Lecture Notes on Numerical Weather Prediction

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Abstract
Notes of Lectures and addional information from books.

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1.1 Introduction to Numerical Weather Prediction

Numerical Weathering Problem (NWP) was first proposed by Bjerkives around 1900. It is mathematical initial value problem (IVP).
Initial value Problem (IVP) $\rightarrow$ simple pendulum.

$$\ddot{\theta} + \omega^2 \theta = 0 \tag{1.1}$$

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0 \tag{1.2}$$

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1.3}$$

Eq.(1.1) and Eq.(1.2) are second order linear ordinary differential equation, whose solution .Eq.(1.3) has 2 constants of integration A and B. Here θ and t are the dependent and independent variable since Eq.(1.1) and Eq.(1.2) have only one independent variable.
Values of A and B will depend on initial condition.
Since ODE is second order, 2 initial condition are needed at initial time, say $t = 0$. Which are:

$$\left.\begin{array}{l} \theta(t=0) = 1 \\ \frac{\theta(t=0)}{dt} = 0 \end{array}\right\} \tag{1.4}$$

Eq.(1.2) and initial conditions Eq.(1.4) are together called **Mathematical IVP**. For any physical system the following two requirements are needed:

1. The equation (ODE or PDE) that governs the evolution of the above system.
2. The initial state of the system.

7 independent variables **(u,v,w,T,ρ,p,q)**.
Surface area of Earth = $4\pi R^2$ = $4\pi(6.37 \times 10^{12}) \approx 5.1 \times 10^{14}$ m^2

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7 independent variables **(u,v,w,T,ρ,p,q)** therefore we need 7 Governing equations (system of 7 coupled non-linear partial differential equations):

1. Conservation of masss (continuity equation).
2. Conservation of momentum in rotating frame of refrence (3 scalar equations, one each corresponding to scalar component of velocity).
3. Conservation of energy (Thermodynamic energy equation).
4. Conservation of moisture (moisture continuity equation).
5. Equation of state (Ideal gas equation).

Euler discription of fluid motion is more convinent becasue of dependence on time and above 7 equations.
Total advective and convective time of lagrangian is given by:

$$\underbrace{\frac{DT}{Dt}}_{\text{Lagrangian Derivative}} = \underbrace{\underbrace{\frac{\partial T}{\partial t}}_{\text{Local derivative}} + \underbrace{\vec{V}\cdot\nabla T}_{\text{Advective Term}}}_{\text{Euler Derivative}} \tag{2.1}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{2.2}$$

Using first law of Thermodynamics, Rate of heat is given by:

$$d\dot{q} = d\dot{u} + d\dot{w}$$
$$\frac{DU}{Dt} = \frac{Dq}{Dt} - \frac{Dw}{Dt}$$
$$C_v\frac{DT}{Dt} = \frac{Dq}{Dt} - p\frac{D\alpha}{Dt}$$

where $\frac{Dq}{Dt}$ is rate at which heating of air parcel due to non-adiabatic process, this change can happen via radiation, convection, conduction, latent heat while phase change.

$$\frac{DU}{Dt} = \vec{F}_{\text{net}} + \vec{F}_{\text{coriolis}} \tag{2.3}$$

This above Eq.(2.3) is convective derivative equation involving non-linear terms (i.e. $u\frac{\partial T}{\partial x}$, $v\frac{\partial T}{\partial y}$, $w\frac{\partial T}{\partial z}$).
Continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0 \tag{2.4}$$

Consider grid on globe as shown in figure(1).
Let grid of following resolutions:

- $1^\circ \times 1^\circ \to 3\times10^6$ grid cells $\therefore$ no. of variables $\to 7\times3\times10^6$.
- $5^\circ \times 5^\circ \to 1.3\times10^5$ grid cells $\therefore$ no. of variables $\to 7\times1.3\times10^5$.
- $20^\circ \times 20^\circ \to 9\times10^3$ grid cells $\therefore$ no. of variables $\to 7\times9\times10^3$.
- $25^\circ \times 25^\circ \to 6\times10^3$ grid cells $\therefore$ no. of variables $\to 7\times6\times10^3$.

These are even larger than entire country, which means that we can't above to find the change of varibles with these grids. This we don't have a way to determine initial condition, if we try to use interpolation, it will cause errors which will grow with time since atmosphere is chaotic and dynamic system.

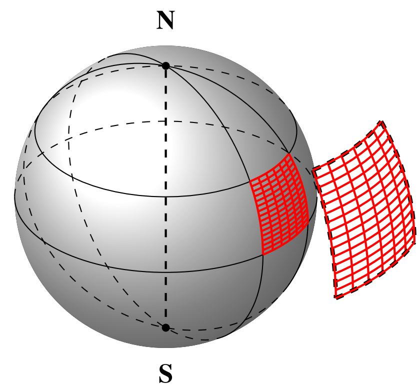

Figure 1. Figure showing the grid

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$$u = \bar{u} + u' \tag{3.1}$$

Here, u is the velocity field, which is decomposed into a mean component $\bar{u}$ and a fluctuating component u'.
Navier-Stokes Equation The general Navier-Stokes equation is given by:

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} \\ + \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\end{aligned} \tag{3.2}$$

Reynolds-Averaged Navier-Stokes (RANS) Equation Applying Reynolds decomposition ($u = \bar{u} + u'$) and averaging leads to the RANS equation:

$$\begin{aligned}\underbrace{\frac{\partial \bar{u}}{\partial t}}_{\text{Local acceleration}} + \underbrace{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z}}_{\text{Advection}} - \underbrace{f\bar{v}}_{\text{Coriolis force}} = \underbrace{\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}}_{\text{Pressure gradient force}} \\ + \underbrace{\gamma\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right)}_{\text{Viscous dissipation}} \\ + \underbrace{\frac{1}{\rho}\left(\frac{\partial\left(-\rho\overline{u'u'}\right)}{\partial x} + \frac{\partial\left(-\rho\overline{u'v'}\right)}{\partial y} + \frac{\partial\left(-\rho\overline{u'w'}\right)}{\partial z}\right)}_{\text{Reynolds stress tensor}}\end{aligned} \tag{3.3}$$

The Reynolds stress tensor represents the transport of momentum due to turbulent fluctuations.
Nonlinear Term Expansion Expanding the nonlinear term $u\frac{\partial u}{\partial x}$ using Reynolds decomposition:

$$\begin{aligned}u\frac{\partial u}{\partial x} &= (\bar{u} + u')\frac{\partial(\bar{u} + u')}{\partial x} \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}.\end{aligned}$$

Appliing Reynolds averaging rules:

$$\begin{aligned}\bar{u} &= \overline{\bar{u} + u'} \\ \bar{u} &= \bar{\bar{u}} + \overline{u'} \\ \bar{u} &= \bar{u} + \overline{u'} \quad \Rightarrow \quad \overline{u'} = 0.\end{aligned}$$

Thus, the fluctuating component u' averages out to zero over time, leaving only the mean component $\bar{u}$ in the averaged equations.
We have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.4}$$

Substituting u, v and w in above Eq.(3.4), we get:

$$\begin{aligned}\overline{\frac{\partial(\bar{u} + u')}{\partial x} + \frac{\partial(\bar{v} + v')}{\partial y} + \frac{\partial(\bar{w} + w')}{\partial z}} = 0 \\ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0\end{aligned} \tag{3.5}$$

The term $\frac{\partial(\overline{u'u'})}{\partial t}$ represents the rate of change of kinetic energy per unit mass due to turbulent fluctuations. It can be expressed as:

$$\frac{\partial(\overline{u'u'})}{\partial t} = \frac{\partial\left(\rho\overline{u'v'}\right)}{\partial y} + \dots$$

This term involves higher-order correlations between velocity fluctuations, which complicates the equation system.
The closure problem arises in the Reynolds-Averaged Navier-Stokes (RANS) equations because the number of dependent variables (unknowns) exceeds the number of equations available. For instance: - $\bar{u}$ is an unknown. - $\overline{u'v'}$ (a Reynolds stress term) introduces additional unknowns.
To resolve this, closure modelsare used, which provide approximations for higher-order terms based on known variables.
For example, consider the term $\overline{u'w'}$. Using a simple closure model:

$$\overline{u'w'} = -k\frac{\partial \bar{u}}{\partial x},$$

where k is a proportionality constant (often related to eddy viscosity). Here, $\bar{u}$ is already an unknown, so no additional variables are introduced, avoiding further complexity.
This is an example of first-order closurewhere higher-order terms are approximated using first-order variables.
Types of Closure Models

1. **First-Order Closure**:
   - Simplifies higher-order terms using known variables and gradients (e.g., eddy viscosity models).
   - Example: $\overline{u'w'} = -k\frac{\partial \bar{u}}{\partial x}$.
   - Advantage: Computationally efficient but may lack accuracy in complex flows.
2. **One-Point Closure**:
   - Approximates turbulence at a single point using local flow properties.
   - Example: Mixing length models, where turbulent viscosity is proportional to local shear.
3. **Second-Order Closure**:
   - Directly models second-order correlations like $\overline{u'u'}$ and $\overline{u'v'}$ by solving additional transport equations.
   - Provides higher accuracy but increases computational cost.
   - Example: Reynolds stress models (RSM), where additional equations are solved for Reynolds stresses.

4. Lecture 4 15/01/2025

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \tag{4.1}$$

Initial conditions:

$$\theta(t = t_0) = \theta_0$$
$$\dot{\theta}(t = t_0) = \omega_0$$

We can rewrite this equation into two 1^{st}-order ODEs,
Let $\frac{\theta}{dt} = p$, then

$$\frac{dp}{dt} + \omega^2\theta = 0 \tag{i}$$
$$\frac{d\theta}{dt} = p \tag{ii}$$

Initial conditions become:

$$\theta(t = t_0) = \theta_0$$
$$p(t = t_0) = \omega_0$$

4.1 Forward Difference

Invoking Taylor series expansion for $f(t+t_0)$:

$$\left.\begin{aligned} f(t+t_0) &= f(t_0) + (t+t_0)\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(t+t_0)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \\ f(t+\Delta t) &= f(t_0) + \Delta t\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(\Delta t)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \end{aligned}\right\} \tag{4.2}$$

we will get;

$$\begin{aligned} p(t+t_0) &\approx p(t_0) + \Delta t\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ p(t+t_0) &\approx p(t_0) + (t-t_0)\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t+t_0) - p(t_0)}{(t-t_0)} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_1) - p(t_0)}{\Delta t} \end{aligned} \tag{4.3}$$

Eq.(4.3) is called **Forward difference**.
Order of Forward difference is $O(\Delta t)$.
Similarly Forward difference for θ,

$$\begin{aligned} \frac{\partial \theta}{\partial t}\Big|_{t=t_0} &\approx \frac{\theta(t+t_0) - \theta(t_0)}{(t-t_0)} \\ \frac{\partial \theta}{\partial t}\Big|_{t=t_0} &\approx \frac{\theta(t_1) - \theta(t_0)}{\Delta t} \end{aligned} \tag{4.4}$$

From Eq.(i) and Eq.(4.3):

$$\begin{aligned} \frac{p(t+t_0) - p(t_0)}{(t-t_0)} + \omega^2\theta(t_0) &= 0 \\ p(t+t_0) &= p(t_0) - (t-t_0)\omega^2\theta(t_0) \\ p(t_1) &= p(t_0) - \Delta t\,\omega^2\theta(t_0) \end{aligned} \tag{4.5}$$

From Eq.(ii) and Eq.(4.4):

$$\begin{aligned} \frac{\theta(t+t_0) - \theta(t_0)}{(t-t_0)} &= p(t_0) \\ \theta(t+t_0) &= \theta(t_0) - (t-t_0)p(t_0) \\ \theta(t_1) &= \theta(t_0) - \Delta t\,p(t_0) \end{aligned} \tag{4.6}$$

4.2 Backward Difference

Invoking Taylor series expansion for $f(t-t_0)$:

$$\left.\begin{aligned} f(t-t_0) &= f(t_0) - (t-t_0)\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(t-t_0)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \\ f(t-\Delta t) &= f(t_0) - \Delta t\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(\Delta t)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \end{aligned}\right\} \tag{4.7}$$

we will get;

$$\begin{aligned} p(t-t_0) &\approx p(t_0) - \Delta t\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ p(t-t_0) &\approx p(t_0) - (t-t_0)\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_0) - p(t-t_0)}{(t-t_0)} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_0) - p(t_{-1})}{\Delta t} \end{aligned} \tag{4.8}$$

Eq.(4.8) is called **Backward difference**.
Order of Backward difference is $O(\Delta t)$.
Similarly Backward difference for θ,

$$\begin{aligned} \frac{\partial \theta}{\partial t}\Big|_{t=t_0} &\approx \frac{\theta(t_0) - \theta(t-t_0)}{(t-t_0)} \\ \frac{\partial \theta}{\partial t}\Big|_{t=t_0} &\approx \frac{\theta(t_0) - \theta(t_{-1})}{\Delta t} \end{aligned} \tag{4.9}$$

From Eq.(i) and Eq.(4.8):

$$\begin{aligned} \frac{p(t+t_0) - p(t_0)}{(t-t_0)} + \omega^2\theta(t_0) &= 0 \\ p(t+t_0) &= p(t_0) - (t-t_0)\omega^2\theta(t_0) \\ p(t_1) &= p(t_0) - \Delta t\,\omega^2\theta(t_0) \end{aligned} \tag{4.10}$$

From Eq.(ii) and Eq.(4.9):

$$\begin{aligned} \frac{\theta(t+t_0) - \theta(t_0)}{(t-t_0)} &= p(t_0) \\ \theta(t+t_0) &= \theta(t_0) - (t-t_0)p(t_0) \\ \theta(t_1) &= \theta(t_0) - \Delta t\,p(t_0) \end{aligned} \tag{4.11}$$

4.3 Current Difference

The current difference method is a combination of the forward and backward difference methods, where we approximate the

values at the current time using both the forward and backward information.
Using forward difference:

$$\frac{d\theta}{dt}\Big|_{t=t_0} = p(t_0) = \text{known} = p(t_0) - \Delta t\,\omega^2\theta(t_0) \quad (4.12)$$

Using backward difference:

$$\frac{d\theta}{dt}\Big|_{t=t_0} = \frac{\theta(t_1) - \theta(t - \Delta t)}{\Delta t} \quad (4.13)$$

From Eq.(4.12) and Eq.(4.13), we can combine both the equations to get:

$$\begin{aligned} \frac{\theta(t_1) - \theta(t - \Delta t)}{\Delta t} &= p(t_0) - \Delta t\,\omega^2\theta(t_0) \\ \theta(t_1) &= \theta(t - \Delta t) + \Delta t[p(t_0) - \Delta t\,\omega^2\theta(t_0)] \end{aligned} \quad (4.14)$$

In Eq.(4.14) all the terms on the RHS are known, allowing us to compute the value of θ at time (t_1).
Thus, we obtain a method to compute the new value of θ based on the current and past time steps.

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$$\left.\begin{aligned} f(x+\Delta x) &= f(x) + \Delta x \frac{\partial f}{\partial x}\Big|_{x=x_0} \\ &+ \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{t=x_0} \\ &+ \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_{t=x_0} + \quad \cdots \end{aligned}\right\} \tag{5.1}$$

$$\left.\begin{aligned} f(x-\Delta x) &= f(x) - \Delta x \frac{\partial f}{\partial x}\Big|_{x=x_0} \\ &+ \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{t=x_0} \\ &- \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_{t=x_0} + \quad \cdots \end{aligned}\right\} \tag{5.2}$$

Using Eq.(5.1),

$$\frac{df}{dx}\Big|_{x=x_0} \simeq \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] + O(\Delta x) \tag{5.3}$$

Similiarly, using Eq.(5.2), one obtain,

$$\frac{df}{dx}\Big|_{x=x_0} \simeq \left[\frac{f(x) - f(x-\Delta x)}{\Delta x}\right] + O(\Delta x) \tag{5.4}$$

Subtracting Eq.(5.2) from Eq.(5.1), we get,

$$\frac{df}{dx} \simeq \left[\frac{f(x+\Delta x) + f(x-\Delta x)}{2\Delta x}\right] + O(\Delta x^2) \tag{5.5}$$

Adding Eq.(5.1) and Eq.(5.2), we get,

$$\frac{d^2 f}{dx^2} \simeq \left[\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}\right] + O(\Delta x^2) \tag{5.6}$$

$$\left.\begin{aligned} f(x+\Delta x,t) &= f(x,t) + \Delta x \frac{\partial f(x,t)}{\partial x}\Big|_{x=x_0} \\ &+ \frac{(\Delta x)^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}\Big|_{t=x_0} \\ &+ \frac{(\Delta x)^3}{3!}\frac{\partial^3 f(x,t)}{\partial x^3}\Big|_{t=x_0} + \cdots \end{aligned}\right\} \tag{5.7}$$

$$\left.\begin{aligned} f(x-\Delta x,t) &= f(x,t) - \Delta x \frac{\partial f(x,t)}{\partial x}\Big|_{x=x_0} \\ &+ \frac{(\Delta x)^2}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}\Big|_{t=x_0} \\ &- \frac{(\Delta x)^3}{3!}\frac{\partial^3 f(x,t)}{\partial x^3}\Big|_{t=x_0} + \cdots \end{aligned}\right\} \tag{5.8}$$

5.1 Space Difference

From Eq.(5.7) and Eq.(5.8) respectively by difference equation w.r.t 1-direction, say x, we obtain **Space difference**:

$$\frac{df}{dx}\Big|_{x=x_0} \simeq \left[\frac{f(x+\Delta x,t) - f(x,t)}{\Delta x}\right] + O(\Delta x) \tag{5.9}$$

$$\frac{df}{dx}\Big|_{x=x_0} \simeq \left[\frac{f(x+\Delta x,t) - f(x,t)}{\Delta x}\right] + O(\Delta x) \tag{5.10}$$

Subtracting Eq.(5.8) from Eq.(5.7), we get,

$$\frac{df}{dx} \simeq \left[\frac{f(x+\Delta x,t) + f(x-\Delta x,t)}{2\Delta x}\right] + O(\Delta x^2) \tag{5.11}$$

Adding Eq.(5.7) and Eq.(5.8), we get,

$$\frac{d^2 f}{dx^2} \simeq \left[\frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{(\Delta x)^2}\right] + O(\Delta x^2) \tag{5.12}$$

5.2 Time Difference

Similiarly, From Eq.(5.7) and Eq.(5.8) respectively, difference equation w.r.t time (t), we obtain **Time difference**:

$$\frac{df}{dt}\Big|_{t=t_0} \simeq \left[\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\right] + O(\Delta t) \tag{5.13}$$

$$\frac{df}{dt}\Big|_{t=t_0} \simeq \left[\frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\right] + O(\Delta t) \tag{5.14}$$

$$\frac{df}{dt} \simeq \left[\frac{f(x,t+\Delta t) + f(x,t-\Delta t)}{2\Delta t}\right] + O(\Delta t^2) \tag{15}$$

$$\frac{d^2 f}{dt^2} \simeq \left[\frac{f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t)}{(\Delta t)^2}\right] + O(\Delta t^2) \tag{5.16}$$

5.3 Explicit Form Of Second Order PDE

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x \partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial x} + Ff = G \tag{5.17}$$

Cases:

1. If A, B, C, D, E, F and G are either constant or function of x and y, Eq.(5.17) is called **Linear PDE**.
2. If A, B and C are function of x, y and f, Eq.(5.17) is called **Quasi-linear PDE**.
3. If A, B and C are function of x and y only, Eq.(5.17) is called **Semi-linear PDE**.

Example: Momentum equation, which is,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

is a quasi-linear PDE.

5.4 Implicit Form Of Second Order PDE

$$G\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2},, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}\right) = 0 \tag{5.18}$$

Eq.(5.18) is Implicit form of PDE. Example: Continuity equation in 1-D, which is,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

1-D linear advection equation:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

in Euler discription.

$$\frac{D\rho}{Dt} = 0$$

in Lagrangian discription, represents change in density (ρ) following the motion.

6. Lecture 6 21/01/2025

6.1 1-D Linear Advection Equation

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial t} = 0 \tag{6.1}$$

where

f is fluid property

u is x-component of fluid velocity

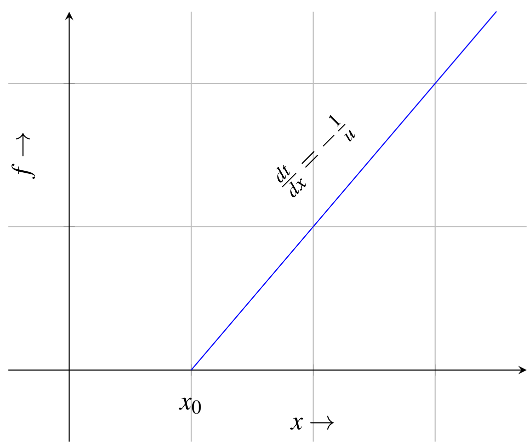


Figure 2. Path line or trajectory of fluid element

$$\frac{dx}{dt} = u = \text{constant} \tag{6.2}$$

Eq.(6.2) is called **Characteristic equation**.
Integrating Eq.(6.2) w.r.t time, we get:

$$x = x_0 + \int_{t_0}^{t} udt \tag{6.3}$$

Eq.(6.3) is called equation of pathline as shown in Fig2.
Substituting, $u = \frac{dx}{dt}$ in Eq.(6.1), we obtain:

$$\frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial t} = 0 \tag{6.4}$$

$$\frac{Df}{Dt} = 0 \tag{6.4}$$

Eq.(6.4) implies the property f is conserved following the motion of fluid element.

6.2 Triangular Property

1. **Hyperbolic:** 1^{st} order partial equation $\rightarrow$ 1 family of characteristic equation in real domain.
2. **Hyperbolic:** 2^{nd} order partial equation $\rightarrow$ 2 distinct and real set of characteristic equations.
3. **Parabolic:** 2^{nd} order partial equation $\rightarrow$ 2 equal and real set of characteristic equations.
4. **Elliptical:** 2^{nd} order partial equation $\rightarrow$ 2 distinct and complex set of characteristic equations.

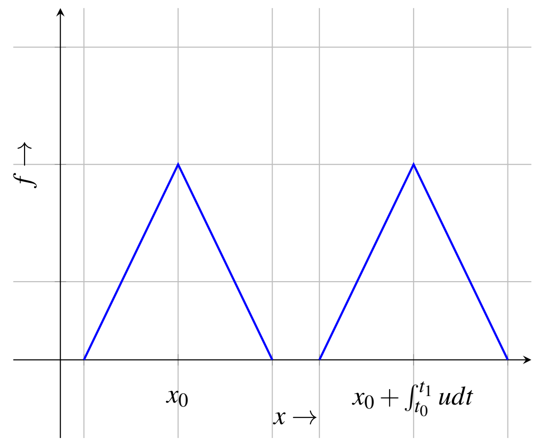


Figure 3. Triangular property distribution

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \begin{cases} u = \text{constant} & \Rightarrow \text{linear PDE ,} \\ u = u(x,t) & \Rightarrow \text{linear PDE ,} \\ u = u(x,t,f) & \Rightarrow \text{quasi-linear PDE} \end{cases}$$

$$a(x,t,f)\frac{\partial f}{\partial t} + b(x,t,f)\frac{\partial f}{\partial x} = 0 \tag{6.5}$$

Above Eq.(6.5) is quasi-linear PDE, having characteristic equation in real domain $\Rightarrow \frac{dx}{dt} = \frac{b}{a}$

$$a(x,t,f)\frac{\partial f}{\partial t} + b(x,t,f)\frac{\partial f}{\partial x} = c(x,t,f) \tag{6.6}$$

Above Eq.(6.6) is semi-linear PDE, having characteristic equation in real domain.

7. Lecture 7 22/01/2025

General quasi-linear 2^{nd} order PDE:

$$A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}+D\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial y}+Ff=G \tag{7.1}$$

i.e. A,B,C can be function of $x,y,f,\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
To find sign of dicriminant B^2-4AC tells the classification of equation:

$$B^2-4AC\begin{cases}>0, & \text{Hyperbolic PDE}\\ =0, & \text{Parabolic PDE}\\ <0, & \text{Elliptical PDE}\end{cases}$$

This is analogus to general equation of conic-section curves. Which is general by Eq.(7.2)

$$Ay^2+Bxy+Cx^2+Dx+Ey+F=0 \tag{7.2}$$

$$B^2-4AC\begin{cases}>0, & \text{Hyperbola}\\ =0, & \text{Parabola}\\ <0, & \text{Ellipse}\end{cases}$$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \tag{7.3}$$

$$d\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}dx+\frac{\partial^2 f}{\partial x\partial y}dy \tag{7.4}$$

$$d\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}dx+\frac{\partial^2 f}{\partial y^2}dy \tag{7.5}$$

Unknown in Eq.(7.3), Eq.(7.4) and Eq.(7.5) are 2^{nd} order derivatives.
From Eq.(7.1),

$$A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}=-D\frac{\partial f}{\partial x}-E\frac{\partial f}{\partial y}-Ff+G \tag{7.6}$$

From Eq.(7.4),

$$\frac{\partial^2 f}{\partial x^2}dx+\frac{\partial^2 f}{\partial x\partial y}dy=d\left(\frac{\partial f}{\partial x}\right) \tag{7.7}$$

From Eq.(7.5),

$$\frac{\partial^2 f}{\partial x\partial y}dx+\frac{\partial^2 f}{\partial y^2}dy=d\left(\frac{\partial f}{\partial y}\right) \tag{7.8}$$

Matrix representation of Eq.(7.6), Eq.(7.7) and Eq.(7.8):

$$\underbrace{\begin{bmatrix}A & B & C\\ dx & dy & 0\\ 0 & dx & dy\end{bmatrix}}_{\det(\mathbf{A})=0}\begin{bmatrix}\frac{\partial^2 f}{\partial x^2}\\ \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial y^2}\end{bmatrix}=\begin{bmatrix}-D\frac{\partial f}{\partial x}-E\frac{\partial f}{\partial y}-Ff+G\\ d\left(\frac{\partial f}{\partial x}\right)\\ d\left(\frac{\partial f}{\partial y}\right)\end{bmatrix}.$$

Determintant of $(\mathbf{A})$ should be equal to zero, i.e.,

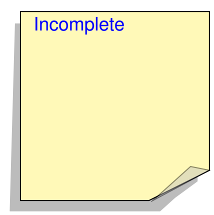
$$\begin{aligned}\det(\mathbf{A})&=0\\ A(dy)^2-B(dxdy)+C(dx)^2&=0\\ A\left(\frac{dy}{dx}\right)^2-B\left(\frac{dy}{dx}\right)+C&=0\end{aligned} \tag{7.9}$$

Solution of above Eq.(7.9) are 2 characteristic equations:

$$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

if $B^2-4AC>0 \Rightarrow$ 2 distinct Real roots
Family of char. curve $\in\mathbb{R}$
Hyperbolic PDE
if $B^2-4AC=0 \Rightarrow$ Real and equal roots
1 Family of char. curve $\in\mathbb{R}$
Parabolic PDE
if $B^2-4AC<0 \Rightarrow$ Imaginary roots
Family of char. curves $\in\mathbb{C}$
Elliptic PDE

Examples:



8. Lecture 08/01/2025

8.1 Heat Equation

$$k\frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0 \tag{8.1}$$

Eq.(8.1) can be decomposed into pair of PDEs:

$$\frac{\partial^2 T}{\partial x \partial t}dx + \frac{\partial^2 T}{\partial t^2}dt = d\left(\frac{\partial T}{\partial t}\right) \tag{8.2}$$

$$\frac{\partial^2 T}{\partial x^2}dx + \frac{\partial^2 T}{\partial x \partial t}dt = d\left(\frac{\partial T}{\partial x}\right) \tag{8.3}$$

Rewriting Eq.(8.1), Eq.(8.2) and Eq.(8.3) into matrix form:

$$\underbrace{\begin{bmatrix} k & 0 & 0 \\ 0 & dx & dt \\ dx & dt & 0 \end{bmatrix}}_{\text{A}} \begin{bmatrix} \frac{\partial^2 T}{\partial x^2} \\ \frac{\partial^2 T}{\partial x \partial t} \\ \frac{\partial^2 T}{\partial t^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial T}{\partial t} \\ d\left(\frac{\partial T}{\partial t}\right) \\ d\left(\frac{\partial T}{\partial x}\right) \end{bmatrix}$$

$$\begin{aligned} \det\,(\text{A}) &= 0 \\ k(0-(dt)^2) - 0(0-dxdt) + 0(0-(dx)^2) &= 0 \\ -k(dt)^2 &= 0 \end{aligned}$$

$$\begin{aligned} \therefore dt &= 0 \\ t &= \text{const.} \end{aligned}$$

Therefore the equation represents Parabolic PDE (2^{n} order quasi-linear PDE).

8.2 1-D Wave Equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{8.4}$$

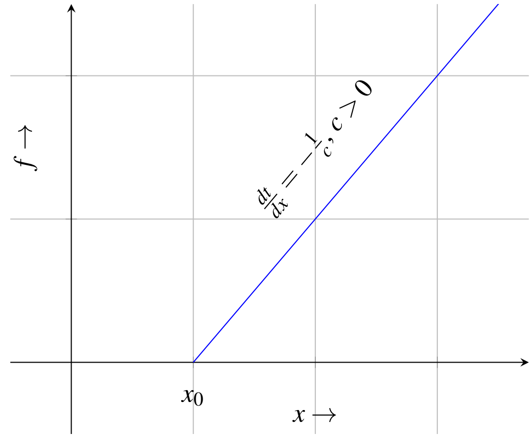
$$\underbrace{\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)}_{\text{Operator acting on f}} f = 0$$

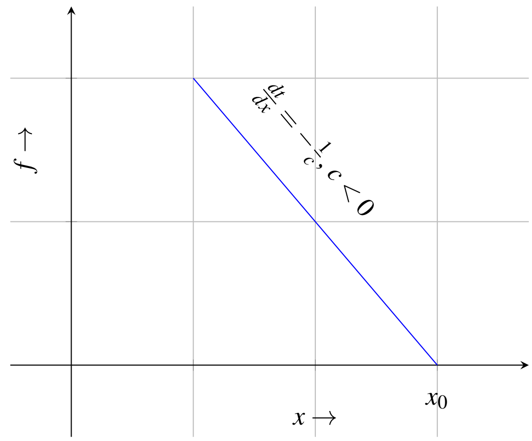
$$\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\right] f = 0 \tag{8.5}$$

Above Eq.(8.5) is pair two 1-D advection equations with constant speed c in **negative** x-direction and **positive** x-direction respectively.
Discrimant of Eq.(8.4), for A = 1, B=0, C=-1:

$$\begin{aligned} B^2 - 4AC \\ 0^2 - 4c^2(-1) \\ 4c^2 > 0 \end{aligned}$$

Therefore Eq.(8.4) represents Hyperbolic PDE.





$$\frac{\partial^2 f}{\partial x \partial t}dx + \frac{\partial^2 f}{\partial t^2}dt = d\left(\frac{\partial f}{\partial t}\right) \tag{8.5}$$

$$\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial t}dt = d\left(\frac{\partial f}{\partial x}\right) \tag{8.6}$$

Rewriting Eq.(8.4), Eq.(8.5) and Eq.(8.6) into matrix form:

$$\underbrace{\begin{bmatrix} c^2 & 0 & -1 \\ 0 & dx & dt \\ dx & dt & 0 \end{bmatrix}}_{\text{A}} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x \partial t} \\ \frac{\partial^2 f}{\partial t^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial t} \\ d\left(\frac{\partial f}{\partial t}\right) \\ d\left(\frac{\partial f}{\partial x}\right) \end{bmatrix}$$

$$\begin{aligned} \det\,(\text{A}) &= 0 \\ c^2(0-(dt)^2) - 0(0-dxdt) + (-1)(0-(dx)^2) &= 0 \\ -c^2(dt)^2 + (dx)^2 &= 0 \\ -c^2\left(\frac{dt}{dx}\right)^2 + 1 &= 0 \\ c^2\left(\frac{dt}{dx}\right)^2 - 1 &= 0 \end{aligned}$$

Slope $\frac{dt}{dx}$ is equal to:

$$\frac{dt}{dx} = \frac{-0 \pm \sqrt{0^2 - 4 \cdot c^2 \cdot (-1)}}{2 \cdot c^2}$$
$$\frac{dt}{dx} = \pm \frac{1}{2c}$$

8.3 Laplace Equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{8.7}$$

$$\frac{\partial^2 f}{\partial x \partial y} dx + \frac{\partial^2 f}{\partial y^2} dy = d\left(\frac{\partial f}{\partial y}\right) \tag{8.8}$$
$$\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial x \partial y} dy = d\left(\frac{\partial f}{\partial x}\right) \tag{8.9}$$

Discrimant of Eq.(8.7), for A = 1, B=0, C=1:

$$\begin{aligned} B^2 - 4AC \\ 0^2 - 4(1)(1) \\ -4 < 0 \end{aligned}$$

Therefore Eq.(8.7) represents Elliptical PDE.
Rewriting Eq.(8.7), Eq.(8.8) and Eq.(0) into matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & dx & dy \\ dx & dy & 0 \end{bmatrix}}_{\text{A}} \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 \\ d\left(\frac{\partial f}{\partial y}\right) \\ d\left(\frac{\partial f}{\partial x}\right) \end{bmatrix}$$

$$\begin{aligned} \det(\text{A}) &= 0 \\ 1(0 - (dy)^2) - 0(0 - dxdy) + 1(0 - (dx)^2) &= 0 \\ -(dy)^2 - (dx)^2 &= 0 \\ \left(\frac{dy}{dx}\right)^2 + 1 &= 0 \\ \frac{dy}{dx} &= \pm\imath \end{aligned}$$

9. Lecture 9 28/01/2025

- Domain of dependence at a given point in the solution domain would corresponding to the region where solution will impact solution will be impacted by the solution at the above given point.
- Range of independence at a given point in the solution domain would corresponding to the region where solution will impact solution will be impacted by the solution at the above given point.

9.1 Wave Equation

Hyperbolic PDE:

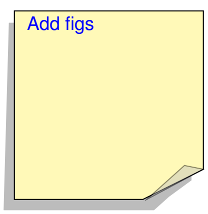
$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{9.1}$$

Parabolic PDE:

$$\frac{\partial T}{\partial t} = c^2 \frac{\partial T}{\partial x} \tag{9.2}$$

Ellipse PDE:

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0 \tag{9.3}$$



$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial x} + c\frac{\partial g}{\partial t} + d\frac{\partial g}{\partial x} = e \tag{9.4}$$

$$A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial x} + C\frac{\partial g}{\partial t} + D\frac{\partial g}{\partial x} = E \tag{9.5}$$

Using chain rule;

$$\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx = df \tag{9.6}$$

$$\frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dx = dg \tag{9.7}$$

Rewriting Eq.(9.4), Eq.(9.5), Eq.(9.6) and Eq.(9.7) into matrix form:

$$\underbrace{\begin{bmatrix} a & b & c & d \\ A & B & C & D \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dy \end{bmatrix}}_{\text{A}} \begin{bmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial t} \\ \frac{\partial g}{\partial x} \end{bmatrix} = \begin{bmatrix} e \\ E \\ df \\ dg \end{bmatrix}$$

$$\det(\text{A}) = 0$$

$$(dx)^2 \underbrace{(aC - Ac)}_{\overline{A}} - (dx)(dt)\underbrace{(aD - Ad + bC - Bc)}_{\overline{B}} + (dt)^2 \underbrace{(bD - Bd)}_{\overline{C}} = 0$$

Discrimant of Eq.(8.7), for $\overline{A} = (aC - Ac)$, $\overline{B} = (aD - Ad + bC - Bc)$, $\overline{C} = (bD - Bd)$:

$$\overline{B}^2 - 4\overline{AC}$$

$$(aD - Ad + bC - Bc)^2 - 4(aC - Ac)(bD - Bd)$$

$$\frac{dx}{dt} = \frac{\overline{B} \pm \sqrt{\overline{B}^2 - 4\overline{AC}}}{2\overline{A}} \tag{9.8}$$

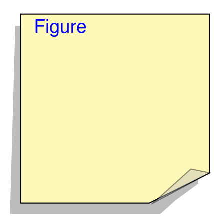
Depending upon sign of $\overline{B}^2 - 4\overline{AC}$ the follwing classification can be made:

- Negative $\rightarrow$ Elliptic
- Zero $\rightarrow$ Parabolic
- positive $\rightarrow$ Hyperbolic

9.2 Heat Transfer Equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{9.9}$$

where, k is coeffeicent of thermal expansion.



Applying FTCS (Forward Time and Central in Space) scheme,

$$t_n = n\Delta t \qquad , n = 0, 1, 2, \ldots$$

$$x_m = x_0 + m\Delta x \qquad , m = 0, 1, 2, \ldots, N$$

Backward difference:

$$\left(\frac{\partial T}{\partial t}\right)_{x,t} = \frac{T_{m,n+1} - T_{m,n}}{\Delta t} \tag{9.10}$$

Central difference:

$$\left(\frac{\partial^2 T}{\partial t^2}\right)_{x,t} = \frac{T_{m+1,n} - 2 \cdot T_{m,n} + T_{m-1,n}}{\Delta t} \tag{9.11}$$

Substituting Eq.(9.10) and Eq.(9.11) into Eq.(9.9), we obtain,

$$\begin{aligned}
\left(\frac{T_{m,n+1}-T_{m,n}}{\Delta t}\right) &= k\left(\frac{T_{m+1,n}-2\cdot T_{m,n}+T_{m-1,n}}{\Delta t}\right) \\
T_{m,n+1} &= T_{m,n}+\underbrace{\frac{k\Delta t}{(\Delta x)^2}}_{\lambda}\left(T_{m+1,n}-2T_{m,n}+T_{m-1,n}\right) \\
T_{m,n+1} &= (1-2\lambda)T_{m,n}+\lambda T_{m-1,n}+\lambda T_{m+1,n}
\end{aligned}$$

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10. Lecture 10 29/01/2025

10.1 Lax Theorem

The **Lax Equivalence Theorem** is a fundamental result in numerical analysis that provides a necessary and sufficient condition for the convergence of finite difference schemes used to approximate solutions of partial differential equations (PDEs).
For a well-posed initial value problem, a consistent finite difference scheme is convergent if and only if it is stable.
Key Concepts:

- **Consistency:** A finite difference scheme is said to be *consistent* if the truncation error tends to zero as the grid spacing $(\Delta x, \Delta t)$ approaches zero.
- **Stability:** A scheme is *stable* if numerical errors do not grow uncontrollably as time progresses.
- **Convergence:** A numerical scheme is *convergent* if the solution obtained from the scheme approaches the exact solution as the grid is refined.

Let $u(x,t)$ be the exact solution of a PDE, and let u_j^n be the numerical solution at time level n and spatial index j. The numerical scheme can be written as:

$$u_j^{n+1} = F(u_{j-1}^n, u_j^n, u_{j+1}^n, \dots),$$

where F is the finite difference operator.
For the scheme to be **consistent**, the local truncation error must satisfy:

$$\lim_{\Delta x, \Delta t \to 0} (\text{Truncation Error}) = 0.$$

By Lax's theorem, if the scheme is consistent and stable, it is guaranteed to be convergent:

$$\lim_{\Delta x, \Delta t \to 0} ||u_j^n - u(x_j, t_n)|| = 0.$$

Significance of Lax's theorem is that it provides a crucial guideline for designing numerical methods:

- Ensuring **consistency** is relatively straightforward by comparing the difference scheme with the PDE.
- **Stability** is often more challenging and requires analysis techniques like Von Neumann stability analysis.
- If a scheme is both **consistent and stable**, it is guaranteed to be **convergent**, meaning it will correctly approximate the true solution.

10.2 Van Neumann

Van Neumann stability analysis is a mathematical technique used to assess the stability of finite difference schemes for solving partial differential equations (PDEs). It is particularly useful in analyzing the behavior of numerical solutions over time.
Key Idea: The method is based on applying a Fourier mode decomposition to the numerical scheme. The numerical solution is expressed as a sum of Fourier modes, and the growth factor

Stability Criterion: A finite difference scheme is **stable** if and only if the magnitude of the amplification factor satisfies Eq.(10.1):

$$|G(k)| \leq 1, \quad \forall k. \tag{10.1}$$

This ensures that numerical errors do not grow uncontrollably as time progresses.

1. **Express the numerical scheme** as a recurrence relation.
2. **Substitute a Fourier mode** of the form:

   $$u_j^n = \hat{u}^n e^{ikj\Delta x},$$

   where i is the imaginary unit, k is the wave number, and Δx is the spatial step size.
3. **Determine the amplification factor** $G(k)$ by solving for:

   $$G(k) = \frac{\hat{u}^{n+1}}{\hat{u}^n}.$$

4. **Check the stability condition** $|G(k)| \leq 1$.

Application : Van Neumann analysis is commonly used for schemes solving hyperbolic PDEs, such as the **heat equation**, **wave equation**, and **advection equation**. It helps determine whether a numerical method will produce bounded solutions over time.
For **stability**, we often use techniques like the **Von Neumann stability analysis** to check that perturbations do not grow indefinitely using Eq.(10.1).

11. Lecture 11 03/02/2025

11.1 Van Neumann Stability Analysis For 1-D Heat Conducting Equation For FTCS Scheme

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{11.1}$$

Appling FTCS scheme:

$$\frac{T_{m,n+1} - T_{m,n}}{\Delta t} = k\left[\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}\right] \tag{11.2}$$

Let $\lambda = k\Delta t/(\Delta x)^2$
Since error $\varepsilon_{m,n}$ identically satisfies difference equations, we can write;

$$\begin{aligned}\varepsilon m,n+1 &= \varepsilon_{m,n} + \lambda\left(\varepsilon_{m+1,n} + \varepsilon_{m-1,n} - 2\varepsilon_{m,n}\right)\\ \varepsilon m,n+1 &= (1-2\lambda)\,\varepsilon_{m,n} + \lambda\left(\varepsilon_{m+1,n} + \varepsilon_{m-1,n}\right)\end{aligned} \tag{11.3}$$

Let $\varepsilon_{m,n} = \exp(at)\cdot\exp(ikx)$
Since, $t = n\Delta t$ and $x = m\Delta x$;

$$\varepsilon_{m,n} = \exp(an\Delta t)\cdot\exp(ikm\Delta x) \tag{11.4}$$

Substituting Eq.(11.4) in Eq.(11.3), we obtain;

$$\begin{aligned}&e^{(a(n+1)\Delta t)}\cdot e^{(ikm\Delta x)} = (1-2\lambda)e^{(an\Delta t)}e^{(ikm\Delta x)}\\ &+\lambda\left[e^{(ik(m+1)\Delta x)} + e^{(ik(m-1)\Delta x)}\right]e^{(an\Delta t)}\\ &e^{(an\Delta t)}\cdot e^{(ikm\Delta x)} = (1-2\lambda)e^{(an\Delta t)}e^{(ikm\Delta x)}\\ &+\lambda\left[e^{(ik(m+1)\Delta x)} + e^{(ik(m-1)\Delta x)}\right]\end{aligned} \tag{11.5}$$

Divide both sides of Eq.(11.5) by $e^{(ikm\Delta x)}$

$$\begin{aligned}e^{(a\Delta t)} &= (1-2\lambda) + \lambda\underbrace{\left[e^{(ik\Delta x)} + e^{-(ik\Delta x)}\right]}_{2\cos(k\Delta x)}\\ e^{(a\Delta t)} &= (1-2\lambda) + 2\lambda\cos(k\Delta x)\\ e^{(a\Delta t)} &= 1 - 2\lambda\left[1-\cos(k\Delta x)\right]\\ e^{(a\Delta t)} &= 1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right]\end{aligned} \tag{11.6}$$

Amplification factor

$$\begin{aligned}G &= \left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{m,n}}\right|\\ &= \left|\frac{e^{a(n+1)\Delta x}\cdot e^{ikm\Delta x}}{e^{an\Delta x}\cdot e^{ikm\Delta x}}\right|\\ &= \left|e^{a\Delta x}\right|\end{aligned} \tag{11.7}$$

FTCS scheme will be stable if

$$\left|e^{a\Delta t}\right| \le 1 \Rightarrow \left|1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right]\right| \le 1$$

The following 2 conditions has to be identically satisfied

$$1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] \le 1 \tag{11.8.i}$$

$$1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] \ge -1 \tag{11.8.ii}$$

For Eq.(11.8.i);

$$\begin{aligned}1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] &\le 1\\ -4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] &\le 0\end{aligned}$$

Therefore, it is always true.
For Eq.(11.8.ii);

$$\begin{aligned}1 - 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] &\ge 1\\ 4\lambda\left[\sin^2\left(\frac{k\Delta x}{2}\right)\right] &\le 2\end{aligned}$$

Maximum magnitude of $\sin^2(k\Delta x/2) = 1$

$$\begin{aligned}4\lambda\cdot 1 &\le 2\\ \lambda &\le \frac{1}{2}\\ \frac{k\Delta t}{(\Delta x)} &\le \frac{1}{2}\\ \Delta t &\le \frac{(\Delta x)^2}{2k}\end{aligned}$$

12. Lecture 12 04/02/2025

12.1 BTCS Scheme

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{12.1}$$

$$\frac{T_{m,n} - T_{m,n-1}}{\Delta t} = k\left[\frac{T_{m+1,n} - 2\cdot T_{m,n} + T_{m-1,n}}{(\Delta x)^2}\right] \tag{12.2}$$

Let $\lambda = k\Delta t/(\Delta x)^2$

$$(1+2\lambda)T_{m,n} - \lambda(T_{m+1,n} + T_{m-1,n}) = T_{m,n-1} \tag{12.3}$$

For stability;

$$(1+2\lambda)\varepsilon_{m,n} - \lambda(\varepsilon_{m+1,n} + \varepsilon_{m-1,n}) = \varepsilon_{m,n-1} \tag{12.4}$$

$$\varepsilon_{m,n} = B^{\mu n\Delta t}\exp(ikm\Delta x) \tag{12.5}$$

Substituting Eq.(12.5) in Eq.(12.4), we obtain;

$$\left.\begin{aligned} &(1+2\lambda)B^{\mu n\Delta t}e^{(ikm\Delta x)} \\ &-\lambda B^{\mu n\Delta t}\left(e^{(ik(m+1)\Delta x)} + e^{(ik(m-1)\Delta x)}\right) \\ &= B^{\mu(n-1)\Delta t}e^{ikm\Delta x} \end{aligned}\right\} \tag{12.6}$$

Divide Eq.(12.6) throughout by $B^{\mu(n-1)\Delta t}\cdot e^{(ikm\Delta x)}$

$$\begin{aligned} (1+2\lambda)B^{\mu\Delta t} - \lambda B^{\mu\Delta t}\left(e^{(ik\Delta x)} + e^{-(ik\Delta x)}\right) &= 1 \\ (1+2\lambda)B^{\mu\Delta t} - 2\lambda B^{\mu\Delta t}\cos(k\Delta x) &= 1 \\ B^{\mu\Delta t}\left(1+2\lambda - 2\lambda\cos(k\Delta x)\right) &= 1 \\ B^{\mu\Delta t}\left(1+2\lambda[1-\cos(k\Delta x)]\right) &= 1 \\ B^{\mu\Delta t}\left(1+4\lambda\sin^2\left(\frac{k\Delta x}{2}\right)\right) &= 1 \end{aligned}$$

$$B^{\mu\Delta t} = \frac{1}{\left(1+4\lambda\sin^2\left(\frac{k\Delta x}{2}\right)\right)} \tag{12.7}$$

Amplification factor

$$G = \left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{m,n}}\right| = \left|B^{\mu\Delta t}\right| \leq 1 \tag{12.8}$$

Eq.(12.8) shows that BTCS scheme is unconditionally stable.

12.2 CTCS Scheme / Richardson Scheme

$$\frac{T_{m,n+1} - T_{m,n-1}}{2\cdot\Delta t} = k\left[\frac{T_{m+1,n} - 2\cdot T_{m,n} + T_{m-1,n}}{(\Delta x)^2}\right] \tag{12.9}$$

Let $\lambda = 2\cdot k\Delta t/(\Delta x)^2$

$$T_{m,n+1} = T_{m,n-1} + \lambda(T_{m+1,n} - 2\cdot T_{m,n} + T_{m-1,n}) \tag{12.10}$$

For stability;

$$\varepsilon_{m,n+1} = \varepsilon_{m,n-1} + \lambda(\varepsilon_{m+1,n} - 2\cdot\varepsilon_{m,n} + \varepsilon_{m-1,n}) \tag{12.11}$$

$$\varepsilon_{m,n} = B^{\mu n\Delta t}\exp(ikm\Delta x) \tag{12.12}$$

$$\begin{aligned} \varepsilon_{m,n} &= B^{\mu n\Delta t}e^{(ikm\Delta x)} \\ &= B^{\mu(n-1)\Delta t}e^{(ikm\Delta x)} \\ &+ \lambda B^{\mu n\Delta t}\left[e^{ik\Delta x} - e^{-ik\Delta x}\right]e^{ikm\Delta x} \end{aligned}$$

Divide throughout by $B^{\mu(n-1)\Delta t}e^{ikm\Delta x}$

$$\begin{aligned} B^{2\cdot\mu(n-1)\Delta t} &= 1 - 2\lambda\cdot B^{\mu\Delta t}\left(1-\cos(k\Delta x)\right) \\ B^{2\cdot\mu(n-1)\Delta t} &= 1 - 2\lambda\cdot B^{\mu\Delta t}\left(2\cdot\sin^2(k\Delta x)\right) \\ \left(B^{\mu(n-1)\Delta t}\right)^2 + 4\lambda\cdot\sin^2(k\Delta x)B^{\mu\Delta t} - 1 &= 0 \end{aligned}$$

$$G^2 + \beta G - 1 = 0 \tag{12.13}$$

Amplification factor

To be checked

$$G = \left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{m,n}}\right| = \left|B^{\mu\Delta t}\right| \leq 1$$

$$G_{1,2} = \frac{-\beta \pm \sqrt{\beta^2+4}}{2}$$

Eq.(12.14) shows that CTCS scheme is stable.

13. Lecture 13 06/02/2025

13.1 Crank Nicholson Scheme

$$\left(\frac{\partial T}{\partial t}\right)_{m,n+1/2} = k\left(\frac{\partial^2 T}{\partial x^2}\right)_{m,n+1/2} \tag{13.1}$$

$$\frac{T_{m,n}-T_{m,n-1}}{\Delta t} = k\left[\frac{T_{m+1,n+1/2}-2T_{m,n+1/2}+T_{m-1,n+1/2}}{(\Delta x)^2}\right] \tag{13.2}$$

Let $\lambda = k\Delta t/2(\Delta x)^2$

$$(1+2\lambda)T_{m,n+1} - \lambda T_{m,n-1} - \lambda T_{m+1,n+1} = (1-2\lambda)T_{m,n} - \lambda T_{m,n} \tag{13.3}$$

For stability;

$$(1+2\lambda)\varepsilon_{m,n+1} - \lambda\varepsilon_{m+1,n+1} - \lambda\varepsilon_{m-1,n+1} = (1-2\lambda)\varepsilon_{m,n} - \lambda\varepsilon_{m+1,n} + \lambda\varepsilon_{m-1,n} \tag{13.4}$$

$$\varepsilon_{m,n} = B^{\mu n\Delta t}\exp(ikm\Delta x) \tag{13.5}$$

Substituting Eq.(13.4) in Eq.(13.3), we obtain;

$$\left.\begin{aligned}&(1+2\lambda)B^{\mu(n+1)\Delta t}e^{(ikm\Delta x)}\\ &-\lambda B^{\mu(n+1)\Delta t}\left[e^{ik(m+1)\Delta x}+e^{-ik(m-1)\Delta x}\right]\\ &=(1-2\lambda)B^{\mu n\Delta t}e^{ikm\Delta x}\\ &+\lambda B^{\mu n\Delta t}\left[e^{ik(m+1)\Delta x}+e^{ik(m-1)\Delta x}\right]\end{aligned}\right\} \tag{13.6}$$

Divide throughout by $B^{\mu n\Delta t}e^{ikm\Delta x}$

$$\begin{aligned}&(1+2\lambda)B^{\mu n\Delta t} - 2\lambda B^{\mu n\Delta t}[\cos(k\Delta x)]\\ &\quad = (1-2\lambda)+2\lambda[\cos(k\Delta x)]\\ &(1+2\lambda-2\lambda\cos(k\Delta x))B^{\mu n\Delta t}\\ &\quad = (1-2\lambda)+2\lambda[\cos(k\Delta x)]\\ &(1+2\lambda(1-\cos(k\Delta x)))B^{\mu n\Delta t}\\ &\quad = 1-2\lambda(1-\cos(k\Delta x))\\ B^{\mu n\Delta t} &= \frac{1-2\lambda(1-\cos(k\Delta x))}{1+2\lambda(1-\cos(k\Delta x))}\\ B^{\mu n\Delta t} &= \frac{1-4\lambda(\sin^2(k\Delta x/2))}{1+4\lambda(\sin^2(k\Delta x/2))}\end{aligned} \tag{13.7}$$

Let $4\lambda(\sin^2(k\Delta x/2)) = \beta$
Amplification factor

$$G = \left|\frac{\varepsilon_{m,n+1}}{\varepsilon_{m,n}}\right| = \frac{1-\beta}{1+\beta} = \left|B^{\mu\Delta t}\right| \le 1 \tag{13.8}$$

13.2 Dufart Frankel Scheme

$$\left(\frac{\partial T}{\partial t}\right)_{m,n} = k\left(\frac{\partial^2 T}{\partial x^2}\right)_{m,n} \tag{13.9}$$

$$\frac{T_{m,n+1}-T_{m,n}}{2\Delta t} = k\left[\frac{T_{m+1,n}-2T_{m,n}+T_{m-1,n}}{(\Delta x)^2}\right] \tag{13.10}$$

Let $\lambda = 2\cdot k\Delta t/(\Delta x)^2$
Taking average $T_{m,n} = \frac{T_{m,n+1}+T_{m,n-1}}{2}$

$$T_{m,n+1}-T_{m,n} = \lambda\left[\frac{T_{m+1,n}-2\left(\frac{T_{m,n+1}+T_{m,n-1}}{2}\right)+T_{m-1,n}}{(\Delta x)^2}\right]$$

$$(1+\lambda)T_{m,n+1} = \lambda T_{m+1,n} + (1-\lambda)T_{m,n-1} + \lambda T_{m-1,n} \tag{13.11}$$

Errors should satisfy the same difference equation identically;

$$(1+\lambda)\varepsilon_{m,n+1} = \lambda\varepsilon_{m+1,n} + (1-\lambda)\varepsilon_{m,n-1} + \lambda\varepsilon_{m-1,n} \tag{13.12}$$

$$\varepsilon_{m,n} = B^{\mu n\Delta t}\exp(ikm\Delta x) \tag{13.13}$$

Substituting Eq.(13.12) in Eq.(13.11), we obtain;

$$\left.\begin{aligned}&(1+\lambda)B^{\mu(n+1)\Delta t}\exp(ikm\Delta x)\\ &=\lambda B^{\mu n\Delta t}\exp(ik(m+1)\Delta x)\\ &+(1-\lambda)B^{\mu(n-1)\Delta t}\exp(ikm\Delta x)\\ &+\lambda B^{\mu n\Delta t}\exp(ik(m-1)\Delta x)\end{aligned}\right\} \tag{13.14}$$

Divide throughout by $B^{\mu n\Delta t}e^{ikm\Delta x}$

$$\begin{aligned}(1+\lambda)\left(B^{\mu\Delta t}\right)^2 &= \lambda B^{\mu\Delta t}(2\cos(k\Delta t)) + (1-\lambda)\\ (1+\lambda)G^2 &= \lambda(2\cos(k\Delta t))G + (1-\lambda)\\ (1+\lambda)G^2 - \lambda(2\cos(k\Delta t))G - (1-\lambda) &= 0\end{aligned} \tag{13.15}$$

$$\begin{aligned}G_{1,2} &= \frac{2\lambda\cos(k\Delta x)\pm\sqrt{4\lambda^2\cos^2(k\Delta x)+4(1-\lambda^2)}}{2(1+\lambda)}\\ G_{1,2} &= \frac{\cancel{2}\lambda\cos(k\Delta x)\pm\cancel{2}\sqrt{\lambda^2(\cos^2(k\Delta x)-1)+1}}{\cancel{2}(1+\lambda)}\\ G_{1,2} &= \frac{\lambda\cos(k\Delta x)\pm\sqrt{1-\lambda^2\sin^2(k\Delta x)}}{(1+\lambda)}\end{aligned} \tag{13.16}$$

For Dufart Frankel Scheme to be stable $G_1 \& G_2$ in Eq.(13.16) should both simultaneously less than 1.
Case i: Quantity under square root is real.

i.e., $1-\lambda^2\sin^2(k\Delta x)\geqslant 0$

$$\begin{aligned}
G &= \frac{\lambda\cos(k\Delta x)\pm\sqrt{1-\lambda^2\sin^2(k\Delta x)}}{(1+\lambda)} \\
&\leqslant \frac{\lambda\cos(k\Delta x)+1}{(1+\lambda)} \\
&\leqslant \frac{\lambda+1}{(1+\lambda)} \\
&\leqslant 1
\end{aligned}$$

Case ii: Quantity under square root is imaginary.
i.e., $1-\lambda^2\sin^2(k\Delta x)\leqslant 0$

$$\begin{aligned}
G_{1,2} &= \frac{\lambda\cos(k\Delta x)\pm\sqrt{1-\lambda^2\sin^2(k\Delta x)}}{(1+\lambda)} \\
&= \frac{\lambda\cos(k\Delta x)\pm\imath\sqrt{\lambda^2\sin^2(k\Delta x)-1}}{(1+\lambda)} \\
|G_{1,2}^2| &= \frac{\lambda^2\cos^2(k\Delta x)+\lambda^2\sin^2(k\Delta x)-1}{(1+\lambda)^2} \\
&= \frac{(\lambda^2-1)}{(1+\lambda)^2} \\
&= \frac{\cancel{(\lambda+1)}(\lambda-1)}{(1+\lambda)^{\cancel{2}}} \\
&= \frac{(\lambda-1)}{(1+\lambda)} \\
&\leqslant 1
\end{aligned}$$

Therefore Dufart Frankel Scheme is unconditionally stable.