11.16.3.15.1

EE24BTECH11064 - Harshil Rathan

Question: If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{8}$, find

i) $P(E \cup F)$

Solution:

Theoretical Solution:

For 2 Boolean variables A and B, the axioms of Boolean Algebra are defined as:

$$A + A' = 1 \tag{0.1}$$
$$A + A = A \tag{0.2}$$
$$AB = BA \tag{0.3}$$
$$A + B = B + A \tag{0.4}$$
$$AA' = 0 \tag{0.5}$$
$$P(1) = 1 \tag{0.6}$$
$$P(A + B) = P(A) + P(B), \text{ if } P(AB) = 0 \tag{0.7}$$

Using these axioms, we will try to prove that

$$P(A + B) = P(A) + P(B) - P(AB) \tag{0.8}$$

We will start by representing A and B as:

$$A = AB + AB' \tag{0.9}$$
$$B = AB + A'B \tag{0.10}$$
$$P(A) = P(AB) + P(AB') \tag{0.11}$$
$$P(B) = P(AB) + P(A'B) \tag{0.12}$$

On adding (12) and (13),

$$A + B = AB + AB + AB' + A'B \tag{0.13}$$
$$A + B = AB + AB' + A'B \tag{0.14}$$
$$P(A + B) = P(AB + AB' + A'B) \tag{0.15}$$
$$P(A + B) = P(AB) + P(AB') + P(A'B) \tag{0.16}$$
$$P(A + B) = P(AB) + P(A) - P(AB) + P(B) - P(AB) \tag{0.17}$$
$$\implies P(A + B) = P(A) + P(B) - P(AB) \tag{0.18}$$

Using the given values of $P(A), P(B)$ and $P(AB)$,

$$P(A + B) = 0.25 + 0.50 - 0.125 \tag{0.19}$$
$$P(A + B) = 0.625 \tag{0.20}$$

Therefore, the value of $P(A+B)$ is 0.625.

Computational Solution:
Let X_1 be an indicator random variable of the event A.
X_1 is defined as:

$$X_1 = \begin{cases} 1, & A \\ 0, & A' \end{cases} \tag{0.21}$$

Let X_2 be the indicator random variable of the event B.
X_2 is defined as:

$$X_2 = \begin{cases} 1, & B \\ 0, & B' \end{cases} \tag{0.22}$$

Let X_3 be the indicator random variable of the event AB.
X_3 is defined as:

$$X_3 = \begin{cases} 1, & AB \\ 0, & (AB)' \end{cases} \tag{0.23}$$

The PMF of the random variable X_1 is:

$$p_{X_1}(n) = \begin{cases} p_1, & n = 1 \\ 1 - p_1, & n = 0 \end{cases} \tag{0.24}$$

The PMF of the random variable X_2 is:

$$p_{X_2}(n) = \begin{cases} p_2, & n = 1 \\ 1 - p_2, & n = 0 \end{cases} \tag{0.25}$$

The PMF of the random variable X_3 is:

$$p_{X_3}(n) = \begin{cases} p_3, & n = 1 \\ 1 - p_3, & n = 0 \end{cases} \tag{0.26}$$

where,

$$p_1 = 0.25 \tag{0.27}$$
$$p_2 = 0.50 \tag{0.28}$$
$$p_3 = 0.125 \tag{0.29}$$
$$\tag{0.30}$$

Let Y be the random variable which is defined as follows:

$$Y = X_1 + X_2 - X_3 \tag{0.31}$$

But we know that X_3 can never be 0 when X_1 and X_2 are 1 and vice versa. So, Y is another indicator random variable whose PMF is defined as:

$$p_Y(n) = \begin{cases} p, & n = 1 \\ 1 - p, & n = 0 \end{cases} \tag{0.32}$$

$$E(Y) = E(X_1 + X_2 - X_3) \tag{0.33}$$

$$E(Y) = E(X_1) + E(X_2) - E(X_3) \tag{0.34}$$

$$1.(p) + 0.(1 - p) = 1.(p_1) + 0.(1 - p_1) + 1.(p_2) + 0.(1 - p_2) - 1.(p_3) - 0.(1 - p_3) \tag{0.35}$$

$$p = p_1 + p_2 - p_3 \tag{0.36}$$

Through our definition, we know that,

$$P(A) = p_1 \tag{0.37}$$

$$P(B) = p_2 \tag{0.38}$$

$$P(AB) = p_3 \tag{0.39}$$

Therefore, by comparison of the axiom

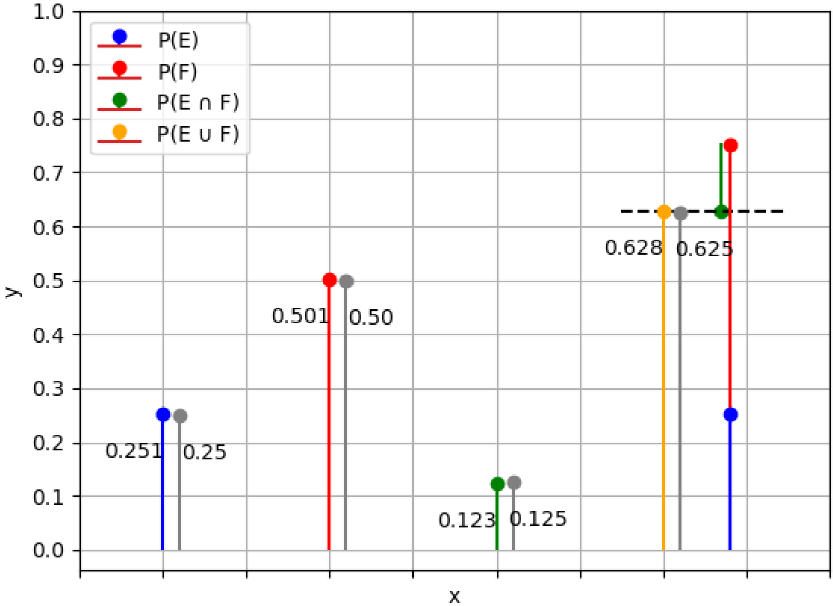
$$P(A + B) = P(A) + P(B) - P(AB) \tag{0.40}$$

and the equation (39),

$$p = P(A + B) \tag{0.41}$$

$$P(A + B) = 0.25 + 0.50 - 0.125 \tag{0.42}$$

$$\implies P(A + B) = 0.625 \tag{0.43}$$


P(E)
P(F)
P(E ∩ F)
P(E ∪ F)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
y
x
0.251
0.25
0.501
0.50
0.123
0.125
0.628
0.625