MTL100 - Introduction to Real Number System

$$\sum_{k=0}^{n} a_k x^k = 0$$

[p/q is a solution, where p & q are integers having no common factor]

$$\sum_{k=0}^{n} a_k \left(p/q\right)^k = 0$$

Multiplying throughout by q^n,

$$\sum_{k=0}^{n} a_k p^k q^{n-k} = 0 \longrightarrow (*)$$

$$\Rightarrow \quad a_n p^n + \sum_{k=0}^{n-1} a_k p^k q^{n-k} = 0$$

$$\Rightarrow \quad a_n p^n = -\left(\sum_{k=0}^{n-1} a_k p^k q^{n-k}\right)$$

$$= -q\left(\sum_{k=0}^{n-1} a_k p^k q^{n-k-1}\right)$$

$\overset{?}{\Rightarrow}$ q divides $a_n p^n$

$\Rightarrow$ q divides a_n.

$\exists$ no rational r s.t. $r^2 = 2$.

$x^2 - 2 = 0$

$a_2 = 1$, $a_1 = 0$ & $a_0 = -2$

If p/q is a solution to $x^2 - 2 = 0$, then

q divides 1 & p divides -2.

$\therefore$ The possibilities for q are 1 & -1

$|||^{ly}$ the possibilities for p are $1, -1, 2$ & -2

But $1, -1, 2$ & -2 are not the solutions.

$\therefore$ the equation doesn't have any rational solution.

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

Real numbers ($\mathbb{R}$).

$\mathbb{R} \supseteq \mathbb{Q}$

- $(\mathbb{R}, +, \cdot)$ is an ordered field
- Extra property : "completeness axiom or LUB axiom".

Let A be a non empty subset of $\mathbb{R}$.

- We say that M is a maximal element of A if
  $M \in A$ & $x \leq M \quad \forall\ x \in A$
- We say that $u \in \mathbb{R}$ is an upper bound of A if
  $x \leq u \quad \forall\ x \in A$
- We say that $l \in \mathbb{R}$ is a least upper bound (LUB) or supremum if
  - i) l is an upper bound for A
  - ii) if l' is any other upper bound for A, then $l \leq l'$.
- We say that A is bounded above if A has an

upper bound.

Example

1) $A = [0, 1]$

Maximum $= 1$ Minimum $= 0$

Upper bound $= m$, any thing greater than or equal to 1

LUB $= 1$.

such that

2) $A = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$

Maximum = doesn't exist

Upper bound $= m$, any number greater than or equal to 1.

LUB $= 1$ (why?)

3) $A = [0, \infty)$ A is unbounded.

Remark. Let A be a non empty subset of $\mathbb{R}$ & let l be a LUB for A. Then, for any given $\varepsilon > 0$ $l - \varepsilon$ is not an upper bound and therefore there exists $x \in A$

such that $l - \varepsilon \leq x \leq l$.

Exercise. Define the terms, minimum, lower bound, & greatest lower bound (GLB) or infimum and illustrate them through examples.

Exercise. Show that $\sup(-A) = -\inf(A)$ for any nonempty subset A of $\mathbb{R}$, where

$$-A = \{-x : x \in \mathbb{R}\}.$$

<u>Example</u>

$$A = \{1 - 1/n : n \in \mathbb{N}\}$$

Minimum $= 0$ Maximum $=$ doesn't exist

GLB $= 0$ LUB $= 1$ (why?)

Completeness axiom or LUB axiom. Every non empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Remark. Completeness axiom doesn't hold for $\mathbb{Q}$. For example, let

$$A = \{ r \in \mathbb{Q} : r^2 < 2 \} \subseteq \mathbb{Q}$$

Show that A has no LUB in $\mathbb{Q}$.

Archimedean property. For any $x \in \mathbb{R}$ $\exists$ $N \in \mathbb{N}$ such that $x < N$

Proof. Suppose that this is false. Then $\exists$ $x \in \mathbb{R}$ such that $x \geq n$ for all $n \in \mathbb{N}$.

$x \geq n$ for all $n \in \mathbb{N}$

$\Rightarrow$ x is an upper bound for $\mathbb{N}$

As $\mathbb{N}$ is non-empty, by LUB axiom, $\mathbb{N}$ has a LUB, say $\alpha \in \mathbb{R}$. Then, $\alpha - 1$ is not an upper bound for $\mathbb{N}$

$\Rightarrow$ there exists $n \in \mathbb{N}$ such that $\alpha - 1 < n$

$\Rightarrow$ $\alpha < n+1 \in \mathbb{N}$

$\Rightarrow$ α is not an upper bound for $\mathbb{N}$, which is a contradiction.

Notations

- $\exists$ - there exists
- $\forall$ - for all
- $\ni$ - such that