8-8.3-3

EE24BTECH11064 - Harshil Rathan

Question:
Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 1$ and $y = 4$.
Solution:
THe equation of the curve is

$$y = 4x^2 \tag{0.1}$$

$$x = \frac{\sqrt{y}}{2} \tag{0.2}$$

w.k.t that 0.1 is an upward parabola symmetric about y-axis
Point of intersection
Expressing the equation of parabola in matrix form $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0 \tag{0.3}$$

The general form of a line equation can be expressed as

$$\mathbf{m}^\top \mathbf{x} = c \tag{0.4}$$

For $y = 1$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = 1 \tag{0.5}$$

For $y = 4$

$$\mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c = 4 \tag{0.6}$$

Intersection of a line and a conic is given by,

$$\kappa_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{0.7}$$

On substituting and solving
The intersection points in the first quadrant are

$$\left(\frac{1}{2}, 1\right), (1, 4) \tag{0.8}$$

Theory
There are two ways to solve the above integral, Theoretically and Computationally

(trapezoid method). We shall compare the results obtained by both methods.

$$\int_1^4 x\,dx = \int_1^4 \left|\frac{\sqrt{y}}{2}\right| dy \tag{0.9}$$

$$\frac{1}{2}\int_1^4 y^{\frac{1}{2}}\,dy = \frac{1}{2}\left|\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^4 \tag{0.10}$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] \tag{0.11}$$

$$= \frac{1}{3}\left(4\sqrt{4} - 1\right) = \frac{1}{3}(8-1) = \frac{7}{3} \ = 2.3333 \text{ sq. units.} \tag{0.12}$$

Solution using Trapezoidal Rule

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \ldots, x_n$ such that they are equally spaced with step-size h.

To apply the trapezoidal rule, we discretize the x-axis into n equally spaced intervals, with $h = \frac{1}{n}$. The points on the x-axis are $x_0, x_1, x_2, \ldots, x_n$, where:

$$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h, \ldots, \quad x_n = 1 \tag{0.13}$$

Sum of all trapezoidal areas is given by

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{0.14}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{0.15}$$

Substituting $y(x_n) = 4x_n^2$, we get:

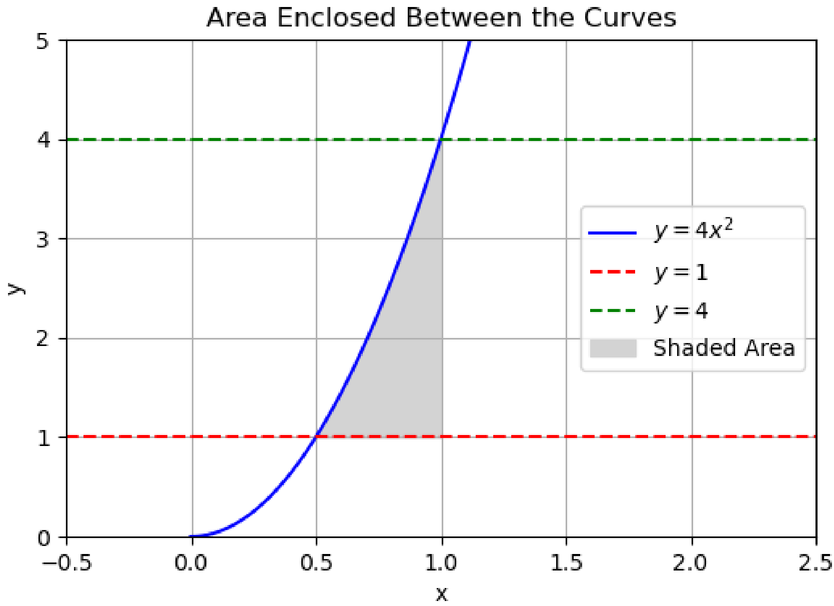
$$A = h\left(\frac{1}{2}\left(4x_0^2 + 4x_n^2\right) + \sum_{i=1}^{n-1} 4x_i^2\right) \tag{0.16}$$

Since $x_0 = 0$ and $y(x_0) = 0$, this simplifies to:

$$A = h\left(\frac{1}{2}\left(4x_n^2\right) + \sum_{i=1}^{n-1} 4x_i^2\right) \tag{0.17}$$

Let $A\left(x_n\right)$ be the area enclosed by the curve $y\left(x\right)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A\left(x_n + h\right) = A\left(x_n\right) + \frac{1}{2}h\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.18}$$



Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{0.19}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(4x_{n+1}^2 + 4x_n^2\right) \tag{0.20}$$

So the final simplified formula is

$$A_{n+1} = A_n + 2h \cdot \left(x_{n+1}^2 + x_n^2\right) \tag{0.21}$$

The solution is hence verified with the theoretical solution
Area required is 2.3333 sq.units