Lecture Notes on Numerical Weather Prediction

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Abstract
Notes of Lectures and addional information from books.

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1. Lecture 1 06/01/2025

Numerical Weathering Problem (NWP) was first proposed by Bjerkives around 1900. It is mathematical initial value problem (IVP).
Initial value Problem (IVP) $\rightarrow$ simple pendulum.

$$\ddot{\theta} + \omega^2 \theta = 0 \tag{1}$$

$$\frac{d^2\theta}{dt^2} + \omega^2 \theta = 0 \tag{2}$$

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t) \tag{3}$$

Eq.(1) and (2) are second order linear ordinary differential equation, whose solution Eq.(3) has 2 constants of integration A and B. Here θ and t are the dependent and independent variable since Eq.(1) and (2) have only one independent variable. Values of A and B will depend on initial condition.
Since ODE is second order, 2 initial condition are needed at initial time, say $t = 0$. Which are:

$$\left.\begin{aligned} \theta(t=0) &= 1 \\ \frac{\theta(t=0)}{dt} &= 0 \end{aligned}\right\} \tag{4}$$

Eq.(2) and initial conditions Eq.(4) are together called **Mathematical IVP**. For any physical system the following two requirements are needed:

1. The equation (ODE or PDE) that governs the evolution of the above system.
2. The initial state of the system.

7 independent variables **(u,v,w,T,ρ,p,q)**.
Surface area of Earth = $4\pi R^2$ = $4\pi(6.37 \times 10^{12}) \approx 5.1 \times 10^{14}$ m^2

2. Lecture 2 07/01/2025

7 independent variables **(u,v,w,T,ρ,p,q)** therefore we need 7 Governing equations (system of 7 coupled non-linear partial differential equations):

1. Conservation of masss (continuity equation).
2. Conservation of momentum in rotating frame of refrence (3 scalar equations, one each corresponding to scalar component of velocity).
3. Conservation of energy (Thermodynamic energy equation).
4. Conservation of moisture (moisture continuity equation).
5. Equation of state (Ideal gas equation).

Euler discription of fluid motion is more convinent becasue of dependance on time and above 7 equations.
Total advective and convective time of lagrangian is given by:

$$\underbrace{\frac{DT}{Dt}}_{\text{Lagrangian Derivative}} = \underbrace{\underbrace{\frac{\partial T}{\partial t}}_{\text{Local derivative}} + \underbrace{\vec{V}\cdot\nabla T}_{\text{Advective Term}}}_{\text{Euler Derivative}}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{5}$$

Using first law of Thermodynamics, Rate of heat is given by:

$$d\dot{q} = d\dot{u} + d\dot{w}$$
$$\frac{DU}{Dt} = \frac{Dq}{Dt} - \frac{Dw}{Dt}$$
$$C_v\frac{DT}{Dt} = \frac{Dq}{Dt} - p\frac{D\alpha}{Dt}$$

where $\frac{Dq}{Dt}$ is rate at which heating of air parcel due to non-adiabatic process, this change can happen via radiation, convection, conduction, latent heat while phase change.

$$\frac{DU}{Dt} = \vec{F}_{\text{net}} + \vec{F}_{\text{coriolis}} \tag{6}$$

This above Eq.(6) is convective derivative equation involving non-linear terms (i.e. $u\frac{\partial T}{\partial x}$, $v\frac{\partial T}{\partial y}$, $w\frac{\partial T}{\partial z}$).
Continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0 \tag{7}$$

Let grid of following resolutions:

- $1^\circ \times 1^\circ \to 3\times10^6$ grid cells $\therefore$ no. of variables $\to 7\times3\times10^6$.
- $5^\circ \times 5^\circ \to 1.3\times10^5$ grid cells $\therefore$ no. of variables $\to 7\times1.3\times10^5$.
- $20^\circ \times 20^\circ \to 9\times10^3$ grid cells $\therefore$ no. of variables $\to 7\times9\times10^3$.
- $25^\circ \times 25^\circ \to 6\times10^3$ grid cells $\therefore$ no. of variables $\to 7\times6\times10^3$.

These are even larger than entire country, which means that we can't above to find the change of varibles with these grids. This we don't have a way to determine initial condition, if we try to use interpolation, it will cause errors which will grow with time since atmosphere is chaotic and dynamic system.

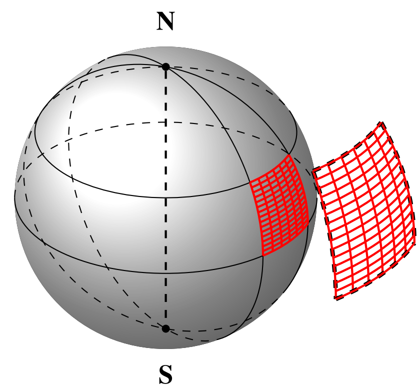

Figure 1. Figure showing the grid

3. Lecture 3 08/01/2025

$$u = \bar{u} + u' \tag{8}$$

Here, u is the velocity field, which is decomposed into a mean component $\bar{u}$ and a fluctuating component u'.
Navier-Stokes Equation The general Navier-Stokes equation is given by:

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} \\ + \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\end{aligned} \tag{9}$$

Reynolds-Averaged Navier-Stokes (RANS) Equation Applying Reynolds decomposition ($u = \bar{u} + u'$) and averaging leads to the RANS equation:

$$\begin{aligned}&\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} - f\bar{v} = \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} \\ &+ \gamma\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) \\ &+ \underbrace{\frac{1}{\rho}\left(\frac{\partial\left(-\rho\overline{u'u'}\right)}{\partial x} + \frac{\partial\left(-\rho\overline{u'v'}\right)}{\partial y} + \frac{\partial\left(-\rho\overline{u'w'}\right)}{\partial z}\right)}_{\text{Reynolds stress tensor}}\end{aligned} \tag{10}$$

The Reynolds stress tensor represents the transport of momentum due to turbulent fluctuations.
Nonlinear Term Expansion Expanding the nonlinear term $u\frac{\partial u}{\partial x}$ using Reynolds decomposition:

$$\begin{aligned}u\frac{\partial u}{\partial x} &= (\bar{u} + u')\frac{\partial(\bar{u} + u')}{\partial x} \\ &= \bar{u}\frac{\partial \bar{u}}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}.\end{aligned}$$

Appliing Reynolds averaging rules:

$$\begin{aligned}&\overline{u} = \overline{\bar{u} + u'} \\ &\overline{u} = \overline{\bar{u}} + \overline{u'} \\ &\overline{u} = \bar{u} + \overline{u'} \quad \Rightarrow \quad \overline{u'} = 0.\end{aligned}$$

Thus, the fluctuating component u' averages out to zero over time, leaving only the mean component $\bar{u}$ in the averaged equations.
We have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{11}$$

Substituting u, v and w in above Eq.(11), we get:

$$\begin{aligned}\overline{\frac{\partial(\bar{u} + u')}{\partial x} + \frac{\partial(\bar{v} + v')}{\partial y} + \frac{\partial(\bar{w} + w')}{\partial z}} = 0 \\ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0\end{aligned}$$

The term $\frac{\partial(\overline{u'u'})}{\partial t}$ represents the rate of change of kinetic energy per unit mass due to turbulent fluctuations. It can be expressed as:

$$\frac{\partial(\overline{u'u'})}{\partial t} = \frac{\partial\left(\rho\overline{u'v'}\right)}{\partial y} + \dots$$

This term involves higher-order correlations between velocity fluctuations, which complicates the equation system.
The closure problem arises in the Reynolds-Averaged Navier-Stokes (RANS) equations because the number of dependent variables (unknowns) exceeds the number of equations available. For instance: - $\overline{u}$ is an unknown. - $\overline{u'v'}$ (a Reynolds stress term) introduces additional unknowns.
To resolve this, closure modelsare used, which provide approximations for higher-order terms based on known variables.
For example, consider the term $\overline{u'w'}$. Using a simple closure model:

$$\overline{u'w'} = -k\frac{\partial \bar{u}}{\partial x},$$

where k is a proportionality constant (often related to eddy viscosity). Here, $\overline{u}$ is already an unknown, so no additional variables are introduced, avoiding further complexity.
This is an example of first-order closurewhere higher-order terms are approximated using first-order variables.
Types of Closure Models

1. **First-Order Closure**:
   - Simplifies higher-order terms using known variables and gradients (e.g., eddy viscosity models).
   - Example: $\overline{u'w'} = -k\frac{\partial \bar{u}}{\partial x}$.
   - Advantage: Computationally efficient but may lack accuracy in complex flows.
2. **One-Point Closure**:
   - Approximates turbulence at a single point using local flow properties.
   - Example: Mixing length models, where turbulent viscosity is proportional to local shear.
3. **Second-Order Closure**:
   - Directly models second-order correlations like $\overline{u'u'}$ and $\overline{u'v'}$ by solving additional transport equations.
   - Provides higher accuracy but increases computational cost.
   - Example: Reynolds stress models (RSM), where additional equations are solved for Reynolds stresses.

4. Lecture 4 13/01/2025

Solar raditation heats up ground/ocean surface, which in turn heats the surface atmosphere which generates unstable convective cells in the atmosphere. where the above air is cold and below layer (near ground) is warmer.
Limits of deterministic predictability of atmoshpere is **approximately 2 weeks**. This is because of the fact that atmoshpere is chatic system, i.e., error in initial condition grows progressively with time.

- Over a season (2-3 months) forcast is called long range.
- $10 \sim 15$ days is called medium range.
- $3 \sim 5$ days is called short range.

Charva and Shukla (1979) stated about the predictibility of atmosphere in their paper the following:

1. Wherever boundary conditions vary slowly and that atmosphere is influenced by slowly varying boundary is more longer predictable ($>$ 2 weeks).
2. They found that since most of ocean are in topical region, therefore atmoshpere over tropics is more predictable but in real-world-scenerio it is less predictable.

5. Lecture 5 15/01/2025

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \tag{5}$$

Initial conditions:

$$\theta(t = t_0) = \theta_0$$
$$\theta(t \dot{=} t_0) = \omega_0$$

We can rewrite this equation into two 1^{st}-order ODEs,
Let $\frac{\theta}{dt} = p$, then

$$\frac{dp}{dt} + \omega^2\theta = 0 \tag{i}$$
$$\frac{d\theta}{dt} = p \tag{ii}$$

Initial conditions becomes:

$$\theta(t = t_0) = \theta_0$$
$$p(t = t_0) = \omega_0$$

Invoking Taylor series expansion for $f(t + t_0)$:

$$\begin{aligned} f(t+t_0) &= f(t_0) + (t+t_0)\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(t+t_0)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \\ f(t+\Delta t) &= f(t_0) + \Delta t\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(\Delta t)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \end{aligned}$$

we will get;

$$\begin{aligned} p(t+t_0) &\approx p(t_0) + \Delta t\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ p(t+t_0) &\approx p(t_0) + (t-t_0)\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t+t_0) - p(t_0)}{(t-t_0)} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_1) - p(t_0)}{\Delta t} \end{aligned} \tag{6}$$

Eq.(6) is called **Forward difference**.
Order of Forward difference is $O(\Delta t)$.
Similiarly,

$$\frac{\partial\theta}{\partial t}\Big|_{t=t_0} \approx \frac{\theta(t+t_0) - \theta(t_0)}{(t-t_0)} \tag{7}$$

From Eq.(i) and Eq.(6):

$$\begin{aligned} \frac{p(t+t_0) - p(t_0)}{(t-t_0)} + \omega^2\theta(t_0) = 0 \\ p(t+t_0) = p(t_0) - (t-t_0)\omega^2\theta(t_0) \\ p(t_1) = p(t_0) - \Delta t\,\omega^2\theta(t_0) \end{aligned}$$

From Eq.(ii) and Eq.(7):

$$\begin{aligned} \frac{\theta(t+t_0) - \theta(t_0)}{(t-t_0)} = p(t_0) \\ \theta(t+t_0) = \theta(t_0) - (t-t_0)p(t_0) \\ \theta(t_1) = \theta(t_0) - \Delta t\, p(t_0) \end{aligned}$$

Invoking Taylor series expansion for $f(t - t_0)$:

$$\begin{aligned} f(t-t_0) &= f(t_0) - (t-t_0)\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(t-t_0)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \\ f(t-\Delta t) &= f(t_0) - \Delta t\frac{\partial f}{\partial t}\Big|_{t=t_0} \\ &\quad + \frac{(\Delta t)^2}{2!}\frac{\partial^2 f}{\partial t^2}\Big|_{t=t_0} + \cdots \end{aligned}$$

we will get;

$$\begin{aligned} p(t-t_0) &\approx p(t_0) - \Delta t\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ p(t-t_0) &\approx p(t_0) - (t-t_0)\frac{\partial p}{\partial t}\Big|_{t=t_0} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_0) - p(t-t_0)}{(t-t_0)} \\ \frac{\partial p}{\partial t}\Big|_{t=t_0} &\approx \frac{p(t_0) - p(t_{-1})}{\Delta t} \end{aligned} \tag{8}$$

Eq.(8) is called **Backward difference**.
Order of Backward difference is $O(\Delta t)$.
Similiarly,

$$\frac{\partial\theta}{\partial t}\Big|_{t=t_0} \approx \frac{\theta(t_0) - \theta(t_{-1})}{\Delta t} \tag{9}$$

incomplete
fix equation no.s