MTL712: Computational Methods for Differential Equations
Assignment 5

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Contents

1 Description

1.1 Problem 1

Apply the Linear Shooting technique with $N = 10$ to the BVP

$$y'' = -\frac{2}{x} + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x}, 1 \leq x \leq 2, y(1) = 1, y(2) = 2,$$

and compare the results to those of the exact solution

$$y = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x),$$

where $c_2 = \frac{1}{70}[8 - 12\sin(\ln 2) - 4\cos(\ln 2)] = -0.03920701320$ and $\frac{11}{10} - c_2 = 1.1392070132$.

1.2 Problem 2

Apply the Linear Shooting technique with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{4}$ to the BVP

$$y'' = 4(y - x), 0 \leq x \leq 1, y(0) = 0, y(1) = 2,$$

and compare the results to those of the exact solution

$$y(x) = x + \frac{e^2\left(e^{2x} - e^{-2x}\right)}{\left(e^4 - 1\right)}.$$

1.3 Problem 3

Apply the Linear Shooting technique with (a) $h = \frac{1}{10}$ and (b) $h = \frac{1}{20}$ to the BVP

$$y'' = 100y, 0 \leq x \leq 1, y(0) = 1, y(1) = e^{-10},$$

and compare the results to those of the exact solution

$$y(x) = e^{-10x}.$$

1.4 Problem 4

Apply the Nonlinear Shooting Method with Newton's Method with $N = 20, M = 10, TOL = 10^{-5}$, to the BVP

$$y'' = \frac{1}{8}\left[32 + 2x^3 - yy'\right], 1 \le x \le 3, y(1) = 17, y(3) = \frac{43}{3},$$

and compare the results to those of the exact solution

$$y = x^2 + \frac{16}{x}$$

1.5 Problem 5

Apply the Nonlinear Shooting Method with $h = 0.5$, to the BVP

$$y'' = -\left(y'\right)^2 - y + \ln x, 1 \le x \le 2, y(1) = 0, y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x.$$

1.6 Problem 6

Apply the Nonlinear Shooting Method with $N = 10, TOL = 10^{-4}$ to the BVP

$$y'' = -e^{-2y}, 1 \le x \le 2, y(1) = 0, y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x$$

1.7 Problem 7

Apply the Nonlinear Shooting Method with $N = 10$, to the BVP

$$y'' = y' \cos x - y \ln y, 0 \le x \le \frac{\pi}{2}, y(0) = 1, y\left(\frac{\pi}{2}\right) = e,$$

and compare the results to those of the exact solution

$$y = e^{\sin x}$$

1.8 Problem 8

Apply the Linear Finite Difference Method with (a) $h = \frac{1}{2}$ and (b) $h = \frac{1}{4}$ to the BVP

$$y'' = 4(y - x), 0 \le x \le 1, y(0) = 0, y(1) = 2,$$

and compare the results to those of the exact solution

$$y(x) = x + \frac{e^2\left(e^{2x} - e^{-2x}\right)}{\left(e^4 - 1\right)}.$$

1.9 Problem 9

Apply the Linear Finite Difference Method with (a) $h = \frac{1}{10}$ and (b) $h = \frac{1}{20}$ to the BVP

$$y'' = 100y, 0 \leq x \leq 1, y(0) = 1, y(1) = e^{-10},$$

and compare the results to those of the exact solution

$$y(x) = e^{-10x}.$$

1.10 Problem 10

Apply the Nonlinear Finite Difference Method with $h = 0.5$, to the BVP

$$y'' = -\left(y'\right)^2 - y + \ln x, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x$$

1.11 Problem 11

Apply the Nonlinear Finite Difference Method with $N = 10, TOL = 10^{-4}$ to the BVP

$$y'' = -e^{-2y}, 1 \leq x \leq 2, y(1) = 0, y(2) = \ln 2,$$

and compare the results to those of the exact solution

$$y = \ln x$$

1.12 Problem 12

Apply the Nonlinear Finite Difference Method with $N = 10$, to the BVP

$$y'' = y' \cos x - y \ln y, 0 \leq x \leq \frac{\pi}{2}, y(0) = 1, y\left(\frac{\pi}{2}\right) = e,$$

and compare the results to those of the exact solution

$$y = e^{\sin x}$$

2 Strategy of solving

The differential equation

$$y'' = f(x, y, y') \quad a \leq x \leq b, \quad y(a) = \alpha, y(b) = \beta$$

is linear when functions $p(x)$, $q(x)$, and $r(x)$ exist with

$$f(x, y, y') = p(x)y' + q(x)y + r(x)$$

1. Problems $1, 2, 3, 8, 9$ are linear and the rest are nonlinear.
2. To numerically approximate $y(x), a \leq t \leq b$, we choose N and define the **step size** $h = \frac{b-a}{N}$ and the points $x_0, x_1, \ldots, x_N$ (total points $N+1$)

$$x_i = a + ih, \quad i = 0, 1, \ldots, N$$

   Notice that $x_N = b$.
3. $y(a) = \alpha$ and $y(b) = \beta$ are given in the problem. The approximate values of $y(x_1), y(x_2), \ldots, y(x_{N-1})$ are denoted as $y_1, y_2, \ldots, y_{N-1}$.
4. $y_1, y_2, \ldots, y_{N-1}$ are iteratively calculated using the iterations rules described in Methods for different methods.
5. For other points in the interval we can use linear interpolation.

3 Methods

3.1 Linear Shooting Method

The linear shooting method is a numerical technique for solving linear boundary-value problems by transforming them into initial-value problems. Consider the boundary-value problem:

$$y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta.$$

This problem is replaced by two initial-value problems:

$$y_1'' = p(x)y_1' + q(x)y_1 + r(x), \quad y_1(a) = \alpha, \quad y_1'(a) = 0,$$

$$y_2'' = p(x)y_2' + q(x)y_2, \quad y_2(a) = 0, \quad y_2'(a) = 1.$$

Solutions $y_1(x)$ and $y_2(x)$ are obtained using numerical methods such as the fourth-order Runge-Kutta technique. The solution to the original boundary-value problem is then approximated by:

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x).$$

This approach adjusts the combination of y_1 and y_2 to satisfy the boundary condition at $x = b$, effectively "shooting" towards the target value β.

3.2 Nonlinear Shooting Method

The nonlinear shooting method solves boundary-value problems by converting them into initial-value problems. For the problem:

$$y'' = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y(b) = \beta,$$

we solve the initial-value problem:

$$y'' = f(x, y, y'), \quad a \leq x \leq b, \quad y(a) = \alpha, \quad y'(a) = t,$$

for different values of t. The goal is to find t such that $y(b, t) \approx \beta$.

To refine t, we use iterative methods like the Secant or Newton method. In the Secant method:

$$t_k = t_{k-1} - \frac{(y(b, t_{k-1}) - \beta)(t_{k-1} - t_{k-2})}{y(b, t_{k-1}) - y(b, t_{k-2})}, \quad k = 2, 3, \ldots$$

Alternatively, Newton's method uses:

$$t_k = t_{k-1} - \frac{y(b, t_{k-1}) - \beta}{\frac{d}{dt} y(b, t_{k-1})}.$$

This process continues until $y(b, t_k)$ is sufficiently close to β.

3.3 Linear Finite-Difference Method

For a linear boundary-value problem:

$$y'' = p(x)y' + q(x)y + r(x), \quad y(a) = \alpha, \quad y(b) = \beta,$$

the interval $[a, b]$ is divided into $N + 1$ equal subintervals, with mesh points $x_i = a + ih$, where $h = \frac{b-a}{N+1}$.

The second derivative is approximated by the centered difference:

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}.$$

The first derivative is:

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h}.$$

Substituting into the differential equation gives a system of equations:

$$\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2} + p(x_i)\frac{w_{i+1} - w_{i-1}}{2h} + q(x_i)w_i = -r(x_i).$$

This results in a tridiagonal matrix system $Aw = b$, which can be solved for w. The method has an error of order $O(h^2)$ and guarantees a unique solution under certain conditions.

3.4 Nonlinear Finite-Difference Method

For a nonlinear boundary-value problem:

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta,$$

we divide the interval $[a, b]$ into $N + 1$ equal subintervals, with mesh points $x_i = a + ih$, where $h = \frac{b-a}{N+1}$.

The second derivative and first derivative are approximated using the centered-difference formulas:

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}, \quad y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h}.$$

This results in the nonlinear system:

$$\frac{-w_{i+1} + 2w_i - w_{i-1}}{h^2} + f\left(x_i, w_i, \frac{w_{i+1} - w_{i-1}}{2h}\right) = 0.$$

The resulting nonlinear system is:

$$2w_1 - w_2 + h^2 f(x_1, w_1, \frac{w_2 - \alpha}{2h}) - \alpha = 0,$$

and similarly for the rest of the points.

To solve this system, we use Newton's Method for nonlinear equations. At each iteration, the Jacobian matrix $J(w_1, \ldots, w_N)$ is tridiagonal, and a linear system is solved:

$$J(w_1, \ldots, w_N) \cdot (v_1, \ldots, v_N) = -\text{residuals}.$$

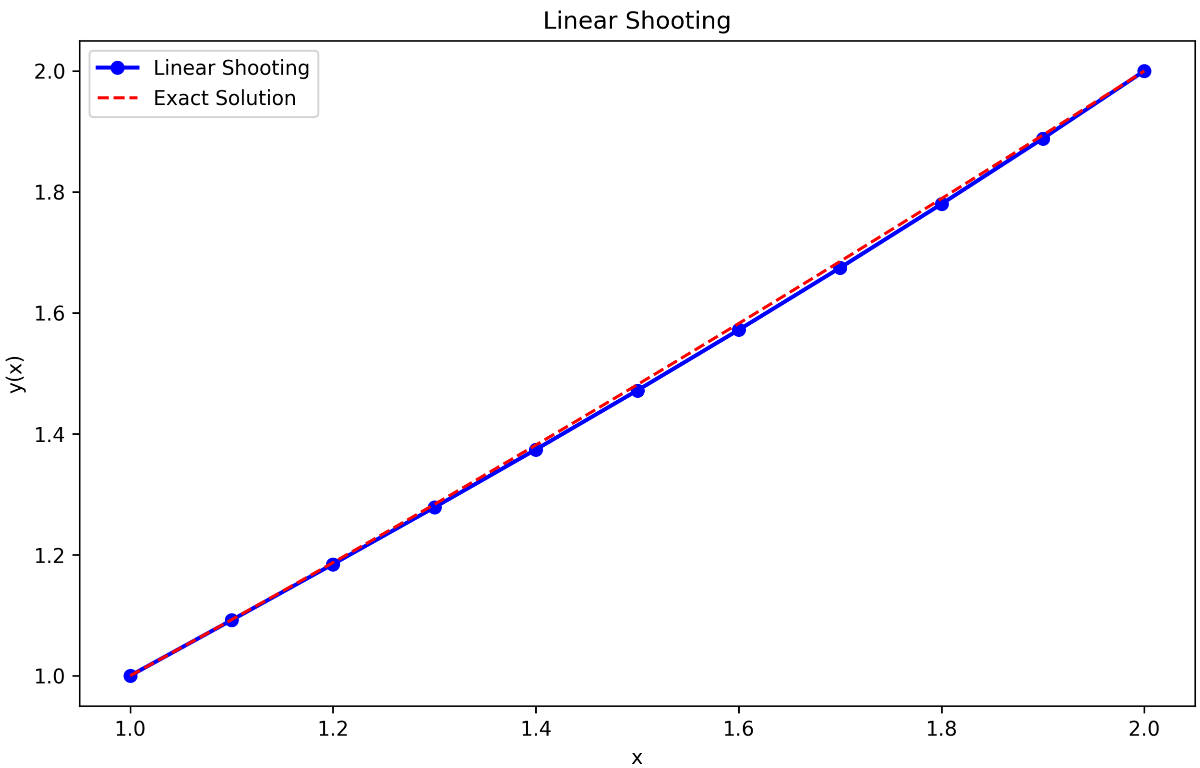
Newton's method iterates as:

$$w_i^{(k)} = w_i^{(k-1)} + v_i, \quad \text{for each } i = 1, 2, \ldots, N.$$

The tridiagonal structure of J allows efficient solution using algorithms like Crout Factorization. The method converges if the initial guess is close enough to the solution, with the solution guaranteed to exist under certain conditions.

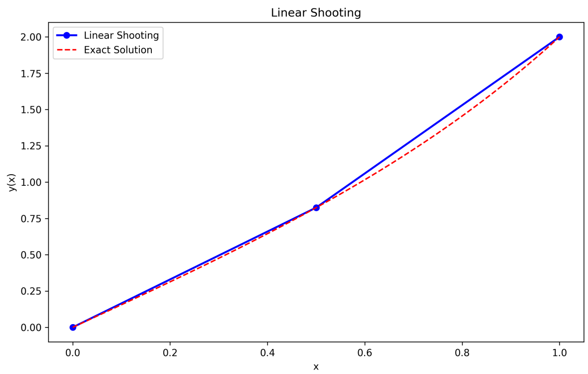
4 Output Plots

4.1 Problem 1

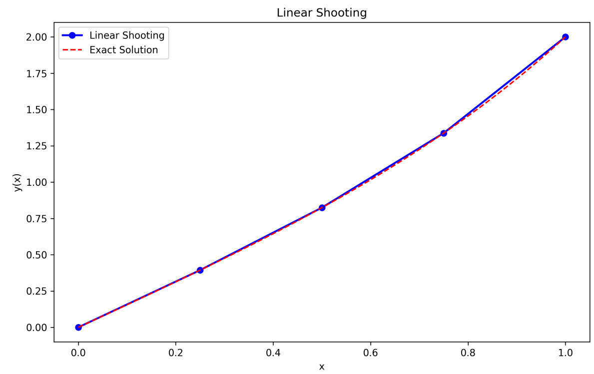


Linear Shooting Method $N = 10$

4.2 Problem 2



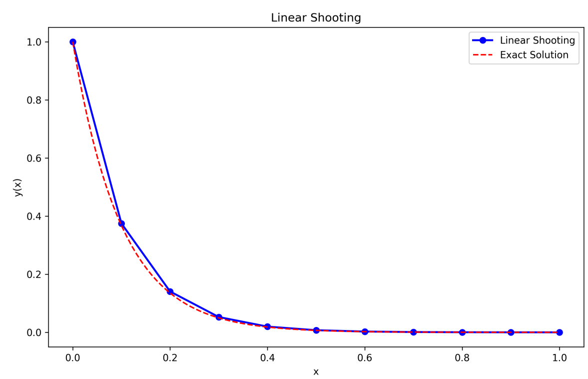
$h = \frac{1}{2}$



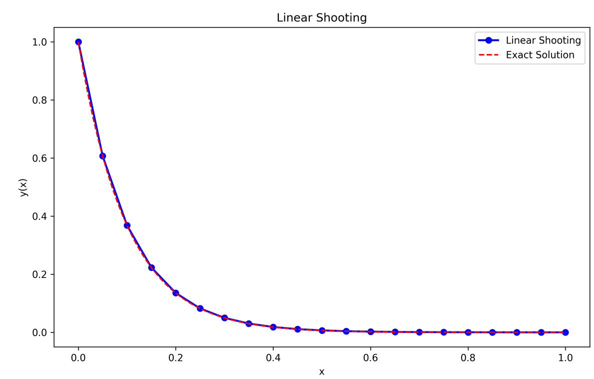
$h = \frac{1}{4}$

Linear Shooting Method

4.3 Problem 3



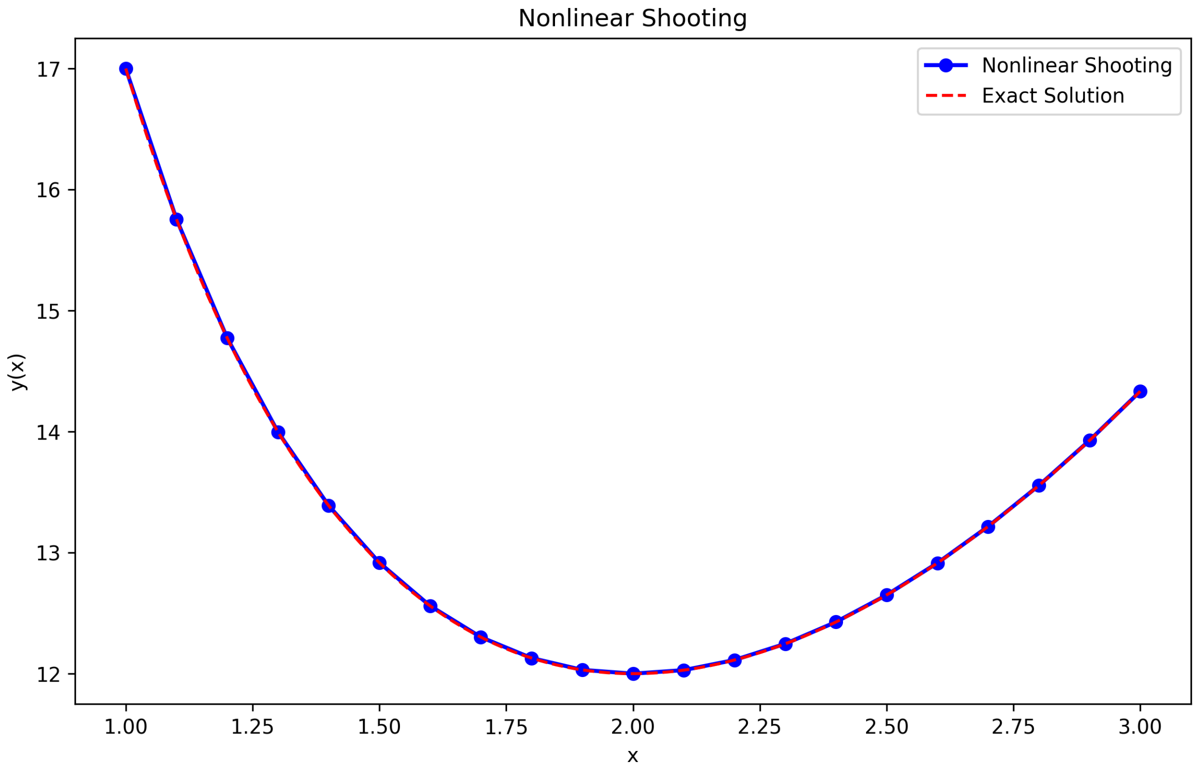
$h = \frac{1}{10}$



$h = \frac{1}{20}$

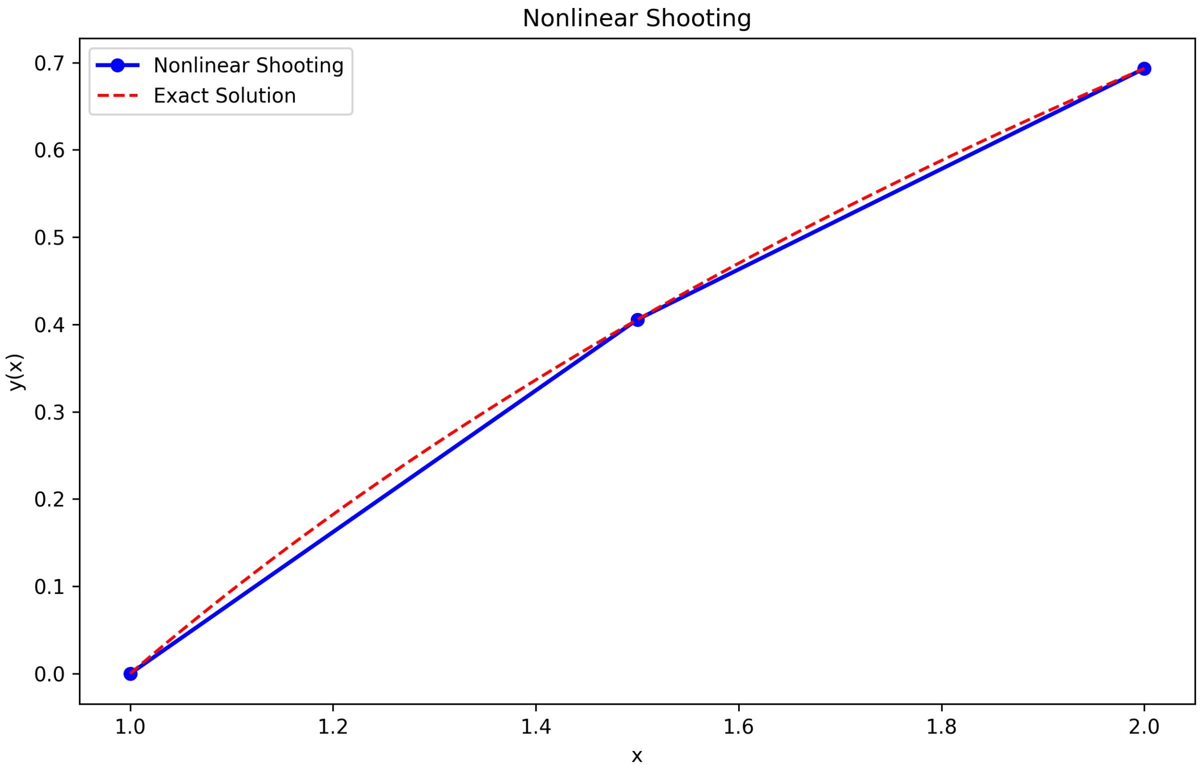
Linear Shooting Method

4.4 Problem 4



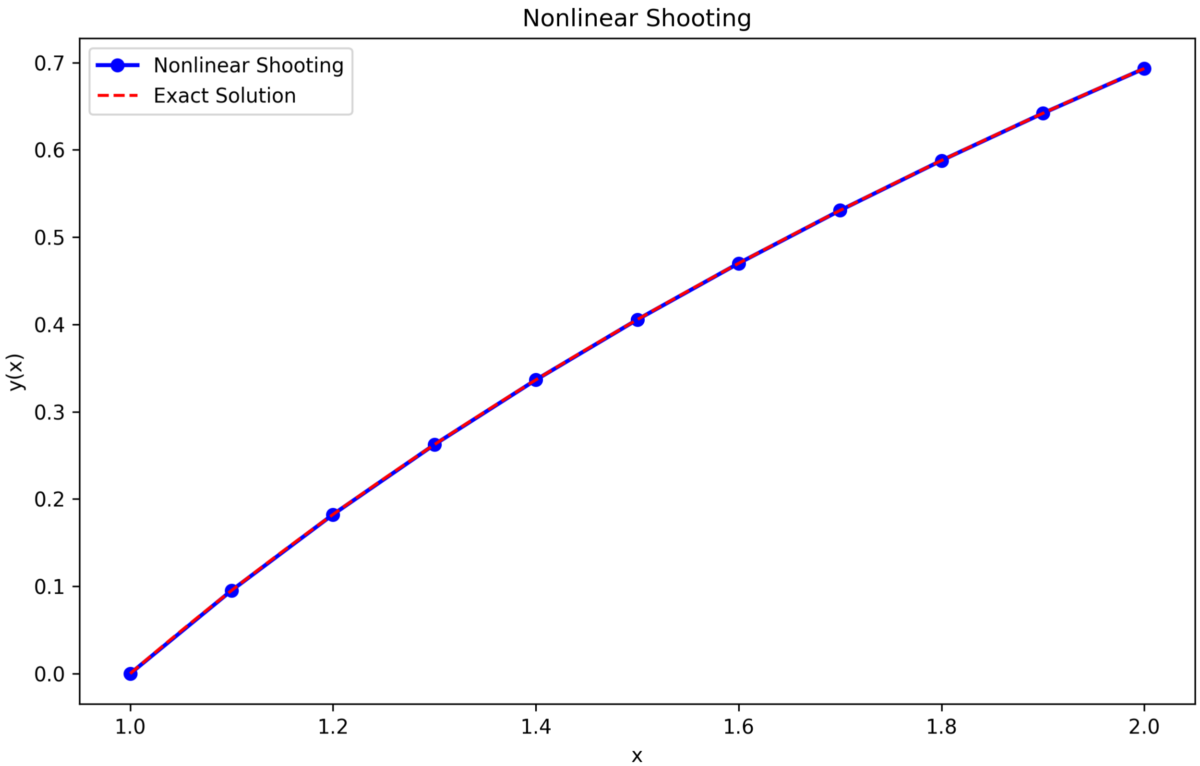
Nonlinear Shooting Method $N = 20$ $M = 10$ $TOL = 10^5$

4.5 Problem 5



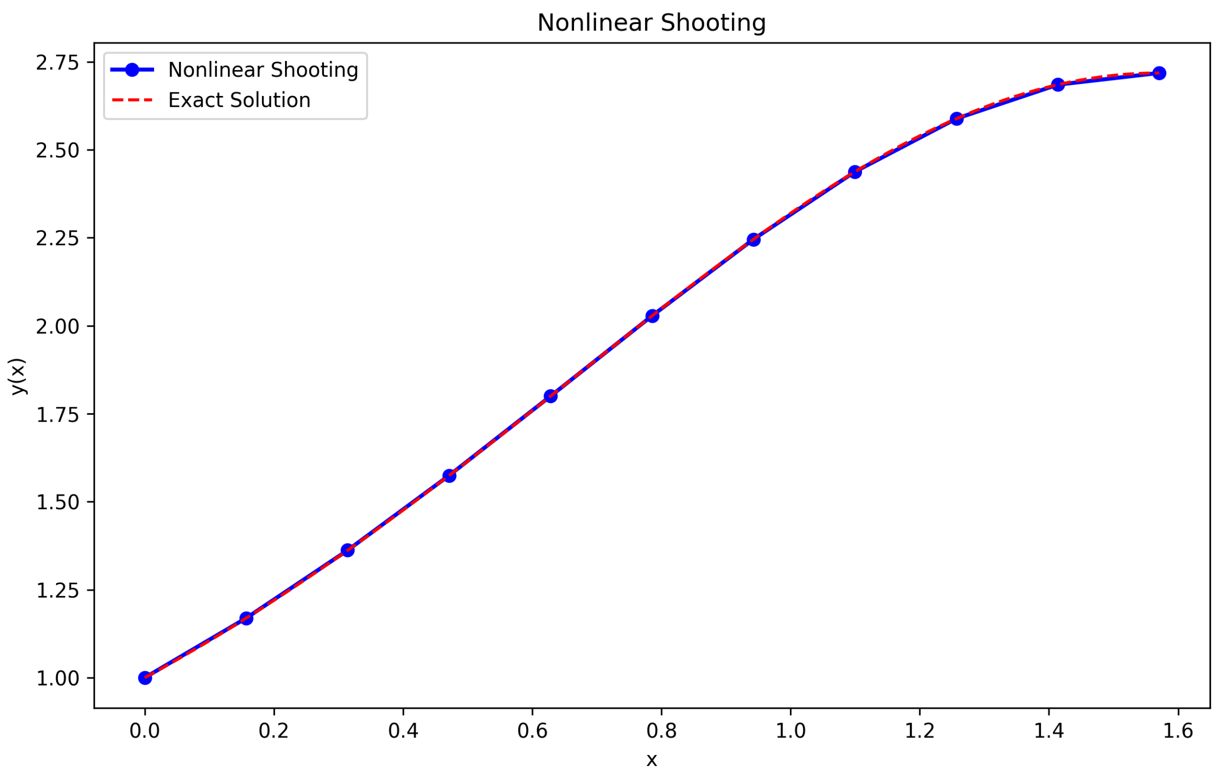
Nonlinear Shooting Method $h = 0.5$

4.6 Problem 6



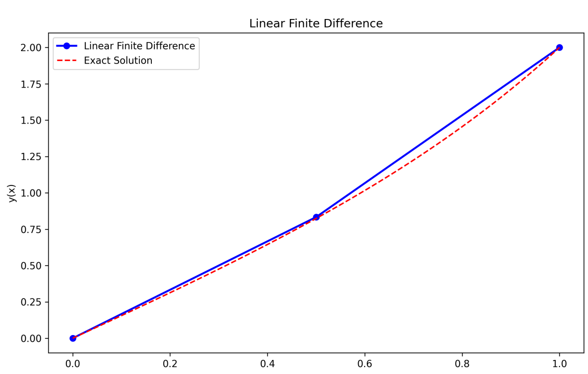
Nonlinear Shooting Method $N = 10$ $TOL = 10^4$

4.7 Problem 7

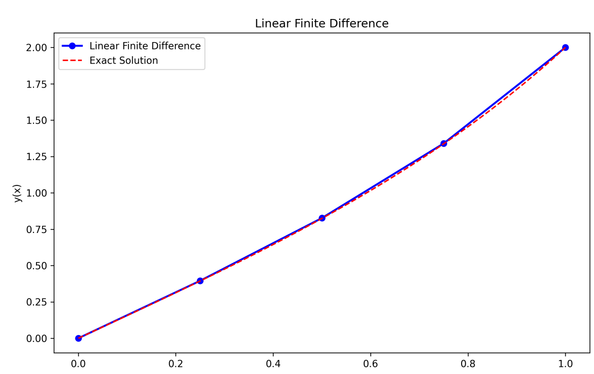


Nonlinear Shooting Method $N = 10$

4.8 Problem 8



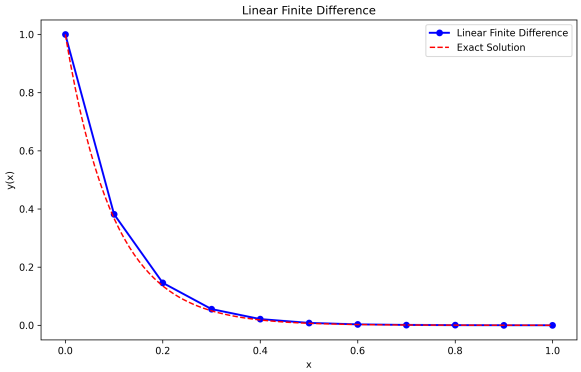
$h = \frac{1}{2}$



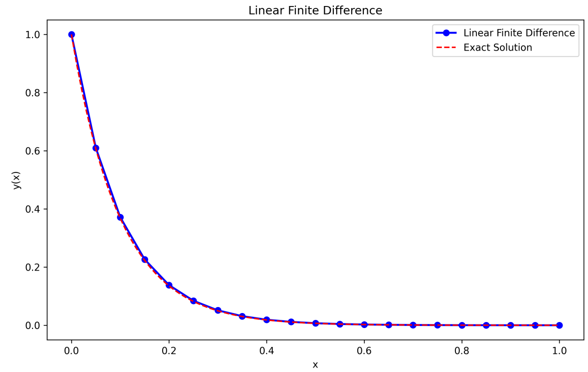
$h = \frac{1}{4}$

Linear Finite Difference Method

4.9 Problem 9



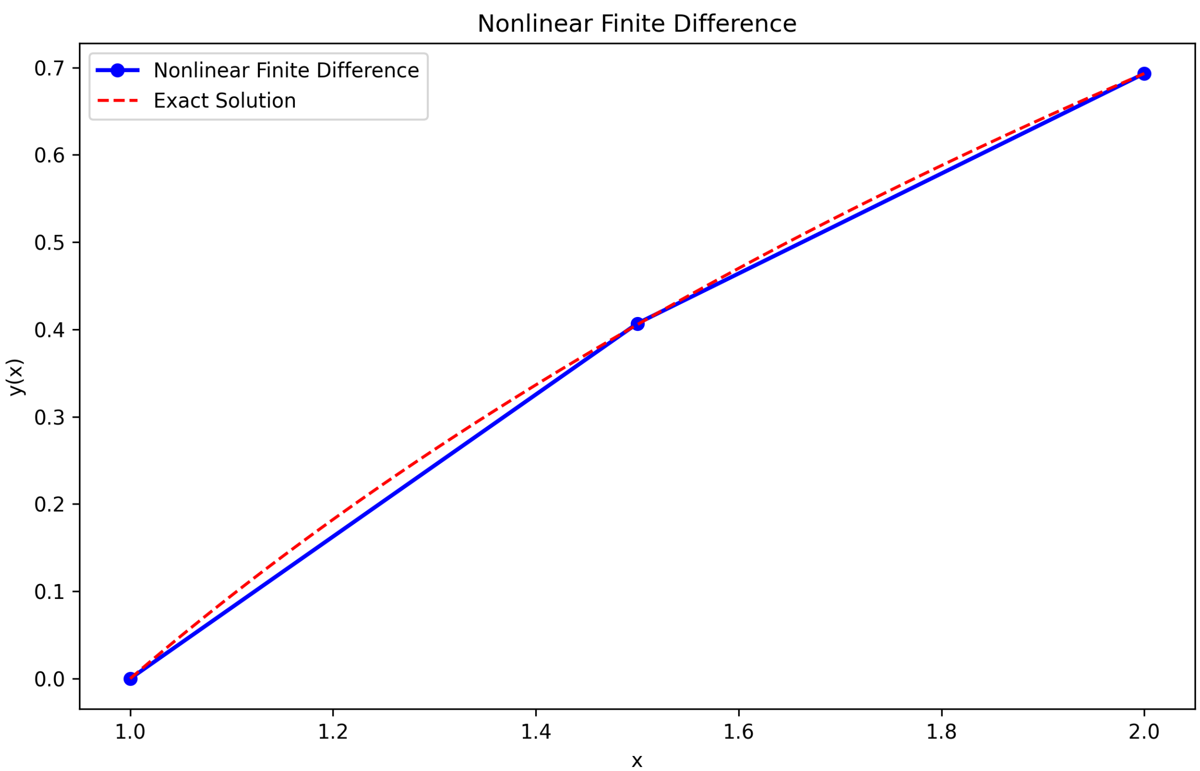
$h = \frac{1}{10}$



$h = \frac{1}{20}$

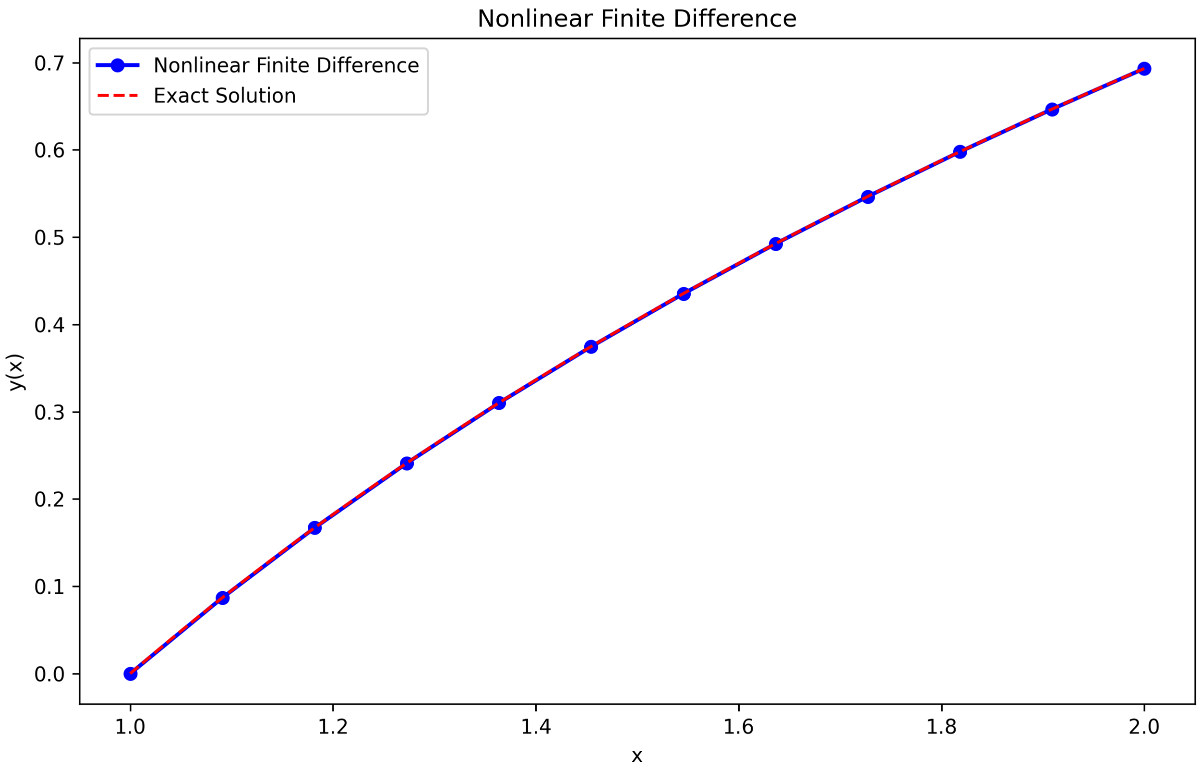
Linear Finite Difference Method

4.10 Problem 10



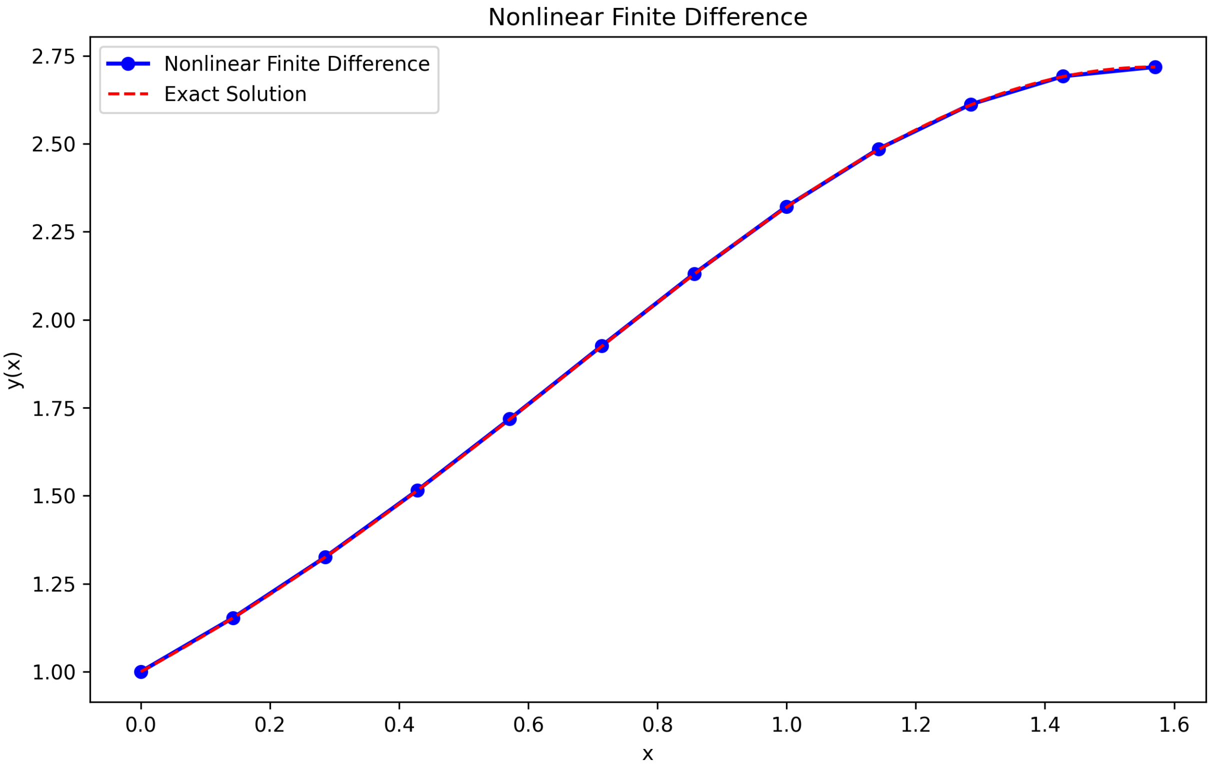
Nonlinear Finite Difference Method $h = 0.5$

4.11 Problem 11



Nonlinear Finite Difference Method $N = 10$ $TOL = 10^{-4}$

4.12 Problem 12



Nonlinear Finite Difference Method $N = 10$

5 Interpretation

1. **Comparison of Methods:** The different numerical methods (linear shooting, non-linear shooting, linear finite difference and non-linear finite difference) demonstrate distinct strengths. *Linear methods* are generally easier to implement and computationally efficient, but have limitations in handling highly nonlinear behavior. *Nonlinear methods*, while more complex, offer robustness and stability, especially in boundary value problems (BVP) with nonlinear terms.

2. **Accuracy and Convergence:** The *Linear Shooting Method* tends to perform well for simple linear BVPs, achieving accurate results with relatively low computational cost. However, for non-linear BVPs, the *nonlinear shooting and non-linear frenzied difference methods* provide better accuracy, since they iteratively adjust the parameters to approximate the solution closely.

3. **Error Analysis:** In both the shooting and finite-difference methods, the choice of step size h significantly affects the error. Smaller values of h tend to reduce the approximation error but increase the computational effort. In Problem 2, 5, 8 and 10 when $h = \frac{1}{2}$ the numerical solution r points is not that accurate. This is fixed when N is increased.