10.3.2.3.3

EE24BTECH11064 - Harshil Rathan

Question:
On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ find out whether the following pair of linear equations are consistent, or inconsistent.

$$\frac{3}{2}x + \frac{5}{3}y = 7$$
$$9x + 10y = 14$$

Solution:
The general form of a pair of linear equations is given as:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Here, the coefficients are:

$$a_1 = \frac{3}{2}, \quad b_1 = \frac{5}{3}, \quad c_1 = 7,$$
$$a_2 = 9, \quad b_2 = 10, \quad c_2 = 14.$$

We calculate the ratios:

$$\frac{a_1}{a_2} = \frac{1}{6}, \quad \frac{b_1}{b_2} = \frac{1}{6}, \quad \frac{c_1}{c_2} = \frac{1}{2}.$$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the given system of equations is inconsistent.
Solution using LU Decomposition
Simplifying and using matrix notation,

$$\frac{3}{2}x + \frac{5}{3}y = 7 \tag{0.1}$$
$$9x + 10y = 14 \tag{0.2}$$
$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 9 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} \tag{0.3}$$

Any non-sigular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{0.4}$$

The upper triangular matrix U is found by row reducing A,

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 9 & 10 \end{pmatrix} \xrightarrow{R_2->R_2-6R_1} \begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 0 & 0 \end{pmatrix} \tag{0.5}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{0.6}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = 6$.

Now,

$$A = \begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 9 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 0 & 0 \end{pmatrix} \tag{0.7}$$

Now we can get the solution to our problem by the two step process,

$$L\mathbf{y} = \mathbf{b} \tag{0.8}$$

$$U\mathbf{x} = \mathbf{y} \tag{0.9}$$

Using forward substitution to solve the first equation,

$$\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} \tag{0.10}$$

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -28 \end{pmatrix} \tag{0.11}$$

Now using back-substitution for the second equation,

$$\begin{pmatrix} \frac{3}{2} & \frac{5}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ -28 \end{pmatrix} \tag{0.12}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix} \tag{0.13}$$

The equation gives

$$0 = -28 \tag{0.14}$$

Which is a contradiction
This means that the system is inconsistent and has no solution. Therefore, there is no solution for x_1 and x_2

Numerical Computation for LU Decomposition

We want to decompose A as the product of a lower triangular matrix L and an upper triangular matrix U

$$A = LU \tag{0.15}$$

L is a lower triangular matrix with ones on the diagonal

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_{21} & 1 & 0 & \cdots & 0 \\ L_{31} & L_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & 1 \end{pmatrix} \tag{0.16}$$

U is an upper triangular matrix

$$\begin{pmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{12} & U_{22} & 0 & \cdots & 0 \\ U_{13} & U_{23} & U_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{1n} & U_{2n} & U_{3n} & \cdots & U_{nn} \end{pmatrix} \tag{0.17}$$

The first row of U is simply the first row of A

$$U_{1j} = A_{1j} \tag{0.18}$$

The first column of L is computed as

$$L_{i1} = \frac{A_{i1}}{U_{11}}, \quad \text{for} \quad i = 2, 3, \ldots, n. \tag{0.19}$$

Subsequent columns of U are computed as

$$U_{kj} = A_{kj} - \sum_{m=1}^{k-1} L_{km} U_{mj} \tag{0.20}$$

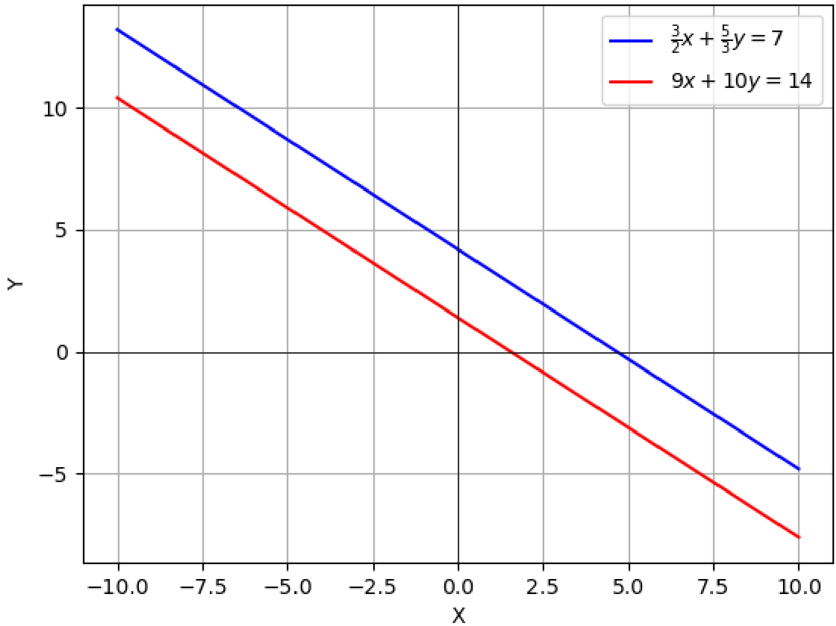
for $j = k, k+1, \ldots, n$

Subsequent columns of L are computed as

$$L_{ik} = \frac{A_{ik} - \sum_{m=1}^{k-1} L_{im} U_{mk}}{U_{kk}} \tag{0.21}$$

for $i = k+1, k+2, \ldots, n$

This systematic approach ensures that the matrix A is decomposed into L and U



$\frac{3}{2}x + \frac{5}{3}y = 7$
$9x + 10y = 14$
10
5
0
−5
Y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
X