MA - 2015

EE24BTECH11064 - Harshil Rathan

1) Let τ_1 be the usual topology on $\mathbb{R}$. Let τ_2 be the topology on $\mathbb{R}$ generated by
$$\mathcal{B} = \{(a, b) \subset \mathbb{R} : -\infty < a < b < \infty\}.$$
Then the set
$$\{x \in \mathbb{R} : 4\sin^2 x \leq 1\} \cup \left\{\frac{n\pi}{2}\right\}_{n\in\mathbb{Z}}$$
is
   a) closed in $(\mathbb{R}, \tau_1)$ but NOT in $(\mathbb{R}, \tau_2)$
   b) closed in $(\mathbb{R}, \tau_2)$ but NOT in $(\mathbb{R}, \tau_1)$
   c) closed in both $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$
   d) neither closed in $(\mathbb{R}, \tau_1)$ nor closed in $(\mathbb{R}, \tau_2)$

2) Let X be a connected topological space such that there exists a non-constant continuous function $f : X \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual topology. Let $f(X) = \{f(x) : x \in X\}$. Then
   a) X is countable but $f(X)$ is uncountable
   b) $f(X)$ is countable but X is uncountable
   c) both $f(X)$ and X are countable
   d) both $f(X)$ and X are uncountable

3) Let d_1 and d_2 denote the usual metric and the discrete metric on *mathbbR*, respectively. Let $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_2)$ be defined by $f(x) = x, x \in \mathbb{R}$. Then
   a) f is continuous but f^{-1} is NOT continuous
   b) f^{-1} is continuous but f is NOT continuous
   c) both f and f^{-1} are continuous
   d) neither f nor f^{-1} is continuous

4) If the trapezoidal rule iwth single interval $[0, 1]$ is exact for approximating the integral $\int_0^1 \left(x^3 - cx^2\right) dx$, then the value of c is equal to ___________.

5) Suppose that the Newton-Raphson method is applied to the equation $2x^2 + 1 - e^{x^2} = 0$ with an initial approximation x_0 sufficiently close to zero. Then, for the root x=0, the order of convergence of the method is equal to ___________.

6) The maximum possible order of a homogeneous linear ordinary differential equation with real constant coefficients having $x^2 \sin x$ as a solution is equal to ___________.

7) The Lagrangian of a system in terms of polar coordinates (r, θ) is given by

$$L = \frac{1}{2}mr^2 + \frac{1}{2}m\left(r^2 + r^2\theta^2\right) - mgr\left(1 - \cos\theta\right)$$

where m is the mass, g is the acceleration dur to gravity and s' denotes the derivative of s with respect to time. Then the equation of motion are

a) $2r' = r\theta'^2 - g\left(1 - \cos\theta\right)$, $\frac{d}{dt}\left(r^2\theta'\right) = -gr\sin\theta$
b) $2r' = r\theta^2 + g\left(1 - \cos\theta\right)$, $\frac{d}{dt}\left(r^2\theta\right) = -gr\sin\theta$
c) $2r' = r\theta^2 - g\left(1 - \cos\theta\right)$, $\frac{d}{dt}\left(r^2\theta\right) = gr\sin\theta$
d) $2r' = r\theta^2 + g\left(1 - \cos\theta\right)$, $\frac{d}{dt}\left(r^2\theta\right) = gr\sin\theta$

8) If $y\left(x\right)$ satisfies the initial value problem

$$\left(x^2 + y\right)dx - xdy, y\left(1\right) = 2$$

then $y\left(2\right)$ is equal to ___________.

9) It is known that Bessel functions $J_n\left(x\right)$, for $n \geq 0$, satisfy the identity

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = J_0\left(x\right) + \sum_{n=1}^{\infty} J_n\left(x\right)\left(t^n + \frac{\left(-1\right)^n}{t^n}\right)$$

for all $t > 0$ and $x \in \mathbb{R}$. The value of $J_0\left(\frac{\pi}{3}\right) + 2\sum_{n=1}^{\infty} J_{2n}\left(\frac{\pi}{3}\right)$ is equal to

10) Let X and Y be two random variables having the joint probability density function

$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the conditional probability $P\left(X \leq \frac{2}{3} | Y - \frac{3}{4}\right)$ is equal to

(A) $\frac{5}{9}$
(B) $\frac{2}{3}$
(C) $\frac{7}{9}$
(D) $\frac{8}{9}$

11) Let $\Omega = (0, 1]$ be the sample space and let $P\left(\right)$ be a probability fucntion defined by

$$P((0, x]) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ x & \text{if} \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then $P\left(\frac{1}{2}\right)$ is equal to ___________.

12) Let X_1, X_2 and X_3 be independent and identically distributed random variables with $E(X_1) = 0$ and $E(X_1^2 = \frac{15}{4})$. If $\phi : (0, \infty \to (0, \infty))$ si deifned through the conditional

expectation

$$\phi t = E(X_1^2|X_1^2 + X_2^2 + X_3^2 = t), t > 0$$

then $E(\phi((X_1 + X_2)^2))$ is equal to ____________.

13) Let $X \sim$ Poisson(λ), where $\lambda > 0$ is unknown. If $\delta(X)$ is the unbiased estimator of $g(\lambda) = e^{-\lambda}\left(3\lambda^2 + 2\lambda + 1\right)$, then $\sum_{k=0}^{\infty} \delta(k)$ is equal to ____________.