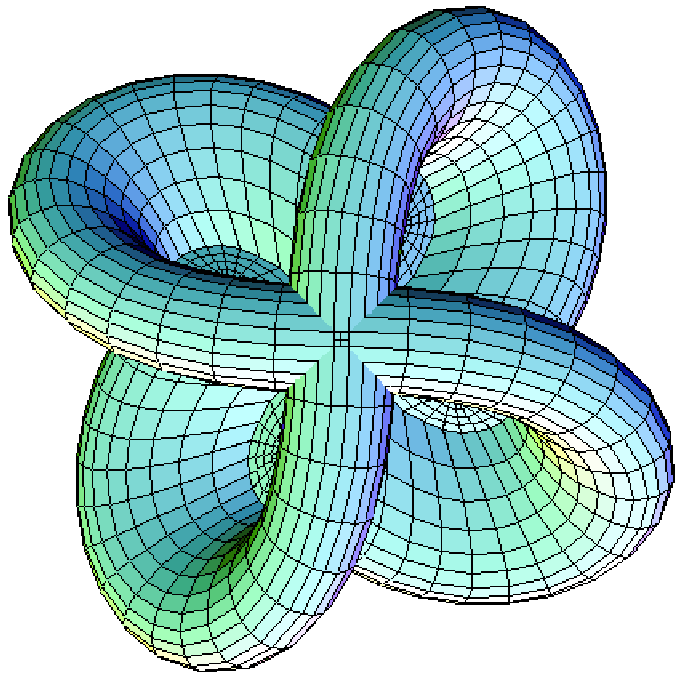

Differential Topology

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VSRP-2021

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§0. Preliminaries

A function from A to B is denoted as $f : A \longrightarrow B$ where A is domain of f and B is codomain. $f : a \mapsto b$ indicates that $f(a) = b$.

A function f is injective if whenever $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$, surjective if for all $b \in B$, there is some $a \in A$ such that f(a) = b, and is bijective if it is both injective and surjective.

We say that $\{U_\alpha\}$ covers covers a set X if X is contained in the union $\cup_\alpha U_\alpha$. An open cover (or open covering) of X is a collection of open sets covering X. A cover $\{V_\beta\}$ is a refinement of the another $\{U_\alpha\}$ if every set V_β is contained in atleast one U_α.

If X is a subset contained in $\mathbb{R}^n$, then a subset V of X is relatively open in X if it can be written as $V = \tilde{V} \cap X$, where $\tilde{V}$ is open in $\mathbb{R}^n$. If Z is a subset of X, we can also speak of open covers of Z in X, meaning coverings of Z by relatively open subsets of X.

By the second countability of $\mathbb{R}^n$, every open cover of X in $\mathbb{R}^n$ has a countable refinement, and hence every every open cover of Z relative to X has a countable refinement.

§1. Manifolds and Smooth Maps

§§1.1. Definitions

Definition 1.1. A mapping f of an open set $U \in \mathbb{R}^n$ into $\mathbb{R}^m$ is called smooth if it has continuous partial derivatives of all orders.

We need a generalization to arbitrary subsets, since it is not possible to define partial derivatives if the domain is not open.

Definition 1.2. A mapping f of an arbitrary subset $X \in \mathbb{R}^n$ into $\mathbb{R}^m$ is called smooth if it can be locally extended to a smooth map on open sets, that is, if around each point $x \in X$ there is an open set $U \in \mathbb{R}^n$ and a smooth map $F : U \to \mathbb{R}^m$ such that F equals f on $U \cap X$

Hence, smoothness is a local property; $f : X \to \mathbb{R}^m$ is smooth if it is smooth in a neighborhood of each point of X.

Definition 1.3. A smooth map $f : X \to Y$ of subsets of two Euclidean spaces is a diffeomorphism if it is one to one and onto, and if the inverse map $f^{-1} : Y \to X$ is also smooth. X and Y are diffeomorphic if such a map exists

Composite of diffeomorphism functions are diffeomorphic. If X is a subset of $\mathbb{R}^n$, then -

Definition 1.4. X is a k-dimensional manifold if it is locally diffeomorphic to $\mathbb{R}^k$, meaning that each point x possesses a neighborhood V in X which is diffeomorphic to an open set U of $\mathbb{R}^k$.

Definition 1.5. A diffeomorphism $\phi : U \to V$ is called a parametrization of the neighborhood V The inverse diffeomorphism $\phi^{-1} : V \to U$ is called a coordinate system on V.

We can prove that every point in X has a neighborhood diffeomorphic to all of $\mathbb{R}^k$. Thus local parametrizations may always be chosen with all of $\mathbb{R}^k$ for their domains.
This is someties represented as $\phi^{-1} = x_1, ...x_k$, and the k smooth functions are called as the coordinate functions, that is a point $v \in V$ is identified with its coordinates $(x_1(v), ...x_k(v)) \in U$.
The dimension k of X is often written dim X.

Example 1.6. $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ is a 1D manifold. We can check that the following 4 functions map (-1,1) bijectively into the upper,down,left and right semicircles respectively, and the inverses are smooth, making these parametrizations -
$\phi_1(x) = (x, \sqrt{1-x^2})$ (y>0)
$\phi_2(x) = (x, -\sqrt{1-x^2})$ (y<0)
$\phi_3(y) = (-\sqrt{1-y^2}, y)$ (x>0)
$\phi_3(y) = (\sqrt{1-y^2}, y)$ (x<0)

We can cover it with just 2 maps using the stereographic projection, but it is impossible to cover with just 1 map, since that would make S^1 homeomorphic to the interval, which cannot be the case since removing a point from the interval disconnects the interval, but not S^1.

More generally, the n-sphere in $\mathbb{R}^n$, $S^n = \{x \in \mathbb{R}^n | |x| = 1\}$, is an n-dimensional manifold.

Theorem 1.7. If X and Y are manifolds, so is $X \times Y$, and dim $X \times Y$ = dim X+ dim Y.

Proof. Suppose that X and Y are manifolds inside $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, so that $X \times Y$ is a subset of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{m+n}$. If dimX = k, thus for each $x \in X$, find an open set $U \in \mathbb{R}^k$ and a local parametrization $\phi : U \to X$ around x. Similarly, if dim Y = l and $y \in Y$, an open $V \in \mathbb{R}^l$ and a parametrization $\psi : V \to Y$ around y. Defining $\phi \times \psi : U \times V \to X \times Y$ as $\phi \times \psi(u, v) = (\phi(u), \psi(v))$. By checking that this map is smooth, bijective and inverse is locally smooth, it is a local parametrization of $X \times Y$. The dimensionality follows from the above. □

Definition 1.8. If X and Z are both manifolds in $\mathbb{R}^N$ and $Z \subset X$, then Z is a submanifold of X.

Any open set of X is a submanifold of X.

§§1.2. Derivatives and Tangent

Definition 1.9. Let f be a smooth map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^m$ and x is any point in its domain. Then for any vector $h \in \mathbb{R}^n$, the derivative of f in the direction h, taken at the point x, is defined by the conventional limit as -$df_x(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$.

Thus $df_x : \mathbb{R}^n \to \mathbb{R}^m$, and it is a linear map, and hence can be represented in the standard basis as the Jacobian Matrix -

Definition 1.10. $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$

The Chain rule is similarly defined as $d(g \circ f)_x = dg_{f(x)} \circ df_x$. Note that if $f : U \to \mathbb{R}^m$ is itself a linear map L, then $df_x = L$ for all $x \in U$.

Definition 1.11. Suppose that X sits in $\mathbb{R}^n$ and that $\phi : U \to X$ is a local parametrization around x, where U is an open set in $\mathbb{R}^k$ and let $\phi(0) = x$. We define a linear approximation to ϕ at 0 as the map $u \mapsto \phi(0) + d\phi_0(u)$. Hence, the tangent space of X at x is defined to be the image of the map $d\phi_0 : \mathbb{R}^k \to \mathbb{R}^n$, and is denoted by $T_x(X)$. A tangent vector to $X \subset \mathbb{R}^n$ at $x \in X$ is a point v of $\mathbb{R}^n$ that lies in the vector subspace $T_x(X)$ of $\mathbb{R}^n$.

Theorem 1.12. $T_x(X)$ is well defined, and another choice of local parametrization produces the same tangent space

Proof. IF $\psi : V \to X$ is another choice with $\psi(0) = x$, shrink U and V such that $\phi(U) = \psi(V)$, which makes $h = \psi^{-1} \circ \phi$ a diffeomorphism. Diffrentiating this, we get $d\phi_0 = d\psi_0 \circ dh_0$, making the image of $d\phi_0$contained in $d\psi_0$, and a similar logic to prove the converse, which shows that their ranges are the same, hence making the tangent space well defined. $\square$

Theorem 1.13. The dimension of the vector space $T_x(X)$ is the dimension k of X.

Proof. Choose a smooth map $\Phi' : \mathbb{R}^n \to \mathbb{R}^k$ which extends ϕ^{-1}. Thus $\Phi' \circ \phi$ is the identity map on U, thus by chain rule, $d\phi_0 \circ d\Phi'$ is the identity map on $\mathbb{R}^k$, from hich it follows that $d\phi_0$ is an isomorphism, proving $dimT_x(X) = k$ $\square$

After deriving these for a smooth map between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, we now construct the best linear approximation of a smooth map of arbitrary manifolds $f : X \to Y$ at a point x. We require this definition to satisfy chain rule and be the same as the usual for Eucledian spaces. Suppose that $\phi : U \to X$ parametrizes X about x and $\psi : V \to Y$ parametrizes Y about y, where $U \in \mathbb{R}^k$ and $V \in \mathbb{R}^l$, and $\phi(0) = x$, $\psi(0) = y$. If U is small enough, then we can draw -

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi^{-1}"'] & Y \arrow[d, "\psi^{-1}"] \\ U \arrow[r, "h=\psi^{-1}\circ f\circ\phi"'] & V \end{tikzcd}$$

Taking derivatives leads us to -

$$\begin{tikzcd} T_x(X) \arrow[r, "df_x"] \arrow[d, "d\phi^{-1}"'] & T_y(Y) \arrow[d, "d\psi_0^{-1}"] \\ \mathbb{R}^k \arrow[r, "dh_0"'] & \mathbb{R}^l \end{tikzcd}$$

Which leads to the definition -

Definition 1.14. $df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$

We can verify this definition of df_x, we must verify that it does not depend on the particular parametrizations ϕ and ψ used.

Theorem 1.15. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps of manifolds, then
$d(g \circ f)_x = dg_{f(x)} \circ df_x$

Proof. Let f be the same map as above and g be another smooth map, as seen from the commutative diagram (where $W \in \mathbb{R}^m$) and $\eta(0) = z$ -

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi^{-1}\big\downarrow & & \big\downarrow\psi^{-1} \\ U & \xrightarrow[h=\psi^{-1}\circ f\circ\phi]{} & V \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \phi^{-1}\big\downarrow & & \big\downarrow\eta^{-1} \\ V & \xrightarrow[j=\eta^{-1}\circ g\circ\psi]{} & W \end{array}$$

from where we get -

$$\begin{array}{ccc} X & \xrightarrow{g\circ f} & Z \\ \phi^{-1}\big\downarrow & & \big\downarrow\eta^{-1} \\ U & \xrightarrow[j\circ h]{} & W \end{array}$$

using this and the chain rule for maps of open subsets of Euclidean spaces, we get
$d(g \circ f)_x = d\eta_0 \circ dj_0 \circ dh_0 \circ d\phi_0^{-1} = dg_y \circ df_x$ □

§§1.3. Inverse Function Theorem and Immersion

Definition 1.16. If X and Y are smooth manifolds of the same dimension, then if a smooth map $f : X \to Y$ maps a neighbourhood around x diffeomorphically onto a neighborhood of $y = f(x)$ we call f a local diffeomorphism at x.

A local diffeomorphism, if it is also one one, is actually a diffeomorphism of X into an open subset of Y.

Theorem 1.17. If f is a local diffeomorphism at x, then its derivative mapping $df_x : T_x(X) \to T_y(Y)$ is an isomorphism

Proof. If f is a diffeomorphism on a neighbourhood of x, f^{-1} is smooth. From the chain rule $di_x = d(f \circ f^{-1})_x = df^{-1}_{f(x)} \circ df_x \implies df^{-1}_{f(x)} = (df_x)^{-1}$. Since di_x is the identity map, df_x is an invertible linear map, making it bijective and hence an isomorphism of the 2 tangent spaces. □

Theorem 1.18. The Inverse Function Thoerem states that if f is a smooth map and df_x is an isomorphism, then f is a local diffeomorphism at x.

(The proof is similar to the ones for Eucledian subsets as X and Y). The criteria of df_x to be an isomorphism reduces to checking that it is non singular (from bijection). Hence, if the determinant of df_x (also called the Jacobian) is non zero, then f is a local diffeomorphism.

Definition 1.19. 2 maps $f : X \to Y$ and $f' : X' \to Y'$ are equivalent (or same upto a diffeomorphism) if there exist diffeomorphisms α and β completing the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha^{-1}"'] & Y \arrow[d, "\beta^{-1}"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

In this terminology, the Inverse Function Theorem says that if df_x is an isomorphism, then f is locally equivalent, at x, to the identity (We can see this by taking the composition of the functions from U to V and V to Y resulting in a new diffeomorphism, and hence f being equivalent to the identity on U). Hence, f is locally equivalent to the identity precisely when df_x is.

Definition 1.20. If dimX < dimY, and $df_x : T_x(X) \to T_y(Y)$ is injective, then f is called an immersion at x. If f is an immersion at every point, it is called an immersion.

When dimX = dimY, the immersions are same as local diffeomorphisms.

Definition 1.21. The canonical immersion is the standard inclusion map of $\mathbb{R}^k$ into $\mathbb{R}^l$ for $l \geq k$, where $(a_1, ..., a_k) \mapsto (a_1, ...a_k, 0, 0, , 0)$. This is the only local immersion upto a diffeomorphism.

Theorem 1.22. The Local Immersion Theorem - Let $f : X \to Y$ is an immersion at x and y = f(x). Then there are local coordinates such that $f(x_1, ..., x_k) = (x_1, ...x_k, 0, ..0)$, that is, f is locally equivalent to the canonical immersion near x.

Proof. Choose the parametrizations such hat the commutative square becomes $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi^{-1}"'] & Y \arrow[d, "\psi^{-1}"] \\ U \arrow[r, "g"'] & V \end{tikzcd}$. Here since dg_0 is an injective map, we can always define a change of basis such that it becomes an $l \times k$ matrix of the form $(\frac{I_k}{0})$. Define $G : U \times \mathbb{R}^{l-k} \to \mathbb{R}^l$ as $G(x, z) = g(x) + (0, z)$. Since dg_0 is an isomorphism (identity) and G maps $\mathbb{R}^l$ to itself, G is a diffeomorphism. Note that $g = G \circ (cannonical\ immersion)$, and $\psi \circ G$ will be a diffeomorphism, shrinking U and V we can see that $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi^{-1}"'] & Y \arrow[d, "(\psi \circ G)^{-1}"] \\ U \arrow[r, "Cannonical\ Immersion"'] & V \end{tikzcd}$ $\square$

We can now also see that if f is an immersion at x, then it is an immersion in a neighborhood of x. This however doesn't necessitate that the map f(X) is a submanifold on Y. This is because by the local immersion theorem, although a neighbourhood of X is mapped diffeomorphically to Y, f(W) need not be open, and hence need not be parametrizable.

Definition 1.23. A map $f : X \to Y$ is called proper if the preimage of every compact set in Y is compact in X

Definition 1.24. An immersion that is injective and proper is called an embedding

Theorem 1.25. An embedding $f : X \to Y$ maps X diffeomorphically onto a submanifold of Y.

Proof. If we show that the image of open W in X is open in Y, then we are done. Let f(W) not be open in Y. Thus there is a sequence of points $y_i \in f(X)$ but not in f(W) but which converge to a point y of f(W). The set $\{y, y_1\}$ is compact, and hence preimage $\{x, x_i\}$ is compact, with x_i converging to x. Since W is open, and $x_i \longrightarrow x$, for large enough i, $x_i \in W$, contradicting that $y_i \notin f(W)$. Thus f(X) is a submanifold of Y. Since f is a local diffeomorphism and bijective, hence f is a diffeomorphism from X to f(X). □

If X is compact, every smooth map is proper, and hence embeddings are just 1-1 immersions.

§§1.4. Submersions

Definition 1.26. Let $dimX \geq dimY$, and $f : X \to Y$ such that $df_x : T_x(X) \to T_y(Y)$ is surjective, then f is called a submersion at x. A map that is a submersion at every point is simply called a submersion

Definition 1.27. The canonical submersion is the standard projection of $\mathbb{R}^k$ onto $\mathbb{R}^l$ for k > l, in which $(a_1, ...a_k) \mapsto (a_1, ...a_l)$

Theorem 1.28. Local Submersion Theorem states that if $f : X \to Y$ is a submersion at x and y=f(x), then there are local coordinates around x and y such that $f(x_1, ...x_k) = (x_1, ...x_l)$, ie f is locally equivalent to the canonical submersion near x.

Proof. The proof is similar to Theorem 1.22, but here we define dg_0 with a linear change of coordinates with an $l \times k$ matrix containg the Identity I_l and define $G = (g(a), a_{l+1}....a_k)$. □

Definition 1.29. Let $f : X \to Y$, then the subset of X which are solutions of $f(x) = y$ for some $y \in Y$, is called the preimage of y denoted as $f^{-1}(y)$.

Definition 1.30. For smooth map $f : X \to Y$, a point $y \in Y$ is called a regular value if df_x : $T_x(X) \to T_y(Y)$ is surjective for every point x such that f(x) = y.

Theorem 1.31. Pre Image Theorem states that if y is a regular value of $f : X \to Y$, then $f^{-1}(y)$ is a submanifold of X, with $dim f^{-1}(y) = dim X - dim Y$

Proof. If y is a regular value, then f is a submersion at $x = f^{-1}(y)$, by the Local Submersion Theorem, choose local coordinates $f(x_1, ...x_k) = (x_1, ...x_l)$, let y correspond to (0,....0), thus $f^{-1}(y) = (0, ...0, x_{l+1}, ...x_k)$. Let V denote the neighborhood of x on which the coordinate system $(x_1, ...x_k$ is defined, and $x_{l+1}....x_k$ form a coordinate system on $f^{-1}(y) \cap V$, a relatively open subset in $f^{-1}(y)$. This shows that it is a submanifold of X, with dimension k-l. □

Definition 1.32. A point $y \in Y$ that is not a regular value of f is called a critical value.

For a y not in the image of X, it is vacously a regular value.

Example 1.33. The eucledian norm map, $f : \mathbb{R}^k \to \mathbb{R}$, has a derivative at a point as $(2a_1, ...2a_k)$, and is surjective unless f(a) = 0. Thus, every non zero real number is a regular value of f, and hence the sphere $S^{k-1} = f^{-1}(1)$ is a k-1 dimensional manifold

Example 1.34. Similarly, if O(n) is the group of orthogonal matrices, with M(n) as the set of all nxn matrices, and S(n) as the set of symmetric matrizes, we define $f; M(n) \to S(n)$ as $f = AA^T$. $df_A(B) = BA^T + AB^T$ can be computed, and seeing that $O(n) = f^{-1}(I)$, we can show that I is a regular value by showing that for any $C \in S(n)$, $B = CA/2$ is the required matrix solving $df_A(B) = C$, proving surjectivity and that I is a regular value, hnce O(n) is a submanifold with dimO(n) = n(n-1)/2.

O(n) is also a group, and a group that is a manifold, and whose group operations are smooth, is called a Lie group.

Proposition 1.35. If smooth real valued functions $g_1, ...g_l$ on X are independent (that is, the l functionals $dg_1(x), ...dg_l(x)$ on $T_x(X)$ are linearly independent) at each point where they vanish, then set Z of these common zeros is a submanifold of X wih dimZ = dim X -l.

Proof. Consider the map $g = (g_1, ...g_l); X \to \mathbb{R}^l$, and let $Z = g^{-1}(0)$, is a submanifold iff 0 is a regular value. We can see that $dg_x : T_x(X) \to \mathbb{R}^l$ is surjective iff the l functionals are linearly independent. □

Definition 1.36. The codimension of any arbitrary submanifold Z of X is given by codimZ = dim X - dim Z.

Thus l independent functions on X cut out a submanifold of codimension l.

Theorem 1.37. If y is a regular value of smooth $f : X \to Y$, then preimage submanifold $f^{-1}(y)$ can be cut out by independent functions

Proof. Let y in Y be a regular value of smooth f. Define a diffeomorphism $h : Y \to \mathbb{R}^l$ in a neighbourhood W of y, and let h(y) = 0. Let $g = h \circ f$. We can see that since dh is surjective at 0, and df at each preimage of y, so is dg, making 0 a regular value of g. Hence, $g^{-1}(0) = f^{-1}(h^{-1}(0)) = f^{-1}(y)$ is a submanifold cut out by g □

Theorem 1.38. Every submanifold of X is locally cut out by independent functions.

Proof. Let Z be a manifold of X of dimension l. Define $f : Z \to X$ as an immersion on Z. Thus, there exists a neighbourhood W in X around any point z in Z such that f is the cannonical immersion on $Z \cap W$, and hence we have k-l independent (due to the definition of g relatng to f) functions $g_{l+1}, ...g_k$ such that it they vanish locally around z. □

Proposition 1.39. Let Z be the preimage of a regular value y in Y under smooth $f : X \to Y$. Then the kernel of the derivative $df_x : T_x(X) \to T_y(Y)$ at any point $x \in Z$ is the tangent space to Z at x, $T_x(Z)$.

Proof. Since f is constant y on Z, df_x is zero on $T_x(Z)$. Thus $T_x(Z)$ is a subspace of the kernel, but $dimT_x(Z) = dimZ = dimX - dimY = dimT_x(X) - dimT_y(Y) = dim\ kernel\ df_x$, and hence, $T_x(Z)$ is the kernel of df_x □

§§1.5. Transversality

Definition 1.40. We call a map f transversal to a submanifold Z written as $f \pitchfork Z$ if at every preimage point of Z, $Image(df_x) + T_y(Z) = T_y(Y)$.

Theorem 1.41. If a smooth map $f : X \to Y$ is transversal to $Z \subset Y$, then $f^{-1}(Z)$ is a submanifold of X. The codimension of $F^{-1}(Z)$ in X is same as the codimension of Z in Y.

Proof. Since the property of being a manifold is local, we need to find a neighbourhood U such that $f^{-1}(Z) \cap U$ is a manifold. At each point f(x) = y for $y \in Z$, using Theorem 1.38, we can find l smooth independent $g_1, ...g_l$ functions which locally cut out Z around y, l being the codimension of Z in Y. Thus $f^{-1}(Z)$ is the zero set of the functions $g_1 \circ f, ...g_l \circ f$, and is a submanifold if 0 is a regular value of $g \circ f$. $d(g \circ f)_x = dg_y \circ df_x : T_x(X) \to \mathbb{R}^l$is surjective if dg_y carries df_x surjectively to $\mathbb{R}^l$, iff f is transversal according to the equation above (from the rank nullity theorem). The codimension follows by noting that there are l composite functions whose 0 set is the inverse map of Z, same as the l original smooth functions cutting out Z. □

If Z is a single point, we are back to the definition of regular value of y.

Definition 1.42. 2 submanifolds X and Z of Y are said to be transversal or $X \pitchfork Z$, iff for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.

Theorem 1.43. The intersection of 2 transversal submanifolds of Y is another submanifold and $codim(X \cap Z) = codimX + codimZ$.

Proof. Consider the inclusion map i of some submanifold X in Z, such that $i^{-1}(Z)$ is the set of points $X \cap Z$. The derivative is also the inclusion map of $T_x(X)$ into $T_y(Y)$. Thus $i \pitchfork Z$ iff Z and X are transversal. The codimensions add since X is the set of say l vanishing functions and Z is of k, then their intersection must be the set of k+l vanishing functions. □

We can also see that if Z and X are transversal submanifolds of Y then for all $y \in X \cap Z$, $T_y(X \cap Z) = T_y(X) \cap T_y(Z)$. The tangent space to the preimage of Z is the preimage of the tangent space of Z. Let $X \to Y \to Z$ be a sequence of smooth maps of manifolds f and g, and assume that g is transversal to a submanifold W of Z. We can show that f is transversal to $g^{-1}(W)$ if and only if $g \circ f$ is transversal to W.

§§1.6. Homotopy and Stability

Let I be the unit interval [0,1] in $\mathbb{R}$.

Definition 1.44. Let f_0, f_1 be smooth maps from X to Y. f_0 and f_1 are called homotopic, written as $f_0 \sim f_1$ if there exists a smooth map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. F is called a homotopy between f_0 and f_1.

Homotopy is clearly an equivalence relation, and the equivalence class to which a mapping belongs to is called a homotopy class. We can define $f_t(x) = F(x, t)$.

Definition 1.45. A property is called stable provided that whenever $f_0 : X \to Y$ possesses that property, there is some $\epsilon > 0$ such that each f_t with $t < \epsilon$ also possesses that property where f_t is a homotopy of f_0

The collection of maps that possess a particular stable property may be referred to as a stable c/ass of maps.

Theorem 1.46. Stability Theorem states that the following classes of smooth maps of a compact manifold X to a manifold Y are stable classes -

- Local diffeomorphisms
- Immersions
- Submersions

- Maps transversal to some submanifold $Z \subset Y$
- Embeddings
- Diffeomorphisms

Proof. Since local diffeomorphisms are just immersions with dimX = dim Y, we can just prove (b). Thus, given f_0 immersion and f_t homotopy of f_0, we need to find an $\epsilon > 0$ such that $(df_t)_x$ is injective for all point $(x,t) \in X \times [0,\epsilon)$. Since X is compact, by the tube lemma, any open neighbourhood of $X \times \{0\}$ in $X \times I$ contains $X \times [0,\epsilon]$ for sufficiently small ϵ. Thus we need only prove that each point $(x_0, 0)$ has a neighborhood U in $X \times I$ such that $(df_t)_x$ is injective for $(x,t) \in U$. By injectivity of df_0, the jacobian $l \times k$ matrix contains a submatrix kxk, with non zero determinant. Since both f_t continous and det continous, the same kxk submatrix must be non singulat for all points (x,t) in a neighbourhood of $(x_0, 0)$, hence f_t must be non singular in some neighbourhood of $(x_0, 0)$.
The proof for c is the exact same, and since transversality wrt Z can be translated to a submersion condition, d follows.
For embeddings we need to to show that if f_0 is one one so is f_t locally. If not, we define the smooth map $G(x,t) = (f_t(x), t)$, and with distinct pairs of x_i, y_i in X, and a sequence of $t_i \longrightarrow 0$, such that $G(x_i, t_i) = G(y_i, t_i)$. From compactness of X, there is a subsequence on which x and y converges. Forming a sequence of such points and letting them converge to $x_0 and y_0$, we note that $x_0 = y_0$ for injectivity on f_0. Locally looking at $dG_{(x_0,0)}$, we have k+1 independent rows, hence making it an injective linear map, hence an immersion and must be one-one on some neighbourhood of $(x_0, 0)$, we obtain a contradiction to our assumption.
To prove for diffeomorphisms, since they are just surjective embeddings, we need to prove surjectivity. If X and hence Y are connected, f_t is a local diffeomorphism for small enough t from (a). Hence $f_t(X)$ is open in Y, but since X is compact, is also closed, and by connectedness is the whole of Y. Hence it is surjective. For non connected, we can find components (finite due to compactness) and we can take the minimum t satisfying for all components. □

Example 1.47. The stability theorem is not true for non compact domains. Consider the counterexample :
Let $\rho : \mathbb{R} \to \mathbb{R}$ be such that $\rho(s) = 1 if |s| < 1$, and $\rho(s) = 0 if |s| > 2$. Define the map $f_t : \mathbb{R} \to \mathbb{R}$ as $f_t(x) = x\rho(tx)$. This doesnt saisfy any of the above, since injectivity is violated for any $|x| > \frac{2}{\epsilon}$, and for the submanifold {0} the definition of transversality doesnt hold.

Definition 1.48. A deformation of a submanifold Z in Y is a smooth homotopy $i_t : Z \to Y$, where i_0 is the inclusion map and each i_t is an embedding. Thus $Z_t = i_t(Z)$ is a smoothly varying submanifold of Y with $Z_0 = Z$.

§§1.7. Sards Theorem and Morse Functions

Definition 1.49. An arbitrary set A in $\mathbb{R}^l$ is said to be of measure 0 if for every $\epsilon > 0$ there is a countable covering of A by solids as $\{S_1, S_2, ...\}$ such that $\sum vol(S_i) < \epsilon$.

S_i can be taken to be cubes. We can extend this to manifolds by -

Definition 1.50. An arbitrary subset C of a manifold Y is said to have measure 0 provided that it can be covered by the images of some collection of local parametrizations ψ_α satisfying the condition that $\psi_\alpha^{-1}(C)$ has measure zero for each α

We need not check for each parametrization since if A in $\mathbb{R}^l$ is of measure 0, then so is g(A) for any smooth map $g : \mathbb{R}^l \to \mathbb{R}^l$.

Definition 1.51. A point $x \in X$ is a regular point of f if $df_x; T_x(X) \to T_y(Y)$ is surjective, and a critical point if not. Hence regular points are points of X and regular values are points of Y, which are only regular if each $x = f^{-1}(y)$ is aregular point, and critical if any one such x is critical.

Theorem 1.52. Sards Theorem states that the set of critical values of a smooth map of manifolds $f : X \to Y$ has measure 0.

Corollary states that the regular values of any smooth map $f : X \to Y$ are dense in Y, and if $f_l : X_i \to Y$ are countable smooth maps, the the points which are regular values for all f_i are dense.
This follows from the fact that no set of measure 0 can contain a open set (apart from the null set), and the countable collection of sets of measure 0 is measure 0 (choose for each set i, solids of volume $\frac{\epsilon}{2^i}$, and hence the collection has volume less than the $\sum \frac{\epsilon}{2^i} = \epsilon$.
We can also see an important application that if dimX<dimY, then the df_x is only surjective if it is trivially true, that is the image of any smooth map $f : X \to Y$ is critical, and hence measure 0 in $\mathbb{R}^l$

Definition 1.53. We define the Hessian matrix of second partials $H = (\frac{\partial^2 f}{\partial x_i \partial x_j})$. If the Hessian matrix is non singular $(detH \neq 0)$ at a critical point x, x is called the nondegenerate critical point of f.

Non degenerate critical points are isolated from other critical points, since the derivative map $g = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$ for a non singular Hessian, diffeomorphically maps a neighbourhood of of x to a neighbourhood of (since dg = H is an isomorphism), and hence no other critical points of f exist locally.

Lemma 1.54. Morse Lemma. If $a \in \mathbb{R}^k$ is a nondegenerate critical point of f, and $(h_{ij}) = (\frac{\partial^2 f(a)}{\partial x_i \partial x_j})$

is the Hessian of f at a. Then there exists a local coordinate system $(x_1, ...x_k)$ around a such that $f = f(a) + \sum h_{ij}x_ix_j$ near a.

Definition 1.55. For manifolds, let $f : X \to \mathbb{R}$ has a critical point at x, then x is nondegenerate for f if for any parametrization ϕ carrying origin to X, if 0 is nondegenerate for $f \circ \phi$

Lemma 1.56. If smooth $f : \mathbb{R}^k \to \mathbb{R}$ with a nondegenerate critical point at 0, and $\psi(0) = 0$, is a diffeomorphism, then $f \circ \psi$ has a nondegenerate critical point at 0.

Proof. Let $f' = f \circ \psi$, then the chain rule implies $\frac{\partial^2 f'}{\partial x_l \partial x_j}(0) = \sum_\alpha \sum_\beta \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}(0)\frac{\partial \psi_\alpha}{\partial x_l}(0)\frac{\partial \psi_\beta}{\partial x_j}(0) + \sum_\alpha \frac{\partial f}{\partial x_\alpha}(0)\frac{\partial^2 \psi_\alpha}{\partial x_l \partial x_j}(0)$, last term being 0 for criticality, and impying that $H' = (d\psi_0)^T H(d\psi_0)$, hence since $detH \neq 0$ (assumption) and $det(d\psi_0) \neq 0$ (diffeomorphism), hence $detH' \neq 0$ □

Definition 1.57. Morse Functions are functions whose critical points are all non degenerate

Let manifold X be in $\mathbb{R}^n$ with $(x_1, ...x_n)$ as the usual coordinate functions, and $a = (a_1, ...a_n)$ as an N-tuple. Define $f_\alpha = f + a_1x_1 + ... + a_nx_n$.

Lemma 1.58. If f is a smooth function on open set U of $\mathbb{R}^k$. The for all k tuples $a = (a_1, ...a_k)$, except for a set of measure 0, f_a is a morse function on U

Proof. Similar to above, let $g = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_k})$, hence $df_a(p) = g(p) + a$ at a point p, and is a critical point if g(p) = -a. The hessian of f at p is dg_p, and if -a is a regular value, dg_p is nonsingular. By Sard, the critical values of g are of measure 0. □

We can generalize this and by using second counability for X to prove

Theorem 1.59. No matter what $f : X \to \mathbb{R}$ is, for all $a \in \mathbb{R}^n$ except for a set of measure 0, f_a is a Morse function on X

Morse functions are stable, that is, if f is a Morse function on a compact manifold X and f_t is homotopic family $f_0 = f$, then f_t is Morse for sufficiently small t.

§§1.8. Embedding manifolds in Euclidean space

Definition 1.60. The tangent bundle of a manifold X in $\mathbb{R}^n$ is $T(X) \subset X \times \mathbb{R}^n$ defined as $T(X) = \{(x, v) \in X \times \mathbb{R}^n : v \in T_x(X)\}$

Any smooth $f : X \to Y$ induces a global derivative map $df : T(X) \to T(Y)$ defined as $df(x, v) = (f(x), df_x(v))$. We can see that df is smooth, since it can be locally extended to a smooth map on an open subset of $\mathbb{R}^{2n}$. The chain rule just functions as $d(g \circ f) = dg \circ df$.
We also notice that diffeomorphic manifolds have diffeomorphic tangent bundles.

Proposition 1.61. The tangent bundle of a manifold is another manifold, and dim T(X) = 2 dim(X)

Proof. Let W be a open set of X, and hence a manifold. Notice that $T(W) = T(X) \cap (W \times \mathbb{R}^n)$, since $W \times \mathbb{R}^n$ is open in T(X), T(W) is open in T(X). Since W is a manifold, it can be parametrized as $\phi : U \to W$, U open in $\mathbb{R}^k$, then $d\phi : T(U) \to T(W)$ is a diffeomorphism. Since $T(U) = U \times \mathbb{R}^k$, is an open subset of $\mathbb{R}^{2k}$, and $d\phi$ parametrizes T(W), hence it is a manifold, having dimension 2k. □

Theorem 1.62. Every k dimensional manifold admits a one to one immersion in $\mathbb{R}^{2k+1}$

Proof. Define a map $h : X \times X \times \mathbb{R} \to \mathbb{R}^m$ as h(x,y,t) = t(f(x) - f(y)), where $f : X \to \mathbb{R}^m$ is an injective immersion and finally define $g : T(X) \to \mathbb{R}^m asg(x, v) = df_x(v)$. Let dim(M)>2k+1, hence both h and g map to measure 0 sets of $\mathbb{R}^m$ (by Sards Theorem), hence there exists a point $a \in \mathbb{R}^m$ belonging to neither the images (and $a \neq 0$). Let π be the projection of $\mathbb{R}^m$ onto the orthogonal complement H of a. Thus $\pi \circ f : X \to H$ is injective, if not then $\pi \circ (f(x) - f(y)) = 0 \implies f(x) - f(y) = ta$, which contradicts that $a \notin Image(h)$. It also must be an immersion, if not, then there is a non zero vector v such that $d(\pi \circ f)_x(v) = 0 \implies \pi \circ df_x(v) = 0 \implies df_x(v) = ta$, contradicting $a \notin Image(g)$. Hence, we have obtained an injective immersion of X into $\mathbb{R}^{m-1}$, also, and this is true for all M>2k+1. □

Here we can see that only taking the g function, and following the exact same proof, we can prove the **Whitney Immersion Theorem**, which states that every k dimensional manifold can be immersed in R^{2k} We can prove the following theorem (which we skip here)

Theorem 1.63. If X is a subset of $\mathbb{R}^n$, let $\{U_\alpha\}$ be an open covering of X, then there is a sequence of smooth functions $\{\theta_i\}$ such that :

- $0 \leq \theta_i(x) \leq 1$ for all x in X and all i
- Each x in X has a neighborhood on which all but finitely many functions θ_i are zero
- Each function θ_i is zero except on some closed set contained in one of the U_α
- For each x in X, $\sum_i \theta_i(x) = 1$

The functions are called a partition of unity subordinate to the open cover $\{U_\alpha\}$

Thus, we can define a proper map $\rho : X \to R$ as $\rho = \sum_{i=1}^{\infty} i\theta_i$. From these, we see that

Theorem 1.64. The Whitney Theorem : Every k dimensional manifold embeds in $\mathbb{R}^{2k+1}$

Proof. We first begin with the injective immersion of X in $\mathbb{R}^{2k+1}$, and compose it with a diffeomorphism into the unit ball $z \mapsto z/(|z|^2+1)$. Take a proper function p, and a new injective function as F(x) = (f(x),p(x)) and compose $\pi \circ F$ to H where H is the linear space perpendicular to some a in $\mathbb{R}^{2k+2}$, which is also an injective immersion. Pick some a that it is neither of the poles of S^{2k+1}. We can see that $\pi \circ F$ is proper, we can prove that $|\pi \circ F| \leq c$ is contained in some $|p(x)| \leq d$, if not, then there is a sequence of points for which $p(x_i)$ blows up, and hence the vector $\frac{F(x_i)-\pi\circ F(x_i)}{p(x_i)}$ is a multiple of a, and so is the limit, which is the north pole, a contradiction to choosing a. $\square$

§2. Transversality and Intersection

§§2.1. Manifolds with Boundary

Definition 2.1. Let the upper half space in $\mathbb{R}^k$ be H^k, consisting of all points with non-negative final coordinates. We call a subset X of $\mathbb{R}^n$ a k dimensional manifold if every point of X possesses a neighbourhood diffeomorphic to an open set in H^k, and such a diffeomorphism is called a parametrization. The boundary ∂X, consists of those points which belong to the image of the boundary of H^k. Int(X), the interior, is just $X - \partial X$.

Proposition 2.2. The product of a manifold without boundary X and a manifold with boundary Y is another manifold with boundary, with the relation $dim(X \times Y) = dimX + dimY$ and $\partial(X \times Y) = X \times \partial Y$

Proof. If $\phi : U \to X$ where $U \subset \mathbb{R}^k$, $\psi : V \to Y$ where $V \subset H^l$ are open subsets then $U \times V$ is open and $\phi \times \psi$ are the required parametrization. □

Tangent spaces and derivatives are defined in the setting of manifolds with boundary in the same way, with the ones on the boundary using the extension of a smooth map, which is still a linear map between the Eucledian spaces, and we can show that the choice of the extension does not matter. Chain rule holds, the tangent space $T_x(X)$ is still a k dimensional linear subspace, and derivative of smooth maps between manifolds are defined in the same way.

Proposition 2.3. If X is a k dim manifold with a boundary, then ∂X is a k-1 dimensional manifold without a boundary

Proof. Let $x \in \partial X$, and $\phi : U \to V$ maps open sets of H^k to X. If we prove $\phi(\partial U) = \partial V$, then we are done since there is a diffemorphism from $\partial U = U \cap \partial H^k$, open in $\mathbb{R}^{k-1}$ (which doesnt contain any boundary) to open neighbourhood $\partial V = \partial X \cap V$, hence, it has no boundary and is k-1 dim manifold. $\phi(\partial U) \subset \partial V$ by defn, for the reverse inclusion, we take ψ as a parametrization W to V, hence, we need to show $\partial U \phi^{-1} \circ \psi(\partial W)$, *call this g*. If we assume a point $w \in \partial W$ maps to an interior point of u=g(w), then using ψ, ϕ, g are diffemorphisms, g(w) contains a neighborhood of u that is open in $\mathbb{R}^k$, and hence w has such a neighbourhood too, contradicting that $w \in \partial W$ □

Definition 2.4. For any smooth f on X, ∂f is the restriction of f to ∂X

If f is a diffeomorphism, then ∂f maps ∂X diffeomorphically into ∂Y.
If X is a manifold with boundary, and $\phi : U \to X$ is a local parametrization such that $\phi(0) = x$, and the upper half space in $T_x(X)$ is defined as $H_x(X) = d\phi_0(H^k)$, and is independent of local parametrization.

Theorem 2.5. Let f be a smooth map from manifold X with boundary to boundaryless manifold Y. Suppose that both f and ∂f both are transversal with respect to a boundaryless submanifold Z in Y. Then preimage $f^{-1}(Z)$ is a manifold with the boundary $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$, and the codimension of $f^{-1}(Z)$ in X equals codimension of Z in Y

The proof is pretty involved, so we wont get into it here, but it uses the useful lemma-

Lemma 2.6. Suppose that S is a manifold without a boundary and $\pi : S \to \mathbb{R}$ is a smooth function with regular value 0. Then $\{s \in S | \pi(s) \geq 0\}$ is a manifold with boundary $\pi^{-1}(0)$

The converse to this lemma can also be proved that: If X is any manifold with a boundary, then there is a smooth non negative function f on X such that $f(\partial X) = 0$ and 0 is a regular value.

Proof. Since $\pi(s) > 0$ is an open set and hence a submanifold of S, and since 0 is a regular value, it is locally equivalent to the cannonical submersion at each preimage of 0, and hence the lemma follows □

Theorem 2.7. For any smooth map f of a manifold X with boundary into boundaryless manifold Y almost every point of Y is a regular value of both f and ∂f

Proof. Since ∂f is just is just the restriction of f, whenever ∂f is regular at x, so is f. So the critical values are the union of those maps on Int(X) and ∂X, both of which are boundaryless, and hence have measure 0, the union of which is also a measure 0 set. □

§§2.2. One Manifolds

Theorem 2.8. The Classification of One Manifolds Every compact, connected 1D manifold with a boundary is diffeomorphic to [0,1] or S^1

The proof is involved, but intuitively we can understand by running along the curve at constant speed, either we hit a boundary or return to the same point on account of compactness.
An important corollary is as follows - The boundary of any compact one-dimensional manifold with boundary consists of an even number of points.

Theorem 2.9. If X is any compact manifold with boundary then there exists no smooth map $G : X \to \partial X$ such that $\partial g : \partial X \to \partial X$ is the identity.

Proof. If such a g exists, let $z \in \partial X$ be a regular point, exists because the set of critical points on ∂X is of measure 0. Then $g^{-1}(z)$ is a submanifold of X with boundary from Theorem 2.5, with codimension in X same as that of {z} in ∂X, ie dim X-1, hence making $g^{-1}(z)$ 1D and compact. Since $\partial g = I$, $\partial g^{-1}(z) = \{z\}$, a single point and contradicting the corollary. □

Theorem 2.10. Brouwer Fixed-Point Theorem : Any smooth map f on the closed unit ball into itself must have a fixed point, that is f(x) = x for some $x \in B^n$

Proof. Let such an f exist without fixed points, there is a unique line passing through x and f(x), and g(x) be the point where this line intersects the boundary. If $x \in \partial B^n$, then on the boundary ∂g is an identity. Since the closed ball is compact and with boundary, we then just need to prove that g is smooth. Write g(x)-f(x) as a multiple t times of x-f(x) with $t \geq 1$. Then g(x) = tx+(1-t)f(x). Using $|g(x)| = 1$, we obtain t smooth and hence g smooth, which is a direct contradiction to the above theorem. □

§§2.3. Transversality

Theorem 2.11. Transversality Theorem : If $F : X \times S \to Y$ is a smooth map of manifolds, where only X has a boundary, and Z be a boundaryless submanifold of Y. If both F and ∂F are transversal to Z, then for almost every s in S, both f_s and ∂f_s are transversal to Z.

Proof. Let $W = f^{-1}(Z)$ is a submanifold of $X \times S$, and boundary $\partial W = W \cap \partial(X \times S)$, and $\pi : X \times S \to S$. We prove that when s is a regular value of for $\pi : W \to S$ then $f_s \bar{\pitchfork} Z$, and same for $\partial \pi$ and ∂f, and applying Sards, the theorem follows. From F transversal to Z we know $dF_{(x,s)}(T_{(x,s)}(X \times S) + T_z(Z) = T_z(Y)$, given some a, there is a b such that $dF_{(x,s)}(b) - a \in T_z(Z)$, we want a vector $v \in T_x(X)\ df_{(x,s)}(v) - a \in T_z(Z)$. We know b=(w,e) and by reguarity assumption on $d\pi$, there is some (u,e) in $T_{(x,s)}(W)$, and hence v=w-u is our required solution. Applying the same for the special case of ∂X and map ∂F, we are done. □

From here, we can also prove the General Position Lemma, which says that if X and Y are submanifolds of $\mathbb{R}^n$, then for almodt every $a \in \mathbb{R}^n$, X+a intersects Y transversally.

Definition 2.12. For each y in Y, define $N_y(Y)$ the normal space of Y at y to be the orthogonal complement of $T_y(Y)$ in $\mathbb{R}^m$. The normal bundle is then $N(Y) = \{(y, v) \in Y \times \mathbb{R}^m | v \in N_y(Y)$, and the projection map $\sigma : N(Y) \to Y$ as $\sigma(y, v) = y$.

Proposition 2.13. If $Y \subset \mathbb{R}^m$, then N(Y) is a manifold of dimension m and σ is a submersion

Proof. Find an open $\tilde(U)$ in $\mathbb{R}^m$ and a submersion (due to the k independent functions) $\phi : \tilde{U} \to \mathbb{R}^k$ with codimension k of Y and $U = Y \cap \tilde{U}$, notice N(U) is open in N(Y) and $d\phi : \mathbb{R}^m \to \mathbb{R}^k$ is surjective and has kernel $T_y(Y)$, and by a result of linear algebra the transpose maps $\mathbb{R}^k$ isomorphically onto $N_y(Y)$. Thus $\psi : U \times \mathbb{R}^k \to N(U)$ as $\psi(y, v) = (y, d\phi_y^t v)$ is a bijective map and hence parametries N(U) with dimension M, and hence N(Y) is a manifold since each point has such a nbd, the submersion follows. □

Theorem 2.14. ϵ **Nbd theorem**: For a compact boundaryless manifold Y in $\mathbb{R}^m$ and a positive number ϵ, let Y^ϵ be the open set of points with distance less than ϵ from Y. Then each point $w \in Y^\epsilon$ possesses a unique closest point in Y, denoted $\pi(w)$, with π as a submersion. For non compact Y, we can make $\epsilon(y)$.

Proof. Let $h : N(Y) \to \mathbb{R}^m$ as $h(y, v) = y + v$, and it is regular at each $Y \times \{0\}$, since the derivative of h at (y,0) maps $Y \times \{0\}$ onto $T_y(Y)$ and the tangent space of $\{y\} \times N_y(Y)$ onto $N_y(Y)$, and hence onto the entirety of $\mathbb{R}^m$. Since h maps $Y \times \{0\}$ diffeomorphically onto Y and is regular at each (y, O), it must map a neighborhood of $Y \times \{0\}$ diffeomorphically onto a neighborhood of Y in $\mathbb{R}^m$. Thus any nbd of Y contains Y^ϵ, and hence $h^{-1} : Y^\epsilon \to N(Y)$ defined and $\pi = \sigma \circ h^{-1}$ is the desired submersion. □

A useful corollary is if we have $f : X \to Y$ be a smooth map, Y being boundaryless, then there is an open ball S in some Euclidean space and a smooth map $F : X \times S \to Y$ such that F(x, 0) = f(x), and for any fixed x in X the map $s \mapsto F(x, s)$ is a submersion $S \to Y$. In particular, both F and ∂F are submersions.
We see this by letting s be the unit ball and defining $F(x, s) = \pi[f(x) + \epsilon s(f(x))]$, and checking that it is smooth, and a submersion for fixed x. F and ∂F are sumersion, since they are on some manifold $\{x\} \times S$.

Theorem 2.15. Transversality Homotopy Theorem : For any smooth map $f : X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$

Proof. This follows immediately from the family of mappings in the above corollary, then we can choose g to be any f_s as given from the transversality theorem. □

A useful application of this is to prove that if a compact manifold X in Y intersects another manifold Z, but dim X+dim Z<dim Y, then an arbitrary small deformation of X can be used to pull it away from Z. Further if U contains $X \cap Z$, then the deformation can be chosen to be constant outside of U.

Theorem 2.16. Extension Theorem : If Z is a closed submanifold of Y, both are boundaryless and C is a closed subset of X. Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then there is a smooth map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$ and on a neighbourhood of C we have g=f.

A corollary for the special case of ∂X is as follows: If for $f : X \to Y$ $\partial f : \partial X \to Y$ is transversal to Z then there is a map $g : X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g = \partial f$. If Z is a submanifold of Y, then the normal bundle of Z in Y is the set $N(Z; Y) = \{(z, v) : z \in Z, v \in T_z(Y)\}$ and that v is perpendicular to $T_z(Z)$. N(Z;Y) is a manifold with same dimension as Y.

§§2.4. Intersection Theory Mod 2

Definition 2.17. 2 submanifolds X and Z have complementary dimension if dim X + dim Z = dim Y. If $X \bar{\pitchfork} Z$ then $X \cap Z$ is a 0 dimensional manifold

Definition 2.18. If X is a compact manifold and $f : X \to Y$ is a smooth map transversal to closed Z in Y and dimX+dimZ = dim Y, Thm 1.43 makes $f^{-1}(Z)$ a closed 0 dim submanifold and hence a finite set. The mod 2 intersection number of the map f with Z, $I_2(f, Z)$ is the number of points in $f^{-1}(Z)$ modulo 2. For arbitrary smooth g, select homotopic map f transversal to Z (from thm 2.15) and define $I_2(f, Z) = I_2(g, Z)$

Theorem 2.19. If $f_0, f_1 : X \to Y$ are homotopic and both transversal to Z then $I_2(f_0, Z) = I_2(f_1, Z)$

Proof. Consider $F : X \times I \to Y$ be a homotopy of f_0 and f_1, with Extension Theorem, choose F $Bar \pitchfork Z$ and $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$, thus ∂F
$Bar \pitchfork Z$ and $F^{-1}(Z)$ is a 1D manifold with boundary $\partial(F^{-1}(Z) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}$, from the c1assification of one-manifolds, $\partial F^{-1}(Z)$ must have an even number of points. □

The corollary obtained is : If g_0 and g_1 are arbitrary homotopic maps, then $I_2(g_0, Z) = I_2(g_1, Z)$.

Definition 2.20. If X is a compact submanifold of Y and Z is a closed submanifold of complementary dimension (needed for finiteness) define the mod 2 intersection number of X with Z by $I_2(X, Z) = I_2(i, Z)$ where i is the inclusion map. When $X \bar{\pitchfork} Z$ then $I_2(X, Z) = X \cap Z$ mod 2. $I_2(X, X)$ is called as the self intersection number.

Theorem 2.21. Boundary Theorem : Suppose that X is the boundary of some compact manifold W and $g : X \to Y$ is a smooth map. If g may be extended to all of W, then $I_2(g, Z) = 0$ for any closed submanifold Z in Y of complementary dimension.

Proof. If $G : W \to Y$ extends g, from the transversality homotopy theorem obtain $F : W \to Y$ with $F \bar{\pitchfork} Z$ and $f = \partial F \bar{\pitchfork} Z$. Then ofcourse f g so from above $I_2(g, Z) = f^{-1}(Z) mod 2$. But $F^{-1}(Z)$ is 1D compact manifold with boundary, so $\partial F^{-1}(Z) = f^{-1}(Z)$ is even. □

Theorem 2.22. If $f : X \to Y$ is a smooth map of compact X into connected Y such that dim X = dim Y, then $I_2(f, \{y\})$ is the same for all y in Y, and called the mod 2 degree of f denoted as $deg_2(f)$

Proof. This is only defined when X is compact (for finiteness) and Y connected (we shall see why). Given y in Y, change f homotopically, to make it transversal to $\{y\}$, thus find a nbd U of y such that $f^{-1}(U)$ is disjoint union of $V_1 \cup \ldots \cup V_n$, each Vi mapps diffeomorphically to U. Thus $I_2(f, \{z\}) = n mod 2$ for all z in U. Thus the function is locally constant, and since Y is connected must be globally constant. □

We also see that

Theorem 2.23. Homotopic maps have the same mod 2 degree

Theorem 2.24. If $X = \partial W$, and $f : X \to Y$ can be extended to all of W, then $deg_2(f) = 0$

Proposition 2.25. If the mod 2 degree of $\frac{p}{|p|} : \partial W \to S^1$ is non zero, where $p : \mathbb{C} \to \mathbb{C}$ is smooth and W is a compact region, the p has a zero in W.

Proof. If p has no zeros on ∂W, only then is the function above defined. If p had no zeroes inside W also, then $\frac{p}{|p|}$ is defined on all of W, and using the last theorem, proves that $deg_2(p/|p|)$ is non zero. □

Theorem 2.26. Every complex polynomial of odd degree has a root.

Proof. We can rescale z and take the monic polynomial $p(z) = z^m + a_1 z^{m-1} + \ldots a_m$. Define a homotopy $p_t(z) = tp(z) + (1-t)z^m$. If W closed ball of sufficiently large radius, then note $p_t(z)/z^m$ tend to 1 under large z limit. Hence ∂W has no zeroes. Also, since p(z) is homotopic to z^m, $deg_2(p/|p|) = deg_2(p_0/|p_0|)$, which is just m mod 2, and we conclude from the above proposition □

Intersection theory is vacous in contractible manifolds, that is if Y is contractable and dim Y > O, then $I_2(f, Z) = 0$ for every $F : X \to Y$, X compact and Z closed, dim X + dim Z = dim Y.

§§2.5. Winding Numbers and the Jordan Brouwer Separation Theorem

A hypersurface in a manifold is a submanifold of codimension one.

Definition 2.27. Let X be a compact connected X and a smooth map $f : X \to \mathbb{R}^n$ and dim X=n-1. Take any point z in $\mathbb{R}^n$ not in f(X). Define $u(x) = \frac{f(x)-z}{|f(x)-z|}$ as a map from X to S^{n-1}. Define the mod 2 winding number of f around z to be $W_2(f, z) = deg_2(u)$

Theorem 2.28. If X is the boundary of D, a compact manifold, let $F : D \to \mathbb{R}^n$ smooth map extending f, $\partial F = f$. If z is a regular value of F not in image of f, then $F^{-1}(Z)$ is finite set and $W_2(f, Z) = F^{-1}(Z)$ mod 2

Proof. We can prove this by showing that if F does not intersect z, then extending it to define on all of D, we see that $W_2(f, z) = 0$. Use this and replace $D' = D - \cup_{i=1}^{l} Int(B_i)$, where $F^{-1}(Z) = \{y_1, \ldots y_l\}$ and B_i be disjoint from one another and let f_i be the restriction of F on ∂B_i, then from above we conclude that $W_2(f,z) = W_2(f_1,z) + \ldots + W_2(f_l,z) mod 2$. We can now use the regularity of z to choose B_i so that $W_2(f_i, Z) = 1$. □

Theorem 2.29. The Jordan Brower Seperation Theorem : The complement of the compact connected hypersurface X in $\mathbb{R}^n$ consists of 2 connected open sets, outside $D_0 = \{z : W_2(X,z) = 0\}$ and inside $D_1 = \{z : W_2(X,z) = 1\}$ where $z \in \mathbb{R}^n - X$. $\bar{D_1}$ is a compact manifold with $\partial\bar{D_1} = X$

Proof.

- We note that if x is any point of X and U is any neighbourhood of x, then there is a point of U that can be joined to z by curve not intersecting X. Let y be the point on X closest to z, and the straight line from y to z obviously intersects any nbd around x, and doesnt intersect X. For other points, we describe the curve from y to x as displaced along the normal by a small amount, tracing along X at a small finite distance, and then y to z as before. We can make sure this doesnt intersect itslef by again displacing it using the stability theorem

- We see that $\mathbb{R}^n - X$ has atmost 2 connected components since by fixing some x in X, and taking any 3 points not in X, and from before X divides a neighbourhood of X into 2 components, and hence 2 of these points must be path connected to the same neighbourhood component of X, and hence there are atmost 2 components

- If z_0, z_1 belong to the same connected component of $\mathbb{R}^n - X$, define the path (from above) between the 2 points as $\gamma : [0,1] \to \mathbb{R}^n - X$, with $\gamma(0) = z_0$ and $\gamma(1) = z_1$, and define $U(x,t) = \frac{x-\gamma(t)}{|x-\gamma(t)|}$, a homotopy between u_0 and u_1, hence $deg_2(u_0) = deg_2(u_1)$, and the realtion on W_2 follows

- Given a point z and a direction vector $\vec{v} \in S^{n-1}$, consider the ray r emanating from z in the direction of $\vec{v}$, and we see that it is transversal to X iff $\vec{v}$ is a regular value of the map $u : X \to S^{n-1}$ (from a previous ex)

- If r emanates from z_0 and intersects X transversally, and z_1 is another point on the ray, with l intersection points between it and z_0, then $\vec{v}$ is regular for both u_0, u_1, thus by definition $W_2(X, z_0) = W_2(X, z_1) + l mod 2$

- Hence, both D_0, D_1 are non empty and are 2 components (since both are connected from above) of $\mathbb{R}^n - X$

- We can also see that if z is very large then $W_2(X,z) = 0$

- We see that $\bar{D_1}$ is a manifold (compactness is obvious from above) since the interior is diffemorophic to $\mathbb{R}^n$, and points on both X and $\bar{D_1}$, from above X divides any neighbourhood of such points into 2 connected components, each of which can be seen to be diffeomorphic to H^n, hence it is a manifold with boundary, the points of boundary are those of X from the above argument.

□

We can also see that from the second last point above, given any $z \in \mathbb{R}^n - X$ and r is a ray from z that is transversal to X, then z is inside X iff r intersects X in an odd number points.

§§2.6. Borsuk-Ulam Theorem

Theorem 2.30. Borsuk-Ulam Theorem : If $f : S^k \to \mathbb{R}^{k-1}$ is a smooth map whose image does not contain the origin and if f satisfies f(-x) = -f(x) for all $x \in S^k$, then $W_2(f, 0) = 1$

Proof. We use induction on k, and can prove for base case of k=1. We can show that if this theorem is true, then it is equivalent to the following: if $f : S^k \to S^k$ carries antipodal to antipodal points, then $deg_2(f) = 1$. We can prove for S^1 as assuming such an f such that $f(cost, sint) = (cosg(t), sing(t))$ (we can always do this). Then the antipodal condition gives $g(s+\pi) = g(s) + \pi q$, where q is odd. Then we note $deg_2(f) = qmod2 = 1$. Now, let S^{k-1} be the equator, embedded in S^k, and the restrcition of f to the equator as g. Use Sard to select a unit vector $\vec{a}$ which is a regular value for both $\frac{g}{|g|} : S^{k-1} \to S^k$ and $\frac{f}{|f|} : S^k \to S^k$. Regularity for g simply means that g never intersects the line l along $\vec{a}$, and regularity for f leads to $f \pitchfork l$. By definition $W_2(f, 0) = (\frac{f}{|f|})^{-1}(\vec{a})mod2$. Note that $+\vec{a}$ and $-\vec{a}$ are hit same amount of times, and f_+ be the restriction of f to the upper hemisphere, thus $f_+^{-1}(l)$. Let V be the orthogonal complement of l, and $\pi : \mathbb{R}^{k+1} \to V$ be the projection. With g being symmetric, so is the projection, and $\pi \circ g : S^{k-1} \to V$, and it cant be 0, since g doesnt intersect l by hypothesis, thus by inductive hypothesis $W_2(\pi \circ g, 0) = 1$. Since $f_+ \pitchfork l$, thus $\pi \circ f_+$ is transversal to $\{0\}$. Now, $W_2(\pi \circ g, 0) = (\pi \circ f_+)^{-1}(0)mod2 = f_+^{-1}(l)mod2 = 1 = W_2(f, 0)$. □

Theorem 2.31. If $f : S^k \to \mathbb{R}^{k+1} - \{0\}$ satisfies f(-x) = -f(x), then f intersects every line through 0 atleast once.

Proof. If not, then use l in the proof (since f transversal to l trivially), which leads to $f^{-1}(l) = 0$, contradicting the theorem above □

Theorem 2.32. Any k smooth functions $f_1, \ldots f_k$ on S^k that all satisfy f(-x) = -f(x), must posses a common zero.

Proof. If not, apply the above theorem to the map $f(x) = (f_1(x), \ldots f_k(x), 0)$, and take l to be the x_{k+1} axis. □

Converting the above by setting $f_i(x) = g_i(x) - g_i(-x)$, thus for these smooth functions there exists a point (which is the common zero of f) satisfying $g_i(p) = g_i(-p)$.

§3. Oriented Intersection Theory

§§3.1. Orientation

Definition 3.1. An equivalent orientation describes an equivalence relation between 2 ordered basis on a finite dimensional vector space V. β and β' are said to be equivalently oriented if the unique linear isomorphism $\beta' = A\beta$ has a positive determinant. This partitions the set of all ordered basis into 2 equivalence classes. An orientation of V is an arbitrary decision to affix a positive sign to the elements of one equivalence class and a negative sign to the others.

If β is an ordered basis for V then replacing one v_i by cv_i is equivalently oriented if c>0 and opposite if c<0. Transposing 2 elements leads to an oppositely oriented basis, and subtracting a linear combination from v_i yields an equivalently ordered basis.

Definition 3.2. An orientation on a manifold with a boundary X is a smooth choice of orientations for all the tangent space $T_x(X)$, e there must be a local paramentrization h : $U \to X$ such that $dh_u : \mathbb{R}^k \to T_{h(u)}(X)$ preserves orientation at each u in U, such an h is called a orientation preserving map. X is orientable if it can be given an orientation.

Proposition 3.3. A connected orientable manifold with boundary admits exactly 2 orientations.

Note that $\partial H^k = (-1)^k \mathbb{R}^{k-1}$.

Proof. If h and h′ are local parametrizations around x, such that dh_u and dh'_u preserve the first and second orientation, then if both agree $d(h^{-1} \circ h')_0$ is orientation preserving, and hence has positive determinant at that point and by continuity of det, also on a nbd. Similarly with the disagreeing case, thus the set of points at which two orientations agree and the set where they disagree are both open, and hence form a seperation, unless one is empty. Hence 2 orientations are identical or opposite □

An oriented manifold is a manifold together with a smooth orientation. If X and Y are oriented and one of them is boundaryless, then $X \times Y$ acquires a product orientation. Let α and β be the ordered bases at $T_x(X), T_y(Y)$. Thus $(\alpha \times 0, 0 \times \beta)$ is the ordered basis for the product, and set $sign(\alpha \times 0, 0 \times \beta) = sign(\alpha)sign(\beta)$

Definition 3.4. An orientation of X natural1y induces a boundary orientation on ∂X. At every point x on the boundary we know that there are 2 - inward (maps onto H^k and outward (maps onto -H^k) normal vectors. Denote outward by n_x, and orient $T_x(\partial X)$ by sign of any ordered basis β to be the sign of ordered basis $\{n_x, \beta\}$.

We see that using the product manifold defintion, for a homotopy, the sign of X_1 is the same, but X_0 is reversed. We denote this as $\partial(I \times X) = X_1 - X_0$.
Here we note that **The sum of the orientation numbers at the boundary points of any compact**

oriented one-dimensional manifold with boundary is zero. Let $V = V_1 \oplus V_2$ be the direct sum, with basis $\beta = (\beta_1, \beta_2)$ and putting the constraint $sign(\beta) = sign(\beta_1) \cdot sign(\beta_2)$.

Definition 3.5. If $f : X \to Y$ is a smooth map with f and ∂f transversal to Z, all manifolds oriented and Z, Y are boundaryless. We define a preimage orientation on $S = f^{-1}(Z)$. Let $N_x(S; X)$ be the orthogonal complement to $T_x(S)$ in $T_x(X)$. Thus -

$$N_x(S; X) \oplus T_x(S) = T_x(X)$$

From transversality, we have $df_x(T_x(X)) + T_z(Z) = T_z(Y)$, and from $T_x(S)$ being the entire preimage of $T_z(Z)$ we get -

$$df_x(N_x(S; X)) \oplus T_z(Z) = T_z(Y)$$

Proposition 3.6. Given a map with the properties above, $\partial[f^{-1}(Z)] = (-1)^{codimZ}(\partial f)^{-1}(Z)$, ie the orientations on the boundary of S, one as a boundary on S and other as a preimage of the map ∂f differ by $(-1)^{codimZ}$

Proof. If H is a subspace of $T_x(\partial X)$ complementary to $T_x(\partial S)$, $H \oplus T_x(\partial S) = T_x(\partial X)$, and note that H is also complementary to $T_x(S)$ in $T_x(X)$, and disjoint from the former. The maps df_x and $d\partial f_x$ must agree on H, and must have the same orientation, ie $H \oplus T_x(S) = T_x(X)$ and $H \oplus T_x(\partial S) = T_x(\partial X)$. Let n_x be the outward normal vector to the boundary of S. Note $R \cdot n_x \oplus T_x(\partial X) = T_x(X)$. Hence we obtain $H \oplus T_x(S) = R \cdot n_x \oplus H \oplus T_x(\partial S)$. Since l=dim H transpositions are required to move it around, the LHS is $(-1)^l$ times $H \oplus R \cdot n_x \oplus T_x(\partial S)$, and using dimH = codim S = codim Z, the result follows. □

We can prove that every compact hypersurface in Euclidean space is orientable. We prove this using the Jordan Brower seperation theorem, an open submanifold of an orientable manifold is orientable, and a manifold with boundary is orientable iff its interior is.

§§3.2. Oriented Intersection number

Definition 3.7. We again consider throughout X,Y,Z to be boundaryless, X compact and Z closed submanifold of Y with dimX+dimZ = dimY. If $f : X \to Y$ is transversal to Z, then $f^{-1}(Z)$ is finite, with an orientation number ± 1 , and we call the Intersection number $I(f, Z)$ to be the sum of these orientation numbers

From 1D manifolds, we can see that -

Proposition 3.8. If $X = \partial$ W with W compact and $f : X \to Y$ extends to W, then I(f,Z) = 0. Homotopic maps always have the same intersection numbers.

Same as before, for any arbitrary map g, find a homotopic map transversal to Z, and define I(f,Z) = I(g,Z).

Definition 3.9. When Y is connected and has the same dimension as X, we define the degree of an arbitrary smooth map f to be the intersection number of I with any point y, $deg(f) = I(f, \{y\})$

Theorem 3.10. The Fundamental Theorem of Algebra- Every non constant complex polynomial has a root, and if W is a compact region where the boundary contains no 0 of p, then total number of zeroes of p inside W is the degree of map $p/|p| : \partial W \to S^1$

Proof. We have shown that for any complex polynomial p(z) of degree m the maps $\frac{p(z)}{|p(z)|}$ and $(\frac{z}{r})^m$ are homotopic, and hence have the same deg, which is m. If $m > 0$, then thus $\frac{p}{|p|}$ cannot be extended to the whole of the disc, thus p must have 0 in the disc. P has only finitely many zeros, let it be $z_1, \ldots, z_n$, and circumscribe disjoint closed disks around each root. Choose $W' = W - \cup D_i$, and $deg(p/|p|)$ is 0 on W′, and hence deg on W is sum of those on D_i. We now show that we do count multiplicities. Let $p(z) = (z - z_i)^l q(z), q(z) \neq 0$, define $g : S^1 \to \partial D_i$ as $g(z) = rz + z_i$ (orientation preserving). Define homotopy $h_t(z) = \frac{z^l q(z_i+trz)}{|q(z_i+trz)|}$ thus h_1 and $h_0 = cz^l$ have the same degree, l. □

Definition 3.11. When X and Z are submanifolds of Y, I(X,Z) is the intersection number of the inclusion map w Z. If X and Z are transversal, then count the points of $X \cap Z$ included with orientation.

Definition 3.12. Let X and Z be compact, boundaryless satisfying dimX+dimZ=dimY, then $f : X \to Y$ and $g; X \to Y$ are said to be transversal if $df_x(T_x(X)) + dg_z(T_z(Z)) = T_y(Y)$ for all points $f(x) = g(z) = y$. We define the local intersection number at (x,z) to be +1 if the direct sum orientation is the sams as that on $T_y(Y)$, and -1 otherwise.

Lemma 3.13. Let U and W be subspaces of the vector space V. Then $U \oplus W = V$ iff $U \times W \oplus \Delta = V \times V$. The product orientation agrees with the direct sum orientation iff W is even dimensional

Proof. If the direct sum of U and W is V, then by definition they are of complementary dimensions. We see that by definition $dimU \times W = codim\Delta$ and $U \cap W = 0 \leftrightarrow U \times W \cap \Delta = 0$, from which the first result follows. Let $\{u_1, \ldots, u_k\}$ and $\{w_1, \ldots, w_l\}$ be positively oriented basis for U and W. Δ has positively oriented basis as $\{(u_1, u_1), \ldots (u_k, u_k), (w_1, w_1) \ldots, (w_l, w_l)\}$. Thus for $U \times W \oplus \Delta$ the basis is $\{(u_1, 0), \ldots, (u_k, 0), (0, w_1), \ldots (0, w_l), (u_1, u_1), \ldots (u_k, u_k), (w_1, w_1), \ldots (w_l, w_l)\}$, which when subtracted by a linear combination preserving orientation and lk transpositions, becomes $\{(u_1, 0), \ldots, (u_k, 0), (0, u_1), \ldots (0, u_k), (0, w_1), \ldots (0, w_l), (w_1, 0), \ldots (w_l, 0)\}$ and on l(l+k) transpositions to $\{(u_1, 0), \ldots, (u_k, 0), (w_1, 0), \ldots (w_l, 0), (0, u_1), \ldots (0, u_k), (0, w_1), \ldots (0, w_l)\}$, hence l(l+2k) transpositions are even iff l is even. □

On substituting $U = df_x(T_x(X)), W = dg_z(T_z(Z)), V = T_y(Y)$

Proposition 3.14. Hence, $f \pitchfork g$ iff $f \times g$ transversal to Δ, then $I(f,g) = (-1)^{dimZ} I(f \times g, \Delta)$.

If Z is a submanifold of Y and i is the inclusion map, then I(f,i) = I(f,Z). Also, if dimX=dimY and Y is connected, then $I(f, \{y\})$ is the same for all y in Y and deg(f) is well defined (since inclusion maps of 2 points in y are homotopic). We see that $I(X,Z) = (-1)^{(dimX)(dimZ)} I(Z,X)$. Hence, the self intersection number is 0 if X is odd dimensional. Hence the Mobius strip is non orientable, since the central circle has self intersection number 1. Note that $I(Z,Z) = I(Z \times Z, \Delta)$, and is invariant and can be defined locally using diffemormophisms even if Z isnt orientable.

Definition 3.15. If Y is a compact and oriented manifold, the Euler characteristic $\chi(Y)$ is defined to be the self intersection number of Δ, $\chi(Y) = I(\Delta, \Delta)$, where Δ is the diagonal $Y \times Y$.

Hence,

Proposition 3.16. The Euler characteristic of an odd-dimensional, compact, oriented manifold is zero.

From the note above, we see that it is well defined even for non orientable manifolds owing to local diffeomorphisms and locally checking for intersection points, and is a diffemorphism invariant.

§§3.3. Lefschetz Fixed - Point Theory

Definition 3.17. Let $f : X \to X$ be a smooth map on a compact oriented manifold, We define the global Lefschetz number of f as $L(f) = I(\Delta, graph(f))$

We can immediately see by contrapositive -

Theorem 3.18. Smooth Lefschetz Fixed-Point Theorem: Let $f : X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a fixed point.

L(f) is clearly a homotopy invariant. If f is homotopic to the identity, then L(f) equals the Euler characteristic of X.

Definition 3.19. Smooth maps $f : X \to X$ such that $graph(f) \pitchfork \Delta$ are called Lefschetz maps, and hence have finitely many fixed points. Note that most maps are Lefschetz

A similar proof to already what is done before gives us -

Proposition 3.20. Every map $f : X \to X$ is homotopic to a Lefschetz map.

Definition 3.21. We call the fixed point x a Lefschetz fixed point of f if df_x has no nonzero fixed point (i.e., if the eigenvalues of df_x are all unequal to + 1).

Thus we see that f is a Lefschetz map if and only if all its fixed points are Lefschetz, since $graph(f)\overline{\pitchfork}\Delta$ iff $graph(df_x) + \Delta_x = T_x(X) \times T_x(X)$ and since they have complementary dim, they span RHS iff $graph(df_x) \cap \Delta_x = 0$ and hence has no eigenvecotr of eigenvalue 1.

Definition 3.22. If x is a Lefschetz fixed point, denote the local Lefschetz number of f at x as the orientation number ± 1 of (x,x) in $\Delta \cap graph(f)$ by $L_x(f)$ (need to just see whether $df_x - I$ preserves or reverses orientation, the sign of the determinant). Hence for Lefschetz maps, $L(f) = \sum_{f(x)=x} L_x(f)$,

Proposition 3.23. The Euler characteristic of S^2 is 2

Proof. Consider the projection onto the unit sphere $\pi : \mathbb{R}^3 - \{0\} \to S^2$ as $x \mapsto x/|x|$ and $f(x) = \pi(x + (0, 0, -1/2))$. This drives all points except on the north pole towards the south pole, hence those 2 being the Lefschetz fixed points, with $L_n(f) = L_s(f) = 1$, Hence $L(f) = 2$ and f is homotopic to the identity by $f_t(x) = \pi(x + (0, 0, -t/2))$ and hence L(f) = $\chi(S^2) = 2$. □

A corollary follows which we can prove otherwise as done in exercises as - Every map of S^2 that is homotopic to the identity must possess a fixed point. In particular, the antipodaI map is not homotopic to the identity.

Theorem 3.24. Classification of Two-Manifolds: Every compact oriented boundaryless two-manifold is diffeomorphic to a surface of genus k.

Proposition 3.25. The surface of genus k admits a Lefschetz map homotopic to the identity, with one source, one sink, and 2k saddles. ConsequentIy, its Euler characteristic is 2 - 2k.

We can prove the splitting proposition which says -

Proposition 3.26. Let U be a neighborhood of the fixed point x that contains no other fixed points of f. Then there exists a homotopy f_t of f such that f_1 has only Lefschetz fixed points in U, and each f_t equals f outside some compact subset of U. If $\rho : [0, 1] \to \mathbb{R}^k$ is 1 on nbd V of 0 and 0 outside K compact in U, then $f_t = f + t\rho a$ suffices for some a with $|a|$ small enough and a regular value of $f(x) - x$.

Definition 3.27. Suppose that x is an isolated fixed point of f. If B is a small closed ball centered at x that contains no other fixed point, then $F : \partial B \to S^{k-1}$ as $z \mapsto \frac{f(z)-z}{|f(z)-z|}$ is a smooth map, and the degree of this map is defined as the local Lefschetz number of I at x, denoted $L_x(f)$. Well defined for all such balls, since if B′ is a smaller ball, then it is defined on the annulus and hence have the same degree since they must sum to 0.

Proposition 3.28. At Lefschetz fixed points, the two definitions of $L_x(f)$ agree.

Proof. Define $f_t = Az + t\epsilon(z)$, such that ϵ is small enough on B. The map $F_t(z) = f_t(z) - z/|f_t(z) - z|$ defines a homtopy on the boundary, and $deg(F_1) = deg(F_0)$ where F_1 is the new defn. Note that this defines the same deg. □

Using this we prove : Suppose that the map f has an isolated fixed point at x, and let B be a closed ball around x containing no other fixed point of f. Choose any map f_1 that equals f outside some compact subset of $Int(B)$ but has only Lefschetz fixed points in B. Then $L_x(f) = \sum_{f_1(z)=z} L_z(f_1)$ for z in B. Similarly, we can prove that -

Theorem 3.29. If $f : X \to X$ be any smooth map on a compact manifold, with only finitely many fixed points. Then the global Lefschetz number (which is a homotopy invariant) equals the sum of the local Lefschetz numbers.