

9-9.5-12

EE24BTECH11064 - Harshil Rathan

Question:

For the following differential equation, find the particular solution satisfying the given condition:

$$x^2 dy + (xy + y^2) dx = 0; \quad y = 1 \quad x = 1$$

Solution:

Let us solve the given equation theoretically and then verify the solution computationally

$$x^2 dy + (xy + y^2) dx = 0 \quad (0.1)$$

$$\frac{dy}{dx} = -\frac{xy + y^2}{x^2} \quad (0.2)$$

$$\frac{dy}{dx} = -\frac{y}{x} - \left(\frac{y}{x}\right)^2 \quad (0.3)$$

Let

$$F(x, y) = -\frac{(xy + y^2)}{x^2}$$

$$F(\lambda x, \lambda y) = \frac{[\lambda x \cdot \lambda y + (\lambda y)^2]}{(\lambda x)^2} = \alpha^0 \times F(x, y) \quad (0.4)$$

Therefore, this is a homogeneous equation. We take the following substitution

$$y = vx \quad (0.5)$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad (0.6)$$

$$x \frac{dv}{dx} = \frac{v - 1}{v + 1} - v \quad (0.7)$$

Substituting values of x and $\frac{dy}{dx}$ in 0.2

$$v + x \frac{dv}{dx} = -\frac{[x \cdot vx + (vx)^2]}{x^2} = -v - v^2 \quad (0.8)$$

$$x \frac{dv}{dx} = -v(v + 2) \quad (0.9)$$

$$\frac{dv}{v(v + 2)} = -\frac{dx}{x} \quad (0.10)$$

$$\frac{1}{2} \frac{(v+2)-v}{v(v+2)} dv = -\frac{dx}{x} \quad (0.11)$$

$$\frac{1}{2} \left(\frac{1}{v} - \frac{1}{v+2} \right) dv = -\frac{dx}{x} \quad (0.12)$$

Integrating on Both sides

$$\frac{1}{2} \int \frac{1}{v} \cdot dv - \frac{1}{2} \int \frac{1}{v+2} \cdot dv = - \int \frac{dx}{x} \quad (0.13)$$

$$\frac{1}{2} \log v - \frac{1}{2} \log(v+2) = -\log x + \log c \quad (0.14)$$

$$\frac{1}{2} \log\left(\frac{v}{v+2}\right) = \log \frac{C}{x} \quad (0.15)$$

$$\frac{v}{v+2} = \left(\frac{C}{x}\right)^2 \quad (0.16)$$

By substituting $v = \frac{y}{x}$,

$$\frac{x^2 y}{y+2x} = C^2 \quad (0.17)$$

at $x=1$ and $y=1$

$$C^2 = \frac{1}{3} \quad (0.18)$$

Substituting $C^2 = \frac{1}{3}$ in 0.17

$$\frac{x^2 y}{y+2x} = \frac{1}{3} \quad (0.19)$$

$$y+2x = 3x^2 y \quad (0.20)$$

Now lets verify the solution computationally from the definition of $\frac{dy}{dx}$

$$y_{n+1} = y_n + \frac{dy}{dx} \cdot h \quad (0.21)$$

From the differential equation given,

$$\frac{dy}{dx} = -\frac{(xy+y^2)}{x^2} \quad (0.22)$$

Substituting 0.22 in 0.21

$$y_{n+1} = y_n + \left(-\frac{(xy+y^2)}{x^2} \right) \cdot h \quad (0.23)$$

The comparison between theoretical and simulation curves is shown in the figure, we can clearly see that both the curves are coincides which verifies our solution

