Lecture Notes on Boundary Layer Meteorology

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Abstract
Notes of Lectures and addional information from books:
An introduction to boundary layer meteorology([1]).

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1. Lecture 1 09/01/2025

1.1 Introduction To Boundary Layer

The Boundary Layer can be defined as part of the troposphere that is directly influenced by the presence of the Earth's surface and responds to surface forcings with a time scale of about an hour or less.

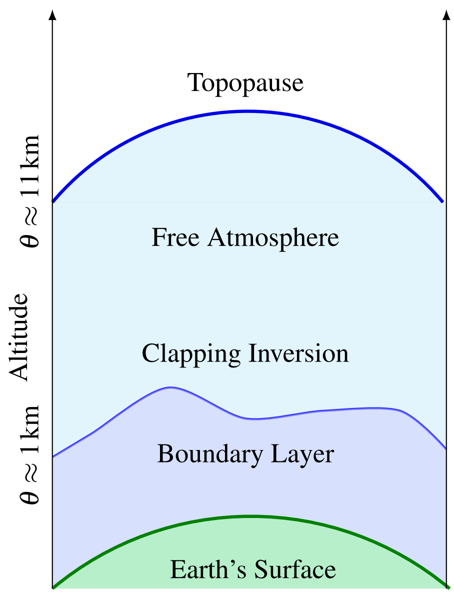


Figure 1.1. Atmosphere can be divided into 2 parts: boundary layer near surface and free atmosphere above it.

1.2 Boundary Layer Forcing Mechanism

What physical process modify boundary layer air parcel?

1. Heat transfer to/from the ground.
2. Frictional drag.
3. Evaporation/transpiration.
4. Terrain-induced flow modification.
5. Pollution emission.

1.3 Types Of Air Flow Or Wind

Air flow or wind can be decomposed into following 3 types:

1. **Mean Wind** $(\bar{u}, \bar{v}, \bar{w})$: Represents the average wind components in the horizontal $(\bar{u}, \bar{v})$ and vertical $(\bar{w})$ directions. It is important for the horizontal transport of quantities such as moisture, heat, momentum, and pollutants, a process known as advection.
2. **Waves**: Atmospheric waves, such as gravity waves, occur mostly at night in the nocturnal boundary layer (NBL). They can influence the structure of the boundary layer and the transport of energy.
3. **Turbulence**: The vertical transport of moisture, heat, momentum, and pollutants is primarily dominated by turbulence, which is characterized by chaotic and irregular motion.

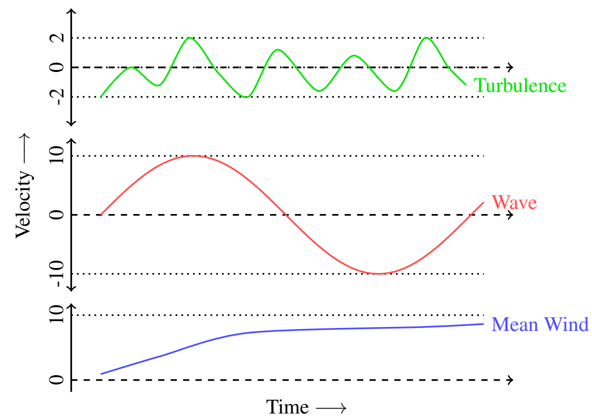


Figure 1.2. Plot showing profiles of Mean, Wave and Turbulent winds

1.4 Eddies

Eddies are formed due to the interaction of currents with obstacles like coastlines, underwater topography, or other currents, as well as from the instability of larger current systems. Eddies exhibit a rotational flow pattern, either clockwise or counterclockwise. Eddies can vary from size 100 to 3000 metres and also can exists as small as few millimetres. Small eddies might last for seconds to minutes, while larger oceanic eddies can persist for weeks, months, or even years.

1.5 Turbulence Generation Mechanisms

- **Solar Heating**: Solar heating generates thermals, which are essentially larger eddies that drive turbulence in the atmospheric boundary layer.
- **Wind Shear**: Variations in wind speed or direction with height create wind shear, which is a significant source of turbulence.
- **Obstacle-Induced Flow**: Deflected flow around obstacles such as trees, buildings, or other structures generates turbulent eddies downstream of these obstacles, creating turbulent wakes.

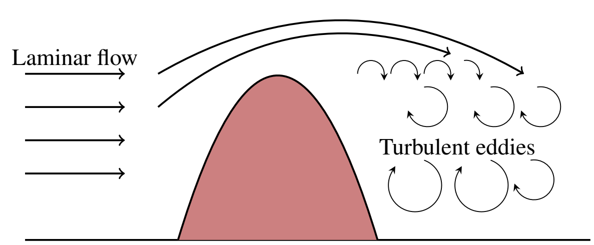


Figure 1.3. Eddy formation due to Turbulence caused by an obstacle

Large eddies will break into smaller eddies after which small eddies dissipates from K.E. to thermal energy.

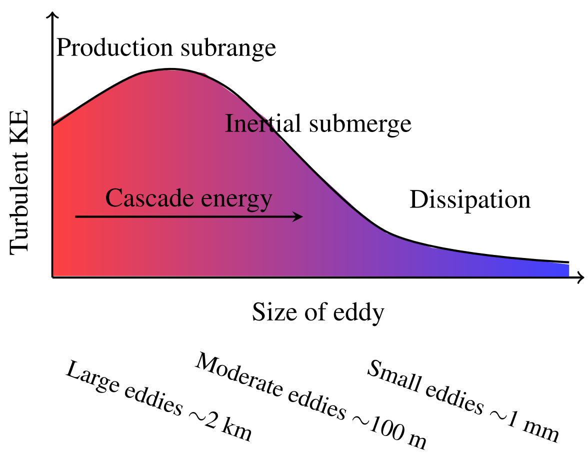


Figure 1.4. Variation of Turbulent Kinetic energy with change in Size of eddies

2. Lecture 2 15/01/2025

2.1 Taylor's Hypothesis

- When studying atmospheric boundary layer (ABL), It is not easy to create a snapshot of turbulence in the Atmosphere.
- Hence it is easier and cheaper to make measurements of point in the atmoshpere for a longer time, then an instantaneous snapshot.
- So we just consider the atmoshpere is frozen.
- ***Taylor suggested that turbulence can be considered frozen as it advects past sensor.***

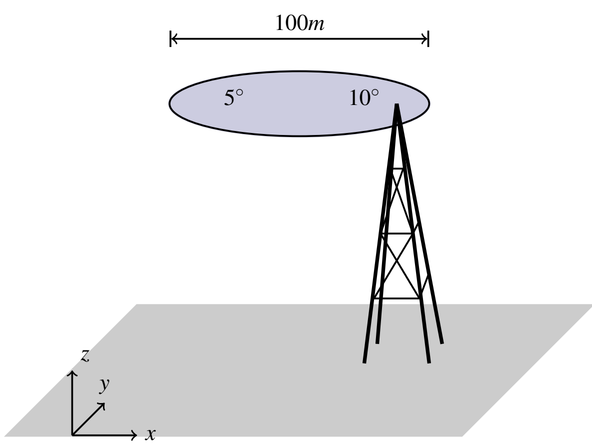


Figure 2.1. Eddy propagation

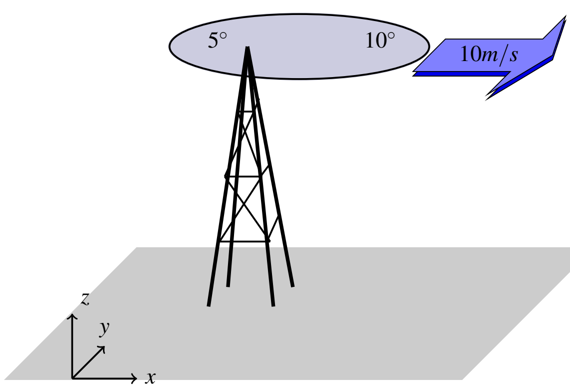


Figure 2.2. Eddy passing by the sensor mounted on tower

$$\frac{\partial T}{\partial x} = 0.05\text{K/m}, \quad \frac{\partial T}{\partial t} = -0.5\text{K/s}$$

$$\underbrace{\cancel{\frac{DT}{Dt}}}_{\substack{\text{Total derivative = 0} \\ \text{(Taylor's hypothesis)}}} = \underbrace{\frac{\partial T}{\partial t}}_{\text{Local derivative}} + \underbrace{u\frac{\partial T}{\partial x}}_{\text{Advective term}} \tag{2.1}$$

2.2 Virtual Potential Temperature

Virtual potential temperature:

$$\theta_v = \theta(1 + 0.61r) \tag{2.2}$$

Virtual temperature:

$$T_v = T(1 + 0.61r) \tag{2.3}$$

The variable r in equations (2.2) and (2.3) represents the **water vapor mixing ratio**, defined as:

$$r = \frac{m_v}{m_d}$$

where:

- m_v is the mass of water vapor in a given volume of air.
- m_d is the mass of dry air in the same volume.

The mixing ratio r is typically expressed in **kg of water vapor per kg of dry air** (kg/kg). It quantifies the amount of moisture in the air, which influences both the virtual temperature and virtual potential temperature by accounting for the effect of water vapor on air density.

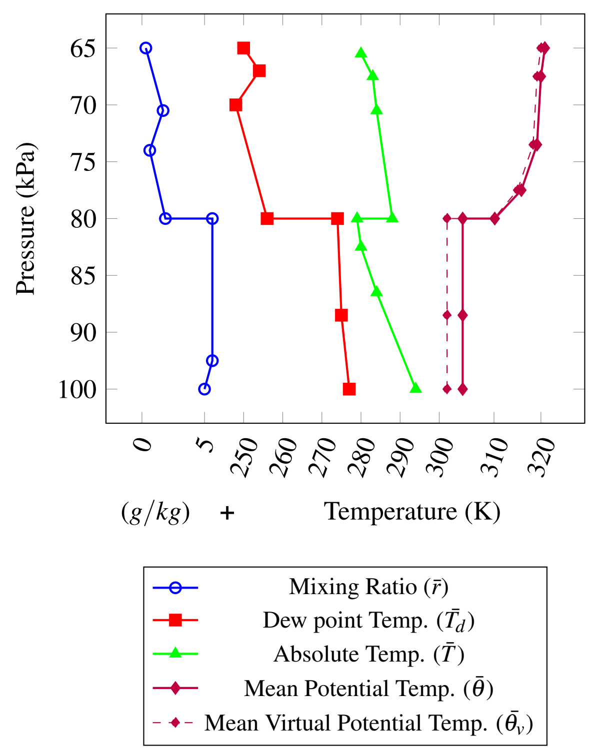


Figure 2.3. Pressure v/s Termperature

Question 2.1:

Given 25°C termperature, mixing ratio $\bar{r}$ is 20g/kg, measured Pressurere at 900hPa, find virtual potential temperature.

Answer 2.1:

Solution:

$$
\begin{aligned}
\theta &= T \times \left(\frac{1000}{P}\right)^{0.286} \\
&= 298 \times \left(\frac{1000}{900}\right)^{0.286} \\
&= 332.222K \\
\theta_v &= \theta \times \left(1 + 0.61r\right) \\
&= 332.22 \times \left(1 + 0.61 \times 0.025\right) \\
&= 336.273K \\
\theta_v - \theta &\approx 4.05K
\end{aligned}
$$

2.3 Boundary Layer Depth and Structure

- Mixed layer
- Residual layer
- Stable Boundary layer
- Capping Inversion
- Nocturnal Boundary layer

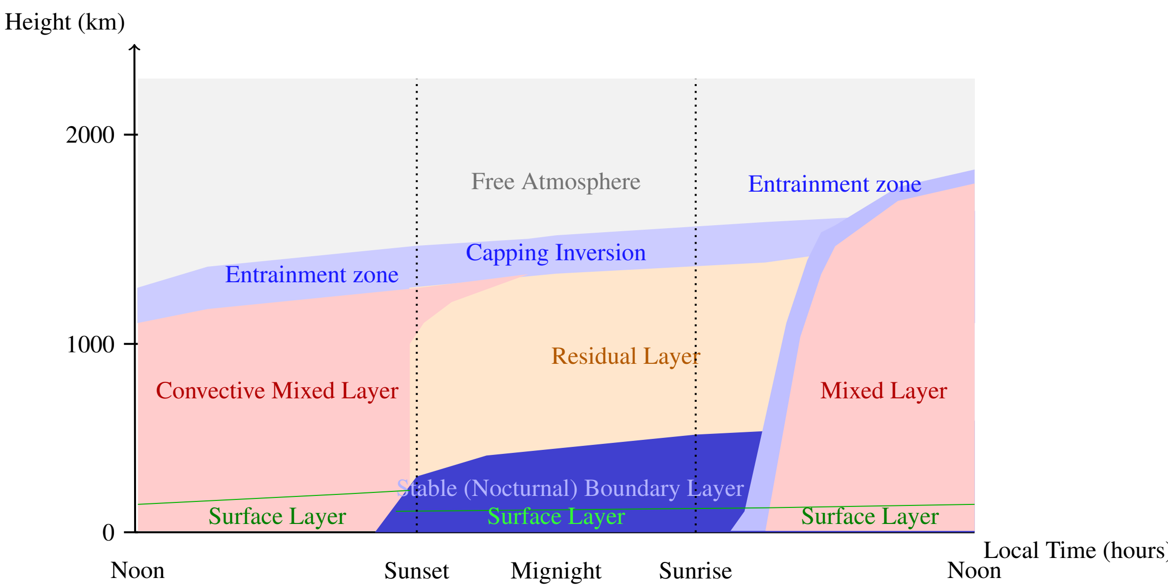


Figure 2.4. Height vs Local Time Diagram of Atmospheric Boundary Layers.

3. Lecture 3 16/01/2025

Ocean: Variations are minimal, with only 10% changes observed over 1000 km. Significant variations occur primarily during weather phenomena.
Land: Day-to-day and diurnal variations are prominent, with distinct boundary layer structures:

1. **Convective Mixed Layer:** Thermodynamically unstable with intense vertical mixing.
2. **Residual Layer:** Neutral stratification with turbulence of equal intensity in all directions.
3. **Stable Boundary Layer (Nocturnal B.L.):**
   - Neutral stratification with nocturnal jets (≈30 m/s, ≈200 m width).
   - Sporadic turbulence and internal gravity waves transporting air parcels vertically.
4. **Capping Inversion:** Found at altitudes between 1.5–3 km, acting as a barrier to upward mixing.
5. **Entrainment Zone:** Transition region from stable to unstable conditions, facilitating energy and mass exchange.

3.1 Stability and Plume Behaviour

1. **Looping plumes:** Occur in highly unstable conditions, usually during the day, when strong surface heating causes warm air to rise and interact turbulently with cooler air. This leads to an oscillatory motion that disperses pollutants in multiple directions, often seen in arid regions.

FIGURE for each to be added later.

   Fanning plumes: Form in stable conditions, typically at night, when surface cooling creates temperature inversions. The plume spreads horizontally, concentrating pollutants close to the surface, which can impact air quality in urban or industrial areas.
3. **Coning plumes:** Develop under neutral conditions, where vertical and horizontal mixing is balanced. The plume takes a cone-like shape, often observed on overcast days or in the early morning and evening, with moderate turbulence.
4. **Lofting plumes:** Occur when the atmosphere is stable near the ground but unstable above. Pollutants rise and disperse above the stable layer, reducing ground-level pollution and minimizing surface concentrations.
5. **Fumigation plumes:** Happen when pollutants are trapped in a stable layer and then forced downward due to rising turbulence. This leads to high concentrations at the surface, posing risks to air quality, especially in industrial areas.

3.2 Importance of Boundary Layer

The boundary layer plays a critical role in regulating interactions between the Earth's surface and the atmosphere. Its study is important in various fields, including:

1. **Agricultural meteorology:** Understanding microclimates within the boundary layer aids in crop management, irrigation planning, and predicting the effects of extreme weather on agriculture.
2. **Air pollution meteorology:** Dispersion and concentration of pollutants are governed by boundary layer processes, making it crucial for air quality monitoring and pollution control strategies.
3. **Cloud nuclei meteorology:** The boundary layer provides a reservoir of aerosols and moisture that act as cloud condensation nuclei, influencing cloud formation, precipitation, and local weather patterns.
4. **Thunderstorms and hurricanes physics:** The exchange of heat, moisture, and momentum in the boundary layer drives the development and intensity of thunderstorms and hurricanes, making it essential for improving weather prediction models.
5. **Urban meteorology:** The boundary layer's interactions with urban landscapes affect local climate, energy balance, and pollutant dispersion, aiding in city planning and sustainability efforts.
6. **Renewable energy:** Wind energy potential and efficiency are heavily dependent on boundary layer dynamics, which dictate wind speed profiles and turbulence levels near the surface.

Property	Boundary Layer	Free Atmosphere
Turbulence	Almost continuously turbulent	Sporadic, CAT, turbulence within clouds
Friction	Strong drag due to surface interaction	Small viscous dissipation
Dispersion	Rapid turbulent mixing	Small molecular diffusion
Winds	Near logarithmic profile	Geostrophic winds
Vertical Transport	Turbulent vertical motion	Horizontal transport by mean wind
Thickness	100m - 3km (variable)	8-16km (less variable)

Table 1. Comparison of Boundary Layer and Free Atmosphere Properties

4. Lecture 4 23/01/2025

4.1 Statistical Tools Required For Turbulence

Turbulence is characteristiced by randomness.

$$U = \overline{u} + u'$$
$$V = \overline{v} + v'$$
$$W = \overline{w} + w'$$
$$C = \overline{c} + c'$$

4.1.1 Mean

$$\overline{A} = \frac{1}{N}\sum_{i=1}^{N} A(i,s)$$
$$\overline{A} = \frac{1}{T}\int_{t=0}^{T} A(t,s)dt$$

4.1.2 Rules for averaging

If A and B are variables dependent on time, then:

$$\overline{A+B} = \overline{A} + \overline{B},$$
$$\overline{\overline{A}} = A,$$
$$\overline{\overline{A}\cdot B} = \overline{A\cdot B},$$
$$\overline{\frac{dA}{dt}} = \frac{d\overline{A}}{dt}.$$

Reynold's averaging rule

$$\overline{\overline{A}} = \overline{\overline{A}+a'} = \overline{A}, \text{since } \overline{a'} = 0$$
$$\overline{\overline{A\cdot B}} = \overline{(\overline{A}+a')(\overline{B}+b')} = \overline{A}\overline{B} + \overline{(a'b')},$$
$$\overline{(a'b')} \neq \overline{a'}\,\overline{b'}$$
$$\overline{a'^2} \neq 0$$
$$\overline{b'^2} \neq 0$$

4.1.3 Variance

$$\sigma_A^2 = \frac{1}{N}\sum_{i=1}^{N}(A_i - \overline{A})^2 = \overline{a'^2}$$

4.1.4 Standard deviation

$$\sigma_A = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(A_i - \overline{A})^2} = \sqrt{\overline{a'^2}}$$

4.1.5 Covariance

$$\sigma_{A,B} = \frac{1}{N}\sum_{i=1}^{N}(A_i - \overline{A})(B_i - \overline{B}) = \overline{a'b'}.$$

4.1.6 Correlation

$$\gamma_{A,B} = \frac{\sigma_{A,B}}{\sigma_A \sigma_B} = \frac{\overline{a'b'}}{\sqrt{\overline{a'^2}}\sqrt{\overline{b'^2}}}$$

Mean Kinetic Energy (MKE) = $\frac{1}{2}(\overline{u}^2 + \overline{v}^2 + \overline{w}^2)$

Turbulent Kinetic Energy (TKE) = $\frac{1}{2}(\overline{u'}^2 + \overline{v'}^2 + \overline{w'}^2)$

Question 4.1:

Suppose we erect instruments with an anemometer to measure u and w components, recording wind speeds every 6 seconds for a minute, resulting in the following 10 points of wind observations shown in Table 2. Calculate the mean, variance, and standard deviation for each component. Also, find the covariance and correlation between them.

u (m/s)	5	6	5	4	7	5	3	5	4	6
w (m/s)	0	-1	1	0	-2	1	2	-1	1	-1

Table 2. Wind observations for u and w components.

Answer 4.1:

Solution:

$$\overline{u} = 5, \qquad \overline{v} = 0$$
$$\sigma_U^2 = 1.2, \qquad \sigma_W^2 = 1.1$$
$$\sigma_U = \sqrt{1.2}, \qquad \sigma_W = \sqrt{1.1}$$
$$\sigma_{U,W} = \overline{u'w'} = -1.1, \qquad \gamma_{U,W} =$$

5. Lecture 5 24/01/2025

5.1 Fluxes

Mass, Heat, Moisture, Momentum, Pollutent, etc.

Quantity	Unit
Mass flux	kg_{air}/m^2s
Heat flux	J/m^2s
Moisture flux	kg_{wv}/m^2s
Momentum flux	$kgms^{-1}/m^2s$
Pollutant flux	$kg_{\text{pollutant}}/m^2s$

5.2 Kinematic Flux

Note: We asssume atmoshpere to be of constant density (ρ_{air}).

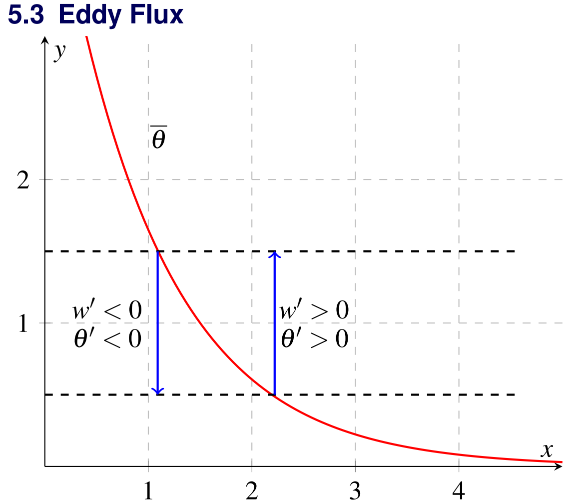
Flux Quantity	Formula	Unit
Mass flux	$\frac{\text{mass}}{\rho_{\text{air}}}$	kg_{air}/m^2s
Heat flux	$\frac{\text{heat}}{\rho C_p}$	J/m^2s
Moisture flux	-	kg_{wv}/m^2s
Momentum flux	-	$kgms^{-1}/m^2s$
Pollutant flux	-	$kg_{\text{pollutant}}/m^2s$

Examples:-

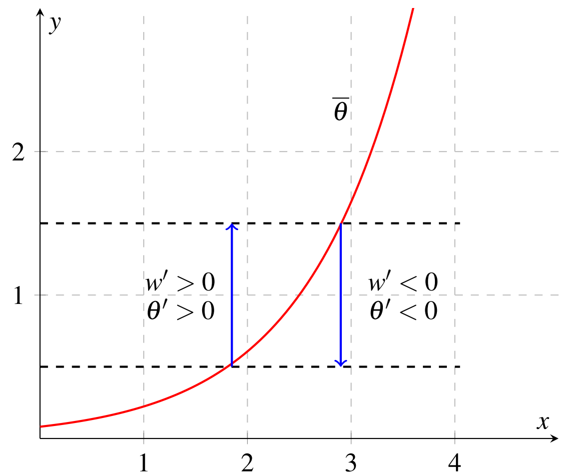
$$\text{Veritcal advective heat flux} = \overline{w}\overline{\theta}$$
$$\text{x-direction advective heat flux} = \overline{u}\overline{\theta}$$
$$\text{x-direction moisture flux} = \overline{uq}$$
$$\text{Veritcal eddy heat flux} = \overline{w\theta}$$
$$\text{x-direction eddy flux} = \overline{uq}$$

x-direction solar flux = $\left(\frac{1/4S_0}{c_p\rho}\right)$ = 0.2773 Km/s

5.3 Eddy Flux



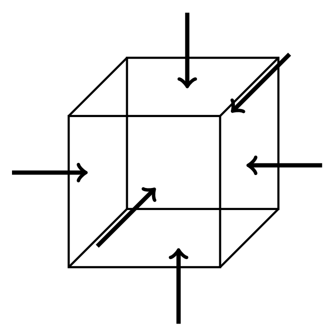
$$\Delta w' = 0$$
$$\Delta(\overline{w'\theta'}) > 0$$



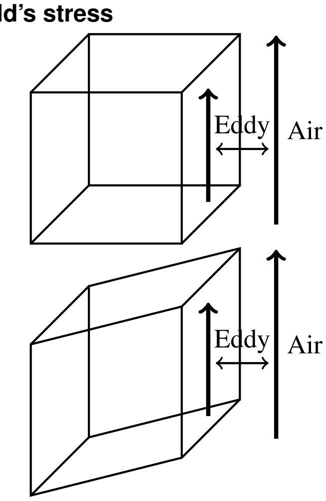
$$\Delta w' = 0$$
$$\Delta(\overline{w'\theta'}) < 0$$

5.4 Stress

Force tending to produce deformation in a body is called **Stress**.



5.4.1 Reynold's stress



6. Lecture 6 29/01/2025

6.1 Summation Notation

Definition: Integer variables m, n, and q can take values $1,2,3$. Then:

- A_m is a velocity vector.
- X_m represents distance.

where,

$$
\begin{aligned}
m &= 1,2,3,\\
n &= 1,2,3,\\
q &= 1,2,3.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A_1 &= u, \qquad & A_2 &= v, \qquad & A_3 &= w,\\
X_1 &= x, \qquad & X_2 &= y, \qquad & X_3 &= z.
\end{aligned}
$$

6.2 Unit vector

Unit vector is represented by δ_m,
where,

$$
\begin{aligned}
\delta_1 &= \hat{i},\\
\delta_2 &= \hat{j},\\
\delta_3 &= \hat{k}
\end{aligned}
$$

6.3 Kronecker Delta

Kronecker Delta is represented by δ_{mn}, such that,

$$
\delta_{mn} = \begin{cases} 1, & \text{for } m = n,\\ 0, & \text{for } m \neq n \end{cases}
$$

6.4 Alternating Unit Tensor

Alternating Unit Tensor is represented by ε_{mn}, such that,

$$
\delta_{mn} = \begin{cases} +1, & \text{for } mnq = 123, 231, 312\\ -1, & \text{for } mnq = 321, 132, 213\\ 0, & \text{for any 2 or more indices alike} \end{cases}
$$

6.5 Rules For Summation Notation

Rule I:

Wherever two identical appear same in one term, it is implied that there is a sum of that term over each value of repreated index.

For example:

1.
$$
\begin{aligned}
A_n \frac{\partial B_m}{\partial X_n} &= A_1 \frac{\partial B_m}{\partial X_1} + A_2 \frac{\partial B_m}{\partial X_2} + A_3 \frac{\partial B_m}{\partial X_3}\\
&= u \frac{\partial B_m}{\partial x} + v \frac{\partial B_m}{\partial y} + w \frac{\partial B_m}{\partial z}
\end{aligned}
$$

2.
$$
\begin{aligned}
\delta_{2n} A_n &= \delta_{21} A_1 + \delta_{22} A_3 + \delta_{23} A_3\\
&= A_2\\
&= v
\end{aligned}
$$

Rule II:

Wherever one index appear unsummed in one term, then the same index appear unsummed in all 3 terms in that equation. Hence that equation effectively represents 3 equations for each value of unsummed index.

$$
\begin{aligned}
A_m &= B_m + \delta_{mn} C_n\\
A_1 &= B_1 + \delta_{1n} C_n = B_1 + C_1\\
A_2 &= B_2 + \delta_{2n} C_n = B_2 + C_2\\
A_2 &= B_3 + \delta_{3n} C_n = B_3 + C_3
\end{aligned}
$$

For example:

1.
$$
\begin{aligned}
\frac{\partial A_m}{\partial t} + B_n \frac{\partial A_m}{\partial X_n} =& -\delta_{m3} g + f \varepsilon_{mn3} B_n\\
& - \frac{1}{\rho} \frac{\partial P}{\partial X_m} + \frac{1}{\rho} \left[\frac{\tau_{mn}}{\partial X_n} \right]
\end{aligned}
$$

Expanding with $n = 1, 2$ and 3, we get:

$$
\begin{aligned}
&\frac{\partial A_m}{\partial t} + B_1 \frac{\partial A_m}{\partial X_1} + B_2 \frac{\partial A_m}{\partial X_2} + B_3 \frac{\partial A_m}{\partial X_3} =\\
&\quad - \delta_{m3} g - f[\varepsilon_{m13} B_1 + \varepsilon_{m23} B_2 + \cancel{\varepsilon_{m33} B_3}]\\
&\quad - \frac{1}{\rho} \left[\frac{\partial P}{\partial X_m} \right] + \frac{1}{\rho} \left[\frac{\tau_{m1}}{\partial X_1} + \frac{\tau_{m2}}{\partial X_2} + \frac{\tau_{m3}}{\partial X_3} \right]
\end{aligned}
$$

Now expanding with $m = 1$, we get following equation:

$$
\begin{aligned}
&\frac{\partial A_1}{\partial t} + B_1 \frac{\partial A_1}{\partial X_1} + B_2 \frac{\partial A_1}{\partial X_2} + B_3 \frac{\partial A_1}{\partial X_3} =\\
&\quad - \cancel{\delta_{13} g} + f[\cancel{\varepsilon_{113} B_1} + \varepsilon_{123} B_2 + \cancel{\varepsilon_{133} B_3}]\\
&\quad - \frac{1}{\rho} \left[\frac{\partial P}{\partial X_1} \right] + \frac{1}{\rho} \left[\frac{\tau_{11}}{\partial X_1} + \frac{\tau_{12}}{\partial X_2} + \frac{\tau_{13}}{\partial X_3} \right]\\
&\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = + f v - \frac{1}{\rho} \left[\frac{\partial P}{\partial x} \right]\\
&\quad + \frac{1}{\rho} \left[\frac{\tau_{xx}}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} \right]
\end{aligned}
$$

Now expanding with $m = 2$, we get following equation:

$$
\begin{aligned}
&\frac{\partial A_2}{\partial t} + B_1 \frac{\partial A_2}{\partial X_1} + B_2 \frac{\partial A_2}{\partial X_2} + B_3 \frac{\partial A_2}{\partial X_3} =\\
&\quad - \cancel{\delta_{23} g} + f[\varepsilon_{213} B_1 + \cancel{\varepsilon_{223} B_2} + \cancel{\varepsilon_{233} B_3}]\\
&\quad - \frac{1}{\rho} \left[\frac{\partial P}{\partial X_2} \right] + \frac{1}{\rho} \left[\frac{\tau_{21}}{\partial X_1} + \frac{\tau_{22}}{\partial X_2} + \frac{\tau_{23}}{\partial X_3} \right]\\
&\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - f u - \frac{1}{\rho} \left[\frac{\partial P}{\partial y} \right]\\
&\quad + \frac{1}{\rho} \left[\frac{\tau_{yx}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{yz}}{\partial z} \right]
\end{aligned}
$$

Now expanding with $m = 3$, we get following equation:

$$\frac{\partial A_3}{\partial t} + B_1 \frac{\partial A_3}{\partial X_1} + B_2 \frac{\partial A_3}{\partial X_2} + B_3 \frac{\partial A_3}{\partial X_3} =$$
$$- \delta_{33} g + f[\cancel{\varepsilon_{313}} B_1 + \cancel{\varepsilon_{323} B_2} + \cancel{\varepsilon_{333} B_3}]$$
$$- \frac{1}{\rho}\left[\frac{\partial P}{\partial X_3}\right] + \frac{1}{\rho}\left[\frac{\tau_{31}}{\partial X_1} + \frac{\tau_{32}}{\partial X_2} + \frac{\tau_{33}}{\partial X_3}\right]$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -g - \frac{1}{\rho}\left[\frac{\partial P}{\partial z}\right]$$
$$+ \frac{1}{\rho}\left[\frac{\tau_{zx}}{\partial x} + \frac{\tau_{zy}}{\partial y} + \frac{\tau_{zz}}{\partial z}\right]$$

7. Lecture 7 30/01/2025

7.1 Governing Equations For Turbulent Flow

1. Mean variable $(\overline{u}, \overline{v}, \overline{w})$.
2. Flux variable $(\overline{u'w'}, \overline{w'\theta'})$.
3. Total Kinetic Energy $(\overline{u'^2}, \overline{v'^2}, \overline{w'^2})$.

Steps:

1. Identify the basic equations.
2. Expand the total derivative into local and advective term.
3. Expand dependent variables into mean and turbulent parts.
4. Apply Reynold's averaging.
5. Add continuity equation to put the result in flux part.

7.2 Governing Equations

1. Equation of state

$$P = \rho RT \tag{7.1}$$

2. Conservation of Mass or Continuity eqaution

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \tag{7.2}$$

If **incompressible**, $\frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = \text{constant}$

$$\begin{aligned} \frac{\partial (\rho U_i)}{\partial x_i} &= 0 \\ \rho \left[\frac{\partial U_1}{\partial X_1} + \frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_3} \right] &= 0 \\ \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] &= 0 \\ \nabla \cdot u &= 0 \end{aligned} \tag{7.3}$$

3. Conservation of Momentum or Momentum equation

$$\left.\begin{aligned} \frac{\partial U_i}{\partial t} + U_i \frac{\partial U_i}{\partial x_i} = &-\delta_{i3} g + f \varepsilon_{ij3} U_j \\ &- \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \underbrace{\gamma}_{\substack{\text{Kinetic} \\ \text{viscousity} \\ \text{coeff.}}} \underbrace{\frac{\partial^2 U_i}{\partial x_i^2}}_{\text{viscousity}} \end{aligned}\right\} \tag{7.4}$$

4. Conservartion of Moisture

$$\left.\begin{aligned} \frac{\partial q}{\partial t} + U_i \frac{\partial q}{\partial x_i} = &\overbrace{\underbrace{\nu_q}_{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{coeff.}}} \cdot \frac{\partial^2 q}{\partial x_i^2}}^{\text{Diffusion term}} + \underbrace{\frac{S}{\rho_{\text{air}}}}_{\text{Residue term}} \\ &+ \underbrace{\frac{E}{\rho_{\text{air}}}}_{\text{Phase change}} \end{aligned}\right\} \tag{7.5}$$

5. Conservartion of Heat or Thermodynamic energy equation

$$\left.\begin{aligned} \frac{\partial \theta}{\partial t} + U_i \frac{\partial \theta}{\partial x_i} = &\underbrace{\overbrace{\nu_\theta}^{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{heat coeff.}}} \cdot \frac{\partial^2 \theta}{\partial x_i^2}}_{\text{Diffusive heat}} - \underbrace{\frac{\partial Q_i}{\partial x_i}}_{\substack{\text{Radiative} \\ \text{divergence}}} \\ &- \underbrace{\frac{L_p E}{\rho_{\text{air}} C_p}}_{\text{Phase change}} \end{aligned}\right\} \tag{7.6}$$

6. Conservartion of Scalar quantity

$$\frac{\partial C}{\partial t} + U_i \frac{\partial C}{\partial x_i} = \underbrace{\overbrace{\nu_c}^{\substack{\text{molecular} \\ \text{diffusivity} \\ \text{coeff.}}} \frac{\partial^2 C}{\partial x_i^2}}_{\text{Diffusion term}} + \underbrace{S_c}_{\text{Phase change}} \tag{7.7}$$

8. Lecture 8 12/02/2025

The governing equations in preivous lecture can not be used for practical applications, since in boundary layer we deal with turbulence which are not included in any of these equations. Now we need to account for turbulence, which can be heat flux, momentum flux, etc., which will added in the governing equations respectively.

8.1 Governing Equations For Turbulent Flow

8.1.1 Equation of state

$$P = \rho R_d T_v \tag{8.1}$$

To account for turbulence we will split equation in mean and turbulence (purturbation) parts, that is,

$$P = \overline{P} + P' \tag{8.2.i}$$
$$\rho = \overline{\rho} + \rho' \tag{8.2.ii}$$
$$T_v = \overline{T_v} + T_v' \tag{8.2.iii}$$

Substituting Eq.(8.2.i), Eq.(8.2.ii) and Eq.(8.2.iii) in Eq.(8.1), we obtain:

$$\begin{aligned}(\overline{P} + P') &= (\overline{\rho} + \rho')\, R_d\, (\overline{T_v} + T_v') \\ \frac{(\overline{P} + P')}{R_d} &= \overline{\rho}\overline{T_v} + \overline{\rho}T_v' + \rho'\overline{T_v} + \rho' T_v' \end{aligned}\tag{8.3}$$

Taking Reynold's average,

$$\begin{aligned}\frac{\overline{\overline{P}} + \overline{P'}}{R_d} &= \overline{\overline{\rho}\overline{T_v}} + \overline{\overline{\rho}T_v'} + \overline{\rho'\overline{T_v}} + \overline{\rho' T_v'} \\ \frac{\overline{P}}{R_d} &= \overline{\rho}\overline{T_v} + \overline{\rho' T_v'} \\ \frac{\overline{P}}{R_d} &\approx \overline{\rho}\overline{T_v} \end{aligned}\tag{8.4}$$

Equation of state remain equation of state in turbulent flow (no addition of turbulent term).
Subtracting Eq.(8.4) from Eq.(8.3), we get turbulent form:

$$\frac{P'}{R_d} = \overline{\rho}T_v' + \rho'\overline{T_v} + \rho' T_v' \tag{8.5}$$

Dividing Eq.(8.5) by Eq.(8.4)

$$\begin{aligned}\frac{P'}{\overline{P}} &= \frac{T_v'}{\overline{T_v}} + \frac{\rho'}{\overline{\rho}} + \underbrace{\frac{\rho' T'}{\overline{\rho}\overline{T_v}}}_{\approx 0} \\ \frac{P'}{\overline{P}} &= \frac{T_v'}{\overline{T_v}} + \frac{\rho'}{\overline{\rho}} \end{aligned}\tag{8.6}$$

For practical purposes term $(P'/\overline{P})$ is negligible as compared to density and temperature terms. Therefore,

$$\frac{\rho'}{\overline{\rho}} \approx \frac{\theta_v'}{\overline{\theta_v}} \tag{8.7}$$

where θ is potential termperature.

8.1.2 Flux Form of Advection Equation

$$u_j \frac{\partial \theta}{\partial x_j} \rightarrow \text{Advection equation for temperature}$$

$$\frac{\partial u_j}{\partial x_j} = 0$$

Multiply above equation by temperature (θ) and add in the previous equation, we obtain;

$$\begin{aligned} u_j \frac{\partial \theta}{\partial x_j} + \theta \frac{\partial u_j}{\partial x_j} &= 0 \\ \frac{\partial (u_j \theta)}{\partial x_j} &= 0 \end{aligned}\tag{8.8}$$

Note: When θ replaced with u_j, the above Eq.(8.8) will become flux form of momentum equation.

8.1.3 Conservation of Mass

$$\frac{\partial u_j}{\partial x_j} = 0$$

Spliting equation in mean and turbulent (purturbation) parts, that is,

$$\begin{aligned} \frac{\partial \left(\overline{u_j} + u_j'\right)}{\partial x_j} &= 0 \\ \frac{\partial \left(\overline{u_j} + u_j'\right)}{\partial x_j} &= 0 \\ \frac{\partial \overline{u_j}}{\partial x_j} + \frac{\partial u_j'}{\partial x_j} &= 0 \end{aligned}\tag{8.9}$$

Applying Reynold's averaging,

$$\begin{aligned} \overline{\frac{\partial \overline{u_j}}{\partial x_j}} + \cancel{\overline{\frac{\partial u_j'}{\partial x_j}}} &= 0 \\ \frac{\partial \overline{u_j}}{\partial x_j} &= 0 \end{aligned}\tag{8.10}$$

Subtracting Eq.(8.10) from Eq.(8.9), we get:

$$\frac{\partial u_j'}{\partial x_j} = 0 \rightarrow \text{Turbulent variable} \tag{8.11}$$

8.1.4 Conservation of Momentum

$$\left.\begin{aligned} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \quad & -\delta_{i3}\left[g - \left(\frac{\theta_v'}{\overline{\theta_v}}\right)\right] + f\varepsilon_{ij3} u_j \\ & -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \gamma \frac{\partial^2 u_i}{\partial x_i^2} \end{aligned}\right\}\tag{8.12}$$

Spliting equation in mean and turbulent (purturbation) parts, that is,

$$\left.\begin{aligned}&\frac{\partial\left(\overline{u_i}+u_i'\right)}{\partial t}+\left(\overline{u_j}+u_j'\right)\frac{\partial\left(\overline{u_i}+u_i'\right)}{\partial x_j}\\&\qquad=-\delta_{i3}\left[g-\left(\frac{\theta_v'}{\theta_v}\right)\right]+f\varepsilon_{ij3}\left(\overline{u_j}+u_j'\right)\\&\qquad\quad-\frac{1}{\left(\overline{\rho}+\rho'\right)}\frac{\partial\left(\overline{P}+P'\right)}{\partial x_j}+\gamma\frac{\partial^2\left(\overline{u_i}+u_i'\right)}{\partial x_j^2}\end{aligned}\right\}\tag{8.13}$$

Assuming density of boundary layer does not change, i.e., $\rho'=0$

$$\left.\begin{aligned}&\frac{\partial\overline{u_i}}{\partial t}+\frac{\partial u_i'}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}+\overline{u_j}\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial\overline{u_i}}{\partial x_j}+u_j'\frac{\partial u_i'}{\partial x_j}\\&\qquad=-\delta_{i3}g-\delta_{i3}\left(\frac{\theta_v'}{\theta_v}\right)+f\varepsilon_{ij3}\overline{u_j}+f\varepsilon_{ij3}u_j'\\&\qquad\quad-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j}-\frac{1}{\overline{\rho}}\frac{\partial P'}{\partial x_j}+\gamma\frac{\partial^2\overline{u_i}}{\partial x_j^2}+\gamma\frac{\partial^2 u_i'}{\partial x_j^2}\end{aligned}\right\}\tag{8.14}$$

Applying Reynold's averaging,

$$\begin{aligned}&\overline{\frac{\partial\overline{u_i}}{\partial t}}+0+\overline{\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}}+0+0+\overline{u_j'\frac{\partial u_i'}{\partial x_j}}\\&\qquad=-\delta_{i3}g-0+f\varepsilon_{ij3}\overline{\overline{u_j}}+0\\&\qquad\quad-\frac{1}{\overline{\rho}}\overline{\frac{\partial\overline{P}}{\partial x_j}}-0+\gamma\overline{\frac{\partial^2\overline{u_i}}{\partial x_j^2}}+0\end{aligned}$$

$$\left.\begin{aligned}&\frac{\partial\overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}+\overline{u_j'\frac{\partial u_i'}{\partial x_j}}=-\delta_{i3}g+f\varepsilon_{ij3}\overline{u_j}\\&\qquad\qquad-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j}+\gamma\frac{\partial^2\overline{u_i}}{\partial x_j^2}\end{aligned}\right\}\tag{8.15}$$

$$\frac{\partial u_j'}{\partial x_j}=0$$

Multiply above equation by velocity (u_i') and take Reynold's average

$$\overline{u_i'\frac{\partial u_j'}{\partial x_j}}=0$$

Add above equation in the Eq.(8.15), we obtain;

$$\begin{aligned}&\frac{\partial\overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}+\overline{u_j'\frac{\partial u_i'}{\partial x_j}}+\overline{u_i'\frac{\partial u_j'}{\partial x_j}}=-\delta_{i3}g\\&\qquad+f\varepsilon_{ij3}\overline{u_j}-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j}+\gamma\frac{\partial^2\overline{u_i}}{\partial x_j^2}\end{aligned}$$

$$\left.\begin{aligned}&\frac{\partial\overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}+\underbrace{\overline{\frac{\partial\left(u_i'u_j'\right)}{\partial x_j}}}_{\text{III}}=-\delta_{i3}g+f\varepsilon_{ij3}\overline{u_j}\\&\qquad\qquad-\frac{1}{\overline{\rho}}\frac{\partial\overline{P}}{\partial x_j}+\gamma\frac{\partial^2\overline{u_i}}{\partial x_j^2}\end{aligned}\right\}\tag{8.16}$$

3^{rd} term marked in Eq.(8.16) is momentum flux (turbulence) which represents eddies.

9. Lecture 9 13/02/2025

9.1 Governing Equations For Turbulent Flow

9.1.1 Turbulent Momentum Flux Equation

General momentum equation:

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\delta_{i3}g+f\varepsilon_{ij3}u_j-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u}{\partial x_i^2} \tag{9.1}$$

Turbulent form of momentum equation:

$$\begin{aligned}\frac{\partial \overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}&=-\delta_{i3}g+f\varepsilon_{ij3}\overline{u_j}-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}\\&\quad+\nu\frac{\partial^2 \overline{u}}{\partial x_i^2}+\frac{\partial \overline{u_i' u_j}}{\partial x_j}\\\frac{\partial \overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}&=-\delta_{i3}g+f\varepsilon_{ij3}\overline{u_j}-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}\\&\quad+\nu\frac{\partial^2 \overline{u}}{\partial x_i^2}+\underbrace{\frac{1}{\overline{\rho}}\frac{\overline{\rho\,\partial u_i' u_j}}{\partial x_j}}_{\text{Reynold's shress}}\end{aligned}$$

9.1.2 Turbulent Heat Flux Equation

General Heat equation:

$$\begin{aligned}\frac{\partial \theta}{\partial t}+u_j\frac{\partial \theta}{\partial x_j}&=\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}-\frac{1}{\rho C_p}\frac{\partial Q_j^*}{\partial x_j}\\&\quad-\underbrace{\frac{L_pE}{\rho C_p}}_{\text{Phase change}}\end{aligned}$$

$$\begin{aligned}\frac{\partial\left(\overline{\theta}+\theta'\right)}{\partial t}+\left(\overline{u_j}+u_j'\right)\frac{\partial\left(\overline{\theta}+\theta'\right)}{\partial x_j}&=\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}\\&\quad-\frac{1}{(\overline{\rho}+\rho')C_p}\frac{\partial Q_j^*}{\partial x_j}-\frac{L_pE}{(\overline{\rho}+\rho')C_p}\end{aligned}$$

Assuming density of boundary layer does not change, i.e., $\rho'=0$

$$\left.\begin{aligned}\frac{\partial\left(\overline{\theta}+\theta'\right)}{\partial t}+\left(\overline{u_j}+u_j'\right)\frac{\partial\left(\overline{\theta}+\theta'\right)}{\partial x_j}&=\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}\\-\frac{1}{\overline{\rho}C_p}\frac{\partial\left(\overline{Q_j^*}+Q_j^{*\prime}\right)}{\partial x_j}-\frac{L_pE}{\overline{\rho}C_p}\end{aligned}\right\} \tag{9.3}$$

$$\left.\begin{aligned}&\frac{\partial\overline{\theta}}{\partial t}+\frac{\partial\theta'}{\partial t}+\overline{u_j}\frac{\partial\overline{\theta}}{\partial x_j}+\overline{u_j}\frac{\partial\theta'}{\partial x_j}+u_j'\frac{\partial\overline{\theta}}{\partial x_j}+u_j'\frac{\partial\theta'}{\partial x_j}\\&=\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}-\frac{1}{\overline{\rho}C_p}\frac{\partial\overline{Q_j^*}}{\partial x_j}+\frac{1}{\overline{\rho}C_p}\frac{\partial Q_j^{*\prime}}{\partial x_j}-\frac{L_pE}{\overline{\rho}C_p}\end{aligned}\right\} \tag{9.4}$$

Apply Reynold's averaging;

$$\left.\begin{aligned}\frac{\partial\overline{\theta}}{\partial t}+\overline{u_j}\frac{\partial\overline{\theta}}{\partial x_j}+\overline{u_j'\frac{\partial\theta'}{\partial x_j}}&=\nu_\theta\frac{\partial^2\overline{\theta}}{\partial x_j^2}-\frac{1}{\overline{\rho}C_p}\frac{\partial\overline{Q_j^*}}{\partial x_j}\\&\quad-\frac{L_pE}{\overline{\rho}C_p}\end{aligned}\right\} \tag{9.5}$$

Add $\overline{\theta'\frac{\partial u_j'}{\partial x_j}}=0$ in above Eq.(9.5)

$$\left.\begin{aligned}\frac{\partial\overline{\theta}}{\partial t}+\overline{u_j}\frac{\partial\overline{\theta}}{\partial x_j}+\frac{\partial\left(\overline{u_j'\theta'}\right)}{\partial x_j}&=\nu_\theta\frac{\partial^2\overline{\theta}}{\partial x_j^2}-\frac{1}{\overline{\rho}C_p}\frac{\partial\overline{Q_j^*}}{\partial x_j}\\&\quad-\frac{L_pE}{\overline{\rho}C_p}\end{aligned}\right\} \tag{9.6}$$

Reynold's number :

$$\begin{aligned}R_d&=\frac{\text{intertial force}}{\text{viscous force}}\\&=\frac{\rho\mathrm{vL}}{\mu}\\&=\frac{\mathrm{vL}}{\nu}\end{aligned}$$

Taking a real-world scenario of a boundary layer, where $v=5$ m/s, $L=100$ m, and $\nu=1.5\times10^{-5}$ m^2/s
$\therefore R_d=\frac{5\times100}{1.5\times10^{-5}}\approx 3.3\times10^7$.

$$\begin{aligned}\nu&=\frac{\mathrm{vL}}{R_d}\\\nu\frac{\partial^2\overline{u_i}}{\partial x_j^2}&=\frac{1}{\underbrace{R_d}_{\text{very large}}}\left[\mathrm{vL}\frac{\partial^2\overline{u_i}}{\partial x_j^2}\right]\approx 0\end{aligned}$$

Therefore viscous term can be neglected for boundary layer in most of the cases.
Hence modified turbulent momentum, heat flux, moisture and pollutant equations can be written respectively as following:

$$\frac{\partial \overline{u_i}}{\partial t}+\overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}=-\delta_{i3}g+f\varepsilon_{i33}\overline{u_j}-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i}+\frac{1}{\overline{\rho}}\frac{\overline{\rho u_i' u_j}}{\partial x_j} \tag{9.7}$$

TODO: To write turbulent heat, moisture, pollutant equations

10. Lecture 10 14/02/2025

10.1 Governing Equations For Turbulent Flow

In the previous lecture, we introduced the turbulent flux terms $\left(\text{i.e.}, \frac{\partial \overline{u'_i \beta'}}{\partial x_j}, \text{ where } \beta \text{ can be } u'_j, \theta', C' \text{ or } Q'\right)$ in the governing equations.

These terms can be expanded using index 'j' of velocity u and cancel out horizontal flucuation because in general, assuming, only vertical transport take place due to turbulence in following way:

$$\text{Heat flux} \Rightarrow \overline{u'_j\theta'} \rightarrow \cancel{\overline{u'\theta'}}, \cancel{\overline{v'\theta'}}, \overline{w'\theta'}$$
$$\text{Kinematic moisture flux} \Rightarrow \overline{u'_jq'} \rightarrow \cancel{\overline{u'q'}}, \cancel{\overline{v'q'}}, \overline{w'q'}$$
$$\text{Pollutent flux} \Rightarrow \overline{u'_jC'} \rightarrow \cancel{\overline{u'C'}}, \cancel{\overline{v'C'}}, \overline{w'C'}$$

***Note*:** Sonic anemometer is used to measure $w'\ (= w - \overline{w})$

10.2 Physical Significance Of Turbulent Terms

$$\overline{w'\theta'} = \begin{cases} > 0 & \text{Heat transport in upward direction} \\ < 0 & \text{Heat transport in downward direction (through inversion layer)} \end{cases}$$

$$\overline{w'q'} = \begin{cases} > 0 & \text{Moisture transport in upward direction (supply of moisture from surface)} \\ < 0 & \text{Moisture transport in downward direction (supply of moisture from TOA)} \end{cases}$$

10.3 Scaling Variable

To account for time delay in measurements and make data invariant of time.
z_i is depth of mixed layer, then,

$$\frac{z}{z_i} \rightarrow \text{scaling variable}$$

$$\frac{(\overline{w'\theta'})}{(\overline{w'q'})} \rightarrow \text{scaling variable}$$

We can intepret term $(\partial u'/\partial t)$ as prediction of every single eddy in atmosphere, but we don't use it becasue ot is not required and very computationally expensive.

10.4 Prognostic Equations For Turbulent Flow

$$\left.\begin{aligned} &\frac{\partial \overline{u_i}}{\partial t} + \frac{\partial u'_i}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_i}\frac{\partial u'_i}{\partial x_j} + u'_j\frac{\partial \overline{u'_i}}{\partial x_j} + u'_j\frac{\partial u'_i}{\partial x_j} \\ &\quad = -\delta_{i3}g + \delta_{i3}\left(\frac{\theta'_v}{\overline{\theta_v}}\right)g + f\varepsilon_{i33}\overline{u_j} + f\varepsilon_{i33}u'_j \\ &\quad -\frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \nu\frac{\partial^2 u'_i}{\partial x_j^2} \end{aligned}\right\} \quad (10.1)$$

Apply Reynold's averaging;

$$\left.\begin{aligned} \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} &= -\delta_{i3}g + f\varepsilon_{i33}\overline{u_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} \\ &+ \nu\frac{\partial^2 \overline{u_i}}{\partial x_j^2} + \nu\frac{\partial^2 u'_i}{\partial x_j^2} - \frac{\partial \overline{(u'_iu'_j)}}{\partial x_i} \end{aligned}\right\} \quad (10.2)$$

Subtracting Eq.(10.2) from Eq.(10.1)

$$\left.\begin{aligned} \frac{\partial u'_i}{\partial t} + \overline{u_i}\frac{\partial u'_i}{\partial x_j} + u'_j\frac{\partial \overline{u'_i}}{\partial x_j} + u'_j\frac{\partial u'_i}{\partial x_j} &= \delta_{i3}\left(\frac{\theta'_v}{\overline{\theta_v}}\right)g \\ + f\varepsilon_{i33}u'_j - \frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i} + \nu\frac{\partial^2 u'_i}{\partial x_j^2} &+ \frac{\partial \overline{(u'_iu'_j)}}{\partial x_i} \end{aligned}\right\} \quad (10.3)$$

To be continued in next lecture...

11. Lecture 11 19/02/2025

11.1 Prognostic Equations For Turbulent Flow

Continuing from previous lecture.
Multiply Eq.(10.3) with $(2u_i')$

$$\left.\begin{aligned} &2u_i'\frac{\partial u_i'}{\partial t}+2u_i'\overline{u_i}\frac{\partial u_i'}{\partial x_j}+2u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}+2u_i'u_j'\frac{\partial u_i'}{\partial x_j}\\ &\quad=2\delta_{i3}u_i'\left(\frac{\theta_v'}{\theta_v}\right)g+2f\varepsilon_{i33}u_i'u_j'\\ &\quad-\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}+2u_i'\nu\frac{\partial^2 u_i'}{\partial x_j^2}+2u_i'\frac{\partial\overline{(u_i'u_j')}}{\partial x_i}\end{aligned}\right\}\tag{11.1}$$

Apply product rule,

$$\left.\begin{aligned} &\frac{\partial u_i'^2}{\partial t}+\overline{u_i}\frac{\partial u_i'^2}{\partial x_j}+2u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}+u_j'\frac{\partial u_i'^2}{\partial x_j}\\ &\quad=2\delta_{i3}u_i'\left(\frac{\theta_v'}{\theta_v}\right)g+2f\varepsilon_{i33}u_i'u_j'\\ &\quad-\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}+2\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}+2u_i'\frac{\partial\overline{(u_i'u_j')}}{\partial x_i}\end{aligned}\right\}\tag{11.2}$$

Applying Reynold's averaging;

$$\left.\begin{aligned} &\frac{\partial \overline{u_i'^2}}{\partial t}+\overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}}+\overline{u_j'\frac{\partial u_i'^2}{\partial x_j}}\\ &\quad=2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\theta_v}\right)}g+2f\varepsilon_{i33}\overline{u_i'u_j'}\\ &\quad-\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}+\cancel{\overline{2u_i'\frac{\partial\overline{(u_i'u_j')}}{\partial x_i}}}\end{aligned}\right\}\tag{11.3}$$

Adding Turbulent continuity equation to Eq.(11.3)

$$\overline{\partial u_i'^2}\frac{u_j'}{\partial x_j}=0$$

$$\left.\begin{aligned} &\frac{\partial \overline{u_i'^2}}{\partial t}+\overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'\frac{\partial \overline{u_i'}}{\partial x_j}}+\overline{u_j'\frac{\partial u_i'^2}{\partial x_j}}+\overline{u_i'^2\frac{\partial u_j'}{\partial x_j}}\\ &\quad=2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\theta_v}\right)}g+2f\varepsilon_{i33}\overline{u_i'u_j'}\\ &\quad-\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}\end{aligned}\right\}\tag{11.4}$$

$$\left.\begin{aligned} &\frac{\partial \overline{u_i'^2}}{\partial t}+\overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}}+\overline{u_j'\frac{\partial\left(u_i'^2u_j'\right)}{\partial x_j}}\\ &\quad=2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\theta_v}\right)}g+2f\varepsilon_{i33}\overline{u_i'u_j'}\\ &\quad-\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+\underbrace{2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}}_{\text{Viscous dissipation}}\end{aligned}\right\}\tag{11.5}$$

Viscous desipation term:

$$\begin{aligned} \nu\frac{\partial\overline{u_i'^2}}{\partial x_j^2}&=\nu\frac{\partial}{\partial x_j}\left[\overline{\frac{\partial u_i'^2}{\partial x_j}}\right]\\ \nu\frac{\partial\overline{u_i'^2}}{\partial x_j^2}&=\nu\frac{\partial}{\partial x_j}\left[2\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right]\\ \nu\frac{\partial\overline{u_i'^2}}{\partial x_j^2}&=2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}+2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}\\ \nu\frac{\partial\overline{u_i'^2}}{\partial x_j^2}&=2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Shear}\\ \text{term}}}+2\nu\underbrace{\overline{\left(\frac{\partial^2 u_i'^2}{\partial x_j^2}\right)}}_{\substack{\text{Diffusion of}\\ \text{velocity}(\approx 0)}}\end{aligned}\tag{11.6}$$

Substitute approximate viscous term Eq.(11.6) in Eq.(11.5)

$$\left.\begin{aligned} &\frac{\partial \overline{u_i'^2}}{\partial t}+\overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}}+\overline{u_j'\frac{\partial\left(u_i'^2u_j'\right)}{\partial x_j}}\\ &\quad=2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\theta_v}\right)}g+2f\varepsilon_{i33}\overline{u_i'u_j'}\\ &\quad-\underbrace{\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}}_{\substack{\text{Pressure}\\ \text{Perturbation}\\ \text{term}}}+2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Viscous}\\ \text{dissipation}\\ \text{term}}}\end{aligned}\right\}\tag{11.7}$$

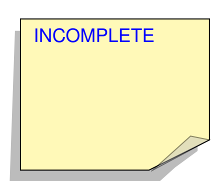
Pressure perturbation term:

$$-\overline{\frac{2u_i'}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}=-\frac{2}{\overline{\rho}}\overline{\left(\frac{\partial(p'u_i')}{\partial x_i}\right)}+\overline{\frac{2p'}{\overline{\rho}}\cdot\underbrace{\left(\frac{\partial u_i'}{\partial x_i}\right)}_{\substack{\nabla u_i'=0\\ \text{Continuity eq.}}}}\tag{11.8}$$

The last term in Eq.(11.8) is called **return-to-isotropy**.
Therefore Eq.(11.7) can be modified to:

$$\left.\begin{aligned} &\frac{\partial \overline{u_i'^2}}{\partial t}+\overline{u_i}\frac{\partial \overline{u_i'^2}}{\partial x_j}+2\overline{u_i'u_j'\frac{\partial \overline{u_i}}{\partial x_j}}+\overline{u_j'\frac{\partial\left(u_i'^2u_j'\right)}{\partial x_j}}\\ &\quad=2\delta_{i3}\overline{u_i'\left(\frac{\theta_v'}{\theta_v}\right)}g+\underbrace{2f\varepsilon_{i33}\overline{u_i'u_j'}}_{\substack{\text{Coriolis}\\ \text{term}}}\\ &\quad-\underbrace{\overline{\frac{2}{\overline{\rho}}\frac{\partial(p'u_i')}{\partial x_i}}}_{\substack{\text{Pressure}\\ \text{Perturbation}\\ \text{term}}}+2\nu\underbrace{\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}}_{\substack{\text{Viscous}\\ \text{dissipation}\\ \text{term}}}\end{aligned}\right\}\tag{11.9}$$

Coriolis term:



12. Lecture 12 20/02/2025

12.1 Governing Equation For Turbulent Flow

$$\frac{\partial q'}{\partial t} + \overline{u_j}\frac{\partial \overline{q}}{\partial x_j} + u'_j\frac{\partial \overline{q}}{\partial x_j} + u'_j\frac{\partial q'}{\partial x_j} = \nu\frac{\partial^2 q'}{\partial x_j^2} + \frac{\partial \overline{(u'_i q')}}{\partial x_i} \tag{12.1}$$

Multiply above Eq.(12.1) with $2q'$

$$\left.\begin{aligned} 2q'\tfrac{\partial q'}{\partial t} + 2q'\overline{u_j}\tfrac{\partial \overline{q}}{\partial x_j} + 2q'u'_j\tfrac{\partial \overline{q}}{\partial x_j} + 2q'u'_j\tfrac{\partial q'}{\partial x_j} \\ = 2q'\nu\tfrac{\partial^2 q'}{\partial x_j^2} + 2q'\tfrac{\partial \overline{(u'_i q')}}{\partial x_i} \end{aligned}\right\} \tag{12.2}$$

$$\left.\begin{aligned} \tfrac{\partial (q')^2}{\partial t} + 2q'\overline{u_j}\tfrac{\partial \overline{q}}{\partial x_j} + 2q'u'_j\tfrac{\partial \overline{q}}{\partial x_j} + u'_j\tfrac{\partial (q')^2}{\partial x_j} \\ = 2q'\nu\tfrac{\partial^2 q'}{\partial x_j^2} + 2q'\tfrac{\partial \overline{(u'_i q')}}{\partial x_i} \end{aligned}\right\} \tag{12.3}$$

Applying Reynold's averaging,

$$\overline{\frac{\partial (q')^2}{\partial t}} + 2\overline{q'u'_j}\frac{\partial \overline{q}}{\partial x_j} + u'_j\overline{\frac{\partial (q')^2}{\partial x_j}} = 2\nu\overline{q'\frac{\partial^2 q'}{\partial x_j^2}} \tag{12.4}$$

Adding flux term $\overline{(q')^2\frac{\partial u'_j}{\partial x_j}} = 0$

$$\overline{\frac{\partial (q')^2}{\partial t}} + 2\overline{q'u'_j}\frac{\partial \overline{q}}{\partial x_j} + \overline{\frac{\partial \left(u_j (q')^2\right)}{\partial x_j}} = 2\nu\overline{q'\frac{\partial^2 q'}{\partial x_j^2}} \tag{12.5}$$

Similiarly,

$$\left.\begin{aligned} \overline{\tfrac{\partial (u'_i)^2}{\partial t}} + 2\overline{u'_i u'_j}\tfrac{\partial \overline{u_i}}{\partial x_j} + \overline{\tfrac{\partial \left(u_j (u'_i)^2\right)}{\partial x_j}} = 2\delta_{i3}g\tfrac{\overline{(u'_i\theta'_v)}}{\partial\overline{\theta_v}} \\ -\tfrac{2}{\overline{\rho}}\tfrac{\partial u'_j p'}{\partial x_j} - 2\varepsilon \end{aligned}\right\} \tag{12.6}$$

Total Kinetic Energy = $\frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \Rightarrow \overline{e} = \frac{1}{2}\overline{u_i'^2}$

$$\overline{\frac{\partial e}{\partial t}} + \overline{u'_i u'_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{\frac{\partial \left(u_j e\right)}{\partial x_j}} = \delta_{i3}g\frac{\overline{(u'_i\theta'_v)}}{\partial\overline{\theta_v}} - \frac{1}{\overline{\rho}}\frac{\partial u'_j p'}{\partial x_j} - \varepsilon \tag{12.7}$$

In general, turbulent act only in vertical direction, i.e.

$$u_j = w, \quad x_j = z$$

$$\left.\begin{aligned} \overline{\tfrac{\partial e}{\partial t}} + \underbrace{\overline{u'_i w'}\frac{\partial \overline{u_i}}{\partial z}}_{\substack{\text{Mechanical}\\ \text{Shear}\\ \text{Term}}} + \underbrace{\frac{\partial \overline{(u'_i e')}}{\partial z}}_{\substack{\text{Turbulent}\\ \text{Advection}\\ \text{Term}}} = \underbrace{g\frac{\overline{u'_i\theta'_v}}{\partial\overline{\theta_v}}}_{\substack{\text{Bouancy}\\ \text{Term}}} - \underbrace{\frac{1}{\overline{\rho}}\frac{\partial \overline{w'p'}}{\partial z}}_{\substack{\text{Pressure}\\ \text{Correlation}\\ \text{Term}}} \\ - \underbrace{\varepsilon}_{\substack{\text{Viscous}\\ \text{Dissipation}\\ \text{Term}}} \end{aligned}\right\} \tag{12.8}$$

Turbulent advection term is neglected because generally mean wind can't carry turbulence to longer distance and get's decipacted.

- Bouancy term : Generation of turbulence due to heat flux. Which can be produce turbulence during unstable conditions (day time), can decipate turbulence during stable conditions (night time).
- Mechanical Shear term : Generation of turbulence due to shear in atmoshpere. Negative sign represents turbulence will generate downward if mean windspeed is high upward, and vice versa. Mechanical Shear is only generation term, i.e., it can only generate turbulence and not decipacte.
- Turbulent advection (transport) term : Niether generates nor decipates turbulence but transport thermal energy. When integrate over entire BL, it goes to zero, but when local BL is considered atmoshpere bring turbulent energy from outside local BL or take away turbulent energy from local BL to outside.
- Pressure correlation term : Represents pressure flucuation, very small term ($\approx 0.01hPa$). At stable BL (night time) it generates internal gravity waves to transport turbulence but in general it is a residual term.
- Viscous decipative term : Only decipates turbulence.

question and answers

13. Lecture 13 27/02/2025

13.1 Turbulent Kinetic Energy

$$\frac{\partial \bar{e}}{\partial t}+\overline{u_j}\frac{\partial \bar{e}}{\partial x_j}=\delta_{i3}g\frac{\overline{(u_i'\theta_v')}}{\overline{\theta_v}}-\overline{u_i'\theta_v'}\frac{\partial \overline{u_i}}{\partial x_j}-\frac{1}{\bar{\rho}}\frac{\partial u_j'p'}{\partial x_j}-\varepsilon \quad (13.1)$$

Taking only vertical direction $u_j = w$, $x_j = z$ and after neglecting turbulent advection term:

$$\frac{\partial \bar{e}}{\partial t}=g\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}}-\overline{w'\theta_v'}\frac{\partial \overline{w}}{\partial z}-\frac{1}{\bar{\rho}}\frac{\partial w'p'}{\partial z}-\varepsilon \quad (13.2)$$

13.2 Types of convection

1. Free convection $\rightarrow g\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}}$ is dominant.
2. Forced convection $\rightarrow \overline{u'w'}\frac{\partial \bar{u}}{\partial z}$ is dominant.

Vertical scaling factor known as **Deodroff velocity scale**:

$$w_* = \left[g_z\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}}\right]^{1/3} \quad (13.4)$$

Time scaling factor:

$$t_* = \left[\frac{z_i}{w_*^3}\right] = \frac{z_i}{\left[g_z\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}}\right]} = \frac{z_i\overline{\theta_v}}{\left[g_z\overline{(w'\theta_v')}\right]} \quad (13.5)$$

Therefore modified Eq.(13.2) after scaling factor will be:

$$\left.\begin{aligned}\frac{z_i}{w_*^3}\frac{\partial \bar{e}}{\partial t} &= \frac{z_i}{w_*^3}g\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}}-\frac{z_i}{w_*^3}\overline{w'\theta_v'}\frac{\partial \overline{w}}{\partial z}- \\ &\frac{z_i}{w_*^3}\frac{1}{\bar{\rho}}\frac{\partial w'p'}{\partial z}-\frac{z_i}{w_*^3}\varepsilon\end{aligned}\right\} \quad (13.6)$$

14. Lecture 14 28/02/2025

14.1 Stability, KH-Waves And Richardson Number

There are 2 types of Stability:

1. Static Stability
   - Determinted by **Bouancy**.
2. Dynamic Stability
   - Determinted by **Shear**.
   - Causes unstabilty and form **Helmon waves**.
   - Clouds formed due to these waves (instability) is called **Below clouds** or **KH clouds**.

Add figure

14.2 Dynamic Stability

$$\frac{\partial \bar{e}}{\partial t} = g\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{w'\theta_v'}\frac{\partial \overline{w}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial w'p'}{\partial z} - \varepsilon \quad (14.1)$$

14.2.1 Richardson Number (R_f)

$$R_f = \frac{g\frac{\overline{w'\theta_v'}}{\overline{\theta_v}}}{\overline{u'w'}\frac{\partial \bar{u}}{\partial z}} \qquad \begin{array}{l} \rightarrow \text{Bouancy term} \\ \rightarrow \text{Mechanical Flux Term} \end{array}$$

Rf conditons and cases

Limitaton of Richardson number : It assumes the flow to be turbulent ($w'\theta_v'$ and $u'w'$, so it can't be applid to laminar flow.

15. Lecture 15 12/03/2025

15.1 Turbulent Kinetic Energy

$$\frac{\partial \bar{e}}{\partial t} = g\frac{\overline{(w'\theta_v')}}{\overline{\theta_v}} - \overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \frac{\partial \overline{w'e}}{\partial z} - \frac{1}{\bar{\rho}}\frac{\partial w'p'}{\partial z} - \varepsilon \tag{15.1}$$

Richarson No.

$$R_f = \frac{\frac{g}{\overline{\theta_v}}\left(\overline{w'\theta_v'}\right)}{\overline{u'w'}\frac{\partial \bar{u}}{\partial z} + \overline{v'w'}\frac{\partial \bar{v}}{\partial z}} \begin{array}{l} \rightarrow \text{Heat flux} \\ \rightarrow \text{Momentum flux} \end{array} \tag{15.2}$$

1. if $R_f < 0$ the flow is always turbulent.
2. if $0 < R_f < 1$ the flow can be laminar or turbulent.
3. if $R_f > 1$ the flow is always laminar.

Flux Richarson number will tell us when a turbulent flow witll become laminar but not vice versa. So need to define new term which can tell both laminar and turbulent flow., i.e,
Case i. Laminar $\rightarrow$ Turbulent
Case ii. Turbulent $\rightarrow$ Laminar

Flux proportional gradients, approximating, using Eddy diffusivity theory,

$$-\overline{w'\theta'} \propto \frac{\partial \bar{\theta}}{\partial z} = k\frac{\partial \bar{\theta}}{\partial z}$$
$$-\overline{u'w'} \propto \frac{\partial \bar{u}}{\partial z}, -\overline{v'w'} \propto \frac{\partial \bar{v}}{\partial z}$$

Replacing all flux terms with gradients in Eq.(15.2), we get
Gradient Richarson's Number

$$R_i = \frac{\frac{g}{\overline{\theta_v}}\left(\overline{w'\theta_v'}\right)}{\left(\frac{\partial \bar{u}}{\partial z}\right)^2 + \left(\frac{\partial \bar{v}}{\partial z}\right)^2} \tag{15.3}$$

- $R_c \approx 0.25$: Critical Richardson's No.
- $R_i < R_c$: Flow is turbulent (dynamically unstable).
- $R_i > R_c$: Flow *not* is laminar.
- $R_i > R_T$: Flow is laminar (dynamically stable)

Practically our measurements do not contain continous data but discrete. Therefore we will change partial differential equations into difference equations, i.e. $\partial \rightarrow \Delta$
Bulk Richardson's Number

$$R_B = \frac{g}{\overline{\theta_v}}\frac{\frac{\Delta\bar{\theta}}{\Delta z}}{\left(\frac{\Delta \bar{u}}{\Delta z}\right)^2 + \left(\frac{\Delta \bar{v}}{\Delta z}\right)^2}$$
$$R_B = \frac{g}{\overline{\theta_v}}\frac{\Delta\bar{\theta}\Delta z}{(\Delta\bar{u})^2 + (\Delta\bar{v})^2} \tag{15.4}$$

16. Lecture 16 13/03/2025

16.1 Governing Equations for Boundary Condition

$$\frac{\partial u_i'}{\partial t}+\overline{u_j}\frac{\partial u_j'}{\partial x_j}+u_j'\frac{\partial \overline{u_j}}{\partial x_j}+u_j'\frac{\partial u_j'}{\partial x_j}=\delta_{i3}g\frac{\overline{\theta_v'}}{\overline{\theta_v}}+f\varepsilon_{ij3}u_j'-\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_j^2}+\frac{\partial \overline{u_i'u_j'}}{\partial x_j} \tag{16.1}$$

Multiply Eq.(16.1) by u_k'

$$u_k'\frac{\partial u_i'}{\partial t}+u_k'\overline{u_j}\frac{\partial u_j'}{\partial x_j}+u_k'u_j'\frac{\partial \overline{u_j}}{\partial x_j}+u_k'u_j'\frac{\partial u_j'}{\partial x_j}=u_k'\delta_{i3}g\frac{\theta_v'}{\overline{\theta_v}}+u_k'f\varepsilon_{ij3}u_j'-u_k'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}+u_k'\nu\frac{\partial^2 u_i'}{\partial x_j^2}+u_k'\frac{\partial \overline{u_i'u_j'}}{\partial x_j} \tag{16.2}$$

Applying Reynold averaging,

$$\overline{u_k'\frac{\partial u_i'}{\partial t}}+\overline{u_k'\overline{u_j}\frac{\partial u_j'}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial \overline{u_j}}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial u_j'}{\partial x_j}}=\overline{u_k'\delta_{i3}g\frac{\theta_v'}{\overline{\theta_v}}}+f\varepsilon_{ij3}\overline{u_k'u_j'}-\overline{u_k'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+\nu\overline{u_k'\frac{\partial^2 u_i'}{\partial x_j^2}} \tag{16.3}$$

Every occurance of 'k' will be replaced with 'i' and every of occurance of 'i' will be replaced with 'k' respectively.

- Summed term will remain summed.
- Unsummed terms continue to represent 3 components.

$$\overline{u_i'\frac{\partial u_k'}{\partial t}}+\overline{u_i'\overline{u_j}\frac{\partial u_j'}{\partial x_j}}+\overline{u_i'u_j'\frac{\partial \overline{u_j}}{\partial x_j}}+\overline{u_i'u_j'\frac{\partial u_j'}{\partial x_j}}=\overline{u_i'\delta_{k3}g\frac{\theta_v'}{\overline{\theta_v}}}+f\varepsilon_{kj3}\overline{u_i'u_j'}-\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_k}}+\nu\overline{u_i'\frac{\partial^2 u_k'}{\partial x_j^2}} \tag{16.4}$$

Add Eq.(16.3) and Eq.(16.4)

$$\begin{aligned}\frac{\partial\overline{u_i'u_k'}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i'u_k'}}{\partial x_j}+\overline{u_i'u_j'\frac{\partial u_k}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial u_i}{\partial x_j}}+\overline{u_j'\frac{\partial u_i'u_k'}{\partial x_j}}\\=\frac{\overline{\theta_v'g}}{\overline{\theta_v}}\left(\delta_{k3}u_i'+\delta_{k3}u_k'\right)+f\varepsilon_{ij3}\overline{u_k'u_j'}+f\varepsilon_{kj3}\overline{u_i'u_j'}\\-\left[\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_k}}\right]+\nu\left[\overline{u_i'\frac{\partial^2 u_k'}{\partial x_j^2}}+\overline{u_k'\frac{\partial^2 u_i'}{\partial x_j^2}}\right]\end{aligned} \tag{16.5}$$

Using continuity equation;

$$\overline{u_iu_k'\frac{\partial u_j'}{\partial x_j}}=0$$

$$\begin{aligned}\frac{\partial\overline{u_i'u_k'}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i'u_k'}}{\partial x_j}+\overline{u_i'u_j'\frac{\partial u_k}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial u_i}{\partial x_j}}+\frac{\partial\overline{u_i'u_j'u_k'}}{\partial x_j}\\=\frac{\overline{\theta_v'g}}{\overline{\theta_v}}\left(\delta_{k3}u_i'+\delta_{k3}u_k'\right)+f\varepsilon_{ij3}\overline{u_k'u_j'}+f\varepsilon_{kj3}\overline{u_i'u_j'}\\-\left[\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_k}}\right]+\nu\left[\frac{\partial^2\overline{u_i'u_k'}}{\partial x_j^2}-2\frac{\partial^2\overline{u_k'u_i'}}{\partial x_j^2}\right]\end{aligned} \tag{16.6}$$

Simplifing pressure gradient term:

$$\left[\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_i}}+\overline{u_i'\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial x_k}}\right]=\frac{1}{\overline{\rho}}\left[\frac{\partial\overline{p'u_k'}}{\partial x_i}+\frac{\partial\overline{p'u_i'}}{\partial x_k}\right]-p'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_j}\right)$$

$$\begin{aligned}\frac{\partial\overline{u_i'u_k'}}{\partial t}+\overline{u_j}\frac{\partial\overline{u_i'u_k'}}{\partial x_j}+\overline{u_i'u_j'\frac{\partial u_k}{\partial x_j}}+\overline{u_k'u_j'\frac{\partial u_i}{\partial x_j}}+\frac{\partial\overline{u_i'u_j'u_k'}}{\partial x_j}\\=\frac{\overline{\theta_v'g}}{\overline{\theta_v}}\left(\delta_{k3}u_i'+\delta_{k3}u_k'\right)+f\varepsilon_{ij3}\overline{u_k'u_j'}+f\varepsilon_{kj3}\overline{u_i'u_j'}\\-\frac{1}{\overline{\rho}}\left[\frac{\partial\overline{p'u_k'}}{\partial x_i}+\frac{\partial\overline{p'u_i'}}{\partial x_k}\right]-p'\left(\frac{\partial u_i'}{\partial x_k}+\frac{\partial u_k'}{\partial x_j}\right)\\+\nu\left[\frac{\partial^2\overline{u_i'u_k'}}{\partial x_j^2}-2\frac{\partial^2\overline{u_k'u_i'}}{\partial x_j^2}\right]\end{aligned} \tag{16.7}$$

References

[1] R. B. Stull. *An Introduction to Boundary Layer Meteorology*. Atmospheric Sciences Library. Kluwer Academic Publishers, 1988.