coupled differential equation for y_{env}:
$$y'' + \left(2 + \frac{H'}{H}\right)y' + \frac{1}{2}\Omega_m(a)(y^{-2} - y) = 0$$
with initial conditions in the matter dominated regime.

- Relating this to the previous parametrization, $a = \frac{p_1 p_3}{p_1 - 1}$, $b = p_1$ and $B = p_2$

Now, we can cast the previous explicit models that we studied in Section 3, however, we will leave the screening scale to be a free parameter, leaving us with just 3 to care about:

- The chameleon model has a non-zero p_7, and hence may involve looking into a changing screening parameter, so we use the values as given in Section 3.
- The Yukawa suppressed regime given a p_1 has $p_2 = \frac{1}{3+2\omega}$, $p_3 = -2$, and p_4 as being density dependent
- The k-mouflage model has $p_2 = 2\beta^2$, $p_3 = \frac{4}{3}$ and p_4 dependent on the model parameters of $\kappa, \mathcal{M}$
- For the nDGP model we have $p_1 = 2$, $p_2 = \frac{1}{3\beta}$, $p_3 = \frac{3}{2}$ and p_4 being environment dependent

Thus, in the case of:
$$\frac{G_{eff}}{G} = A + \sum_i^{N_0} B_i \prod_j^{N_i} b_{ij} \left(\frac{r}{r_{0ij}}\right)^{a_{ij}} \left(\left[1 + \left(\frac{r_{0ij}}{r}\right)^{a_{ij}}\right]^{\frac{1}{b_{ij}}} - 1\right)$$

and following the same method of Section 5, to find $\vec{k} \cdot (\nabla\tilde{v} \star \bar{P}_\delta)(k)$ and thus the interaction term with this G_{eff}, we simply end up with more additional terms: Thus,
$$\vec{k} \cdot (\nabla\tilde{v} \star P_\delta)(k) = \frac{-iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\int y^2 dy \bar{P}_\delta(ky)J(y, y_0) - \frac{iA_\varphi^{GR}}{n}\frac{k^3}{(2\pi)^2}\sum_i^{N_0}\sum_j^{N_i} B_i b_{ij}(y_{0ij})^{a_{ij}}\int y^2 dy \bar{P}_\delta(ky)J'_{ij}(y, y_0, y_{ij})$$

where
$$J'(y, y_0, y_{ij}) = \int_{-1}^{1} d\mu \left(\frac{1 - y\mu}{1 + y_0^2 + y^2 - 2y\mu}\right)\left(\frac{1}{(1 + y^2 - 2y\mu)^{a_{ij}/2}}\right)\left[\left(1 + \left(\frac{(1 + y^2 - 2y\mu)^{a_{ij}/2}}{y_{0ij}^{a_{ij}}}\right)\right)^{1/b_{ij}} - 1\right]$$

which gives us the interaction term:
$$\langle S_I \rangle = 3\int_0^t dt' g_H(t, t')\frac{a}{m}D_+^2\left[\sigma_{J,y_0}^2 + \sum_i^{N_0}\sum_j^{N_i} B_i b_{ij}(y_{0ij})^{a_{ij}}\sigma_{J'_{ij}, y_0, y_{ij}}^2\right]$$

We now add a screeningParameters.cpp part to take in the 8 parameters p_i, caluclate a, B, b, and then the G_{eff} using it (this is for $N_0 = N_1 = 1$).

```
double gScreening::operator () (double a, double k)
{
  if (std::isnan(k) == false)
    this->k_internal = k;
  screeningParameters q (cos_model,a);
  double k_star = 1.0; /* till now, free parameter*/
  double Geff_Gn = 1.0 + q.param_B*q.param_b*pow (k_star/this->k_internal,q.param_a)
    *(pow(1.0 + pow(this->k_internal/k_star,q.param_a),1/q.param_b)-1.0);
  return Geff_Gn;
}
```

and the additional interaction term:

```
double forceTerm::interaction_term_hamilton (double k, double a)
{ screeningParameters q (cosmological_model,a);
  astro::integrator integrate
    ([&] (double x)
    {
      return
        1./a_min*propagator_h->g_qp(a,x)*
        pow(Dplus(x,k),2.)*(sigma_J_sq_table(k,x)+ q.param_B*q.param_b*sigma_J_prime_sq_table(k,x)*pow(k_star/k,q.param_a))/pow(x/a_min,2.)/E(x); },1.0e-2);
  return 3.0*integrate (a_min, a);
}
```

The parameter to ensure GR takes the unchanged interaction term and the rest of the implementation remains the same.

6.1 Implementing the Chameleon model

Instead of using the single parametrization in this general case which has a non zero p_7, we use the 2 screening mechanisms which have been given in Section 3 as: $B = 1/3, a_1 = 8.34, b_1 = 6.21, r_{01} = 0.52, a_2 = -5.82, b_2 = 1.52, r_{02} = 28.1$. We also use the approximation that $k_{*1/2} \approx \frac{1}{r_{01/02}}$. To incorporate this, we have to include 2 J' integrals, and 2 additional terms in the interaction. The code is a straightforward extension of the above.

One of the additional J' integral is well behaved using the second parameters, however, the first parameter J' oscillates:

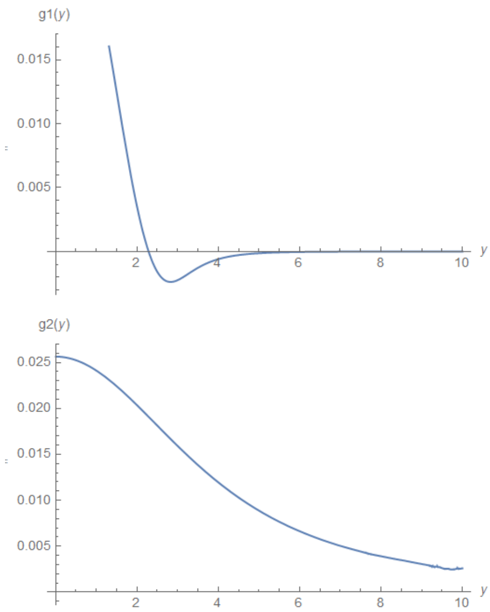

Figure 7: The $Bb_2(y_{*2})^{a_2}J_2', Bb_1(y_{*1})^{a_1}J_1'$ integrals respectively as a function of y

This results in an oscillatory non-linear power spectrum, which also might be due to numerical interpolation in the tables:

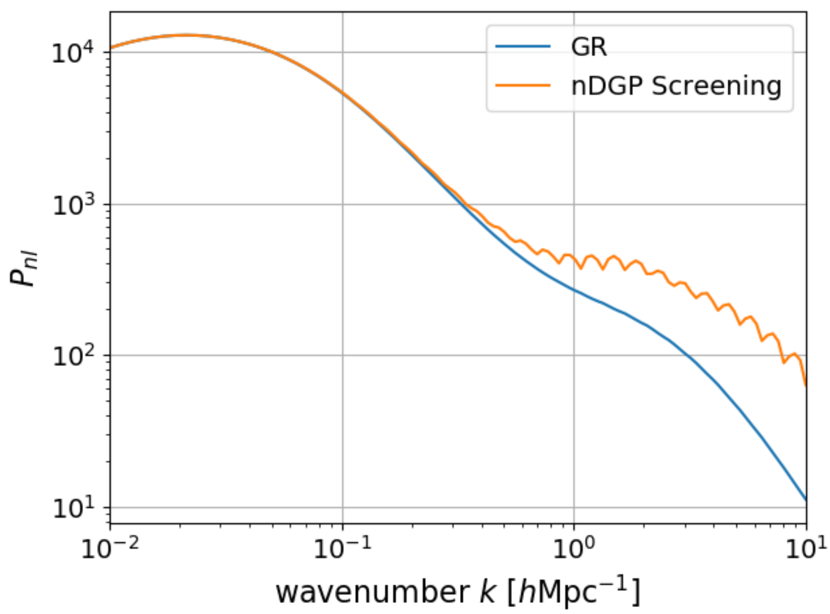

Figure 8: The non-linear power spectrum for the Chameleon model in Brans-Dicke representation

6.2 Implementing the Yukawa suppression model

Here, since we are not aware of the proper p_1 values to use for the model, we start with 2.0, and we can see the dependency of this parameter further by running over it. Using the parameters as defined above, we get the following non-linear power spectrum:

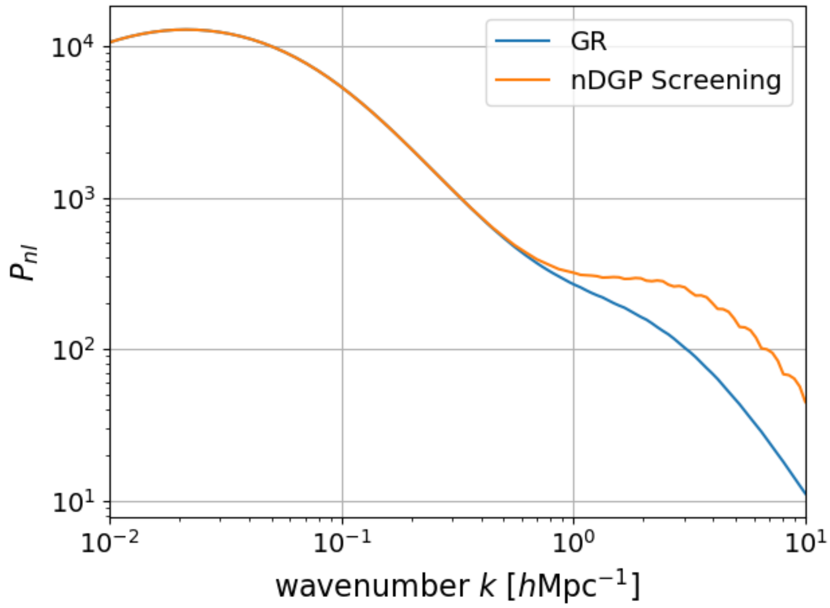

Figure 9: The non-linear power spectrum for the Yukawa suppression model $p_1 = 2.0$

Increasing the p_1 to 10.0 smoothens the curve out a little bit as seen:

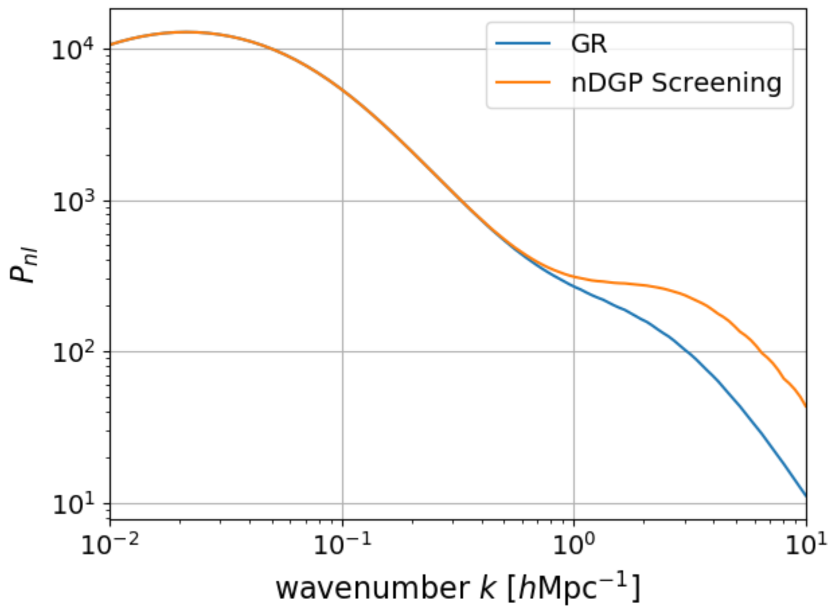

Figure 10: The non-linear power spectrum for the Yukawa suppression model $p_1 = 10.0$

6.3 Implementing the k-mouflage model

Here, since we are not aware of the proper p_1 values to use for the model, we start with 2.0, and we can see the dependency of this parameter further by running over it. Using the parameters as defined above, we get the following non-linear power spectrum:

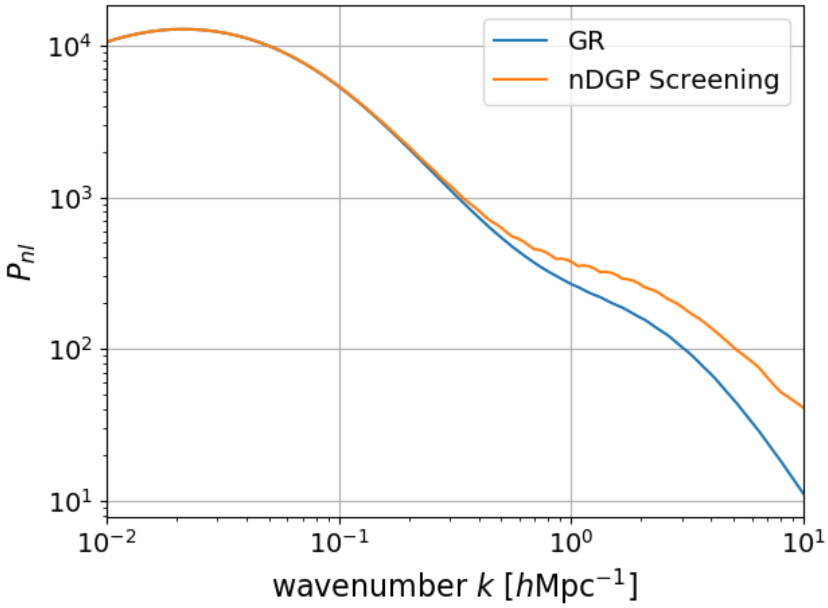

Figure 11: The non-linear power spectrum for the k-mouflage model $p_1 = 2.0$

Increasing the p_1 to 10.0 smoothens the curve out a little bit as seen:

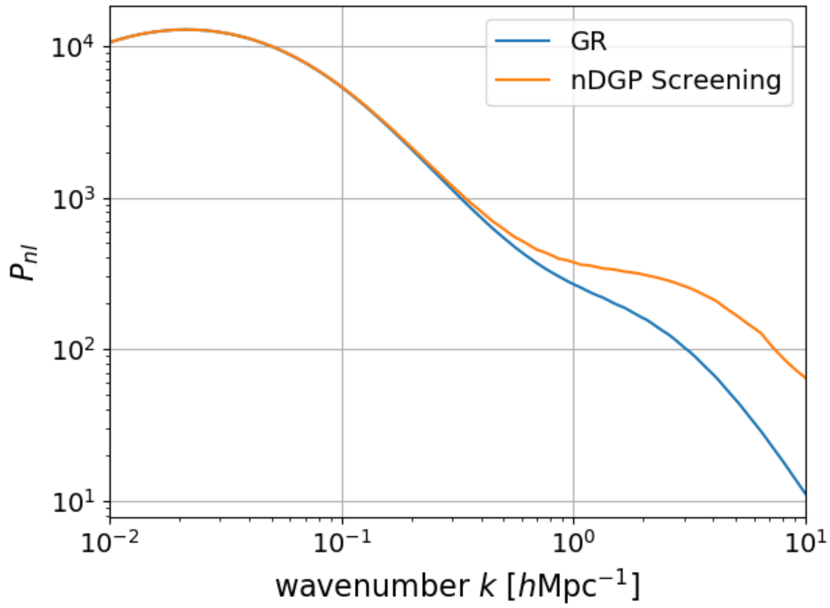

Figure 12: The non-linear power spectrum for the k-mouflage model $p_1 = 10.0$

References

[1] Farbod Hassani and Lucas Lombriser. N-body simulations for parametrized modified gravity. *Monthly Notices of the Royal Astronomical Society*, 497(2):1885–1894, jul 2020.

[2] Kazuya Koyama, Atsushi Taruya, and Takashi Hiramatsu. Nonlinear evolution of the matter power spectrum in modified theories of gravity. *Physical Review D*, 79(12), jun 2009.

[3] Lucas Lombriser. A parametrisation of modified gravity on nonlinear cosmological scales. *Journal of Cosmology and Astroparticle Physics*, 2016(11):039–039, nov 2016.