MTL 100 - Introduction to Real Number System

Groups 6 - 10

1) Neither start or stop the recording
2) Unless & until necessary donot unmute

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Course

- Website - https://web.iitd.ac.in/~siva/2021_MTL100.html
- K. A. Ross, Elementary analysis: The theory of Calculus

  G. B. Thomas & R. L. Finney, Calculus

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$$\lim_{n \to \infty} f(n) \qquad \lim_{x \to a} f(x)$$

$$\lim_{n \to \infty} a_n \qquad \lim_{x \to \infty} f(x)$$

$\mathbb{N}$ - set of natural numbers

$\mathbb{Z}$ - set of integers

$\mathbb{Q}$ - set of rational numbers

$\mathbb{R}$ - set of real numbers

$\mathbb{C}$ - set of complex numbers - we will not deal with this

$\mathbb{N}$

- $\mathbb{N}$ is an infinite set
- 1 is the least element of $\mathbb{N}$
- $\mathbb{N}$ has no maximal element

On each of the above sets we have two binary operations:

$+$ – addition

$\cdot$ – multiplication

$\exists$ an ordering defined on the above sets: given two x, y, any one of the following holds:

$x < y$, $x = y$, $x > y$.

Well-ordering property of $\mathbb{N}$: Every nonempty subset S of $\mathbb{N}$ has a least element.

Principle of Mathematical induction. If S is a nonempty subset of $\mathbb{N}$ such that $1 \in S$ & if $n \in S \Rightarrow n+1 \in S$, then $S = \mathbb{N}$.

Remark. Well-ordering principle & the principle of Mathe-
-matical induction are equivalent.

$\mathbb{Z}$

- $\mathbb{Z} \supseteq \mathbb{N}$ & hence an infinite set
- $\mathbb{Z}$ has an additive identity '0' : $0+a = a = a+0$ for all $a \in \mathbb{Z}$.
- Every element in $\mathbb{Z}$ has an additive inverse, i.e., for any $a \in \mathbb{Z}$ $\exists$ $-a \in \mathbb{Z}$ such that
$$a + (-a) = 0 = (-a) + a.$$

$\mathbb{Q}$

- $\mathbb{Q} \supseteq \mathbb{Z} \supseteq \mathbb{N}$
- It has an additive identity

- Every element has an additive inverse
- $\mathbb{Q} \setminus \{0\}$ has a multiplicative identity $\underline{1}$
- Every element of $\mathbb{Q} \setminus \{0\}$ has a multiplicative inverse.

  $x \cdot 1/x = 1$
- Distributive law:

$$a \cdot (b+c) = a \cdot b + a \cdot c$$

- $\mathbb{Q}$ is an ordered field
  - $a < b$ $\Rightarrow a+c < b+c$ for all c
  - $a < b$ $\Rightarrow ac < bc$ whenever $c > 0$.

<u>Short comings</u>

$\mathbb{Q}$ has "gaps" in some sense.

<u>Example</u>. $\nexists$ no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Rational root theorem. Consider the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

where $n \in \mathbb{N}$, $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, $a_0 \neq 0$, $a_n \neq 0$. Suppose that a rational number p/q satisfies this equation, where p & q have no common factors. Then p must divide a_0 & q must divide a_n.

<u>Example</u> $\exists$ no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

This is equivalent to saying that the polynomial equation $x^2 - 2 = 0$ has no solution in $\mathbb{Q}$.

Suppose that $x = p/q$ satisfies the equation $x^2 - 2 = 0$. By Rational root theorem, then p should belong to $\{1, -1, 2, -2\}$ & q should belong to $\{1, -1\}$. But none of the combinations will be a solution to $x^2 - 2 = 0$.

Proof of the theorem. We have

$$a_n (p/q)^n + a_{n-1} (p/q)^{n-1} + \cdots + a_1 (p/q) + a_0 = 0$$

$\Rightarrow a_n p \;+ a_{n-1} p \;\; q + \cdots + a_1 p q \;\; + a_0 q = 0$

$\Rightarrow a_n p^n = -q \left(a_{n-1} p^{n-1} + \cdots + a_1 p q^{n-2} + a_0 q^{n-1} \right)$

$\overset{?}{\Rightarrow}$ q divides $a_n p^n$

$\Rightarrow$ q divides a_n $\quad$ ($\because$ p & q have no common factor)

|||rly, p divides a_0 (Exercise).