Fair Division of a Graph

CS656 : Algorithmic Game Theory

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Definitions and Notations

Problem: Fair allocation of indivisible goods where each allocated bundle is connected in an underlying graph.

Definition: An instance of the connected fair division problem (CFD) is a triple I = (G, N, U) where

- G = (V, E) is an undirected graph,
- N = {1, . . . , n} is a set of players, or agents,
- U is an n-tuple of utility functions $u_i : V \to R_{\geq 0}$, where $\Sigma_{v \in V} u_i(v) = 1$ for each $i \in N$.

Elements of V are referred as items and number of items are denoted by m.

For each $X \subseteq V$, we set $u_i(X) = \Sigma_{v \in X} u_i(v)$, i.e. utility functions are additive.

Definitions and Notations

Two players $i, j \in N$ are of the same type if $u_i(v) = u_j(v)$ for all $v \in V$. Number of player types in a given instance I is denoted by p.

Definition: An allocation is a function $\pi : N \to 2^V$ assigning each player a bundle of items. An allocation π is valid if for each player $i \in N$ the bundle $\pi(i)$ is connected in G and no item is allocated twice, so that $\pi(i) \cap \pi(j) = \varnothing$ for each pair of distinct players $i, j \in N$. We say that a valid allocation π is

- proportional if $u_i(\pi(i)) > 1/n$ for all $i \in N$,
- envy-free if $u_i(\pi(i)) > u_i(\pi(j))$ for all $i, j \in N$, and
- complete if $\cup_{i \in N} \pi(i) = V$.

Definitions and Notations

We also consider maximin share (MMS) allocations , adapting the usual definition to our setting as follows. Given an instance I = (G, N, U) of CFD with G = (V, E), let Π_n denote the space of all partitions of V into n connected pieces. The maximin share guarantee of a player i ∈ N is

$$mms_i(I) = \max_{(P_1,\dots,P_n) \in \Pi_n} \min_{j \in \{1,\dots,n\}} u_i(P_j).$$

A valid allocation π is a maximin share (MMS) allocation if we have $u_i(\pi(i)) > mms_i(I)$ for each player i ∈ N.

Definitions and Notations

We consider the following computational problems that all take an instance $I = (G, N, U)$ of the connected fair division problem as input. For computational purposes, we assume utilities take values in rational numbers.

- PROP-CFD: Does I admit a proportional valid allocation?
- COMPLETE-EF-CFD: Does I admit a complete envy-free valid allocation?
- MMS-CFD: Does I admit an MMS allocation?

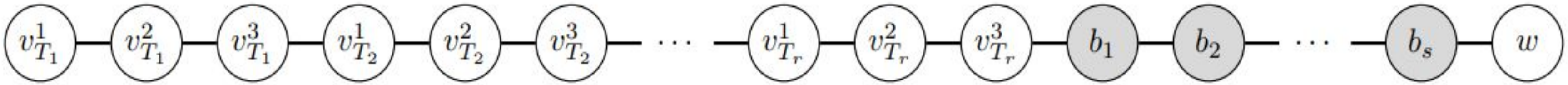
We assume that the number of items m is at least as large as the number of players n, so that no player is left behind. Also, given a positive integer k, we write $[k]$ to denote the set $\{1, \ldots, k\}$.

Proportionality

Consider an instance J = (X, $\mathcal{T}$) of X3C; for each T $\in \mathcal{T}$, we denote the elements of T by x_T^1, x_T^2, x_T^3. Now let us construct an instance I of PROP-CFD with vertices:

- Three vertices v_T^1, v_T^2, v_T^3 for each set T $\in \mathcal{T}$
- A set of vertices B = $\{b_1, b_2, \ldots, b_s\}$
- A dummy vertex w.

Graph looks like



Proportionality

The Players are:

- One player i_T for each $T \in \mathcal{T}$
- One player i_x for each $x \in X$
- One dummy player d

Total number of players are n= 3s + r + 1

Proportionality

The utilities are:

- $u_{iT}(v) = 1/(3n)$, if $v = v_T^k$
- $u_{iT}(v) = 1/n$, if $v \in B$
- $u_{iT}(v) = (n\text{-}s\text{-}1)/n$, if $v = w$
- $u_{iT}(v) = 0$, otherwise
- $u_{ix}(v) = 1/n$, if $v = v_T^k$ and $x \in T$
- $u_{ix}(v) = (n\text{-}3p_x)/n$, if $v = w$
- $u_{ix}(v) = 0$, otherwise
- $u_d(v) = 1$ if $v = w$, otherwise 0.

Proportionality

Now $u_{iT}(V) = 3/(3n) + s/n + (n\text{-}s\text{-}1)/n = 1$.

$u_{ix}(V) = 3p_x/n + (n\text{-}3p_x)/n = 1$ and $u_d(V) = 1$.

For proportional allocation:

- d must receive vertex w .
- i_x must receive $v_T^{\,k}$ such that $x \in T$.
- i_T must receive either $v_T^{\,1}, v_T^{\,2}, v_T^{\,3}$ (a triple interval) or a vertex from B

Proportionality

Theorem 3.1: PROP-CFD is NP-complete even if G is a path.
Proof: We describe a polynomial-time reduction from the NP-complete problem EXACT-3-COVER (X3C) to PROP-CFD.
An instance of X3C is given by a set of elements $X = \{x_1, x_2, \ldots, x_{3s}\}$ and a family $\mathcal{T} = \{T_1, T_2, \ldots, T_r\}$ of three-element subsets of X.
It is an 'yes'-instance if and only if X can be covered by s sets from $\mathcal{T}$.
This problem remains NP-complete if for each element $x \in X$, its frequency $p_x = |\{T \in \mathcal{T} : x \in T\}|$ is at most 3.

Proportionality

If J has a cover $\mathcal{T}^{\#}$ of size s we can get a proportional allocation as follows:

- Let f be a bijective mapping from $\mathcal{T}^{\#}$ to B
- For each T $\in \mathcal{T}^{\#}$ player i_T gets f(T)
- For each T $\notin \mathcal{T}^{\#}$ player i_T gets a triple interval v_T^1, v_T^2, v_T^3
- Player i_x gets v_T^k such that $x = x_T^k$ and T $\in \mathcal{T}^{\#}$.

Now each player is assigned one connected piece with value at least 1/n.

Proportionality

Conversely, if I has a valid proportional allocation then the number of T-players assigned to triple intervals is $r - s$ as $|B| = s$. So, the number of triple intervals left for players of type i_x are s. Respective sets constitute an exact cover.

Finding proportional allocation for a star graph is easy.

Proportionality

Theorem: PROP-CFD is solvable in polynomial time if G is a star.

Proof: let c denote the center of the star. now we will search for a valid allocation assigning c to each player $i \in N$.

For each assignment of c we will do as follows:

Construct a bipartite graph $H = (Z, Z_0, L)$ with $Z = N \setminus \{i\}$, $Z_0 = V \setminus \{c\}$.
$\{j, v\} \in L$ if and only if $u_j(v) \geq 1/n$. The weight of this edge is $u_i(v)$.

Let us say matching in H is perfect if all the vertices in Z are mapped and is one to one.

Proportionality

Now if I has a proportional valid allocation that assigns c to i if and only if H admits a perfect matching M with $w(M) \leq (n-1)/n$.
This is because player i gets c along with remaining vertices of Z_0, say set y.
For $u_i(y) \geq 1/n$, $w(M)$ should be $\leq (n-1)/n$ as $u_i(V) = 1$.
A minimum-weight perfect matching can be computed in polynomial time

Proportionality

If underlying graph is a path and all the players are of same type, then we can find the allocation greedily, if possible in linear time.

Definition: A problem is slice-wise polynomial (XP) with respect to a parameter k if each instance I of this problem can be solved in time $|I|^{f(k)}$ where f is a computable function.

Proportionality

Theorem: PROP-CFD is in XP with respect to the number of player types p if G is a path.

Proof: Let $G = (V, E)$, where $V = \{v_1, \ldots, v_m\}$, $E = \{\{v_i, v_{i+1}\} : i \in [m - 1]\}$
Suppose there are n_t players of type t, for $t \in [p]$
Let $V_0 = \phi$ and $V_i = \{v_1, \ldots, v_i\}$, $i > 1$
For $i = 0, \ldots, m$, and a collection of indices $j_1, \ldots, j_p$ such that $0 \leq k \leq n$ for each $k \in [p]$
Let $Ai [j_1, \ldots, j_p] = 1$ if there exists a valid partial allocation π of V_i with j_k happy agents of type k, $k \in [p]$, 0 otherwise. For $i = 1, \ldots, m$, we have $A_i [j_1, \ldots, j_p] = 1$ if and only if there exists a value $s < i$ and $t \in [p]$ such that $A_s[j_1, \ldots, j_{t-1}, \ldots, j_p] = 1$ and a player of type t values the set of items $\{v_{s+1}, \ldots, v_i\}$ at 1/n or higher

Proportionality

A proportional allocation exists if $A_m[j_1, . . . , j_p] = 1$ for some collection of indices $j_1, . . . , j_p$ such that $j_t > n_t$ for all $t \in [p]$. There are at most $(m + 1)(n + 1)^p$ values to compute.

Each value can be found in $O(mt)$ time. Thus, PROP-CFD is in XP with respect to p

Proportionality

If the number of agents n is bounded by a constant and graph is a tree, then
We can make n partitions of tree by cutting off n-1 edges which mean a total of ${}^{m-1}C_{n-1}$ possible partitions. We can check a proportional valid allocation among these in polynomial time. So, PROP-CFD on trees is in XP wrt n.

Definition: A problem is fixed parameter tractable (FPT) with respect to a parameter k if each instance I of this problem can be solved in time f(k)poly(|I|) where f is a function that depends only on k.

Proportionality

Proposition: When utilities are encoded in binary, PROP-CFD is NP-complete even for $n = 2$, $p = 1$, and even if the underlying graph G is bipartite

Proof: An instance of PARTITION is given by a set of integers $J = \{a_i : i \in H\}$ such that $i \in H\ a_i = 2k$. It is a ‘yes’-instance if and only if there exists a subset of indices $H' \subset H$ such that $i \in H'\ a_i = i \in H \setminus H'\ a_i = k$

Let an instance I derived from above PARTITION with $G = (V, E)$ where $V = \{v_i : i \in H\} \cup \{w1, w2\}$ and $E = \{\{v_i, w_1\}, \{v_i, w_2\} : i \in H\}$. Let the two players have utility $u(v_i) = a_i/(2k)$ for $i \in H$ and $u(w_1) = u(w_2) = 0$. Therefore, I has a proportional valid allocation iff J is a yes instance of partition.

Envy-freeness

Theorem 4.1 : COMPLETE-EF-CFD is NP-complete even if G is a star

Proof : We describe a reduction from INDEPENDENT SET.

An instance of INDEPENDENT SET is given by an undirected graph (W, L) and an integer k.

It is a 'yes'-instance if and only if (W, L) contains an independent set of size k

Envy-freeness

Given an instance (W, L) of INDEPENDENT SET, we construct an instance of COMPLETE-EF-CFD as follows.

1. For each vertex $w \in W$, we create an item w and a player i_w
2. For each edge $l \in L$ we create an item l and a player i_l
3. Create a set of dummy items D with $|D| = k$, as well as an item c and a player i_c

The graph G is a star with center c and set of leaves $W \cup L \cup D$

Envy-freeness

Utilities

1. For each $w \in W$, $u_{iw}(w) = 1/(k + 1)$ and $u_{iw}(d) = 1/(k + 1)$ for each $d \in D$.
2. For each $l \in L$ with $l = \{x, y\}$, we set $u_{il}(l) = 3/7$, $u_{il}(x) = u_{il}(y) = 2/7$
3. $u_{ic}(c)=1$

All other utilities are 0.

Envy-freeness

If there exists an independent set $X \subseteq W$ of size k,

An allocation π can be constructed as:

1. Player i_c receives $X \cup \{c\}$
2. For $w \in W \setminus X$, player i_w receives w
3. For $w \in X$, player i_w receives one item in D
4. For $l \in L$, player i_l receives item l

Envy-freeness

π is a complete valid allocation

Player i_c does not envy any other player

Vertex players i_w don't envy any other player

Edge players i_l don't envy any other player

Hence π is a complete envy-free valid allocation.

Envy-freeness

Conversely let there be a complete envy-free allocation π .

Then player i_c would receive the item c and every other player receives atmost one item.

Since π is complete, i_c gets at least k leaf items and since π is envy-free, $\pi(i_l) = \{l\}$

The bundle of player i_c can't contain more than one dummy item , hence it has atleast one item $w \in W$

$\pi(i_c)$ cannot contain any dummy item

$\pi(i_c)$ consists of c and k vertex items

No two vertices with their vertex items in $\pi(i_c)$ can have an edge between them

Hence, $\pi(i_c) \setminus \{c\}$ forms an independent set of size k in (W, L)

Envy-freeness

Theorem 4.2： The problem COMPLETE-EF-CFD is NP-complete even if G is a path

Proof： We describe a reduction from the NP-complete problem EXACT-3-COVER (X3C)

An instance of X3C is given by a set of elements $X = \{x_1, x_2, \dots, x_{3s}\}$ and a family $\mathcal{T} = \{T_1, T_2, \dots, T_r\}$ of three-element subsets of X

It is a ‘yes’-instance if and only if X can be covered by s sets from $\mathcal{T}$

This problem remains NP-complete if for each element $x \in X$, its frequency $p_x = |\{T_i \in \mathcal{T} : x \in T_i\}|$ is at most 3.

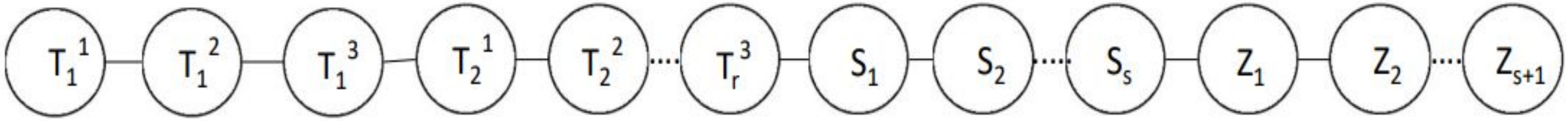
Envy-freeness

Consider an instance $J = (X, \mathcal{T})$ of X3C; for each $T_i \in \mathcal{T}$, we denote the elements of T_i by x_i^1, x_i^2, x_i^3.

We construct an instance I of COMPLETE-EF-CFD with **vertices**:

1. Three vertices T_i^1, T_i^2, T_i^3 for each set $T_i \in \mathcal{T}$
2. A set of vertices $S = \{S_1, S_2, \ldots, S_s\}$
3. A set of dummy vertices $Z = \{Z_1, Z_2, \ldots, Z_{s+1}\}$

The edges between the vertices are as follows



Envy-freeness

Players:

1. One player t_i for each $T_i \in \mathcal{T}$
2. One player y_j for each $x \in X$
3. One dummy player z_k for each $Z_i \in Z$

Envy-freeness

For each y_j , let H_j be an arbitrary set of $s+1-p_j$ vertices among the dummy vertices Z

- $u_{yj}(v)=3s$ if $v= T_i^k$ and $y_j= x_i^k$
- $u_{yj}(v)=3s$ if $v \in H_j$
- $u_{yj}(v)=0$ otherwise

- $u_{ti}(v)=s$ if $v= T_i^k$ for $k=1,2,3$
- $u_{ti}(v)=3s$ if $v = S_j$ for $k=1,2,..,s$
- $u_{ti}(v)=0$ otherwise

- $u_{zk}(v)=3s$ if $v= Z_1, Z_2,\dots, Z_{s+1}$
- $u_{zk}(v)=0$ otherwise

Envy-freeness

If J has a cover $\mathcal{T}'$ of size s we can get a complete envy-free allocation as follows:

- Player y_j gets T_i^k if $y_j = x_i^k$ and $T_i \in \mathcal{T}'$
- Let f be a bijective mapping from $\mathcal{T}'$ to S
- For each $T_i \in \mathcal{T}'$ player t_i gets $f(T_i)$
- For each $T \notin \mathcal{T}'$ player t_i gets a triple interval T_i^1, T_i^2, T_i^3
- Players z_k get $Z_1, Z_2, \ldots, Z_{s+1}$ in any order

Envy-freeness

Each player is assigned one connected piece of value 3s

None can envy y_j and t_i receiving S_k

Players t_i cannot envy each other

Player y_j cannot envy t_i receiving T_i^1 , T_i^2, T_i^3

Hence this allocation is envy-free

Envy-freeness

Conversely, if there exists a valid complete envy-free allocation π in I

Each player should receive atleast 3s

Player t_i either receives one S-vertex or T_i^1 , T_i^2, T_i^3

As the number of S-vertices is only s, the number of t-players assigned to triple intervals is r − s. So the number of T-vertices available for y-players is 3s and they constitute an exact cover

Envy-freeness

Theorem 4.3 : COMPLETE-EF-CFD is in XP with respect to the number of player types p if G is a path

Proof : For an allocation to be envy-free, all pieces assigned to players of a given type should have the same value to players of that type

There are at most m^2 many possibilities for the utility of a player.

In the algorithm,for each player type,it guesses the utility that players of that type assign to their pieces (there are at most $(m^2)^p$ possibilities) and proceeds similar to dynamic programming algorithm

Maximin share guarantee

Proposition: Let I = (G, N, U) be an instance of CFD where G is a tree and let $(q_i)_{i \in N}$ be an n-tuple of rational numbers. If $mms_i(I) > q_i$ for all $i \in N$, then there exists a valid allocation π such that each player $i \in N$ receives the bundle of value at least q_i , i.e., $u_i(\pi(i)) > q_i$. Moreover, one can compute such an allocation in polynomial time.

Notation: For each $X \subseteq V$, we let $G \setminus X$ denote the subgraph induced by $V \setminus X$; also, we denote the restriction of u_i to X by $u_i|_X$.

π can be computed by a recursive algorithm.

Maximin share guarantee

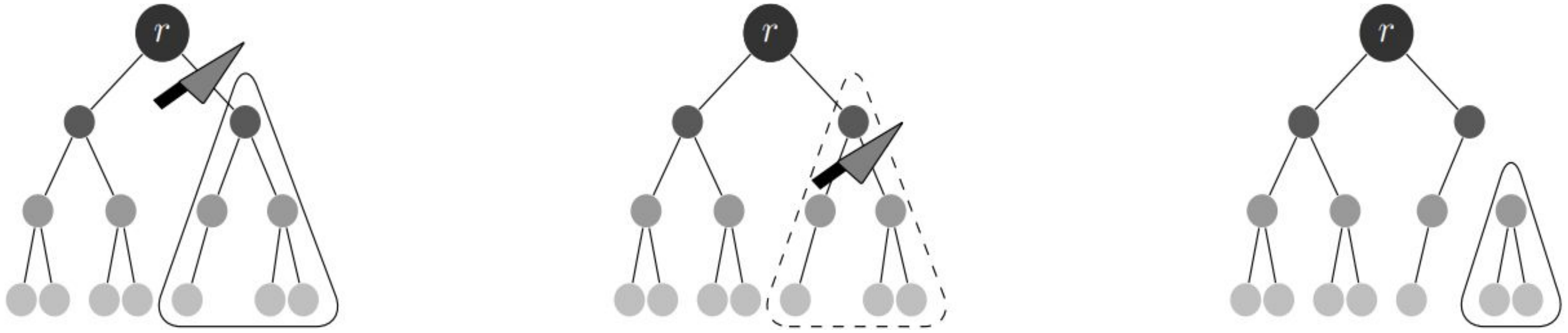
Algorithm A: first check whether input graph G' has a value of at least q_i for each player $i \in N'$; if this is not the case, A fails. Then, if there is only one player, return the allocation that assigns all items to that player.

When there are at least two players, turn the graph into a rooted tree by choosing an arbitrary node as its root;

Let D(v) be the set of descendants of a vertex v in this rooted tree. Then each player i finds a vertex v_i such that his value for $D(v_i)$ is at least q_i , but for each child w of v his value for D(w) is less than q_i .

Maximin share guarantee

Now allocate $D(v_i)$ to the last-diminisher i whose vertex v_i has minimal height (such a pair (i, v_i) can be found by starting at the root of the tree and moving downwards). The player i exits with the bundle D(vi), and call same algorithm A on the remaining instance.



Maximin share guarantee

It is obvious that A runs in polynomial time.

Let $I_n, \ldots, I_1$ be the sequence of instances constructed by A when called on I and $(q_i)i \in N$, where $I_k = (G_k, N_k, U_k)$ and $|N_k| = k$ (i.e., $I = I_n$).

If A does not fail on any of these instances, then $A(I,(q_i)i \in N)$ returns a desired allocation: each agent is allocated a bundle that she values at least as highly as her given value q_i.

We need to show that none of the recursive calls fails. To this end, we will prove the following lemma.

Maximin share guarantee

Lemma: . $mms_j(I_k) \geq q_j$ for all $k \in [n]$ and all $j \in N_k$.

Let us try to prove this by backward induction.Let this is true for n. Suppose that lemma is true for k, we will try to prove it for k-1. Let $i \in N_k - N_{k-1}$, then for each $j \in N_{k-1}$ we have,

$mms_j (I_k) \geq q_j$. Let this partition be $\boldsymbol{P} = (P_1, \ldots, P_k)$. So, $u_j(P_l) \geq q_j$ for all $l \in [k]$.

Let $v_i \in P_1$, let us remove this P_1 and get the new graph G_{k-1} with k-1 connected components. Let this partition be $\boldsymbol{P}^\#$. By construction itself, we have $u_j(P^\#) \geq q_j$ for each $P^\# \in \boldsymbol{P}^\#$ i.e., $mms_j(I_{k-1}) \geq q_j$.

Now when A is called on I_k , we have $u_i(V_k) \geq mms_i(I_k) \geq q_i$. V_k is set of vertices in G_{k-1}. So, this algorithm never fails.

Maximin share guarantee

Lemma: For an instance $I = (G, N, U)$ of CFD where G is a tree, and a player $i \in N$, we can compute $mms_i(I)$ in polynomial time.

Proof: Fix a player $i \in N$. If $u_i(v)$ is represented as x_v/y_v, where x_v and y_v are integers, set $u_i'(v) = u_i(v)\prod_{v\in V} y_v$. Let $mms_i'(I)$ be the maximin share of player i with respect to these new utilities. Then $mms_i'(I)$ is an integer between 0 and mL^{m+1}, where L $= \max_{v\in V} \max\{x_v, y_v\}$ and

$$mms_i(I) = mms_i'(I) /(\prod_{v\in V} y_v)$$

Calculating $mms_i(I)$ is the same as maximizing the worst payoff for the instance I'' where all players are copies of player i.

Maximin share guarantee

So, $I'' = (G, N'', u'')$ where N'' is set of n copies of i, $u''_j = u'_i$ for all $j \in N''$. Let this has a valid allocation π with $u''_j(\pi(j)) \geq 1/4$ for each $j \in N''$. If such allocation exists, then by recursive algorithm we can get the partition, we can get the value of q by doing the binary search in its range found before. This can be found in $O((m+1) \log L)$ calls to function where each function call may take $O(mt)$. So, overall running time is in polynomial of m and log L.

Our next example shows that an MMS allocation may not exist on a cycle of 8 vertices.

Maximin share guarantee

Example: Consider an instance I = (G, N, U) of CFD where G = (V, E) with $V = \{ v_i \mid i = 1, 2, \ldots, 8 \}$, $E = \{ \{v_i, v_{i+1}\} \mid i = 1, 2, \ldots, 7 \} \cup \{\{v_1, v_8\}\}$, $N = \{1, 2, 3, 4\}$, and the utilities are given as follows

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Players 1 & 2	1	4	4	1	3	2	2	3
Players 3 & 4	4	4	1	3	2	2	3	1

To normalize to 1, each utility is divided by 20.

Maximin share guarantee

With partition P1 = $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}\}$,

$mms_1(I) = mms_2(I) \geq ¼$

With partition P2 = $\{\{v_2, v_3\}, \{v_4, v_5\}, \{v_6, v_7\}, \{v_8, v_1\}\}$,

$mms_3(I) = mms_4(I) \geq ¼$

Now suppose I has an MMS allocation π, then each player should get atleast two vertices because no one vertex has a value ≥ ¼ for any player. So, the partitions possible are only P1 and P2

Maximin share guarantee

Suppose π cuts graph into P1, then Players 3 & 4 has value ≥ ¼ for (v_1 , v_2), So one has to get the value less than ¼. Same goes with P2 . So, at least one player is getting less than maxmin share in all allocations. So, there exist no MMS allocation.

Thank You