Lecture Notes on Numerical Weather Prediction

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Abstract
Notes of Lectures and addional information from books.

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1. Lecture 1 06/01/2025

Numerical Weathering Problem (NWP) was first proposed by Bjerkives around 1900. It is mathematical initial value problem (IVP).

Initial value Problem (IVP) $\rightarrow$ simple pendulum.

$$\ddot{\theta} + \omega^2 \theta = 0 \tag{1}$$

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0 \tag{2}$$

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{3}$$

Eq.(1) and (2) are second order linear ordinary differential equation, whose solution Eq.(3) has 2 constants of integration A and B. Here θ and t are the dependent and independent variable since Eq.(1) and (2) have only one independent variable.

Values of A and B will depend on initial condition.

Since ODE is second order, 2 initial condition are needed at initial time, say $t = 0$. Which are:

$$\left.\begin{aligned} \theta(t=0) &= 1 \\ \frac{\theta(t=0)}{dt} &= 0 \end{aligned}\right\} \tag{4}$$

Eq.(2) and initial conditions Eq.(4) are together called **Mathematical IVP**. For any physical system the following two requirements are needed:

1. The equation (ODE or PDE) that governs the evolution of the above system.
2. The initial state of the system.

7 independent variables **(u,v,w,T,ρ,p,q)**.

Surface area of Earth = $4\pi R^2$ = $4\pi(6.37 \times 10^{12}) \approx 5.1 \times 10^{14}$ m^2

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7 independent variables **(u,v,w,T,ρ,p,q)** therefore we need 7 Governing equations (system of 7 coupled non-linear partial differential equations):

1. Conservation of masss (continuity equation).
2. Conservation of momentum in rotating frame of refrence (3 scalar equations, one each corresponding to scalar component of velocity).
3. Conservation of energy (Thermodynamic energy equation).
4. Conservation of moisture (moisture continuity equation).
5. Equation of state (Ideal gas equation).

Euler discription of fluid motion is more convinent because of dependance on time and above 7 equations.

Total advective and convective time of lagrangian is given by:

$$\underbrace{\frac{DT}{Dt}}_{\text{Lagrangian Derivative}} = \underbrace{\underbrace{\frac{\partial T}{\partial t}}_{\text{Local derivative}} + \underbrace{\vec{V}\cdot\nabla T}_{\text{Advective Term}}}_{\text{Euler Derivative}}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{5}$$

Using first law of Thermodynamics, Rate of heat is given by:

$$\begin{aligned} d\dot{q} &= d\dot{u} + d\dot{w} \\ \frac{DU}{Dt} &= \frac{Dq}{Dt} - \frac{Dw}{Dt} \\ C_v\frac{DT}{Dt} &= \frac{Dq}{Dt} - p\frac{D\alpha}{Dt} \end{aligned}$$

where $\frac{Dq}{Dt}$ is rate at which heating of air parcel due to non-adiabatic process, this change can happen via radiation, convection, conduction, latent heat while phase change.

$$\frac{DU}{Dt} = \vec{F}_{\text{net}} + \vec{F}_{\text{coriolis}} \tag{6}$$

This above Eq.(6) is convective derivative equation involving non-linear terms (i.e. $u\frac{\partial T}{\partial x}$, $v\frac{\partial T}{\partial y}$, $w\frac{\partial T}{\partial z}$).

Continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0 \tag{7}$$

Let grid of following resolutions:

- $1^\circ \times 1^\circ \rightarrow 3\times 10^6$ grid cells $\therefore$ no. of variables $\rightarrow 7\times 3\times 10^6$.
- $5^\circ \times 5^\circ \rightarrow 1.3\times 10^5$ grid cells $\therefore$ no. of variables $\rightarrow 7\times 1.3\times 10^5$.
- $20^\circ \times 20^\circ \rightarrow 9\times 10^3$ grid cells $\therefore$ no. of variables $\rightarrow 7\times 9\times 10^3$.
- $25^\circ \times 25^\circ \rightarrow 6\times 10^3$ grid cells $\therefore$ no. of variables $\rightarrow 7\times 6\times 10^3$.

These are even larger than entire country, which means that we can't above to find the change of varibles with these grids. This we don't have a way to determine initial condition, if we try to use interpolation, it will cause errors which will grow with time since atmosphere is chaotic and dynamic system.

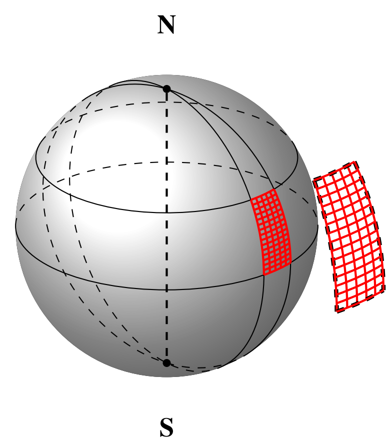

Figure 1. Figure showing the grid

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$$u = \bar{u} + u' \tag{8}$$

Here, u is the velocity field, which is decomposed into a mean component $\bar{u}$ and a fluctuating component u'.

Navier-Stokes Equation The general Navier-Stokes equation is given by:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} \\ + \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \end{aligned} \tag{9}$$

Reynolds-Averaged Navier-Stokes (RANS) Equation Applying Reynolds decomposition ($u = \bar{u} + u'$) and averaging leads to the RANS equation:

$$\begin{aligned} &\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} - f\bar{v} = \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} \\ &\quad + \gamma\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) \\ &\quad + \underbrace{\frac{1}{\rho}\left(\frac{\partial\left(-\rho\overline{u'u'}\right)}{\partial x} + \frac{\partial\left(-\rho\overline{u'v'}\right)}{\partial y} + \frac{\partial\left(-\rho\overline{u'w'}\right)}{\partial z}\right)}_{\text{Reynolds stress tensor}} \end{aligned} \tag{10}$$

The Reynolds stress tensor represents the transport of momentum due to turbulent fluctuations.

Nonlinear Term Expansion Expanding the nonlinear term $u\frac{\partial u}{\partial x}$ using Reynolds decomposition:

$$\begin{aligned} u\frac{\partial u}{\partial x} &= (\bar{u} + u')\frac{\partial(\bar{u} + u')}{\partial x} \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}. \end{aligned}$$

Appliing Reynolds averaging rules:

$$\begin{aligned} &\overline{u} = \overline{\bar{u} + u'} = \overline{\bar{u}} + \overline{u'} \\ &\overline{u} = \bar{u} + \overline{u'} \quad \Rightarrow \quad \overline{u'} = 0. \end{aligned}$$

Thus, the fluctuating component u' averages out to zero over time, leaving only the mean component $\bar{u}$ in the averaged equations.