MTL712: Computational Methods for Differential Equations
Assignment 4

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Contents

1 Description

1.1 Problem 1

Solve the following initial-value problems using Euler's method:

1. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with step size $h = 0.5$.
2. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with step size $h = 0.5$.
3. $y' = 1 + \frac{y}{t}, \quad 1 \le t \le 2, \quad y(1) = 2,$ with step size $h = 0.25$.
4. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with step size $h = 0.25$.

The exact solutions to the above IVP's are, respectively,

1. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$
2. $y(t) = t + \frac{1}{1-t}$
3. $y(t) = t \ln t + 2t$
4. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

1.2 Problem 3

Solve the following initial-value problems using Euler's method:

1. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1,$ with step size $h = 0.1$.
2. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0,$ with step size $h = 0.2$.
3. $y' = -(y + 1)(y + 3), \quad 0 \le t \le 2, \quad y(0) = -2,$ with step size $h = 0.2$.
4. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3},$ with step size $h = 0.1$.

The exact solutions to the above IVP's are, respectively,

1. $y(t) = \frac{t}{1+\ln t}$
2. $y(t) = t\tan(\ln t)$
3. $y(t) = -3 + \frac{2}{1+e^{-2t}}$
4. $y(t) = t^2 + \frac{1}{3}e^{-5t}$

1.3 Problem 5

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0$$

is $y(t) = t^2(e^t - e)$.

1. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
2. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values:
   (a) $y(1.04)$
   (b) $y(1.55)$
   (c) $y(1.97)$
3. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using

   $$|y(t_i) - w_i| \le \frac{hM}{2L}[e^{L(t_i-1)} - 1]$$

   where L is the Lipschitz constant and M is a constant satisfying $|y''(t)| \le M$ for all $t \in [1, 2]$.

1.4 Problem 9

Solve the following initial-value problems using Taylor's method of order two:

1. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with step size $h = 0.5$.
2. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with step size $h = 0.5$.
3. $y' = 1 + \frac{y}{t}, \quad 1 \le t \le 2, \quad y(1) = 2,$ with step size $h = 0.25$.
4. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with step size $h = 0.25$.

The exact solutions to the above IVP's are given in Problem 1 in the same order.

1.5 Problem 11

Solve the following initial-value problems using Taylor's method of order two:

1. $y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1,$ with step size $h = 0.1$.
2. $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0,$ with step size $h = 0.5$.
3. $y' = (y^2 + y)/t, \quad 1 \le t \le 3, \quad y(1) = -2,$ with step size $h = 0.5$.
4. $y' = -ty + 4ty^{-1}, \quad 0 \le t \le 1, \quad y(0) = 1,$ with step size $h = 0.25$.

1.6 Problem 13

The exact solution for the IVP

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0$$

is $y(t) = t^2(e^t - e)$.

1. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
2. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values of y:
   (a) $y(1.04)$
   (b) $y(1.55)$
   (c) $y(1.97)$
3. Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
4. Use the answers generated in part (c) and linear interpolation to approximate the following values of y, and compare them to the actual values of y:
   (a) $y(1.04)$
   (b) $y(1.55)$
   (c) $y(1.97)$

1.7 Problem 15

Solve the following initial-value problems using the Modified Euler's method, the Midpoint method, Heun's method, and Runge-Kutta method of order four:

1. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1,$ with step size $h = 0.1,$ actual solution $y(t) = \frac{t}{1+\ln t}$
2. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0,$ with step size $h = 0.2,$ actual solution $y(t) = t\tan(\ln t)$
3. $y' = -(y+1)(y+3), \quad 0 \le t \le 2, \quad y(0) = -2,$ with step size $h = 0.2,$ actual solution $y(t) = -3 + \frac{2}{1+e^{-2t}}$
4. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3},$ with step size $h = 0.1,$ actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$

2 Strategy of solving

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

- To numerically approximate $y(t), t_0 \leq t \leq T$, we choose N and define the **step size** $h = \frac{T-t_0}{N}$ and the points, $t_0, t_1, \ldots, t_N$ (total $N+1$ points)

  $$t_i = t_0 + ih \quad i = 0, 1, \ldots, N$$

  Notice that $t_N = T$.

- $y(t_0) = y_0$ is given in the problem. The approximate values of $y(t_1), y(t_2), \ldots, y(t_N)$ are denoted as $y_1, y_2, \ldots, y_N$.

- $y_1, y_2, \ldots, y_N$ are iteratively calculated using the iterations rules described in Methods for different methods.

- For other points in the interval we can use linear interpolation, as we'll see in Problems 5 and 13.

3 Methods

3.1 Euler's method

The update rule is shown below. For $n = 0, 1, \ldots, N-1$

$$y_{n+1} = y_n + hf(t_n, y_n)$$

3.2 Modified Euler's method

The update rule is shown below. For $n = 0, 1, \ldots, N-1$

$$y^*_{n+1} = y_n + hf(t_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})]$$

3.3 Taylor's method of order k

The update rule is shown below. For $n = 0, 1, \ldots, N-1$

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2!}f'(t_n, y_n) + \cdots + \frac{h^k}{k!}f^{(k-1)}(t_n, y_n)$$

3.4 Midpoint method

The update rule is shown below. For $n = 0, 1, \ldots, N-1$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2}f(t_n, y_n)$$
$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_{n+\frac{1}{2}})$$

3.5 Heun's method

This is same as the Modified Euler's method. The update rule is shown below. For $n = 0, 1, \ldots, N-1$

$$
\begin{aligned}
y^*_{n+1} &= y_n + hf(t_n, y_n) \\
y_{n+1} &= y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})]
\end{aligned}
$$

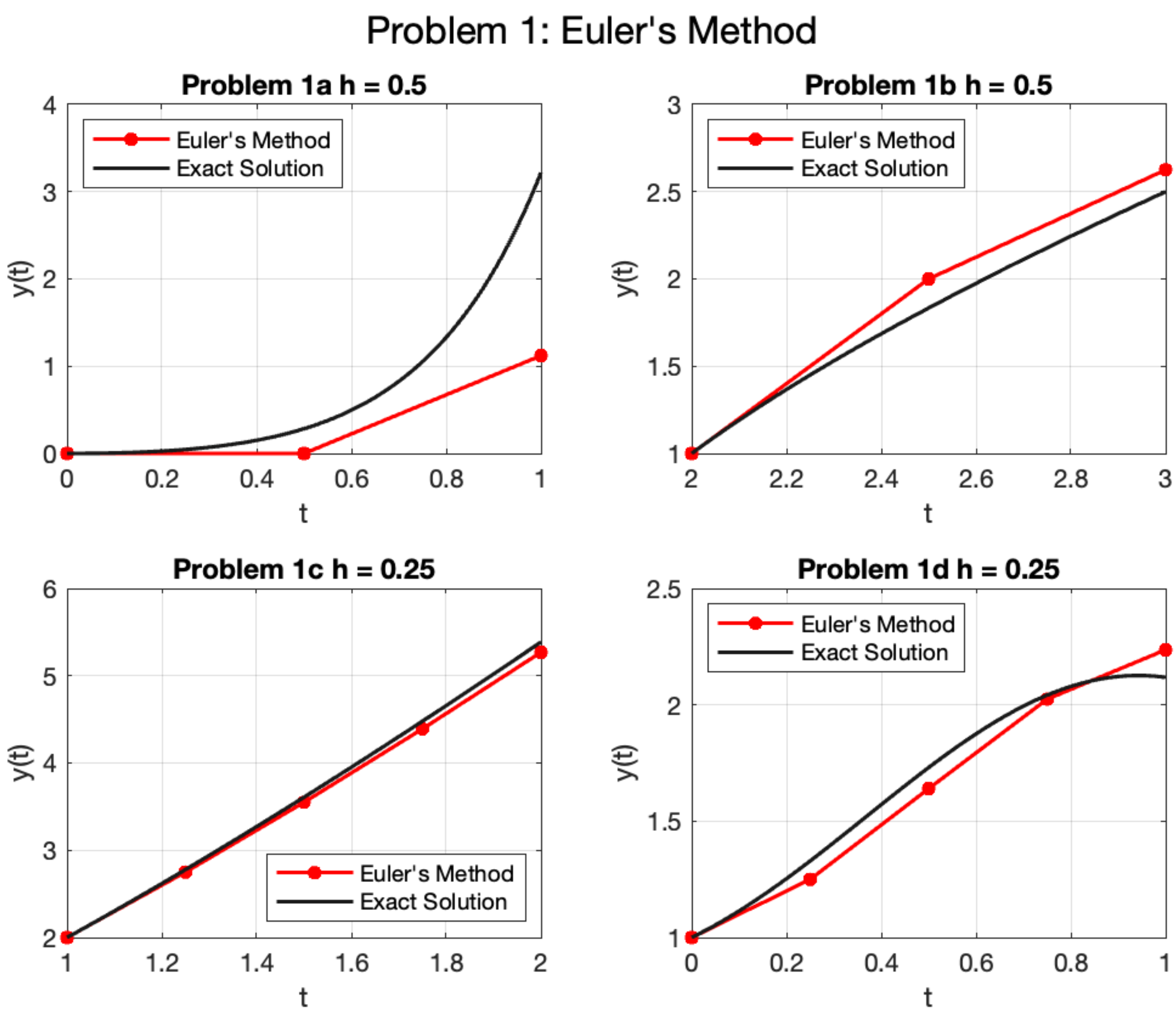
3.6 Runge-Kutta method of order four

The update rule is shown below. For $n = 0, 1, \ldots, N-1$:

$$
\begin{aligned}
k_1 &= hf(t_n, y_n) \\
k_2 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\
k_3 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\
k_4 &= hf(t_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

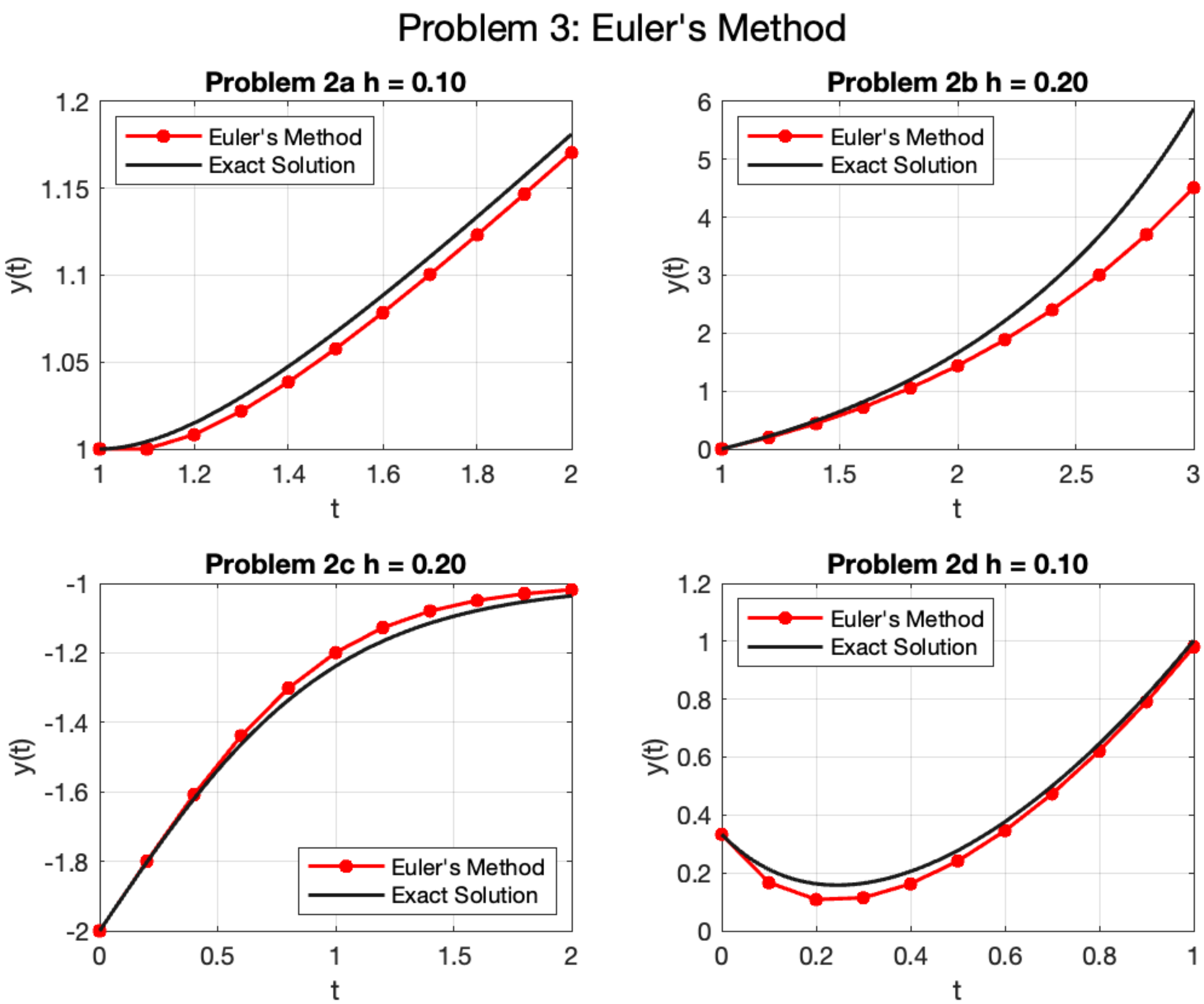
4 Output Plots

4.1 Problem 1



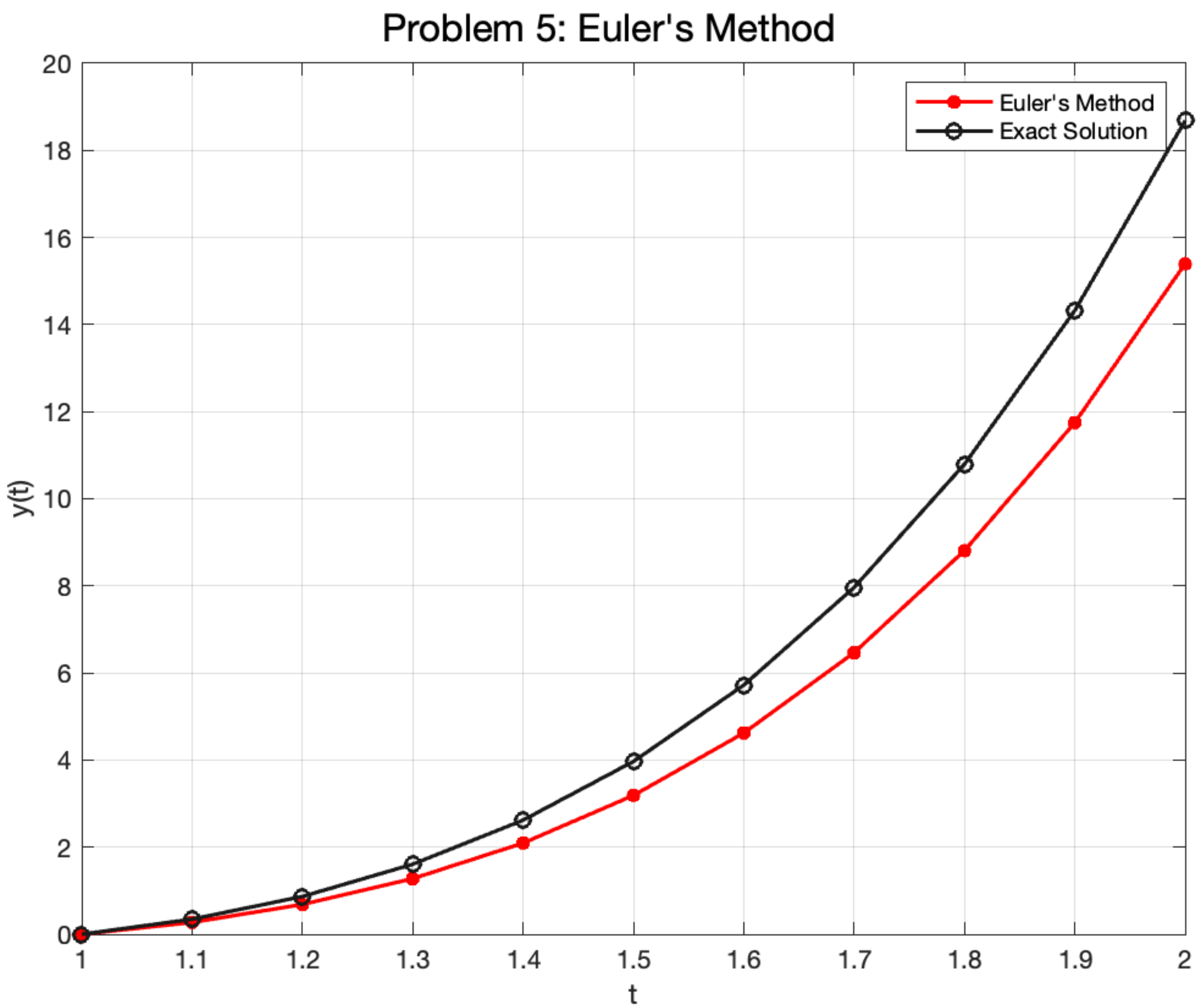
Problem 1

4.2 Problem 3



Problem 3

4.3 Problem 5



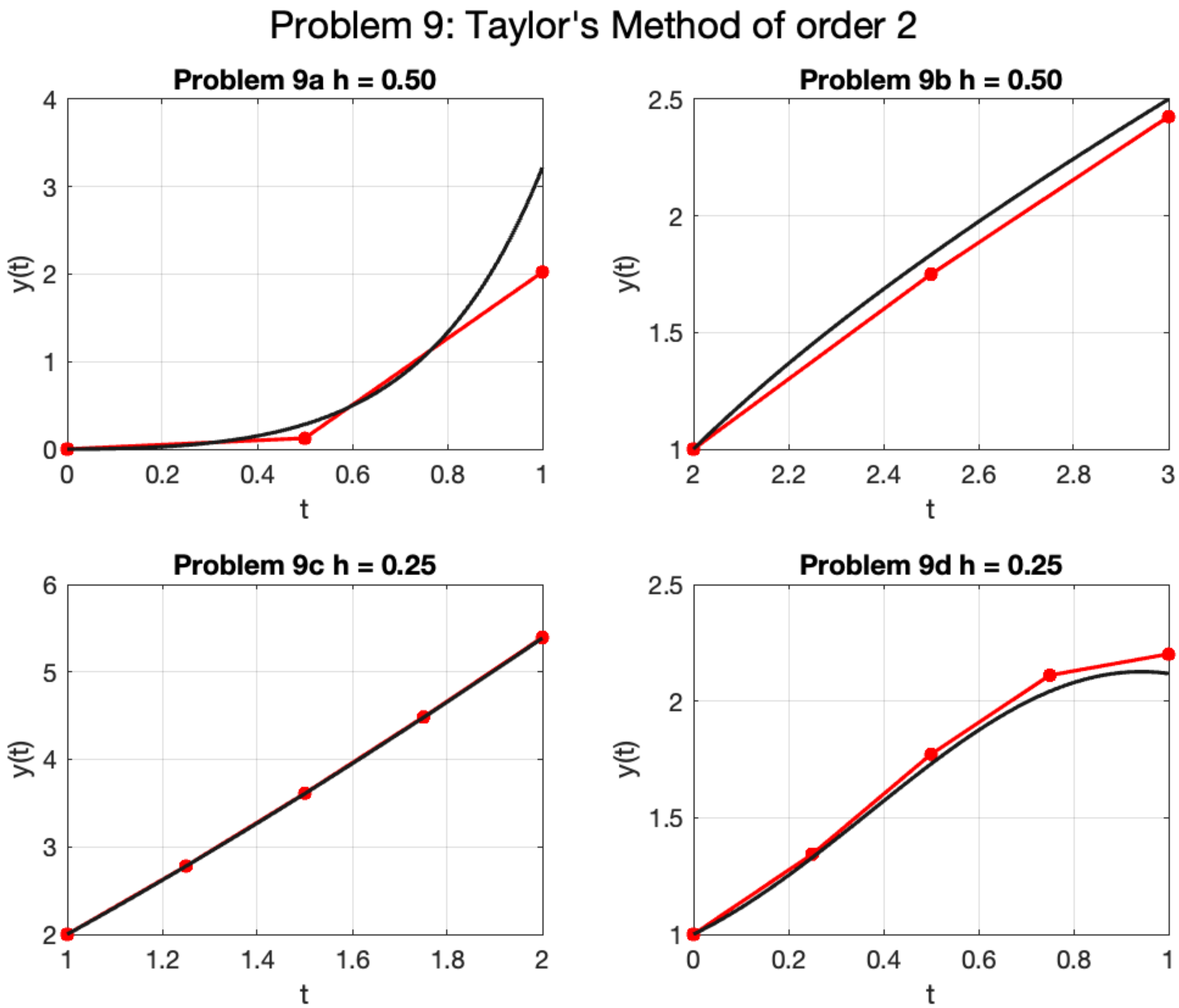
Problem 5

Table 1: Interpolated values for Problem 5b

t	Interpolated Value	Actual Value
1.04	0.10873	0.1200
1.55	3.9041	4.7886
1.97	14.303	17.2793

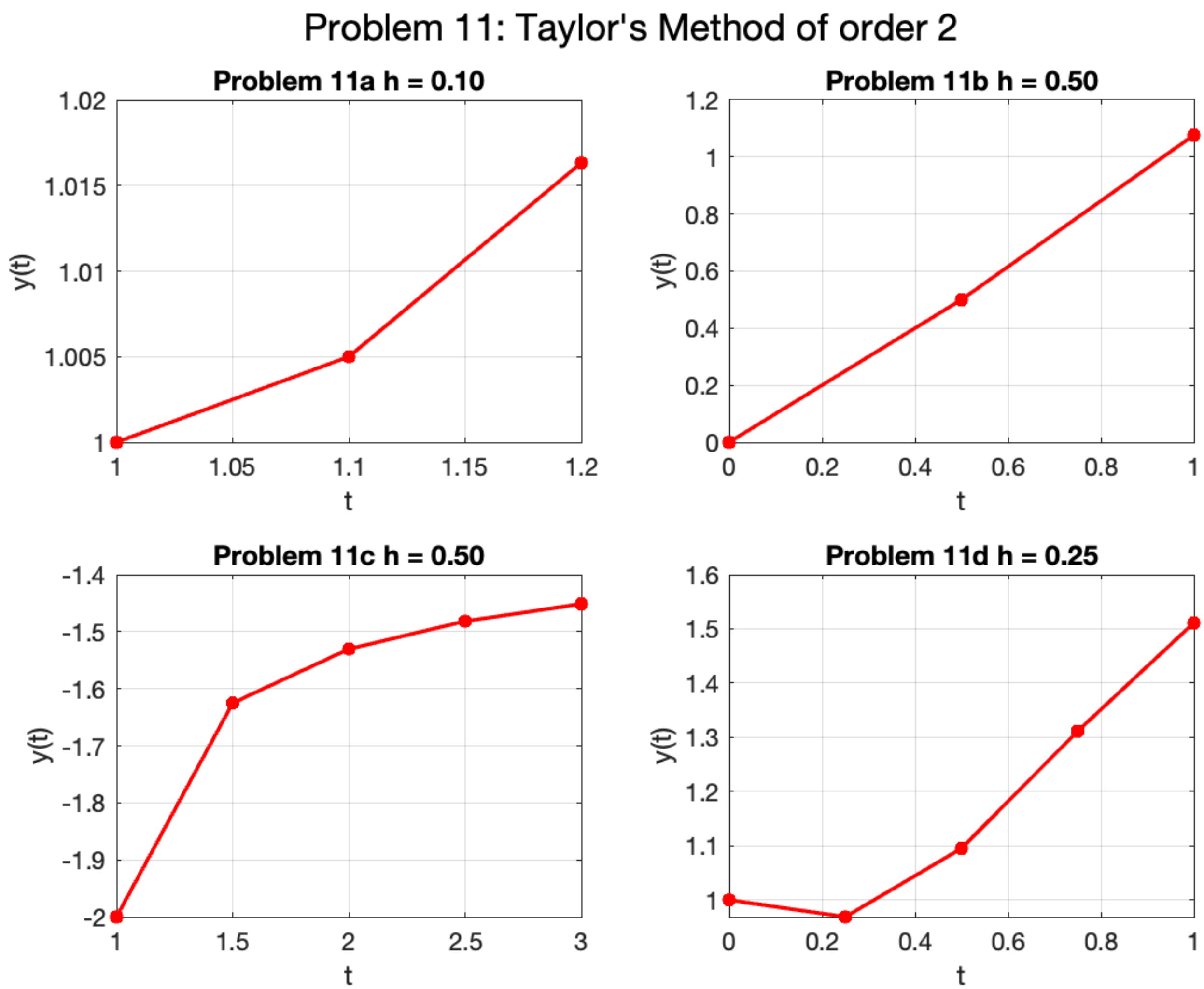
Problem 5c: The required h so that $|y(t_i) - y_i| < 0.1$ is $h \leq 0.0067089$.

4.4 Problem 9



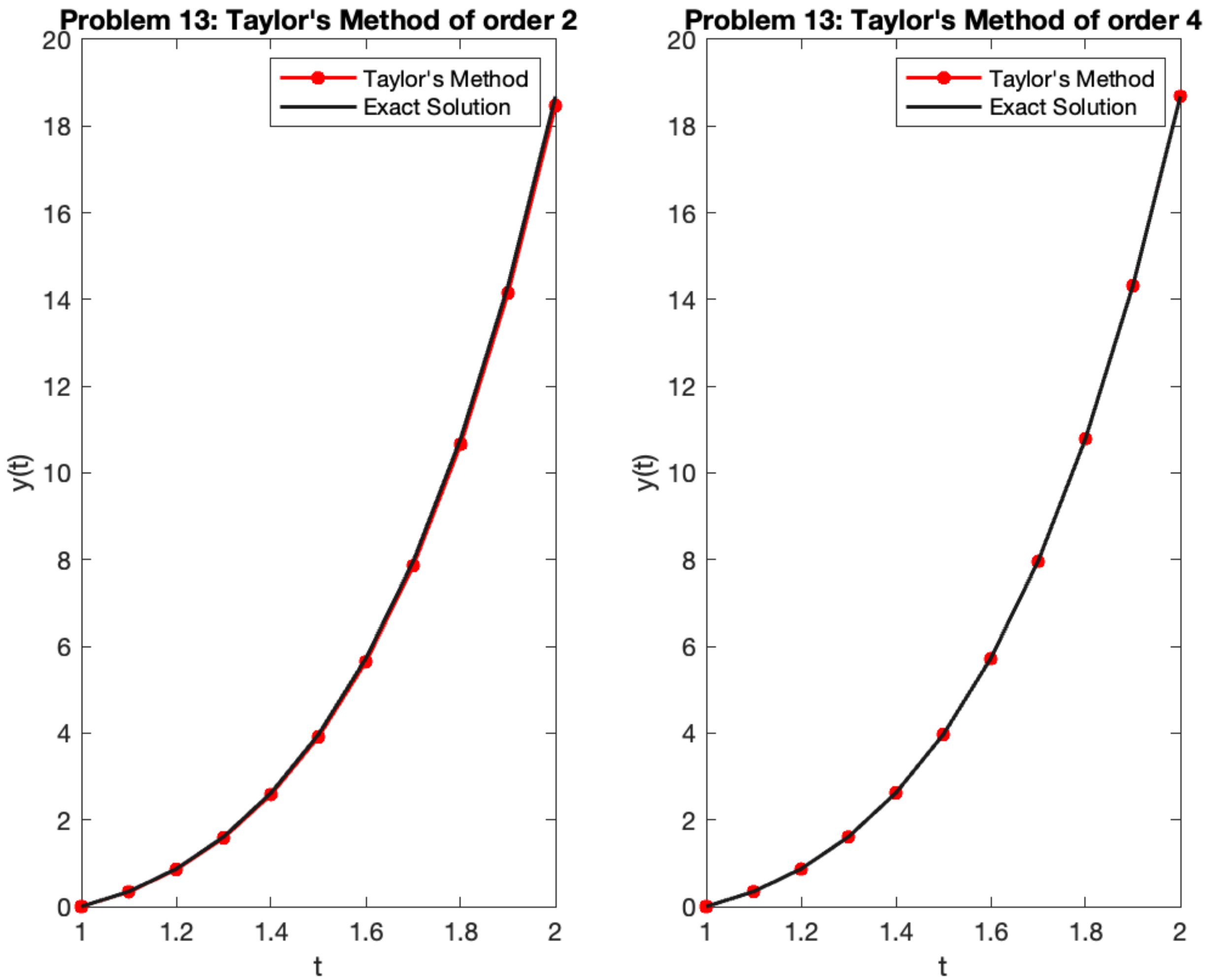
Problem 9

4.5 Problem 11



Problem 11

4.6 Problem 13



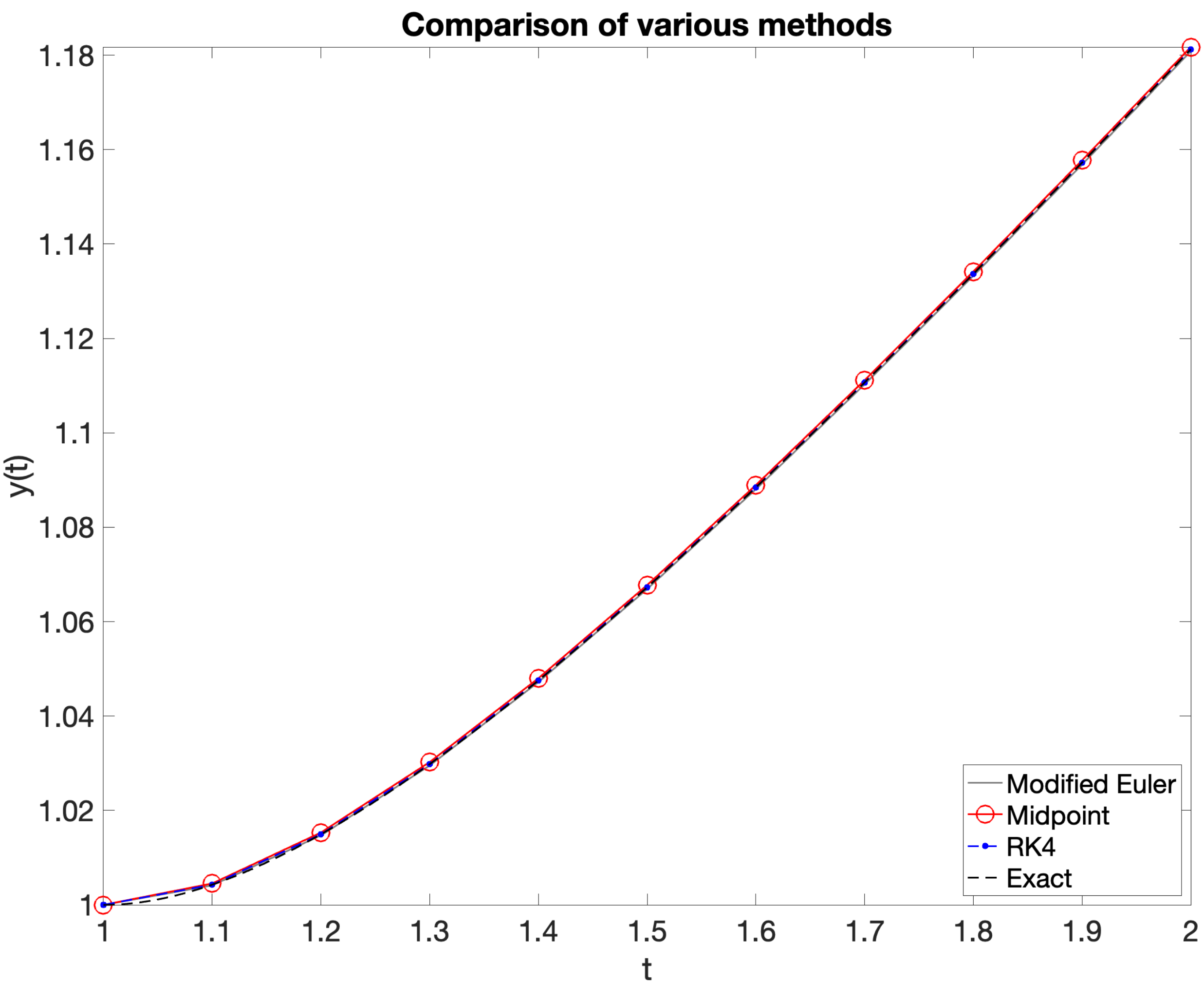
Problem 13 $h = 0.1$

Table 2: Interpolated values Problem 13b and 13d

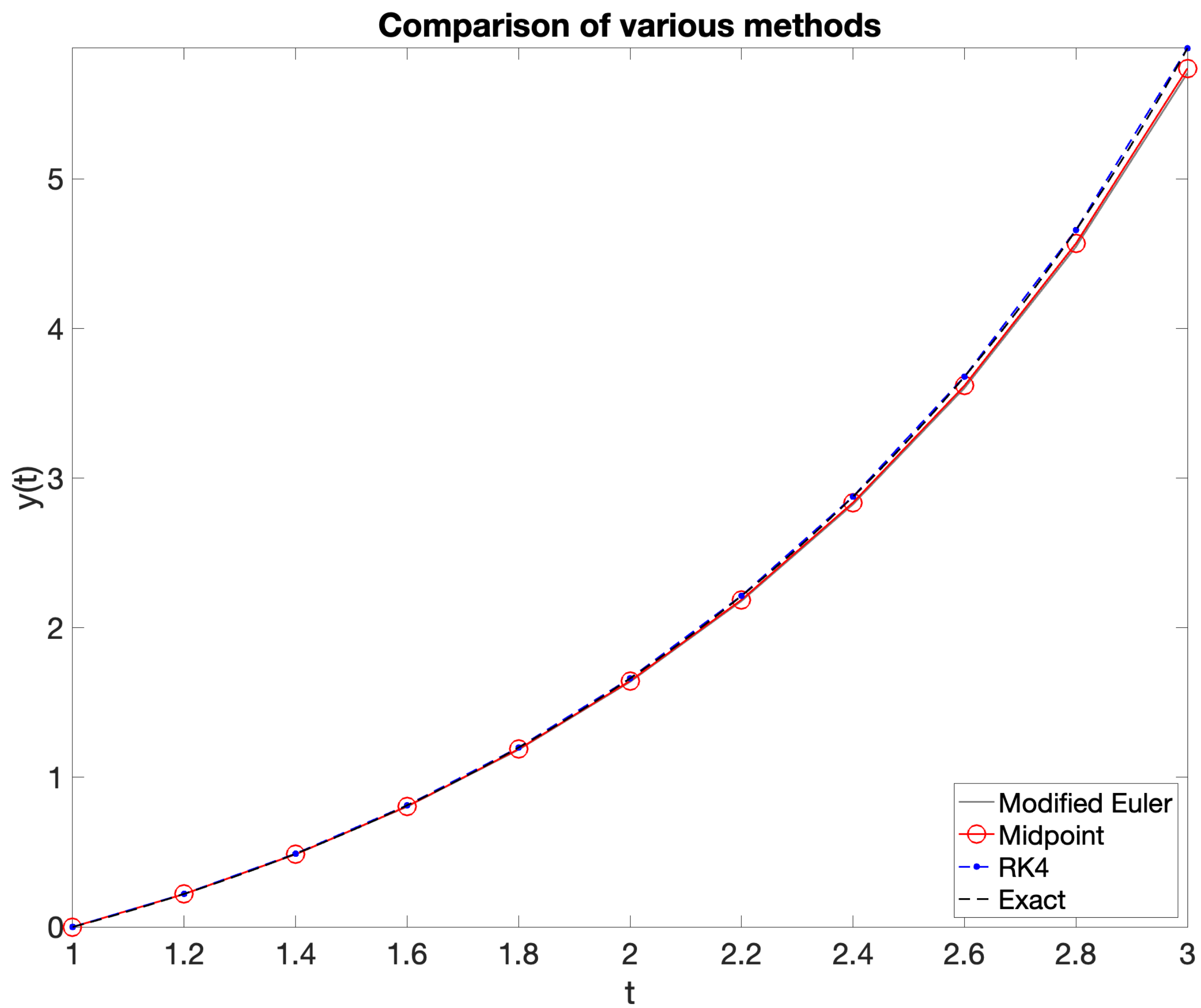
t	Taylor Order 2	Taylor Order 4	Actual Value
1.04	0.1359	0.1384	0.119987
1.55	4.7770	4.8442	4.788635
1.97	17.1748	17.3749	17.279298

4.7 Problem 15

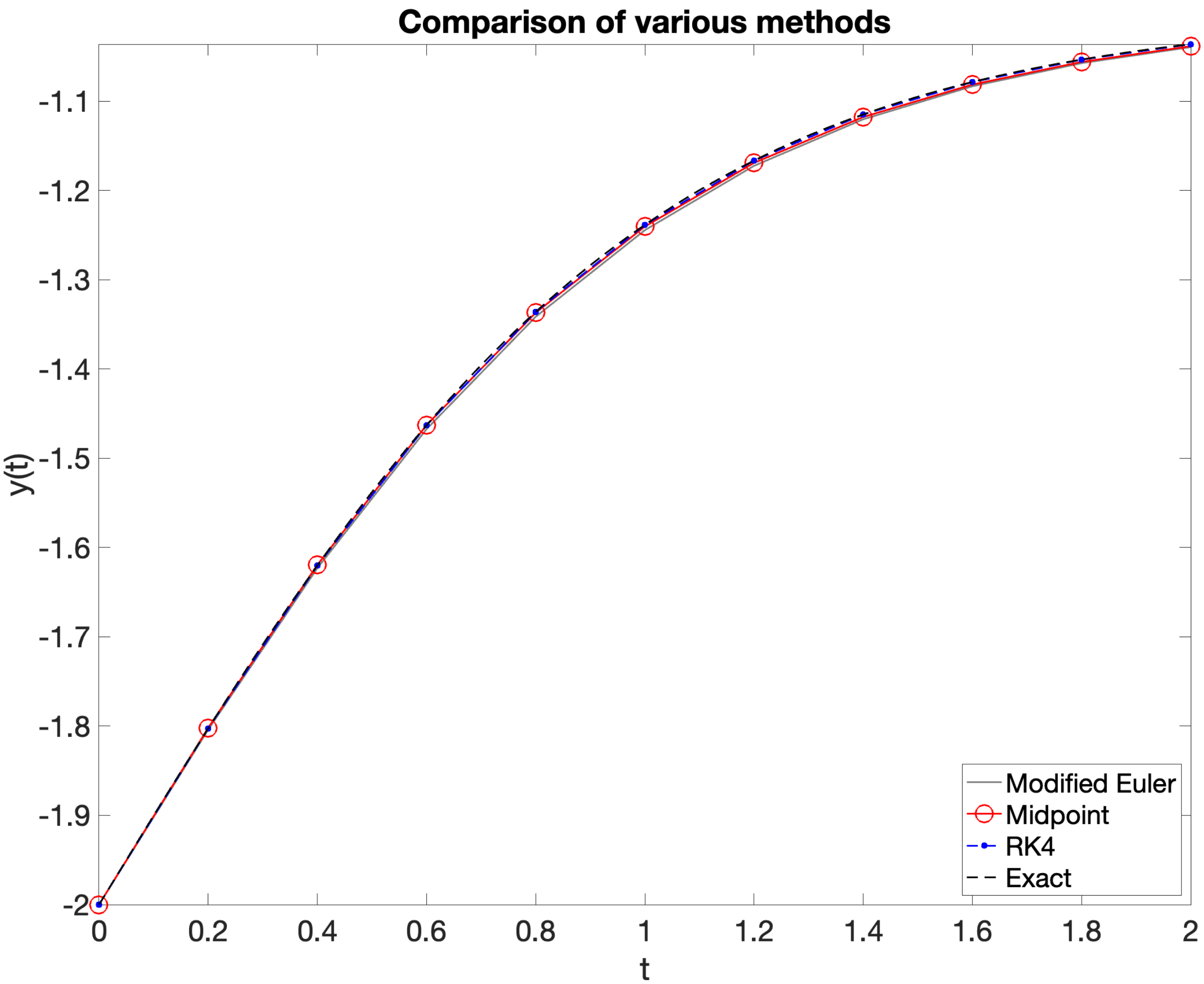
Heun's method has not been tabulated/ plotted as it is same as the Modified Euler's method.


Comparison of various methods
y(t)
1.18
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
t
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
Modified Euler
Midpoint
RK4
Exact

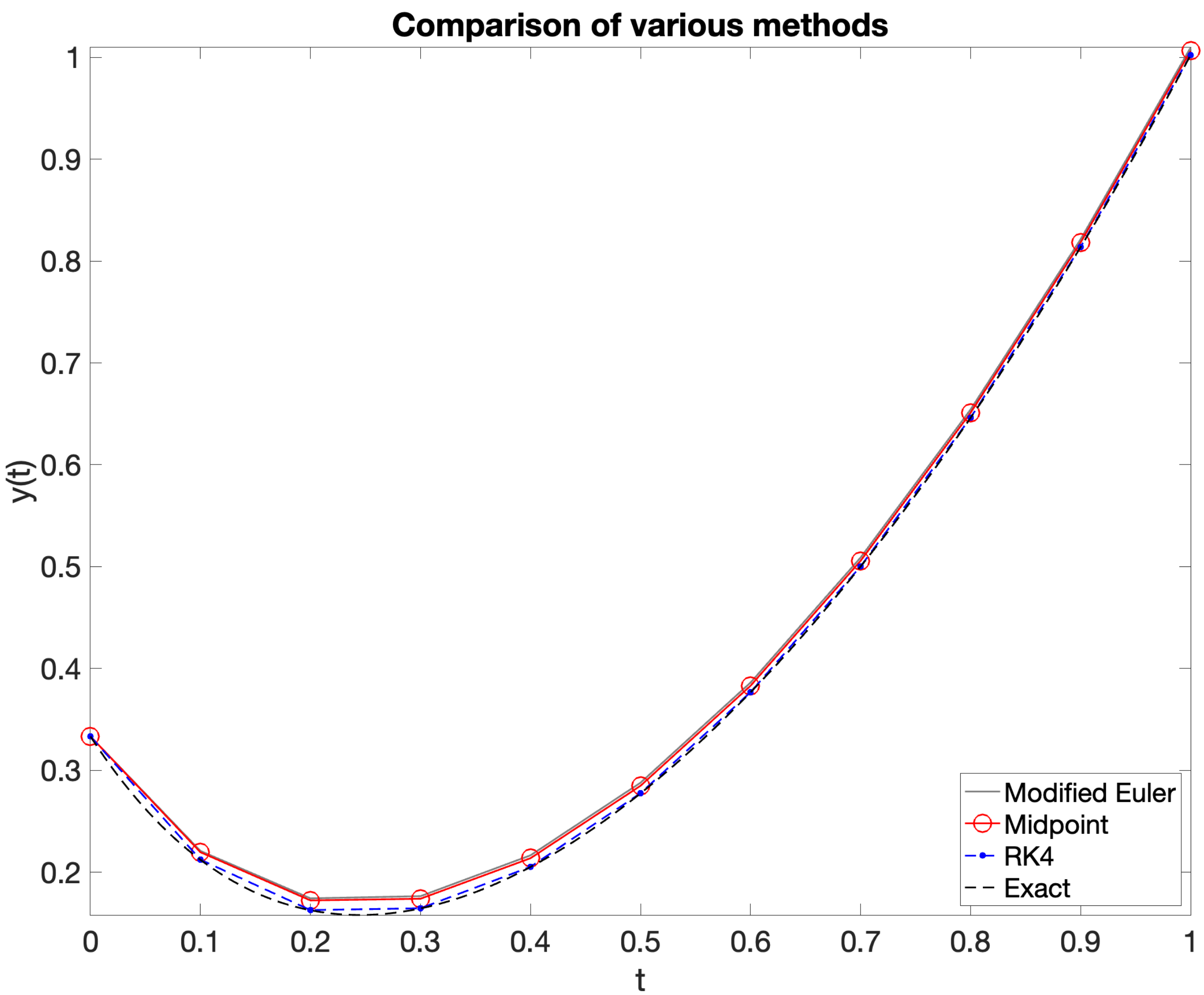
Problem $15a$


Comparison of various methods
y(t)
0
1
2
3
4
5
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3
t
Modified Euler
Midpoint
RK4
Exact

Problem $15b$


Comparison of various methods
y(t)
t
-1.1
-1.2
-1.3
-1.4
-1.5
-1.6
-1.7
-1.8
-1.9
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Modified Euler
Midpoint
RK4
Exact

Problem 15c


Comparison of various methods
y(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
Modified Euler
Midpoint
RK4
Exact

Problem $15d$

Table 3: Problem 15a

t	Modified Euler's	Midpoint	RK4	Exact
1.00	1.000000	1.000000	1.000000	1.000000
1.10	1.004132	1.004535	1.004282	1.004282
1.20	1.014714	1.015326	1.014952	1.014952
1.30	1.029520	1.030247	1.029813	1.029814
1.40	1.047204	1.047998	1.047534	1.047534
1.50	1.066909	1.067743	1.067262	1.067262
1.60	1.088064	1.088921	1.088432	1.088432
1.70	1.110275	1.111148	1.110655	1.110655
1.80	1.133266	1.134148	1.133653	1.133653
1.90	1.156835	1.157724	1.157228	1.157228
2.00	1.180834	1.181727	1.181232	1.181232

Table 4: Problem 15b

t	Modified Euler's	Midpoint	RK4	Exact
1.00	0.000000	0.000000	0.000000	0.000000
1.20	0.219444	0.219835	0.221246	0.221243
1.40	0.485049	0.486177	0.489684	0.489682
1.60	0.804012	0.806185	0.812752	0.812753
1.80	1.184856	1.188439	1.199432	1.199439
2.00	1.638423	1.643889	1.661265	1.661282
2.20	2.178877	2.186861	2.213469	2.213502
2.40	2.825065	2.836436	2.876494	2.876551
2.60	3.602525	3.618493	3.678379	3.678475
2.80	4.546614	4.568894	4.658506	4.658665
3.00	5.707570	5.738647	5.873839	5.874100

Table 5: Problem 15c

t	Modified Euler's	Midpoint	RK4	Exact
0.00	-2.000000	-2.000000	-2.000000	-2.000000
0.20	-1.804000	-1.802000	-1.802627	-1.802625
0.40	-1.622921	-1.619297	-1.620058	-1.620051
0.60	-1.467240	-1.462767	-1.462963	-1.462950
0.80	-1.341320	-1.336790	-1.335982	-1.335963
1.00	-1.244290	-1.240247	-1.238431	-1.238406
1.20	-1.172211	-1.168898	-1.166374	-1.166345
1.40	-1.120076	-1.117516	-1.114677	-1.114648
1.60	-1.083078	-1.081179	-1.078358	-1.078331
1.80	-1.057170	-1.055799	-1.053218	-1.053194
2.00	-1.039194	-1.038223	-1.035992	-1.035972

Table 6: Problem 15d

t	Modified Euler's	Midpoint	RK4	Exact
0.00	0.333333	0.333333	0.333333	0.333333
0.10	0.220833	0.219583	0.212283	0.212177
0.20	0.174271	0.172240	0.162765	0.162626
0.30	0.176419	0.173899	0.164517	0.164377
0.40	0.216512	0.213687	0.205241	0.205112
0.50	0.287820	0.284805	0.277477	0.277362
0.60	0.386138	0.383003	0.376698	0.376596
0.70	0.508836	0.505627	0.500158	0.500066
0.80	0.654272	0.651017	0.646105	0.646105
0.90	0.821420	0.818135	0.813703	0.813703
1.00	1.009638	1.006335	1.002246	1.002246

5 Interpretation

1. We can see that decreasing h increases the accuracy of all the methods However this comes with increased computational cost.

2. In Problem 5 we can see that Modified Euler's method is more accurate than Euler's method in all 4 cases.

3. In Problem 5, the value of h necessary so that $|y(t_i) - y_i| \leq 0.1$ is computed as $h \leq 0.0067089$.

4. In Problem 13 we see from the plot Taylor's method of order 4 is more accurate than that of order 2 but the error on the interpolated points is higher in the former.

5. The Heun's method is same as the Modified Euler's methid so it has not beem tabulated/ plotted in Problem 15.

6. In Problem 15 we see that the Runge-Kutta method is the most accurate method among the implemented methods.

7. Euler's Method is a first-order method, simple to implement but has limited accuracy, and error becomes noticable with h is large.

8. Taylor's method of orders 2 and 4 utilizes higher derivatives, improving accuracy by capturing more of the solution's curvature. This, however, increases computational cost and is difficult when higher-order derivatives are not available or are difficult to calculate.

9. Runge-Kutta method of order 4 (RK4) requires only the first derivative but still offers more accuracy as it takes multiple intermediate slope calculations within each step. It is in between Euler's method and Taylor's method in computational cost and achieves nearly exact results.