# Lecture Notes on Boundary Layer Meteorology

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**Abstract**

Notes of Lectures and addional information from books:
*An introduction to boundary layer meteorology* ([1]).

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# 1. Lecture 1 09/01/2025

## 1.1 Introduction To Boundary Layer

The Boundary Layer can be defined as part of the troposphere that is directly influenced by the presence of the Earth's surface and responds to surface forcings with a time scale of about an hour or less.
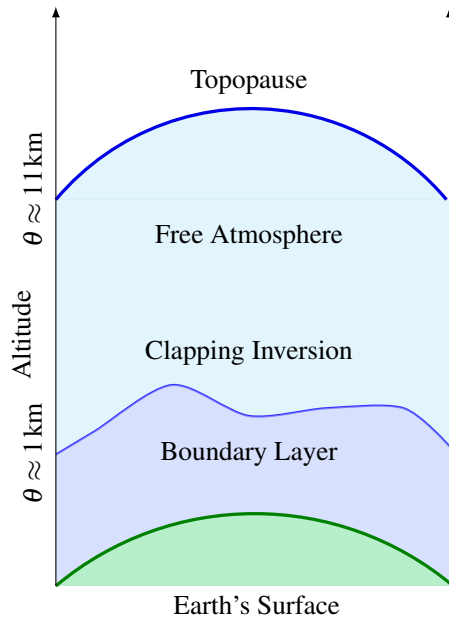


**Figure 1.1.** Atmosphere can be divided into 2 parts: boundary layer near surface and free atmosphere above it.

## 1.2 Boundary Layer Forcing Mechanism

What physical process modify boundary layer air parcel?

1. Heat transfer to.from the ground.
2. Frictional drag.
3. Evaporation/transpiration.
4. Terrain-induced flow modification.
5. Pollution emission.

## 1.3 Types Of Air Flow Or Wind

Air flow or wind can be decomposed into following 3 types:

1. **Mean Wind** $(\bar{u}, \bar{v}, \bar{w})$: Represents the average wind components in the horizontal $(\bar{u}, \bar{v})$ and vertical $(\bar{w})$ directions. It is important for the horizontal transport of quantities such as moisture, heat, momentum, and pollutants, a process known as advection.
2. **Waves**: Atmospheric waves, such as gravity waves, occur mostly at night in the nocturnal boundary layer (NBL). They can influence the structure of the boundary layer and the transport of energy.
3. **Turbulence**: The vertical transport of moisture, heat, momentum, and pollutants is primarily dominated by turbulence, which is characterized by chaotic and irregular motion.
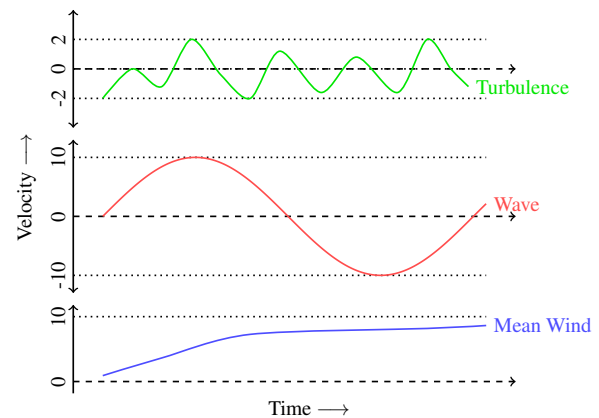


**Figure 1.2.** Plot showing profiles of Mean, Wave and Turbulent winds

## 1.4 Eddies

Eddies are formed due to the interaction of currents with obstacles like coastlines, underwater topography, or other currents, as well as from the instability of larger current systems. Eddies exhibit a rotational flow pattern, either clockwise or counterclockwise. Eddies can vary from size 100 to 3000 metres and also can exists as small as few millimetres. Small eddies might last for seconds to minutes, while larger oceanic eddies can persist for weeks, months, or even years.

## 1.5 Turbulence Generation Mechanisms

- **Solar Heating**: Solar heating generates thermals, which are essentially larger eddies that drive turbulence in the atmospheric boundary layer.
- **Wind Shear**: Variations in wind speed or direction with height create wind shear, which is a significant source of turbulence.
- **Obstacle-Induced Flow**: Deflected flow around obstacles such as trees, buildings, or other structures generates turbulent eddies downstream of these obstacles, creating turbulent wakes.
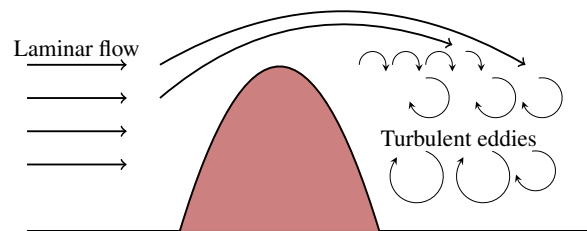


**Figure 1.3.** Eddy formation due to Turbulence caused by an obstacle

Large eddies will break into smaller eddies after which small eddies dissipates from K.E. to thermal energy.
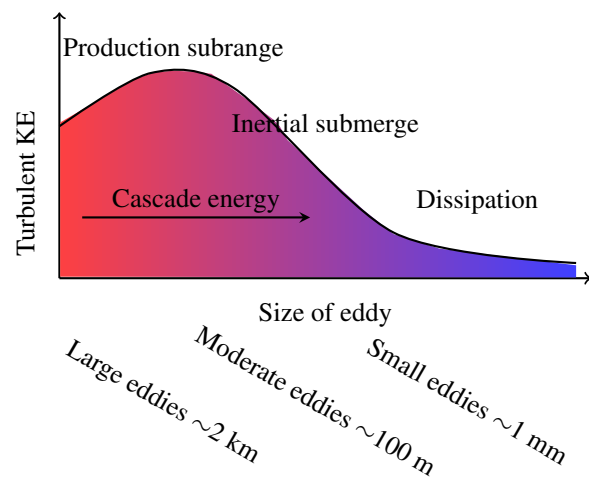
**Figure 1.4.** Variation of Turbulent Kinetic energy with change in Size of eddies

## 2. Leture 2 15/01/2025

### 2.1 Taylor's Hypothesis

- When studying atmospheric boundary layer (ABL), It is not easy to create a snapshot of turbulence in the Atmosphere.
- Hence it is easier and cheaper to make measurements of point in the atmoshpere for a longer time, then an instantaneous snapshot.
- So we just consider the atmoshpere is frozen.
- *Taylor suggested that turbulence can be considered frozen as it advects past sensor.*
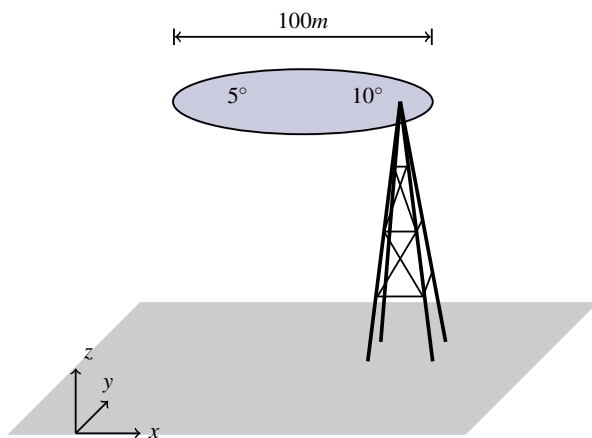


**Figure 2.1.** Eddy propagation
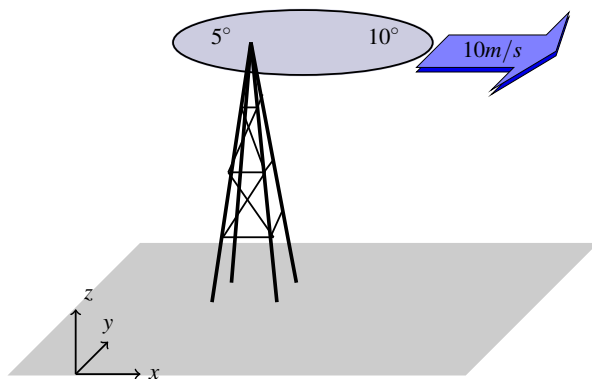


**Figure 2.2.** Eddy passing by the sensor mounted on tower

$$\frac{\partial T}{\partial x} = 0.05\text{K/m}, \quad \frac{\partial T}{\partial t} = -0.5\text{K/s}$$

$$\underbrace{\frac{DT}{Dt}}_{\substack{\text{Total derivative} = 0 \\ \text{(Taylor's hypothesis)}}} = \underbrace{\frac{\partial T}{\partial t}}_{\text{Local derivative}} + \underbrace{u\frac{\partial T}{\partial x}}_{\text{Advective term}} \tag{2.1}$$

### 2.2 Virtual Potential Temperature

Virtual potential temperature:

$$\theta_v = \theta(1 + 0.61r) \tag{2.2}$$

Virtual temperature:

$$T_v = T(1 + 0.61r) \tag{2.3}$$
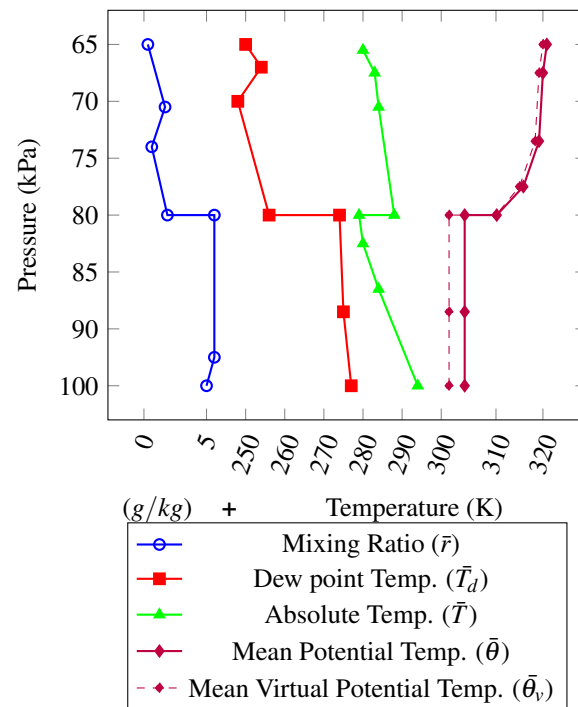


**Figure 2.3.** Pressure v/s Termperature

> **Question 2.1:**
>
> Given 25°C termperature, mixing ratio $\bar{r}$ is $20 g/kg$, measured Pressurere at $900 hPa$, find virtual potential temperature.

> **Answer 2.1:**
>
> Solution:
>
> $$\theta = T \times \left(\frac{1000}{P}\right)^{0.286}$$
> $$= 298 \times \left(\frac{1000}{900}\right)^{0.286}$$
> $$= 332.222K$$
> $$\theta_v = \theta \times (1 + 0.61r)$$
> $$= 332.22 \times (1 + 0.61 \times 0.025)$$
> $$= 336.273K$$
> $$\theta_v - \theta \approx 4.05K$$

### 2.3 Boundary Layer Depth and Structure

- Mixed layer
- Residual layer
- Stable Boundary layer

- Capping Inversion
- Nauturnal Boundary layer
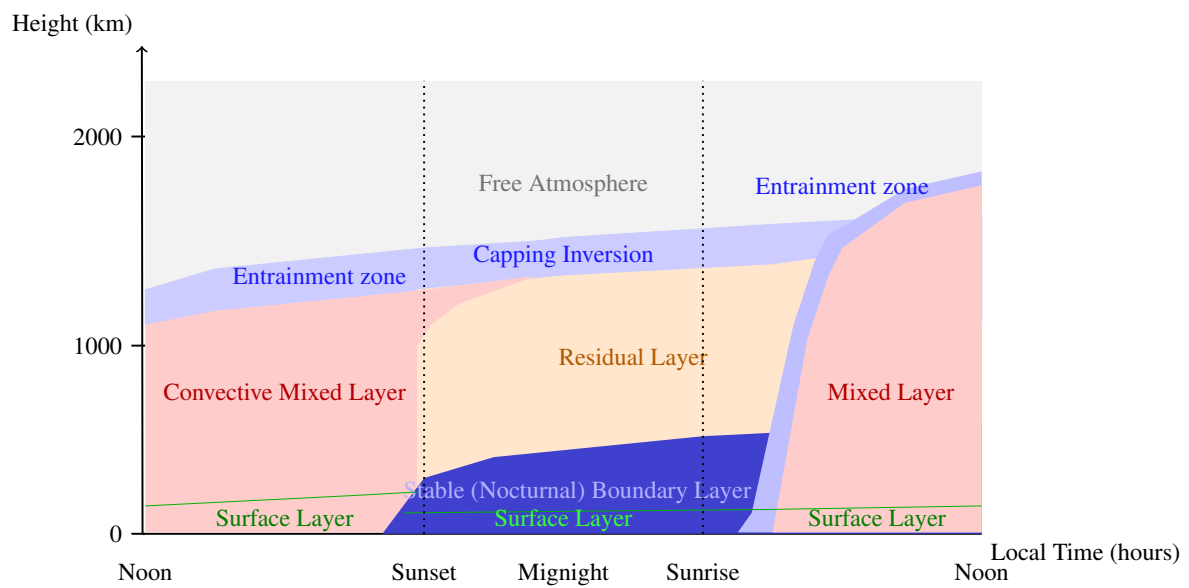
**Figure 2.4.** Height vs Local Time Diagram of Atmospheric Boundary Layers.

## 3. Lecture 3 16/01/2025

**Ocean:** Variations are minimal, with only 10% changes observed over 1000 km. Significant variations occur primarily during weather phenomena.

**Land:** Day-to-day and diurnal variations are prominent, with distinct boundary layer structures:

1. **Convective Mixed Layer:** Thermodynamically unstable with intense vertical mixing.
2. **Residual Layer:** Neutral stratification with turbulence of equal intensity in all directions.
3. **Stable Boundary Layer (Nocturnal B.L.):**
   - Neutral stratification with nocturnal jets ( 30 m/s, 200 m width).
   - Sporadic turbulence and internal gravity waves transporting air parcels vertically.
4. **Capping Inversion:** Found at altitudes between 1.5–3 km, acting as a barrier to upward mixing.
5. **Entrainment Zone:** Transition region from stable to unstable conditions, facilitating energy and mass exchange.

### 3.1 Stability and Plume Behaviour

1. **Looping plumes:** Occur in highly unstable conditions, usually during the day, when strong surface heating causes warm air to rise and interact turbulently with cooler air. This leads to an oscillatory motion that disperses pollutants in multiple directions, often seen in arid regions.

   **Fanning plumes:** Form in stable conditions, typically at night, when surface cooling creates temperature inversions. The plume spreads horizontally, concentrating pollutants close to the surface, which can impact air quality in urban or industrial areas.
3. **Coning plumes:** Develop under neutral conditions, where vertical and horizontal mixing is balanced. The plume takes a cone-like shape, often observed on overcast days or in the early morning and evening, with moderate turbulence.
4. **Lofting plumes:** Occur when the atmosphere is stable near the ground but unstable above. Pollutants rise and disperse above the stable layer, reducing ground-level pollution and minimizing surface concentrations.
5. **Fumigation plumes:** Happen when pollutants are trapped in a stable layer and then forced downward due to rising turbulence. This leads to high concentrations at the surface, posing risks to air quality, especially in industrial areas.

FIGURE for each to be added later.

1. **Agricultural meteorology:** Understanding microclimates within the boundary layer aids in crop management, irrigation planning, and predicting the effects of extreme weather on agriculture.
2. **Air pollution meteorology:** Dispersion and concentration of pollutants are governed by boundary layer processes, making it crucial for air quality monitoring and pollution control strategies.
3. **Cloud nuclei meteorology:** The boundary layer provides a reservoir of aerosols and moisture that act as cloud condensation nuclei, influencing cloud formation, precipitation, and local weather patterns.
4. **Thunderstorms and hurricanes physics:** The exchange of heat, moisture, and momentum in the boundary layer drives the development and intensity of thunderstorms and hurricanes, making it essential for improving weather prediction models.
5. **Urban meteorology:** The boundary layer's interactions with urban landscapes affect local climate, energy balance, and pollutant dispersion, aiding in city planning and sustainability efforts.
6. **Renewable energy:** Wind energy potential and efficiency are heavily dependent on boundary layer dynamics, which dictate wind speed profiles and turbulence levels near the surface.

| Property | Boundary Layer | Free Atmosphere |
|---|---|---|
| **Turbulence** | Almost continuously turbulent | Sporadic, CAT, turbulence within clouds |
| **Friction** | Strong drag due to surface interaction | Small viscous dissipation |
| **Dispersion** | Rapid turbulent mixing | Small molecular diffusion |
| **Winds** | Near logarithmic profile | Geostrophic winds |
| **Vertical Transport** | Turbulent vertical motion | Horizontal transport by mean wind |
| **Thickness** | 100m - 3km (variable) | 8-16km (less variable) |

**Table 1.** Comparison of Boundary Layer and Free Atmosphere Properties

### 3.2 Importance of Boundary Layer

The boundary layer plays a critical role in regulating interactions between the Earth's surface and the atmosphere. Its study is important in various fields, including:

# 4. Lecture 4 23/01/2025

## 4.1 Statistical Tools Required For Turbulence

Turbulence is characteristiced by randomness.

$$U = \bar{u} + u'$$
$$V = \bar{v} + v'$$
$$W = \bar{w} + w'$$
$$C = \bar{c} + c'$$

### 4.1.1 Mean

$$\bar{A} = \frac{1}{N} \sum_{i=1}^{N} A(i,s)$$
$$\bar{A} = \frac{1}{T} \int_{t=0}^{T} A(t,s)dt$$

### 4.1.2 Rules for averaging

If $A$ and $B$ are variables dependent on time, then:

$$\overline{A+B} = \bar{A} + \bar{B},$$
$$\bar{\bar{A}} = A,$$
$$\overline{\bar{A} \cdot B} = \bar{A} \cdot \bar{B},$$
$$\overline{\frac{dA}{dt}} = \frac{d\bar{A}}{dt}.$$

**Reynold's averaging rule**

$$\bar{\bar{A}} = \overline{\bar{A} + a'} = \bar{A}, \text{since } \bar{a}' = 0$$
$$\overline{A \cdot B} = \overline{(\bar{A} + a')(\bar{B} + b')} = \bar{A}\bar{B} + \overline{(a'b')},$$
$$\overline{(a'b')} \neq \overline{a'b'}$$
$$\overline{a'^2} \neq 0$$
$$\overline{b'^2} \neq 0$$

### 4.1.3 Variance

$$\sigma_A^2 = \frac{1}{N} \sum_{i=1}^{N} (A_i - \bar{A})^2 = \overline{a'^2}$$

### 4.1.4 Standard deviation

$$\sigma_A = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (A_i - \bar{A})^2} = \left(\overline{a'^2}\right)^2$$

### 4.1.5 Covariance

$$\sigma_{A,B} = \frac{1}{N} \sum_{i=1}^{N} (A_i - \bar{A})(B_i - \bar{B}) = \overline{a'b'}.$$

### 4.1.6 Correlation

$$\gamma_{A,B} = \frac{\overline{a'b'}}{\sigma_A \sigma_B}$$

Mean Kinetic Energy (MKE) = $\frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2)$

Turbulent Kinetic Energy (TKE) = $\frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

---

**Question 4.1:**

Suppose we erect instruments with an anemometer to measure $u$ and $w$ components, recording wind speeds every 6 seconds for a minute, resulting in the following 10 points of wind observations shown in Table 2. Calculate the mean, variance, and standard deviation for each component. Also, find the covariance and correlation between them.

---

| u (m/s) | 5 | 6 | 5 | 4 | 7 | 5 | 3 | 5 | 4 | 6 |
|---------|---|---|---|---|---|---|---|---|---|---|
| w (m/s) | 0 | -1 | 1 | 0 | -2 | 1 | 2 | -1 | 1 | -1 |

**Table 2.** Wind observations for $u$ and $w$ components.

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**Answer 4.1:**

Solution:

$$\bar{u} = 5, \qquad \bar{v} = 0$$
$$\sigma_U^2 = 1.2, \qquad \sigma_W^2 = 1.1$$
$$\sigma_U = \sqrt{1.2}, \quad \sigma_W = \sqrt{1.1}$$
$$\sigma_{U,W} = \overline{u'w'} = -1.1, \quad \gamma_{U,W} =$$

## 5. Lecture 5 24/01/2025

### 5.1 Fluxes

Mass, Heat, Moisture, Momentum, Pollutent, etc.

| Quantity | Unit |
|----------|------|
| Mass | $kg_{\text{air}}/m^2 s$ |
| Heat | $J/m^2 s$ |
| Moisture | $kg_{\text{wv}}/m^2 s$ |
| Momentum | $kgms^{-1}/m^2 s$ |
| Pollutant | $kg_{\text{pollutant}}/m^2 s$ |

### 5.2 Kinematic Flux

**Note:** We asssume atmoshpere to be of constant density ($\rho_{\text{air}}$).

| Flux Quantity | Formula | Unit |
|---------------|---------|------|
| Mass flux | $\frac{\text{mass}}{\rho_{\text{air}}}$ | $kg_{\text{air}}/m^2 s$ |
| Heat flux | $\frac{\text{heat}}{\rho C_p}$ | $J/m^2 s$ |
| Moisture flux | - | $kg_{\text{wv}}/m^2 s$ |
| Momentum flux | - | $kgms^{-1}/m^2 s$ |
| Pollutant flux | - | $kg_{\text{pollutant}}/m^2 s$ |

Examples:-

$$\text{Veritcal advective heat flux} = \overline{w}\,\overline{\theta}$$
$$\text{x-direction advective heat flux} = \overline{u}\,\overline{\theta}$$
$$\text{x-direction moisture flux} = \overline{u}\overline{q}$$
$$\text{Veritcal eddy heat flux} = \overline{w\theta}$$
$$\text{x-direction eddy flux} = \overline{uq}$$

$$\text{x-direction solar flux} = \left(\frac{1/4 S_0}{c_p \rho}\right) = 0.2773 \text{ Km/s}$$

### 5.3 Eddy Flux
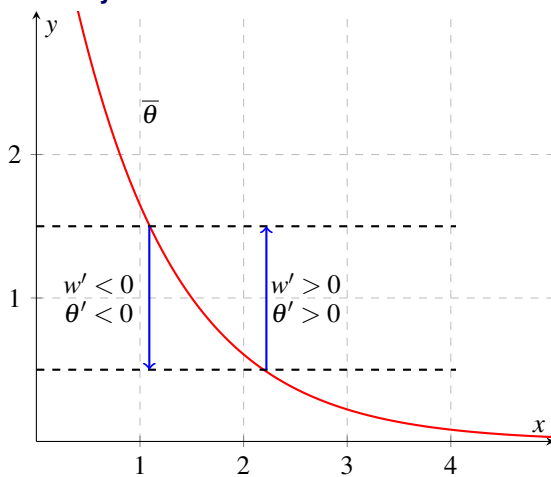


$$\Delta w' = 0$$
$$\Delta(\overline{w'\theta'}) > 0$$
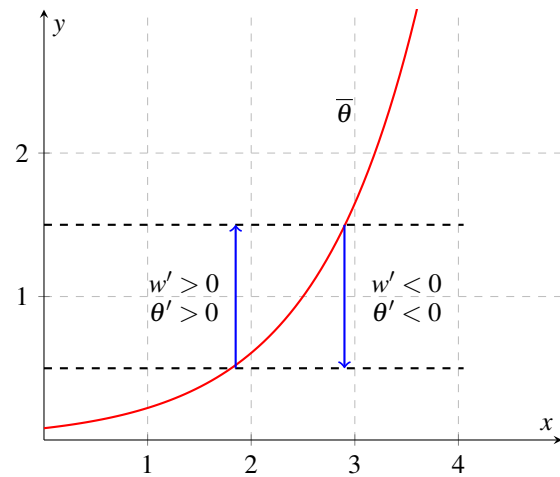


$$\Delta w' = 0$$
$$\Delta(\overline{w'\theta'}) < 0$$

### 5.4 Stress

Force tending to produce deformation in a body is called **Stress**.
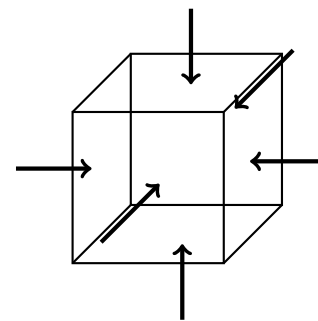


#### 5.4.1 Reynold's stress
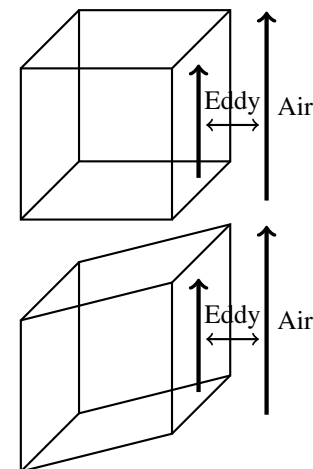
## 6. Lecture 6 29/01/2025

### 6.1 Summation Notation
**Definition:** Integer variables $m, n$, and $q$ can take values $1, 2, 3$. Then:

- $A_m$ is a velocity vector.
- $X_m$ represents distance.

where,

$$m = 1, 2, 3,$$
$$n = 1, 2, 3,$$
$$q = 1, 2, 3.$$

Hence,

$$A_1 = u, \qquad A_2 = v, \qquad A_3 = w,$$
$$X_1 = x, \qquad X_2 = y, \qquad X_3 = z.$$

### 6.2 Unit vector
Unit vector is represented by $\delta_m$,
where,

$$\delta_1 = \hat{i},$$
$$\delta_2 = \hat{j},$$
$$\delta_3 = \hat{k}$$

### 6.3 Kronecker Delta
Kronecker Delta is represented by $\delta_{mn}$, such that,

$$\delta_{mn} = \begin{cases} 1, & \text{for } m = n, \\ 0, & \text{for } m \neq n \end{cases}$$

### 6.4 Alternating Unit Tensor
Alternating Unit Tensor is represented by $\varepsilon_{mn}$, such that,

$$\delta_{mn} = \begin{cases} +1, & \text{for } mnq = 123, 231, 312 \\ -1, & \text{for } mnq = 321, 132, 213 \\ 0, & \text{for any 2 or more indices alike} \end{cases}$$

### 6.5 Rules For Summation Notation
**Rule I:**
Wherever two identical appear same in one term, it is implied that there is a sum of that term over each value of repreated index.
For example:
1.
$$A_n \frac{\partial B_m}{\partial X_n} = A_1 \frac{\partial B_m}{\partial X_1} + A_2 \frac{\partial B_m}{\partial X_2} + A_3 \frac{\partial B_m}{\partial X_3}$$
$$= u \frac{\partial B_m}{\partial x} + v \frac{\partial B_m}{\partial y} + w \frac{\partial B_m}{\partial z}$$

2.
$$\delta_{2n} A_n = \delta_{21} A_1 + \delta_{22} A_3 + \delta_{23} A_3$$
$$= A_2$$
$$= v$$

**Rule II:**
Wherever one index appear unsummed in one term, then the same index appear unsummed in all 3 terms in that equation. Hence that equation effectively represents 3 equations for each value of unsummed index.

$$A_m = B_m + \delta_{mn} C_n$$
$$A_1 = B_1 + \delta_{1n} C_n = B_1 + C_1$$
$$A_2 = B_2 + \delta_{2n} C_n = B_2 + C_2$$
$$A_2 = B_3 + \delta_{3n} C_n = B_3 + C_3$$

For example:
1.
$$\frac{\partial A_m}{\partial t} + B_n \frac{\partial A_m}{\partial X_n} = -\delta_{m3} g + f \varepsilon_{mn3} B_n$$
$$-\frac{1}{\rho} \frac{\partial P}{\partial X_m} + \frac{1}{\rho} \left[ \frac{\tau_{mn}}{\partial X_n} \right]$$

Expanding with $n = 1, 2$ and 3, we get:

$$\frac{\partial A_m}{\partial t} + B_1 \frac{\partial A_m}{\partial X_1} + B_2 \frac{\partial A_m}{\partial X_2} + B_3 \frac{\partial A_m}{\partial X_3} =$$
$$-\delta_{m3} g - f[\varepsilon_{m13} B_1 + \varepsilon_{m23} B_2 + \varepsilon_{m33} B_3]$$
$$-\frac{1}{\rho} \left[ \frac{\partial P}{\partial X_m} \right] + \frac{1}{\rho} \left[ \frac{\tau_{m1}}{\partial X_1} + \frac{\tau_{m2}}{\partial X_2} + \frac{\tau_{m3}}{\partial X_3} \right]$$

Now expanding with $m = 1$, we get following equation:

$$\frac{\partial A_1}{\partial t} + B_1 \frac{\partial A_1}{\partial X_1} + B_2 \frac{\partial A_1}{\partial X_2} + B_3 \frac{\partial A_1}{\partial X_3} =$$
$$-\delta_{13} g + f[\varepsilon_{113} B_1 + \varepsilon_{123} B_2 + \varepsilon_{133} B_3]$$
$$-\frac{1}{\rho} \left[ \frac{\partial P}{\partial X_1} \right] + \frac{1}{\rho} \left[ \frac{\tau_{11}}{\partial X_1} + \frac{\tau_{12}}{\partial X_2} + \frac{\tau_{13}}{\partial X_3} \right]$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = +fv - \frac{1}{\rho} \left[ \frac{\partial P}{\partial x} \right]$$
$$+\frac{1}{\rho} \left[ \frac{\tau_{xx}}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} \right]$$

Now expanding with $m = 2$, we get following equation:

$$\frac{\partial A_2}{\partial t} + B_1 \frac{\partial A_2}{\partial X_1} + B_2 \frac{\partial A_2}{\partial X_2} + B_3 \frac{\partial A_2}{\partial X_3} =$$
$$-\delta_{23} g + f[\varepsilon_{213} B_1 + \varepsilon_{223} B_2 + \varepsilon_{233} B_3]$$
$$-\frac{1}{\rho} \left[ \frac{\partial P}{\partial X_2} \right] + \frac{1}{\rho} \left[ \frac{\tau_{21}}{\partial X_1} + \frac{\tau_{22}}{\partial X_2} + \frac{\tau_{23}}{\partial X_3} \right]$$
$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -fu - \frac{1}{\rho} \left[ \frac{\partial P}{\partial y} \right]$$
$$+\frac{1}{\rho} \left[ \frac{\tau_{yx}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{yz}}{\partial z} \right]$$

Now expanding with $m = 3$, we get following equation:

$$\frac{\partial A_3}{\partial t} + B_1 \frac{\partial A_3}{\partial X_1} + B_2 \frac{\partial A_3}{\partial X_2} + B_3 \frac{\partial A_3}{\partial X_3} =$$

$$- \delta_{33} g + f \left[ \varepsilon_{313} B_1 + \varepsilon_{323} B_2 + \varepsilon_{333} B_3 \right]$$

$$- \frac{1}{\rho} \left[ \frac{\partial P}{\partial X_3} \right] + \frac{1}{\rho} \left[ \frac{\tau_{31}}{\partial X_1} + \frac{\tau_{32}}{\partial X_2} + \frac{\tau_{33}}{\partial X_3} \right]$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -g - \frac{1}{\rho} \left[ \frac{\partial P}{\partial z} \right]$$

$$+ \frac{1}{\rho} \left[ \frac{\tau_{zx}}{\partial x} + \frac{\tau_{zy}}{\partial y} + \frac{\tau_{zz}}{\partial z} \right]$$

## 7. Lecture 7 30/01/2025

### 7.1 Governing Equations For Turbulent Flow

1. Mean variable $(\bar{u}, \bar{v}, \bar{w})$.
2. Flux variable $(\overline{u'w'}, \overline{w'\theta'})$.
3. Total Kinetic Energy $(\overline{u'^2}, \overline{v'^2}, \overline{w'^2})$.

**Steps:**

1. Identify the basic equations.
2. Expand the total derivative into local and advective term.
3. Expand dependent variables into mean and turbulent parts.
4. Apply Reynold's averaging.
5. Add continuity equation to put the result in flux part.

### 7.2 Governing Equations

1. Equation of state

$$P = \rho R T \qquad (7.1)$$

2. Conservation of Mass or Continuity eqaution

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \qquad (7.2)$$

If **incompressible**, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = \text{constant}$

$$\frac{\partial (\rho U_i)}{\partial x_i} = 0$$

$$\rho \left[ \frac{\partial U_1}{\partial X_1} + \frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_3} \right] = 0$$

$$\left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] = 0$$

$$\nabla \cdot u = 0 \qquad (7.3)$$

3. Concservation of Momentum or Momentum equation

$$\frac{\partial U_i}{\partial t} + U_i \frac{\partial U_i}{\partial x_i} = -\delta_{i3}g + f\varepsilon_{ij3}U_j$$

$$- \frac{1}{\rho}\frac{\partial P}{\partial x_i}$$

$$+ \underbrace{\gamma}_{\substack{\text{Kinetic} \\ \text{viscousity} \\ \text{coeff.}}} \underbrace{\frac{\partial^2 U_i}{\partial x_i^2}}_{\text{viscousity}} \qquad (7.4)$$

4. Conservartion of Moisture

$$\frac{\partial q}{\partial t} + U_i \frac{\partial q}{\partial x_i} = \underbrace{\underbrace{v_q}_{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{coeff.}}} \cdot \frac{\partial^2 q}{\partial x_i^2}}_{\text{Diffusion term}}$$

$$+ \underbrace{\frac{S}{\rho_{\text{air}}}}_{\text{Residue term}} + \underbrace{\frac{E}{\rho_{\text{air}}}}_{\text{Phase change}} \qquad (7.5)$$

5. Conservartion of Heat or Thermodynamic equation

$$\frac{\partial \theta}{\partial t} + U_i \frac{\partial \theta}{\partial x_i} = \underbrace{\overbrace{v_\theta}^{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{heat coeff.}}} \cdot \frac{\partial^2 \theta}{\partial x_i^2}}_{\text{Diffusive heat}} - \underbrace{\frac{\partial Q_i^2}{\partial x_i}}_{\substack{\text{Radiative} \\ \text{divergence}}}$$

$$- \underbrace{\frac{L_p E}{\rho_{\text{air}} C_p}}_{\text{Phase change}} \qquad (7.6)$$

6. Conservartion of Scalar quantity

$$\frac{\partial C}{\partial t} + U_i \frac{\partial C}{\partial x_i} = \underbrace{\overbrace{v_c}^{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{coeff.}}} \frac{\partial^2 C}{\partial x_i^2}}_{\text{Diffusion term}} + \underbrace{S_c}_{\text{Phase change}} \qquad (7.7)$$

# 8. Lecture 8 12/02/2025

The governing equations in preivous lecture can not be used for practical applications, since in boundary layer we deal with turbulence which are not included in any of these equations. Now we need to account for turbulence, which can be heat flux, momentum flux, etc., which will added in the governing equations respectively.

## 8.1 Governing Equations For Turbulent Flow

### 8.1.1 Equation of state

$$P = \rho R_d T_v \tag{8.1}$$

To account for turbulence we will split equation in mean and turbulence (purturbation) parts, that is,

$$P = \overline{P} + P' \tag{8.2.i}$$
$$\rho = \overline{\rho} + \rho' \tag{8.2.ii}$$
$$T_v = \overline{T_v} + T_v' \tag{8.2.iii}$$

Substituting Eq.(8.2.i), Eq.(8.2.ii) and Eq.(8.2.iii) in Eq.(8.1), we obtain:

$$(\overline{P} + P') = (\overline{\rho} + \rho') R_d (\overline{T_v} + T_v')$$
$$\frac{(\overline{P} + P')}{R_d} = \overline{\rho}\,\overline{T_v} + \overline{\rho}T_v' + \rho'\overline{T_v} + \rho'T_v' \tag{8.3}$$

Taking Reynold's average,

$$\frac{\overline{\overline{P}} + \overline{P'}}{R_d} = \overline{\overline{\rho}\,\overline{T_v}} + \overline{\overline{\rho}T_v'} + \overline{\rho'\overline{T_v}} + \overline{\rho'T_v'}$$
$$\frac{\overline{P}}{R_d} = \overline{\rho}\,\overline{T_v} + \overline{\rho'T_v'}$$
$$\frac{\overline{P}}{R_d} \approx \overline{\rho}\,\overline{T_v} \tag{8.4}$$

Equation of state remain equation of stae in turbulent flow (no addition of termulent term).
Subtract Eq.(8.4) from Eq.(8.3), we get turbulent form:

$$\frac{P'}{R_d} = \overline{\rho}T_v' + \rho'\overline{T_v} + \rho'T_v' \tag{8.5}$$

Dividing Eq.(8.5) by Eq.(8.4)

$$\frac{P'}{\overline{P}} = \frac{T_v'}{\overline{T_v}} + \frac{\rho'}{\overline{\rho}} + \underbrace{\frac{\rho'T'}{\overline{\rho}\,\overline{T_v}}}_{\approx 0}$$
$$\frac{P'}{\overline{P}} = \frac{T_v'}{\overline{T_v}} + \frac{\rho'}{\overline{\rho}} \tag{8.6}$$

For practical purposes term $(P'/\overline{P})$ is negligible as compared to density and temperature terms. Therefore,

$$\frac{\rho'}{\overline{\rho}} \approx \frac{\theta_v'}{\theta_v} \tag{8.7}$$

where $\theta$ is potential termperature.

### 8.1.2 Flux Form of Adection Equation

$$u_j \frac{\partial \theta}{\partial x_j} \rightarrow \text{Advection equation for tremperature}$$

$$\frac{\partial u_j}{\partial x_j} = 0$$

Multiply above equation by temperature ($\theta$) and add in the previous equation, we obtain;

$$u_j \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial u_j}{\partial x_j} = 0$$
$$\frac{\partial (u_j \theta)}{\partial x_j} = 0 \tag{8.8}$$

**Note:** When $\theta$ replaced with $u_j$, the above Eq.(8.8) will become flux form of momentum equation.

### 8.1.3 Conservation of Mass

$$\frac{\partial u_j}{\partial x_j} = 0$$

Spliting equation in mean and turbulence (purturbation) parts, that is,

$$\frac{\partial \left( \overline{u_j} + u_j' \right)}{\partial x_j} = 0$$
$$\frac{\partial \left( \overline{u_j} + u_j' \right)}{\partial x_j} = 0$$
$$\frac{\partial \overline{u_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} = 0 \tag{8.9}$$

Applying Reynold's averaging,

$$\overline{\frac{\partial \overline{u_j}}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial x_j}} = 0$$
$$\frac{\partial \overline{u_j}}{\partial x_j} = 0 \tag{8.10}$$

Subtracting Eq.(8.10) from Eq.(8.9), we get:

$$\frac{\partial u_j'}{\partial x_j} = 0 \rightarrow \text{Turbulent variable} \tag{8.11}$$

### 8.1.4 Conservation of Momentum

$$\left. \begin{array}{ll} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} & = -\delta_{i3}\left[ g - \left( \frac{\theta_v'}{\theta_v} \right) \right] + f\varepsilon_{ij3}u_j \\ & -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \gamma \frac{\partial^2 u_i}{\partial x_i^2} \end{array} \right\} \tag{8.12}$$

Spliting equation in mean and turbulence (purturbation) parts, that is,

$$\left.\begin{aligned}
\frac{\partial(\overline{u_i}+u_i')}{\partial t} + \left(\overline{u_j}+u_j'\right)\frac{\partial(\overline{u_i}+u_i')}{\partial x_j} \\
= -\delta_{i3}\left[g-\left(\frac{\theta_v'}{\theta_v}\right)\right] + f\varepsilon_{ij3}\left(\overline{u_j}+u_j'\right) \\
- \frac{1}{(\overline{\rho}+\rho')}\frac{\partial(\overline{P}+P')}{\partial x_j} + \gamma\frac{\partial^2(\overline{u_i}+u_i')}{\partial x_j^2}
\end{aligned}\right\} \quad (8.13)$$

Assuming density of boundary layer does not change, i.e., $\rho' = 0$

$$\left.\begin{aligned}
\frac{\partial\overline{u_i}}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j} + \overline{u_j}\frac{\partial u_i'}{\partial x_j} + u_j'\frac{\partial\overline{u_i}}{\partial x_j} + u_j'\frac{\partial u_i'}{\partial x_j} \\
= -\delta_{i3}g - \delta_{i3}\left(\frac{\theta_v'}{\theta_v}\right) + f\varepsilon_{ij3}\overline{u_j} + f\varepsilon_{ij3}u_j' \\
- \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j} - \frac{1}{\overline{\rho}}\frac{\partial P'}{\partial x_j} + \gamma\frac{\partial^2\overline{u_i}}{\partial x_j} + \gamma\frac{\partial^2 u_i'}{\partial x_j^2}
\end{aligned}\right\}$$

$$(8.14)$$

Applying Reynold's averaging,

$$\overline{\frac{\partial\overline{u_i}}{\partial t}} + 0 + \overline{\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}} + 0 + 0 + \overline{u_j'\frac{\partial u_i'}{\partial x_j}}$$

$$= -\delta_{i3}g - 0 + f\varepsilon_{ij3}\overline{u_j} + 0$$

$$- \frac{1}{\overline{\rho}}\overline{\frac{\partial\overline{P}}{\partial x_j}} - 0 + \gamma\overline{\frac{\partial^2\overline{u_i}}{\partial x_j}} + 0$$

$$\left.\begin{aligned}
\frac{\partial\overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} = -\delta_{i3}g + f\varepsilon_{ij3}\overline{u_j} \\
- \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j} + \gamma\frac{\partial^2\overline{u_i}}{\partial x_j}
\end{aligned}\right\} \quad (8.15)$$

$$\frac{\partial u_j'}{\partial x_j} = 0$$

Multiply above equation by velocity ($u_i'$) and take Reynold's average

$$\overline{u_i'\frac{\partial u_j'}{\partial x_j}} = 0$$

Add above equation in the Eq.(8.15), we obtain;

$$\frac{\partial\overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} + \overline{u_i'\frac{\partial u_j'}{\partial x_j}} = -\delta_{i3}g$$

$$+ f\varepsilon_{ij3}\overline{u_j} - \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j} + \gamma\frac{\partial^2\overline{u_i}}{\partial x_j}$$

$$\left.\begin{aligned}
\frac{\partial\overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j} + \underbrace{\frac{\overline{\partial\left(u_i'u_j'\right)}}{\partial x_j}}_{\text{III}} = -\delta_{i3}g \\
+ f\varepsilon_{ij3}\overline{u_j} - \frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j} + \gamma\frac{\partial^2\overline{u_i}}{\partial x_j}
\end{aligned}\right\} \quad (8.16)$$

$3^{rd}$ term marked in Eq.(8.16) is momentum flux (turbulence) which represents eddies.

## 9. Lecture 9 13/02/2025

### 9.1 Governing Equations For Turbulent Flow

#### 9.1.1 Turbulent Momentum Flux Equation

General momentum equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3}g + f\varepsilon_{i33}u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i}$$
$$+ v\frac{\partial^2 u}{\partial x_i^2}$$

Turbulent form of momentum equation:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\delta_{i3}g + f\varepsilon_{i33}\overline{u_j} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}$$
$$+ v\frac{\partial^2 \overline{u}}{\partial x_i^2} + \frac{\overline{u_i'u_j}}{\partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\delta_{i3}g + f\varepsilon_{i33}\overline{u_j} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}$$
$$+ v\frac{\partial^2 \overline{u}}{\partial x_i^2} + \underbrace{\frac{1}{\overline{\rho}}\frac{\overline{\rho u_i'u_j}}{\partial x_j}}_{\text{Reynold's shress}}$$

#### 9.1.2 Turbulent Heat Flux Equation

General Heat equation:

$$\frac{\partial \theta}{\partial t} + u_j\frac{\partial \theta}{\partial x_j} = v_\theta\frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho C_p}\frac{\partial Q_j^*}{\partial x_j}$$
$$- \underbrace{\frac{L_pE}{\rho C_p}}_{\text{Phase change}}$$

$$\frac{\partial (\overline{\theta}+\theta')}{\partial t} + (\overline{u_j}+u_j')\frac{\partial (\overline{\theta}+\theta')}{\partial x_j} = v_\theta\frac{\partial^2 \theta}{\partial x_j^2}$$
$$- \frac{1}{(\overline{\rho}+\rho')C_p}\frac{\partial Q_j^*}{\partial x_j} - \frac{L_pE}{(\overline{\rho}+\rho')C_p}$$

Assuming density of boundary layer does not change, i.e., $\rho' = 0$

$$\left.\begin{array}{l}\frac{\partial (\overline{\theta}+\theta')}{\partial t} + \left(\overline{u_j}+u_j'\right)\frac{\partial (\overline{\theta}+\theta')}{\partial x_j} = v_\theta\frac{\partial^2 \theta}{\partial x_j^2}\\[2mm] - \frac{1}{\overline{\rho}C_p}\frac{\partial \left(\overline{Q_j^*}+Q_j^{*\prime}\right)}{\partial x_j} - \frac{L_pE}{\overline{\rho}C_p}\end{array}\right\} \quad (9.3)$$

$$\left.\begin{array}{l}\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \overline{u_j}\frac{\partial \overline{\theta}}{\partial x_j} + \overline{u_j}\frac{\partial \theta'}{\partial x_j} + u_j'\frac{\partial \overline{\theta}}{\partial x_j} + u_j'\frac{\partial \theta'}{\partial x_j}\\[2mm] = v_\theta\frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\overline{\rho}C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j} + \frac{1}{\overline{\rho}C_p}\frac{\partial Q_j^{*\prime}}{\partial x_j}\\[2mm] - \frac{L_pE}{\overline{\rho}C_p}\end{array}\right\} \quad (9.4)$$

Apply Reynold's averaging;

$$\left.\begin{array}{l}\frac{\partial \overline{\theta}}{\partial t} + \overline{u_j}\frac{\partial \overline{\theta}}{\partial x_j} + \overline{u_j'\frac{\partial \theta'}{\partial x_j}} = v_\theta\frac{\partial^2 \overline{\theta}}{\partial x_j^2} - \frac{1}{\overline{\rho}C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j}\\[2mm] - \frac{L_pE}{\overline{\rho}C_p}\end{array}\right\} \quad (9.5)$$

Add $\overline{\theta'\frac{\partial u_j'}{\partial x_j}} = 0$ in above Eq.(9.5)

$$\left.\begin{array}{l}\frac{\partial \overline{\theta}}{\partial t} + \overline{u_j}\frac{\partial \overline{\theta}}{\partial x_j} + \frac{\partial \overline{\left(u_j'\theta'\right)}}{\partial x_j} = v_\theta\frac{\partial^2 \overline{\theta}}{\partial x_j^2} - \frac{1}{\overline{\rho}C_p}\frac{\partial \overline{Q_j^*}}{\partial x_j}\\[2mm] - \frac{L_pE}{\overline{\rho}C_p}\end{array}\right\} \quad (9.6)$$

**Reynold's number :**

$$R_d = \frac{\text{intertial force}}{\text{viscous force}}$$
$$= \frac{\rho vL}{\mu}$$
$$= \frac{vL}{v}$$

Taking a real-world scenario of a boundary layer, where $v = 5$ m/s, $L = 100$ m, and $v = 1.5 \times 10^{-5}$ m²/s
$\therefore R_d = \frac{5\times 100}{1.5\times 10^{-5}} \approx 3.3 \times 10^7$.

$$v = \frac{vL}{R_d}$$

$$v\frac{\partial^2 \overline{u_i}}{\partial x_j^2} = \underbrace{\frac{1}{R_d}}_{\text{very large}}\left[vL\frac{\partial^2 \overline{u_i}}{\partial x_j^2}\right] \approx 0$$

Therefore viscous term can be neglected for boundary layer in most of the cases.

Hence modified turbulent momentum, heat flux, moisture and pollutant equations can be written respectively as following:

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = -\delta_{i3}g + f\varepsilon_{i33}\overline{u_j} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{\overline{\rho}}\frac{\overline{\rho u_i'u_j}}{\partial x_j} \quad (9.7)$$

TODO: To write turbulent heat, moisture, pollutant equations

## 10. Lecture 10 14/02/2025

### 10.1 Governing Equations For Turbulent Flow

In the previous lecture, we introduced the turbulent flux terms $\left(\text{i.e., } \frac{\partial \overline{u'_j \beta'}}{\partial x_j}, \text{ where } \beta \text{ can be } u'_j, \theta', C' \text{ or } Q'\right)$ in the governing equations.

These terms can be expanded using index '$j$' of velocity $u$ and cancel out horizontal flucuation because in general, assuming, only vertical transport take place due to turbulence in following way:

$$\text{Heat flux} \Rightarrow \overline{u'_j \theta'} \rightarrow \overline{\cancel{u'\theta'}}, \overline{\cancel{v'\theta'}}, \overline{w'\theta'}$$

$$\text{Kinematic moisture flux} \Rightarrow \overline{u'_j q'} \rightarrow \overline{\cancel{u'q'}}, \overline{\cancel{v'q'}}, \overline{w'q'}$$

$$\text{Pollutant flux} \Rightarrow \overline{u'_j C'} \rightarrow \overline{\cancel{u'C'}}, \overline{\cancel{v'C'}}, \overline{w'C'}$$

***Note***: Sonic anemometer is used to measure $w' (= w - \overline{w})$

### 10.2 Physical Significance Of Turbulent Terms

$$\overline{w'\theta'} = \begin{cases} > 0 & \text{Heat transport in upward direction} \\ < 0 & \text{Heat transport in downward direction (through inversion layer)} \end{cases}$$

$$\overline{w'q'} = \begin{cases} > 0 & \text{Moisture transport in upward direction (supply of moisture from surface)} \\ < 0 & \text{Moisture transport in downward direction (supply of moisture from TOA)} \end{cases}$$

### 10.3 Scaling Variable

To account for time deplay in measurements and make data invariant of time.
$z_i$ is depth of mixed layer, then,

$$\frac{z}{z_i} \rightarrow \text{scaling variable}$$

$$\frac{(w'\theta')}{(\overline{w'q'})} \rightarrow \text{scaling variable}$$

We can intepret term $(\partial u'/\partial t)$ as prediction of every single eddy in atmosphere, but we don't use it becasue ot is not required and very computationally expensive.

### 10.4 Prognostic Equations For Turbulent Flow

$$\left. \begin{aligned} &\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial u'_i}{\partial t} + \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_i} \frac{\partial u'_i}{\partial x_j} + u'_j \frac{\partial \overline{u_i}}{\partial x_j} + u'_j \frac{\partial u'_i}{\partial x_j} \\ &= -\delta_{i3}g + \delta_{i3}\left(\frac{\theta'_v}{\overline{\theta}_v}\right)g + f\varepsilon_{i33}\overline{u_j} + f\varepsilon_{i33}u'_j \\ &-\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \nu \frac{\partial^2 u'_i}{\partial x_j^2} \end{aligned} \right\} \quad (10.1)$$

Apply Reynold's averaging;

$$\left. \begin{aligned} \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} \quad &= -\delta_{i3}g + f\varepsilon_{i33}\overline{u_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} \\ &+ \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \nu \frac{\partial^2 u'_i}{\partial x_j^2} - \frac{\partial \overline{(u'_i u'_j)}}{\partial x_i} \end{aligned} \right\} \quad (10.2)$$

Subtracting Eq.(10.2) from Eq.(10.1)

$$\left. \begin{aligned} \frac{\partial u'_i}{\partial t} + \overline{u_i}\frac{\partial u'_i}{\partial x_j} + u'_j\frac{\partial \overline{u_i}}{\partial x_j} + u'_j\frac{\partial u'_i}{\partial x_j} &= \delta_{i3}\left(\frac{\theta'_v}{\overline{\theta}_v}\right)g \\ + f\varepsilon_{i33}u'_j - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u'_i}{\partial x_j^2} + \frac{\partial \overline{(u'_i u'_j)}}{\partial x_i} \end{aligned} \right\} \quad (10.3)$$

To be continued in next lecture...

## 11. Lecture 11 19/02/2025

### 11.1 Prognostic Equations For Turbulent Flow

Continuing from previous lecture.

Multiply Eq.(10.3) with $(2u_i')$

$$
\left.
\begin{aligned}
2u_i'\frac{\partial u_i'}{\partial t} &+ 2u_i'\overline{u_i}\frac{\partial u_i'}{\partial x_j} + 2u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j} + 2u_i'u_j'\frac{\partial u_i'}{\partial x_j} \\
&= 2\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + 2f\varepsilon_{i33}u_i'u_j' \\
&- \frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + 2u_i'\nu\frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i'\frac{\partial \overline{(u_i'u_j')}}{\partial x_i}
\end{aligned}
\right\}
\tag{11.1}
$$

Apply product rule,

$$
\left.
\begin{aligned}
\frac{\partial u_i'^2}{\partial t} &+ \overline{u_i}\frac{\partial u_i'^2}{\partial x_j} + 2u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j} + u_j'\frac{\partial u_i'^2}{\partial x_j} \\
&= 2\delta_{i3}u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)g + 2f\varepsilon_{i33}u_i'u_j' \\
&- \frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + 2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2} + 2u_i'\frac{\partial \overline{(u_i'u_j')}}{\partial x_i}
\end{aligned}
\right\}
\tag{11.2}
$$

Applying Reynold's averaging;

$$
\left.
\begin{aligned}
\frac{\partial \overline{u_i'^2}}{\partial t} &+ \overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'^2}{\partial x_j}} \\
&= 2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + 2f\varepsilon_{i33}\overline{u_i'u_j'} \\
&- \overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}} + \cancel{2u_i'\frac{\partial \overline{(u_i'u_j')}}{\partial x_i}}
\end{aligned}
\right\}
\tag{11.3}
$$

Adding Turbulent continuity equation to Eq.(11.3)

$$
\overline{\partial u_i'^2}\frac{u_j'}{\partial x_j} = 0
$$

$$
\left.
\begin{aligned}
\frac{\partial \overline{u_i'^2}}{\partial t} &+ \overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial u_i'^2}{\partial x_j}} + \overline{u_i'^2\frac{\partial u_j'}{\partial x_j}} \\
&= 2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + 2f\varepsilon_{i33}\overline{u_i'u_j'} \\
&- \overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}
\end{aligned}
\right\}
\tag{11.4}
$$

$$
\left.
\begin{aligned}
\frac{\partial \overline{u_i'^2}}{\partial t} &+ \overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial \left(u_i'^2u_j'\right)}{\partial x_j}} \\
&= 2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + 2f\varepsilon_{i33}\overline{u_i'u_j'} \\
&- \overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} + \underbrace{2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}}_{\text{Viscous disipation}}
\end{aligned}
\right\}
\tag{11.5}
$$

**Viscous desipation term:**

$$
\nu\frac{\partial \overline{u_i'^2}}{\partial x_j^2} = \nu\frac{\partial}{\partial x_j}\left[\overline{\frac{\partial u_i'^2}{\partial x_j}}\right]
$$

$$
\nu\frac{\partial \overline{u_i'^2}}{\partial x_j^2} = \nu\frac{\partial}{\partial x_j}\left[\overline{2u_i'\frac{\partial u_i'}{\partial x_j}}\right]
$$

$$
\nu\frac{\partial \overline{u_i'^2}}{\partial x_j^2} = 2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + 2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}
$$

$$
\nu\frac{\partial \overline{u_i'^2}}{\partial x_j^2} = 2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Shear}\\\text{term}}} + 2\nu\underbrace{\overline{\left(\frac{\partial^2 u_i'^2}{\partial x_j^2}\right)}}_{\substack{\text{Diffusion of}\\\text{velocity}(\approx 0)}}
\tag{11.6}
$$

Substitute approximate viscous term Eq.(11.6) in Eq.(11.5)

$$
\left.
\begin{aligned}
\frac{\partial \overline{u_i'^2}}{\partial t} &+ \overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial \left(u_i'^2u_j'\right)}{\partial x_j}} \\
&= 2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + 2f\varepsilon_{i33}\overline{u_i'u_j'} \\
&- \underbrace{\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}}_{\substack{\text{Pressure}\\\text{Perturbation}\\\text{term}}} + 2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Viscous}\\\text{disipation}\\\text{term}}}
\end{aligned}
\right\}
\tag{11.7}
$$

**Pressure perturbation term:**

$$
-\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}} = -\frac{2}{\overline{\rho}}\overline{\left(\frac{\partial (p'u_i')}{\partial x_i}\right)} + \frac{2p'}{\overline{\rho}}\cdot\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_i}\right)}}_{\substack{\nabla u_i'=0\\\text{Continuity eq.}}}
\tag{11.8}
$$

The last term in Eq.(11.8) is called **return-to-isotropy**. Therefore Eq.(11.7) can be modified to:

$$
\left.
\begin{aligned}
\frac{\partial \overline{u_i'^2}}{\partial t} &+ \overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}} + \overline{u_j'\frac{\partial \left(u_i'^2u_j'\right)}{\partial x_j}} \\
&= 2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\overline{\theta_v}}\right)}g + \underbrace{2f\varepsilon_{i33}\overline{u_i'u_j'}}_{\substack{\text{Coriolis}\\\text{term}}} \\
&- \underbrace{\frac{2}{\overline{\rho}}\overline{\frac{\partial (p'u_i')}{\partial x_i}}}_{\substack{\text{Pressure}\\\text{Perturbation}\\\text{term}}} + 2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Viscous}\\\text{disipation}\\\text{term}}}
\end{aligned}
\right\}
\tag{11.9}
$$

**Coriolis term:**

INCOMPLETE

## 12. Lecture 12 20/02/2025

### 12.1 Governing Equation For Turbulent Flow

$$\frac{\partial q'}{\partial t} + \overline{u}_j \frac{\partial \overline{q}}{\partial x_j} + u'_j \frac{\partial \overline{q}}{\partial x_j} + u'_j \frac{\partial q'}{\partial x_j} = \nu \frac{\partial^2 q'}{\partial x_j^2} + \frac{\partial \overline{(u'_i q')}}{\partial x_i} \quad (12.1)$$

Multiply above Eq.(12.1) with $2q'$

$$\left.\begin{array}{r} 2q'\frac{\partial q'}{\partial t} + 2q'\overline{u}_j \frac{\partial \overline{q}}{\partial x_j} + 2q'u'_j \frac{\partial \overline{q}}{\partial x_j} + 2q'u'_j \frac{\partial q'}{\partial x_j} \\[2mm] = 2q'\nu \frac{\partial^2 q'}{\partial x_j^2} + 2q' \frac{\partial \overline{(u'_i q')}}{\partial x_i} \end{array}\right\} \quad (12.2)$$

$$\left.\begin{array}{r} \frac{\partial (q')^2}{\partial t} + 2q'\overline{u}_j \frac{\partial \overline{q}}{\partial x_j} + 2q'u'_j \frac{\partial \overline{q}}{\partial x_j} + u'_j \frac{\partial (q')^2}{\partial x_j} \\[2mm] = 2q'\nu \frac{\partial^2 q'}{\partial x_j^2} + 2q' \frac{\partial \overline{(u'_i q')}}{\partial x_i} \end{array}\right\} \quad (12.3)$$

Applying Reynold's averaging,

$$\frac{\overline{\partial (q')^2}}{\partial t} + 2\overline{q'u'_j} \frac{\partial \overline{q}}{\partial x_j} + \overline{u'_j \frac{\partial (q')^2}{\partial x_j}} = 2\nu \overline{q' \frac{\partial^2 q'}{\partial x_j^2}} \quad (12.4)$$

Adding flux term $\overline{(q')^2 \frac{\partial u'_j}{\partial x_j}} = 0$

$$\frac{\overline{\partial (q')^2}}{\partial t} + 2\overline{q'u'_j} \frac{\partial \overline{q}}{\partial x_j} + \frac{\overline{\partial (u_j (q')^2)}}{\partial x_j} = 2\nu \overline{q' \frac{\partial^2 q'}{\partial x_j^2}} \quad (12.5)$$

Similarly,

$$\left.\begin{array}{r} \frac{\overline{\partial (u'_i)^2}}{\partial t} + 2\overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\overline{\partial (u_j (u'_i)^2)}}{\partial x_j} = 2\delta_{i3} g \frac{\overline{(u'_i \theta'_v)}}{\partial \overline{\theta}_v} \\[3mm] - \frac{2}{\rho} \frac{\overline{\partial u'_i p'}}{\partial x_j} - 2\varepsilon \end{array}\right\} \quad (12.6)$$

Total Kinetic Energy $= \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \Rightarrow \overline{e} = \frac{1}{2}\overline{u'^2_i}$

$$\frac{\overline{\partial e}}{\partial t} + \overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\overline{\partial (u_j e)}}{\partial x_j} = \delta_{i3} g \frac{\overline{(u'_i \theta'_v)}}{\partial \overline{\theta}_v} - \frac{1}{\rho} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon \quad (12.7)$$

In general, turbulent act only in vertical direction, i.e.

$$u_j = w, \quad x_j = z$$

$$\left.\begin{array}{r} \frac{\overline{\partial e}}{\partial t} + \underbrace{\overline{u'_i w'} \frac{\partial \overline{u}_i}{\partial z}}_{\substack{\text{Mechanical} \\ \text{Shear} \\ \text{Term}}} + \underbrace{\frac{\overline{\partial (u'_i e')}}{\partial z}}_{\substack{\text{Turbulent} \\ \text{Advection} \\ \text{Term}}} = \underbrace{g \frac{\overline{u'_i \theta'_v}}{\partial \overline{\theta}_v}}_{\substack{\text{Bouancy} \\ \text{Term}}} - \underbrace{\frac{1}{\rho} \frac{\partial \overline{w' p'}}{\partial z}}_{\substack{\text{Pressure} \\ \text{Correlation} \\ \text{Term}}} \\[5mm] - \underbrace{\varepsilon}_{\substack{\text{Viscous} \\ \text{Dissipation} \\ \text{Term}}} \end{array}\right\} \quad (12.8)$$

Turbulent advection term is neglected because generally mean wind can't carry turbulence to longer distance and get's decipacted.

- **Bouancy term :** Generation of turbulence due to heat flux. Which can be produce turbulence during unstable conditions (day time), can decipate turbulence during stable conditions (night time).
- **Mechanical Shear term :** Generation of turbulence due to shear in atmosphere. Negative sign represents turbulence will generate downward if mean windspeed is high upward, and vice versa. Mechanical Shear is only generation term, i.e., it can only generate turbulence and not decipacte.
- **Turbulent advection (transport) term :** Niether generates nor decipates turbulence but transport thermal energy. When integrate over entire BL, it goes to zero, but when local BL is considered atmoshpere bring turbulent energy from outside local BL or take away turbulent energy from local BL to outside.
- **Pressure correlation term :** Represents pressure flucuation, very small term ($\approx 0.01 hPa$). At stable BL (night time) it generates internal gravity waves to transport turbulence but in general it is a residual term.
- **Viscous decipative term :** Only decipates turbulence.

question and answers
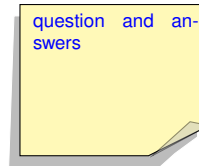
## References

[1] R. B. Stull. *An Introduction to Boundary Layer Meteorology*. Atmospheric Sciences Library. Kluwer Academic Publishers, 1988.