This assignment is based on state estimation techniques and consists of two parts.

Please use Python for implementation, using only standard python libraries (e.g., numpy, math). **Do not use any third party libraries/implementations for algorithms**.

Please prepare a report to accompany your implementation. The report should contain detailed responses to questions. Briefly describe the key findings/insights for the graphs. **Ensure the reproducibility of graphs** (modulo probabilistic execution) for your submission. Submit the report along with your code used to generate your graphs/observations.

For plotting, please use plotly library as it can generate interactive html plots. Here is an example showing how to plot 3D plots in plotly. Export all your plots in html format and upload them to a drive folder. Ensure that no changes are made to the folder after the deadline. Aptly name your plots to aid evaluation. Include the link to the folder in your report, and ensure that the link is working. Also provide snapshots of the plots in the report wherever required.

Additional readings: Artificial Intelligence: A Modern Approach (Ch. 15) and Probabilistic Robotics (Ch. 2 and Ch. 3).

Please refer to the submission instructions on the course webpage. We encourage you to seek clarifications in the problem statements on Piazza should you have any.

As always, submission will be thoroughly analysed for plagiarism; any cases found will be heavily penalised and dealt with as per the course policy.

1. State Estimation of 1D motion using Kalman Filters (40 points)

Consider a train going along a straight track from stations A to B. Since the Track is a conductor of sound, we can send a sharp pulse periodically, wait for it to echo back, and use the time taken (z_t) as a measure for the train's distance (x_t) from A. This obeys the standard relation $z_t = 2 * x_t / v_{sound}$. We control the motion of the train by providing it relevant accelerations/decelerations, $u_t = \ddot{x}_t$.

The uncertainty in motion is characterized by a presence of a Gaussian noise $\epsilon_t \sim \mathcal{N}(0, R)$. This may be interpreted as a noisy reaction to the control provided - inaccurate change in velocity despite an intended acceleration (engine/power fluctuations), and the inaccurate change in distance despite the velocity (skidding on track, track non-linearity etc.).

Similarly, the measurement uncertainty is characterized by presence of Gaussian noise $\delta_t \sim \mathcal{N}(0, Q)$. This may have arisen due to temperature non-uniformity, equipment errors, stray noise and so on.

Our goal is to estimate the train's position x_t and velocity $\dot{x}_t$ at time instant t, given noisy observations $z_{t'}|_{t'=0}^{t}$. Unless otherwise stated, make the following assumptions for parts (a) through (f).

1. Time is measured in hour units and distance in kilometer units.
2. Filter updates have to be performed for $N = 326$ time steps, every 0.01 hour.
3. The train, a point object, is initially at rest and at starting position 0. Assume that this initial belief is known with the covariance $10^{-4} * \mathbb{I}_2$, where the $\mathbb{I}_2$ is the 2x2 Identity matrix.
4. The Observation noise is a Gaussian distribution with a standard deviation $\sigma_s = 0.01$.
5. Noise in the motion is parameterised by: $\sigma_x = 0.1$ and $\sigma_{\dot{x}} = 0.5$ forming the covariance matrix R as $diag(\sigma_x^2, \sigma_{\dot{x}}^2)$ where $diag$ denotes the diagonal elements of R with off-diagonals as zero.
6. Assume $v_{sound} = 3000.0$ km/h for the medium of the track.
7. The control policy provides the accelerations, u_t as per the following schedule :

$$u(t) = \begin{cases} +400, & \text{if } t < 0.25 \\ -400, & \text{if } 3 < t < 3.25 \\ 0, & \text{otherwise} \end{cases}$$

   These numbers are such that the train comes to a stop at the end of the simulation.

Environment Setup and Filtering (15 points)

(a) Consider state X_t to be the vector $[x_t, \dot{x}_t]^T$. Formally describe the motion and observation models for this problem. Plot the (x_t, t), and ($\dot{x}_t$, t) obtained.

(b) Formally describe the Kalman Filter model for estimating the train state. Implement the same, and plot the estimated state (x_t^e, t), and ($\dot{x}_t^e$, t).

(c) Jointly plot the actual trajectory x_t, and the estimated trajectory x_t^e along with it's uncertainty bars, vs time. An uncertainty bar denotes the points that are one standard deviation away from the mean. Briefly discuss your observations.

Experiments (25 points)

(d) We have described 3 types of noise parameters: the noise in position update σ_x, the noise in velocity update $\sigma_{\dot{x}}$ and the noise in the sensor measurements σ_s. Qualitatively describe how the trajectories (actual and estimated) and uncertainty bars change on varying the each of the 3 noise parameters. Explain your observations.

(e) Kalman Gain (K_t) is defined in the lecture slides. Plot (K_t^x, t) and $(K_t^{\dot{x}}, t)$ for the simulation, where $K_t = [K_t^x, K_t^{\dot{x}}]^T$ How do you interpret these quantities? Vary the noise parameters as in part(d), and present how these change. Explain your observations.

(f) Due to some mechanical failure, we are unable to receive any observations for time $t \in [1.5, 2.5]$. Plot the actual trajectory x_t, the estimated trajectory x_t^e and the uncertainty bars. Contrast with the plots from part (c) and explain your observations.

2. State Estimation of 3D motion using Kalman Filters (60 points)

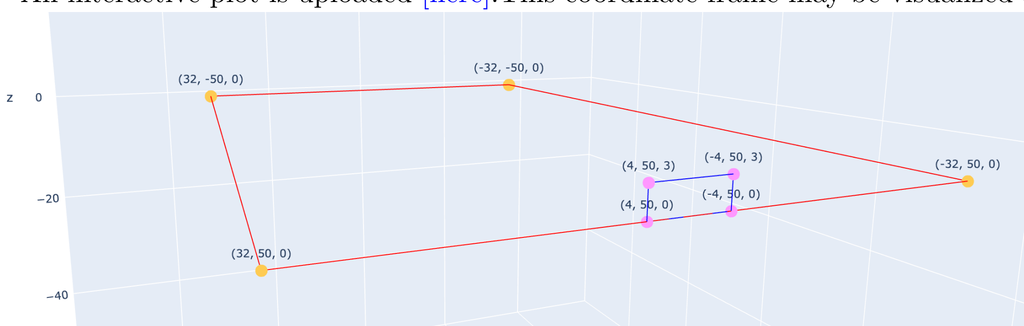
Our goal is to estimate a football's motion as a soccer game proceeds.

Measurements z_t are provided by 3 classes of noisy sensors: a GPS system, a set of base stations or an IMU sensor fixed inside the ball. The football is subject to a constant gravity $g = -10.0m/s^2$, which get added to the velocity components $\dot{x}_t$, $\dot{y}_t$ and $\dot{z}_t$ at each time step.

The uncertainty in the motion is characterized by a presence of Gaussian noise $\epsilon_t \sim \mathcal{N}(0, R)$. This may be attributed to the spin of the ball, interaction with the wind so on. Similarly, the measurement uncertainty of each sensor family is characterized by presence of a Gaussian noise $\delta_t \sim \mathcal{N}(0, Q)$.

Unless otherwise stated, make the following assumptions for parts (a) through (f).

1. Time is measured in seconds and distance in meters.
2. Filter updates have to be performed for $N = 130$ time steps, every 0.01 seconds.
3. The field is 100 * 64 meters. The goal post is 8*3 meters, erected along one of the sides. An interactive plot is uploaded [here].This coordinate frame may be visualized as follows:



4. The observation noise in all systems are isotropic Gaussian distributions : The GPS system with $\sigma_{GPS} = 0.1$, the base-station $\sigma_{BS} = 0.1$, and the ball's IMU $\sigma_{IMU} = 0.1$.
5. Noise in the motion is parameterised by: σ_x, σ_y, $\sigma_z = 0.01$ and $\sigma_{\dot{x}}$, $\sigma_{\dot{y}}$, $\sigma_{\dot{z}} = 0.1$ forming the covariance matrix Q as $diag(\sigma_x^2, \sigma_y^2, \sigma_z^2\sigma_{\dot{x}}^2, \sigma_{\dot{y}}^2, \sigma_{\dot{z}}^2)$ where $diag$ denotes the diagonal elements of Q with off-diagonals as zero.
6. We will only consider estimating an in-air trajectory and will not deal with the physics of the ball hitting the ground. Thus, the control u_t is just g acting along along the z-axis.
7. Initial State of the Ball $= [24.0, 4.0, 0.0, -16.04, 36.8, 8.61]^T$, known with covariance $10^{-4} * \mathbb{I}_6$. One may think of this as a case where a player has attempted to shoot at the top-right corner of the goal. An interactive plot is uploaded [here].

Environment Setup and Filtering (20 points)

(a) Consider the state $X_t = [x_t, y_t, z_t, \dot{x}_t, \dot{y}_t, \dot{z}_t]^T$. Formally describe the motion model for this problem.

(b) The inputs from sensor systems are described as follows:

- GPS : $L_t = [x_t^o, y_t^o, z_t^o]^T$, the noisy positions of the football.
- Base-Stations : $D_t = [D_t^1, D_t^2, D_t^3, D_t^4]^T$, the noisy distances observed from the base stations located on each of the corners of the field at z=10.

- IMU : $I_t = [x_t^o, y_t^o, z_t^o, \dot{x}_t^o, \dot{y}_t^o, \dot{z}_t^o]^T$, the noisy positions and velocities of the football.

For each system at a time, formally describe the model for estimating the state of the ball. Implement the Kalman/EKF filters as relevant. For the same ground-truth trajectory, estimate the trajectories via each of the three systems. Plot and explain your observations.

Experiments (40 points)

(c) Our goal now is to build an *automated referee system* for declaring if a goal is scored. In the following sub-parts, we will study how the decision for declaring a goal can be made if we had access to the ground truth trajectory, the raw (noisy) observations and from estimates derived from a filtering process. For each part below, justify your choices and observations:

- Devise an approach (a decision rule) to determine whether a goal has been scored or not if you have direct access to the *ground truth* trajectory. Note that the coordinates of the goal region have been provided before. Perform 1000 simulations and evaluate the fraction of times a goal is scored from the ground truth trajectory.
- Next, determine the fraction of times we can declare a goal scored from the *raw observations* (that is from the GPS sensor and the IMU sensor).
- Next, we consider the problem of determining if goals are scored from the *estimates* from a filtering process. Note that filtering would produce a mean and a variance. Determine how to you extend your approach (decision rule) to predict if a goal is score given a estimates. What fraction of *estimates* are declared as goals?

(d) We have described 3 types of noise parameters : noise in position update (σ_x, σ_y, σ_z), noise in velocity update ($\sigma_{\dot{x}}$, $\sigma_{\dot{y}}$, $\sigma_{\dot{z}}$) and noise in the sensor measurements (σ_s). Please repeat part (c) for the GPS system, varying each of the 3 noise parameters (you may still keep $\sigma_x = \sigma_y = \sigma_z$ and similarly for velocity). Tabulate and explain your observations.

(e) Simulate a set of actual trajectory $[x_t, y_t, z_t]$, the noisy observations $[x_t^o, y_t^o, z_t^o]$ and the trajectory estimated by the filter $[x_t^e, y_t^e, z_t^e]$ for the GPS sensors. Plot their projections onto the XY plane, along with the uncertainty ellipses of the estimated trajectory. Report and explain the variations observed with different position update noise values (keeping $\sigma_x = \sigma_y = \sigma_z$) and observation noise values(σ_{GPS}). An uncertainty ellipse denotes the locus of points that are one standard deviation away from the mean.

(f) For this case, we consider a 2D version of this problem, i.e., the state is now $X_t = [x_t, y_t, \dot{x}_t, \dot{y}_t]^T$. Assume the initial state to be $[0.0, -50.0, 0.0, 40.0]^T$, known with covariance $\mathbb{I}_4$. There is no additional control ($|u_t| = 0$), all other parameters are the same as earlier. This may be interpreted to be a goalkeeper directly aiming for the opposite goal.

We now have 4 Base Stations : $B1([-32, 50])$, $B2([-32, -50])$, $B3([-32, 0])$, $B4([32, 0])$ that measure the (noisy, each with $\sigma_{BS} = 0.1$) euclidean distances from the ball. Due to cost constraints, we can get observations from exactly one of $B1$, $B2$, $B3$, or the combination $[B3, B4]$.

1. For each of these 4 cases, plot the trajectory, the estimates and the uncertainty ellipses (all in the XY plane) as in part (e) for 250 time-steps.
2. Comment on the orientation of the ellipses (Hint : Draw some straight lines from the Base-Stations to the trajectory to get a clearer indication). Why is this happening?
3. Plot the lengths of the major and minor axis of the uncertainty ellipses vs time. What do these quantities represent? Compare and explain the observations from the 4 cases.