MTL712: Computational Methods for Differential Equations
Assignment 3

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Contents

1 Description

1.1 Problem 1

Use Forward Difference method to approximate $u(x, 0.5)$ where

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < 1, \quad 0 \leq t$$

with the boundary conditions

$$u(0,t) = 0 = u(1,t) \quad 0 \leq t$$

and the initial condition

$$u(x,0) = \sin(\pi x) \quad 0 \leq x \leq 1$$

Exact solution

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

1.2 Problem 2

Use Backward Difference method to approximate $u(x, 0.5)$ where

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < 1, \quad 0 \leq t$$

with the boundary conditions

$$u(0,t) = 0 = u(1,t) \quad 0 \leq t$$

and the initial condition

$$u(x,0) = \sin(\pi x) \quad 0 \leq x \leq 1$$

Exact solution

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

1.3 Problem 3

Use Backward Difference method to approximate $u(x, 0.5)$ where

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < 2, \quad 0 \leq t$$

with the boundary conditions

$$u(0,t) = 0 = u(2,t) \quad 0 \leq t$$

and the initial condition

$$u(x,0) = \sin(\frac{\pi}{2} x) \quad 0 \leq x \leq 2$$

Exact solution

$$u(x,t) = e^{-\frac{\pi^2}{4} t} \sin(\frac{\pi}{2} x)$$

1.4 Problem 4

Use Forward-Difference, Backward-Difference and Crank-Nicolson method to approximate $u(x, 0.5)$ where

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < 2, \quad 0 \leq t$$

with the boundary conditions

$$u(0,t) = 0 = u(2,t) \quad 0 \leq t$$

and the initial condition

$$u(x,0) = \sin(2\pi x) \quad 0 \leq x \leq 2$$

Exact solution

$$u(x,t) = e^{-4\pi^2 t} \sin(2\pi x)$$

1.5 Problem 5

Use Forward-Difference, Backward-Difference and Crank-Nicolson method to approximate $u(x, 0.5)$ where

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \quad 0 < x < \pi, \quad 0 \leq t$$

with the boundary conditions

$$u(0,t) = 0 = u(\pi,t) \quad 0 \leq t$$

and the initial condition

$$u(x,0) = \sin(x) \quad 0 \leq x \leq \pi$$

Exact solution

$$u(x,t) = e^{-t} \sin(x)$$

2 Strategy of solving

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < l, \quad t > 0$$

subject to the conditions

$$u(0,t) = u(l,t) = 0, \quad t > 0 \quad \text{and} \quad u(x,0) = f(x), \quad 0 \leq x \leq l.$$

1. Select an integer $m > 0$ and define the x-axis step size h and time-step size k. The grid points for this situation are (x_i, t_j).

$$\begin{aligned} h &= \frac{l}{m}, \quad k = \frac{T}{N} \\ x_i &= ih \text{ for } i = 0, 1, \ldots, m \\ t_j &= jk \text{ for } j = 0, 1, \ldots, N \end{aligned}$$

2. Compute λ and $\mathbf{w}^{(0)}$ as defined below.

$$\begin{aligned} \lambda &= \alpha^2 \frac{k}{h^2} \\ \mathbf{w}^{(0)} &= (f(x_1), f(x_2), \ldots, f(x_{m-1}))^t \end{aligned}$$

3. $w_{i,j}$ is the numerical solution of $u(x_i, t_j)$. For each j, $w_{0,j} = w_{m,j} = 0$ from the boundary condition. Define $\mathbf{w}^{(j)}$ as shown below. For each $j \geq 1$, $\mathbf{w}^{(j)}$ is computed using the iterative rules described in section Methods.

$$\mathbf{w}^{(j)} = (w_{1,j}, w_{2,j}, \ldots, w_{m-1,j})^t$$

4. In the expressions given in the next section, A and B are $(m-1) \times (m-1)$ matrices.

3 Methods

3.1 Forward Difference Method

$$A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ 0 & \lambda & 1-2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\lambda \end{bmatrix}$$

$$\mathbf{w}^{(j)} = A\mathbf{w}^{(j-1)}, \quad \text{for each } j = 1, 2, \ldots, N$$

3.2 Backward Difference Method

$$A = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \cdots & 0 \\ 0 & -\lambda & 1+2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\lambda \end{bmatrix}$$

$$A\mathbf{w}^{(j)} = \mathbf{w}^{(j-1)}, \quad \text{for each } j = 1, 2, \ldots, N$$

3.3 Crank-Nicolson Method

$$A = \begin{bmatrix} 1+\lambda & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & 1+\lambda & -\frac{\lambda}{2} & \cdots & 0 \\ 0 & -\frac{\lambda}{2} & 1+\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\lambda \end{bmatrix} \quad B = \begin{bmatrix} 1-\lambda & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & 1-\lambda & \frac{\lambda}{2} & \cdots & 0 \\ 0 & \frac{\lambda}{2} & 1-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\lambda \end{bmatrix}$$

$$A\mathbf{w}^{(j)} = B\mathbf{w}^{(j-1)}, \quad \text{for each } j = 1, 2, \ldots, N$$

4 MATLAB Code

```
function u_final = FTCS(h, k, t_final, u_initial)
    lambda = k / (h * h);
    steps = floor(t_final / k);
    m = length(u_initial) - 1;
    u_curr = u_initial(2 : m)';
    main_diag = (1 - 2 * lambda) * ones(m - 1, 1);
    off_diag = lambda * ones(m - 2, 1);
    A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);
    for i = 1: steps
        u_curr = A * u_curr;
    end
    u_final = [0; u_curr; 0]';
end
```

Listing 1: Code for Forward Difference Method

```
function u_final = BTCS(h, k, t_final, u_initial)
    lambda = k / (h * h);
    steps = floor(t_final / k);
    m = length(u_initial) - 1;
    u_curr = u_initial(2 : m)';
    main_diag = (1 + 2 * lambda) * ones(m - 1, 1);
    off_diag = -lambda * ones(m - 2, 1);
    A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);
    for i = 1: steps
        u_curr = A \ u_curr;
    end
    u_final = [0; u_curr; 0]';
end
```

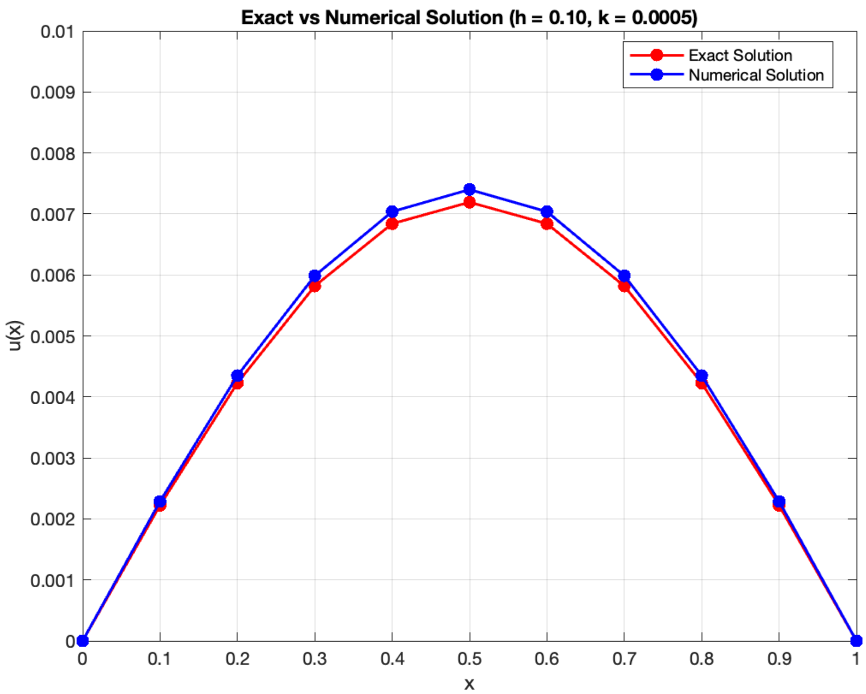
Listing 2: Code for Backward Difference Method

```
function u_final = Crank(h, k, t_final, u_initial)
    lambda = k / (h * h);
    steps = floor(t_final / k);
    m = length(u_initial) - 1;
    u_curr = u_initial(2 : m)';
    main_diag = (1 + lambda) * ones(m - 1, 1);
    off_diag = -lambda * ones(m - 2, 1) / 2;
    A = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);

    main_diag = (1 - lambda) * ones(m - 1, 1);
    off_diag = lambda * ones(m - 2, 1) / 2;
    B = diag(main_diag) + diag(off_diag, 1) + diag(off_diag, -1);

    for i = 1: steps
        u_curr = A \ (B * u_curr);
    end
    u_final = [0; u_curr; 0]';
end
```

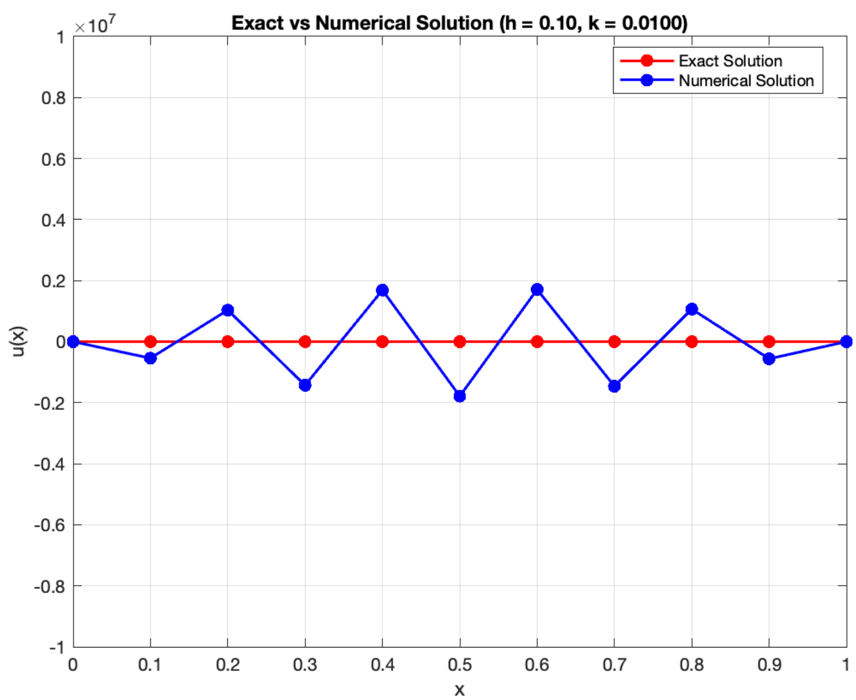
Listing 3: Code for Crank-Nicolson Method

5 Output Plots

5.1 Problem 1

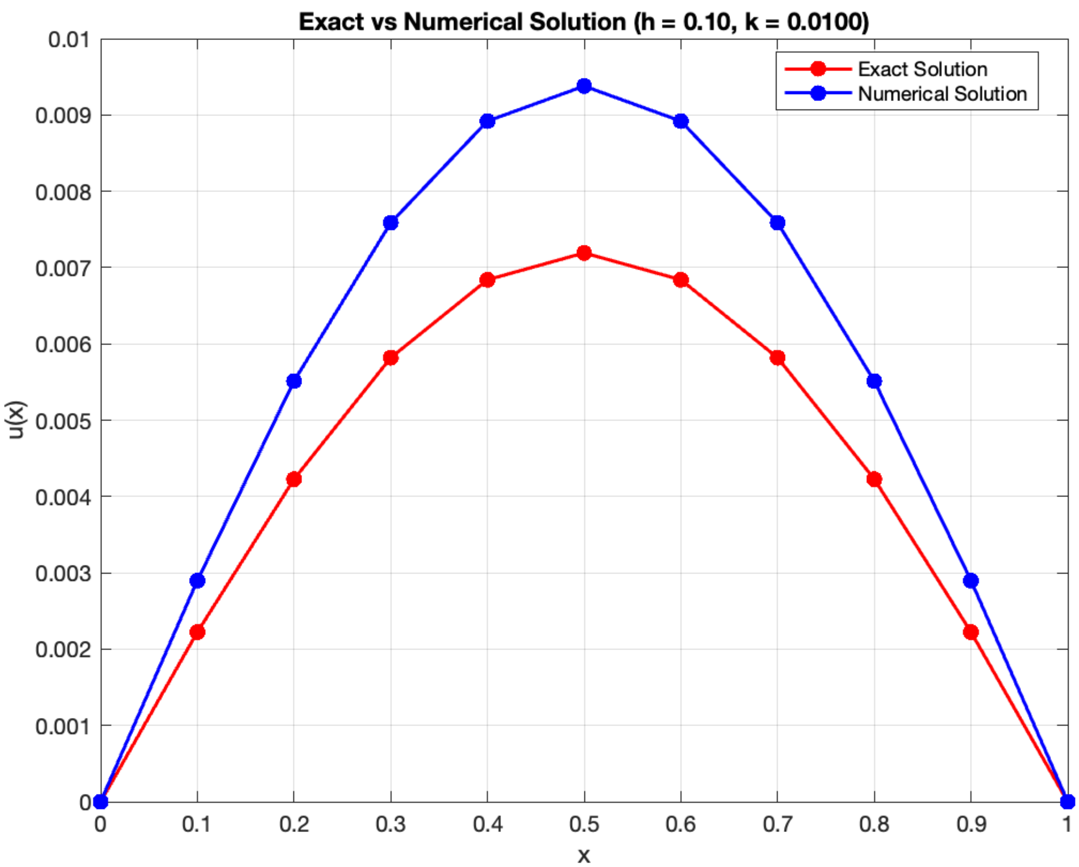


Problem 1: Forward Difference Method $h = 0.1 \quad k = 0.0005 \quad t = 0.5$



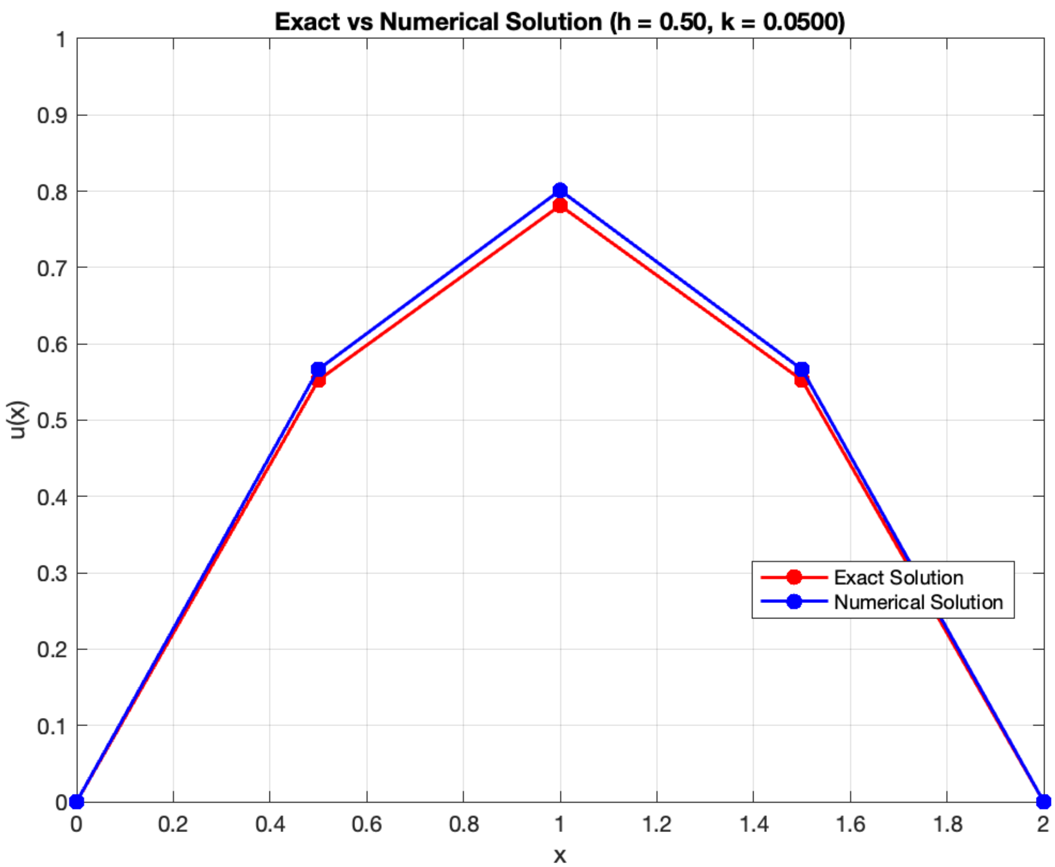
Problem 1: Forward Difference Method $h = 0.1 \quad k = 0.01 \quad t = 0.5$

5.2 Problem 2



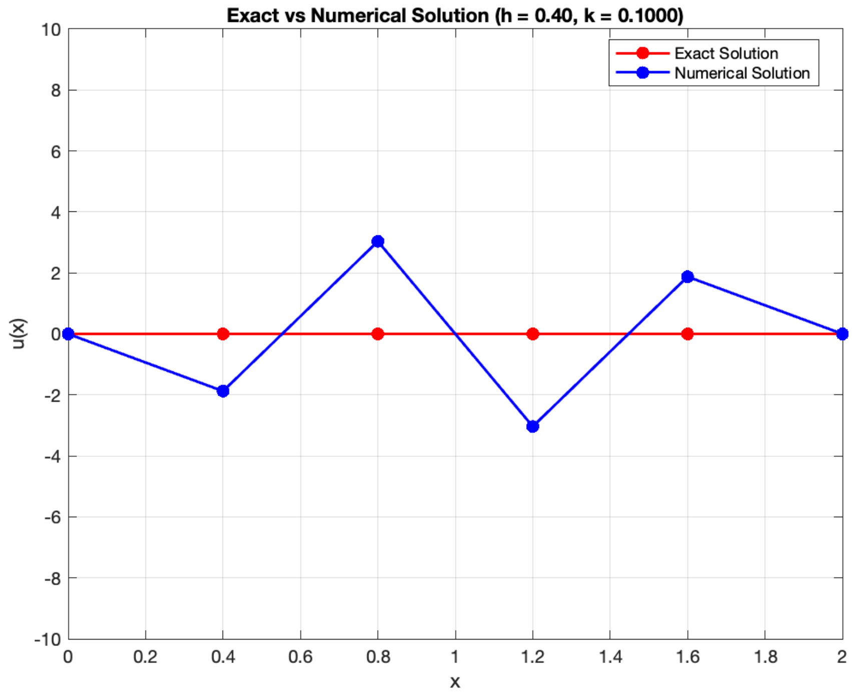
Problem 2: Backward Difference Method $h = 0.1$ $k = 0.01$ $t = 0.5$

5.3 Problem 3

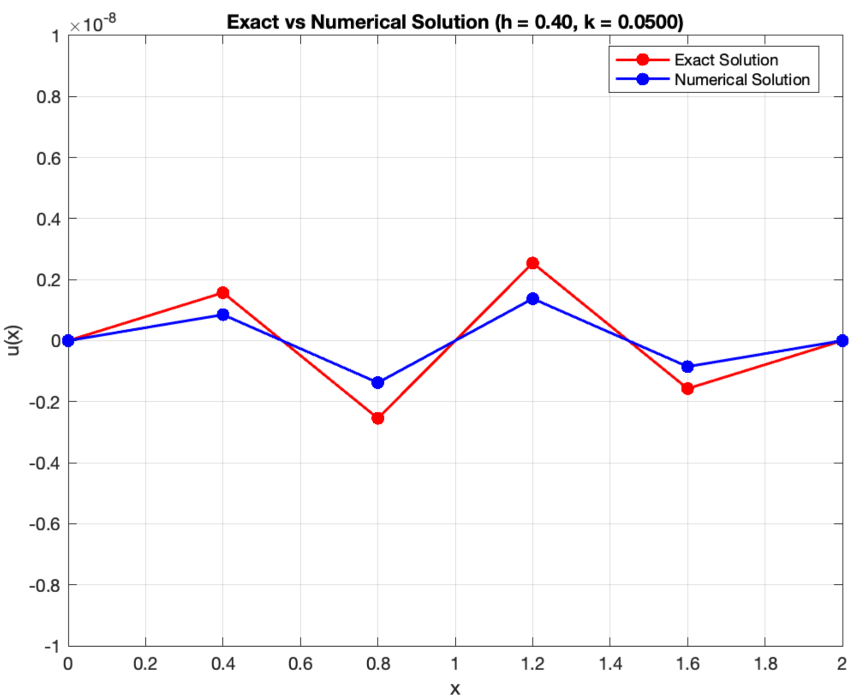


Problem 3: Backward Difference Method $m = 4$ $T = 0.1$ $N = 2$

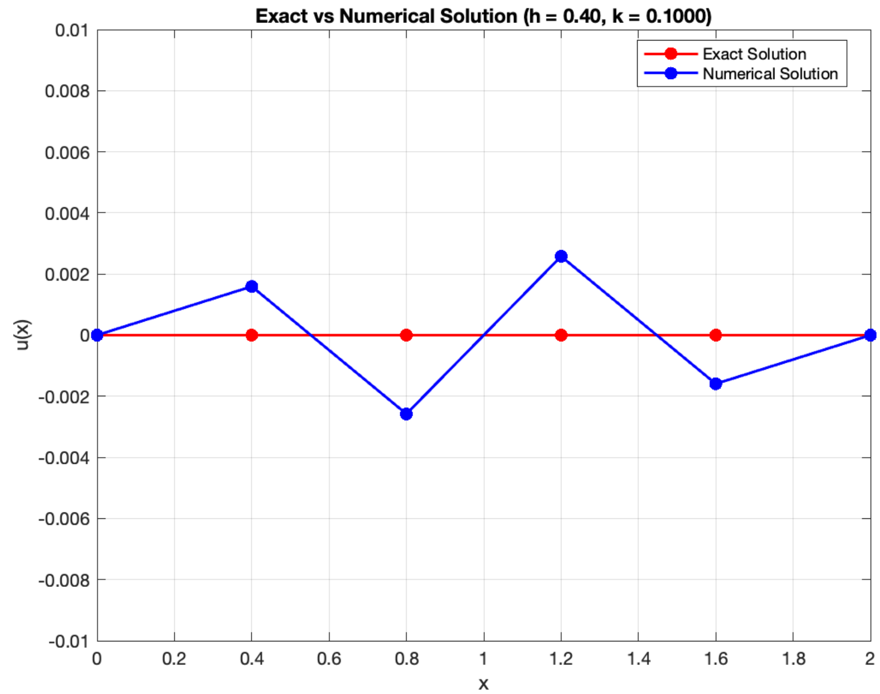
5.4 Problem 4



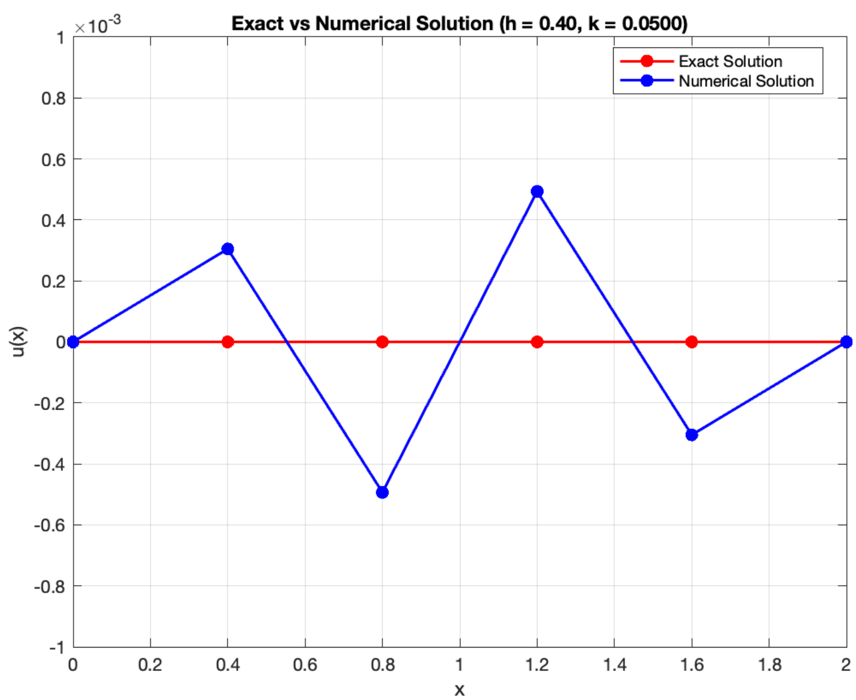
Problem 4: Forward Difference Method $h = 0.4 \;\; k = 0.1 \;\; t = 0.5$



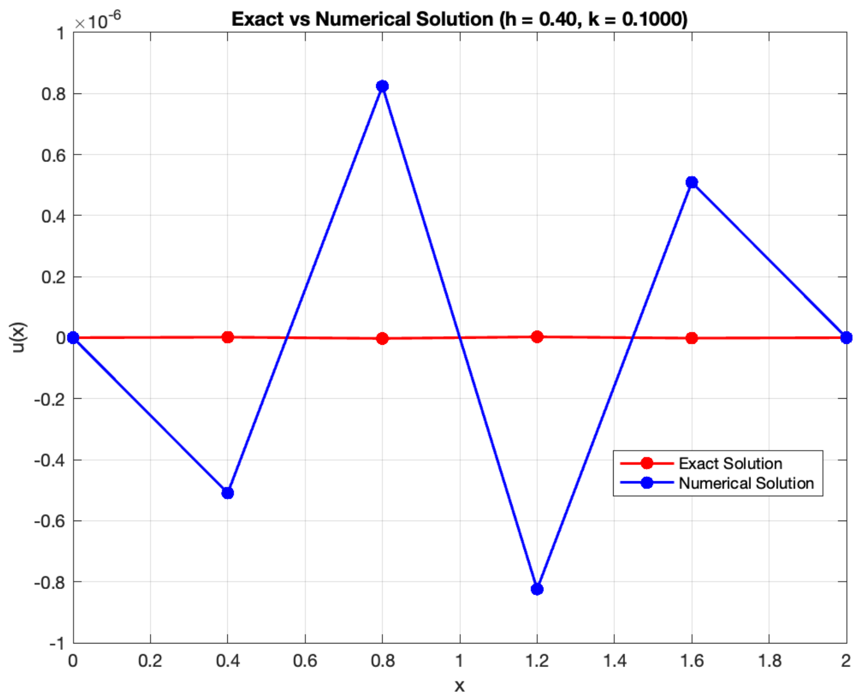
Problem 4: Forward Difference Method $h = 0.4 \;\; k = 0.05 \;\; t = 0.5$



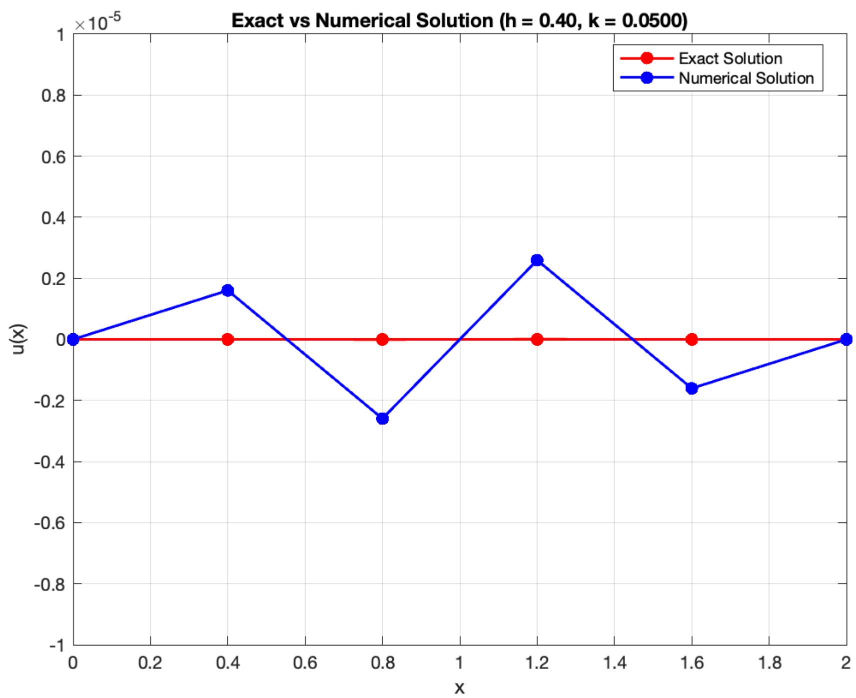
Problem 4: Backward Difference Method $h = 0.4 \quad k = 0.1 \quad t = 0.5$



Problem 4: Backward Difference Method $h = 0.4 \quad k = 0.05 \quad t = 0.5$

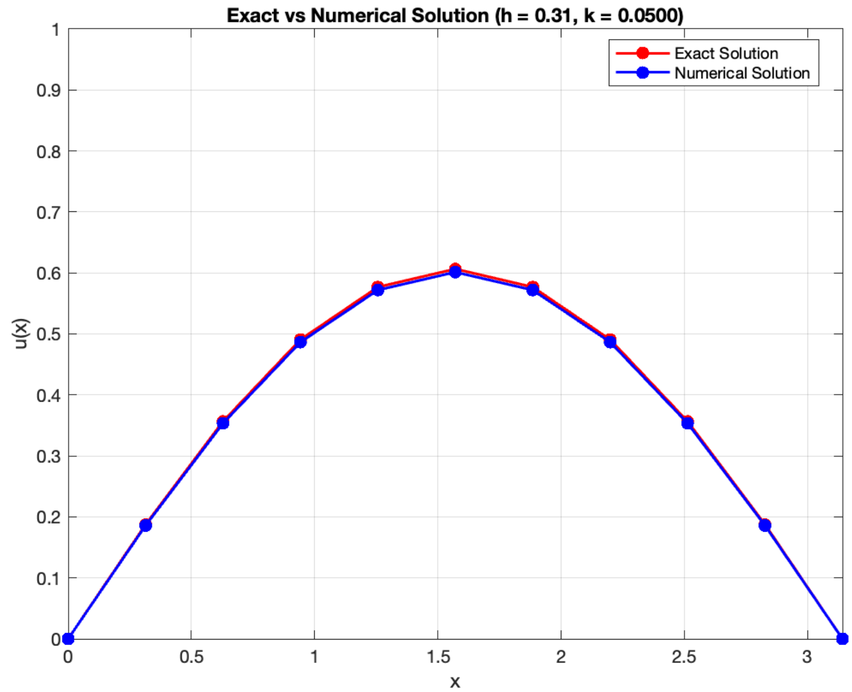


Problem 4: Crank-Nicolson Method $h = 0.4 \quad k = 0.1 \quad t = 0.5$

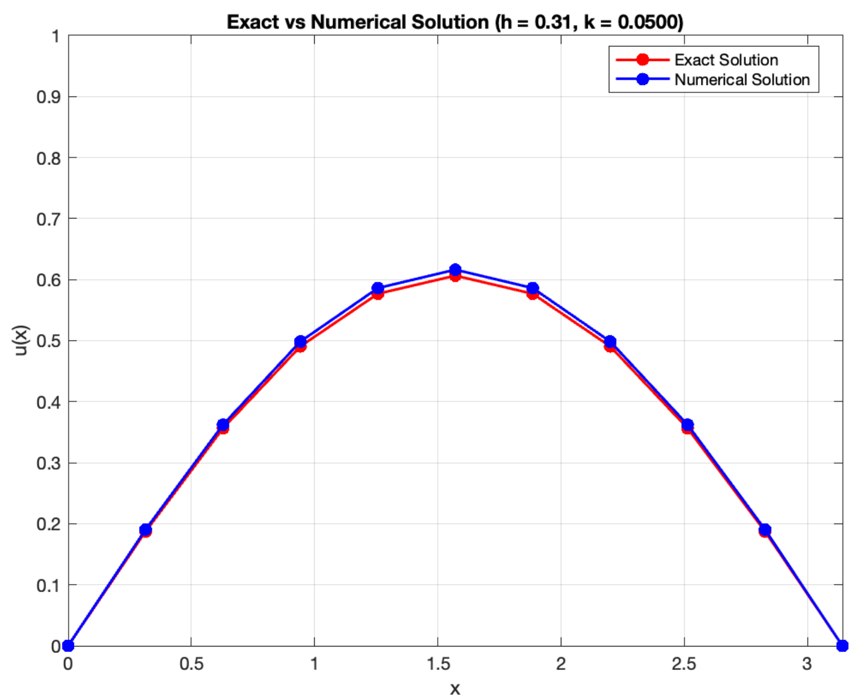


Problem 4: Crank-Nicolson Method $h = 0.4 \quad k = 0.05 \quad t = 0.5$

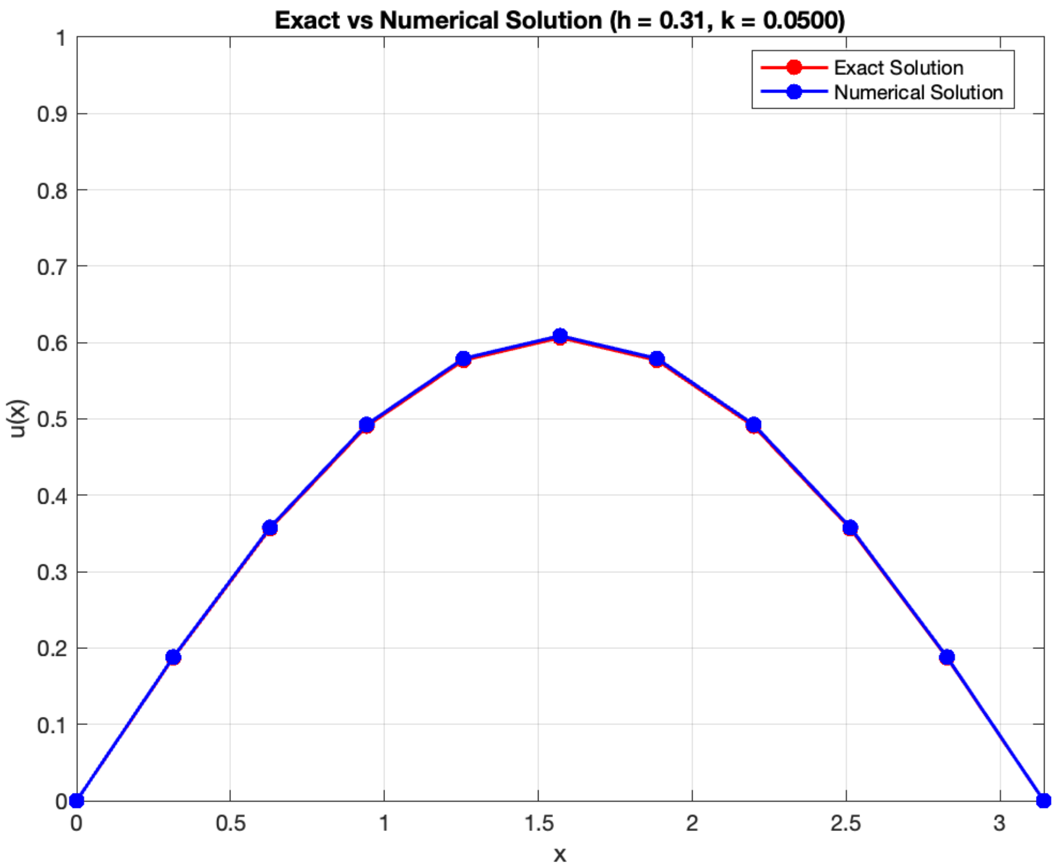
5.5 Problem 5



Problem 5: Forward Difference Method $h = \frac{\pi}{10}$ $k = 0.05$ $t = 0.5$

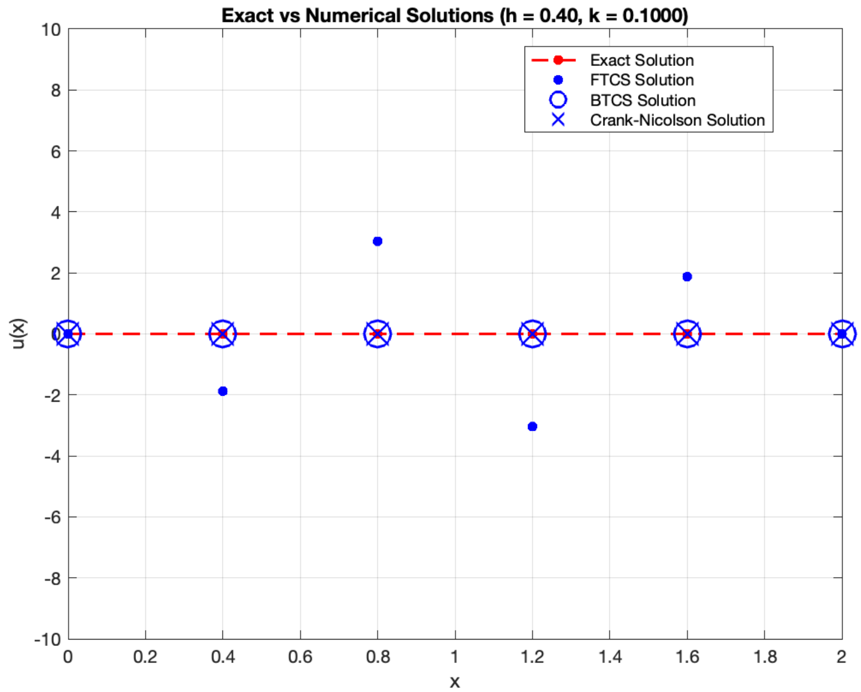


Problem 5: Backward Difference Method $h = \frac{\pi}{10}$ $k = 0.05$ $t = 0.5$

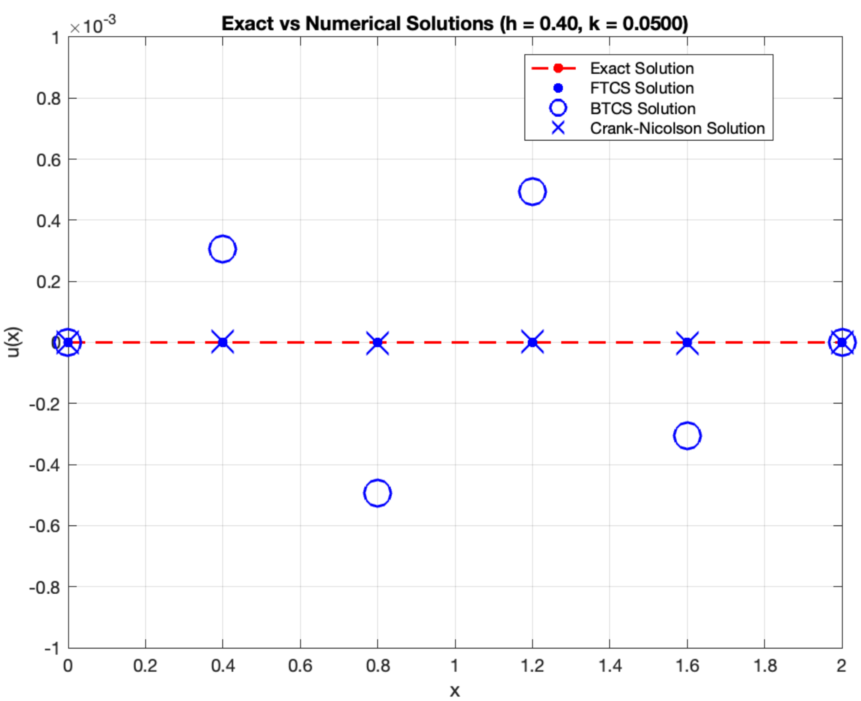


Problem 5: Crank-Nicolson Method $h = \frac{\pi}{10}$ $k = 0.05$ $t = 0.5$

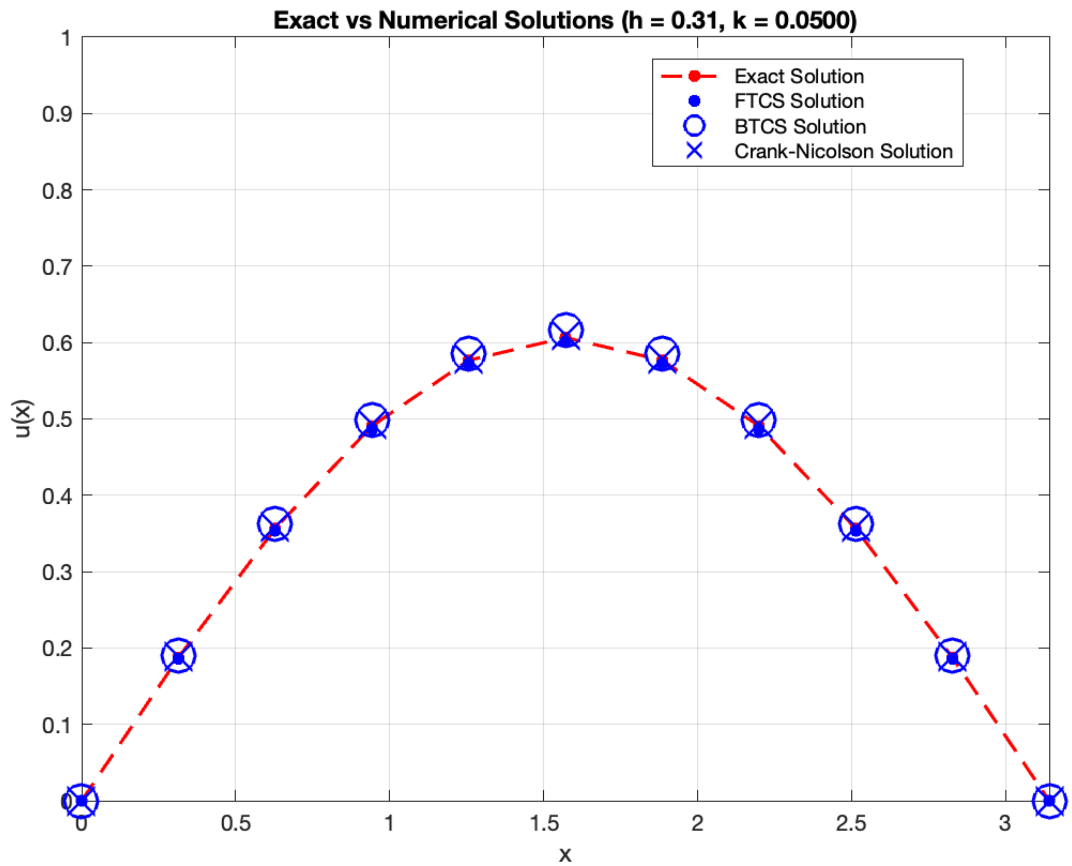
6 Comparison of Methods



Problem 4a: $h = 0.4 \quad k = 0.1 \quad t = 0.5$



Problem 4b: $h = 0.4 \quad k = 0.05 \quad t = 0.5$



Problem 5: $h = \frac{\pi}{10}$ $k = 0.05$ $t = 0.5$

7 Tables

x_m	**Exact Solution**	**FCTS h = 0.1, k = 0.0005**	**FCTS h = 0.1, k = 0.01**
0.0	0.00000000	0.00000000	0.00000000 $\times 10^6$
0.1	0.00222241	0.00228652	-0.54195315 $\times 10^6$
0.2	0.00422728	0.00434922	1.03365351 $\times 10^6$
0.3	0.00581836	0.00598619	-1.42869928 $\times 10^6$
0.4	0.00683989	0.00703719	1.68842147 $\times 10^6$
0.5	0.00719188	0.00739934	-1.78566825 $\times 10^6$
0.6	0.00683989	0.00703719	1.70812264 $\times 10^6$
0.7	0.00581836	0.00598619	-1.46057644 $\times 10^6$
0.8	0.00422728	0.00434922	1.06553067 $\times 10^6$
0.9	0.00222241	0.00228652	-0.56165432 $\times 10^6$
1.0	0.00000000	0.00000000	0.00000000 $\times 10^6$

Table 1: Problem 1 $t = 0.5$

x_m	**Exact Solution**	**BTCS**
0.0	0.00000000	0.00000000
0.1	0.00222241	0.00289802
0.2	0.00422728	0.00551236
0.3	0.00581836	0.00758711
0.4	0.00683989	0.00891918
0.5	0.00719188	0.00937818
0.6	0.00683989	0.00891918
0.7	0.00581836	0.00758711
0.8	0.00422728	0.00551236
0.9	0.00222241	0.00289802
1.0	0.00000000	0.00000000

Table 2: Problem 2 $h = 0.1 \quad k = 0.01 \quad t = 0.5$

8 Interpretation

1. The **Forward Difference** method is stable only if

$$\alpha^2 \frac{k}{h^2} \leq \frac{1}{2}$$

   Hence this method is conditionally stable.

2. For Problem 1a, this value is 0.05 (stable), and for Problem 1b this is 1 (unstable). This aligns with the plots shown in Output Plots.

x_m	**Exact Solution**	**BTCS**
0.0	0.00000000	0.00000000
0.5	0.55249345	0.56657363
1.0	0.78134373	0.80125611
1.5	0.55249345	0.56657363
2.0	0.00000000	0.00000000

Table 3: Problem 3 $m = 4 \quad N = 2 \quad T = 0.1$

x_m	**Exact Solution**	**FTCS**	**BTCS**	**Crank**
0.0	$0.00000000 \times 10^{-8}$	0.00000000	0.00000000	$0.00000000 \times 10^{-6}$
0.4	$0.15724948 \times 10^{-8}$	-1.87612234	0.00159325	$-0.50930841 \times 10^{-6}$
0.8	$-0.25443501 \times 10^{-8}$	3.03562971	-0.00257793	$0.82407832 \times 10^{-6}$
1.2	$0.25443501 \times 10^{-8}$	-3.03562971	0.00257793	$-0.82407832 \times 10^{-6}$
1.6	$-0.15724948 \times 10^{-8}$	1.87612234	-0.00159325	$0.50930841 \times 10^{-6}$
2.0	$0.00000000 \times 10^{-8}$	0.00000000	0.00000000	$0.00000000 \times 10^{-6}$

Table 4: Problem 4a $h = 0.4 \quad k = 0.1 \quad t = 0.5$

x_m	**Exact Solution**	**FTCS**	**BTCS**	**Crank**
0.0	$0.00000000 \times 10^{-8}$	$0.00000000 \times 10^{-8}$	$0.00000000 \times 10^{-3}$	$0.00000000 \times 10^{-5}$
0.4	$0.15724948 \times 10^{-8}$	$0.08508144 \times 10^{-8}$	$0.30487854 \times 10^{-3}$	$0.16029201 \times 10^{-5}$
0.8	$-0.25443501 \times 10^{-8}$	$-0.13766466 \times 10^{-8}$	$-0.49330383 \times 10^{-3}$	$-0.25935792 \times 10^{-5}$
1.2	$0.25443501 \times 10^{-8}$	$0.13766467 \times 10^{-8}$	$0.49330383 \times 10^{-3}$	$0.25935792 \times 10^{-5}$
1.6	$-0.15724948 \times 10^{-8}$	$-0.08508144 \times 10^{-8}$	$-0.30487854 \times 10^{-3}$	$-0.16029201 \times 10^{-5}$
2.0	$0.00000000 \times 10^{-8}$	$0.00000000 \times 10^{-8}$	$0.00000000 \times 10^{-3}$	$0.00000000 \times 10^{-5}$

Table 5: Problem 4b $h = 0.4 \quad k = 0.05 \quad t = 0.5$

x_m	**Exact Solution**	**FTCS**	**BTCS**	**Crank**
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.31415927	0.18742828	0.18581972	0.19045178	0.18817896
0.62831853	0.35650978	0.35345011	0.36226081	0.35793766
0.94247780	0.49069361	0.48648234	0.49860923	0.49265892
1.25663706	0.57684494	0.57189429	0.58615031	0.57915530
1.57079633	0.60653066	0.60132525	0.61631491	0.60895992
1.88495559	0.57684494	0.57189429	0.58615031	0.57915530
2.19911486	0.49069361	0.48648234	0.49860923	0.49265892
2.51327412	0.35650978	0.35345011	0.36226081	0.35793766
2.82743339	0.18742828	0.18581972	0.19045178	0.18817896
3.14159265	0.00000000	0.00000000	0.00000000	0.00000000

Table 6: Problem 5 $h = \frac{\pi}{10} \quad k = 0.05 \quad t = 0.5$

3. The **Backward Difference** method is an unconditionally stable method (i.e., stable independent of choice of k, h). We can see that it gives good solutions in all cases, but Problem 4.

4. Both **Forward Difference** and **Backward Difference** methods have truncation error of order $O(k + h^2)$.

5. The **Crank-Nicolson** method is unconditionally stable and has a truncation error of the order $O(k^2 + h^2)$. This is smaller than the previous two methods. We can see that it gives the best approximation in Problem 5, where all three methods converge.

6. In Problem 2 the Backward Difference method doesn't explode but the amplitude is significantly larger than in the exact solution.

7. In Problem 3 we can see that the approximation of the Backward Difference method is very close to the exact solution, despite the small values of m and N.

8. In Problem 4, almost all methods explode, except for the forward difference method with $h = 0.4 \quad k = 0.05$.

9. In Problem 5, Forward Difference method has slightly greater amplitude than the exact solution, the Backward Difference method has slightly greater amplitude and the Crank-Nicolson method is almost exact.