COURSE MATERIAL	
SUBJECT	LINEAR ALGEBRA AND CALCULUS (20A54101)
UNIT	1
COURSE	B.TECH
DEPARTMENT	S&H
SEMESTER	11

TABLE OF CONTENTS – UNIT 1			
S. NO	CONTENTS		PAGE NO.
1	COURSE OBJECTIVES		1
2	PREREQUISITES		1
3	SYLLABUS		1
4	COURSE OUTCOMES		1
5	CO - PO/PSO MAPPING		1
6	LESSON PLAN		2
7	ACTIVITY BASED LEARNING		2
8	LECTURE NOTES		2
	1.1	Introduction	2
	1.2	Rank of a matrix by echelon form	12
	1.3	Solving system of homogeneous and nonhomogeneous equations linear equations	19
	1.4	Solving system of homogeneous equations linear equations	26
	1.5	Eigen values and Eigenvectors and their properties	41
	1.6	Cayley-Hamilton theorem (without proof) finding inverse and power of a matrix by Cayley-Hamilton theorem	53
	1.7	diagonalisation of a matrix.	55
9	PRACTICE QUIZ		60
10	ASSIGNMENTS		61
11	PART A QUESTIONS & ANSWERS (2 MARKS QUESTIONS)		61
12	PART B QUESTIONS		63
13	SUPPORTIVE ONLINE CERTIFICATION COURSES		63
14	REAL TIME APPLICATIONS		63
15	CONTENTS BEYOND THE SYLLABUS		64
16	PRESCRIBED TEXT BOOKS & REFERENCE BOOKS		64

1. Course Objectives

The objectives of this course is to

At the end of the course, the students will be able to:

1. • This course will illuminate the students in the concepts of calculus and linear algebra.
2. To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real world problems and their applications.

2. Prerequisites

Students should have knowledge on

1. Basic knowledge on English grammer.
2. Basic Mathematics

3. Syllabus

UNIT I

Rank of a matrix by echelon form, normal form. Solving system of homogeneous and nonhomogeneous equations linear equations. Eigen values and Eigenvectors and their properties, CayleyHamilton theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton theorem, diagonalisation of a matrix.

4. Course outcomes

After completing this course the student must demonstrate the knowledge and ability to:

CO1: Solving systems of linear equations, using technology to facilitate row reduction determine the rank, eigen values and eigenvectors.

CO2: Identify special properties of a matrix, such as positive definite, etc., and use this information• to facilitate the calculation of matrix characteristics

5. Co-PO / PSO Mapping

Machine Tools	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1				2	2									
CO2						2								
CO3									1			2		
CO4											3			
CO5														

6. Lesson Plan

Lecture No.	Weeks	Topics to be covered	References
1	1	Introduction	T1
2		Rank of a matrix by echelon form	T1, R1
3		Solving system of homogeneous and nonhomogeneous equations linear equations	T1, R1
4		Solving system of nonhomogeneous equations linear equations	T1, R1
5	2	Eigen values and Eigenvectors and their properties	T1, R2
6		Cayley-Hamilton theorem (without proof), finding inverse and power of a matrix by Cayley-Hamilton theorem	T1, R1
7	3	diagonalisation of a matrix.	T1, R1

7. Activity Based Learning

1. Problems on Matrices properties
2. Problems on Power of matrices.

8. Lecture Notes

1.1 INTRODUCTION

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order mxn.

Eg: $$\begin{bmatrix} a_{11} & a_{12}\cdots\cdots a_{1n} \\ a_{21} & a_{12}\cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2}\cdots\cdots a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m\times n} \text{ where } 1 \le i \le m,\ 1 \le j \le n.$$

Types of matrices:

1. Square matrix : A square matrix A of order n x n is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2nd order matrix

2. Rectangular matrix: A matrix which is not a square matrix is called a rectangular matrix,

Eg: $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

3. Row matrix: A matrix of order 1xm is called a row matrix

Eg: $\begin{bmatrix}1 & 2 & 3\end{bmatrix}_{1\times 3}$

4. Column matrix: A matrix of order nx1 is called a column matrix

Eg: $\begin{bmatrix}1\\1\\2\end{bmatrix}_{3\times 1}$

5. Unit matrix: if A= $[a_{ij}]_{n\times n}$ such that $a_{ij} = 1$ for i = j and $a_{ij} = 0$ for i≠j, then A is called a unit matrx.

Eg: $I_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $I_3 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

6. Zero matrix : if A = $[a_{ij}]_{m\times n}$ such that $a_{ij} = 0$ ∀ I and j then A is called a zero matrix (or) null matrix

Eg: $O_{2\times 3} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

7. Diagonal elements in a matrix: A= $[a_{ij}]_{n\times n}$, the elements a_{ij} of A for which i = j. i.e. $(a_{11}, a_{22}....a_{nn})$ are called the diagonal elements of A

Eg: A= $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ diagonal elements are 1, 5, 9

Note: the line along which the diagonal elements lie is called the principle diagonal of A

8. Diagonal matrix: A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If $d_1, d_2..... d_n$ are diagonal elements of a diagonal matrix A, then A is written as A = diag $(d_1, d_2....d_n)$

E.g. : A = diag (3, 1,-2)= $\begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & -2\end{bmatrix}$

9. Scalar matrix: A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. **Equal matrices :** Two matrices A = $[a_{ij}]$ and b= $[b_{ij}]$ are said to be equal if and only if (i) A and B are of the same type(order) (ii) $a_{ij} = b_{ij}$ for every i & j

11. **The transpose of a matrix:** The matrix obtained from any given matrix A, by interchanging its rows and columns is called the transpose of A. It is denoted by A^1 (or) A^T.

If A = $[a_{ij}]_{m \times n}$ then the transpose of A is $A^1 = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$ Also $(A^1)^1 = A$

Note: A^1 and B^1 be the transposes of A and B respectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1+B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1= B^1A^1$

12. **The conjugate of a matrix:** The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$

Note: if $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then,

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{A+B} = \overline{A} + \overline{B}$

(iii) $(\overline{KA}) = \overline{K}\,\overline{A}$ (K is a any complex number)

(iv) $\overline{AB} = \overline{A}\,\overline{B}$

Eg: if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2\times3}$ then $\overline{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2\times3}$

13. **The conjugate Transpose of a matrix**

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ

Thus $A^\theta = (\overline{A^1})$ where A^1 is the transpose of A. Now A = $[a_{ij}]_{m \times n} \Rightarrow A^\theta = [b_{ij}]_{n \times m}$, where bij = $\bar{a}$ij

i.e. the $(i,j)^{th}$ element of A^θ conjugate complex of the $(j, i)^{th}$ element of A.

Eg: if $A = \begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2X3}$

then $A^{\theta} = \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3x2}$

14.

(i) Upper Triangular matrix: A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an upper triangular matrix

(ii) **Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e, $a_{ij=0}$ for i< j

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$

(iii) **Triangular matrix:** A matrix is said to be triangular matrix it is either an upper triangular matrix
or a lower triangular matrix

15. Symmetric matrix: A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus A is a symmetric matrix if $A^T = A$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric: A square matrix $A = [a_{ij}]$ is said to be skew – symmetric if $a_{ij} = -a_{ji}$ for every i and j.

E.g : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus A is a skew – symmetric iff $A = -A^1$ or $-A = A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

17. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtain by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA (or) AK

Thus: A + $[a_{ij}]_{m\times n}$ then KA = $[ka_{ij}]_{m\times n}$ = k$[a_{ij}]_{m\times n}$

18. Sum of matrices:

Let A = $[a_{ij}]_{m\times n}$,B = $[b_{ij}]_{m\times n}$ be two matrices. The matrix C = $[c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij}+b_{ij}$ is called

the sum of the matrices A and B.

The sum of A and B is denoted by A+B.

Thus $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$ and $[a_{ij}+b_{ij}]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

19. The difference of two matrices: If A, B are two matrices of the same type then A+ (-B) is taken as

A – B

20. Matrix multiplication: Let A = $[a_{ik}]_{m\times n}$, B = $[b_{kj}]_{n\times p}$ then the matrix C = $[c_{ij}]_{m\times p}$ where c_{ij} is called the

product of the matrices A and B in that order and we write C = AB.

The matrix A is called the pre-factor & B is called the post – factor

Note: If the number of columns of A is equal to the number of rows in B then the matrices are said

to be conformable for multiplication in that order.

21. Positive integral powers of a square matrix:

Let A be a square matrix. Then A^2 is defined A.A

Now, by associative law $A^3 = A^2.A = (AA)A$

$= A(AA) = AA^2$

Similarly we have $A^{m-1}A = A\,A^{m-1} = A^m$ where m is a positive integer

Note 1: Multiplication of matrices is distributive w.r.t. addition of matrices.

i.e, A (B+C) =AB + AC

(B+C)A =BA+CA

Note 2: If A is a matrix of order mxn then $AI_n = I_nA = A$

22. <u>**Trace of A square matrix :**</u> Let A = $[a_{ij}]_{n\times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$.

and is denoted by 'tr A'

Thus trA = $\sum_{i=1}^{n} a_{ii}$ = $a_{11}+a_{22}+ \ldots\ldots a_{nn}$

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then trA = a+b+c

Properties: If A and B are square matrices of order n and λ is any scalar, then

(i) tr (λ A) = λ tr A

(ii) tr (A+B) = trA + tr B

(iii) tr(AB) = tr(BA)

23. Idempotent matrix: If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix: If A is a square matrix such that $A^m=0$ where m is a +ve integer then A is called

nilpotent matrix.

Note: If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involuntary: If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^T = A^TA = I$

Examples:

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta+\sin^2\theta & -\cos\theta\sin\theta+\cos\theta\sin\theta \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & \cos^2\theta+\sin^2\theta \end{bmatrix}$$

$$A\,A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol: Given A = $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A.A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A.A^T = I$

Similarly $A^T.A = I$

Hence A is orthogonal Matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol: - For orthogonal matrix $AA^T = I$

So $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$ $\quad a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ $(c^2=2b^2)$

10

$\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3b}$

$= \pm\frac{1}{\sqrt{2}}$

$b = \pm\frac{1}{\sqrt{6}}$

$c = \pm\sqrt{2b}$

$= \pm\frac{1}{\sqrt{3}}$

27. Determinant of a square matrix:

If $A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{i1} & a_{i2} \cdots\cdots a_{in} \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{bmatrix}_{n\times n}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{i1} & a_{i2} \cdots\cdots a_{in} \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{vmatrix}$

28. Minors and cofactors of a square matrix

Let $A = [a_{ij}]_{n\times n}$ be a square matrix when form A the elements of i^{th} row and j^{th} column are

deleted the determinant of (n-1) rowed matrix [Mij] is called the minor of aij of A and is denoted

by $|M_{ij}|$

The signed minor $(-1)^{i+j}\,|M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}..

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$|A| = a_{11}\,|M_{11}| + a_{12}\,|M_{12}| + a_{13}\,|M_{13}|$ (or)
$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

E.g.: Find Determinant of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by using minors and co-factors.

Sol: A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

det A = $1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$

=1(-12-12)-1(-4-6)+3(-4+6)

= -24+10+6 = -8

Similarly, we find det A by using co-factors also.

Note 1: If A is a square matrix of order n then $|KA| = K^n|A|$, where k is a scalar.

Note 2: If A is a square matrix of order n, then $|A| = |A^T|$

Note 3: If A and B be two square matrices of the same order, then $|AB| = |A|\,|B|$

29. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A^{-1}.

Note:1 $(A^{-1})^{-1} = A$

Note 2: $I^{-1} = I$

30. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A by replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note: For any scalar k, adj(kA) = k^{n-1} adj A

Note: The necessary and sufficient condition for a square matrix to posses' inverse is that $|A| \neq 0$

Note: if $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adj A)$

31. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$.

If $|A| \neq 0$ then 'A' is said to be non-singular.

Note: 1. only non-singular matrices possess inverses.

2. The product of non-singular matrices is also non-singular.

Theorem: If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A')^{-1} = (A^{-1})'$

Proof:

(i). we have $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}(I\,B)$

$= B^{-1}B$

$= I$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii). $A^{-1}A = AA^{-1} = I$

Consider $A^{-1}A = I$

$\Rightarrow (A^{-1}A)' = I'$

$\Rightarrow A'.(A^{-1})' = I$

$\Rightarrow (A')^{-1} = (A^{-1})'$

Unitary matrix:

A square matrix A such that $(\overline{A})^T = A^{-1}$

i.e $(\overline{A})^T A = A(\overline{A})^T = I$

If $A^\theta A = I$ then A is called Unitary matrix

Note: The transpose of a unitary matrix is unitary.

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given matrix is

called is sub Matrix.

E.g.: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$ is a sub matrix of A obtained by deleting first row

and 4th column of A.

Minor of a Matrix: Let A be an m x n matrix. The determinant of a square sub matrix of A is called a

minor of the matrix.

13

Note: If the order of the square sub matrix is't' then its determinant is called a minor of order is't'.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4X3}$

$\rightarrow B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2'

$\rightarrow C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

detC = 2(7-12)-1(21-10) +(18-5)

= 2(-5)-1(11) +1(13)

= -10-11+13 = -8 is a minor of order '3'

***Rank of a Matrix:**

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is '0' (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note:

1. It is denoted by ρ (A)
2. Rank of a matrix is unique.
3. Every matrix will have a rank.
4. If A is a matrix of order mxn,

   Rank of A $\le$ min (m,n)
5. If $\rho(A)$ = r then every minor of A of order r+1, or more is zero.
6. Rank of the Identity matrix I_n is n.
7. If A is a matrix of order n and A is non-singular then $\rho(A)$ = n

Important Note:

1. The rank of a matrix is $\le$r if all minors of $(r+1)^{th}$ order are zero.

2. The rank of a matrix is ≥r, if there is at least one minor of order 'r' which is not equal to zero.

PROBLEMS

1. **Find the rank of the given matrix** $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$\rightarrow$ det A = 1(48-40)-2(36-28) +3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3

$\rho(A) = 3$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank ≤ min (3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is = 1(-49)-2(-56) +3(35-48)

= -49+112-39 = 24 ≠ 0.

Hence rank of the given matrix is '3'.

*** Elementary Transformations on a Matrix:**

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow K R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th}

row then it is denoted by $R_j \rightarrow R_j + KR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \to k\, c_j$ $\quad c_j \to c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A and it is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). the first non-zero entry in each non-zero row is equal to '1'.

(iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note:

1. The number of non-zero rows in echelon form of A is the rank of 'A'.
2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

E.g.:

1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

PROBLEMS

1. **Find the rank of the matrix A =** $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

Sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A

$$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix} R_2 \rightarrow R_2 - 3R_1,\ R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_2 \rightarrow R_2/7,\ R_3 \rightarrow R_3/9$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 - R_2$$

This is the Echelon form of matrix A.

The rank of a matrix A = Number of non-zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

By applying $R_2 \rightarrow 4R_2 - R_1$, $R_3 \rightarrow 4R_3 - kR_1$, $R_4 \rightarrow 4R_4 - 9R_1$

$$\text{We get } A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

Since Rank A = 3 ⇨ det A =0

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

⇨ 1[(8-4k)3]-1(8-4k) (4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k) (-24-4k) =0

$\Rightarrow$ (2-k) (6+k) =0

$\Rightarrow$ k =2 or k = -6

3. Find the rank of the matrix by using Echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 4R_1;\ R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

By applying $R_2 \to \frac{R_2}{-1};\ R_3 \to \frac{R_3}{-1};\ R_4 \to \frac{R_4}{-3}$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$$

By applying $R_3 \to R_3 - R_2;\ R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$A is in Echelon form

Rank of A= Number of non zero rows

$\therefore \rho(A) = 2$

4. Find the rank of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ **by reducing it to Echelon form.**

Sol: Given $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 3R_1;\ R_4 \to R_4 + R_1$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

By applying $R_3 \to R_3 - R_2;\ R_4 \to R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ A is in Echelon form

Rank of A= Number of non zero rows

$\therefore \rho(A) = 2$

5. Find the rank of the matrix $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$

By applying $R_1 \to -R_1$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

By applying $R_2 \to R_2 - 2R_1 ; R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

By applying $R_2 \to 8R_2$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 40 & 8 & 128 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

By applying $R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 40 & 8 & 128 \end{bmatrix}$$

By applying $R_3 \to R_3 - 5R_2$

$$A \sim \begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 8 & 4 & 31 \\ 0 & 0 & -12 & -27 \end{bmatrix}$$

∴ A is in Echelon form

Rank of A= Number of non zero rows

$\therefore \rho(A) = 3$

6. Find the rank of the matrix $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

By applying $R_2 \to R_2 + 2R_1 ; R_3 \to R_3 + R_1 ; R_4 \to R_4 + 3R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

19

By applying $R_3 \to R_3 - 2R_3 ; R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A is in Echelon form

Rank of A= Number of non zero rows

$\therefore \rho(A) = 2$

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (or) (I_r) (or) $\begin{pmatrix} I_r & \\ 0 & \end{pmatrix}$ (or) $\begin{pmatrix} I_r & 0 \\ & \end{pmatrix}$ by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ =

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Normal form another name is "canonical form"

E.g.: By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **into normal form, find its rank.**

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

By applying $R_3 \to R_3/-2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$$

By applying $R_3 \to R_3+R_2$

20

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

By applying $c_2 \to c_2 - 2c_1$, $c_3 \to c_3 - 3c_1$, $c_4 \to c_4 - 4c_1$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

By applying $c_3 \to 3c_3 - 2c_2$, $c_4 \to 3c_4 - 5c_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$$

By applying $c_2 \to c_2/-3$, $c_4 \to c_4/18$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By applying $c_4 \leftrightarrow c_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

SYSTEM OF HOMOGENEOUS AND NON HOMOGENEOUS LINEAR EQUATIONS

An equation of the form $a_1x_1 + a_2x_2 + \ldots\ldots\ldots + a_nx_n = b$ where $x_1, x_2 \ldots\ldots x_n$ are unknowns and $a_1, a_2 \ldots\ldots\ldots a_n$, b are constants is called linear equation in n unknowns .

Definition: Consider the system of m linear equations in n unknowns $x_1, x_2 \ldots\ldots x_n$ as given below:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2$$

....

$$a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots\ldots + a_{in}x_n = b_i$$

....

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n = b_m$$

The number a_{ij}'s are known as coefficient and $b_1, b_2, \ldots\ldots b_m$ are constants. An ordered n-tuple $(x_1, x_2 \ldots\ldots x_n)$ satisfying all the equations simultaneously is called a solution of system.

Homogeneous system:

If all $b_i = 0$ for $i = 1, 2, \ldots\ldots\ldots\ldots m$.

Non-Homogeneous system:

If all $b_i \neq 0$ i.e.at least one $b_i \neq 0$.

Matrix Representation:

The above system of linear non Homogeneous equations can be written in Matrix form as AX=B

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

Augmented Matrix:

It is denoted by [A/B] or [A B] is obtained by Augmenting A by the column B.

$$\therefore [A/B] = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots\ldots & a_{2n} & b_2 \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

By reducing [A /B] into its row echelon form the existence and uniqueness of solution AX = B exists.

NOTE:

Given a system, we do not know in general whether it has a solution or not .If there is atleast one solution , then the system is said to be consistent .If does not have any solution then the system is inconsistent.

1. CONSISTENT: A system is said to be consistent if it has at least one solution

a) If $\rho(AB) = \rho(A) = r = n$ (total number of unknowns) then the system is consistent and it has unique solution.

b) If $\rho(AB) = \rho(A) = r < n$ (total number of unknowns) then the system is consistent and it has Infinitely many solutions.

NOTE: To obtain solutions set (n – r) variables any arbitrary value & solve for remaining unknowns.

2. INCONSISTENT: If $\rho(AB) \neq \rho(A)$ then the system is inconsistent and it has no solution.

NOTE: Here rank is denoted by ρ

For Non Homogeneous System, The system AX = B is consistent i.e it has a solution .

The system is inconsistent i.e. it has no solution.

NOTE: Find the rank A and rank [A /B] by reducing the augmented matrix [A /B] to Echelon form by elementary row operations . Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

Example 1: Find whether the following equations are consistent, if so solve them

$x + y + 2z = 4; \; 2x - y + 3z = 9; \; 3x - y - z = 2.$

Solution: The given equations can be written in the matrix form as $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

i.e .AX = B

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A / B] = 3

Since the number of non- zero rows of matrix A is 3

Since the number of non- zero rows of matrix [A /B] is 3

$\therefore$ Rank of A = Rank of [A B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent. So, it has a solution.

Since Rank of A = Rank of [A /B] = Number of unknowns (n)

$\therefore$ The given system has a unique solution.

23

$$\begin{bmatrix}1 & 1 & 2\\0 & -3 & -1\\0 & 0 & -17\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\1\\-34\end{bmatrix}$$

$\Rightarrow$ $x + y + 2z = 4 \rightarrow (1)$

$-3y - z = 1 \rightarrow (2)$

$-17z = -34 \rightarrow (3)$

From (3) $z = 2$

Substituting $z = 2$ in eq(2), we get $y = -1$

Substituting $z = 2$ & $y = -1$ in eq(1), we get $x = 1$.

$\therefore$ $x = 1$, $y = -1$, $z = 2$ is the solution.

Example 2: Find whether the following system of equations is consistent. If so solve them.

$x + 2y + 2z = 2;\ 3x - 2y - z = 5;\ 2x - 5y + 3z = -4;\ x + 4y + 6z = 0.$

Solution: The given equations can be written in the matrix form as AX = B

i.e. $\begin{bmatrix}1 & 2 & 2\\3 & -2 & -1\\2 & -5 & 3\\1 & 4 & 6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\5\\-4\\0\end{bmatrix}$

The Augmented matrix [A/ B] = $\begin{bmatrix}1 & 2 & 2 & 2\\3 & -2 & -1 & 5\\2 & -5 & 3 & -4\\1 & 4 & 6 & 0\end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 3R_1;\ R_3 \rightarrow R_3 - 2R_1;\ R_4 \rightarrow R_4 - R_1$

[A /B] = $\begin{bmatrix}1 & 2 & 2 & 2\\0 & -8 & -7 & -1\\0 & -9 & -1 & -8\\0 & 2 & 4 & -2\end{bmatrix}$

Applying $R_3 \rightarrow 8R_3 - 9R_2;\ R_4 \rightarrow 4R_4 + R_2$,we get

$$[A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

Applying $R_3 \to R_3/55$; $R_4 \to R_4/9$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A/ B] = 3

∴ Rank of A = Rank of [A /B]

i.e. $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

Since Rank of A = Rank of [A /B] = Number of unknowns (n)

∴ The given system has a unique solution.

We have $\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

$\Rightarrow$ $x + 2y + 2z = 2 \to (1)$

$-8y - 7z = -1 \to (2)$

$z = -1 \to (3)$

From (3) $z = -1$

Substituting $z = -1$ in eq (2), we get $y = 1$

Substituting $z = -1$ & $y = 1$ in eq (1), we get $x = 2$.

∴ $x = 2$, $y = 1$, $z = -1$ is the solution.

25

Example3: Show that the equations x + y + z = 4 , 2x + 5y -2z = 3 , x + 7y – 7z = 5 are not consistent.

Solution: The given equations can be written in the matrix form as AX = B

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

The Augmented matrix $[A /B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$

$$[A /B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$[A /B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

We can see $\rho(AB) = 3$, since the number of non- zero rows is 3

Applying the same row operations on A , we get from above

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here the number of non- zero rows is 2 so the rank of A = 2

Here we have $\rho(AB) \neq \rho(A)$.

$\therefore$ The given system is not consistent.

Example 4: Discuss for what values of λ, μ the simultaneous equations x + y + z = 6 ;

x + 2y + 3z =10; x + 2y + λ z = μ have

i) no solution

ii) a unique solution

iii) an infinite number of solutions.

26

Solution: The given equations can be written in the matrix form as AX = B

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

We have the Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ we get

$$[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case i): Let $\lambda \neq 3$ then rank of A =3 and rank of [A/B]=3. So that they have same rank.

Then the system of equations is consistent .Here the number of unknowns is 3 which is

same as the rank of A.The system of equations will have a unique solution. This is true

for any value of μ.

Thus if $\lambda \neq 3$ and μ has any value , the given system of equations will have a

Unique solution.

Case ii): Suppose $\lambda = 3 \,\&\, \mu \neq 10$, then we can see that rank of A = 2 and rank of [A/B]=3.

Since the ranks of A and [A/B] are not equal , we say that the system of equations has no

solution (inconsistent).

Case iii): Suppose $\lambda = 3 \,\&\, \mu = 10$.Then we have rank of A = rank of [A/B] = 2

$\therefore$ The given system of equations will be consistent.

But here the number of unknowns = 3 > rank of A

Hence the system has infinitely many solutions.

Some more problems:

1 .Test for the consistency of $x + y + z = 1$; $x - y + 2z = 1$;$x - y + 2z = 5$; $2x - 2y + 3z = 1$

Solution: Not consistent

2. Find whether the following system of equations is consistent. If so solve them.

$X + y + 2z = 9$; $x - 2y + 2z = 3$; $2x - y + z = 3$; $3x - y + z = 4$.

Solution: Not consistent

3. Determine whether the following equations will have a solution, if so solve them

$x_1 + 2x_2 + x_3 = 2$; $3x_1 + x_2 - 2x_3 = 1$; $4x_1 - 3x_2 - x_3 = 3$; $2x_1 + 4x_2 + 2x_3 = 4$.

Solution: $x_1 = 1$; $x_2 = 0$; $x_3 = 1$.

4. Find the values of a & b for which the equations

$x + y + z = 3$; $x + 2y + 2z = 6$; $x + 9y + a z = b$ have

i) No Solution ii) A unique solution iii) Infinite number of solutions.

Solution : i) Let $a = 17$; $b \neq 27$ ii) Let $a \neq 17$; $b \neq 27$; iii) Let $a = 17$; $b = 27$.

Homogeneous System of Linear Equations:

Definition:

Consider the system of m homogeneous linear equations in n unknowns $x_1, x_2 .... x_n$ as given below:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$
$$..$$
$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = 0$$
$$..$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \rightarrow (1)$$

Matrix Representation:

The above system of linear Homogeneous equations can be written in Matrix form as $AX = 0$

$$\begin{bmatrix} a_{11} & a_{12} \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots\dots & a_{2n} \\ a_{m1} & a_{m2} \dots\dots\dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here A is called coefficient Matrix. It is clear that $x_1 = 0 = x_2 = x_3 = \dots\dots\dots\dots = x_n$ is a solution of equation (1)

This is called Trivial solution of AX = 0.

Definitions:

1. The system AX = 0 is always consistent since X = 0 i.e.($x_1 = 0 = x_2 = x_3 = \dots\dots\dots\dots = x_n$) is a Solution of AX = 0. This solution is called a trivial solution of the system. The Trivial solution is called zero solution.

2. Given AX = 0, we try to decide whether it has a solution $X \neq 0$. Such a solution, if exists, is called a non -Trivial solution.

Note: 1. If A is non-Singular matrix i.e.(det A $\neq 0$) then the linear system AX = 0 has only the zero solution.

2. The system AX = 0 possesses a non – zero solution $\Leftrightarrow$ A is singular matrix.

Working Rule For Finding The Solutions Of The Equation AX = 0 :

I .i) If r = n (number of variables) $\Rightarrow$ the system of equations have only Trivial solution (Zero solution)

ii) If r< n $\Rightarrow$ the system of equations have an infinite number of Non – Trivial solutions, we shall have n – r linearly independent solutions.

To obtain infinite solutions, set (n – r) variables any arbitrary value and solve for the remaining Unknowns.

II. If the number of equations is less than number of variables, the solution is always other than a Trivial Solution.

III. If the number of equations is equal to number of variables, the necessary and sufficient condition for the solutions other than a Trivial solution is that the determinant of the coefficient matrix is zero.

Example 1: Solve completely the system of equations:

$$x + 2y + 3z = 0;\ 3x + 4y + 4z = 0;\ 7x + 10y + 12z = 0.$$

Solution: Taking $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We get the system of equations as AX = 0

Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 7R_1$),we get

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

This is in Echelon form. Number of non -zero rows is 3.

$\therefore$ The rank of A = 3

$\therefore$ No. of variables = 3

$\therefore$ Number of non-zero solutions is = n - r =3 - 3 = 0.

$\therefore$ x = 0, y = 0, z = 0 is the only solution.

Example2: Solve completely the system of equations :

$$x + y + w = 0;\ y + z = 0;\ x + y + z + w = 0;\ x + y + 2z = 0.$$

Solution: The given equations can be written in the matrix form as AX = 0

Where $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Consider $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

Applying $R_3 \rightarrow R_3 - R_1$, $R_4 \rightarrow R_4 - R_1$,we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Applying $R_4 \rightarrow R_4 - 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ Rank of A =4 and Number of variables = 4

$\therefore$ There is no non -zero solution

Hence x = y = z = w = 0 is the only solution.

Example 3: Find all the solutions of system of equations.

$x + 2y - z = 0;\ 2x + y + z = 0;\ x - 4y + 5z = 0.$

Solution: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix}$ .Then det A = 1 (5 +4) -2(10-1)-1(-8 - 1)

=9 -18+9 = 0

$\therefore$ The rank of A = 2 < 3(no. of variables)

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of A = 2

$\therefore$ Number of non – zero solutions = n – r = 3 - 2 = 1.

From the above matrix , $-3y + 3z = 0$

$\Rightarrow y = z$

Let $z = k$. Then $y = k$

From the above matrix , $x + 2y - z = 0$

$\Rightarrow x = z - 2y$

$\Rightarrow x = k - 2k = -k$

$\therefore$ The solutions are given by $x = -k, y = k, z = k$.

Example 4: Solve completely the system of equations:

$x + y - 2z + 3w = 0; x - 2y + z - w = 0; 4x + y - 5z + 8w = 0; 5x - 7y + 2z - w = 0.$

Solution: The given equations can be written in the matrix form as AX = 0

$$AX = \begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 4R_1$, $R_4 \rightarrow R_4 - 5R_1$,),we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix}$$

Applying $R_4 \to R_4/4$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_4 \to R_4 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in Echelon form. We have

Rank A = The number of non -zero rows in this echelon form = 2(r) and number of unknowns = 4

Since $r < n$. The given system has infinite number of non – trivial solutions.

$\therefore$ Number of independent solutions = $4 - 2 = 2$ Now we shall assign arbitrary values to 2 variables and the remaining 2 variables shall be found in terms of these. The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives the equations $x + y - 2z + 3w = 0$; $-3y + 3z - 4w = 0$.

Taking $z = k_1$ and $w = k_2$ We see that $x = k_1 - \frac{5}{3}k_2$, $Y = k_1 - \frac{4}{3}k_2$, $Z = k_1$, $w = k_2$ constitutes the general solution of the given system.

Example 5: show that the only real number λ for which the system

$x + 2y + 3z = \lambda x$; $3x + y + 2z = \lambda y$; $2x + 3y + z = \lambda z$ has non -zero solution is 6 and

solve them when $\lambda = 6$.

Solution: The given equations can be written in the matrix form as $AX = 0$

Where A = $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 0 = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here the number of variables = n = 3

The given system of equations possesses a non-zero (non-trivial) solution, if rank of A < no. of unknowns i.e. Rank of A < 3

For this we must have detA = 0

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $R_1 \rightarrow R_1+R_2+R_3$, we get $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ we get

$$(6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\,[(-2-\lambda)(-1-\lambda)+1] = 0$$

$$\Rightarrow (6-\lambda)\,(\lambda^2+3\lambda+3) = 0$$

$\Rightarrow \lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow 5R_2 + 3R_1$, $R_3 \rightarrow 5R_3 + 2R_1$, we get

34

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 19 & -19\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$, we get

$$\sim\begin{bmatrix}-5 & 2 & 3\\ 0 & -19 & 19\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow$ $-5x + 2y + 3z = 0$

$\Rightarrow$ $-19y + 19z = 0$

$\Rightarrow y = z$

Since Rank of A < Number of unknowns. i.e. $r = 2 < n = 3$

$\therefore$ The given system has infinite number of non-trivial solutions.

Set $n - r = 3 - 2 = 1$. Let $z = k$ $\Rightarrow y = k$ and from the above equations $-5x + 2k + 3k = 0$

$\Rightarrow x = k$

$\therefore$ $x = k$, $y = k$, $z = k$ is the solution.

Eigen Values & Eigen Vectors

Def: Characteristic vector of a matrix:

Let $A = [a_{ij}]$ be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that $AX=\lambda X$.

Note: If $AX=\lambda X$ $(X\neq 0)$, then we say 'λ' is the Eigen value (or) characteristic root of 'A'.

Eg: Let $A = \begin{bmatrix}5 & 4\\ 1 & 3\end{bmatrix}$ $X = \begin{bmatrix}1\\ 1\end{bmatrix}$

$AX = \begin{bmatrix}5 & 4\\ 1 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix} = 1.\begin{bmatrix}1\\ 1\end{bmatrix}$

$AX = 1.X$

Here Characteristic vector of A is $\begin{bmatrix}1\\ 1\end{bmatrix}$ and Characteristic root of A is "1".

Note: We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

Method of finding the Eigen vectors of a matrix.

Let A = [a_{ij}] be a nxn matrix. Let X be an Eigen vector of A corresponding to the Eigen value λ.

Then by definition AX = λX.

AX = λIX

AX –λIX = 0

(A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns. Will have a non-zero solution X if and only | A-λI | = 0

A-λI is called characteristic matrix of A

| A-λI | is a polynomial in λ of degree n and is called the characteristic polynomial of A

| A-λI | =0 is called the characteristic equation

Solving characteristic equation of A, we get the roots, $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots, \lambda_n$, these are called the characteristic roots or Eigen values of the matrix.

Corresponding to each one of these n Eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a given square matrix

Characteristic matrix of A is (A-λI)

$$i.e., A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A - \lambda I|$

$$say\,\phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\phi(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called Eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

PROBLEMS

1. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic equation of A *is* $|A-\lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(2-\lambda)+8=0$$

$$\Rightarrow 16+\lambda^2-10\lambda+8=0$$

$$\Rightarrow \lambda^2-10\lambda+24=0$$

$$\Rightarrow (\lambda-6)(\lambda-4)=0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

Eigen vector corresponding to λ=4:

Put $\lambda = 4$ in the above system, we get

⇨ $\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

⇨ $4x_1 - 4x_2 = 0 - - - (1)$

⇨ $2x_1 - 2x_2 = 0 - - - (2)$

from (1)*and* (2)*we have* $x_1 = x_2$

Let $x_1 = \alpha$, *Eigen vector is* $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

37

$\begin{bmatrix}1\\1\end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value λ=4

Eigen vector corresponding to λ=6:

put $\lambda = 6$ in the above system, we get

$$\begin{bmatrix}2 & -4\\2 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow 2x_1 - 4x_2 = 0 - - - (1)$

$2x_1 - 4x_2 = 0 - - - (2)$

from (1) and (2) we have $x_1 = 2x_2$

Say $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix}2\alpha\\\alpha\end{bmatrix} = \alpha\begin{bmatrix}2\\1\end{bmatrix}$$

$\begin{bmatrix}2\\1\end{bmatrix}$ is a Eigen vector of matrix A corresponding to the Eigen value λ=6

2. Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$

Sol: Let $A = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$

The characteristic equation of matrix A is |A-λI|=0

i.e. $|A-\lambda I| = \begin{vmatrix}2-\lambda & 0 & 1\\0 & 2-\lambda & 0\\1 & 0 & 2-\lambda\end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$

$\Rightarrow (2-\lambda)^3 - (\lambda-2) = 0$

$\Rightarrow \lambda - 2\,[-(\lambda-2)^2 - 1] = 0$

$\Rightarrow \lambda - 2\,[-\lambda^2 + 4\lambda - 3] = 0$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$

$\therefore$ λ=1, 2, 3

The Eigen values of A is 1, 2, 3

To find the Eigen vector consider the system $(A - \lambda I)X = 0$

38

$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to λ=1:

$$\begin{bmatrix} 2-1 & 0 & 1 \\ 0 & 2-1 & 0 \\ 1 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_3 = 0$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$say\ x_3 = \alpha$

$x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigenvector to the corresponding Eigen value λ=1.

Eigen vector corresponding to λ=2:

$$\begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ *and* $x_3 = 0$ *and we can take any arbitary value* x_2 *i.e* $x_2 = \alpha$ *(say)*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigen vector corresponding to λ=3:

39

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

here by solving we get $x_1 = x_3, x_2 = 0$ *say* $x_3 = \propto$

$x_1 = \propto$, $x_2 = 0$, $x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation of matrix A is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)[(3-\lambda)^2 - 1] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$\Rightarrow (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 2, 2, 8$

The Eigen values of A is 2, 2, 8

<u>**The Eigen vector of matrix A corresponding to $\lambda = 2$:**</u>

To find the Eigen vector consider the system $(A-\lambda I) \Longleftrightarrow 0$ $\quad (A-2I)X = 0$

40

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

By observing that three equations are identical

Therefore we have to take two arbitrary constant for any two variables

Let $x_1 = k_1, x_2 = k_2$

From third equation $2k_1 - k_2 + x_3 = 0$

$$x_3 = -2k_1 + k_2$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 + k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ -2k_1 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_1\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to$\lambda = 8$;

To find the Eigen vector consider the system $(A - \lambda I)X \Longrightarrow 0$ $\quad (A - 8I)X = 0$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

By solving above three equations we get $x_1 = 2, x_2 = -1, x_3 = 1$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

41

The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The characteristic equation of matrix A is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)]=0$

$\Rightarrow (8-\lambda)[\lambda^2-10\lambda+5]+6[6\lambda-10]+2[2\lambda-10]=0$

$\Rightarrow \lambda^3-18\lambda^2+45\lambda=0$

$\lambda = 0, 3, 15$

The Eigen values of A is 0, 3, 15

The Eigen vector of matrix A corresponding to $\lambda = 0$:

To find the Eigen vector consider the system $(A-\lambda I) \Longleftrightarrow 0$ $(A-0I)X=0$

$\Rightarrow \begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 8x_1-6x_2+2x_3=0$

$-6x_1+7x_2-4x_3=0$

$2x_1-4x_2+3x_3=0$

By solving above any of the two equations we get $x_1=1, x_2=2, x_3=2$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

The Eigen vector of matrix A corresponding to $\lambda = 3$:

To find the Eigen vector consider the system $(A-\lambda I) \Longleftrightarrow 0$ $(A-3I)X=0$

$\Rightarrow \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

42

$\Rightarrow\ 5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 = 0$

By solving above any of the two equations we get $x_1 = -2, x_2 = -1, x_3 = 2$

$$\text{Eigen vector } X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

The Eigen vector of matrix A corresponding to $\lambda = 15$:

To find the Eigen vector consider the system $(A - \lambda I) \Longleftrightarrow 0$ $\qquad (A - 15I)X = 0$

$$\Rightarrow \begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow\ -7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$

By solving above any of the two equations we get $x_1 = 2, x_2 = -2, x_3 = 1$

$$\text{Eigen vector } X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\ X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ $\ X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Properties of Eigen Values:

1. The sum of the Eigen values of a matrix A is same as trace of the matrix A.

Proof: To prove this theorem, we consider third order square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The characteristic polynomial of A is

$$= |A - \lambda I|$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22-\lambda} & a_{23} \\ a_{31} & a_{32} & a_{33-\lambda} \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda\,(a_{21}a_{12} + a_{13}a_{31} + \cdots \ldots) + (a_{11}a_{23}a_{32} + a_{12}a_{23}a_{32} + \cdots)$$

43

We have

$$\text{Sum of the roots} = \frac{-\lambda^2 \, coefficient}{\lambda^3 \, coefficient}$$
$$= \frac{-(a_{11}+a_{22}+a_{33})}{-1}$$
$$= a_{11} + a_{22} + a_{33}$$
$$= \text{Trace of A}$$

Hence the result.

2. The product of the Eigen values of a matrix A is equal to its determinant.

Proof: Let $\lambda_1, \lambda_2 \dots\dots\dots \lambda_n$ be the Eigen values of $A_{n\times n}$ .Then

$$|A - \lambda I| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2) \dots\dots\dots (\lambda - \lambda_n) = 0$$

Put $\lambda = 0$

$$|A| = (-1)^n(-\lambda_1)(-\lambda_2) \dots\dots\dots (-\lambda_n)$$
$$= (-1)^n(-1)^n(\lambda_1\lambda_2 \dots\dots\dots \lambda_n)$$
$$= (-1)^{2n}(\lambda_1\lambda_2 \dots\dots\dots \lambda_n)$$
$$|A| = \lambda_1\lambda_2 \dots\dots\dots \lambda_n$$

Hence the result.

3. If λ is an Eigen value of A corresponding to the Eigen vector X then λ^n is the Eigen value of A^n corresponding to the Eigen vector X.

Proof:

Since λ is an Eigen value of A corresponding to the Eigen vector X, we have

$$AX = \lambda X \quad \text{------------------ } 1$$

Pre multiplying equation 1 by A

$$A(AX) = A\,(\lambda X)$$

$$(AA)X = \lambda\,(A X)$$
$$A^2X = \lambda\,(\lambda X)$$
$$A^2X = \lambda^2X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.

Thus the theorem is true for $n = 2$.

Let the result be true for $n = k$.

Then $A^kX = \lambda^kX$

Pre multiplying this by A

$$A\,(A^kX) = A(\lambda^kX)$$
$$A^{k+1}X = \lambda^k(AX)$$
$$= \lambda^k(\lambda X) \qquad \text{(Since } AX = \lambda X \text{)}$$
$$= \lambda^{k+1}X$$

$$A^{k+1}X = \lambda^{k+1}X$$

Which implies that λ^{k+1} is Eigen value of A^{k+1} with X itself as the corresponding Eigen

Vector.

Hence by the principle of mathematical induction, the theorem is true for all positive integers n.

4. A square matrix A and it's transpose A^T have the same Eigen values.

Proof:

We have $(A - \lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A - \lambda I)^T| = |A^T - \lambda I|$ or

$|A - \lambda I| = |A^T - \lambda I|$ **[Since $|A^T| = |A|$]**

$\therefore$ $|A - \lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

i.e., λ is an Eigen value of A if and only if λ is an Eigen value of A^T.

Thus the Eigen values of A and A^T are same.

5. If λ is an Eigen value of the matrix A corresponding to the Eigen vector X then $k + \lambda$ is an Eigen value of the matrix $A + kI$.

Proof:

Let λ be an Eigen value of the matrix A corresponding to the Eigen vector X then

$$AX = \lambda X \quad \text{--- } 1$$

Now

$$(A + kI)\, X = AX + kIX$$

$= \lambda X + kX$ (**Since $AX = \lambda X$**)

$= (\lambda + k)X$

$$(A + kI)\, X = (\lambda + k)X \quad \text{--- } 2$$

From Equation , we see that that the scalar $k + \lambda$ is an Eigen value of the matrix $A + kI$ and X is a corresponding Eigen vector.

6. If $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n$ are the Eigen values of a matrix A, then $k\lambda_1, k\lambda_2 \ldots\ldots\ldots k\lambda_n$ are the Eigen values of the matrix KA, where k is a non-zero scalar.

Proof:

Let A be a square matrix of order n.

Then $|kA - \lambda kI| = |k(A - \lambda I)|$

$= k^n|A - \lambda I|$

Since $k \neq 0$, therefore $|kA - \lambda kI| = 0$ if and only if $|A - \lambda I| = 0$

i.e., $k\lambda$ is an Eigen value of kA if and only if λ is an Eigen value of A.

Thus $k\lambda_1, k\lambda_2 \ldots\ldots\ldots k\lambda_n$ are the Eigen Values of the Matrix kA if $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n$ are the Eigen values of the matrix A.

7. **If $\lambda_1, \lambda_2 \dots \dots \lambda_n$ are the Eigen Values of A, then $(\lambda_1 - k)(\lambda_2 - k) \dots \dots \dots (\lambda_n - k)$ are the Eigen Values of the matrix $(A - kI)$, where k is a non-zero scalar.**

Proof:

Since $\lambda_1, \lambda_2 \dots \dots \lambda_n$ are the Eigen values of A, then the characteristic polynomial of A is

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots \dots \dots (\lambda_2 - \lambda) \text{ ------------------------- } 1$$

Thus the Characteristic polynomial of $(A - kI)$ is

$|A - kI - \lambda I| = |A - (k + \lambda)I|$

$= [\lambda_1 - (\lambda + k)]\ [\lambda_2 - (\lambda + k)] \dots\dots [\lambda_n - (\lambda + k)]$ **(Sin 1 rom)**

$= [(\lambda_1 - k) - \lambda]\ [(\lambda_2 - k) - \lambda] \dots\dots\dots\dots\dots\dots\dots\dots [(\lambda_n - k) - \lambda]$

This shows that the Eigen values of $(A - kI)$ are $(\lambda_1 - k)(\lambda_2 - k) \dots \dots \dots (\lambda_n - k)$.

8. **Prove that the Eigen values of A^{-1} are the reciprocals of the Eigen values of A.**

(Or)

If λ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X. Then λ^{-1} is an Eigen value of A^{-1} and corresponding Eigen vector X itself.

Proof:

Since A is non-singular and product of the Eigen values is equal to $|A|$, it follows that none of

the Eigen values of A is 0.

If λ is an Eigen value of the non-singular matrix A and X is the corresponding Eigen vector,

$\lambda \neq 0$ and $AX = \lambda X$

Pre multiplying this with A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$
$$(A^{-1}A)X = \lambda(A^{-1}X)$$
$$IX = \lambda(A^{-1}X)$$
$$X = \lambda(A^{-1}X)$$
$$\lambda^{-1}X = A^{-1}X$$
$$A^{-1}X = \lambda^{-1}X \qquad \textbf{Since } \lambda \neq 0$$

Hence by the definition, it follows that λ^{-1} is an Eigen value of A^{-1} and X is the corresponding Eigen vector.

9. **If λ is an Eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix $adj\ A$.**

Proof:

Since λ is an Eigen value of a non-singular matrix, therefore $\lambda \neq 0$.

Also λ is an Eigen value of A implies that there exist a non-zero vector X such that

$$AX = \lambda X \text{ ------------- } 1$$

Pre multiplying (1) by $adj\ A$

$$(adj\ A)\ AX = (adj\ A)\ \lambda X$$

$$[(adj\ A)\ A]X = \lambda[(adj\ A)\ X]$$
$$|A|\ I\ X = \lambda(adj\ A)X \qquad \text{[Since } (adj\ A)A = |A|\ I\ ]$$
$$|A|\ X = \lambda(adj\ A)X$$
$$\frac{|A|}{\lambda} X = (adj\ A)X$$

$$(adj\ A)X = \frac{|A|}{\lambda} X$$

Since X is a non-zero vector, therefore from the 1 ation . It is clear that $\frac{|A|}{\lambda}$ is an Eigen value of the matrix $adj\ A$.

10. If λ is an Eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also an Eigen value.

Proof:

We know that if λ is an Eigen value of a matrix A, then $\frac{1}{\lambda}$ is an Eigen value of A^{-1}.

Since A is an orthogonal matrix, therefore

$$A^{-1} = A^T$$

$\frac{1}{\lambda}$ is an Eigen value of A^T.

But the matrices A and A^T have the same Eigen values. Since the determinants $|A - \lambda I|$ and $|A^T - \lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an Eigen value of A.

11. The Eigen values of a diagonal matrix are its diagonal elements.

Proof: Let

$$D = \begin{bmatrix} a_{11} & 0 & 0 & \dots\dots & 0 \\ 0 & a_{22} & 0 & \dots\dots & 0 \\ 0 & 0 & a_{33} & \dots\dots & 0 \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn} \end{bmatrix}$$

The characteristic equation of D is

$$|D - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 & \dots\dots & 0 \\ 0 & a_{22}-\lambda & 0 & \dots\dots & 0 \\ 0 & 0 & a_{33}-\lambda & \dots\dots & 0 \\ . & . & . & \dots\dots & . \\ . & . & . & \dots\dots & . \\ 0 & 0 & 0 & \dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\dots\dots\dots(a_{nn}-\lambda) = 0$$

$\Rightarrow \lambda = a_{11}, a_{22}, a_{33} \dots\dots\dots\dots a_{nn}$ are the Eigen values of D.

47

12. The Eigen values of a triangular matrix are the diagonal elements.

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots\cdots & a_{3n} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ 0 & 0 & 0 & \cdots\cdots & a_{nn} \end{bmatrix}$ be the upper triangular matrix.

The Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \cdots\cdots & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \cdots\cdots & a_{3n} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ 0 & 0 & 0 & \cdots\cdots & a_{nn-\lambda} \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\ldots\ldots\ldots.(a_{nn}-\lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}a_{33}\ldots\ldots\ldots\ldots a_{nn}$$

Hence the Eigen values are the diagonal elements.

13. Eigen values of two similar matrices are same.

Proof:

Let A and B be two similar matrices so that

$$B = P^{-1}AP$$

To prove that A and $P^{-1}AP$ have the same Eigen values.

Let λ be Eigen value of B then,

$$|B - \lambda I| = 0$$
$$|P^{-1}AP - \lambda P^{-1}P| = 0$$
$$|P^{-1}P(A - \lambda I)| = 0$$
$$|P^{-1}||P|\,|A - \lambda I| = 0$$
$$|A - \lambda I| = 0 \quad \textbf{since } |\boldsymbol{P^{-1}}||\boldsymbol{P}| = \mathbf{1}$$
$$|B - \lambda I| = 0 \Rightarrow |A - \lambda I| = 0$$

Hence A and B have the same Eigen values.

14. If A and B are square matrices and if A is invertible then the matrices $A^{-1}B$ and BA^{-1} have the same Eigen values.

Proof:

Given A is invertible, then A^{-1} exists.

Now $A^{-1}B = A^{-1}B\,I$

$= A^{-1}B\,(A^{-1}A)$

$= A^{-1}(BA^{-1})A$

1

$\Rightarrow A^{-1}B = A^{-1}(BA^{-1})A$ ---------------

By using the above property, matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have the same Eigen values.

Now b 1 , the matrices $A^{-1}B$ and BA^{-1} have the same Eigen values.

15. The Eigen values of a real symmetric matrix are always real.

Proof:

Let λ be an Eigen value of a real symmetric matrix A and let X be the corresponding Eigen vector. Then 1 $AX = \lambda X$ ------------------

Taking the Conj 1 te of

$$\bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

Taking the transpose on both sides

$$(\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$$

$$(\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T$$

Since A is symmetric, we have

$$\bar{A} = A \text{ and } A^T = A$$

$$\therefore (\bar{X})^T A = \bar{\lambda}(\bar{X})^T$$

Post multiplying by X, we get

$$(\bar{X})^T AX = \bar{\lambda}(\bar{X})^T X \text{ --------- } 2$$

Pre multip 1 g with $(\bar{X})^T$ we get

$$(\bar{X})^T AX = (\bar{X})^T \lambda X \text{ --------- } 3$$

2 3

$$\Rightarrow (\lambda - \bar{\lambda})(\bar{X})^T X = 0$$

$$\Rightarrow (\lambda - \bar{\lambda}) = 0 \quad \textit{since } (\bar{X})^T X \neq 0$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\Rightarrow \lambda$ is real

Hence the result.

16. If λ is an Eigen value of A, then prove that the Eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$.

Proof:

If X be the Eigen vector corresponding to the Eigen value λ , then

$$AX = \lambda X \text{ -------------- } 1$$

Pre multiplying by A on b 1 sides of

$$A(AX) = A(\lambda X)$$

$$(AA)X = \lambda(AX)$$

$$A^2X = \lambda(\lambda X) \qquad (\textbf{Since } AX = \lambda X)$$

$$A^2X = \lambda^2 X$$

Hence λ^2 is Eigen value of A^2 with X itself as the corresponding Eigen Vector.
We have

$$B = a_0A^2 + a_1A + a_2I$$
$$BX = (a_0A^2 + a_1A + a_2I)X$$
$$BX = a_0A^2X + a_1AX + a_2X$$
$$BX = a_0\lambda^2X + a_1\lambda X + a_2X$$
$$BX = (a_0\lambda^2 + a_1\lambda + a_2)X$$

This shows that $a_0\lambda^2 + a_1\lambda + a_2$ is an Eigen value of B and the corresponding Eigen Vector of B is X.

17. Prove that the two Eigen vectors corresponding to the two different Eigen Values are linearly independent.

Proof:

Let A be a square matrix. Let $X_1$and X_2 be the two Eigen vectors of A corresponding to two distinct Eigen values λ_1 and λ_2 .Then

$$AX_1 = \lambda_1 X_1 \text{ And } AX_2 = \lambda_2 X_2 \text{ ------------ } (1)$$

Now we shall prove that the Eigen vectors $X_1$and X_2 are linearly independent.
Let us assume that the $X_1$and X_2 are linearly dependent.
Then for two scalars $k_1$and k_2 not both zeroes such that $k_1X_1 + k_2X_2 = 0$ --(2)----------

Multiplying both sides of (2) by A, we get

$$A(k_1X_1 + k_2X_2) = A(0)$$
$$A(k_1X_1) + A(k_2X_2) = 0$$
$$k_1(AX_1) + k_2(AX_2) = 0$$
$$k_1(\lambda_1X_1) + k_2(\lambda_2X_2) = 0 \text{ ------------ } (3) \quad \textbf{(Since from (1))}$$

(3) - λ_2(1) $\Rightarrow k_1(\lambda_1 - \lambda_2)X_1 = 0$

$\Rightarrow k_1 = 0$ **Since $\lambda_1 \neq \lambda_2$ & $X_1 \neq 0$**

Similarly $k_2 = 0$

But this contradicts our assumption that k_1, k_2 are not zeroes. Hence our assumption that X_1 & X_2 are Linearly independent is wrong.

Hence the statement is true.

18. For a real symmetric matrix, the Eigen vectors corresponding to two distinct Eigen values are orthogonal.

Proof :

Let λ_1 and λ_2 be Eigen values of a real symmetric matrix A and let X_1 & X_2 be the corresponding Eigen vectors.

Let $\lambda_1 \neq \lambda_2$, we have to show that $X_1^T X_2 = 0$

We have $AX_1 = \lambda_1 X_1$---(1)------------

$AX_2 = \lambda_2 X_2$ -------------- 2

Pre multiplying 1 by X_2^T

$X_2^T AX_1 = X_2^T \lambda_1 X_1$

Taking Transpose, we have

$(X_2^T AX_1)^T = (X_2^T \lambda_1 X_1)^T$

$\Rightarrow X_1^T A^T X_2 = \lambda_1 X_1^T X_2$

$\Rightarrow X_1^T AX_2 = \lambda_1 X_1^T X_2$ -------------- 3 **(Since $A^T = A$)**

Pre multiplying 2 by X_1^T

$X_1^T AX_2 = \lambda_2 X_1^T X_2$ -------------- 4

3 from 4

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Rightarrow X_1^T X_2 = 0$ **Since $\lambda_1 \neq \lambda_2$**

$\therefore X_1$ is orthogonal to X_2

5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse, where A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: Given A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Characteristic roots are 1, 2, 3

Eigen vector corresponding to λ=1

$$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2 + 4x_3 = 0$$

$$x_2 + 5x_3 = 0$$

$$2x_3 = 0$$

$x_2 = 0, x_3 = 0$

and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where α is arbitrary constant

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 1$

Eigen vector corresponding to λ=2

$$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$$

$$5x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$$

$Let\ x_2 = k$

$x_1 = 3k$

$$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 2$

Eigen vector corresponding to λ=3

$$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0$$
$$-x_2 + 5x_3 = 0$$

52

$$Say\ x_3 = K \implies x_2 = 5K$$

$$x_1 = \frac{19}{2}K$$

$$X = \begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the Eigen vector corresponding to $\lambda = 3$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\Rightarrow$ *Eigen values of* A^{-1} *are* $1, \frac{1}{2}, \frac{1}{3}$

We know Eigen vectors of A^{-1} are same as Eigen vectors of A.

6. *Find the Eigen values of* $3A^3 + 5A^2 - 6A + 2I$ *where* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

$$i.e, \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\implies [(1-\lambda)(3-\lambda)(-2-\lambda) - 0] = 0$$

$$\implies (1-\lambda)(3-\lambda)(2+\lambda) = 0 \qquad \lambda = 1, 3, -2$$

Eigen values of A are $1, 3, -2$

We know that if λ *is an eigen value of A and* $f(A)$ *is a polynomial in A.*

then the eigen value of $f(A)$ *is* $f(\lambda)$

Let $f(A) = 3A^3 + 5A^2 - 6A + 2I$

Then Eigen values of f(A) are f(1), f(3) and f(-2)

$f(1) = 3(1)^3+5(1)^2-6(1)+2(1) = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4, 110, 10

53

7. Find the Eigen values of A=$\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A}=\begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T=\begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A}=-A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow A^T=\begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^2-2i\lambda+8=0$$

$\Rightarrow$ $\lambda=4i,-2i$ are the Eigen values of A

8. Find the Eigen values of $A=\begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$$Now\,\overline{A}=\begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and$$

$$\left(\overline{A}\right)^T=\begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

We can see that $\overline{A}^T.A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

Thus A is a unitary matrix

$\therefore$ The characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix} = 0$$

Which gives $\lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2} and \frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and

$$\lambda = 1/2\sqrt{3} + 1/2i$$

Hence above λ values are Eigen values of A.

Cayley - Hamilton Theorem:

Statement: Every square matrix satisfies its own characteristic equation

PROBLEMS

1. **Show that the matrix A =** $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ **satisfies its characteristic equation Hence find A^{-1}**

Sol: The Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

C2 → C2+C3

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have $A^3-A^2+A-I=0$------(1)

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

55

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying equation (1) with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2) **Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix A =**
$\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$

$$i.e., \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$.....(1)

Multiply equation (1) with A^{-1} we get

$$A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right]$$

56

$$A^2=\begin{bmatrix}25 & 8 & -8\\ -24 & -7 & 8\\ 24 & 8 & -7\end{bmatrix}\quad A^3=\begin{bmatrix}79 & 26 & -26\\ -78 & -25 & 26\\ 78 & 26 & -25\end{bmatrix}$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}-3 & -2 & 2\\ 6 & 5 & -2\\ -6 & -2 & 5\end{bmatrix}$$

Multiply equation (1) with A , we get

$$A^4-5A^3+7A^2-3A=0$$

$$A^4=5A^3-7A^2+3A$$

$$A^4=\begin{bmatrix}395 & 130 & -130\\ -390 & -125 & 130\\ 390 & 130 & -125\end{bmatrix}-\begin{bmatrix}175 & 56 & -56\\ -168 & -49 & 56\\ 168 & 56 & -69\end{bmatrix}+\begin{bmatrix}21 & 6 & -6\\ -18 & -3 & 6\\ 18 & 6 & -3\end{bmatrix}$$

$$A^4=\begin{bmatrix}241 & 80 & -80\\ -240 & -79 & 80\\ 240 & 80 & -79\end{bmatrix}$$

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors $(X_1,X_2\ldots X_n)$ corresponding to the n eigen values $\lambda_1,\lambda_2\ldots\lambda_n$ respectively then a matrix P can be found such that

$P^{-1}AP$ is a diagonal matrix.

Proof: Given that $(X_1,X_2\ldots X_n)$ be eigen vectors of A corresponding to the eigen values $\lambda_1,\lambda_2\ldots\lambda_n$ respectively and these eigen vectors are linearly independent Define P = $(X_1,X_2\ldots X_n)$

Since the n columns of P are linearly independent $|P|\neq 0$

Hence P^{-1} exists

Consider $AP = A[X_1,X_2\ldots X_n]$

$= [AX_1, AX_2\ldots AX_n]$

$= [\lambda X_1, \lambda_2X_2\ldots\lambda_nX_n]$

$$=[X_1,X_2\ldots X_n]\begin{bmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}$$

$$= PD$$

Where D = diag $(\lambda_1,\lambda_2,\lambda_3,\ldots\ldots\ldots\lambda_n)$

AP=PD

$P^{-1}(AP) = P^{-1}(PD) \Rightarrow P^{-1}AP = (P^{-1}P)D$

$\Rightarrow P^{-1}AP = (I)D$

$= D$

= diag $(\lambda_1,\lambda_2,\lambda_3,\ldots\ldots\ldots\lambda_n)$

Hence the theorem is proved.

Modal and Spectral matrices:

The matrix P in the above result which diagonalizable the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Note 1: If $X_1,X_2\ldots X_n$ are not linearly independent this result is not true.

2: Suppose A is a real symmetric matrix with n pair wise distinct Eigen values $\lambda_1,\lambda_2\cdots\lambda_n$ then the corresponding Eigen vectors $X_1,X_2\ldots X_n$ are pair wise orthogonal.

Hence if $P = (e_1,e_2\ldots e_n)$

Where $e_1 = (X_1 / ||X_1||)$, $e_2 = (X_2 / ||X_2||)\ldots.e_n = (X_n)/ ||X_n||$

then P will be an orthogonal matrix.

i.e, $P^TP=PP^T=I$

Hence $P^{-1} = P^T$

$P^{-1}AP = D \Rightarrow P^TAP=D$

Calculation of powers of a matrix:

We can obtain the power of a matrix by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2 = (P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Similarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$........(1)

To obtain A^n, Pre-multiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$

$= (PP^{-1})A^n(PP^{-1}) = A^n$

$$\Rightarrow A^n = PD^nP^{-1}$$

$$\text{Hence } A^n = P\begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix}P^{-1}$$

PROBLEMS

1. Determine the modal matrix P of A= $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$**. Verify that** $P^{-1}AP$ **is a diagonal matrix.**

Sol: The characteristic equation of A is $|A-\lambda I| = 0$

$$\text{i.e. } \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the Eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

$$\text{when } \lambda=5 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given Eigen value $\lambda=-3$ we can have two linearly independent Eigen vectors

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$P = (X_1 \; X_2 \; X_3)$$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = modal\ matrix\ of\ A$$

$$Now \det P = 1(-1) - 2(2) + 3(0-1) = -8$$

$$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= -\tfrac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$P^{-1}AP = -\tfrac{1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = diag\ (5, -3, -3)$$

Hence $P^{-1}AP$ is a diagonal matrix.

2. Find a matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ **too diagonal form. Hence Calculate** A^4 **.**

Sol: Characteristic equation of A is given by $|A-\lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 0 - 1[2 - 2(2-\lambda 0\} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, \lambda = 2, \lambda = 3$$

Thus the eigen values of A are 1, 2, 3

If x_1, x_2, x_3 be the components of an Eigen vector corresponding to the Eigen value λ, we have

$$[A-\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$, eigen vectors are given by

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ i.e. $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an Eigen vector corresponding to $\lambda=1$

For $\lambda=2$, $\lambda=3$ we can obtain Eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The Matrix P is called modal matrix of A . $P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

$$Now\,P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - D\ (say)$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

61

9. Practice Quiz

1. The Characteristic equation of $\begin{bmatrix}1 & 1\\2 & 5\end{bmatrix}$ is
   a) $\lambda^2 - 6\lambda + 3 = 0$ b) $\lambda^2 - 6\lambda + 5 = 0$ c) $\lambda^2 - 5\lambda + 3 = 0$ d) None
2. The eigen values of $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ are
   a) i, -i b) 1, 1 c) 1,-1 d) i, -1
3. A Square matrix A is said to be _________ if det(A)=0
   a) Non singular b) Singular c) Idempotent d) Nilpotent
4. A Square matrix A is said to be _________ if $A^T = -A$
   a) Skew –symmetric b) Symmetric c) Orthogonal d) None
5. The rank of matrix 3x2 matrix with two zero rows is
   a) 1 b) 2 c) 0 d) 3
6. The minimum possible rank of non zero matrix is
a) 1 b) 2 c) 0 d) 3
7. The Eigen values of $A = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ are
   a) 0, 2 b) 1, 1 c) 2, 2 d) 0, 3
8. The sum of the Eigen values of a square matrix A is
   a) trace (A) b) det (A) c) 0 d) none
9. The product of the Eigen values of a square matrix A is
   a) trace (A) b) det (A) c) 0 d) none
10. Cayley-Hamilton theorem states that
    a) Every matrix satisfies its characteristic equation
    b) Every square matrix satisfies its own characteristic equation
    c) Every square matrix doesn't satisfies its own characteristic equation
    d) Every matrix satisfies its own characteristic equation
11. If $\rho(A) = \rho(A/B) = n$ then the system $Ax = b$ have
a) Unique solution, b) infinite solutions, c) no solution d) none
12. If $\rho(A) = \rho(A/B)$ then the system $Ax = b$ is
    a) Consistent b) singular c) in consistent d) none
13. If $\rho(A) = \rho(A/B) < n$ then the system $Ax = b$ have
a) Unique solution, b) infinite solutions c) no solution d) none
14. If $\rho(A) \neq \rho(A/B)$ then the system $Ax = b$ is
    a) Consistent, b) singular c) in consistent d) none
15. The rank of matrix $\begin{pmatrix}-4 & 1 & -1\\-1 & -1 & -1\\7 & -3 & 1\end{pmatrix}$ is
    a) 1 b) 2 c) 3 d) 0

1. Assignments

S. No	Question	BL	CO
1	Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.	2	1
2	Solve completely the system of equations: x +2y + 3z = 0; 3x + 4y + 4z = 0 ; 7x + 10y + 12z =0.	3	1
3	Discuss for what values of λ, μ the simultaneous equations $x+y+z=6, x+2y+3z=10, x+2y+\lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.	4	2
4	Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	2	1
5	Prove that The sum of the Eigen values of a matrix A is same as trace of the matrix A.	3	2
6	Verify Cayley-Hamilton theorem for $A=\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ *and hence find* A^{-1}.	4	2

11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	Define Rank of Matrix	1	1
2	Discuss about the solution of homogeneous system of equations.	1	1
3	Makes a truth table for the statement $(p \wedge q)\ (\sim p)$	1	1
4	Discuss about the solution of non-homogeneous system of equations.	1	1
5	Find the Eigen value of	1	1

63

	$\begin{bmatrix}2 & 3 & 7\\3 & -2 & 4\\1 & -3 & -1\end{bmatrix}$		
6	The product of the Eigen values of a matrix A is equal to its determinant.	**1**	**1**
7	A square matrix A and it's transpose A^T have the same Eigen values.	**2**	**1**
8	Prove that the Eigen values of A^{-1} are the reciprocals of the Eigen values of A.	**2**	**1**
9	If λ is an Eigen value of a non-singular matrix A, then $\frac{\|A\|}{\lambda}$ is an Eigen value of the matrix $adj\ A.$	**2**	**1**
10	The Eigen values of a triangular matrix are the diagonal elements.	**2**	**1**
11	The sum of the Eigen values of a matrix A is equal to its trace.	**2**	**1**
12	State Cayley-Hamilton theorem	**2**	**1**
13	Define Model matrix and Orthogonal matrix	**1**	**1**
14	Find the rank of the matrix $A = \begin{bmatrix}1 & -2 & 0 & 1\\2 & -1 & 1 & 0\\3 & -3 & 1 & 1\\-1 & -1 & -1 & 1\end{bmatrix}$	**2**	**1**
15	For what values of k the matrix $\begin{bmatrix}4 & 4 & -3 & 1\\1 & 1 & -1 & 0\\k & 2 & 2 & -2\\9 & 9 & k & 3\end{bmatrix}$ has rank '3'.	**1**	**1**
16	By reducing the matrix $\begin{bmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\\3 & 0 & 5 & -10\end{bmatrix}$ into normal form, find its rank.	**2**	**1**

12. Part B- Questions

S.No	Question	BL	CO
1	Discuss for what values of λ, μ the simultaneous equations $x+y+z=6, x+2y+3z=10, x+2y+\lambda z=\mu$ have (i) no solution (ii) (ii) a unique solution (iii) (iii) an infinite number of solutions.	1	1
2	Find the Eigen values and the corresponding Eigen vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	2	1
3	Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Normal form.	2	1
4	Find a matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ too diagonal form. Hence Calculate A^4 .	3	1
5	Determine the modal matrix P of A= $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal	3	1
6	Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ *and hence find* A^{-1}.	3	1

13. Supportive Online Certification Courses

LINEAR ALGEBRA -NPTEL online programme-SWAYAM.

14. Real Time Applications

S.No	Application
1	Games especially 3D
2	Encryption

15. Contents Beyond the Syllabus

1. Real time Applications
2. Quadratic Forms-Nature, signature, Rank

16. Prescribed Text Books & Reference Books

Text Book

1. "B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. "Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

References:

1. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.
2. George B. Thomas, Maurice D. Weir and Joel Hass, Thomas Calculus, 13/e, Pearson Publishers, 2013.
3. Glyn James, Advanced Modern Engineering Mathematics, 4/e, Pearson publishers, 2011.
4. Micheael Greenberg, Advanced Engineering Mathematics, 9th edition, Pearson edn
5. Dean G. Duffy, Advanced Engineering Mathematics with MATLAB, CRC Press
6. Peter O'neil, Advanced Engineering Mathematics, Cengage Learning.
7. R.L. Garg Nishu Gupta, Engineering Mathematics Volumes-I &II, Pearson Education

COURSE MATERIAL	
SUBJECT	LINEAR ALGEBRA AND CALCULUS (20A54101)
UNIT	2
COURSE	B.TECH
DEPARTMENT	SCIENCE & HUMANITIES
SEMESTER	11
PREPARED BY (Faculty Name/s)	Mr. G. MOHAN BABU Dr. K ANANTH KUMAR
VERSION	V-I
PREPARED / REVISED DATE	17-05-2021

TABLE OF CONTENTS – UNIT 4		
SNo	Contents	Page No.

1. **Course Objectives**

The objectives of this course are to

- This course will illuminate the students in the concepts of calculus and linear algebra.
- To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real world problems and their applications.

2. **Prerequisites**
   1. Students should have knowledge on continuity and derivatives
   2. Students should have knowledge on Calculus

3. **Syllabus**

UNIT II

Mean Value Theorems

Rolle's Theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Taylor's and Maclaurin theorems with remainders (without proof);

4. **Course outcomes**

At the end of the course, the student will be able to

- Translate the given functions as series of Taylor's and Maclaurin's with remainders (L3)
- Utilize mean value theorems to real life problems (L3)
- Analyze the behavior of functions using mean value theorems. (L4)

69

5. Co-PO / PSO Mapping

MAD	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
CO3	3													

6. Lecture Plan

1	1	Introduction	T1
2		Rolle's Theorem	TB1/RB2
3		Problems on Rolle's Theorem	TB1/RB2
4		Problems on Rolle's Theorem	TB1/RB2
5	2	Mean value Theorem	TB1/RB2
6		Problems on Mean value Theorem	TB1/RB2
7		Problems on Mean value Theorem	TB1/RB2
8		Cauchy's Mean value Theorem	TB1/RB2
9		Problems on Cauchy's Mean value Theorem	TB1/RB2
10	3	Taylor's and Maclaurin theorems	TB1/RB2
11		Taylor's and Maclaurin theorems with remainders	TB1/RB2
12		Problems on Taylor's and Maclaurin theorems with remainders	TB1/RB2

7. Activity Based Learning

1. To find the Taylor's and Maclaurin series expansions for different functions.

8. Lecture Notes

Mean Value Theorems

I Rolle's Theorem:

Let f(x) be a function such that

(i). It is continuous in closed interval [a,b]

(ii). It is differentiable in open interval (a,b) and

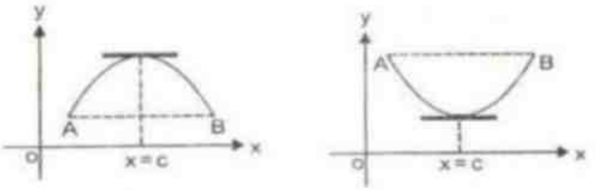
(iii). f(a) = f(b).

Then there exists at least one point 'c' in (a,b) such that

$f^1(c) = 0$.

Geometrical Interpretation of Rolle's Theorem:

Let $f:[a,b] \rightarrow R$ be a function satisfying the three conditions of Rolle 's Theorem. Then the graph.



1. y=f(x) in a continuous curve in [a,b].
2. There exist a unique tangent line at every point x=c, where a<c<b
3. The ordinates f(a), f(b) at the end points A,B are equal so that the points A and B are equidistant from the X-axis.
4. By Rolle's Theorem, There is at least one point x=c between A and B on the curve at which the tangent line is parallel to the x-axis and also it is parallel to chord of the curve.

1. Verify Rolle's theorem for the function $f(x) = \sin x/e^x$ or $e^{-x}\sin x$ in $[0,\pi]$

Sol: i) Since sinx and e^x are both continuous functions in $[0, \pi]$.

Therefore, $\sin x/e^x$ is also continuous in $[0,\pi]$.

ii)Since sinx and e^x be derivable in $(0,\pi)$, then f is also derivable in $(0,\pi)$.

iii) $f(0) = \sin 0/e^0 = 0$ and $f(\pi) = \sin \pi/e^{\pi} = 0$

$\therefore$ $f(0) = f(\pi)$

Thus all three conditions of Rolle 's Theorem are satisfied.

$\therefore$ There exists $c \in (0, \pi)$ such that $f^1(c)=0$

Now $f^1(x) = \dfrac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \dfrac{\cos x - \sin x}{e^x}$

$f^1(c) = 0 \Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0$

$\cos c = \sin c \Rightarrow \tan c = 1$

3

71

$c = \pi/4 \in (0,\pi)$

Hence Rolle's theorem is verified.

2. Verify Rolle's theorem for the functions $\log\left(\frac{x^2+ab}{x(a+b)}\right)$ in[a,b] , a>0, b>0.

Sol: Let $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$

$= \log(x^2+ab) - \log x - \log(a+b)$

(i). Since f(x) is a composite function of continuous functions in [a,b], it is continuous in [a,b].

(ii). $f^1(x) = \frac{1}{x^2+ab}.2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$f^1(x)$ exists for all $x \in (a,b)$

(iii). $f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

$f(a) = f(b)$

Thus f(x) satisfies all the three conditions of Rolle 's Theorem.

So, $\exists\ c \in (a, b) \Rightarrow f^1(c) = 0$.

$f^1(c) = 0, \Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem verified.

3. Verify whether Rolle 's Theorem can be applied to the following functions in the intervals.

i) $f(x) = \tan x$ in$[0, \pi]$ and ii) $f(x) = 1/x^2$ in $[-1,1]$

(i) f(x) is discontinuous at $x = \pi/2$ as it is not defined there. Thus condition (i) of Rolle 's Theorem is not satisfied. Hence we cannot apply Rolle 's Theorem here.

$\therefore$ Rolle's theorem cannot be applicable to $f(x) = \tan x$ in $[0,\pi]$.

(ii). $f(x) = 1/x^2$ in $[-1,1]$

f(x) is discontinuous at x= 0.Hence Rolle 's Theorem cannot be applied.

4. Verify Rolle's theorem for the function $f(x) = (x-a)^m(x-b)^n$ where m,n are positive integers in [a,b].

Sol: (i). Since every polynomial is continuous for all values, f(x) is also continuous in[a,b].

(ii) $f(x) = (x-a)^m(x-b)^n$

$f^1(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m.n(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$

$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$

Which exists

4

72

Thus f(x) is derivable in (a,b)

(iii) f(a) = 0 and f(b) = 0

$\therefore$ f(a) =f(b)

Thus three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists c ϵ(a,b) such that $f^1(c)=0$

$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)]=0$

$\Rightarrow$ (m+n)c-(mb+na)=0

=> (m+n)c = mb+na

$\Rightarrow c = \frac{mb+na}{m+n} \epsilon (a,b)$

$\therefore$ Rolle 's Theorem verified.

5. Using Rolle 's Theorem, show that $g(x) = 8x^3-6x^2-2x+1$ has a zero between 0 and 1.

Sol: $g(x) = 8x^3-6x^2-2x+1$ being a polynomial, it is continuous on [0,1] and differentiable on (0,1)

Now g(0) = 1 and g(1)= 8-6-2+1 = 1

Also g(0)=g(1)

Hence, all the conditions of Rolle's theorem are satisfied on [0,1].

Therefore, there exists a number cϵ(0,1) such that $g^1(c)=0$.

Now $g^1(x) = 24x^2-12x-2$

$\therefore g^1(c) = 0 \Rightarrow 24c^2-12c-2 = 0$

$\Rightarrow c = \dfrac{3 \pm \sqrt{21}}{12}$ *ie* c= 0.63 or -0.132

only the value c = 0.63 lies in (0,1)

Thus there exists at least one root between 0 and 1.

6. Verify Rolle's theorem for $f(x) = x^{2/3} - 2x^{1/3}$ in the interval (0, 8).

Sol: Given $f(x) = x^{2/3} - 2x^{1/3}$

f(x) is continuous in [0,8]

$f^1(x) = 2/3 \cdot 1/x^{1/3} - 2/3 \cdot 1/x^{2/3} = 2/3(1/x^{1/3} - 1/x^{2/3})$

Which exists for all x in the interval (0,8)

$\therefore$ f is derivable (0,8).

Now f(0) = 0 and $f(8) = (8)^{2/3}-2(8)^{1/3} = 4-4 = 0$

i.e., f(0) = f(8)

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one value of c in(0,8) such that $f^1(c)=0$

ie. $\dfrac{1}{c^{\frac{1}{3}}} - \dfrac{1}{c^{\frac{2}{3}}} = 0 \Rightarrow c = 1 \epsilon (0,8)$

Hence Rolle's Theorem is verified.

7. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3,0].

Sol: - (i). Since x(x+3) being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x, their product $x(x+3)e^{-x/2} = f(x)$ is also continuous for every value of x and in particular f(x) is continuous in the [-3,0].

(ii). we have $f^1(x) = x(x+3)(-1/2\ e^{-x/2})+(2x+3)e^{-x/2}$

$$= e^{-x/2}\left[2x+3-\frac{x^2+3x}{2}\right]$$

$$=e^{-x/2}[6+x-x^2/2]$$

Since $f^1(x)$ doesnot become infinite or indeterminate at any point of the interval(-3,0).

f(x) is derivable in (-3,0)

(iii) Also we have f(-3) = 0 and f(0) =0

$\therefore f(-3)=f(0)$

Thus f(x) satisfies all the three conditions of Rolle's theorem in the interval [-3,0].

Hence there exist at least one value c of x in the interval (-3, 0) such that $f^1(c)=0$

i.e., $\frac{1}{2}e^{-c/2}(6+c-c^2)=0 \Rightarrow 6+c-c^2=0$ ($e^{-c/2}\neq0$ for any c)

$\Rightarrow c^2+c-6 = 0 \Rightarrow (c-3)(c+2)=0$

c=3,-2

Clearly, the value c= -2 lies within the (-3,0) which verifies Rolle's theorem.

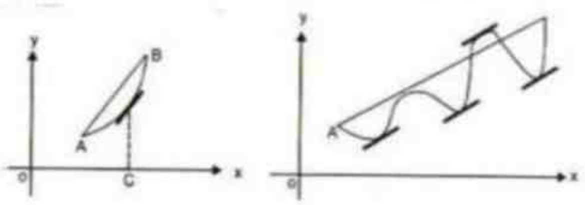
II. Lagrange's mean value Theorem

Let f(x) be a function such that (i) it is continuous in closed interval [a,b] & (ii) differentiable in (a,b). Then ∃ at least one point c in (a,b) such that

$$f^1(c) = \frac{f(b)-f(a)}{b-a}$$

Geometrical Interpretation of Lagrange's Mean Value theorem:

Let $f:[a,b]\rightarrow R$ be a function satisfying the two conditions of Lagrange's theorem. Then the graph.



1.y=f(x) is continuous curve in [a,b]

2. At every point x=c, when a<c<b, on the curve y=f(x), there is unique tangent to the curve. By Lagrange's theorem there exists at least one point $c\in(a,b) \ni f^1(c)=\frac{f(b)-f(a)}{b-a}$

6

74

Geometrically there exist at least one point c on the curve between A and B such that the tangent line is parallel to the chord $\overleftrightarrow{AB}$

1. Verify Lagrange's Mean value theorem for $f(x) = x^3-x^2-5x+3$ in [0, 4]

Sol: Let $f(x) = x^3-x^2-5x+3$ is a polynomial in x.

$\therefore$ It is continuous & derivable for every value of x.

In particular, f(x) is continuous [0,4] & derivable in (0,4)

Hence by Lagrange's Mean value theorem $\exists$ $c \in (0,4) \ni$

$$f^1(c) = \frac{f(4)-f(0)}{4-0}$$

i.e., $3c^2-2c-5 = \dfrac{f(4)-f(0)}{4}$ ....(1)

Now $f(4) = 4^3-4^2-5.4+3 = 64-16-20-3 = 67-36 = 31$ & $f(0)=3$

$$\frac{f(4)-f(0)}{4} = \frac{(31-3)}{4} = 7$$

From equation (1), we have

$3c^2-2c-5 = 7 \Rightarrow 3c^2-2c-12 = 0$

$$c = \frac{2 \pm \sqrt{4+144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

We see that $\dfrac{1+\sqrt{37}}{3}$ lies in open interval (0,4) & thus Lagrange's Mean value theorem is verified.

2. Verify Lagrange's Mean value theorem for $f(x) = \log_e x$ in [1,e]

Sol: - $f(x) = \log_e x$

This function is continuous in closed interval [1,e] & derivable in (1,e).Hence L.M.V.T is applicable here. By this theorem, $\exists$ a point c in open interval (1,e) such that

$$f^1(c) = \frac{f(e)-f(1)}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

But $f^1(c) = \dfrac{1}{e-1} \Longrightarrow \dfrac{1}{c} = \dfrac{1}{e-1}$

$\therefore c = e - 1$

Note that (e-1) is in the interval (1, e).

Hence Lagrange's mean value theorem is verified.

3. Give an example of a function that is continuous on [-1, 1] and for which mean value theorem does not hold with explanations.

Sol:- The function f(x) = $|x|$ is continuous on [-1,1]

But Lagrange Mean value theorem is not applicable for the function f(x) as its derivative does not exist in (-1,1) at x=0.

4. If a<b, P.T $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$ using Lagrange's Mean value theorem. Deduce the following.

i). $\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii). $\frac{5\pi+4}{20} < Tan^{-1}2 < \frac{\pi+2}{4}$

Sol: consider f(x) = $Tan^{-1}x$ in [a,b] for 0<a<b<1

Since f(x) is continuous in closed interval [a,b] & derivable in open interval (a,b).

We can apply Lagrange's Mean value theorem here.

Hence there exists a point c in (a,b)∋

$f^1(c) = \frac{f(b)-f(a)}{b-a}$

Here $f^1(x) = \frac{1}{1+x^2}$ & hence $f^1(c) = \frac{1}{1+c^2}$

Thus ∃ c, a<c<b ∋

$\frac{1}{1+c^2} = \frac{Tan^{-1}b - Tan^{-1}a}{b-a}$ ------- (1)

We have $1+a^2<1+c^2<1+b^2$

$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ .... (2)

From (1) and (2), we have

$\frac{1}{1+a^2} > \frac{Tan^{-1}b - Tan^{-1}a}{b-a} > \frac{1}{1+b^2}$

or

$\frac{b-a}{1+a^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+b^2}$ ....(3)

Hence the result

Deductions: -

8

76

(i)We have $\dfrac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \dfrac{b-a}{1+a^2}$

Take $b = \dfrac{4}{3}$ & a=1, we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}(\frac{4}{3}) - Tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2} \Longrightarrow \frac{\frac{4-3}{3}}{\frac{25}{9}} < Tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{\frac{4-3}{3}}{2}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < Tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) Taking b=2 and a=1, we get

$$\frac{2-1}{1+2^2} < Tan^{-1}2 - Tan^{-1}1 < \frac{2-1}{1+1^2} \Rightarrow \frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < Tan^{-1}2 < \frac{2+\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} + < Tan^{-1}2 < \frac{2+\pi}{4}$$

5. Show that for any x > 0, 1 + x < e^x < 1 + xe^x.

Sol: - Let f(x) = e^x defined on [0, x]. Then f(x) is continuous on [0, x] & derivable on (0, x).

By Lagrange's Mean value theorem ∃ a real number c ∈ (0, x) such that

$$\frac{f(x)-f(0)}{x-0} = f^1(c)$$

$$\Rightarrow \frac{e^x - e^0}{x-0} = e^c \Rightarrow \frac{e^x - 1}{x} = e^c \text{ ....(1)}$$

Note that 0<c<x => e^0<e^c<e^x (e^x is an increasing function)

=>$1 < \dfrac{e^x-1}{x} < e^x$ From (1)

=> x<e^x-1<xe^x

=> 1+x<e^x<1+xe^x.

6. Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T.

Sol: - Let f(x) = $\sqrt[5]{x}$ =$x^{1/5}$& a=243, b=245

Then $f^1(x) = 1/5\, x^{-4/5}$& $f^1(c) = 1/5c^{-4/5}$

By L.M.V.T, we have

$$\frac{f(b)-f(a)}{b-a} = f^1(c)$$

$$=> \frac{f(245)-f(243)}{245-243} = \frac{1}{5}c^{\frac{-4}{5}}$$

9

77

=> f (245) =f (243) +2/5c$^{-4/5}$

=> c lies b/w 243 & 245 take c= 243

=> $\sqrt[5]{245}$ = (243) $^{1/5}$ +2/5(243)$^{-4/5}$= $(3^5)^{\frac{1}{5}} + \frac{2}{5}(3^5)^{\frac{-4}{5}}$

= 3+ (2/5) (1/81) = 3+2/405 = 3.0049

7. Find the region in which f(x) = 1-4x-x² is increasing &the region in which it is decreasing using M.V.T.

Sol: - Given $f(x) = 1-4x-x^2$

f(x) being a polynomial function is continuous on [a,b] & differentiable on (a,b) ∀ a,b ∈R

∴ f satisfies the conditions of L.M.V.T on every interval on the real line.

$f^1(x) = -4-2x = -2(2+x)\ \forall\ x \in R$

$f^1(x) = 0$ if $x = -2$

for $x<-2$, $f^1(x) >0$ & for $x>-2$, $f^1(x)<0$

Hence f(x) is strictly increasing on (-∞, -2) & strictly decreasing on (-2,∞)

8. Using Mean value theorem prove that Tan x > x in 0<x<π/2

Sol: - Consider f(x) = Tan x in $[\xi, x]$ where $0< \xi <x<\pi/2$

Apply L.M.V.T to f(x)

∃ a points c such that $0< \xi <c<x<\pi/2$ such that

$$\frac{Tan\,x - Tan\,\xi}{x-\xi} = \sec^2 c ==>$$

$$\text{Tan x - Tan } \xi = (\text{x} - \xi)\sec^2 \text{c}$$

$$Take\xi \to 0+0 then Tan\,x = x\sec^2 x$$

But $\sec^2 c>1$.

Hence Tan x > x

9. If f¹(x) = 0 Through out an interval [a, b], prove using M.V.T f(x) is a constant in that interval.

Sol: - Let f(x) be function defined in [a, b] & let $f^1(x) = 0\ \forall$ x in [a, b].

Then $f^1(t)$ is defined & continuous in [a, x] where a≤x≤b.

& f(t) exist in open interval (a, x).

By L.M.V.T ∃ a point c in open interval (a, x) ∍

$$\frac{f(x)-f(a)}{x-a} = f^1(c)$$

But it is given that $f^1(c) = 0$

$$\therefore \text{f(x) - f(a)} = 0$$

$$\therefore \text{f(x)} = \text{f(a)}\ \forall\ \text{x}$$

Hence f(x) is constant.

10 Using mean value theorem

i) $x > \log(1+x) > \frac{x}{1+x}$ $x > 0$

ii) $\pi/6 + (\sqrt{3}/15) < \sin^{-1}(0.6) < \pi/6 + (1/6)$

iii) $1+x < e^x < 1+xe^x \forall x > 0$

iv) $\frac{v-u}{1+v^2} < \tan^{(-1)}v - \tan^{(-1)}u < \frac{v-u}{1+u^2}$ where $0 < u < v$ hence deduce

a) $\pi/4 + (3/25) < \tan^{(-1)}(4/3) < \pi/4 + (1/6)$

III. Cauchy's Mean Value Theorem

If $f: [a,b] \to R$, $g:[a,b] \to R \ni$ (i) f,g are continuous on [a,b] (ii) f,g are differentiable on (a,b)

(iii) $g^1(x) \neq 0 \forall x \in (a,b)$, *then*

$$\exists a \text{ point } c \in (a,b) \ni \frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

1. Find c of Cauchy's mean value theorem for

$$f(x) = \sqrt{x} \ \ \& \ g(x) = \frac{1}{\sqrt{x}}$$

in [a, b] where 0<a<b

Sol: - Clearly f, g are continuous on $[a, b] \subseteq R^+$

We have $f^1(x) = \frac{1}{2\sqrt{x}}$ and $g^1(x) = \frac{-1}{2x\sqrt{x}}$ which exits on (a, b)

$\therefore$ f, g are differentiable on $(a, b) \subseteq R^+$

Also $g^1(x) \neq 0$, $\forall x \in (a, b) \subseteq R^+$

Conditions of Cauchy's Mean value theorem are satisfied on (a, b) so $\exists c \in (a, b) \ni$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} \Longrightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}} \Rightarrow \sqrt{ab} = c$$

Since a, b >0 , $\sqrt{ab}$ is their geometric mean and we have $a < \sqrt{ab} < b$

$c \in (a,b)$ which verifies Cauchy's mean value theorem.

2. Verify Cauchy's Mean value theorem for $f(x) = e^x$ & $g(x) = e^{-x}$ in [3, 7] &

Find the value of c.

Sol: We are given $f(x) = e^x$ & $g(x) = e^{-x}$

11

79

f(x) & g(x) are continuous and derivable for all values of x.

=>f & g are continuous in [3, 7]

=> f & g are derivable on (3,7)

Also $g^1(x) = e^{-x} \neq 0 \ \forall \ x \in (3, 7)$

Thus f & g satisfies the conditions of Cauchy's mean value theorem.

Consequently, $\exists$ a point $c \in (3, 7)$ such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f^1(c)}{g^1(c)} \Longrightarrow \frac{e^7 - e^3}{e^{-7} - e^{-3}} = \frac{e^c}{-e^{-c}} \Longrightarrow \frac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}} = -e^{2c}$$

=> $-e^{7+3} = -e^{2c}$

=> 2c = 10

=> $c = 5 \in (3, 7)$

Hence C.M.T. is verified

TAYLOR'S SERIES

The series $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + ... + \frac{(x-a)^n}{n!}f^n(a)$

is called **Taylor series expansion of** $f(x)$ **of degree n about x = a** assuming that $f(x)$ has successive derivatives of all orders for $x \in [a,b]$

MACLAURIN 'S SERIES

The series $f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^n}{n!}f^n(0) + ...$

is called **maclaurin series expansion of** $f(x)$ **degree n about x = 0** assuming that $f(x)$ has successive derivatives of all orders for $x \in [0,b]$ and the remainder after nth term in Lagrange's form of remainder

$\frac{x^n}{n!}f^n(\theta x)$. $(0 < \theta < 1)$

PROBLEMS

1).Obtain the Taylor's series expansion of sin x in powers of $x - \pi/4$

Sol: Given that f (x) = sin x and $x - \pi/4$

The Taylor's series expansion of f (x) in powers of x – a is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + ...$$

here f(x) = sin x and a = π/4

$f(x) = \sin x \qquad f(a) = f(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Now differentiate f(x) w.r.t. x successively , we get

$f'(x) = \cos x \qquad f'(a) = f'(\frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

12

80

$$f''(x) = -\sin x \qquad f''(a) = f''(\frac{\pi}{4}) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(x) = -\cos x \qquad f'''(a) = f'''(\frac{\pi}{4}) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f^{iv}(x) = \sin x \qquad f^{iv}(a) = f^{iv}(\frac{\pi}{4}) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Hence the Taylor's series expansion is

$$Sin x = \frac{1}{\sqrt{2}} + (x-\frac{\pi}{4})\frac{1}{\sqrt{2}} + \frac{(x-\frac{\pi}{4})^2}{2!}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^3}{3!}(-\frac{1}{\sqrt{2}}) + \frac{(x-\frac{\pi}{4})^4}{4!}(\frac{1}{\sqrt{2}}) + \ldots.$$

$$Sin x = \frac{1}{\sqrt{2}}\left[1 + (x-\frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^2}{2!} - \frac{(x-\frac{\pi}{4})^3}{3!} + \frac{(x-\frac{\pi}{4})^4}{4!} + \ldots..\right]$$

2). Find the Taylors series expansion of sin2x about $x = \pi/4$

Sol: Write $x - \frac{\pi}{4} = t \Rightarrow x = \frac{\pi}{4} + t$

$\therefore$ Sin2x = Sin2($\frac{\pi}{4}$ + t) = sin ($\frac{\pi}{2}$ + 2t) = cos2t

= 1 - $\frac{2^2 t^2}{2!}$ + $\frac{2^4 t^4}{4!}$ - ...., for all values of t

=1 - $\frac{2^2}{2!}(x-\frac{\pi}{4})^2 + \frac{2^4}{4!}(x-\frac{\pi}{4})^4 - \ldots\ldots\ldots\ldots\ldots$ for all values of x

3). Expand $e^{x\sin x}$ in powers of x

Sol. f(x)= $e^{x\sin x}$

$\Rightarrow$ f(0) = e^0 = 1

$f^1(x) = e^{x\sin x}(x\cos x + \sin x) = f(x)(x\cos x + \sin x)$

$f^1(0) = f(0)(0+0) = 1(0) = 0$

$f^{11}(x) = f^1(x)(x\cos x + \sin x) + f(x)(-x\sin x + 2\cos x)$

$f^{11}(0) = 0 + 1(0+2) = 2$

$f^{111}(x) = f^{11}(x)(x\cos x + \sin x) + f^1(x)(-x\sin x + 2\cos x) + f^1(x)(-x\sin x + 2\cos x) + f(x)(-x\cos x - 3\sin x)$

$f^{111}(0) = 2(0+0) + 0 + 0 + 1(0-0) = 0$

And so on.

Substituting the values of f(0) , f^1(0) , etc ., in the Maclaurin's series , we obtain

$e^{x\sin x}$= f(x) = f(0) $+\frac{x}{1!}f^1(0) + \frac{x^2}{2!}f^{11}(0) + \frac{x^3}{3!}f^{111}(0) + \ldots\ldots\ldots.$

$= 1 + \frac{x^2}{2!}(2) + \ldots\ldots\ldots\ldots$

$= 1 + x^2 + \ldots\ldots\ldots$

81

4) *Show that* $\log\left(1+e^x\right) = \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+....$ *and hencededucethat*

$$\frac{e^x}{e^x+1} = \frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+..........$$

Sol:

We know that the maclaurin's series expansion of $f(x)$ is given by

$$f(x) = f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+.........+\frac{x^n}{n!}f^n(0)+.......$$

Here $f(x)=\log\left(1+e^x\right)$ *and* $f(a) = f(0) = \log 2$

Now differentiate $f(x)$ *w.r.t.x successivdy we get*

$$f'(x) = \frac{e^x}{e^x+1} \qquad f'(0) = \frac{e^0}{e^0+1} = \frac{1}{2}$$

$$f''(x) = \frac{(e^x+1)e^x - e^x.e^x}{(e^x+1)^2} = \frac{e^x}{(e^x+1)^2} \qquad f''(0) = \frac{e^0}{(e^0+1)^2} = \frac{1}{4}$$

$$f'''(x) = \frac{(e^x+1)^2e^x - 2e^x(1+e^x)e^x}{(e^x+1)^4} = \frac{(1+e^x)\left[e^x+e^{2x}-2e^{2x}\right]}{(e^x+1)^4} = \frac{e^x-e^{2x}}{(e^x+1)^3}$$

$$f'''(0) = \frac{e^0 - e^0}{(e^0+1)^3} = 0$$

$$f^{iv}(x) = \frac{(e^x+1)^3(e^x-2e^{2x})-(e^x-e^{2x})3(1+e^x)^2e^x}{(e^x+1)^6}$$

$$= \frac{(1+e^x)\left[e^x-2e^{2x}\right]-3(e^x-e^{2x})e^x}{(e^x+1)^4}$$

$$f^{iv}(0) = \frac{(1+e^0)\left[e^0-2e^0\right]-3(e^0-e^0)e^0}{(e^0+1)^4} = -\frac{2}{16} = -\frac{1}{8}$$

Substitute these values in the Maclaurin's series

$$\log\left(1+e^x\right) = \log 2 + x\frac{1}{2}+\frac{x^2}{2!}\frac{1}{4}+\frac{x^3}{3!}(0)+\frac{x^4}{4!}\left(-\frac{1}{8}\right)+..........$$

$$\log\left(1+e^x\right) = \log 2 + \frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+.................. \qquad \text{\_\_\_\_\_\_(1)}$$

Deduction: Differentiate equation (1) w.r.t. x ,we get

$$\frac{e^x}{e^x+1} = \frac{1}{2}+\frac{2x}{8}-\frac{4x^3}{192}+.................$$

$$\frac{e^x}{e^x+1} = \frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+..............$$

14

82

5. Obtain Taylors series expansion of e^x about $x = -1$.

Sol: Let $f(x) = e^x$

Put $x+1 = t \Rightarrow x = t-1$

$$f(x) = e^x = e^{t-1} = e^{-1}.e^t = \frac{1}{e}[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...] \text{ for all values of } t$$

$$\therefore f(x) = \frac{1}{e}[1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + ...] \text{ for all values of } x$$

Lagrange's Form of remainder:

$$R_n = \frac{(b-a)^n f^{(n)}(c)}{n!}$$

6).Verify Taylor`s theorem for f(x) = $(1-x)^{5/2}$ with Lagrange`s form of remainder upto 2 terms in the interval $[0,1]$.

Sol: Consider $f(x) = (1-x)^{5/2}$ in $[0,1]$

(i) $f(x)$, $f'(x)$ are continuous in $[0,1]$

(ii) $f''(x)$ is differentiable in (0, 1)

Thus f(x) satisfies the conditions of Taylor's theorem

Now we consider Taylor's theorem with Lagrange's form of remainder

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \text{ with } 0<\theta<1 \rightarrow (1)$$

Here $a = 0$ and $x = 1$, $n = p = 2$, $b = 1$

Now $f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}(1-0)^{3/2} = \frac{-5}{2}$$

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2} \Rightarrow f^{11}(\theta) = \frac{15}{4}(1-\theta)^{\frac{1}{2}}$$

And $f(1) = 0$

From (1), We have $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

Substituting the values, we get $0 = 1 + 1(\frac{-5}{2}) + \frac{1^2(15)}{2!(4)}(1-\theta)^{\frac{1}{2}}$

$$\Rightarrow \theta = \frac{9}{25} = 0.36$$

$\therefore \theta$ lies in between 0 and 1.

Thus Taylor's theorem is verified.

7. Obtain the Maclaurin's series expansion of coshx

Sol : Given $f(x) = \cosh x$

$f^1(x) = \sinh x \Rightarrow f(0) = f^{11}(0) = ...f^{2n}(0) = \cosh 0 = 1$

$f^{11}(x) = \cosh x$

$f^{111}(x) = \sinh x$

$\Rightarrow f(0) = f^{11}(0) = ...= f^{2n}(0) = \cosh 0 = 1$

$f^1(0) = f^{111}(0) .....= f^{2n+1}(0) = \sinh 0 = 0$

Hence by the Maclaurin's series expansion of coshx is given by

83

coshx $= f(0) + xf^{1}(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{111}(()) + ...$

$$= 1 + + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... + \frac{x^{2n}}{(2n)!} + ...$$

8. Find Maclaurin's theorem with Lagrange's form of remainder for f(x) = cosx.

Sol: Maclaurin's theorem with Lagrange's form of remainder is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

Consider f(x) = cosx

$$\therefore f^n(x) = \frac{d^n}{dx^n}(\cos x) = \cos(\frac{n\pi}{2} + x)$$

At x=0 , $f^n(0) = \cos(\frac{n\pi}{2})$

Thus f(0) = cos0=1

$$f^{2n}(0) = \cos\frac{2n\pi}{2} = \cos n\pi = (-1)^n$$

And $f^{(2n+1)}(0) = \cos[\frac{(2n+1)\pi}{2}] = 0$

If n is even, coefficient of x will be $(-1)^n$. If n is odd, coefficients of x are all zero.

Substituting these values, we have

$$\cos x = f(x) = 1 + 0 + \frac{x^2}{2!}(-1) + ... + \frac{x^4}{4!}(1) + ... + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... + \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Worked out problems:

1. Obtain the Maclaurin's series for the following functions

a) f(x) = (1+x) b) f(x) = $\log_e(1+x)$ c) f(x) = sinhx d) f(x) = e^x

2. Show that

$$i)\ \frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + ...$$

$$ii)\ \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + ... for\ 0 < x < 2$$

9. Practice Quiz

1. The value of c of Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ in (0, π) is

   a) 0 b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

2. Using which mean value theorem we can calculate approximately the value of $(65)^{\frac{1}{6}}$ in an easier way

   a) Cauchy's b) Lagrange's c) Taylor's d) Rolle's

3. The value of Cauchy's mean value theorem for $f(x)=e^x$ & $g(x)=e^{-x}$

defined on [a, b] 0<a<b is

a) $\sqrt{ab}$ b) $\frac{a-b}{2}$ c) $\frac{a+b}{2}$ d) $\frac{2ab}{a+b}$

4. If f(x) is continuous in[a, b], exists for every value of x in (a, b), f(a)=f(b)

then there exists at least one value of c in(a, b) such that $f'(c)=$

a)1 b)0 c) a d)b

5. Lagrange's mean value theorem for $f(x)=\sec x$ in $(0,2\pi)$is

a) applicable b) not applicable due to non differentiable

c) applicable, $c=\pi$ d) not applicable due to discontinuity

6. If $f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+---$ then the series is called

a)Maclaurin's b)Taylor's c) Cauchy's d)none

7. The value of c of Rolle's theorem for $f(x)=\sin x$ in (0, π) is

b) 0 b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) π

8. The value of c of Rolle's theorem in (-1,1) for $f(x)=x^3-x$ is

a) 0 b) $\frac{1}{\sqrt{2}}$ c) $\sqrt{\frac{1}{3}}$ d)none

9. Rolle's theorem is not applicable for $f(x)=1/x^2$ in $[-1,1]$ because

a) is not defined at 0 b) it is defined at 0 c) is differentiable at 0 d) None

10. Rolle's theorem is not applicable to $f(x)=x^2$ in [1, 2] because

a) is not defined at x=1 b) is not defined at x=2 c) $f(1)\neq f(2)$ d) None

11. The value of c of Lagrange's mean value theorem for $f(x)=x^2-3x+2$ in [-2, 3] is

a) -1/2 b) 1/2 c) 1/3 d) -1/3

12. The Taylor's series expansion of f(x) at x=0 is

a) $f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+--$ b) $f(x)=\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+--$

c) $f(x)=f(0)+\frac{x}{1!}f(0)+\frac{x^2}{2!}f''(0)+--$ d) $f(x)=f(0)+f(0)+f''(0)+--$

13. The Taylor series expansion of $f(x)=sinx$ about x=0

a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}+---$ b) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+---$

c) $x-\frac{x^3}{3!}-\frac{x^5}{5!}+---$ d)) $1-\frac{x^2}{2!}-\frac{x^4}{4!}+---$

14. Lagrange's mean value theorem for $f(x)=\tan x$ is not applicable in the interval

a) $[0,\frac{\pi}{2}]$ b) $[0,\pi]$

c) $[0,2\pi]$ d) All the above

17

10 Assignments

1. State Rolle's Theorem and verify it for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in [a, b] , a>0,b>0
2. Verify Taylor's theorem for $f(x) = (1-x)^{\frac{5}{2}}$ with Lagrange's form of reminder up to 2 terms in[0,1].
3. Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$.
4. If a<b, prove that $\frac{b-a}{(1+b^2)} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{(1+a^2)}$ using Lagrange's Mean value theorem. Deduce $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$
5. State Cauchy's Mean value theorem. Find **c** for $f(x) = e^x$ *and* $g(x) = e^{-x}$ in [a, b].

11. PART-A QUESTIONS

1. State Rolle's theorem

2. State Mean value theorem.

3. State Cauchy's Mean value theorem.

State Taylor's theorem with remainders

5 State Maclaurin theorem with remainders

12. PART-B QUSTIONS

1. State Rolle's Theorem and verify it for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in [a, b] , a>0,b>0

2. Verify Taylor's theorem for $f(x) = (1-x)^{\frac{5}{2}}$ with Lagrange's form of reminder up to 2 terms in[0,1].

3. Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$.

4. If a<b, prove that $\frac{b-a}{(1+b^2)} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{(1+a^2)}$ using Lagrange's Mean value theorem. Deduce

$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

5 State Cauchy's Mean value theorem. Find **c** for $f(x) = e^x$ *and* $g(x) = e^{-x}$ in [a, b].

13. Supportive Online Certification Courses:

- Basic Calculus (NPTEL)
- Engineering Mathematics-I (NPTEL)

14. Real Time Applications:

- Application of Mean value theorems are used mainly to find the behavior of various functions. And their expansions

15. Content beyond the Syllabus:

- Find the Taylor series expansions of functions of two variables. .

16. Prescribed Text Books and Reference Books:

Text Books

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.

2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

Reference Books:

1. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.

2. George B. Thomas, Maurice D. Weir and Joel Hass, Thomas Calculus, 13/e, Pearson Publishers, 2013.

17. Mini Project Suggestion:

COURSE MATERIAL	
SUBJECT	LINEAR ALGEBRA AND CALCULUS (20A54101)
UNIT	3
COURSE	B.TECH
DEPARTMENT	CSE
SEMESTER	I-I
PREPARED BY (Faculty Name/s)	Dr. P. SREENIVASULU REDDY Professor of Mathematics DEPARTMENT OF S & H
Version	V-1
PREPARED / REVISED DATE	24-02-2021

1. **Learning Outcomes:**

The objective of this unit

- Find partial derivatives numerically and symbolically and use them to analyze and interpret the way a function varies. (L3)
- Acquire the Knowledge maxima and minima of functions of several variable (L1)
- Utilize Jacobian of a coordinate transformation to deal with the problems in change of variables

1. Prerequisites

   Students should have knowledge on
   1. Differentiation
   2. Basic Mathematics

2. Syllabus

Multivariable Calculus

- ✓ Partial derivatives,
- ✓ total derivatives,
- ✓ chain rule,
- ✓ change of variables,
- ✓ Jacobians, maxima and minima of functions of two variables,
- ✓ method of Lagrange multipliers.

Multivariable Calculus

We have already studied the notion of limit, continuity and differentiation in relation with functions of a single variable. In this chapter we introduce the notion of a function of several variables, (i.e) function of two or more variables.

In day -to-day life we deal with things which depend on two or more quantities. For example, the area of a room which is a rectangle consists of two variables; length(say a) and breadth (say b) is given by A = ab. Hence A has a definite value for a pair of values a and b, Similarly , the volume of rectangular parallelepiped consists of three variables: length a ,breadth b and height h is given by V = abh.

Partial Differentiation:

A Partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant

PROBLEMS:

1. If $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right]$, *prove that* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol: Given $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right]$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{1 + \frac{4x^2y^2}{(x^2 - y^2)^2}} \cdot \frac{\partial}{\partial x}\left[\frac{2xy}{x^2 - y^2}\right]$$

2. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ *for the function* $u = \tan^{-1}\left[\frac{x}{y}\right]$

Sol: Let $u = \tan^{-1}\left[\frac{x}{y}\right]$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + (\frac{x}{y})^2} \times \frac{1}{y} = \frac{y}{y^2 + x^2}$$

(Here y is a constant)

And $$\frac{\partial^2 u}{\partial y \partial x} = \frac{(y^2 + x^2)1 - y(2y)}{(y^2 + x^2)^2} = \frac{x^2 - y^2}{(y^2 + x^2)^2} \rightarrow (1)$$

Now $$\frac{\partial u}{\partial y} = \frac{1}{1 + (\frac{x}{y})^2} \times (\frac{-x}{y^2}) = \frac{-x}{y^2 + x^2}$$

(Here x is a constant)

$$\frac{\partial^2 u}{\partial x \partial y} = -[\frac{(y^2 + x^2)1 - x(2x)}{(y^2 + x^2)^2}] = -\frac{y^2 - x^2}{(y^2 + x^2)^2} = \frac{x^2 - y^2}{(y^2 + x^2)^2} \rightarrow (2)$$

From (1) & (2), we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

90

3. If $z = f(x+ay)+\phi(x-ay)$, Prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol: Given $z = f(x+ay)+\phi(x-ay)$

$$\therefore \frac{\partial z}{\partial x} = f^1(x+ay)+\phi^1(x-ay),$$

$$\frac{\partial^2 z}{\partial x^2} = f^{11}(x+ay)+\phi^{11}(x-ay) \to (1)$$

$$\frac{\partial z}{\partial y} = f^1(x+ay).a+\phi^1(x-ay)(-a)$$

$$= a(f^1(x+ay)-\phi^1(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a(f^{11}(x+ay).a+\phi^{11}(x-ay)a)$$

$$= a^2(f^{11}(x+ay)+\phi^{11}(x-ay))$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2} (by (1))$$

4 .If $r^2 = x^2+y^2+z^2$ and u = r^m then Prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$

Sol: We have $r^2 = x^2+y^2+z^2 \to (1)$

Differentiating partially w.r.t. x, we get

$2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r} \to (2)$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Also we have u = r^m

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} = m.r^{m-1}.\frac{x}{r} = mr^{m-2}.x \quad \text{(Using (2))}$$

And $\frac{\partial^2 u}{\partial x^2} = m[x.(m-2)r^{m-3}.\frac{\partial r}{\partial x} + r^{m-2}.1]$

$= mr^{m-2}[(m-2).\frac{x^2}{r^2}+1]$ (using (2))

$= \frac{mr^{m-2}}{r^2}[(m-2).x^2+r^2]$

Hence $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{mr^{m-2}}{r^2}[(m-2)\sum x^2+\sum r^2]$

$= \frac{mr^{m-2}}{r^2}[(m-2)r^2+3r^2]$

= mrm-2[(m-2) +3]

= m (m+1) rm-2

5. If w = (y-z) (z-x) (x-y) find the value of $\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}$

Sol: Given w = (y-z) (z-x) (x-y)

$\therefore \frac{\partial w}{\partial x}$ =(y -z) $\frac{\partial}{\partial x}$ (z-x) (x-y)] [Since y, z are constants]

= (y-z) [(z-x). 1 + (x-y). (-1)]

= (y-z) [z-x - x + y]

= (y-z) (y+z-2x)

= (y-z) (y+z)-2x(y-z)

= y2-z2-2x(y-z) $\rightarrow (1)$

By symmetry, we have

$\frac{\partial w}{\partial y}$ = z2-x2-2y(z-x) $\rightarrow (2)$

And $\frac{\partial w}{\partial z}$ = x2-y2-2z(x-y) $\rightarrow (3)$

(1) +(2) + (3) gives

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z} = (y2\text{-}z2)\text{-}2x(y\text{-}z) + (z2\text{-}x2)\text{-}2y(z\text{-}x) + (x2\text{-}y2)\text{-}2z(x\text{-}y)$$

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Hence the result.

6. If U = $\frac{1}{\sqrt{x^2+y^2+z^2}}, x^2+y^2+z^2 \neq 0$ then Prove that $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0$

Sol: Given U = $(x^2+y^2+z^2)^{\frac{-1}{2}}$

$$\frac{\partial U}{\partial x}=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}\times\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$=\frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}}\times(2x)$$ Since y, z are constants

$$=-(x^2+y^2+z^2)^{\frac{-3}{2}}\times(x)$$

$$\frac{\partial^2 U}{\partial x^2} = -[(x^2+y^2+z^2)^{\frac{-3}{2}}+x(\frac{-3}{2})(x^2+y^2+z^2)^{\frac{-5}{2}}.2x]$$

$$=-(x^2+y^2+z^2)^{\frac{-5}{2}}[x^2+y^2+z^2-3x^2]$$

$$=(x^2+y^2+z^2)^{\frac{-5}{2}}[2x^2-y^2-z^2]\rightarrow(1)$$

Similarly, we get

$$\frac{\partial^2 U}{\partial y^2} = (x^2+y^2+z^2)^{\frac{-5}{2}}[-x^2+2y^2-z^2]\rightarrow(2)$$

$$\frac{\partial^2 U}{\partial z^2} = (x^2+y^2+z^2)^{\frac{-5}{2}}[-x^2-y^2+2z^2]\rightarrow(3)$$

(1)+ (2) + (3) gives

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0$$

93

7. If $U = \log(x^3 + y^3 + z^3 - 3xyz)$ then show that $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 U = \frac{-9}{(x+y+z)^2}$

Sol. Given that $U = \log(x^3 + y^3 + z^3 - 3xyz)$

Differentiate U w.r.t. 'x' partially, we get

$$\frac{\partial U}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

Similarly

$$\frac{\partial U}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

And $\frac{\partial U}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3}{(x+y+z)}$$

Now $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 U = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})U$

$$= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z})$$

$$= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})\frac{3}{(x+y+z)}$$

$$= 3(\frac{\partial}{\partial x}\frac{1}{(x+y+z)} + \frac{\partial}{\partial y}\frac{1}{(x+y+z)} + \frac{\partial}{\partial z}\frac{1}{(x+y+z)})$$

$$= 3(\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2})$$

$$= \frac{-9}{(x+y+z)^2}$$

$$\therefore (\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 U = \frac{-9}{(x+y+z)^2}$$

HOMOGENEOUS FUNCTION:

Definitions:

1. A function f(x,y) is said to be homogeneous function of degree n if the degree of each term in f(x,y) is n , where n is a real number.

For example: If f(x,y) $= a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ... + a_{n-1}xy^{n-1} + a_ny^n \rightarrow (1)$

Then f(x,y) is a homogeneous function of degree n ,since the degree of each term is n.

The above definition of homogeneity applies to polynomial functions only.

2. (i) A function f(x,y) is said to be homogeneous function of degree or order n in variables

94

x,y if f(kx,ky) = kn f(x,y), n is a real number.

(ii) A function f(x,y) is said to be a homogeneous function of degree or order n in variables

x,y,z if f(kx,ky,kz) = kn f(x,y,z) ,n isa real number .

Further a homogeneous function of order n in x&y can be expressed as xnf(y/x) or yn f(x/y).

Examples :

1. Let f(x, y) = x2+y2.

Now f (kx, ky) = k2x2+k2y2 = k2 (x2+y2) = k2 f(x, y)

∴ f(x, y) is a homogeneous function of order 2 .

2. Let f(x, y) = logy +2logx.

Now f(kx, ky) = log(ky) + 2log(kx) = 3logk +f(x,y) ≠ f(x,y)

∴ f(x,y) is not a homogeneous function.

3. Let f(x, y) = $\frac{3x}{y}+\log(\frac{y}{x})$

Now f (kx, ky) = $\frac{3kx}{ky}+\log(\frac{ky}{kx})=\frac{3x}{y}+\log(\frac{y}{x})=k^0 f(x,y)$

∴ f(x, y) is a homogeneous function of degree 0 .

4. Let f(x,y) = sin (xy) is not a homogeneous function since f(kx,ky) = sin (k2 xy) ≠ Kn f(x,y).

EULER'S THEOREM OR HOMOGENEOUS FUNCTIONS:

Statement: If z = f(x, y) is a homogeneous function of degree n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz, \forall x, y$$

in the domain of the function.

Corollary: If u is a homogeneous function of x, y & z of degree n ,then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

Examples:

1. If f(x, y) = $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$,then f(kx, ky) = $\frac{\sqrt{kx}-\sqrt{ky}}{\sqrt{kx}+\sqrt{ky}}=k^0 f(x,y)$

$\therefore$ f (x, y) is a homogeneous function of degree 0 .

By Euler's theorem $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0.f=0$

$\Rightarrow xf_x+yf_y=0$

2. If u = $(\frac{x}{z})^2+(\frac{y}{z})^2+\frac{x}{y}$, then u (kx, ky, kz) = $(\frac{kx}{kz})^2+(\frac{ky}{kz})^2+\frac{kx}{ky}$

= $k^0[(\frac{x}{z})^2+(\frac{y}{z})^2+\frac{x}{y}]$

= k0 u

$\therefore$ u is a homogeneous function of degree 0 in x , y , z.

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$ = 0. (u) = 0

Deductions from Euler's Theorem:

If z is a homogeneous function of x, y of degree n and z= f(u), then

i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f^1(u)}$

ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g^1(u)-1]$ where g (u) = $n\frac{f(u)}{f^1(u)}$

1. Find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ if $u=\frac{x^3y^3}{x^3+y^3}$

Sol: Given u (x,y) = $\frac{x^3y^3}{x^3+y^3}$

Now u (kx , ky) = $\frac{(k^3x^3)(k^3y^3)}{k^3x^3+k^3y^3}=\frac{k^6x^3y^3}{k^3(x^3+y^3)}=k^3u(x,y)$

$\therefore$ u is a homogeneous function of degree 3.

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$.

2. If u = f (x,y,z) = ($x^2+y^2+z^2)^{\frac{-1}{2}}$ Show that $\sum x\frac{\partial u}{\partial x}=-u$

Sol: f(kx, ky, kz) = (k $k^2x^2+k^2y^2+k^2z^2)^{\frac{-1}{2}}=k^{-1}(x^2+y^2+z^2)^{\frac{-1}{2}}=k^{-1}f(x,y,z)$

$\therefore$ u = f(x, y, z) is a homogeneous function in x, y, z of degree -1.

By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=(-1)u$.

$\therefore \sum x\frac{\partial u}{\partial x}=-u$

3. Verify Euler's theorem for z = ax2 + 2hxy + by2.

Sol: Given z = ax2 + 2hxy + by2 $\rightarrow (1)$

Z is a homogeneous function in x & y of degree 2, since the degree of each term is 2

∴ To Verify Euler's Theorem we have to show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$

Now $\frac{\partial z}{\partial x}=2ax+2hy \rightarrow (2)$

$\frac{\partial z}{\partial y}=2hx+2by \rightarrow (3)$

Multiplying (2) by x and (3) by y and adding we get

$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2ax^2+2hxy+2hxy+2by^2$

= 2 (ax2 + 2hxy + by2)

= 2z using (1)

Hence Euler's theorem is verified.

4. If $u=\log\frac{x^2+y^2}{x+y}$ Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$

Sol: Given $u=\log\frac{x^2+y^2}{x+y} \rightarrow (1)$

Since u is not a homogeneous function, we write (1) as

$$e^u=\frac{x^2+y^2}{x+y}$$

$$e^u=\frac{x^2(1+\frac{y^2}{x^2})}{x(1+\frac{y}{x})}$$

$= x\phi(\frac{y}{x})$

$= f(x, y)$ (say)

$\therefore$ f is a homogeneous function of x and y of degree 1.

By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1.f$

$$\Rightarrow x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = e^u$$

$$\Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

5. If $u = \tan^{-1}(\frac{x^3 + y^3}{x + y})$ Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Sol: Here u is not a homogeneous function

So we write $\tan u = \frac{x^3 + y^3}{x + y} = x^2[\frac{(1 + (\frac{y}{x})^3}{(1 + (\frac{y}{x})}]$

$= x^2\phi(\frac{y}{x})$

$= f(x, y)$ (say)

$\therefore$ f(x,y) is a homogeneous function of x and y of degree 2.

By Euler's theorem $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$

$$\Rightarrow x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 2\tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \sin 2u$$

6. If $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$ Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

98

Sol: $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$ = $\dfrac{\sin^{-1}(\frac{1}{\frac{y}{x}}) + \tan^{-1}(\frac{y}{x})}{}$ = $f(\frac{y}{x}) = x^0 f(\frac{y}{x})$

$\therefore$ u is a homogeneous function of x and y of degree 0.

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u = 0$.

Alternate Method:

Given $u = \sin^{-1}(\frac{x}{y}) + \tan^{-1}(\frac{y}{x})$

Now $\frac{\partial u}{\partial x}$ = $\frac{1}{\sqrt{1-(\frac{x}{y})^2}}(\frac{1}{y}) + \frac{1}{1+(\frac{y}{x})^2}(\frac{-y}{x^2})$

$\frac{\partial u}{\partial x}$ = $\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \rightarrow (1)$

Similarly $\frac{\partial u}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2} \rightarrow (2)$

Multiplying (1) by x and (2) by y and adding, we get

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$ = 0.

7. If $u = \sec^{-1}(\frac{x^3-y^3}{x+y})$ Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$ then evaluate

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Sol: We have $u = \sec^{-1}(\frac{x^3-y^3}{x+y}) \rightarrow (1)$

But u is not a homogeneous function. So we write (1) as

Z = secu = $\frac{x^3-y^3}{x+y}$

Now z is a homogeneous function of x & y of degree 2.

By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

$$x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}=2z$$

$$x\sec u\tan u\frac{\partial u}{\partial x}+y\sec u\tan u\frac{\partial u}{\partial y}=2\sec u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2}{\tan u}=2\cot u=g(u)$$

By Euler's theorem of second order, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g^1(u)-1]$$

= 2cotu [-2cosec2u - 1]

=-2 cotu [2 cosec2 u + 1]

Exact differential:

In thermo dynamics, we come across more number of variables such as temperature (T), Volume (V), Pressure (P), Work (W), etc. In this case any one f these can be expressed as a function of remaining two variables. For example: T = f (P, V), T = f (P,W).Then we have the following results.

$$dT=\frac{\partial T}{\partial p}dp+\frac{\partial T}{\partial v}dv \rightarrow (1) \quad dT=\frac{\partial T}{\partial p}dp+\frac{\partial T}{\partial w}dw \rightarrow (2)$$

In (1) ,the value of $\frac{\partial T}{\partial p}$ where T is a function of P& V is obtained treating V as a constant and we can write $(\frac{\partial T}{\partial p})_v$ .In the same way we can have $(\frac{\partial T}{\partial p})_w$ in (2) . This dT is Exact Differential of T. In general, if $\phi=\phi(x,y)$, $d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy$ is called Exact Differential of ϕ.

The Chain Rule of Partial Differentiation:

1. If z = f(x, y) where x = ϕ (t), y = $\psi(t)$ then z is called composite function of a variable t.

2. If z = f(x,y) where x = ϕ (u,v) ,y = $\psi(u,v)$ then z is called a composite function of two variables u &v

Theorem: Let z = f (u, v) where u = ϕ (x, y) and v = g(x, y) .Then

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

These are referred to as the chain rule of partial differentiation .The above rule can be extended to functions of more than two independent variables.

Total Differential Coefficient:

Let z = f(x, y) where x = ϕ (t) and y = g(t)

Substituting x and y in z = f(x, y), z becomes a function of a single variable t.

Then $\frac{dz}{dt}$ is called the total differential coefficient or total derivative of z.

$$\therefore \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

1. If $\boldsymbol{u=u(y-z,z-x,x-y)}$ then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Sol: Given $u=u(y-z,z-x,x-y)$

Let $X=y-z, Y=z-x, Z=x-y$ then $u=u(X,Y,Z)$ where X, Y, Z are functions of x, y, z.

Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$$

$$=\frac{\partial u}{\partial X}.0+\frac{\partial u}{\partial Y}.(-1)+\frac{\partial u}{\partial Z}.1$$

$$=-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z} \quad \text{---------------- (i)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}$$

$$=\frac{\partial u}{\partial X}.1+\frac{\partial u}{\partial Y}.0+\frac{\partial u}{\partial Z}.(-1)$$

$$=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \quad \text{---------------- (ii)}$$

And $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}$

$$=\frac{\partial u}{\partial X}.(-1)+\frac{\partial u}{\partial Y}.1+\frac{\partial u}{\partial Z}.0$$

$$= -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \quad \text{---------------- (iii)}$$

By adding (i), (ii) and (iii) we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

2. If u = x logxy where x3+y3+3xy = 1 .Find $\frac{du}{dx}$

Sol: We have u = x logxy = x(logx+logy) = xlogx+xlogy

$$\therefore \frac{\partial u}{\partial x} = x.\frac{1}{x} + \log x(1) + \log y = 1 + \log x + \log y$$

$$= 1+\log xy \quad \rightarrow (1)$$

And $\frac{\partial u}{\partial y} = x.\frac{1}{y} = \frac{x}{y} \rightarrow (2)$

Also given $x^3 + y^3 + 3xy = 1$

Differentiating , $3x^2 + 3y^2\frac{dy}{dx} + 3(x\frac{dy}{dx} + y) = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x^2+y)}{(y^2+x)} \rightarrow (3)$$

Now $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$

$$= 1 + \log xy + \frac{x}{y}\left[\frac{-(x^2+y)}{y^2+x}\right] \quad \text{from (1), (2), (3)}$$

$$= 1 + \log xy - \frac{x(x^2+y)}{y(y^2+x)}$$

JACOBIAN:

Let u=u(x,y), v=v(x,y) then these two simultaneous relations constitute a transformation from (x,y) to (u,v)

The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} or \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is called the Jacobian of the transformation and the determinant value is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

102

If u=u(x,y,z), v=v(x,y,z), w=w(x,y,z) then the jacobian of u, v, w, w.r.to x, y, z is given by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \ or \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

PROPERTIES OF JACOBIAN:

1. If J = $\frac{\partial(u,v)}{\partial(x,y)}$ and JI = $\frac{\partial(x,y)}{\partial(u,v)}$ then JJI = 1

2. If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

3. If u & v are functions of the independent variables x & y and the Jacobin

$J\left(\frac{u,v}{x,y}\right)=0$ *then* u & v *are said to be functionally dependent otherwise they are called functionally independent.*

PROBLEMS:

1.

$$If\ x=r\cos\theta,\ y=r\sin\theta,\ find\ \frac{\partial(x,y)}{\partial(r,\theta)}\ and\ \frac{\partial(r,\theta)}{\partial(x,y)}.\ Also\ show that\ \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)}=1$$

Sol: Given that

$$x = r\cos\theta \qquad ....(1)$$

$$y = r\sin\theta \qquad ....(2)$$

Now differentiate eqn (1) & (2) w.r.to r and θ partially we get

$$\frac{\partial x}{\partial r}=\cos\theta\ ; \frac{\partial x}{\partial \theta}=-r\sin\theta\ ; \frac{\partial y}{\partial r}=\sin\theta\ ; \frac{\partial y}{\partial \theta}=r\cos\theta$$

$$\therefore J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}=\begin{vmatrix} \cos\theta & -r\sin\theta \\ -\sin\theta & r\cos\theta \end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r$$

Now we have to find $\frac{\partial(r,\theta)}{\partial(x,y)}$

From eqn (1) & (2)

$$x^2 = r^2 \cos^2 \theta$$

$$y^2 = r^2 \sin^2 \theta$$

$$x^2 + y^2 = r^2 \qquad \text{......(3)}$$

From these $\dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \qquad \text{......(4)}$

Now differentiate eqn (3) and (4) w.r.to x & y partially we get

$$\frac{\partial r}{\partial x} = \frac{x}{r};\frac{\partial r}{\partial y} = \frac{y}{r};\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2};\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

$$\therefore J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{r(x^2+y^2)} + \frac{y^2}{r(x^2+y^2)} = \frac{1}{r}$$

$$\therefore JJ' = \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r.\frac{1}{r} = 1$$

2. *If* $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, *show that* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ *and find* $\dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

Sol: *Given*

$$x = r\sin\theta\cos\phi \qquad \text{........(1)}$$

$$y = r\sin\theta\sin\phi \qquad \text{........(2)}$$

$$z = r\cos\theta \qquad \text{........(3)}$$

Now differentiate eqn (1), (2) and (3) w.r.to r, θ,ϕ partially we get

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi;\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi;\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi;\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi;\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta;\frac{\partial z}{\partial \theta} = -r\sin\theta;\frac{\partial z}{\partial \phi} = 0$$

104

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$=\sin\theta\cos\phi\,(0+r^2\sin^2\theta\cos\phi)$$
$$-r\cos\theta\cos\phi(-r\sin\theta\cos\theta\cos\phi)$$
$$-r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi-r\cos^2\theta\sin\phi)$$
$$=r^2\sin^3\theta\cos^2\phi+r^2\sin\theta\cos^2\theta\cos^2\phi+r^2\sin^3\theta\sin^2\phi$$
$$+r^2\sin\theta\cos^2\theta\sin^2\phi$$
$$=r^2\sin^3\theta+r^2\sin\theta\cos^2\theta$$

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=r^2\sin\theta$$

$$we\ know\ that\ JJ'=1\quad\Rightarrow\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\cdot\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}=1\Rightarrow\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}=\frac{1}{r^2\sin\theta}$$

3. If $x+y+z=u$, $y+z=uv$, $z=uvw$, then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: Given $u=x+y+z\rightarrow(1)$

$uv=y+z\rightarrow(2)$

$uvw=z\rightarrow(3)$

From eq (2), $uv=y+z\Rightarrow y=uv-z$

$\Rightarrow y=uv-uvw$ (from (3))

From eq (1), $u=x+y+z$

$\Rightarrow x=u-(y+z)$

$\Rightarrow x=u-uv$ (using (2))

$$\therefore\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}(u-uv)=1-v\ ,\quad \frac{\partial x}{\partial v}=-u\ ,\quad \frac{\partial x}{\partial w}=0$$

$$\frac{\partial y}{\partial u}=v-vw\ ,\quad \frac{\partial y}{\partial v}=u-uw\ ,\quad \frac{\partial y}{\partial w}=-uv$$

$$\frac{\partial z}{\partial u}=vw\ ,\quad \frac{\partial z}{\partial v}=uw\ ,\quad \frac{\partial z}{\partial w}=uv$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

= (1-v)[(u-uw) (uv)+uv(uw)]+u [(v-vw(uv)+(uv)(vw)]

$\frac{\partial(x,y,z)}{\partial(u,v,w)}$ = u2v.

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta, y = r\sin\theta$, show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Sol: Given $u = x^2 - y^2$, $v = 2xy$

Since $x = r\cos\theta, y = r\sin\theta$, we have

$$u = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta \to (1)$$

Ansd $v = 2(r\cos\theta)(r\sin\theta) = r^2\sin 2\theta \to (2)$

Differentiate (1) & (2) partially w.r.t r & θ, we have

$\therefore \frac{\partial u}{\partial r} = 2r\cos 2\theta$, $\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta$

& $\frac{\partial v}{\partial r} = 2r\sin 2\theta$, $\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix} = 4r^3(\cos^2 2\theta + \sin^2 2\theta) = 4r^3$$

5. Show that the functions u = xy + yz + zx , v = x2 +y2 +z2 and w = x + y+ z are Functionally related. Find the relation between them.

Sol: Given u = xy + yz + zx , v = x2 +y2 +z2 and w = x + y+ z ....(1)

Now differentiate eqn (1) w.r.to x, y & z partially we get

$$\frac{\partial u}{\partial x} = y + z \,; \frac{\partial u}{\partial y} = z + x \,; \frac{\partial u}{\partial z} = x + y$$

$$\frac{\partial v}{\partial x} = 2x \,; \frac{\partial v}{\partial y} = 2y \,; \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial w}{\partial x} = 1 \,; \frac{\partial w}{\partial y} = 1 \,; \frac{\partial w}{\partial z} = 1$$

$$\therefore J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (y+z)(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$$

$$= 2(y+z)(y-z) - 2(x+z)(x-z) + 2(x+y)(x-y)$$

$$= 2y^2 - 2z^2 - 2x^2 + 2z^2 + 2x^2 - 2y^2$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Hence u, v and w are functionally dependent. That is the functional relationship exists between u, v and w

Now w2 = (x+ y+ z)2 = x2 + y2 + z2 + 2(xy + yz + zx)

W2= v + 2u is the functional relation between u, v & w

6. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1} x + \tan^{-1} y$ .Find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence Prove that u & v are Functionally dependent. Find the functional relation between them.

Sol: Given $u = \frac{x+y}{1-xy}$, $v = \tan^{-1} x + \tan^{-1} y$

$$\therefore \frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}, \quad \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

107

Now $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$

$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$

Hence u & v are functionally dependent.

Now $v = \tan^{-1} x + \tan^{-1} y$

$= \tan^{-1}(\frac{x+y}{1-xy})$

$v = \tan^{-1} u$

$\therefore v = \tan^{-1} u$ is the functional relation between them.

** Maximum & Minimum for function of a single Variable:

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find f1(x) and equate it to zero

(ii) Solve the above equation we get x0,x1 as roots.

(iii) Then find f11(x).

If f11(x)(x = x0)> 0, then f(x) is minimum at x0

If f11(x)(x = x0) , < 0, f(x) is maximum at x0 . Similarly we do this for other stationary points.

1. Find the max & min of the function f(x) = x5 -3x4 + 5 ('08 S-1)

Sol: Given f(x) = x5 -3x4 + 5

f1(x) = 5x4 – 12x3

for maxima or minima f1(x) =0

5x4 – 12x3 = 0

x =0, x= 12/5

f11(x) = 20 x3 – 36 x2

At x = 0 => f11(x) = 0. So f is neither maximum nor minimum at x = 0

At x = (12/5) => f11(x) =20 (12/5)3 – 36(12/5)

=144(48-36) /25 =1728/25 > 0

So f(x) is minimum at x = 12/5

The minimum value is f (12/5) = (12/5)5 -3(12/5)4 + 5

108

**Maxima & Minima for functions of two Variables:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a1, b1) (a2,b2) (a3 ,b3) ....

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x \partial y}$, n = $\frac{\partial^2 f}{\partial y^2}$

3. i. If l n –m2> 0 and l < 0 at (a1,b1) then f(x ,y) is maximum at (a1,b1) and maximum value is f(a1,b1)

   ii. If l n –m2> 0 and l > 0 at (a1,b1) then f(x ,y) is minimum at (a1,b1) and minimum value is f(a1,b1) .

   iii. If l n –m2< 0 and at (a1, b1) then f(x, y) is neither maximum nor minimum at (a1, b1). In this case (a1, b1) is saddle point.

   iv. If l n –m2 = 0 and at (a1, b1), no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

*Extremum: A function which has a maximum or minimum or both is called 'Extremum'

*Extreme value- The maximum value or minimum value or both of a function is Extreme value.

*Stationary points: - To get stationary points we solve the equations $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0 i.e the pairs (a1, b1), (a2, b2) .... are called Stationary.

PROBLEMS:

1. Locate the stationary points & examine their nature of the following function.

u =x4 + y4 -2x2 +4xy -2y2, (x > 0, y > 0)

Sol: Given u(x, y) = x4 + y4 -2x2 +4xy -2y2

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x}$ = 4x3 -4x + 4y = 0 ⇒x3 – x + y = 0 -------------------> (1)

$\frac{\partial u}{\partial y}$ = 4y3 +4x - 4y = 0 ⇒ y3 + x – y = 0 -------------------> (2)

Adding (1) & (2),

x3 + y3 = 0

109

$\Rightarrow (x+y)(x^2-xy+y^2)=0$

$\Rightarrow (x+y)=0$ or $(x^2-xy+y^2)=0$

$\Rightarrow$ x = – y ----------------------> (3)

(1) $\Rightarrow$ x3 – 2x $\Rightarrow$ x = $0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow$ y = 0, $-\sqrt{2}, \sqrt{2}$

l = $\frac{\partial^2 u}{\partial x^2}$ = 12x2 – 4, m = $\frac{\partial^2 u}{\partial x \partial y}$ = $\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})$ = 4 & n = $\frac{\partial^2 u}{\partial y^2}$ = 12y2 – 4

ln – m2 = (12x2 – 4)(12y2 – 4) -16

At ($-\sqrt{2}$, $\sqrt{2}$), ln – m2 = (24 – 4) (24 -4) -16 = (20) (20) – 16 > 0 and l=20>0

The function has minimum value at ($-\sqrt{2}$, $\sqrt{2}$)

At (0, 0), ln – m2 = (0– 4)(0 -4) -16 = 0

(0, 0) is not a extreme value.

2. Investigate the maxima & minima, if any, of the function f(x) = x3y2 (1-x-y).

Sol: Given f(x) = x3y2 (1-x-y) = x3y2- x4y2 – x3y3

$\frac{\partial f}{\partial x}$ = 3x2y2 – 4x3y2 -3x2y3

$\frac{\partial f}{\partial y}$ = 2x3y – 2x4y -3x3y2

For maxima & minima $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0

⇨ 3x2y2 – 4x3y2 -3x2y3 = 0 => x2y2(3 – 4x -3y) = 0 ----------------> (1)

⇨ 2x3y – 2x4y -3x3y2 = 0 => x3y(2 – 2x -3y) = 0 ----------------> (2)

From (1) & (2) 4x + 3y – 3 = 0

2x + 3y - 2 = 0

2x = 1 => x = ½

4 (½) + 3y – 3 = 0 => 3y = 3 -2, y = (1/3)

l = $\frac{\partial^2 f}{\partial x^2}$ = 6xy2-12x2y2 -6xy3

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,\ 1/3)}$ = 6(1/2)(1/3)2 -12 (1/2)2(1/3)2 -6(1/2)(1/3)3 = 1/3 – 1/3 - 1/9 = -1/9

110

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ =6x2y -8 x3y – 9x2y2

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (1/2 , 1/3) = 6(1/2)2(1/3) -8 (1/2)3(1/3) -9(1/2)2(1/3)3= $\frac{6-4-3}{12}$ = $\frac{-1}{12}$

$n = \frac{\partial^2 f}{\partial y^2}$ = 2x3 -2x4 -6x3y

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (1/2, 1/3) = 2(1/2)3 -2(1/2)4 -6(1/2)3(1/3) = $\frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

ln- m2 =(-1/9)(-1/8) –(-1/12)2 = $\frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144}$> 0 and l = $\frac{-1}{9} < 0$

The function has a maximum value at (1/2, 1/3)

∴ Maximum value is $f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{8}\times\frac{1}{9}\right)\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}$

3. Find three positive numbers whose sum is 100 and whose product is maximum.

Sol: Let x, y, z be three +ve numbers.

Then x + y + z = 100

⇨ z = 100 – x – y

Let f(x,y) = xyz =xy(100 – x – y) =100xy –x2y-xy2

For maxima or minima $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0

$\frac{\partial f}{\partial x}$ =100y –2xy-y2 = 0 => y(100- 2x –y) = 0 ----------------> (1)

$\frac{\partial f}{\partial y}$ = 100x –x2 -2xy = 0 => x(100 –x -2y) = 0 ------------------> (2)

From (1) & (2)

100 -2x –y = 0

200 -2x -4y =0

-100 + 3y = 0 => 3y =100 => y =100/3

100 – x –(200/3) = 0 => x = 100/3

$l = \frac{\partial^2 f}{\partial x^2}$ =- 2y

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(100/3,\ 100/3)} = -200/3$

m = $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 100 - 2x - 2y$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(100/3,\ 100/3)} = 100 - (200/3) - (200/3) = -(100/3)$

n = $\frac{\partial^2 f}{\partial y^2} = -2x$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(100/3,\ 100/3)} = -200/3$

$ln - m^2 = (-200/3)(-200/3) - (-100/3)^2 = (100)^2/3$

The function has a maximum value at (100/3, 100/3)

i.e. at x = 100/3, y = 100/3 $\quad z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1-x^2) = 0 \Rightarrow x = 0,\ x = \pm 1$

$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0 \Rightarrow -4y(1-y^2) = 0 \Rightarrow y = 0,\ y = \pm 1$

l = $\left(\frac{\partial^2 f}{\partial x^2}\right) = 4 - 12x^2$

m = $\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

n = $\left(\frac{\partial^2 f}{\partial y^2}\right) = -4 + 12y^2$

we have $ln - m^2 = (4 - 12x^2)(-4 + 12y^2) - 0$

$= -16 + 48x^2 + 48y^2 - 144x^2y^2$

$= 48x^2 + 48y^2 - 144x^2y^2 - 16$

i) At $(0, \pm 1)$

$ln - m^2 = 0 + 48 - 0 - 16 = 32 > 0$

112

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

f (x ,y) = 2(x2 –y2) –x4 +y4

f (0 , ± 1) = 0 – 2 – 0 + 1 = -1

The minimum value is '-1 '.

ii) At (± 1 ,0)

ln – m2 = 48 + 0 - 0 -16 =32 > 0

l = 4-12 = - 8 < 0

f has maximum value at (± 1 ,0)

f (x ,y) = 2(x2 –y2) –x4 +y4

f (± 1 , 0) =2 -0 -1 + 0 = 1

The maximum value is '1 '.

iii) At (0,0) , (± 1 , ± 1)

ln – m2< 0

l = 4 -12x2

(0 , 0) & (± 1 , ± 1) are saddle points.

f has no max & min values at (0 , 0) , (± 1 , ± 1).

5. A rectangular box open at the top is to have volume of 32 cubic feets. Find the

Dimensions of the box requiring least material for its construction

Sol: let 'x' feet 'y' feet, 'z' feet are dimensions of the box. The surface of the box is

s=2xy+2yz+zx................................ (1)

The volume of the box is xyz=32.......................... (2)

From (2), z= $\frac{32}{xy}$

'z' value is substitute in eq(1)

S= $2xy + 2y(\frac{32}{xy}) + (\frac{32}{xy})x$

S= $2xy + \frac{64}{x} + \frac{32}{y}$ .... (3)

Now differentiate equation (3) w.r.to x and y we get

$$\frac{\partial s}{\partial x}=2y-\frac{64}{x2}$$
$$\frac{\partial s}{\partial y}=2x-\frac{32}{y2}$$

for maximum and minimum ,we are taking

$$\frac{\partial s}{\partial x}=0 \qquad \frac{\partial s}{\partial y}=0$$

$$2y-\frac{64}{x^2}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$2x-\frac{32}{y^2}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$from(4)\Rightarrow 2y=\frac{64}{x^2} \Rightarrow y=\frac{32}{x^2}$$

$$from\ (5)\Rightarrow 2x=\frac{32}{y^2}\Rightarrow 2xy^2=32 \Rightarrow 2x(\frac{32}{x^2})^2=32\Rightarrow 2x.\frac{32.32}{x^4}=32$$

$$2.\frac{32}{x^3}=1\Rightarrow x^3=64\Rightarrow x=4$$

substitute x in eqn (5)

$$(5)\Rightarrow 2x=\frac{32}{y^2}\Rightarrow x=\frac{16}{y^2}\Rightarrow 4=\frac{16}{y^2}\Rightarrow y^2=\frac{16}{4}\Rightarrow y=2$$

substitute x& y in z we get

$$z=\frac{32}{xy}\Rightarrow z=\frac{32}{4.2}\Rightarrow z=\frac{32}{8}\Rightarrow z=4$$

x=4 feet, y=2 feet, z=4feet are the dimensions of the box required for its construction.

6. Find the extreme value of f(x,y)=sinx+siny+sin(x+y)

Sol: Given that f(x,y)=sinx+siny+sin(x+y)

$$\frac{\partial f}{\partial x}=\cos x+\cos(x+y)$$
$$\frac{\partial f}{\partial y}=\cos y+\cos(x+y)$$
$$\frac{\partial f}{\partial x}=0 \qquad \frac{\partial f}{\partial x}=0$$

$$\cos x+\cos(x+y)=0 \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

cosy+cos(x+y)=0 ....(2)

cosx=cosy $\Rightarrow$ x=y

Substitute x=y in eqn (1), we get

cosx+cos2x=0

$$2\cos\frac{3x}{2}\cos\frac{x}{2}=0 \quad \Rightarrow\cos\frac{3x}{2}=0 \qquad (or) \qquad \cos\frac{x}{2}=0$$

$$\Rightarrow\frac{3x}{2}=\cos^{-1}(0) \qquad \frac{x}{2}=\cos^{-1}(0)$$

$$\Rightarrow\frac{3x}{2}=\pm\frac{\pi}{2} \qquad \frac{x}{2}=\pm\frac{\pi}{2}$$

$$\Rightarrow 3x=\pm\pi \qquad x=\pm\pi$$

$$\Rightarrow x=\pm\frac{\pi}{3} \qquad x=-\pi,+\pi$$

$\therefore$ *the stationary points are*

$$(-\pi,-\pi)(\pi,\pi)(-\frac{\pi}{3},-\frac{\pi}{3}),(\frac{\pi}{3},\frac{\pi}{3})$$

Now

$$r=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial x}[\cos x+\cos(x+y)]=-\sin x-\sin(x+y)$$

$$s=\frac{\partial^2 f}{\partial x\partial f}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial x}[\cos y+\cos(x+y)]=-\sin(x+y)$$

$$t=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial y}(\cos y+\cos(x+y)=-\sin y-\sin(x+y)$$

Case1: At (-π,-π)

r=0 ; s=0 ; t=0

rt-s2=0

No conclusion

case2: At (π,π)

r=0, s=0,t=0

rt-s2=0

No conclusion

115

$case3: At\ (-\frac{\pi}{3},-\frac{\pi}{3})$

$r=+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\sqrt{3}\ ,s=\frac{\sqrt{3}}{2},t=\sqrt{3}$

$rt-s^2=\sqrt{3}.\sqrt{3}-(\frac{\sqrt{3}}{2})^2=3-\frac{3}{4}$

$rt-s^2=\frac{9}{4}>0\ \ and\ \ r=\sqrt{3}>0$

$(-\frac{\pi}{3},-\frac{\pi}{3})$ *is a point minimum*

The minimum value is $=f(-\frac{\pi}{3},-\frac{\pi}{3})=-\frac{3\sqrt{3}}{2}$

$case4: At(\frac{\pi}{3},\frac{\pi}{3})$

$r=-\sqrt{3},s=-\frac{\sqrt{3}}{2},t=-\sqrt{3}$

$rt-s^2=\frac{9}{4}>0\ and\ \ r=-\sqrt{3}<0$

$(\frac{\pi}{3},\frac{\pi}{3})$ *is a point of maximum*

$\therefore$ *The maximum value is* $=f(\frac{\pi}{3},\frac{\pi}{3})=\sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin(\frac{\pi}{3}+\frac{\pi}{3})$

$$f(\frac{\pi}{3},\frac{\pi}{3})=\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$$

LEGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Working rule:

Suppose it is required to find the extreme for the function f(x,y,z) subject to the condition $\phi(x,y,z)=0.$ ....(1)

STEP1: *From Lagrangean function* $F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$ *where* λ *is called the lagrange multiplier, which is determined by the following conditions.*

STEP2: *obtain the equations*

$$\frac{\partial F}{\partial x}=0 \quad i.e. \quad \frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \qquad \text{...............(2)}$$

$$\frac{\partial F}{\partial y}=0 \quad i.e. \quad \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0 \qquad \text{................(3)}$$

$$\frac{\partial F}{\partial z}=0 \quad i.e. \quad \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \qquad \text{..................(4)}$$

STEP3: *solve the equations* (1), (2), (3) *and* (4).

the values of x, y, z *so obtained will give the stationary point of* $f(x,y,z)$.

Problem:

1. Find the minimum value of x2 +y2 +z2, given x + y + z =3a

Sol: u = x2 +y2 +z2

Ø = x + y + z - 3a = 0

Using Lagrange's function

F(x , y , z) = u(x , y , z) + λ Ø(x , y , z)

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \lambda\frac{\partial Ø}{\partial x} = 2x + \lambda = 0 \text{ ------------ (1)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \lambda\frac{\partial Ø}{\partial y} = 2y + \lambda = 0 \text{ ------------ (2)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \lambda\frac{\partial Ø}{\partial z} = 2z + \lambda = 0 \text{ ------------ (3)}$$

From (1) , (2) & (3)

λ= -2x = -2y = -2z

$x = y = z$

⇨ Ø = x + x + x - 3a = 0

⇨ X=a

x= y =z = a

Minimum value of u = a2 + a2 + a2 =3 a2

117

2. Find the maximum value of $x^m y^n z^p$ given that $x+y+z=a$.

Sol: *Given that* $f(x,y,z)=x^m y^n z^p$

given condition $x+y+z=a \Rightarrow x+y+z-a=0$

$\therefore \phi(x,y,z)=x+y+z-a$ ....(1)

By lagrange's method

$f(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$

$F=x^m y^n z^p+\lambda(x+y+z-a)$ ....(2)

Differentiate equation (1) *w.r.to* x,y,z *partially, we get*

$$\frac{\partial F}{\partial x}=y^n z^p m x^{m-1}+\lambda(1)$$
$$\frac{\partial F}{\partial y}=x^m z^p n y^{n-1}+\lambda(1)$$
$$\frac{\partial F}{\partial z}=x^m y^n p z^{p-1}+\lambda(1)$$

these three are equating to zero, we get

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$$

$y^n z^p m x^{m-1}+\lambda=0 \Rightarrow m y^n z^p x^{m-1}=-\lambda$ ....(3)

$x^m z^p n y^{n-1}+\lambda=0 \Rightarrow n x^m z^p y^{n-1}=-\lambda$ ....(4)

$x^m y^n p z^{p-1}+\lambda=0 \Rightarrow p x^m y^n z^{p-1}=-\lambda$ ....(5)

from (3) & (4)

$$m x^{m-1} y^n z^p = n x^m z^p y^{n-1} \Rightarrow m x^{m-1} y^n - n x^m y^{n-1}=0$$
$$\Rightarrow x^m y^n[m x^{-1}-n y^{-1}]=0$$
$$\Rightarrow \frac{m}{x}=\frac{n}{y} \Rightarrow x=\frac{my}{n} \Rightarrow y=\frac{nx}{m}$$

from (4) & (5)

$$n x^m z^p y^{n-1}=p x^m y^n z^{p-1} \Rightarrow \frac{n}{y}=\frac{p}{z} \Rightarrow y=\frac{nz}{p} \Rightarrow z=\frac{py}{n} \Rightarrow z=\frac{p.\frac{nx}{m}}{n}$$
$$\Rightarrow z=\frac{pnx}{mn} \Rightarrow z=\frac{px}{m}$$

118

$Now\ substitute\ y\ and\ z\ in\ (1), we\ get$

$$x+y+z=a \quad \Rightarrow x+\frac{nx}{m}+\frac{px}{m}=a \Rightarrow mx+nx+px=ma$$

$$\Rightarrow (m+n+p)x=ma \Rightarrow x=\frac{ma}{m+n+p}$$

$$x \Rightarrow y=\frac{nz}{p} \Rightarrow y=\frac{na}{m+n+p}$$

$$x \Rightarrow z=\frac{px}{m} \Rightarrow z=\frac{pa}{m+n+p}$$

$$\therefore \text{ stationary point is } \left(\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}\right)$$

$$The \quad \max imum\ value\ of \quad f(x,y,z)=x^m y^n z^p=\left(\frac{ma}{m+n+P}\right)^m\left(\frac{na}{m+n+P}\right)^n\left(\frac{pa}{m+n+p}\right)^p$$

$$f(x,y,z)=\frac{a^{m+n+p}.m^m.n^n.p^p}{(m+n+p)^{m+n+p}}$$

3. Find a point on the plane 3x+2y+z-12=0 which is nearest to the origin.

Sol: Let p(x,y) be a point on the plane which is nearest to the origin.

$op = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$op^2 = x^2 + y^2 + z^2$

$let\ op^2 = f(x,y,z)$

$\therefore f(x,y,z) = x^2 + y^2 + z^2$ ....(1)

$given\ that\ \ 3x + 2y + z + 2 = 0$

$\phi(x,y,z) = 3x + 2y + z - 12$ ....(2)

$By\ Lagrange's\ method$

$F(x,y,z) = f(x,y,z) + \lambda(3x + 2y + z - 12)$ ....(3)

$differentiate\ equation(3)\ w.r.to\ x,y,z\ partially\ and\ equating\ to\ zero$

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$$

$2x + 3\lambda = 0$ ....(4)

$2y + 2\lambda = 0$ ....(5)

$2z + \lambda = 0$ ....(6)

$from\ (4)\ x = \frac{-3\lambda}{2}$; $from\ (5)\ y = -\lambda$; $from\ (6)\ z = \frac{-\lambda}{2}$

$substitute\ x,y,z\ values\ in\ (2)$

$$3(\frac{-3\lambda}{2}) + 2(-\lambda) - \frac{\lambda}{2} - 12 = 0 \Rightarrow \lambda = \frac{-12}{7}$$

$$\lambda \Rightarrow x = \frac{18}{7}\ ;\ y = \frac{12}{7}\ ;\ z = \frac{6}{7}$$

$\therefore The\ stationary\ point\ is\ (\frac{18}{7}, \frac{12}{7}, \frac{6}{7})$

$\therefore The\ point\ is\ p(\frac{18}{7}, \frac{12}{7}, \frac{6}{7})$

4. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Sol: Let 2x, 2y, 2z be the length, breadth and height of the rectangular parallelepiped that

can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

120

Then the centroid of the parallelepiped coincides with the centre O (0, 0, 0) of the ellipsoid and the corners of the parallelepiped lie on the surface of the ellipsoid (1).

If(x, y, z) is any corner of the parallelepiped then it satisfies condition (1).

Let v be the volume of the parallelepiped i.e., v=8xyz.we have to find the maximum value of v subject to the condition (1).

$$F(x,y,z)=v+(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1) \qquad \text{..............(2)}$$

Where λ is the multiplier to be determined such that

$$\frac{\partial F}{\partial x}=0,\frac{\partial F}{\partial y}=0,\frac{\partial F}{\partial z}=0$$

$$8yz+\frac{2x}{a^2}\lambda=0, \qquad \text{................(3)}$$

$$8zx+\frac{2y}{b^2}\lambda=0, \qquad \text{..................(4)}$$

$$and \quad 8xy+\frac{2z}{c^2}\lambda=0 \qquad \text{...............(5)}$$

Now (3),(4),(5) are combined as:

$$\frac{a^2yz}{x}=\frac{b^2zx}{y}=\frac{c^2xy}{z}=-\frac{\lambda}{4} \qquad \text{.................(6)}$$

from first two fractions, we have

$$\frac{a^2y}{x}=\frac{b^2x}{y}\Rightarrow\frac{x^2}{a^2}=\frac{y^2}{b^2} \qquad \text{...................(7)}$$

$$similarly \quad \frac{y^2}{b^2}=\frac{z^2}{c^2}$$

substituting (7)*and*(8) *in* (1), *we get*

$$x=\frac{a}{\sqrt{3}},y=\frac{b}{\sqrt{3}},z=\frac{c}{\sqrt{3}}$$

hence the possible extreme point is $p(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})$

Thus v is maximum at $p(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}})and$ its maximum value is v=8xyz $=\frac{8abc}{3\sqrt{3}}c.u.$

2-Marks Questions:

121

1.If $u = \tan^{-1}\left[\frac{2xy}{x^2-y^2}\right]$, then find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

2. If w = (y-z) (z-x) (x-y) find the value of $\frac{\partial w}{\partial x}$

3. If $u = \log\frac{x^2+y^2}{x+y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

4. If $x = r\cos\theta, y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$

5/10 - Marks Questions:

1. If $U = \log(x^3+y^3+z^3-3xyz)$ then show that $(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z})^2 U = \frac{-9}{(x+y+z)^2}$

2. If U = $\frac{1}{\sqrt{x^2+y^2+z^2}}, x^2+y^2+z^2 \neq 0$ then Prove that $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0$

3. If x +y + z = u , y + z = uv, z = uvw , then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta, y = r\sin\theta$, show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

5. Investigate the maxima & minima, if any, of the function f(x) = x3y2 (1-x-y).

6. Find the extreme value of f(x,y)=sinx+siny+sin(x+y).

7. Find the maximum value of $x^m y^n z^p$ given that $x+y+z=a$.

8. Find a point on the plane 3x+2y+z-12=0 which is nearest to the origin.

9. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

122

COURSE MATERIAL	
SUBJECT	LINEAR ALGEBRA AND CALCULUS (MA20ABS101)
UNIT	4
COURSE	B.TECH
DEPARTMENT	SCIENCE & HUMANITIES
SEMESTER	11
PREPARED BY (Faculty Name/s)	K.Siva Kumar Asst. Professor
VERSION	V-1
PREPARED / REVISED DATE	28-04-2021

BTECH-SEM 11

123

TABLE OF CONTENTS – UNIT 4			
SNo	Contents		Page No.
1	Course Objectives		1
2	Prerequisites		1
3	Syllabus		1
4	Course Outcomes		1
5	CO - PO/PSO Mapping		2
6	Lecture Plan		2
7	Activity Based Learning		2
8	4.1	Double integrals	3
	4.2	Double integrals in polar co-ordinates	9
	4.3	Change of order of Integration	10
	4.4	Change of variables	14
	4.5	Triple integrals	16
	4.6	Change of Variables in a Triple Integral	18
9	Practice Quiz		24
10	Assignments		25
11	Part A Questions & Answers (2 Marks Questions)		26
12	Part B Questions		26
13	Supportive Online Certification Courses		27
14	Real Time Applications		27
15	Contents Beyond the Syllabus		27
16	Prescribed Text Books & Reference Books		27
17	Mini Project Suggestion		27

124

1. **Course Objectives**

   The objectives of this course are to

- This course will illuminate the students in the concepts of calculus and linear algebra.
- To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real world problems and their applications.

2. **Prerequisites**
   1. Students should have knowledge on Linear Algebra
   2. Students should have knowledge on Calculus

3. **Syllabus**

   UNIT IV

   Multiple Integrals

   Double integrals, change of order of integration, change of variables. Evaluation of triple integrals, change of variables between Cartesian, cylindrical and spherical polar co-ordinates.

4. **Course outcomes**

   At the end of the course, the student will be able to

- Develop the use of matrix algebra techniques that is needed by engineers for practical applications (L6)
- Utilize mean value theorems to real life problems (L3)
- Familiarize with functions of several variables which is useful in optimization (L3)
- Apply multiple integrals to find the area and volumes for different functions. (L3)
- Analyze the concepts of Beta and Gamma special function for different functions. (L4)

125

5. Co-PO / PSO Mapping

MAD	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
CO3	3													

6. Lecture Plan

1	1	Introduction to Multiple Integrals	T1
2		Double integrals	TB1/RB2
3		Problems on Double integrals	TB1/RB2
4		Problems on Double integrals	TB1/RB2
5		change of variables in Double integrals	TB1/RB2
6		Problems on change of variables in Double integrals	TB1/RB2
7	2	change of order of integration	TB1/RB2
8		Problems on change of order of integration	TB1/RB2
9		Problems on change of order of integration	TB1/RB2
10		Problems on change of order of integration	TB1/RB2
11		Evaluation of triple integrals	TB1/RB2
12		Problems on triple integrals	TB1/RB2
13	3	Problems on triple integrals	TB1/RB2
14		Problems on change of variables between Cartesian and cylindrical	TB1/RB2
15		Problems on change of variables between Cartesian and cylindrical	TB1/RB2
16		Problems on change of variables between Cartesian and spherical polar co-ordinates	TB1/RB2
17		Problems on change of variables between Cartesian and spherical polar co-ordinates	TB1/RB2
18		Problems on change of variables between Cartesian and spherical polar co-ordinates	TB1/RB2

7. Activity Based Learning

1. To find Area and Volume for different functions.

126

8. Lecture Notes

Multiple Integrals

4.1 Double Integrals:

I. When y_1, y_2 are functions of x and x_1 and x_2 are constants. f(x,y)is first integrated w.r.t y keeping 'x' fixed between limits y1,y2and then the resulting expression is integrated w.r.t 'x' with in the limits x_1,x_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=x_1}^{x=x_2}\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx$$

II. When x_1,x_2 are functions of y and y_1 ,y_2 are constants, f(x,y)is first integrated w.r.t 'x' keeping 'y' fixed, within the limits x_1,$x_2$and then resulting expression is integrated w.r.t 'y' between the limits y_1,y_2 i.e.,

$$\iint_R f(x,y)\,dx\,dy = \int_{y=y_1}^{y=y_2}\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy$$

III. When x_1, x_2, y_1, y_2 are all constants. Then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

Problems

1. Evaluate $\int_1^2\int_1^3 xy^2\,dx\,dy$

Sol. $\int_1^2\left[\int_1^3 xy^2\,dx\right]dy = \int_1^2\left[y^2.\frac{x^2}{2}\right]_1^3 dy = \int_1^2 \frac{y^2}{2}dy[9-1]$

$$=\frac{8}{2}\int_1^2 y^2dy = 4.\int_1^2 y^2dy$$

$$=4.\left[{y^3}/{3}\right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

2. Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol. $\int_{x=0}^2\int_{y=0}^x y\,dy\,dx = \int_{x=0}^2\left[\int_{y=0}^x y\,dy\right]dx$

$$=\int_{x=0}^2\left[\frac{y^2}{2}\right]_0^x dx = \int_{x=0}^2 \frac{1}{2}(x^2-0)\,dx = \frac{1}{2}\int_{x=0}^2 x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate $\int_0^5\int_0^{x^2} x(x^2+y^2)\,dx\,dy$

127

Sol. $\int_{x=0}^{5}\int_{y=0}^{x^2} x(x^2+y^2)\,dy\,dx = \int_{x=0}^{5}\left[x^3y+\frac{xy^3}{3}\right]_{y=0}^{x^2} dx$

$$= \int_{x=0}^{5}\left[x^3.x^2+\frac{x(x^2)^3}{3}\right]dx = \int_{x=0}^{5}\left(x^5+\frac{x^7}{3}\right)dx = \left[\frac{x^6}{6}+\frac{1}{3}.\frac{x^8}{8}\right]_0^5 = \frac{5^6}{6}+\frac{5^8}{24}$$

4. Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

Sol: $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}dy\right]dx$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2+y^2}dy\right]dx = \int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}}dx$$

$$[\because \int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(x/a)]$$

$$= \int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}1-Tan^{-1}0\right]dx \; or \; \frac{\pi}{4}(\sinh^{-1}x)_0^1 = \frac{\pi}{4}(\sinh^{-1}1)$$

$$= \frac{\pi}{4}\int_{x=0}^{1}\frac{1}{\sqrt{1+x^2}}dx = \frac{\pi}{4}\left[\log(x+\sqrt{x^2+1})\right]_{x=0}^{1}$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

5. Evaluate $\int_0^4\int_0^{x^2} e^{y/x}dydx$

Answer:

$3e^4-7$

6. Evaluate $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)dxdy$

Answer:

3/35

7. Evaluate $\int_0^2\int_0^x e^{(x+y)}dydx$

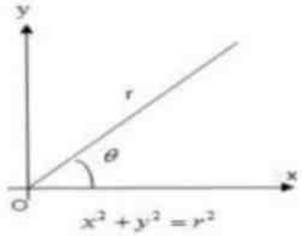
Answer: $\frac{e^4-e^2}{2}$

8. Evaluate $\int_0^{\frac{\pi}{2}}\int_{-1}^{1} x^2 y^2 dxdy$

Answer: $\frac{\pi^3}{36}$

9. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy$

Sol: $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy = \int_0^{\infty} e^{-y^2}\left[\int_0^{\infty} e^{-x^2} dx\right] dy$



$$x^2 + y^2 = r^2$$

$$= \int_0^{\infty} e^{-y^2} \frac{\sqrt{\pi}}{2} dy \quad \because \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2}\int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

Alter:

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2} r dr d\theta \quad (\because x^2+y^2=r^2)$$

(Changing to polar coordinates taking $x = r\cos\theta, y = r\sin\theta$)

$$= \int_0^{\pi/2}\left[\frac{e^{-r^2}}{-2}\right]_0^{\infty} d\theta = \int_0^{\pi/2}\left[\frac{0-1}{-2}\right] d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\pi/2 - 0\right)$$

$$= \frac{\pi}{4}$$

10. Evaluate $\iint xy(x+y)dxdy$ over the region R bounded by $y=x^2$ and $y=x$

Sol: $y=x^2$ is a parabola through (0,0) symmetric about y-axis y=x is a straight line through (0,0) with slope1.

Let us find their points of intersection solving $y=x^2$, y=x we get x^2=x $\Rightarrow$ x=0,1 Hence y=0, 1

$\therefore$The point of intersection of the curves are (0,0), (1,1)

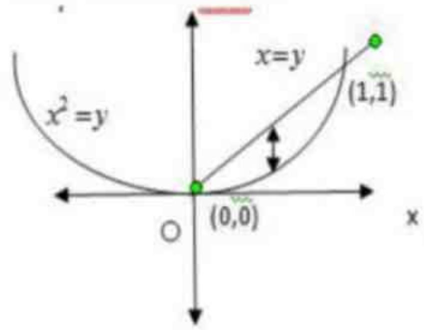
Consider $\iint_R xy(x+y)dxdy$

For the evaluation of the integral, we first integrate w.r.t 'y' from $y=x^2$ to $y=x$

and then w.r.t. 'x'

from x=0 to x=1

$$\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy(x+y)\,dy\right]dx=\int_{x=0}^{1}\left[\int_{y=x^2}^{x}\left(x^2y+xy^2\right)dy\right]dx$$



$$=\int_{x=0}^{1}\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x}dx$$

$$=\int_{x=0}^{1}\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$=\int_{x=0}^{1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$=\left(\frac{5}{6}\cdot\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$=\frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{28-12-7}{168}=\frac{28-19}{168}=\frac{9}{168}=\frac{3}{56}$$

11. Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis and x=2a and the curve

$x^2=4ay$.

Sol. The line x=2a and the parabola $x^2=4ay$ intersect at B(2a,a)

∴The given integral = $\iint_R xy\,dx\,dy$

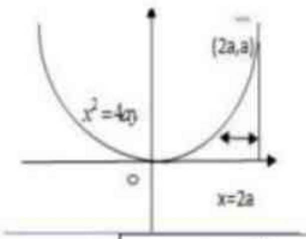
Let us fix 'y'

For a fixed 'y', x varies from $2\sqrt{ay}$ to 2a. Then y varies from 0 to a.

Hence the given integral can also be written as

130

$$\int_{y=0}^{a}\int_{x=2\sqrt{ay}}^{x=2a} xy\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=2\sqrt{ay}}^{x=2a} x\,dx\right] y\,dy$$



$$= \int_{y=0}^{a}\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{2a} y\,dy$$

$$= \int_{y=0}^{a}\left[2a^2 - 2ay\right] y\,dy$$

$$= \left[\frac{2a^2y^2}{2} - \frac{2ay^3}{3}\right]_0^a = a^4 - \frac{2a^4}{3} = \frac{3a^4 - 2a^4}{3} = \frac{a^4}{3}$$

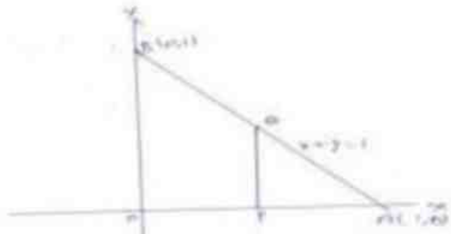
12. Evaluate $\int_0^1 \int_0^{\pi/2} r\sin\theta\, d\theta\, dr$

Sol. $\int_{r=0}^{1} r\left[\int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\right] dr = \int_{r=0}^{1} r(-\cos\theta)_{\theta=0}^{\pi/2}\, dr$

$$= \int_{r=0}^{1} -r\left(\cos \pi/2 - \cos 0\right) dr$$

$$= \int_{r=0}^{1} -r(0-1)\,dr = \int_0^1 r\,dr = \left(\frac{r^2}{2}\right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

13. Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the positive quadrant For Which $x+y\le 1$



Sol. $\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1} dx \int_{y=0}^{y=1-x} (x^2+y^2)\,dy$

$$= \int_{x=0}^{1}\left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_{x=0}^{1}\left(x^2 - x^3 + \frac{1}{3}(1-x)^3\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{6}$$

14. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

131

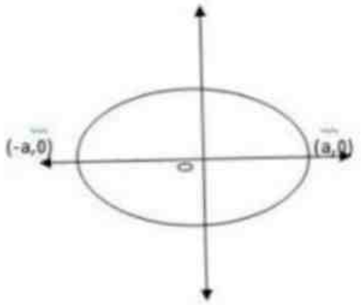
$$i.e., \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2}\left(a^2 - x^2\right) (or)\ y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$\therefore y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Hence the region of integration R can be expressed as

$$-a \le x \le a, \frac{-b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore \iint_R \left(x^2 + y^2\right) dx\,dy = \int_{x=-a}^{a} \int_{y=-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} \left(x^2 + y^2\right) dx\,dy$$



$$= 2\int_{-a}^{a}\left[x^2 . \frac{b}{a}\sqrt{a^2 - x^2} + \frac{b^3}{3a^3}\left(a^2 - x^2\right)^{3/2}\right] dx$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2 - x^2} + \frac{b^3}{3a^3}\left(a^2 - x^2\right)^{3/2}\right] dx$$

Changing to polar coordinates

$$putting\ x = a\sin\theta$$
$$dx = a\cos\theta d\theta$$

$$\frac{x}{a} = \sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$$
$$x \to 0, \theta \to 0$$
$$x \to a, \theta \to \frac{\pi}{2}$$

$$= 4\int_{0}^{\pi/2}\left[a^3 b \sin^2\theta\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right] d\theta = 4\left[a^3 b . \frac{1}{4}\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}.........\frac{\frac{1}{2}.\frac{\pi}{2}}{m}\right]$$

$$= \frac{4\pi}{16}\left(a^3 b + ab^3\right) = \frac{\pi ab}{4}\left(a^2 + b^2\right)$$

15. Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

132

Sol: Given integral $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \int_0^1\left[\int_0^1 \frac{1}{\sqrt{1-y^2}}dy\right]\frac{dx}{\sqrt{1-x^2}}$

$$= \int_0^1\left[\int_0^1 d(\sin^{-1}y)\right]\frac{dx}{\sqrt{1-x^2}}$$

$$= \int_0^1(\sin^{-1}y)_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$= \left(\frac{\pi}{2}\right)\int_0^1 d(\sin^{-1}x)\,dx$$

$$= \frac{\pi}{2}[\sin^{-1}x]_0^1$$

$$= \frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$\therefore \quad \int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}} = \frac{\pi^2}{4}$

4.2 Double integrals in polar co-ordinates:

1. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol. $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}dr\right\}d\theta = -\frac{1}{2}\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}dr\right\}d\theta$

$$= \frac{-1}{2}\int_0^{\pi/4} 2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta = (-1)\int_0^{\pi/4} 2\left[\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right]d\theta$$

$$= (-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta = (-a)(\sin\theta-\theta)_0^{\pi/4}$$

$$= (-a)\left[\left[\sin\frac{\pi}{4}-\frac{\pi}{4}\right]-(0-0)\right]$$

$$= (-a)\left[\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right] = 2\left[\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right]$$

1. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$ Ans: $\frac{a^2\pi}{4}$

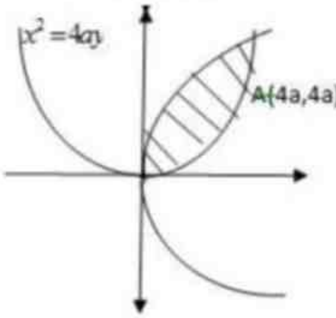
2. Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\,r\,d\theta\,dr$ Ans: $\frac{\pi}{4}$

3. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$ Ans: $\frac{3\pi a^2}{4}$

4.2 Change of order of Integration:

1. Change the order of Integration and evaluate $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol. In the given integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to4a. Let us draw the curves $y=\frac{x^2}{4a}$ and $y=2\sqrt{ax}$



The region of integration is the shaded region in diagram.

The given integral is $=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$

Changing the order of integration, we must fix y first, for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$ and then y varies from 0 to 4a.

Hence the integral is equal to

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy=\int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$=\int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}\,dy=\int_{y=0}^{4a}\left[2\sqrt{ay}-{y^2}/{4a}\right]dy$$

$$=\left[2\sqrt{a}.\frac{y^{3/2}}{3/2}-\frac{1}{4a}.\frac{y^3}{3}\right]_0^{4a}$$

$$=\frac{4}{3}.\sqrt{a}.4a\sqrt{4a}-\frac{1}{12a}.64a^3$$

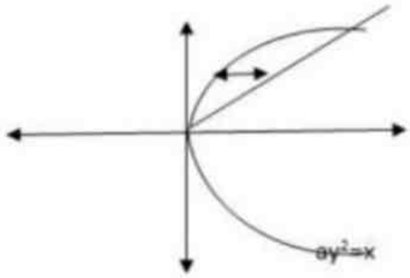
$$=\frac{32}{3}a^2-\frac{16}{3}a^2=\frac{16}{3}a^2$$

2. Change the order of integration and evaluate $=\int_0^a\int_{x/a}^{\sqrt{x/a}}\left(x^2+y^2\right)dx\,dy$

Sol. In the given integral for a fixed x, y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$ and then x varies from 0 to a

Hence we shall draw the curves $y=\frac{x}{a}$ and $y=\sqrt{\frac{x}{a}}$

134



i.e ay=x and $ay^2=x$

We get $\boxed{ay = ay^2}$

$\boxed{\Rightarrow ay - ay^2 = 0}$

$\boxed{\Rightarrow ay(1-y) = 0}$

$\boxed{\Rightarrow y = 0, y = 1}$

If y=0, x=0 if y=1, x=a

The shaded region is the region of integration. The given integral is

$$\boxed{\int_{x=0}^{a}\int_{y=x/a}^{y=\sqrt{x/a}} (x^2+y^2)\,dx\,dy}$$

Changing the order of integration, we must fix y first. For a fixed y, x varies from ay^2 to ay and then y varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$$\boxed{\int_{y=0}^{1}\int_{x=ay^2}^{ay} (x^2+y^2)\,dx\,dy} = \boxed{\int_{y=0}^{1}\left[\int_{x=ay^2}^{ay} (x^2+y^2)\,dx\right]dy}$$

$$\boxed{= \int_{y=0}^{1}\left(\frac{x^3}{3} + xy^2\right)_{x=ay^2}^{ay} dy}$$

$$\boxed{= \int_{y=0}^{1}\left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right)dy}$$

$$\boxed{= \left(\frac{a^3y^4}{12} + \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5}\right)_{y=0}^{1}}$$

$$\boxed{= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20}}$$

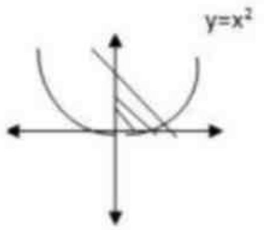
3. Change the order of integration in $\boxed{\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy}$ and hence evaluate the double integral.

Sol. In the given integral for a fixed x, y varies from x^2 to 2-x and then x varies from 0 to 1.

Hence we shall Draw the curves $y=x^2$ and $y=2-x$

The line $y=2-x$ passes through (0, 2), (2, 0)

Solving $y=x^2$,$y=2-x$



Then we get $\boxed{x^2=2-x}$

$$\boxed{\Rightarrow x^2+x-2=0}$$

$$\boxed{\Rightarrow x^2+2x-x-2=0}$$

$$\boxed{\Rightarrow x(x+2)-1(x+2)=0}$$

$$\boxed{\Rightarrow (x-1)(x+2)=0}$$

$$\boxed{\Rightarrow x=1,-2}$$

$$\boxed{If\ x=1, y=1}$$

$$\boxed{If\ x=-2, y=4}$$

Hence the points of intersection of the curves are (-2, 4), (1, 1)

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix y, for the region within OACO for a fixed

y, x varies From 0 to $\boxed{\sqrt{y}}$

Then y varies from 0 to 1

For the region within CABC, for a fixed y, x varies from 0 to 2-y , then y varies from 1 to 2

Hence $\boxed{\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \iint\limits_{OACO} xy\,dx\,dy + \iint\limits_{CABC} xy\,dx\,dy}$

$$\boxed{=\int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{y}} x dx\right] y\,dy + \int_{y=1}^{2}\left[\int_{x=0}^{2-y} x\,dx\right] y\,dy}$$

$$\boxed{=\int_{y=0}^{1}\left(\frac{x^2}{2}\right)_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2}\left(\frac{x^2}{2}\right)_{x=0}^{2-y} y\,dy}$$

$$=\int_{y=0}^{1}\frac{y}{2}.y\,dy+\int_{y=1}^{2}\frac{(2-y)^2}{2}y\,dy$$

$$=\frac{1}{2}\int_{y=0}^{1}y^2dy+\frac{1}{2}.\int_{y=1}^{2}\left(4y-4y^2+y^3\right)dy$$

$$=\frac{1}{2}.\left(\frac{y^3}{3}\right)_0^1+\frac{1}{2}.\left[\frac{4y^2}{2}-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$=\frac{1}{2}.\frac{1}{3}+\frac{1}{2}\left[2.4-2.1-\frac{4}{3}(8-1)+\frac{1}{4}(16-1)\right]$$

$$=\frac{1}{6}+\frac{1}{2}\left[6-\frac{28}{3}+\frac{15}{4}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{72-112+45}{12}\right]=\frac{1}{6}+\frac{1}{2}\left[\frac{5}{12}\right]=\frac{4+5}{24}=\frac{9}{24}=\frac{3}{8}$$

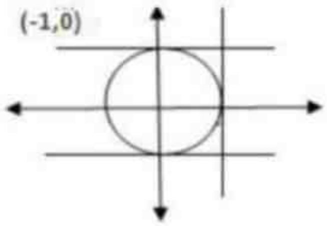
4. Change of the order of integration $\int_0^1\int_0^{\sqrt{1-x^2}}y^2dx\,dy \quad Ans:\frac{\pi}{16}$

Hint: Now limits are $y=0\,to\,1$ and $x=0\,to\,\sqrt{1-y^2}$

$$put\ y=\sin\theta$$
$$\sqrt{1-y^2}=\cos\theta$$
$$dy=\cos\theta d\theta$$

$$=\int_0^1 y^2\sqrt{1-y^2}\,dy$$

$$=\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta=\int_0^{\pi/2}\sin^2\theta d\theta-\int_0^{\pi/2}\sin^4\theta d\theta$$



$$=\frac{1}{2}\left(\pi/2\right)-\frac{3}{4}.\frac{1}{2}\left(\pi/2\right)=\pi/16$$

4.4 Change of variables:

The variables x, y in $\iint_R f(x,y)dx\,dy$ are changed to u, v with the help of the relations

$x = f_1(u,v), y = f_2(u,v)$ then the double integral is transferred into

$$\iint_{R^1} f\left[f_1(u,v), f_2(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Where R1is the region in the uv plane, corresponding to the region R in the xy-plane.

Changing from Cartesian to polar co-ordinates

$x = r\cos\theta, y = r\sin\theta$

$$\partial\left(\frac{(x,y)}{(r,\theta)}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r \quad \therefore \iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Note: In polar form dxdy is replaced by $r\,dr\,d\theta$

Problems:

1. Evaluate the integral by changing to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy$

Sol. The limits of x and y are both from 0 to ∞.

$\therefore$ The region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

Substituting $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Hence $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$

$$\begin{aligned} &Put\ r^2 = t \\ &\Rightarrow 2rdr = dt \\ &\Rightarrow r\,dr = dt/2 \end{aligned}$$

$Where\ r = 0 \Rightarrow t = 0\ and\ r = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dx\,dy = \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} \frac{1}{2}e^{-t}\,dt\,d\theta$$

$$= \int_0^{\pi/2} \frac{-1}{2}\left(e^{-t}\right)_0^\infty d\theta$$

138

$$= \frac{-1}{2}\int_0^{\pi/2}(0-1)\,d\theta \Rightarrow \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\pi/2 = \pi/4$$

3. Evaluate the integral by changing to polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$

Sol. The limits for x are x=0 to

$$x = \sqrt{a^2-y^2}$$
$$\Rightarrow x^2+y^2 = a^2$$

$\therefore$ The given region is the first quadrant of the circle.

By changing to polar co-ordinates

$$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$$

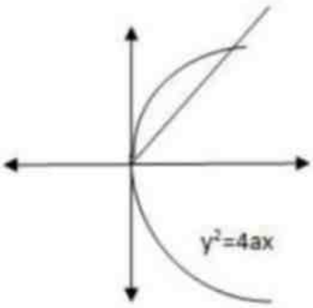
Here 'r' varies from 0 to a and 'θ' varies from 0 to $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^a d\theta = \frac{a^4}{4}(\theta)_0^{\pi/2}$$

$$= \pi/8\, a^4$$

4. Show that $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

Sol. The region of integration is given by $x = y^2/4a,\ x = y$ and y=0, y=4a



i.e., the region is bounded by the parabola $y^2=4ax$ and the straight line x=y.

Let $x = r\cos\theta,\ y = r\sin\theta.\ Then\ dx\,dy = r\,dr\,d\theta$

The limits for r are r=0 at O and for P on the parabola

$$r^2\sin^2\theta = 4a(r\cos\theta) \Rightarrow r = \frac{4a\cos\theta}{\sin^2\theta}$$

For the line y=x, slope m=1 i.e., $Tan\theta = 1, \theta = \pi/4$

The limits for $\theta : \pi/4 \to \pi/2$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{4a\cos\theta/\sin^2\theta}(\cos^2\theta-\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\left(\frac{r^2}{2}\right)_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}(\cos^2\theta-\sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta}d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}(\cos^4\theta-\cot^2\theta)\,d\theta = 8a^2\left[\frac{3\pi-8}{12}+\frac{\pi}{4}-1\right] = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$$

4.5 Triple integrals:

If x_1, x_2 are constants. y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then f(x, y, z) is first integrated w.r.t. 'z' between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t 'y' between the limits y1 and y2 keeping x constant. The resulting expression is integrated w.r.t. 'x' from x_1 to x_2

$$\therefore \iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x=a}^{b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\;dy\;dx$$

Problems

1. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol. $\int_{x=0}^1\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz = \int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} dy\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz$

$$= \int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{1-x^2-y^2}} dy$$

$$= \frac{1}{2}\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy$$

$$= \frac{1}{2}\int_{x=0}^1 dx\int_{y=0}^{\sqrt{1-x^2}} x\left[(1-x^2)y-y^3\right]dy$$

$$= \frac{1}{2}\int_{x=0}^1 x\left[(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\cdot\int_{x=0}^1 x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{8}\int_{x=0}^{1} x\left[2\left(1-x^2\right)-2x^2\left(1-x^2\right)-\left(1-x^2\right)^2\right]dx$$

$$=\frac{1}{8}\int_{x=0}^{1}\left(x-2x^3+x^5\right)dx=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}$$

1. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$

Sol: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+y)\,dx\,dydz=\int_{-1}^{1}\int_{0}^{z}\left[\left(xy+{y^2}/{2}+zy\right)_{x-z}^{x+z}\right]dx\,dz$

$$=\int_{-1}^{1}\int_{0}^{z} x(x+z)-x(x-z)+\left[\frac{x+z}{2}\right]^2-\left[\frac{x-z}{2}\right]^2+z(x+z)-z(x-z)dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[2z(x+z)+\frac{1}{2}4xz\right]dx\,dz$$

$$=2\int_{-1}^{1}\left[z.\frac{x^2}{2}+z^2x+z.\frac{x^2}{2}\right]_0^z dz$$

$$=2.\int_{-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz=4.\left(\frac{z^4}{4}\right)_{-1}^{1}=0$$

3. Evaluate $\int_0^1 dx\int_1^2 dy\int_1^3 xyzdz$

Sol: $\int_0^1 dx\int_1^2 dy\int_1^3 xyzdz=\int_0^1 dx\int_1^2 dy\left[xy\left(\frac{z^2}{2}\right)_1^3\right]$

$$=\int_0^1 dx\int_1^2 dy\left[\frac{1}{2}xy(9-1)\right]$$

$$=\int_0^1 dx\int_1^2 dy\left[\frac{8}{2}xy\right]$$

$$=\int_0^1 dx\int_1^2 4xydy$$

$$=\int_0^1 dx(4x)\left(\frac{y^2}{2}\right)_1^2$$

141

$= 4$

4. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dzdxdy$

Sol: $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dzdxdy = \int_1^e \int_1^{\log y} [z \log z - z]_1^{e^x} dxdy$

$$= \int_1^e \int_1^{\log y} [(e^x \log_e e^x - e^x) - (0 - 1)] dxdy$$

$$= \int_1^e \left[\int_1^{\log y} (xe^x - e^x + 1) dx\right] dy$$

$$= \int_1^e (xe^x - e^x(1) - e^x + x)_1^{\log y} dy$$

$$= \int_1^e (xe^x - 2e^x + x)_1^{\log y} dy$$

$$= \int_1^e [(\log y \, e^{\log y} - 2e^{\log y} + \log y) - (1e - 2e + 1)] dy$$

$$= \int_1^e [(\log y \, (y) - 2y + \log y) - (1 - e)] dy$$

$$= \left[\log y \left(\frac{y^2}{2}\right) - \int_1^e \frac{y^2}{2}\left(\frac{1}{y}\right) dy - 2\frac{y^2}{2} + (y \log y - y) - (1 - e)y\right]_1^e$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

5. Evaluate $\iiint_V (xy + yz + zx) dxdydz$ where V is the region of space bounded by the planes

x=0, x=1, y=0, y=2 and z=0, z=3.

Sol: $\iiint_V (xy + yz + zx) dxdydz = \int_{z=0}^3 \int_{y=0}^2 \int_{x=0}^1 \iiint_V (xy + yz + zx) dxdydz$

$$= \int_{z=0}^3 \int_{y=0}^2 [\frac{x^2}{2} y + xyz + \frac{x^2}{2} z]_0^1 dydz$$

$$= \int_{z=0}^3 \int_{y=0}^2 [\frac{y}{2} + yz + \frac{z}{2}] dydz$$

$$= \int_{z=0}^3 [\frac{y^2}{4} + \frac{y^2}{2} z + \frac{yz}{2}]_0^2 dz$$

$$= \int_{z=0}^{3} [1 + 3z]dz$$

$$= [z + \frac{3z^2}{2}]_0^3 = \frac{33}{2}$$

Volume as a triple integral:

Suppose a three dimensional solid is cut in to elemental rectangular parallelepiped by drawing planes parallel to the coordinate planes. The volume of an elemental parallelepiped δv is $\delta x, \delta y, \delta z$. Hence the total volume of the solid is $\iiint_v dv = \iiint_v dxdydz$ where the integration is carried over the entire volume.

1. Evaluate $\iiint_R (x+y+z)\, dzdydx$ where R is the region bounded by the planes x =0,x =1, y=0, y=1,z =0,z =1.

Sol: $\iiint_R (x+y+z)\, dzdydx = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(x+y+z)\, dzdydx$

$$= \int_0^1\int_0^1 (xz + yz + \frac{z^2}{2})_0^1\, dydx$$

$$= \int_0^1\int_0^1 (x + y + \frac{1}{2})dydx$$

$$= \int_0^1 (xy + \frac{y^2}{2} + \frac{y}{2})_0^1\, dx$$

$$= \int_0^1 (x + \frac{1}{2} + \frac{1}{2})dx$$

$$= \int_0^1 (x+1)dx$$

$$= (\frac{x^2}{2} + x)_0^1$$

$$= \frac{1}{2} + 1 = \frac{3}{2}.$$

4.6 Change of Variables in a Triple Integral:

Let the functions $x = \phi_1(u,v,w), y = \phi_2(u,v,w)\ \&\ z = \phi_3(u,v,w)$ be the transformations from Cartesian coordinates to the curvilinear coordinate's u, v, and w.

The Jacobin for this transformation is given by

$$J = \boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)}}$$

Then $\boxed{\iiint_V f(x,y,z)dxdydz = \iiint_{V'} f(\phi_1,\phi_2,\phi_3)|J|dudvdw}$

Where v is the corresponding domain in the curvilinear coordinates u, v, w.

Change of variables from Cartesian to spherical polar coordinate system :

In problems having symmetry w.r.t a point O, it would be convenient to use spherical coordinates with this point chosen as origin.

The relation between Cartesian coordinates x,y,z and spherical polar coordinates $\boxed{\rho,\theta,\phi(i.e\, u=\rho, v=\theta, w=\phi)}$ are given by

$$\boxed{\begin{aligned} x &= \rho\sin\theta\cos\phi((or)r\sin\theta\cos\phi) \\ y &= \rho\sin\theta\sin\phi((or)r\sin\theta\sin\phi) \\ z &= \rho\sin\theta((or)r\cos\theta) \end{aligned}}$$

And $\boxed{dxdydz = |J|d\rho d\theta d\phi (or) |J|drd\theta d\phi}$

$$\text{Where } J = \boxed{\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{vmatrix}$$

$$= \boxed{\rho^2\sin\theta(or)r^2\sin\theta}$$

$$\boxed{\iiint_V f(x,y,z)dxdydz = \iiint_{V'} f(\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)\rho^2\sin\theta d\rho d\theta d\phi}$$

The region V in (x, y, z) is to be covered by the limits of $\boxed{\rho,\theta,\phi}$ and is denoted as V^1

Change of variables from Cartesian to cylindrical coordinate system :

The relation between the Cartesian coordinates x, y, z and cylindrical coordinates $\boxed{\rho,\theta,z(i.e\, u=\rho, v=\theta, w=z)}$ are given by

$$\boxed{\begin{aligned} x &= \rho\cos\theta((or)r\cos\theta) \\ y &= \rho\sin\theta((or)r\sin\theta) \\ z &= z \end{aligned}}$$

144

And $J = \dfrac{\partial(x,y,z)}{\partial(\rho,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$

$$\iiint_V f(x,y,z)dxdydz = \iiint_V f(\rho\cos\theta, \rho\sin\theta, z)\rho d\rho d\theta dz$$

1. Evaluate $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \dfrac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}}$ by changing in to spherical polar coordinates.

Sol: Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

We have J = $r^2\sin\theta$, $x^2+y^2+z^2 = r^2$ & $dxdydz = r^2\sin\theta dr d\theta d\phi.$

Also given region of integration is the volume of the sphere $x^2+y^2+z^2 = 1$

In the positive octant for which r varies from 0 to 1, θ varies from 0 to $\dfrac{\pi}{2}$ and ϕ Varies from 0 to $\dfrac{\pi}{2}$.

$$\therefore \int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}} = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^2\sin\theta dr d\theta d\phi}{\sqrt{(1-r^2)}}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 \frac{1-(1-r^2)dr}{\sqrt{(1-r^2)}} \right\} \sin\theta d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 \frac{1}{\sqrt{(1-r^2)}} - \sqrt{(1-r^2)} \right\} dr \sin\theta d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\sin^{-1} r - \left\{ \frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1} r \right\} \right]_0^1 \sin\theta d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left[\frac{\pi}{2}-\frac{1}{2}\frac{\pi}{2}\right]\sin\theta d\theta d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}}\sin\theta d\theta\right]d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}(-\cos\theta)_0^{\frac{\pi}{2}}\,d\phi$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}d\phi = \frac{\pi}{4}\left(\frac{\pi}{2}-0\right)=\frac{\pi^2}{8}.$$

2. Evaluate $\iiint(x^2+y^2+z^2)dxdydz$ taken over the volume enclose by the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates.

Sol: Changing to spherical polar coordinates by putting

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

We have J = $r^2\sin\theta$, $x^2+y^2+z^2=r^2$ & $dxdydz = r^2\sin\theta dr d\theta d\phi.$

Now $\iiint(x^2+y^2+z^2)dxdydz = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2(r^2\sin\theta dr d\theta d\phi)$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{r^5}{5}\sin\theta d\theta d\phi$$

$$= \frac{1}{5}\int_0^{2\pi}(-\cos\theta)_0^{\pi}\,d\phi$$

$$= \frac{2}{5}\int_0^{2\pi}d\phi = \frac{2}{5}(\phi)_0^{2\pi}$$

$$= \frac{4\pi}{5}$$

3. Using cylindrical coordinates, find the volume of cylinder with base radius a and height h.

Sol: The region of integration is bounded by $x^2+y^2\le a^2, 0\le z\le h$.

The same region in cylindrical coordinates will be as follows:

$$r:0\to a, \theta:0\to 2\pi, z:0\to h$$

$\therefore$ Required volume = $\iiint dxdydz = \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r dr d\theta dz$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi}\left(\frac{r^2}{2}\right)_0^a d\theta dz$$

$$= \boxed{\frac{a^2}{2}\int_{z=0}^{h}(\theta)_0^{2\pi}dz}$$

$$= \boxed{\frac{a^2}{2}(2\pi)\int_{z=0}^{h}dz}$$

$$= \boxed{a^2\pi(z)_0^h}$$

$$= \boxed{\pi a^2 h}.$$

4. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Sol: The sphere $x^2 + y^2 + z^2 = a^2$ is cut into 8 equal parts by three co-ordinate planes.

Hence the volume of the sphere is equal to 8 times the volume of the solid bounded by

x=0, y=0, z=0 and $x^2 + y^2 + z^2 = a^2$.

Z-varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

Y-varies from 0 to $\sqrt{a^2 - x^2}$

X-varies from 0 to a

Required volume V= $8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8\int_0^a [\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{(a^2-x^2)}{2}\sin^{-1}(\frac{y}{\sqrt{a^2-x^2}})]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \frac{(a^2-x^2)}{2}\frac{\pi}{2}dx$$

$$= 2\pi \int_0^a (a^2 - x^2)dx$$

$$= 2\pi[a^2x - \frac{x^3}{3}]_0^a$$

$$= \frac{4\pi a^3}{3}$$

9. Practice Quiz

1. $\int_0^1\int_0^x e^x dxdy=$

   a) 2
   b) -2
   c) 1
   d) -1

2. $\int_0^2\int_0^x (x+y)dxdy=$

   a) 1
   b) 2
   c) 3
   d) 4

3. $\int_0^1 \int_1^2 x\, y\, dy\, d\, x =$

   a) 3
   b) 4
   c) $\frac{3}{4}$
   d) $\frac{4}{3}$

4. In polar coordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, d\, x\, dy =$

   a) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, dr\, d\theta$

   b) $\int_0^{\frac{\pi}{4}} \int_0^\infty e^{-r^2}\, .r.\, dr\, d\theta$

   c) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\, dr\, d\theta$

   d) $\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r}\, dr\, d\theta$

5. $\int_1^0\int_0^1 (x+y)dx\ dy\ =$

   a) -2
   b) 2
   c) -1
   d) 1

148

6. $\int_0^a\int_0^b (x^2+y^2)dydx=$

a) $\frac{ab}{3}(a^2+b^2)$

b) $\frac{ab}{2}(a^2-b^2)$

c) $\frac{ab}{3}(a^2-b^2)$

d) none

7. The value of $\iint_R xy\ dx\ dy =$ _______where R is the region in the positive quadrant of the circle $x^2+y^2=a^2$.

a) $\frac{a^2}{8}$

b) $\frac{a^3}{8}$

c) $\frac{a^4}{8}$

d) $\frac{a^4}{2}$

8. $\int_0^1\int_0^1\int_0^1 xyz\ dx\,dy\,dz =$

a) $\frac{1}{5}$

b) $\frac{1}{3}$

c) $\frac{1}{8}$

d) $\frac{1}{12}$

9. $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\ dz =$

a) 38

b) 48

c) -48

d) none

10. The volume of the tetrahedron bounded by the surfaces x=0, y=0, z=0 and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is _____

a) $\frac{abc}{2}$

b) $\frac{abc}{4}$

c) $\frac{abc}{6}$

d) $\frac{abc}{3}$

Assignments

1. Evaluate $\iint(x^2 + y^2)dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)dydx$ by changing into polar coordinates.
3. Change the order of integration and evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$.
4. Evaluate by changing the order of integration $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}}$.
5. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dzdxdy$.

11. PART-A QUESTIONS

1. Evaluate $\boxed{\int_0^4 \int_0^{x^2} e^{y/x} dydx}$
2. Evaluate $\boxed{\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dxdy}$
3. Evaluate $\boxed{\int_0^2 \int_0^x e^{(x+y)} dydx}$
4. Evaluate $\boxed{\int_0^{\frac{\pi}{2}} \int_{-1}^1 x^2 y^2 dxdy}$
5. Evaluate $\int_0^1 dx \int_1^2 dy \int_1^3 xyzdz$

12. PART-B QUSTIONS

1. Evaluate $\iint(x^2 + y^2)dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)dydx$ by changing into polar coordinates.
3. Change the order of integration and evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$.
4. Evaluate by changing the order of integration $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}}$.
5. Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dzdxdy$.
6. Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} dzdydx$
7. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

150

13. Supportive Online Certification Courses:

- ➤ Advanced Linear Algebra (NPTEL)
- ➤ Engineering Mathematics (NPTEL)

14. Real Time Applications:

- Application of Triple integrals are used mainly to calculate the volume of a three dimensional solid.

15. Content beyond the Syllabus:

- Find the volumes by using double integrals.

16. Prescribed Text Books and Reference Books:

Text Books

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.

2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

Reference Books:

1. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.

2. George B. Thomas, Maurice D. Weir and Joel Hass, Thomas Calculus, 13/e, Pearson Publishers, 2013.

17. Mini Project Suggestion:

151

COURSE MATERIAL	
SUBJECT	LINEAR ALGEBRA AND CALCULUS (20A54101)
UNIT	5
COURSE	B.TECH
DEPARTMENT	CSE
SEMESTER	11
PREPARED BY (Faculty Name/s)	Dr B RAMOORTHY REDDY Assistant Professor
Version	V-1
PREPARED / REVISED DATE	25-03-2021

TABLE OF CONTENTS – UNIT 1			
S. NO	CONTENTS		PAGE NO.
1	COURSE OBJECTIVES		1
2	PREREQUISITES		1
3	SYLLABUS		1
4	COURSE OUTCOMES		1
5	CO - PO/PSO MAPPING		1
6	LESSON PLAN		2
7	ACTIVITY BASED LEARNING		2
8	LECTURE NOTES		2
	1.1	Beta function	2
	1.2	Gamma functions and their properties	14
	1.3	Relation between beta and gamma functions	17
	1.4	Evaluation of definite integrals using beta and gamma functions.	21
9	ASSIGNMENTS		25
10	PART A QUESTIONS & ANSWERS (2 MARKS QUESTIONS)		26
11	PART B QUESTIONS		27
12	SUPPORTIVE ONLINE CERTIFICATION COURSES		27
13	REAL TIME APPLICATIONS		27
14	CONTENTS BEYOND THE SYLLABUS		28
15	PRESCRIBED TEXT BOOKS & REFERENCE BOOKS		28

153

1. Course Objectives

The objectives of this course is to

At the end of the course, the students will be able to:

1. • This course will illuminate the students in the concepts of calculus and linear algebra.
2. To equip the students with standard concepts and tools at an intermediate to advanced level mathematics to develop the confidence and ability among the students to handle various real world problems and their applications.

2. Prerequisites

Students should have knowledge on

1. Basic knowledge on English grammer.
2. Basic Mathematics

3. Syllabus

UNIT 5

Beta and Gamma functions and their properties, relation between beta and gamma functions, evaluation of definite integrals using beta and gamma functions.

4. Course outcomes

After completing this course the student must demonstrate the knowledge and ability to:

CO3: Understand beta and gamma functions and its relations.

CO4: Conclude the use of special function in evaluating definite integrals•

5. Co-PO / PSO Mapping

Machine Tools	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1				2	2									
CO2						2								
CO3									1			2		
CO4											3			
CO5														

6. Lesson Plan

Lecture No.	Weeks	Topics to be covered	References
1	1	**Beta function**	T1

154

2		**Gamma functions and their properties**	T1, R1
3		**Relation between beta and gamma functions**	T1, R1
4		**Evaluation of definite integrals using beta and gamma functions.**	T1, R1

7. Activity Based Learning

1. Problems on Beta functions
2. Problems on gamma functions.

8. Lecture Notes

1.1 Beta Function:

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta function and is denoted by $\beta(m,n)$ and read as "Beta of m,n".The above integral converges for $m > 0, n > 0$.

Thus $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ where $m > 0, n > 0$

- Beta function is also called Eulerian integral of the first kind.

Properties of Beta Function:

1. Symmetry of Beta function: i.e., $\beta(m,n) = \beta(n,m)$

Proof: By the def , we have

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $(1-x) = y \Rightarrow x = (1-y)$ so $dx = -dy$

when $x = 0 \Rightarrow y = 1$

$x = 1 \Rightarrow y = 0$

$$\therefore \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= -\int_1^0 y^{n-1}(1-y)^{m-1}dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy \qquad \therefore \int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx \qquad \therefore \int_a^b f(x)dx = \int_a^b f(y)dy$$

$$= \beta(n,m)$$

$$\therefore \beta(m,n) = \beta(n,m)$$

Alternate Proof:

We know that $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

From the properties of definite integrals, we have

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\therefore \beta(m,n) = \int_0^1 (1-x)^{m-1}(1-(1-x))^{n-1}dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$= \beta(n,m)$$

$$\therefore \beta(m,n) = \beta(n,m)$$

2. Prove that $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta\, d\theta$

Proof: By the def , we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = sin^2\theta \implies dx = 2sin\theta cos\theta d\theta$

when $x = 0 \implies \theta = 0$

$x = 1 \implies \theta = \frac{\pi}{2}$

$$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}(1-sin^2\theta)^{n-1}\, 2sin\theta cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (sin^2\theta)^{m-1}(cos^2\theta)^{n-1}\, sin\theta cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-2}\theta\, cos^{2n-2}\theta\, sin\theta cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta\, d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta\, d\theta$$

Note: $\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$

3. Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Proof: By the def , we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$\beta(m+1,n) = \int_0^1 x^m (1-x)^{n-1} dx$

$\beta(m,n+1) = \int_0^1 x^{m-1} (1-x)^n dx$

Now $\beta(m+1,n) + \beta(m,n+1) = \int_0^1 x^m (1-x)^{n-1} dx + \int_0^1 x^{m-1}(1-x)^n dx$

$= \int_0^1 [x^{m-1}(1-x)^n + x^{m-1}(1-x)^n] dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)] dx$

$= \int_0^1 x^{m-1}(1-x)^{n-1} dx$

$= \beta(m,n)$

Hence $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

4. If m and n are positive integers then $\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$

Proof: By the def , we have

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ ---------------(1)

$= [x^{m-1} \frac{(1-x)^n}{n(-1)}]_0^1 - \int_0^1 \frac{(1-x)^n}{n(-1)} (m-1) x^{m-2} dx$ (By integration by parts)

$= \frac{(m-1)}{n} \int_0^1 x^{m-2}(1-x)^n dx$

$= \frac{(m-1)}{n} \beta(m-1,n+1)$ ---------------(2)

Now we have to find $\beta(m-1,n+1)$

To obtain this put $m = m-1$ and $n = n+1$ in equation (2), we have

$\beta(m-1,n+1) = \frac{(m-2)}{(n+1)} \beta(m-2,n+2)$

From equation (2)

$\beta(m,n) = \frac{(m-1)}{n} . \frac{(m-2)}{(n+1)} \beta(m-2,n+2)$ ---------------(3)

Changing $m = m-2$ and $n = n+2$ in equation (2), we have

157

$$\beta(m-2,n+2) = \frac{(m-3)}{(n+1)}\beta(m-2,n+2)$$

From equation (3)

$$\beta(m,n) = \frac{(m-1)}{n}.\frac{(m-2)}{(n+1)}.\frac{(m-3)}{(n+1)}\beta(m-2,n+2) \text{ ------------------(4)}$$

Proceeding like this, we get

$$\beta(m,n) = \frac{(m-1)}{n}.\frac{(m-2)}{(n+1)}.\frac{(m-3)}{(n+1)}\dots\dots\dots.\frac{(m-(m-1))}{(n+m-2)}\beta(m-(m-1),n+(m-1))$$

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3)\dots\dots\dots1}{n(n+1)(n+2)\dots\dots\dots(n+m-2)}\beta(1,n+m-1) \text{----------------------(5)}$$

$$\beta(1,n+m-1) = \int_0^1 x^0(1-x)^{n+m-2}dx = \int_0^1(1-x)^{n+m-2}dx$$

$$= [\frac{(1-x)^{n+m-1}}{(n+m-1)(-1)}]_0^1 = \frac{1}{n+m-1}$$

From equation (5), we have

$$\beta(m,n) = \frac{(m-1)(m-2)(m-3)\dots\dots\dots1}{n(n+1)(n+2)\dots\dots\dots(n+m-2)(n+m-1)}$$

$$\beta(m,n) = \frac{(m-1)!}{(n+m-1)(n+m-2)\dots(n+2)(n+1)n}$$

Multiplying the numerator and denominator by $(n-1)!$, we have

$$\beta(m,n) = \frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2)\dots(n+2)(n+1)n(n-1)!} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

$$\therefore \beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Other forms of Beta Function:

1. Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$ (or) $\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$

Proof: By the def , we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ --------------------(1)}$$

Put $x = \frac{1}{1+y}$ so that $dx = \frac{-1}{(1+y)^2}dy$

when $x = 0 \Rightarrow y \to \infty$

$x = 1 \Rightarrow y = 0$

From equation (1), we have

$$\beta(m,n) = \int_{\infty}^{0} (\frac{1}{1+y})^{m-1}(1 - \frac{1}{1+y})^{n-1} \frac{(-1)}{(1+y)^2} dy$$

$$\beta(m,n) = \int_{\infty}^{0} \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{(-1)}{(1+y)^2} dy$$

$$\beta(m,n) = \int_{0}^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\therefore \beta(m,n) = \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Since Beta function is symmetrical in m and n, we also have

$$\beta(m,n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Hence $\beta(m,n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_{0}^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

2.Show that $\beta(m,n) = \int_{0}^{1} \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

Proof: We have $\beta(m,n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_{0}^{1} \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_{1}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \text{ -------------(1)}$$

Now consider $\int_{1}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

Put $x = \frac{1}{y}$ so that $dx = \frac{-1}{y^2} dy$

when $x = 1 \Rightarrow y = 1$

$x \to \infty \Rightarrow y = 0$

$$\therefore \int_{1}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_{1}^{0} \frac{(\frac{1}{y})^{m-1}}{(1+\frac{1}{y})^{m+n}} \cdot \frac{-1}{y^2} dy$$

$$= \int_{0}^{1} \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}} \cdot \frac{1}{y^2} dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Hence equation (1) becomes

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$$

3. Show that $\beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

Proof: We have, $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx = a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}(\frac{a}{b}x+1)^{m+n}} dx$

Put $y = \frac{ax}{b} \Rightarrow x = \frac{by}{a}$ so that $dx = \frac{b}{a} dy$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\therefore a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx = \frac{a^m b^n}{b^{m+n}} \int_0^\infty \frac{(\frac{by}{a})^{m-1}}{(\frac{a}{b}\frac{by}{a}+1)^{m+n}} \cdot \frac{b}{a} dy$$

$$= a^m b^{-m} \frac{b^m}{a^m} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

We know that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

Hence $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \beta(m,n)$

$$\therefore \beta(m,n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$$

4.Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

Proof: By the definition , we have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad \text{------------------}(1)$$

Put $x = \frac{(1+a)y}{(y+a)}$ so that $dx = \frac{a(1+a)}{(y+a)^2} dy$

when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

Now equation (1) becomes

$$\beta(m,n) = \int_0^1 \frac{(1+a)^{m-1}y^{m-1}}{(y+a)^{m-1}}.(1-\frac{(1+a)y}{(y+a)})^{n-1}.\frac{a(1+a)}{(y+a)^2}dy$$

$$= \int_0^1 \frac{(1+a)^{m-1}y^{m-1}}{(y+a)^{m-1}}\left(\frac{y+a-y-ay}{y+a}\right)^{n-1}\frac{a(1+a)}{(y+a)^2}dy$$

$$= \int_0^1 \frac{a(1+a)^m y^{m-1}}{(y+a)^{m-1+n-1+2}}.(a-ay)^{n-1}dy$$

$$= \int_0^1 \frac{a(1+a)^m y^{m-1}}{(y+a)^{m+n}}.a^{n-1}(1-y)^{n-1}dy$$

$$= a^n(1+a)^m\int_0^1 \frac{y^{m-1}(1-y)^{n-1}}{(y+a)^{m+n}}dy = a^n(1+a)^m\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx$$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx = \frac{\beta(m,n)}{a^n(1+a)^m}$$

5. Show that $\int_a^b (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\beta(m,n), m>0, n>0.$

Proof: By the definition

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = \frac{y-b}{a-b}$ so that $dx = \frac{1}{a-b}dy$

when $x = 0 \Rightarrow y = b$

$x = 1 \Rightarrow y = a$

$$\therefore \beta(m,n) = \int_a^b (\frac{y-b}{a-b})^{m-1}(1-(\frac{y-b}{a-b}))^{n-1}.\frac{1}{a-b}dy$$

$$= \int_a^b \frac{(y-b)^{m-1}}{(a-b)^{m-1}}.\frac{(a-y)^{n-1}}{(a-b)^{n-1}}.\frac{1}{a-b}dy$$

$$= \int_a^b \frac{(y-b)^{m-1}(a-y)^{n-1}}{(a-b)^{m+n-1}}dy$$

$$= \int_a^b \frac{(x-b)^{m-1}(a-x)^{n-1}}{(a-b)^{m+n-1}}dx$$

$$\int_a^b \frac{(x-b)^{m-1}(a-x)^{n-1}}{(a-b)^{m+n-1}}dx = \beta(m,n)$$

Hence $\int_a^b (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\beta(m,n)$

6. Show that $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\beta(\frac{m+1}{2},\frac{n+1}{2})$

Sol: $\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\cos^{n-1}\theta(\sin\theta\cos\theta)d\theta$

$$= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{m-1}{2}}(\cos^2\theta)^{\frac{n-1}{2}}(\sin\theta\cos\theta)d\theta$$

Put $\sin^2\theta = x$ so that $\sin\theta\cos\theta d\theta = \frac{dx}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\int_0^1 x^{\frac{m-1}{2}}(1-x)^{\frac{n-1}{2}}dx$$

$$= \frac{1}{2}\int_0^1 x^{(\frac{m+1}{2})-1}(1-x)^{(\frac{n+1}{2})-1}dx$$

$$= \frac{1}{2}\beta(\frac{m+1}{2},\frac{n+1}{2})$$

Aliter:

We have $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n) \to (1)$

Put p= 2m-1, q = 2n-1

$$m = \frac{(p+1)}{2}, n = \frac{(q+1)}{2}$$

From Eq (1)

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\beta(\frac{p+1}{2},\frac{q+1}{2})$$

Or

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta d\theta = \frac{1}{2}\beta(\frac{m+1}{2},\frac{n+1}{2})$$

162

Hence the Result.

Problems:

1. Express the following integral in terms of Beta function $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

Sol: Given $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$

$$put\ x^2 = y$$

$$\Rightarrow dx = \frac{dy}{2x} = \frac{1}{2}y^{\frac{-1}{2}}dy$$

When x= 0, y = 0;

When x = 1, y = 1.

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \int_0^1 \frac{y^{\frac{1}{2}}}{\sqrt{1-y}}\frac{1}{2}y^{\frac{-1}{2}}dy$$

$$= \frac{1}{2}\int_0^1 y(1-y)^{\frac{-1}{2}}dy = \frac{1}{2}\int_0^1 y^{1-1}(1-y)^{\frac{1}{2}-1}dy = \frac{1}{2}\beta(1,\frac{1}{2}).$$

2. Express the following integral in terms of Beta function $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Sol: Given $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Put $x^2 = 9y$

$$\Rightarrow dx = \frac{3}{2}y^{\frac{-1}{2}}dy$$

When x= 0, y = 0;

When x = 3, y = 1.

$$\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \int_0^3 (9-x^2)^{\frac{-1}{2}}dx = \int_0^1 (9-9y)^{-\frac{1}{2}}\frac{3}{2}y^{\frac{-1}{2}}dy$$

$$= \frac{3}{2}\int_0^1 (1-y)^{-\frac{1}{2}}\frac{1}{3}y^{\frac{-1}{2}}dy$$

$$= \frac{1}{2}\int_0^1 (1-y)^{\frac{1}{2}-1} y^{\frac{1}{2}-1} dy$$

$$= \frac{1}{2}\beta(\frac{1}{2},\frac{1}{2}).$$

3. Show that $\int_0^1 x^m (1-x^n)^p dx = \frac{1}{n}\beta(\frac{m+1}{n}, p+1)$

Sol: Given that $\int_0^1 x^m (1-x^n)^p dx$

Put $x^n = y$

$$\Rightarrow dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$$

When x= 0, y = 0;

When x = 1, y = 1.

$$\int_0^1 x^m (1-x^n)^p dx = \int_0^1 y^{\frac{m}{n}} (1-y)^p \frac{1}{n} y^{\frac{1-n}{n}} dy = \frac{1}{n}\int_0^1 y^{\frac{m+1-n}{n}} (1-y)^p dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1} (1-y)^{(p+1)-1} dy$$

$$= \frac{1}{n}\beta(\frac{m+1}{n}, p+1)$$

4. Show that $\int_{-1}^1 (1+x)^{m-1}(1-x)^{n-1} dx = 2^{m+n-1}\beta(m,n)$

Sol: we have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Put $x = \frac{1+y}{2} \Rightarrow dx = \frac{1}{2}dy$

When x= 0, y = -1;

When x= 1, y = 1

$$\beta(m,n) = \int_{-1}^1 \frac{(1+y)^{m-1}}{2^{m-1}}(1-\frac{1+y}{2})^{n-1} \frac{1}{2} dy$$

$$= \int_{-1}^{1} \frac{(1+y)^{m-1}(1-y)^{n-1}}{2^{m+n-1}} dy$$

$$= \frac{1}{2^{m+n-1}} \int_{-1}^{1} (1+x)^{m-1}(1-x)^{n-1} dx$$

$$\therefore \int_{-1}^{1} (1+x)^{m-1}(1-x)^{n-1} dx = 2^{m+n-1} \beta(m,n).$$

5. Prove that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$

Sol: Given that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$

Put $x^5 = y \Rightarrow x = y^{\frac{1}{5}}$

$$\Rightarrow dx = \frac{1}{5} y^{\frac{1}{5}-1} dy = \frac{1}{5} y^{\frac{-4}{5}} dy$$

Also x= 0, y =0; when x= 1, y =1

$$\therefore \int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \int_0^1 \frac{y^{\frac{1}{5}}}{\sqrt{1-y}} \frac{1}{5} y^{\frac{-4}{5}} dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{-3}{5}} (1-y)^{\frac{-1}{2}} dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{2}{5}-1} (1-y)^{\frac{1}{2}-1} dy$$

$$= \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2}).$$

6. Evaluate $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}}$ in terms of Beta function.

Sol: Given $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}} = \int_0^1 x^2 (1-x^5)^{\frac{-1}{2}} dx$

$$= \int_0^1 \frac{x^2}{x^4}(1-x^5)^{\frac{-1}{2}} x^4 dx$$

$$= \int_0^1 x^{-2}(1-x^5)^{\frac{-1}{2}} x^4 dx$$

Put $x^5 = y \Rightarrow 5x^4 dx = dy$

$$\Rightarrow x^4 dx = \frac{dy}{5}$$

Also x= 0, y =0; when x= 1,y =1

$$\therefore \int_0^1 \frac{x^2 dx}{\sqrt{1-x^5}} = \int_0^1 y^{\frac{-2}{5}}(1-y)^{\frac{-1}{2}} \frac{dy}{5}$$

$$= \frac{1}{5}\int_0^1 y^{\frac{3}{5}-1}(1-y)^{\frac{1}{2}-1} dy$$

$$= \frac{1}{5}\beta(\frac{3}{5},\frac{1}{2}).$$

7. Show that $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1}\beta(m+1,n+1)$

Sol: Given that $\int_a^b (x-a)^m (b-x)^n dx$

Put x = a+ (b-a) y (or $x = \frac{t-a}{b-a}$) so that dx = (b-a)dy

Also when x = a, y = 0 and x = b, y = 1

$$\int_a^b (x-a)^m (b-x)^n dx = \int_0^1 [(b-a)y]^m [b-a-(b-a)y]^n (b-a)dy$$

$$= \int_0^1 (b-a)^m y^m (b-a)^n (1-y)^n (b-a)dy$$

$$= (b-a)^{m+n+1}\int_0^1 y^m (1-y)^n dy$$

$$= (b-a)^{m+n+1}\int_0^1 y^{(m+1)-1}(1-y)^{(n+1)-1} dy$$

$$= (b-a)^{m+n+1}\beta(m+1,n+1).$$

8. Show that $\int_0^\infty \frac{x^{m-1}dx}{(x+a)^{m+n}} = a^{-n}\beta(m,n)$

Sol: we have $\beta(m,n) = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$

Put $x = \frac{t}{a} \Rightarrow dx = \frac{dt}{a}$

Also when x = 0, t= 0 and x =∞, t = ∞

$$\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{a^{m-1}(1+\frac{t}{a})^{m+n}}\frac{dt}{a} = \frac{1}{a^m}\int_0^\infty \frac{t^{m-1}a^{m+n}}{(a+t)^{m+n}}dt = a^n\int_0^\infty \frac{t^{m-1}}{(t+a)^{m+n}}dt$$

$$\frac{1}{a^n}\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(x+a)^{m+n}}dx$$

Hence $\int_0^\infty \frac{x^{m-1}dx}{(x+a)^{m+n}} = a^{-n}\beta(m,n)$

1.2 Gamma Function:

The definite integral $\int_0^\infty e^{-x}x^{n-1}dx$ is called the Gamma function and it's denoted by $\Gamma(n)$ and read as "Gamma of n".The integral converges only for $n > 0$.

Thus, $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$ where $n > 0$

Gamma function is also called Eulerian integral of the second kind.

Note: The integral $\int_0^\infty e^{-x}x^{n-1}dx$ does not converge if $n \le 0$

Properties of Gamma function:

1. Show that $\Gamma(1) = 1$

Sol: By the def of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$\therefore \Gamma(1) = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}x^0 dx = \int_0^\infty e^{-x}dx$

$$= [\frac{e^{-x}}{-1}]_0^\infty = -[e^{-\infty} - e^{-0}] = -[0-1] = 1$$

$\therefore \Gamma(1) = 1$

2. Show that $\Gamma(n) = (n-1)\Gamma(n-1)$ where $n > 1$

Sol: By the def of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$= [x^{n-1}\frac{e^{-x}}{-1}]_0^\infty - \int_0^\infty (n-1)x^{n-2}(\frac{e^{-x}}{-1})dx \quad \text{(Integrate by parts)}$$

$$= (n-1)\int_0^\infty e^{-x}x^{n-2}dx$$

$$= (n-1)\Gamma(n-1)$$

$\therefore \Gamma(n) = (n-1)\Gamma(n-1)$

Note: 1. $\Gamma(n+1) = n\Gamma(n)$

2. If n is a positive fraction then we can write

$\Gamma(n) = (n-1)(n-2) \ldots \ldots \ldots (n-r)\Gamma(n-r)$ Where $(n-r) > 0$

3. If n is non negative integer then $\Gamma(n+1) = n!$

Sol: From property 2, we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) \quad \text{(Again by property 2)}$$

$$= n(n-1)(n-2)\Gamma(n-2) \quad \text{(Again by property 2)}$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= n(n-1)(n-2)(n-3) \ldots \ldots \ldots 3.2.1\,\Gamma(1)$$

$$= n(n-1)(n-2)(n-3) \ldots \ldots \ldots 3.2.1$$

$$= n!$$

$\therefore \Gamma(n+1) = n!$

Problems:

1. Find $\Gamma\left(\frac{9}{2}\right)$

Sol: $\Gamma\left(\frac{9}{2}\right) = \left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$$= \frac{7}{2}.\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{7}{2}.\frac{5}{2}.\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi} \qquad \left[\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ \right]$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$$

2. Solve Find $\Gamma\left(\frac{13}{3}\right)$

Sol: $\Gamma\left(\frac{13}{3}\right) = \frac{13}{3}.\frac{10}{3}.\frac{7}{3}.\frac{4}{3}.\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$

Note: When n is a negative fraction

We have $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

3. Compute $\Gamma\left(-\frac{1}{2}\right)$

Sol: We know that $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Put $n = -\frac{1}{2}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\sqrt{\pi}}{-\frac{1}{2}}$$

$$\therefore \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

4. Compute $\Gamma\left(-\frac{5}{2}\right)$

Sol: We know that $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

169

$$\Gamma\left(-\frac{5}{2}\right)=\frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}}=-\frac{2}{5}\Gamma\left(-\frac{3}{2}\right)$$

$$=-\frac{2}{5}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}=\frac{4}{15}\Gamma\left(-\frac{1}{2}\right)$$

$$=\frac{4}{15}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}=-\frac{8}{15}\Gamma\left(\frac{1}{2}\right)=-\frac{8}{15}\sqrt{\pi}$$

$$\therefore \Gamma\left(-\frac{5}{2}\right)=\frac{-8\sqrt{\pi}}{15}$$

1.3 Relation between β & Γ functions:

Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Proof: By the definition of Γ - function

$$\Gamma(m)=\int_0^\infty e^{-x}x^{m-1}dx$$

Put $x=t^2 \Rightarrow dx=2tdt$

When x = 0, t = 0 & x = ∞, t = ∞

$$\Gamma(m)=\int_0^\infty e^{-t^2}(t^2)^{m-1}2dt=\int_0^\infty e^{-t^2}t^{2m-2+1}2dt$$

$$=2\int_0^\infty e^{-t^2}t^{2m-1}dt$$

$$\therefore \Gamma(m)=2\int_0^\infty e^{-x^2}x^{2m-1}dx$$

Similarly $\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dxdy$$

Transform it into polar coordinates

x= $r\cos\theta$,y = $r\sin\theta$,dxdy = $rdrd\theta$

r varies from 0 to ∞,θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta drd\theta$$

$$= 2\int_0^\infty e^{-r^2}\ r^{2(m+n)-1} dr.\ 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta$$

$$= \Gamma(m+n).\beta(m,n)$$

$$\therefore\ \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Problems:

1. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof: We know that $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Taking m = n = ½, we have

$$\beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = (\Gamma(\frac{1}{2}))^2 \text{.......................... (1)} \qquad \text{since } \Gamma(1) = 1$$

But $\beta(\frac{1}{2},\frac{1}{2}) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1} dx = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta d\theta$

171

Also when x =0, $\theta = 0$; when x =1, $\theta = \frac{\pi}{2}$

$$\therefore \beta(\frac{1}{2},\frac{1}{2}) = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta}\frac{1}{\cos\theta} 2\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta = 2(\theta)_0^{\frac{\pi}{2}} = 2(\frac{\pi}{2} - 0) = \pi \text{ ....(2)}$$

From (1) & (2) we have

$$[\Gamma(\frac{1}{2})]^2 = \pi$$

$$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

2. To show that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof: We have $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

Taking n = ½, we have $\Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-x} x^{\frac{-1}{2}} dx$

Put $x = t^2 \Rightarrow dx = 2tdt$

Also when x=0, t = 0; when $x \to \infty, t \to \infty$

$$\therefore \Gamma(\frac{1}{2}) = \int_0^{\infty} e^{-t^2} (t^2)^{\frac{-1}{2}} 2tdt = 2\int_0^{\infty} e^{-t^2} dt$$

$$\Rightarrow 2\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\therefore \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Note:

1. $\int_{-\infty}^{0} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

2. $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

3. $\Gamma(n)$ is defined when n > 0.

4. $\Gamma(n)$ is defined when n is a negative fraction but $\Gamma(n)$ is not defined when n = 0 and n is a negative integer.

3. Show that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Proof: We know that $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx$ (by one of the forms of Beta function)

Also we have $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Taking m+n = 1 so that m = 1-n, we get

$$\int_0^{\infty} \frac{x^{n-1}}{(1+x)}dx = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)}$$

$$\Gamma(1-n)\Gamma(n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)}dx \quad \text{.................(1)}$$

We have $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}dx = \frac{\pi}{2n}\operatorname{cosec}(\frac{2m+1}{2n})\pi$ where m > 0, n > 0 and n > m

Put $x^{2n} = t$ & $\frac{2m+1}{2n} = s$ we have

$$\int_0^{\infty} \frac{t^{(\frac{2m}{2n})} t^{\frac{1}{2n}}}{2n(1+t)t}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\frac{1}{2n}\int_0^{\infty} \frac{t^{(\frac{2m}{2n})} t^{\frac{1}{2n}-1}}{(1+t)}dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$\int_0^{\infty} \frac{t^{[(2m+1)/2n]-1}}{(1+t)}dt = \pi \operatorname{cosec} s\pi$$

$$\int_0^{\infty} \frac{t^{s-1}}{(1+t)}dt = \frac{\pi}{\sin s\pi}$$

173

$$\int_0^\infty \frac{x^{s-1}}{(1+x)}dx = \frac{\pi}{\sin s\pi} \; or \int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\pi}{\sin n\pi} \quad \text{.................(2)}$$

From (1) & (2) we have

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

1.4 Evaluation of definite integrals

4. To show that $\Gamma(n) = \int_0^1 (\log\frac{1}{x})^{n-1} dx$, $n > 0$

Sol: We have $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \rightarrow (1)$

Put $x = \log\frac{1}{y} = -\log y \Rightarrow y = e^{-x}$

$$\Rightarrow dy = -e^{-x} dx or dx = -\frac{1}{y} dy$$

Also when x=0, y= 1; when x=∞, y=0

Now (1) becomes

$$\Gamma(n) = -\int_1^0 (\log\frac{1}{y})^{n-1} y.\frac{1}{y} dy = \int_0^1 (\log\frac{1}{y})^{n-1} dy$$

$$\Gamma(n) = \int_0^1 (\log\frac{1}{x})^{n-1} dx$$

5. Evaluate $\int_0^1 x^4(1-x)^2 dx$

Sol: Given $\int_0^1 x^4(1-x)^2 dx = \int_0^1 x^{5-1}(1-x)^{3-1} dx$

174

$$=\beta(5,3)=\frac{\Gamma(5)\Gamma(3)}{\Gamma(8)}$$

$$=\frac{4!2!}{7!}$$

$$=\frac{2}{7\times6\times5}$$

$$=\frac{1}{105}$$

6. Evaluate $\int_0^2 x(8-x^3)^{\frac{1}{3}}dx$

Sol: Given that $\int_0^2 x(8-x^3)^{\frac{1}{3}}dx$

Put $x^3=8y \Rightarrow x=2y^{\frac{1}{3}} \Rightarrow dx=\frac{2}{3}y^{\frac{-2}{3}}dy$

When x = 0, y = 0; When x = 2, y = 1

$$\int_0^2 x(8-x^3)^{\frac{1}{3}}dx = \int_0^1 2y^{\frac{1}{3}}(8-8y)^{\frac{1}{3}}\frac{2}{3}y^{\frac{-2}{3}}dy$$

$$=\frac{8}{3}\int_0^1 y^{-\frac{1}{3}}(1-y)^{\frac{1}{3}}dy$$

$$=\frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{4}{3}-1}dy$$

$$=\frac{8}{3}\beta(\frac{2}{3},\frac{4}{3})=\frac{8}{3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3}+\frac{4}{3})}=\frac{8}{3}\frac{\Gamma(\frac{2}{3})(\frac{4}{3}-1)\Gamma(\frac{4}{3}-1)}{\Gamma(2)} \because \Gamma(n)=(n-1)\Gamma(n-1)$$

$$=\frac{8}{3}\frac{1}{3}\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})=\frac{8}{9}\Gamma(\frac{1}{3})\Gamma(1-\frac{1}{3})$$

$$=\frac{8}{9}\frac{\pi}{\sin(\frac{\pi}{3})}=\frac{16\pi}{9\sqrt{3}} \qquad [\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}]$$

7. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^{\frac{7}{2}}\theta d\theta$

Solution: We have $\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta = \frac{1}{2}\beta(m,n) \to (1)$

Put 2m – 1 = 5 & 2n - 1 = 7/2 so that m = 3 , n = 9/4

Eq (1) becomes

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^{\frac{7}{2}}\theta d\theta = \frac{1}{2}\beta(3,\frac{9}{4}) = \frac{1}{2}\frac{\Gamma(3)\Gamma(\frac{9}{4})}{\Gamma(3+\frac{9}{4})}$$

Since $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma(\frac{9}{4})}{\Gamma(\frac{21}{4})} = \frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{21}{4})} \qquad \because \Gamma(3) = 2! = 2$$

$$\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^{\frac{7}{2}}\theta d\theta = \frac{\Gamma(\frac{9}{4})}{\frac{17}{4}.\frac{13}{4}.\frac{9}{4}\Gamma(\frac{9}{4})} = \frac{64}{1989}$$

8. Evaluate $\int_0^{\infty} 3^{-4x^2} dx$

Solution: Given $\int_0^{\infty} 3^{-4x^2} dx$

$$\Rightarrow 3^{-4x^2} = e^{-4x^2 \log 3}$$

$$\Rightarrow \int_0^{\infty} 3^{-4x^2} dx = \int_0^{\infty} e^{-4x^2 \log 3} dx \qquad \because x = e^{\log x}$$

Put $2x\sqrt{\log 3} = y \Rightarrow dx = \frac{dy}{2\sqrt{\log 3}}$

$$\int_0^{\infty} 3^{-4x^2} dx = \int_0^{\infty} e^{-y^2} \frac{dy}{2\sqrt{\log 3}} = \frac{1}{2\sqrt{\log 3}}\int_0^{\infty} e^{-y^2} dy$$

$$\int_0^{\infty} 3^{-4x^2} dx = \frac{1}{2\sqrt{\log 3}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}} = \sqrt{\frac{\pi}{16\log 3}}$$

9. When n is a positive integer , Prove that $2^n\Gamma(n+\frac{1}{2}) = 1.3.5......(2n-1)\sqrt{\pi}$

Proof: We know that $\Gamma(n+1) = n\Gamma(n) \rightarrow (1)$

$$\Gamma(n+\frac{1}{2}) = \Gamma(n-\frac{1}{2}+1) = (n-\frac{1}{2})\Gamma(n-\frac{1}{2})$$

$$= (n-\frac{1}{2})\Gamma(n-\frac{3}{2}+1) = (n-\frac{1}{2})(n-\frac{3}{2})\Gamma(n-\frac{3}{2}), by (1)$$

$$= (n-\frac{1}{2})(n-\frac{3}{2})(n-\frac{5}{2})\Gamma(n-\frac{5}{2})$$

$$= \frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}\Gamma(\frac{2n-5}{2})$$

$$= \frac{2n-1}{2}.\frac{2n-3}{2}.\frac{2n-5}{2}..........\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2})$$ (by repeated use of (1) n times)

$$= \frac{(2n-1)(2n-3)(2n-5).........3.1}{2^n}\sqrt{\pi}$$

Hence $2^n\Gamma(n+\frac{1}{2}) = 1.3.5......(2n-1)\sqrt{\pi}$

10. Show that $2^{2n-1}\Gamma(n)(n+\frac{1}{2}) = \Gamma(2n)\sqrt{\pi}$

Proof : Since we have $\beta(m,\frac{1}{2}) = 2^{2m-1}\beta(m,m)$

Writing the above result in terms of Gamma function we have

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} or \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)}$$

$$\frac{\sqrt{\pi}}{\Gamma(m+\frac{1}{2})} = 2^{2m-1}\frac{\Gamma(m)}{\Gamma(2m)} or \Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

Hence $2^{2n-1}\Gamma(n)(n+\frac{1}{2}) = \Gamma(2n)\sqrt{\pi}$

11. Prove that $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = 0$ using β - Γ functions.

Proof:

177

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx = \int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}dx \qquad \text{Since } \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= \beta(9,15) - \beta(15,9)$$
$$= \beta(9,15) - \beta(9,15)$$
$$= 0$$

12. Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ Where n is a positive integer and m > -1.

Proof: Let $\int_0^1 x^m (\log x)^n dx$

Put $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$

Also $x = 0, t = \infty \;\&\; x = 1, t = 0$

$$\int_0^1 x^m (\log x)^n dx = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}dt)$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t} t^n dt = (-1)^n \int_0^\infty e^{-(m+1)t} t^{(n+1)-1} dt$$

$$= (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}} \qquad \because \int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}, n > 0, k > 0$$

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}$$

9. Assignments

S. No	Question	BL	CO
1	Derive the relation between Beta and Gamma functions.	2	3
2	Express the following integral in terms of Beta function $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$	3	4

178

3	Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ Where n is a positive integer and m > -1.	4	3
4	Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^{\frac{7}{2}} \theta d\theta$	2	3
5	Prove that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$	3	3
6	Show that $\Gamma(n) = (n-1)\Gamma(n-1)$ where $n > 1$	4	4

10. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	What is a beta function	1	3
2	Prove that $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$	1	3
3	Using definition of beta function prove $\beta(m,n) = \beta(n,m)$	1	3
4	What is a gamma function	3	3
5	Find $\Gamma\left(\frac{9}{2}\right)$	1	3
6	Prove that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$	1	4
7	Show that $\int_{-\infty}^0 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$	2	3
8	show that $\Gamma(n) = \int_0^1 (\log\frac{1}{x})^{n-1} dx$, n > 0	2	3

179

9	Evaluate $\int_0^\infty 3^{-4x^2}\,dx$	2	3
10	Evaluate $\int_0^2 x(8-x^3)^{\frac{1}{3}}\,dx$	2	3

11. Part B- Questions

S.No	Question	BL	CO
1	Derive the relation between Beta and Gamma functions.	1	1
2	Express the following integral in terms of Beta function $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$	2	1
3	Prove that $\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ Where n is a positive integer and m > -1.	2	1
4	Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^{\frac{7}{2}}\theta d\theta$	3	1
5	Prove that $\int_0^1 \frac{xdx}{\sqrt{1-x^5}} = \frac{1}{5}\beta(\frac{2}{5},\frac{1}{2})$	3	1
6	show that $\Gamma(n) = \int_0^1 (\log\frac{1}{x})^{n-1}\,dx$, n > 0	3	1

12. Supportive Online Certification Courses

LINEAR ALGEBRA -NPTEL online programme-SWAYAM.

13. Real Time Applications

S.No	Application
1	Games especially 3D
2	Encryption

14. Contents Beyond the Syllabus

1. Real time Applications
2. Quadratic Forms-Nature, signature, Rank

15. Prescribed Text Books & Reference Books

Text Book

1. "B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. "Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.

References:

1. R. K. Jain and S. R. K. Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.
2. George B. Thomas, Maurice D. Weir and Joel Hass, Thomas Calculus, 13/e, Pearson Publishers, 2013.
3. Glyn James, Advanced Modern Engineering Mathematics, 4/e, Pearson publishers, 2011.
4. Micheael Greenberg, Advanced Engineering Mathematics, 9th edition, Pearson edn
5. Dean G. Duffy, Advanced Engineering Mathematics with MATLAB, CRC Press
6. Peter O'neil, Advanced Engineering Mathematics, Cengage Learning.
7. R.L. Garg Nishu Gupta, Engineering Mathematics Volumes-I &II, Pearson Education

181