UNIT - 2

Relation & Functions

Let X and Y be 2 sets then a relation R from X to Y $(X \to Y)$ is a subset of the cartesian product of X and Y, ie. $X \times Y$

The set R contains all ordered paired in which the first element is from the set X and the second element is from the set Y. and the first element is related to the second element by a definite relation R,

* Let $X = \{1, 2, 3\}$, $Y = \{3, 4, 5\}$

Cartesian product $(X \times Y) = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5)\}$

* Let $x \in X$, $y \in Y$ and $x < y$

then $R = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5)\}$

Ques Let (a) $X = \{1, 2, 3, 4\}$
S and R are a relations from X to X.

$R = \{(x, y) \mid x \in X \cap y \in X \cap ((x-y))$ is an integral non-zero multiple of 2 $\}$.

$S = \{(x, y) \mid x \in X \cap y \in Y \cap ((x-y))$ is an integral non-zero multiple of 3 $\}$.

Find $R \cup S$ and $R \cap S$?

Sol. $R = \{(3,1), (1,3), (2,4), (4,2)\}$
$S = \{(1,4), (4,1)\}$

$\therefore R \cup S = \{(3,1), (1,3), (2,4), (4,2), (1,4), (4,1)\}$

$R \cap S = \phi$

Ques (b) $X = \{1, 2, 3, ...\}$
Relation is $R \cap S$
$R, S \rightarrow$ same as previous ques.

$R \cap S = \{(x,y) \mid x \in X \wedge y \in X \wedge ((x-y))$ is an integral non-zero multiple of $6\}$.

$\therefore R \cap S = \phi$.

PROPERTIES OF RELATION

1) Reflexive Relation

A binary relation R in a set X is reflexive if for every $x \in X$ $(x,x) \in R$ or $x R x$ (x is Related with x).

If Reflexive $\Rightarrow (x)(x \in X \rightarrow x R x)$

Ex- $X = \{1,2,3\}$

$R = \{\underbrace{(1,1), (2,2), (3,3)}_{\text{it matters}}, \underbrace{(1,2), (2,2)}_{\text{doesn't matter}}\}$

$\leq$ is a reflexive Relation

2) Symmetric Relation

A relation R in a set X is symmetric, if for every $x, y \in X$, whenever $x R y$ then $y R x$

ie. R is symmetric $\Leftrightarrow (x)(y) (x \in X \wedge y \in X \wedge x R y \rightarrow y R x)$

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Ex — $X = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,1), (3,3), (3,4), (4,3)\}$

3) Transitive Relation

A relation R in a set X is transitive if for every x, y and z whenever x R y and y R z, then x R z.

$\Leftrightarrow (x)(y)(z)\ (x \in X \cap y \in X \cap z \in X \cap xRy \cap yRz \rightarrow xRz)$

eg $X = \{1, 2, 3, 4\}$
$R = \{(1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

↳ ~~this is not transitive~~

∵ (1, 2) but ~~(2, _)~~

i.e. x is related with y (xRy) but y is not related with z (yRz).

4) Irreflexive - A relation R in a set X is irreflexive if for every $x \in X$, $x \not R x$,

OR $\boxed{(x, x) \notin R}$

Ex- $X = \{1, 2, 3\}$
$R = \{(1,1), (2,3), (3,2), (1,3)\}$

due to this it is neither reflexive nor irreflexive

5) Antisymmetric

A relation R in a set X is antisymmetric if for every $x, y \in X$, whenever x R y and y R x then x = y

$(x)(y)\ (x \in X \cap y \in X \cap (xRy \cap yRx) \rightarrow x = y)$

Ex- $X = \{1, 2, 3\}$,
$R = \{(1,1); (2,2)\}$

Note - 1) In any case if antecedent is true then true.

2) If R is null, then it is symmetric as well as antisymmetric.

Ques (1) $X = \{1, 2, 3, 4, 5, 6\}$
$R = \{(1,2), (2,3), (1,3), (5,6)\}$
Irreflexive, transitive & Antisymmetric

(2) $R = \{(5,6)\}$
Irreflexive, transitive & Antisymmetric.

Matrix and Graph (pictorial Representation) of Relation

(1) Matrix Representation:
$X = \{x_1, x_2, \ldots x_m\}$ and $Y = \{y_1, y_2 \ldots y_n\}$

R from X to Y. $(m \times n)$

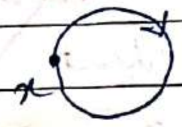
$$r_{ij} = \begin{cases} 1 & \text{if } x_i R y_j \\ 0 & \text{if } x_i \not R y_j \end{cases}$$

Ex- $X = \{x_1, x_2, x_3\}$
$Y = \{y_1, y_2\}$
$R = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2)\}$

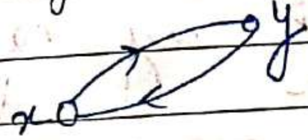
$$R_M = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

(2) Graphical Representation

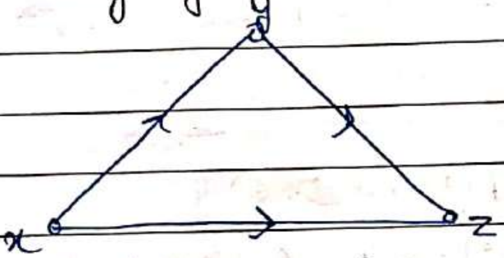
xRx,



$xRy \cap yRx$

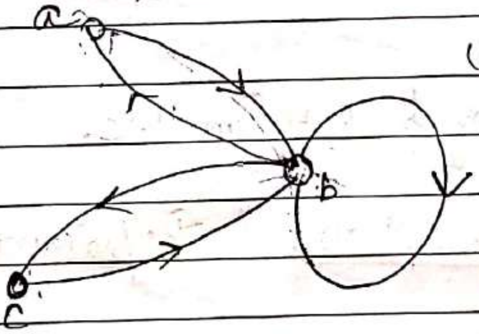


$xRy \cap yRz \cap xRz$

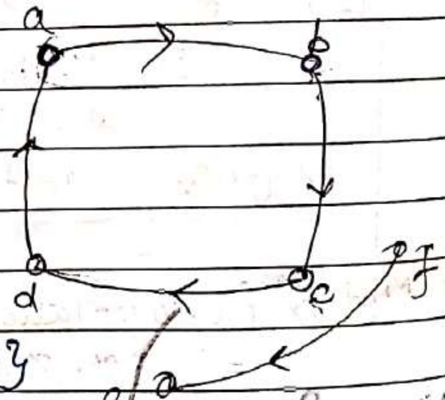


Ques

(a)



(b)



Irreflexive also $(x,x) \notin R$

(a) $R = \{(a,b), (b,a), (b,c)(c,b)(b,b)\}$

↓

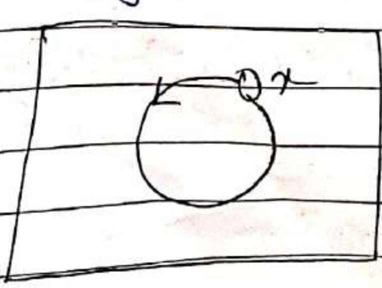
Symmetric

(b) $R = \{(a,b), (b,c), (c,d), (d,a), (f,e)\}$

→ x is related with y but y is not related with any x so, antisymmetric.

→ Antisymmetric

(3)



Reflexive, symmetric
transitive, antisymmetric

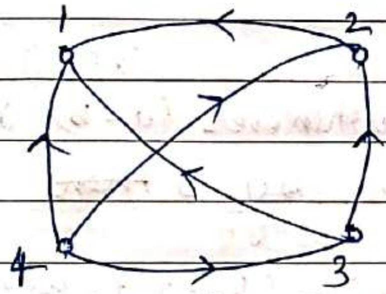
Ques Let $X = \{1,2,3,4\}$ and R is a relation from X to X i.e. $R = \{(x,y) \mid x > y\}$

Draw the graph of R and also give its matrix?

Soln

$$R = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$

1) Graphical:-



2) Matrix:-

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

Equivalence Relation

The Relation is a equivalence relation if it is reflexive, symmetric and transitive.

Ques $X = \{1,2,3,\ldots,7\}$

$R = \{(x,y) \mid x-y$ is divisible by $3\}$

Show that R is an equivalence Relation. Draw the graph of R?

Sol

$(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)$

$R = \{\overset{x\ y}{(4,1)}, (7,1), (5,2), (6,3), (7,4)\} \cup \{(1,4),(1,7),(2,5),(3,6),(4,7)\}$

Note: If there are n elements &

for Reflexive: For any $a \in X$, $a-a$ is always div by 3 i.e. a is related with a (aRa) or $(a,a) \in R$.

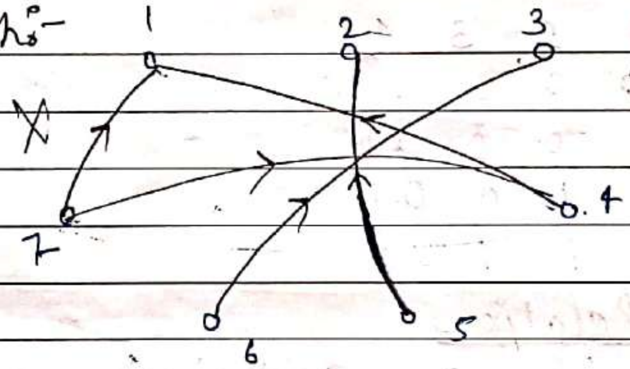
General

For Symmetric:

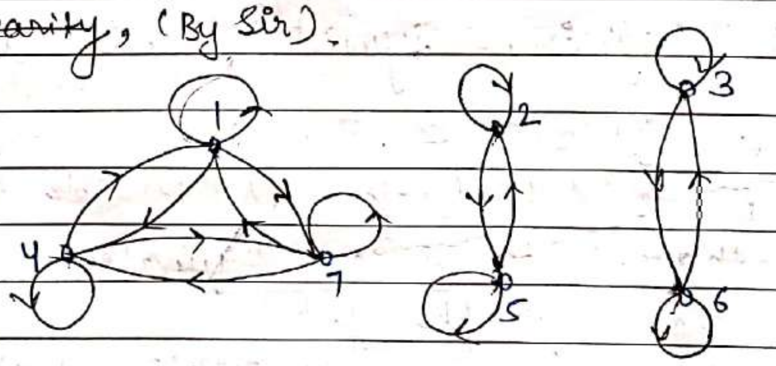
For any $a, b \in X$, whenever $(a-b)$ is div by 3 then $(b-a)$ is also div by 3.
i.e. xRy then yRx

For transitive.
for any $a, b, c \in X$, whenever $(a-b)$ (aRb) is div by 3 and (bRc) $(b-c)$ is also div by 3 ~~then~~.
So that
$a-c = (a-b)+(b-c)$ is also div. by 3.

Graph:-



For clearity, (By Sir).



Ques Let R denote a relation on a set of ordered pair of +ve Integers such that $(x, y) R (u, v)$ if and only if (iff) $xv = yu$. Show that R is an equivalence Relation

Sol Reflexive: $(x, y) R (x, y)$

$xy = yx$ i.e. $xv = yu$,

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Symmetric: Let $(x,y) \in N$, $(u,v) \in N$ and $(x,y)\,R\,(u,v)$.

$\Rightarrow xv = yu$

$\Rightarrow uy = xv$

$\Rightarrow (u,v)\,R\,(y,x)$

Transitive: Let $(x,y) \in N$, (u,v), $(w,z) \in N$ and $(x,y)\,R\,(u,v)$ and $(u,v)\,R\,(w,z)$

$\Rightarrow \quad xv = yu \quad \text{---(1)}$

$uz = vw \quad \text{---(2)}$

$(x\cancel{v})(\cancel{u}z) = (y\cancel{u})(\cancel{v}w)$

$xz = yw$

or

$xz = wy$

$\boxed{(x,y)\,R\,(z,w)}$

Inverse or Converse of a Relation

If a given relation R from x to y, a Relation R^{-1} from y to x is called converse or inverse of R where the ordered pair of R^{-1} are obtained by interchanging the members in each of ordered pair of R.

i.e. $\quad X \to Y$

$R = \{(1,2), (5,6)\}$

then, $R^{-1} = \{(2,1), (6,5)\}$

Composition of Binary Relation

Let R is a relation from X to Y and S be a relation from Y to Z then their composition of relation ~~if R is relation from~~ $R_o S$ is related from X to Z where ~~$R_o S$ is~~.

$$\boxed{R_o S = \{(x,z) \mid x \in X \cap z \in Z \cap (\exists y)(y \in Y \cap (x,y) \in R \cap (y,z) \in S)\}}$$

Composite Relation

eg. let $R = \{(1,2), (3,4), (2,2)\}$
$S = \{(4,2), (2,5), (3,1), (1,3)\}$
find R_oS, S_oR, $R_o(S_oR)$, R_oR.

Sol
(i) $R_o S = \{(1,5), (3,2), (2,5)\}$
1 is related with 2 & 2 is related with 5.

S_oR, =
(ii) $S_o R = \{(4,2), (3,2), (1,4)\}$

Here, $\boxed{R_oS \neq S_oR}$ → not commutative.

(iii) $R_o(S_oR) = \{(3,2)\}$.

(iv) $R_oR = \{(1,2)\}$.

Note :- Composition is always associative.
i.e. $\boxed{R_o(S_o W) = (R_o S)_o W}$

Theorem 1:-

Let R is a relation from the set A to set B and S is a relation from set B to set C. then we have to prove that $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

To prove:- $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

Proof:- Let $(c,a) \in (R \circ S)^{-1}$

$\Rightarrow (a,c) \in (R \circ S)$

$\forall\, a \in A$ and $c \in C$

So that, there exist an element $b \in B$ with $(a,b) \in R$ and $(b,c) \in S$

$\Rightarrow (b,a) \in R^{-1}$ and $(c,b) \in S^{-1}$

$\Rightarrow (c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$

$\Rightarrow (c,a) \in S^{-1} \circ R^{-1}$

$\Rightarrow \boxed{(R \circ S)^{-1} = S^{-1} \circ R^{-1}}$

Ques Let A be the set i.e. $A = \{1, 2, 3\}$

Doubt Define the following type of binary relation on A?

(1) A relation ie both symmetric and anti-symmetric

(2) A relation that is neither symmetric nor anti?

Sol^n ~~$R = \{(1,2),(2,1),(2,3),(3,2),(1,1),(2,2),(3,3)\}$~~

(1) $R = \{(1,1), (2,2), (3,3)\}$

↳ Both Symm. & anti-Sym.

(11) $S = \{(1,2), (2,1), (2,3)\}$

or

$\{(1,2),(2,3),(3,2)\}$

↳ Neither Sym. nor anti symm. [∵ (2,3) [illegible] but (3,2) [illegible]]

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Theorem - 2

Let R be an equivalence relation on the set A then prove that R^{-1} is also an equivalence relation on the set A.

~~Reflexive proof~~

Reflexive :-

Let $x \in A$, then $(x, x) \in R$
$\Rightarrow (x, x) \in R^{-1}$.

Symmetric :-

let $x, y \in A$, since R is symmetric,
$\therefore$, $(x, y) \in R \Rightarrow (y, x) \in R$
$\Rightarrow (y, x) \in R^{-1}$ and $(x, y) \in R^{-1}$
$\Rightarrow$ if $(y, x) \in R^{-1} \Rightarrow (x, y) \in R^{-1}$.

Transitive :-

let $x, y \in A$ and $y, z \in A$
Since R is transitive.
$\therefore$, $(x, z) \in R$
$\Rightarrow (z, x) \in R^{-1}$ and $(y, x) \in R^{-1}$ and $(z, y) \in R^{-1}$
$\Rightarrow$ ~~$(x, z) \in R^{-1}$ and $(z, x) \in R^{-1}$~~
$\Rightarrow$ ~~if $(x, z) \in R^{-1} \Rightarrow (z, x)$~~

$\Rightarrow \boxed{(z, y) \in R^{-1}, (y, x) \in R^{-1} \Rightarrow (z, x) \in R^{-1}}$

x.y y.x

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Ques If R and S are equivalence relation on the set A show that $R \cap S$ is also equivalence relation on the set A and $R \cup S$ is not necessary an equivalence relation?

Sol (I) Reflexive:

$\forall a \in A$, $(a,a) \in R$, $(a,a) \in S$

$\Rightarrow (a,a) \in R \cap S$.

$\Rightarrow R \cap S$ is ~~equivalence~~ reflexive.

Symmetric:

Let $\boxed{x,y \in R \cap S}$ ~~Since~~

$\Rightarrow (x,y) \in R$ and $(x,y) \in S$

$\Rightarrow (y,x) \in R$ and $(y,x) \in S$

$\Rightarrow \boxed{(y,x) \in R \cap S}$

Transitive:

let $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$

$\Rightarrow (x,y) \in R$ and $(x,y) \in S$ and $(y,z) \in R$ and $(y,z) \in S$

$\Rightarrow (x,z) \in R$ and $(x,z) \in S$

$\Rightarrow \boxed{(x,z) \in R \cap S}$.

(II) Let $A = \{a,b,c\}$

$R = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

$S = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$

$R \cup S = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

$\therefore$ $R \cup S$ is not ~~an~~ necessarily equivalence Rel. but $R \cap S$ is.

Ques R is a set of Real number, let $A = R \times R$.
Define the following relation on A.
$(a,b) R (c,d)$
$\Rightarrow a^2+b^2 = c^2+d^2$, Show that R is an equivalence relation.

Sol:

Reflexive:

$\forall (a,b) \in A$

$a^2+b^2 = a^2+b^2$

$\Rightarrow (a,b) R (a,b)$

$\therefore$ R is reflexive.

Symmetric:

$\forall (a,b), (c,d) \in A$

$a^2+b^2 = c^2+d^2$

$\Rightarrow c^2+d^2 = a^2+b^2$

$\Rightarrow (c,d) R (a,b)$

$\therefore$ R is symmetric

Transitive:

$\forall (a,b), (c,d), (e,f) \in A$

$a^2+b^2 = c^2+d^2$ and $c^2+d^2 = e^2+f^2$

$\Rightarrow a^2+b^2 = e^2+f^2$

$\Rightarrow (a,b) R (e,f)$

$\therefore$ R is transitive.

$\therefore$ R is an equivalence Relation

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Reflexive closure :

Let R be a relation on the set A and R is not reflexive. A relation R_1 is the reflexive closure of the relation R if R_1 is the smallest relation containing R and R is reflexive.

Let $A = \{a, b, c\}$

$R = \{(a,a), (a,b), (b,c)\}$

$$\boxed{R_1 = \{(a,a), (b,b), (c,c), (a,b), (b,c)\}}$$

reflexive closure of R (Smallest containing R & R is reflexive)

Symm. ~~Transitive~~ Closure:

$R_2 = \{(a,a), (a,b), (b,c), (b,a), (c,b)\}$

(b, a), (c, b) by Addi

Transitive ~~Symmetric~~ closure:

$R_3 = \{(a,a), (a,b), (b,c), (a,c)\}$

Partial Ordering

A partial ordering R in a set P is called ~~p.o~~ a partial order relation or a partial ordering iff R is reflexive, antisymm. & transitive.

It is conventional to denote partial ordering or partial ordered rel^n by symbol $(\leq)$. This symbol does not necessarily means "less than equal to" as used for real number. If $(\leq)$ is a partial ordering on P(set) then the ordered pair $(P, \leq)$

is called partially ordered set or POSET.

Ex- (1) $(P, \leq)$ ✓
(2) $(P, \subseteq)$ ✓
(3) $(P, \subset)$ ✗
(4) (P, divide) ✓
(5) $(P, =)$ ✗ ∵ it is antisym, reflexive and transitive but symmetric also-

Partially Ordered Sets

Representation & associated Terminology

In a partially ordered set $(P, \leq)$, an element $y \in P$ is called to cover an element $x \in P$ if $x < y$ and if $\nexists$ an element $z \in P$ such that $x \leq z \leq y$

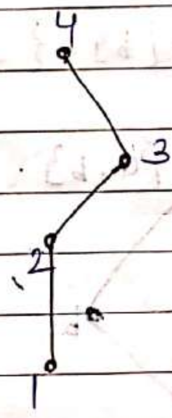
Hasse Diagram

A partially ordering $(\leq)$ on a set P can be represented by means of diagram known as Hasse diagram or a partially ordered set diagram of POSET $(P, \leq)$. In such a diagram, each element is represented by a small circle or a dot. The circle for $(x = P)$ is draw below circle for $(y = P)$ if $x < y$ and a line is drawn b/w x & y if y cover x. ~~if $x < y$~~
if $x < y$ but y doesn't cover x then x and y are not connecting directly

by a straight line, However they are connecting through one or more elements of P.

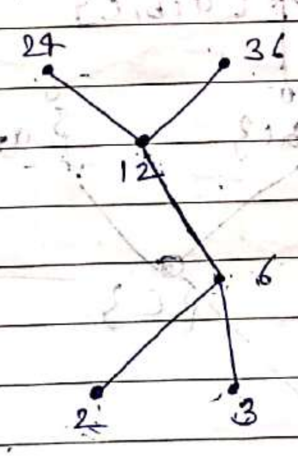
Example: Let $P = \{1, 2, 3, 4\}$ and partial ordering is "less than equalto". Draw Hasse diagram→

Sol



Ques Let $X = \{2, 3, 6, 12, 24, 36\}$ Relation ($\leq$) be such that $x \leq y$ if x divides y or $(x|y)$. Draw Hasse diag?

Sol

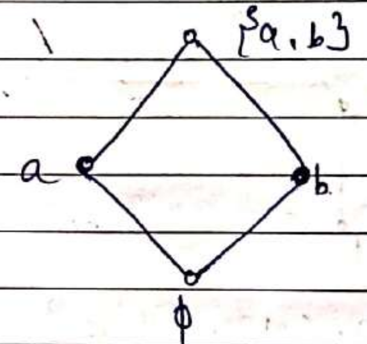


Ques Let A be a given finite set and P(A) is a power set of A. Let ⊆ be a inclusion relation on the power set of A.
Draw Hasse diagram of (P(A), ⊆) for

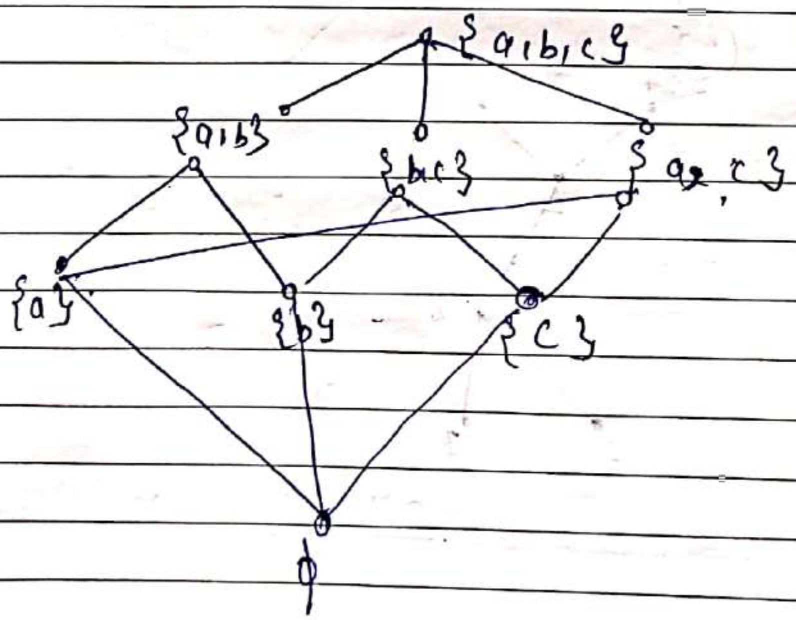
a) A = {a, b}
b) A = {a, b, c}

(a) P(A) = {φ, {a}, {b}, {a, b}}

Soln. a)



b) P(A) = {φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}



Ques Let A be the set of factors of a particular +ve integers m and let ≤ be the relation divides.
Draw the Hasse diagram.

(a) m = 12 (c) m = 45
(b) m = 30

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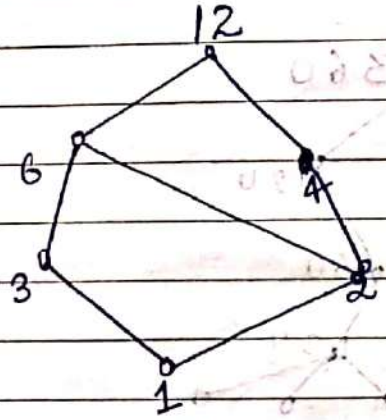
$\mu = 12$

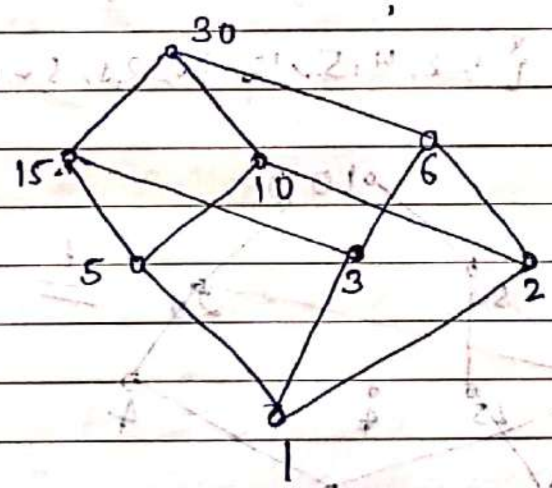
(a) $A = \{1, 2, 3, 4, 6, 12\}$

partial ordering → divide.



(b) $\mu = 30$

$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$



(c) $\mu = 45$

$A = \{1, 3, 5, 9,$

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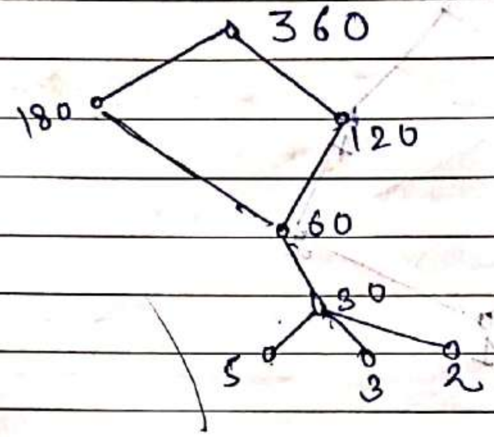
Draw Hasse diagram of set

1) A = { 2, 3, 5, 30, 60, 120, 180, 360 } '|' }

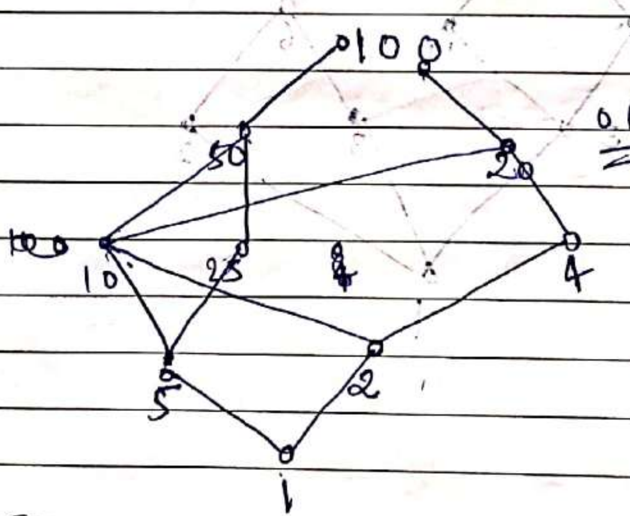
↓

partial ordering.

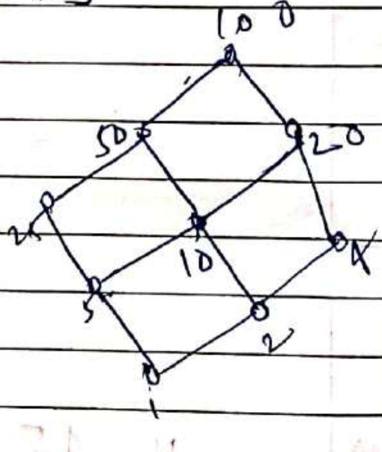


2) Draw Hasse diagram of (D100, '|')

A = { 1, 2, 4, 5, 10, 20, 25, 50, 100 }



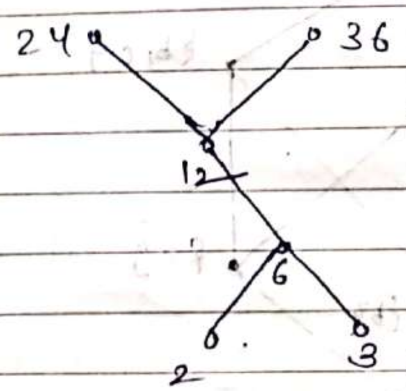
OR



Totally Ordered OR Simply Ordered Set

Let $(P, \leq)$ be POSET. If for every $(x, y) \in P$ we have either $x \leq y$ or $y \leq x$ then '$\leq$' is called a simple ordering or linear ordering on P. and partially ordered set is called totally ordered or simply ordered on the set P or ~~for~~ (chain on the set P).

Ex-1 $X = \{2, 3, 6, 12, 24, 36\}$

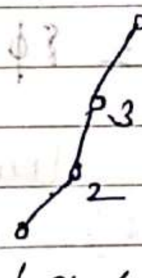


This POSET $(X, |)$ is not a totally ordered set.

set — partially ordering

Ex-2 $A = \{1, 2, 3, 4\}$

$\leq$



→ This POSET $(X, \leq)$ is a totally ordered set

→ It is always a chain.

Upper Bound and Lower Bound

Let POSET $(P, \subseteq)$ be a POSET and $A \subseteq P$. Any element $x \in P$ is an upper bound for the set A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is a lower bound for the set A if for all $a \in A$ $(\forall a \in A)$, $x \leq A$

P.O.

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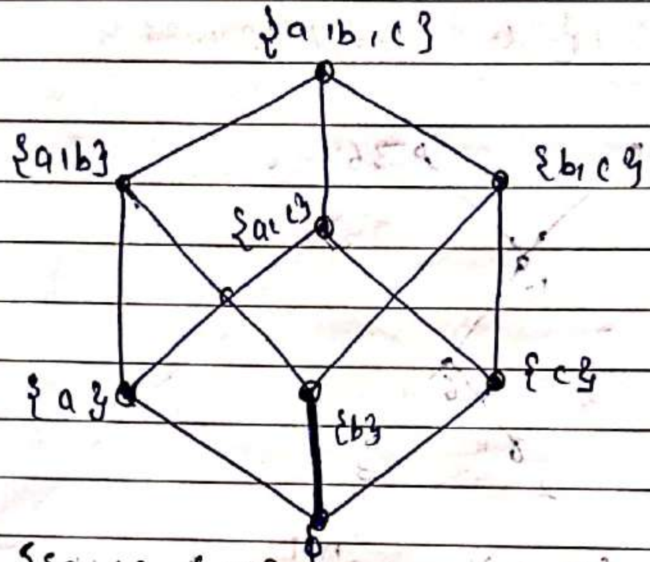
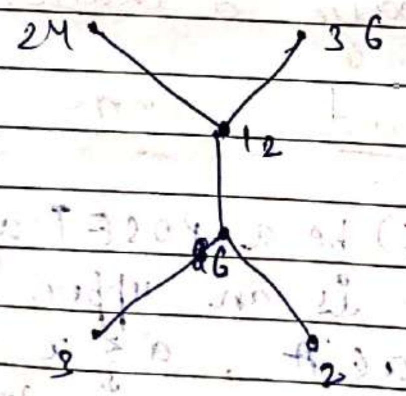
<u>for example</u> - Let us consider the POSET
A = {a, b, c}
POSET is (P(A), ⊆) (Power set of A).



1) B = {{a, c}, {c}}
upper bond = {a, b, c}, {a, c}
lower bound = φ and c.

2) B = {{a, b}, {a, c}, {c}}
upper bound = {a, b, c}
lower bound = {φ}

<u>Que</u> find upper and lower bond for A where A = {2, 3, 6}
P.O. is divide.



upper bounds : 6, 12, 24, 36
Lower bound doesn't exist

(LUB)
Least Upper Bound or (Supremum) and Greatest lower bound or (Infimum)
(GLB)

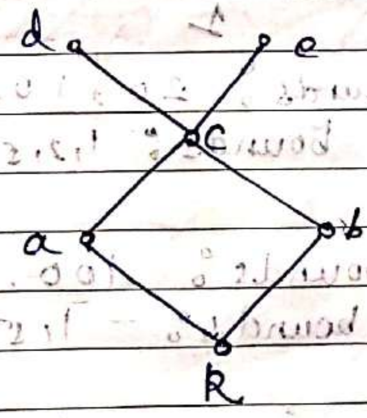
LUB → Least element in ~~lower~~ upper bound.
GLB → Greatest element in lower bound

Ex – Upper bound = $\{a,c\}, \{a,b,c\}$
lower bound = $\phi, \{c\}$

$\therefore$ $\boxed{\begin{array}{l} LUB = \{a,c\} \\ GLB = \{c\} \end{array}}$

Ques find out LUB and GLB of $B = \{a, b, c\}$ if they exist of the POSET whose diagram is shown.

Sol



Sol. Assume po is divide.

upper bounds : c, d, e

LUB : c

Lower bound : k

GLB = k

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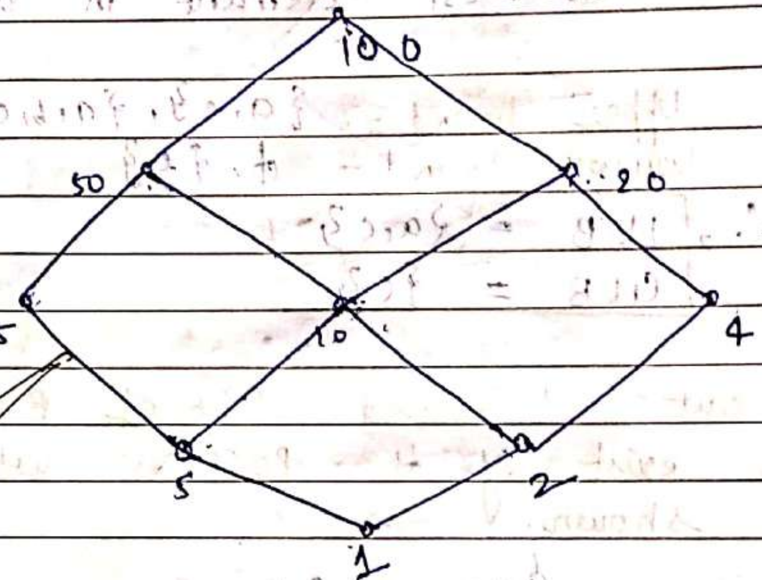
Q. Draw Hasse diagram of $(D_{100}, '|')$ Determine LUB and GLB of B & C
a) $B = \{10, 20\}$
b) $C = \{5, 10, 20, 25\}$

$A = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$



a) upper bounds: 20, 100   LUB = 20
lower bounds: 1, 2, 5, 10   GLB = 10

b) upper bounds: 100.   LUB = 100
Lower bounds: 1, 5.   GLB = 5.

Lattice

A lattice is a POSET $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound (GLB) and a least upper bound (LUB) and they are unique.

Note: If a Greatest lower bounds & least upper bounds exist for every pair then lattices and they must be unique.

This is a lattice ∵ every pair of it has lower & upper bounds

Note: 1) Lowest Upper bound:

Join, LUB, $a \cup b$, Sup (a, b)

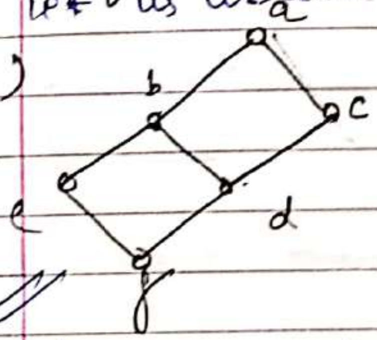
2) Greatest ~~upp~~ lower bound:

Meet, GLB, $a \cap b$, Inf (a, b)

Ques find which POSET is lattice:

let us assume p.o. is "$\leq$".

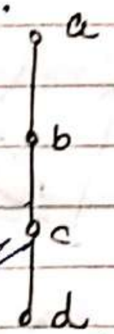
(1)



for every pair LB & UB exists.

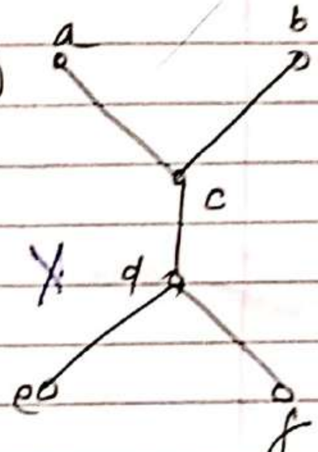
$\Rightarrow$ Lattice.

(2)



Lattice.

(3)

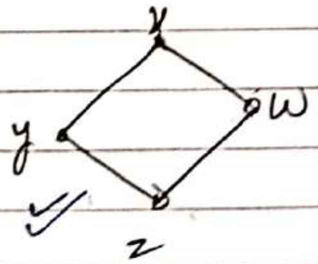


$(e, f) \rightarrow$ upper bounds a, b, c, d

but LB doesn't exist

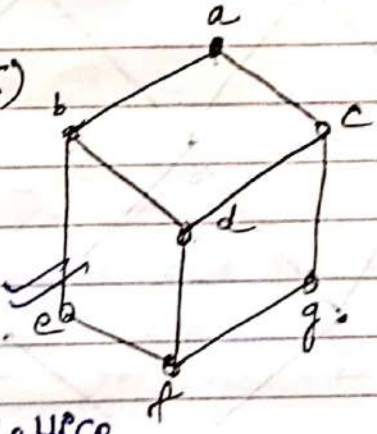
Not a lattice.

(4)



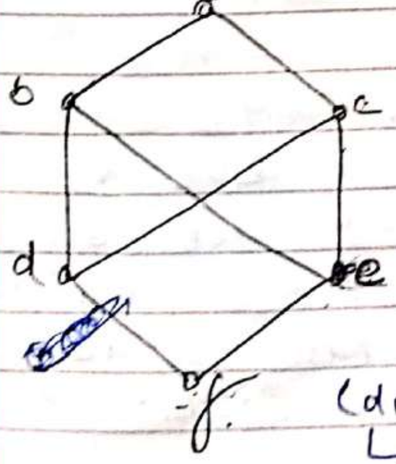
Lattice

(5)



lattice

(6)



lower bound = {f}

GLB = f.

(d, e).

$\hookrightarrow$ upper bound = {a, b, c}

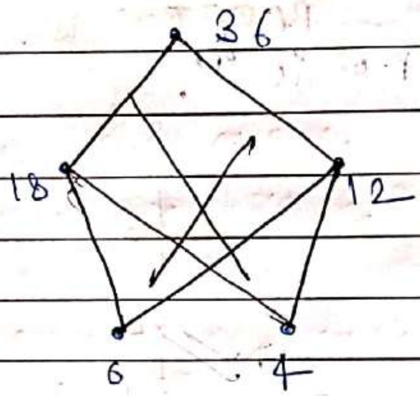
but least U bound does not exist

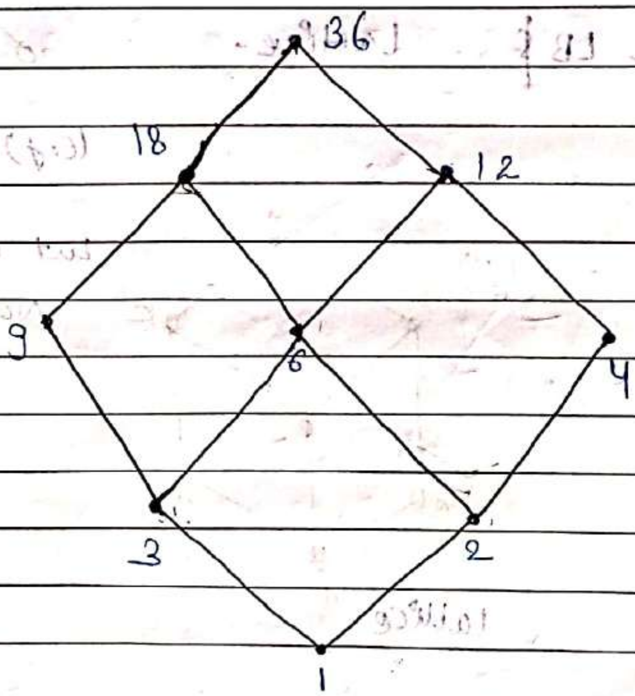
as they are 2 same of same priority.

So, not a lattice.

Ques Check, whether this is lattice or not
$(D_{36}, '|')$

Sol $A = \{36, 12, 9, 6, 4, 3, 2, 1, 18\}$





Let us take a pair (6, 9)
upper bound = 18, 36 ⇒ LUB = 18
lower bound = 3, 1 ⇒ GUB = 3
ie all the pair has LUB and GUB in this hasse diagram
So, it is a lattice.

Note:- $(L, \cup, \cap)$ — $L \to$ set, $\cup \to$ LUB, $\cap \to$ GLB

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GREATEST ELEMENT (1)

Let $(P, \leq)$ be a POSET and element $x \in P$ is called greatest element of P if $\forall\ a \in P, a \leq x$, where '$\leq$' ~~be~~ is the given partial ordering.

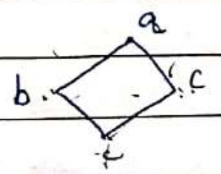
LEAST Element (0)

Let $(P, \leq)$ be a partially ordered set (POSET) and an element $y \in P$ is called least element of P if $\forall\ a \in P$, $y \leq a$.

Bounded Lattice

A lattice with ~~leas~~ LUB & GLB $(L, \cup, \cap)$ is set to be bounded if it has a greatest element as well as a least element

Ex- $A = \{\alpha, \beta, \gamma\}$ $\{P(A), \subset\}$



Eg The set of +ve integers under binary relation $(\leq)$ is not a bounded lattice.

* $\boxed{\text{Infact every finite lattice is bounded.}}$

$\boxed{\text{lattice but not bounded}}$

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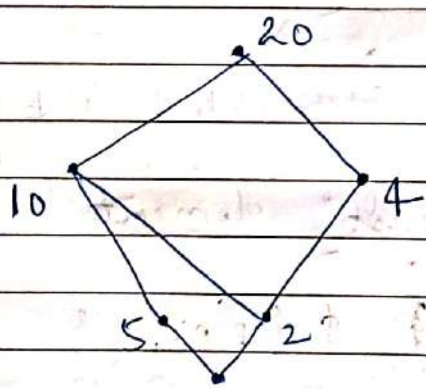
Complement:

In a bounded lattice $\langle L, \leq \rangle$ or $\langle L, \cup, \cap \rangle$ with greatest element (1) and least element (0) then for the element $x, y \in L$ element y is called complement of x iff

$$GLB(x,y) \text{ or } (x \cap y) = 0 \quad \text{(least element)}$$
$$LUB(x,y) \text{ or } (x \cup y) = 1 \quad \text{(Greatest element)}$$

Ques. find complement of each element of in the lattice $\langle D_{20}, | \rangle$

Sol: $A = \{20, 10, 5, 4, 2, 1\}$



To find comp. of 1:

~~Let us take~~

LB of (1,20) is 1 so, GLB = 1 ← least element

UB of (1,20) is 20 so, LUB = 20 ← greatest element

So, 1 is complement of 20
20 " " " 1.

To find comp. of 4:

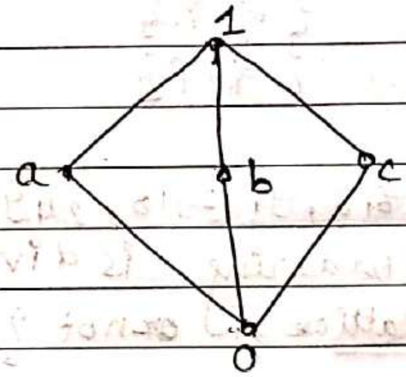
For comp of 4 is 5 means comp of 5 is 4

∵ UB of (4,5) is 20 ⇒ LUB = 20 Greatest element

LB " " " ⇒ GLB = 1 least element

Both condition satisfies in this case.
i.e. LUB is a greatest element and GLB is a least element.

Ques. find complement of a?



→ (a, b)
UB = 1 ⇒ LUB is 1 (greatest element)
LB = 0 ⇒ GLB is 0 (least element)

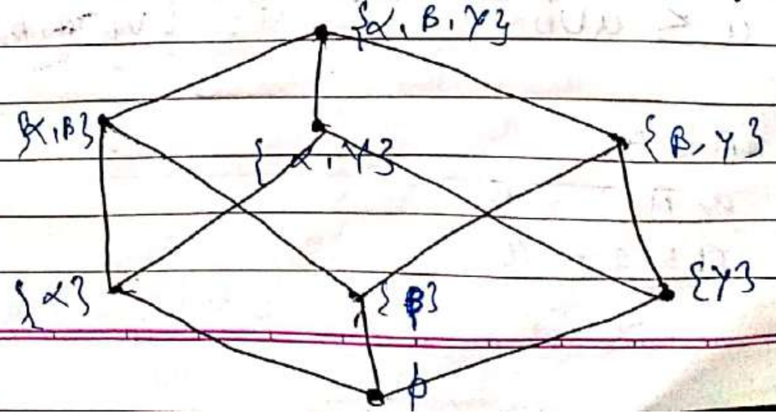
It is possible that an element may have more than one complement or no complement

Complemented Lattice

A Lattice (L, ∪, ∩) is said to be a complemented lattice if 'L' is bounded and every element should have a complement.

Ques Show that POSET < P(X), ⊆ > where X = {α, β, γ} is a complemented lattice?

Sol.



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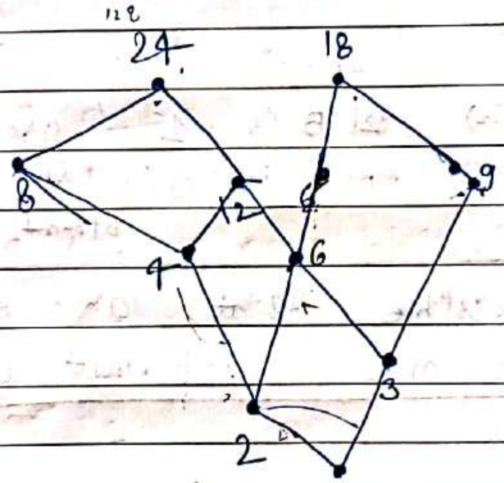
Sol

S.No.	element	Complement	Verification
1	ϕ	$\{\alpha, \beta, \gamma\}$	LUB $= \{\alpha, \beta, \gamma\}$ & GLB $= \{\phi\}$
2	α	$\{\beta, \gamma\}$	LUB $(\alpha, \{\beta, \gamma\}) = \{1\}$ & GLB $= 0$
3	β	$\{\alpha, \gamma\}$	
4	γ	$\{\alpha, \beta\}$	

Ques Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$
Partial ordering is divide whether it is a lattice or not?

Soln



$(4, 9) \rightarrow$ upper bound doesn't exist so, LUB also doesn't exist
$\rightarrow$ lower bound is 1.
So, it is not a lattice $\because$ bcz every pair of lattice must have LUB and GLB.

Some Properties of lattice

Let $\langle L, \leq \rangle$ be a lattice, then $\forall a, b \in L$.

(i) $a \leq a \cup b \Rightarrow a \leq LUB(a, b)$

(ii) $a \cap b \leq a$

(iii) $a \cap a = a$
$a \cup a = a$

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(iv) $a \cup b = b \cup a$

$a \cap b = b \cap a$

(v) Associative law-

(1) $\boxed{a \cap (b \cap c) = (a \cap b) \cap c}$

Let $x = a \cap (b \cap c)$ and $y = (a \cap b) \cap c$

Let $x = a \cap (b \cap c) \Rightarrow x \leq a, x \leq (b \cap c)$

~~GLB~~ GLB GLB $\Rightarrow x \leq a, x \leq b, x \leq c$

$\Rightarrow x \leq a \cap b, x \leq c$

$\Rightarrow x \leq (a \cap b) \cap c$

$\Rightarrow x \leq y$ ---- (1)

Let $y = (a \cap b) \cap c \Rightarrow y \leq (a \cap b), y \leq c$

$\Rightarrow y \leq a, y \leq b, y \leq c$

$\Rightarrow y \leq a, y \leq (b \cap c)$

$\Rightarrow y \leq a \cap (b \cap c)$

$\Rightarrow y \leq x$ --- (2)

from eqn (1) and eqn (2) we can say that $\boxed{x = y}$

ie $\boxed{a \cap (b \cap c) = (a \cap b) \cap c}$

(2) $\boxed{a \cup (b \cup c) = (a \cup b) \cup c}$

Proof:-

Let $x = a \cup (b \cup c)$ and $y = (a \cup b) \cup c$

let $x = a \cup (b \cup c) \Rightarrow x \leq a, x \leq (b \cup c)$

$\Rightarrow x \leq a, x$

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Isomorphic POSET

Let $<P, \leq>$ and $<P_1, \leq_1>$ be two POSET and let $f: P \to P_1$ be a bijective mapping (one to one onto) b/w P and P_1 the function f is called isomorphic from $<P, \leq>$ to the POSET $<P_1, \leq_1>$ then we say if $\forall (a, b) \in P$

$\Rightarrow \boxed{f(a) \cdot (b) \leq P_1}$

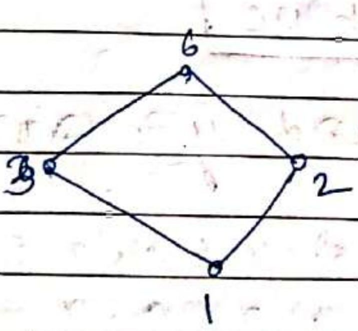
For example

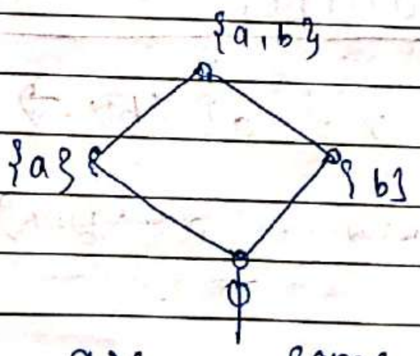
Let $A = \{1, 2, 3, 6\}$, '1' $\}$, $B = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Sh. that $<A, '1'>$, $<B ; \subseteq>$ is isomorphic

inclusion relation

Sol





Both of the Hasse diagrams are same so isomorphic

ie.

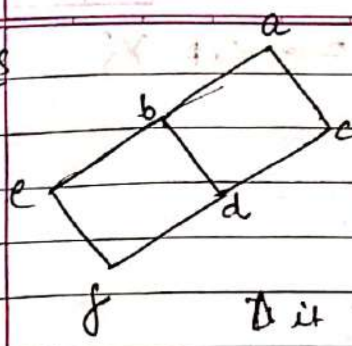
$f(1) = \phi$

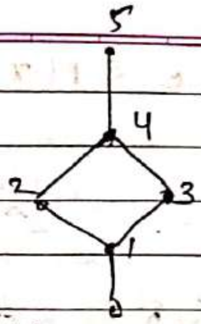
$f(3) = \{a\}$

$f(2) = \{b\}$

$f(6) = \{a, b\}$

Ques





Is it isomorphic?

No, because Hasse diagrams are different.

Distributive lattice (Every chain is a distributive lattice)

A lattice is called a distributive lattice if $\forall$ element (a, b, c) of lattice L it satisfies the following distributive properties.

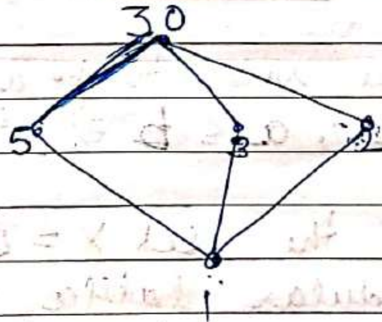
i.e. (i) $a \overset{GLB}{\cap} (b \overset{LUB}{\cup} c) = (a \cap b) \cup (a \cap c) \quad \forall\ a, b, c \in L$

(ii) $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

Ques Check whether it is a D-lattice or not under the partial order relation 'divide'.

$L = \{1, 2, 3, 5, 30\}$

Sol



Let a pair $(2, 3, 5)$

LHS $2 \cap (3 \cup 5)$ RHS $(2 \cap 3) \cup (2 \cap 5)$

$2 \cap 30$ $1 \cup 1$

2 $= 1$

LHS $\neq$ RHS

So, it is not a distr. lattice.

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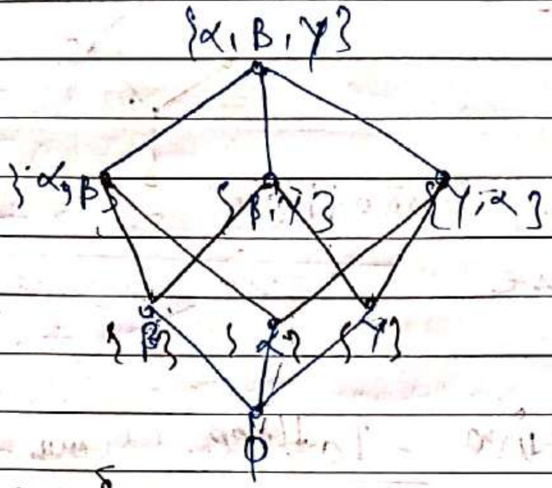
Ques The lattice $\langle P(X), \subseteq \rangle$, $X = \{\alpha, \beta, \gamma\}$

Sol



Let $\{\alpha, \beta, \gamma\}$

LHS

$\{\alpha\} \cap (\{\beta\} \cup \{\gamma\})$

$= \{\alpha\} \cap \{\beta, \gamma\}$ $\because$ LUB of $\{\beta\}$ & $\{\gamma\}$ is $\{\beta, \gamma\}$

$= \phi$

RHS

$(\{\alpha\} \cap \{\beta\}) \cup (\{\alpha\} \cap \{\gamma\})$

$\phi \cup \phi$

$= \phi$

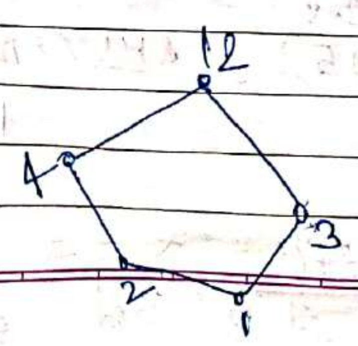
$\Rightarrow$ It is a distr. lattice

Modular lattice

A lattice is said to be a modular if $\forall (a, b, c) \in L$, $a \leq b \Rightarrow a \cup (b \cap c) = (a \cup b) \cap c$

Ques Whether the set $X = \{1, 2, 3, 4, 12, '1'\}$ Is it a modular lattice or not?

Sol



Let $\{1, 2, 3\}$

$a \leq b = \underline{1 \leq 2}$

LHS

$a \cup (b \cap c) = 1 \cup (2 \cap 3)$

$= 1 \cup 1$

$= 1$

RHS $(a \cup b) \cap c$

$(1 \cup 2) \cap 3$

$2 \cap 3$

3

LHS $\neq$ RHS $\Rightarrow$ this is not a modular lattice.

Algebraic Structure

1) **Semi group-** Let G be a non empty set and * be a binary operation on G then $(G, *)$ is called a semigroup if it satisfies the following postulates-

1st postulate

a) Closure property:-

$$\boxed{\forall a, b \in G} \Rightarrow \boxed{a * b \in G}$$

2nd postulate:-

b) Associative law :-

$\forall a, b, c \in G$

$$\boxed{(a * b) * c = a * (b * c)}$$

Note :- If these both conditions are satisfied then G is a semigroup w.r.t binary ope. '*'.

Ex- $(G, *)$

$(N, +)$ ✓ $\qquad (N, -)$ ✗

Natural numbers $\qquad$ ∵ the subtraction of 2 numbers is not always a natural number.

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Monoid

→ (1) and (2) condition should be satisfied.

(3) Existence of identity element :-

$\forall a \in G$

$$\boxed{a + e = a = e + a}$$

Ex- $(R, +)$ ✓

(1),(2) conditions satisfied.

(3) → $a + 0 = a = 0 + a$ → 3rd cond. " ∵ identity element exists.

(4) Existence of Inverse :-

$\forall a \in G$

$$\boxed{a * a^{-1} = e = a^{-1} * a}$$

Identity element

Ex – $(R, +)$ → (1), (2), (3) cond. satisfied ✓

(4) → $a + (-a) = 0 = (-a) + a$ ← (4) also satisfied.

'5' Abelian of Commutative group

$\forall a, b \in G$

$$\boxed{a + b = b + a}$$

Ques Show that set of integer Z form an abelian wrt addition of integers.

Sol 1) closure property :-

Since the sum of 2 integers is also an integer, the set Z is closed wrt addition

i.e.

$$\boxed{\forall a, b \in Z \Rightarrow a + b \in Z}$$

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2. Associative law :-
We know that set of integers satisfied associative law w.r.t sum of integers.
i.e.
$$\boxed{(a+b)+c = a+(b+c) \; ; \; \forall\, a,b,c \in Z}$$

3. Existance of identity :-
The integer 0 belongs to Z is the identity element as for all $(\forall)$ $a \in Z$
$$\boxed{a+0 = a = 0+a}$$

4. Existance of inverse :-
$\forall\, a \in Z$, there exists $(-a) \in Z$ such that
$$\boxed{a+(-a) = 0 = (-a)+a}$$

5) Commutative law :-
As we know sum of 2 integers is always commutative,
i.e. $\forall\, a, b \in G$
$$\boxed{a+b = b+a}$$

Ques. P. that the set of matrices A_α

i.e. $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ where α is a real number

form an ∞ abelian group under multiplication.

Sol
1) ~~Closure property~~
Let G be the set of matrices $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

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Closure property: Let $A_\alpha, A_\beta \in G$

Now,

i.e. $A_\alpha * A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$A_\alpha * A_\beta = \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\beta\cos\alpha - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$

$$\boxed{A_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}}$$

from this we can say,

$$\boxed{\alpha, \beta \in R \Rightarrow \alpha+\beta \in R}$$

$\therefore$, $\boxed{\forall A_\alpha, A_\beta \in G \Rightarrow A_\alpha * A_\beta \in G}$

(b) Associative law

Multiplication of 2×2 matrices is always associative.

(3) Existance of identity element

$\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

This is an identity matrix and we multiply any matrix with this than the matrix will remain same

$\therefore$, 0 is an identity element.

ie $\boxed{a + 0 = 0 = 0 + a}$

(4) Existance of inverse

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A(-\alpha) = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Now,
$$A\alpha * A(-\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cancel{\cos\alpha} - \sin\alpha\cancel{\cos\alpha} \\ \sin\alpha\cancel{\cos\alpha} - \sin\alpha\cancel{\cos\alpha} & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Que which of the following table define a semigroup or a monoid on A ?

$A = \{a, b\}$

(1)

*	a	b
a	b	a
b	a	b

✓ semigroup and monoid

(2)

*	a	b
a	b	b
b	a	a

→ semigroup ✓ not a monoid ✗

Sol^n (1)

1) ~~closure~~ closure :- $\forall a, b \in A$

$a * b \in A$

So, ~~it is a~~ closure prop. satisfied

(2) Associative

$a * b = a * b$

So, associative ✓

(3) Identity element

$a * b = a$

$b * b = b$

$\Rightarrow \boxed{a * b = a = b * a}$

and

$\boxed{\cancel{b} * b = b = b * b}$ ✓

So, identity element is b.

All the conditions are satisfied so ~~the~~ table (1) is a ~~sem~~ monoid as well as a semigroup

(2) a) Closure :-

$\forall a, b \in A$

$a * b \in A$

So, closed,

b) Associative :-

associative

c) Identity element :- doesnot exist

So, it is a semigroup only not a monoid

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Properties of groups

1.) The identity element in a group is unique.
Let e and e_1 be two identity element of group then

if e is $ae = a = ea$ --- (1)
if e_1 is $ae_1 = a = e_1 a$ --- (2)

$$\cancel{a} e = \cancel{a} e_1$$

$$\Rightarrow \boxed{e = e_1} \quad \text{[left cancellation law]}$$

2) The inverse of every element of a group is unique.
Let $a \in G$, let b and c are two inverse of a then

if $a \cdot b = e = b \cdot a$ — (1)
$a \cdot c = e = c \cdot a$ — (2)

$$a \cdot b = a \cdot c$$

$$\Rightarrow \boxed{b = c} \quad \text{[left cancellation law]}$$

3.) Let G be a group then $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

any composition

Let $a, b \in G$ and a^{-1}, b^{-1} are the inverse of a and b respectively.

$$a \cdot a^{-1} = e = a^{-1} \cdot a$$
$$b \cdot b^{-1} = e = b^{-1} \cdot b$$

Now,
$$(a \cdot b)(b^{-1} \cdot a^{-1}) = a \cdot (b b^{-1}) a^{-1} \quad \text{[Associative law]}$$
$$= (a \cdot e) \cdot a^{-1}$$
$$= a \cdot a^{-1}$$
$$= e$$

Now,
$$(b^{-1} \cdot a^{-1})(a \cdot b) = b^{-1} (a^{-1} a) b \quad \text{[by Ass. law]}$$
$$= b^{-1} e b$$

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$= b^{-1} b$

$= e$

$\Rightarrow (a.b)(b^{-1}.a^{-1}) = e = (b^{-1}.a^{-1})(a.b)$

$\Rightarrow \boxed{(a \cdot b)^{-1} = b^{-1}a^{-1}}$

Ques If $(G, *)$ is an abelian group then show that $(a*b)^3 = b^3 * a^3$.

$(a*b)^2 = b^2 * a^2$

Soln $G = [\infty, \dots -5, -3, -2, -1, 0, 1, 2, 3 \dots]$

and composition is addition

Sol To check:

Let us take any pair $(2,3)$

closure

we know, $(2+3)^2 = (2+3) + (2+3)$ [$\because$ composition is addn]

$\Rightarrow (2+3) + (2+3) = (2+2) + (3+3)$

$10 = 10.$

Ques Show that any group of elements less than or equals to 3 must be abelian?

Sol

1) Let G be a group having only one element. $\therefore$, there should be an identity element as we are considering that it is a group.

$G = \{e\}$

$e \cdot e = e = e.e$ [$\because$ comm. is a abelian]

2) Assume that we have 2 elements in a group

i.e. $G = \{e, a\}$

where e is the identity element of the group.

$\therefore$, $\boxed{ea = a = ae}$

$a^2 = a * a$ ✗ $a^2 = a$ composition a

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$\because$ commutative so abelian.

3.) ~~Let~~ Assume that we have 3 elements in a group.

ie $G = \{e, a, b\}$

$\cdot$	e	a	b
e	e	a	b
a	a	a^2	$a \cdot b$
b	b	$b a$	b^2

$b^2 = e$, $b \cdot a = e$, $b^2 = a$

There should be an atleast one in a row and column also.

So,

In ~~first~~ second row -

either $a^2 = e$ or $a \cdot b = e$

In third row - $ba = e$ or $b^2 = a$

Let $a^2 = e$ it means $a \cdot b = b$.

~~and~~ but if $a \cdot b = b \Rightarrow a = e$ [which is not possible]

$\Rightarrow \boxed{ab = e, \ a^2 = b}$

So, comp. table should look like this:

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	ba	b^2

again we know every element should exist once in one row and column :-

So, comp. table should be like this:-

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$$\boxed{\begin{matrix} ba = e \\ b^2 = a \end{matrix}}$$

Sub Groups

A ~~[illegible]~~ non-empty subset H of a group G is called subgroup of G if H is also a group for the composition in G.

Let $G = \{-\infty \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \infty\}$

take a subset $H = \{0, 1, 2, 3, \ldots \infty\}$

$\downarrow$

inverse of any element doesn't exist

So, H is not a group wrt addition

Consider any another example where H is a subset of G and also a group.

Note:- The identity element of the subgroup H is same as that of G.

Proof:-

Assume e and e' are identity element of Group G and Group H respectively.

$\forall \; a \in H \Rightarrow \boxed{a \cdot e' = a = e' \cdot a}$ —(1)

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Now, $\boxed{\forall a \in H \Rightarrow a \in G}$ $\because$ $\boxed{H \subseteq G}$
Subset

$\Rightarrow \boxed{a \cdot e = a = e \cdot a}$ — (2)

from eqn (1) & (2)

$a \cdot e' = a \cdot e$

$\Rightarrow \boxed{e' = e}$ [By left cancellation law]

Theorem-1

The non-empty Subset H of a group G is a subgroup of G if and only if

(1) $a \in H, b \in H \Rightarrow a \cdot b \in H$

(2) $a \in H \Rightarrow a^{-1} \in H$

where a^{-1} is the Inverse of a in G.

Necessary condn :-

1) Let H be a subgroup of G then it must be closed wrt a given composition in G. $\Rightarrow$ $\boxed{a \in H, b \in H \Rightarrow a \cdot b \in H}$

2) if H is a subgroup of G then every element in H should have its inverse wrt a comp. in G.

$\Rightarrow \boxed{a \in H \Rightarrow a^{-1} \in H}$

Sufficient condn :-

1) Let H is a non empty subset of G satisfying the condition (1) and (2).

Now, we have to prove that H is a subgroup of G.

(1) Closure property:
from condition (1) it is clearly ~~said~~ seen that it is closed wrt any composition

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(2) **Associative:**

$\forall a, b, c \in H \Rightarrow a, b, c \in G$

$\Rightarrow a(bc) = (ab)c$ [$\because$ G is a group]

(3) ~~identity element:~~

Existance of Inverse:-

from cond^n (2)

if $a \in H \Rightarrow a^{-1} \in H$

So, inverse of every element exists.

(4) Existance of identity element

We know, $a \in H \Rightarrow a^{-1} \in H$ [cond 2]

Now,

$a \in H,\ a^{-1} \in H$

$\Rightarrow a \cdot a^{-1} \in H$ [using cond 1]

$\Rightarrow \boxed{e \in H}$

Previous

Ques. Show that the set 4th root of unity mainly $(1, -1, i, -i)$ from an abelian group wrt multiplication.

Sol

understanding

$x = 1^{1/4} \longrightarrow$ fourth unity

$x^4 = 1$

$(x^4 - 1) = 0$

$(x^2+1)(x^2-1) = 0$

$x^2 + 1 = 0 \Rightarrow \boxed{x = \pm i}$

$x^2 - 1 = 0 \Rightarrow \boxed{x = \pm 1}$

X	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	+i
i	i	−i	−1	1
−i	−i	i	1	−1

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1) Closure property :- Since all the entries in composition table are the element of the set G so it is closed.

2) Associative property :-
Multiplication of complex number is always associative.

3) Existance of identity element :-
from the first row of composition table it is clear that identity element exist i.e. 1.

4) Inverse :-

$$a * a^{-1} = e$$
$$1^{-1} = 1$$
$$(-1)^{-1} = -1$$
$$(i)^{-1} = -i$$
$$(-i)^{-1} = i$$

5) Commutative group :-
Multiplication of complex number is always commutative.

HW
Ques Show that the set of cube root of unity is an abelian group w.r.t multiplication

Sol

$1, \omega, \omega^2$

$$x = 1^{1/3}$$
$$\Rightarrow x^3 = 1$$
$$(x^3 - 1) = 0$$
$$(x-1)(x^2 + x + 1) = 0$$
$$x - 1 = 0 \Rightarrow \boxed{x = 1}$$
$$x^2 + x + 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

X	1	$\frac{-1+\sqrt{3}i}{2}$	$\frac{-1-\sqrt{3}i}{2}$
1	1	$\frac{-1+\sqrt{3}i}{2}$	$\frac{-1-\sqrt{3}i}{2}$
$\frac{-1+\sqrt{3}i}{2}$			
$\frac{-1-\sqrt{3}i}{2}$			

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Order of a finite group

The no. of elements in a finite group is called order of the group. and we denote it by this symbol '$O(G)$'.

Ques Pv- that the set $G = \{0,1,2,3,4,5\}$ is a finite group of order 6 wrt addition modulo six as the composition in the set G.

Sol eg $2 +_6 7 = 3$ $(2+7 = (9 \% 6) = 3)$
$2 +_6 5 = 1$ $(2+5 = (7 \% 6) = 1)$

from truth table:

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(1) Closure property:-
Since all the element in the composition table are of the set G so it is closed wrt addition modulo 6.

(2) Associative:-
Multiplication of Addition modulo is always associative.

(3) Existance of Identity:-
from first column it is clear that $\boxed{e=0}$
So identity element exist.

(4) Inverse:-

$(0)^{-1} = 0$

$(1)^{-1} = 5$

$(2)^{-1} = 4$

$(3)^{-1} = 3$

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$(4)^{-1} = 2$

$(5)^{-1} = 1$

$\therefore$ inverse exist.

Que $(\{a, b\}, *)$ be the semigroup where $a*a = b$. Show that $a*b = b*a$.

Sol $\cancel{(a*b)*c = a*(b*c)}$ → Given

$$a*b = a*(a*a) \quad \text{[}\because \text{ it is a semigroup}$$
$$= (a*a)*a$$
$$= b*a$$

H.P.

Order of an element of a group

Let G be a group and we denote the composition by multiplication and $(a \in G)$ is arbitrary element of the group G by the order of the element $(a)\ o(a)$.

also mean the least +ve integer n (if exist) such that $\boxed{a^n = e}$ where e is the identity element in G.

if there doesn't exist an integer n satisfying $\boxed{a^n = e}$ then we say that the element a is of infinite order. The order of the element a of a group is denoted by '$O(a)$'.

→ If any group G then the identity element order of identity element is always 1.

Ques find the order of all element of the multiplication group

$$G = \{1, -1, i, -i\}$$

Sol $o(1) = 1$

$(-1)^1 = -1, (-1)^2 = 1 \to e$

$\Rightarrow o(-1) = 2.$

$(i)^4 = 1$

$\Rightarrow O(i) = 4$

$O(-i) = 4$

Ques find the order of each element of the group $G = \{0, 1, 2, 3, 4, 5\}$

find order of all element of group G

composition $\rightarrow$ addition modulo 6.

So so from power ques

$\boxed{e = 0}$

$\Rightarrow$ ① $\boxed{O(0) = 1}$ $\because 0^1 = 0$

② $(1)^1 = 1$

$(1)^2 = 1 +_6 1 = 2$

$(1)^3 = 1 +_6 1 +_6 1 = 3$

$(1)^5 = 1 +_6 1 +_6 1 +_6 1 +_6 1 = 5$

$(1)^6 = 1 +_6 1 +_6 1 +_6 1 +_6 1 +_6 1 = 0$

$\Rightarrow \boxed{O(1) = 6}$

③ $2^1 = 2$

$2^2 = 2 +_6 2 = 4$

$2^3 = 2 +_6 2 +_6 2 = 0$

$\boxed{O(2) = 3}$

④ $4^1 = 4$

$4^2 = 4 +_6 4 = 2$

$4^3 = 4 +_6 4 +_6 4 = 0$

$\boxed{O(4) = 3}$

$5^1 = 5$

$5^2 = 5 +_6 5 = 4$

$5^3 = 5 +_6 5 +_6 5 = 3$

$5^4 = 5 +_6 5 +_6 5 +_6 5 = 2$

$5^5 = 5 +_6 5 +_6 5 +_6 5 +_6 5 = 1$

$5^6 = 5 +_6 5 +_6 5 +_6 5 +_6 5 +_6 5 = 0$

$\Rightarrow \boxed{O(5) = 6}$

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Ques Let $(G, *)$ is an abelian group then s.t. that $(a*b)^2 = a^2 * b^2$

Sol $(a*b)^2 = (a*b) * (a*b)$

$= a * (b*a) * b$ [∵ it is associative group]

$= a * (a*b) * b$ [∵ abelian so comm. law hold]

$= (a*a) * (b*b)$ [associative]

$\boxed{(a*b)^2 = a^2 * b^2}$ H.P.

Continued

Ques Necessary and sufficient condition for a non-empty set H of a group to be a subgroup is that $\forall a \in H, b \in H \Rightarrow a.b^{-1} \in H$?

Cond^n is necessary:-

Sol ∵ $b \in H \Rightarrow b^{-1} \in H$ [2nd cond^n]

and $\boxed{a \in H,\ b^{-1} \in H \Rightarrow a \cdot b^{-1} \in H}$ [1st cond^n]

Since H is itself a group wrt composition given in G so it must each element of H must have its inverse. Now,

$a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$

$\Rightarrow a.b^{-1} \in H$

Cond^n is sufficient:-

Let H be a non empty subset of G satisfying cond^n $\boxed{a \in H, b \in H \Rightarrow a \cdot b^{-1} \in H}$ --- (i)

Now, we have to prove that H is a subgroup of G.

So,

(1) Identity element exist:

Let $a \in H$, $a \in H \Rightarrow a.a^{-1} \in H$

$\Rightarrow \boxed{e \in H}$

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2) Existance of Inverse:-

$e \in H, a \in H \Rightarrow e a^{-1} \in H$ [using cond^n 1 in suffix ler]

$\Rightarrow a^{-1} \in H$

3) closure property:-

$a \in H$ $\quad$ ~~$e \in H$~~

$b \in H$ $\quad$ ~~$a^{-1} \in H$~~

$\Rightarrow a \in H, b^{-1} \in H$

$\Rightarrow a(b^{-1})^{-1} \in H$ [Using cond 1 in this question]

$\Rightarrow a \cdot b \in H$

4) Associativity

$a, b, c \in H \Rightarrow a, b, c \in G$

$\Rightarrow \boxed{a(bc) = (ab)c}$ [$\because$ G is a group]

Theorem:-

If H_1 and H_2 are 2 subgroup of a group G then $H_1 \cap H_2$ is also a subgroup of G

~~st~~ Both H_1 and H_2 are subgroups so atleast identity element must exist $\therefore$, H_1 and H_2 are non-empty sets.

i.e, $H_1 \cap H_2 \neq \phi$

To prove that it is a subgroup of G we prove that

If $a \in H_1 \cap H_2$, $b \in H_1 \cap H_2 \Rightarrow a \cdot b^{-1} \in H_1 \cap H_2$

$a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

Now,

$a \in H_1, b \in H_1 \Rightarrow a \cdot b^{-1} \in H_1$ [Since H_1 is a

$a \in H_2, b \in H_2 \Rightarrow a \cdot b^{-1} \in H_2$ subgroup of G]

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$\Rightarrow \boxed{(a \cdot b^{-1}) \in H_1 \cap H_2}$

So, ∴, it is a subgroup of G.

Theorem:-

Union of 2 subgroup of a group G is not necessarily a subgroup of G.

Let G be a additive group of integers.

$$G = \{-\infty, \ldots\ldots -2, -1, 0, 1, 2, 3, \ldots\ldots \infty\}$$

Let us find 2 subgroup of G.

i.e.

$$H_1 = \{0, \pm 2, \pm 4, \pm 6, \pm 8 \ldots\}$$

$$H_2 = \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\}$$

Now, $\boxed{H_1 \cup H_2 = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9 \ldots\}}$

Here, 3 is a element of $H_1 \cup H_2$ and 4 is a element of $H_1 \cup H_2$ but $3+4=7$ is not an element of $H_1 \cup H_2$

So, union of 2 subgroup is not necessary a subgroup of G. but intersection is always a subgroup of G.

Cyclic Group

→ A group $(G, *)$ is called cyclic if for $a \in G$, every element $x \in G$ is of the form a^n where n is some integer.

→ The element a is called the generator of the group G. We may have more than one generator of a cyclic group.

If G is a cyclic group generated by a. i.e.

$$\boxed{G = \{a\}}$$

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Ex- $G = \{1, -1, i, -i\}$

this is a cyclic group $\because$ G can be written as

$\boxed{G = \{i, i^2, i^3, i^4\}}$

and G is a cyclic group of i

i.e. $G = \{i\}$

→ Every element of a group can be written in the form of a^n.

and it has 2 generators

1) i and -i.

↳ $\boxed{G = \{-i, (-i)^2, (-i)^3, (-i)^4\}}$

Que. Show that the group $G(\{0,1,2,3,4,5\}, +_6)$ is cyclic? If yes find all generators?

Sol for 1:

$1^1 = 1$

$1^2 = 1 +_6 1 = 2$

$1^3 = 1 +_6 1 +_6 1 = 3$

$1^4 = 1 +_6 1 +_6 1 +_6 1 = 4$

$1^5 = 1 +_6 1 +_6 1 +_6 1 +_6 1 = 5$

$1^6 = 1 +_6 1 +_6 1 +_6 1 +_6 1 +_6 1 = 0$

i.e. $G = \{1^6, 1^1, 1^2, 1^3, 1^4, 1^5\}$

So This is a cyclic group generated by 1.

for 2:

$2^1 = 2$

$2^2 = 2 +_6 2 = 4$

$2^3 = 2 +_6 2 +_6 2 = 0$

$2^4 = 2 +_6 2 +_6 2 +_6 2 = 2$

$2^5 = 2 +_6 2 +_6 2 +_6 2 +_6 2 = 4$

No, $\because$ 4 exist twice.

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and $5^1 = 5$

$5^2 = 4$ $\quad\therefore\quad G = \{5^6, 5^5, 5^4, 5^3, 5^2, 5^1\}$

$5^3 = 3$

$5^4 = 2$

$5^5 = 1$

$5^6 = 0$

So, it has 2 generators i.e. 2 and 5.

Theorem

learn

Every cyclic group is an abelian group.

Let G be a cyclic group generated by a, let $x, y \in G$ arbitrarily. and R and s are some integers, such that

$$x = a^r \text{ and } y = a^s$$

~~As~~ If it is an abelian group commutative property will be holds

$$x \cdot y = a^r \cdot a^s = a^{r+s}$$
$$= a^{s+r}$$
$$= a^s \cdot a^r$$
$$= y \cdot x$$

comm prop. hold $\therefore$ abelian group

learn

Theorem

If a is a generator of a cyclic group G then a^{-1} is also a generator of G.

Let $G = \{a\}$ G is a cyclic group gen by a

Let $a^r \in G$ where r is some integer. be an element of G arbitrarily

If r is an integer, $-r$ is also an integer
Thus,
if any element is written in the form a^r
then any element can be written in the form of $\boxed{a^r = (a^{-1})^{-r}}$

$\therefore$ if a is a generator, then a^{-1} is also a generator

Ques Show that the group $G = (\{1,2,3,4,5,6\}, \times_7)$ and if yes, find generators?

Sol

$1^1 = 1$ $\quad$ $5^1 = 5$
$1^2 = 1 \times_6 1 = 1$ $\quad$ $5^2 = 5 \times_7 5 = 4$
$1^3 = 1 \times_6 1 \times_6 1 = 1$ $\quad$ $5^3 = 5 \times_7 5 \times_7 5 = 6$
$5^4 = 2$
3^1 $\quad$ $5^5 = 3$
3^2 $\quad$ $5^6 = 1$
3^3
3^4
3^5
3^6

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

ie $1^{-1} = 1$ $\quad$ $6^{-1} = 6$
$2^{-1} = 4$
$3^{-1} = 5$
$4^{-1} = 2$
$5^{-1} = 3$

So, we can say that if 3 is a generator then 3^{-1} will also be a generator.
i.e. 5 is also a generator $\because 3^{-1} = 5$

Ques Suppose that G be a additive grp of integer.
$G = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$ and H is a non empty subset of G i.e. $H = \{-1, 0, 1\}$
Check whether H is a subgroup of G or not?

Sol

Ques Pr that any group of ordered 3 is cyclic
The exist exactly 3 elements in the group $G = \{e, a, b\}$ where e is the Identity element of the group?

Sol —

	e	a	b
e	e	a	b
a	a	a^2	ab
b	b	ba	b^2

$a^2 = e$ or $a \cdot b = e$

if $a^2 = e$,
$\Rightarrow a \cdot a = e \Rightarrow a = e$ which is contradiction

$\Rightarrow \boxed{a \cdot b = e}$ --- ①
$\Rightarrow \boxed{b = a^2}$

Group Homomorphism

Let $(G, *)$ and (G, Δ) be any 2 groups. A mapping 'F' from G to G' $(G \to G')$ is called a homomorphism of $G \to G'$ if $f(a*b) = f(a) \Delta f(b)$

for eg.

Let G be a group of integer under addition and $G' = G$

and $f(x) = 3x$ for every $(\forall)$ $x \in G$ then

$$f(x+y) = 3(x+y)$$
$$= 3x + 3y$$
$$= f(x) + f(y)$$

Such a mapping is called group homomorphism.

Isomorphism of Group

Let $(G, *)$ and (G', Δ) be 2 groups. A mapping 'F' from G to G' denoted by $f(a*b) = f(a) \Delta f(b)$ is called Isomorphism if f is one to one onto mapping

for ex- prove that $f_a = G \to G'$ which is defined as. $f_a(x) = a^{-1}xa$ is a group homomorphism

Let $x, y \in G$

$$f_a(xy) = a^{-1}(xy)a \quad \text{[from given cond}^n\text{]}$$
$$= (a^{-1}xa)(a^{-1}ya)$$
$$= f_a(x) f_a(y)$$

$\therefore$, it is a group homo

Ques Let R be a addition group of Real no's and R+ the multiplicative group of the Real numbers. Show that $f: R \to R^+$ defined by $f(x) = e^x \; \forall x \in R$ is a group homomorphism?

Sol Let $x_1, x_2 \in R$

$$f(x_1 + x_2) = e^{(x_1 + x_2)}$$
$$= e^{x_1} \cdot e^{x_2}$$
$$= f(x_1) \cdot f(x_2)$$

$\Rightarrow \boxed{f(x_1 + x_2) = f(x_1) \cdot f(x_2)}$ So this is an homomorphism.

Ring

An algebraic structure $(R, +, \cdot)$ is called a Ring. If the binary operation addition and multiplication satisfies the following properties-

1) $(R, +)$ is an abelian group.
2) $(R, \cdot)$ is an semi group.
3) The operation $(\cdot)$ is distributive over addition, for $\forall a, b, c \in R$

i.e. $a \cdot (b + c) = ab + ac$
$= (b + c) \cdot a$

For ex-
the set of integers wrt to addition and multiplication is a ring

Type of Rings

1) If in a ring R there exist an element denoted by 1 such that

$$\boxed{1 \cdot a = a = a \cdot 1}$$

$\forall a \in R$ then R is called Ring with unique element the element $1 \in R$ is called unique element of the ring.

2) Commutative Ring OR

The ring R is said to be commutative ring or abelian Ring if it satisfies the commutative law.

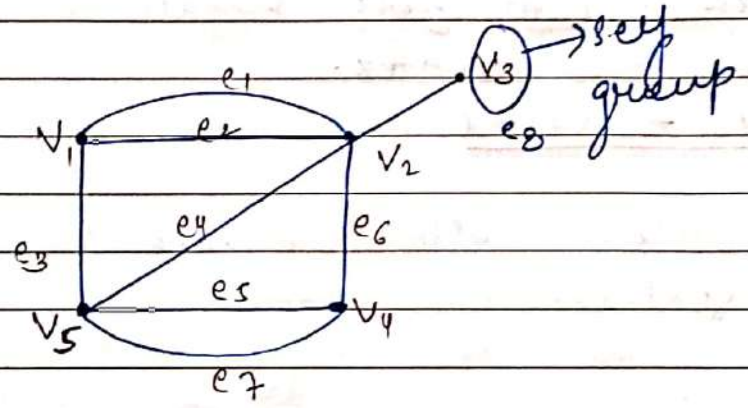
$$\boxed{a \cdot b = b \cdot a \quad \forall a, b \in R}$$

3) Ring without unity

A ring R which doesn't contain multiplicati identity is called a Ring without unity

Graph theory

A graph $G = (V, E)$ consist of a finite set denoted by G or $V(G)$ and $E(G)$ of ordered pair (u, v)
Each element of V is called a vertex and each element
Each edge



An edge associated with vertex pair $(V_i, V$

Parallel edges

If 2 or more edges in a graph G are associated with a given pair say (V_i, V_j) then the edges are referred as parallel edges.

Eg → 1) e_1 and e_2 are ‖ edges
∵ it has same end vertex
also 2) e_5 and e_7 are ‖ edges
same Reason

Self loop

No of edges incident ~~of~~ on a vertex v is called Degree of a vertex. Each self loop is counted twice. Degree of a vertex is always +ve and denoted by d(v).

$d(v_3) = 3$

$d(v_5 = 4)$

Isolated vertex

Vertex with zero degree is called isolated vertex.

Pendant vertex

A vertex with degree 1 is called pendant vertex.

Theorem

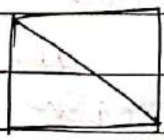
The sum of degree of all vertices in a graph G is twice the number of edges.

Proof – Since each edge contribute 2 degree so total degree of graph is always twice the number of edges.

Simple graph

A graph without self group and parallel edges is called simple graph

Ex –



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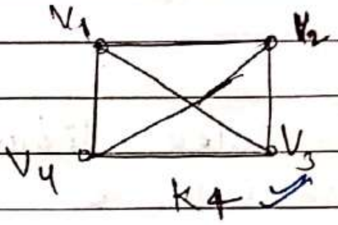
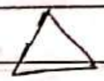
Complete graph / Full graph

→ If in a graph there exist an edge b/w each and every pair of vertices, then the graph is called complete/full graph.

→ A complete graph of n vertices is denoted by K_n.

Eg — 

, 

Ques A graph G has 21 edges, 3 vertices of degree 4 and other vertices are of degree 3. find the number of vertices in G?

Sol

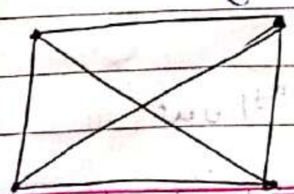
Using theorem,

$$3 \times 4 + (n-3)3 = 42$$

$$\boxed{n = 13}$$

Regular graph

A simple graph is said to be regular if all vertices of graph G are of equal degree. if each vertex having degree R. then G is said to be regular of degree R or simply R regular graph.



degree of all the vertices is 3 so, this is 3-regular graph.

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Theorem

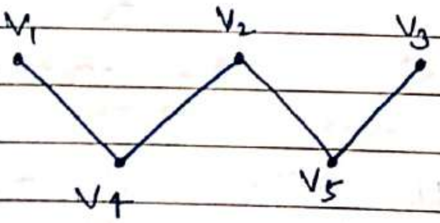
Prove that the no. of vertices having odd degree in a Graph G is always even.

Proof :- $\sum_{odd} v_k + \sum_{even} d_i = 2e$

Bipartite Graph

A graph G is called bipartile graph if its Vertex set (V, G) can be partition into two non empty disjoint subset $V_1 G$ and $V_2 G$) in such a way each edge belongs to E(G) i.e. $e \in E(G)$ has its one end point (vertex) in $V_1 G$ and other end point in $V_2 G$, the partion V $\boxed{V = V_1 \cup V_2}$ is called a bipartion in G.

Eg -

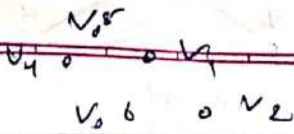


One end point is in the side v_1, v_2, v_3 and other end point are in the side v_4, v_5 so bipartion Graph.

$V_1 = \{v_1, v_2, v_3\}$
$V_2 = \{v_4, v_5\}$.

Null Graph

A graph without any edge is called null graph.



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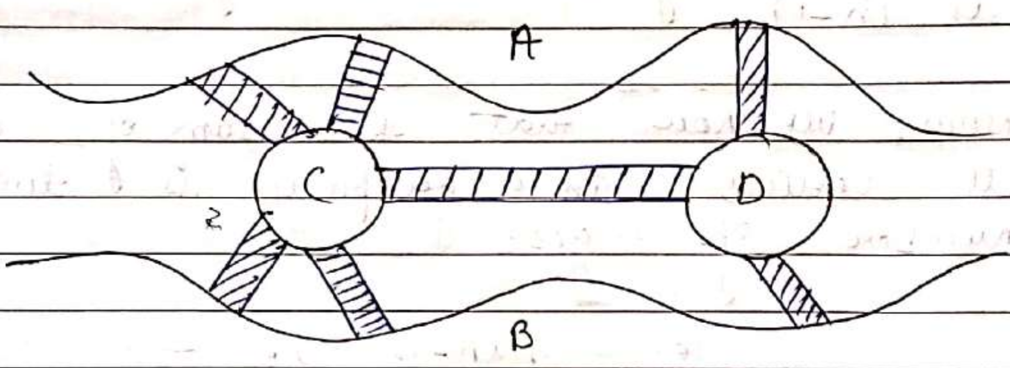
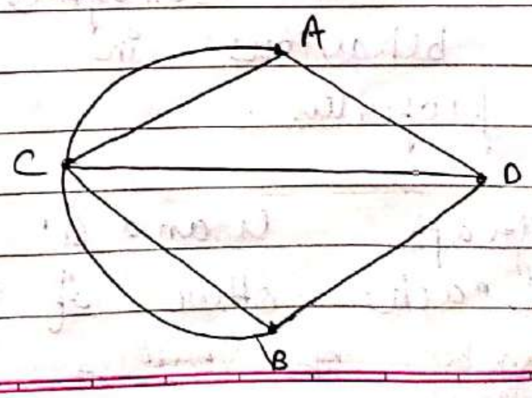
Konigsberg Bridge Problem

Graph theory was born in 1736 with Euler's paper in which he solved a very long awaiting problem known as Konigsberg bridge problem.

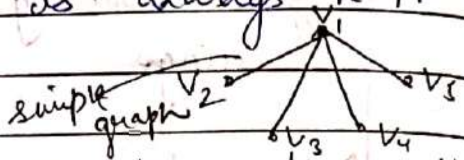
The problem was





Ques Show the max^m no. of edges in a simple graph with n Vertices is $\frac{1}{2} n(n-1)$.

max^m degree of a vertex in a simple graph with n vertices is always n-1.



So The max^m degree of a simple graph of n vertices is $(n-1)$.

Now, we know that the sum of degree of all vertices in a graph G is twice the number of edges.

ie. $n(n-1) = 2e$

$$\Rightarrow \boxed{e = \frac{1}{2} n(n-1)}$$

→ max no. of edges

Path and Circuits

1) Isomorphism :-

In geometry, to 2 figures are equivalent are called congruent if they have identical behaviour in terms of geometric properties likewise the 2 graphs are thought of equivalent (and called isomorphic) if they have identical behaviours in terms of graph theoretical properties.

The 2 graph G and G' are said to be isomorphic to each other if graphs must had

1) the same number of vertices

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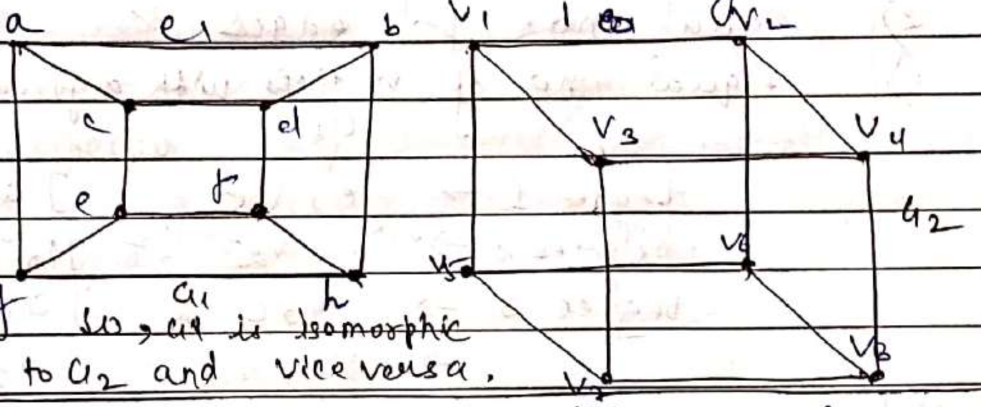
2) the Same number of edges
3) an equal no. of vertices with a given degree
4) if there is one to one correspondence b/w their vertices & b/w their edges such that incidence relationships is preserved.

For eg
Let us consider these 2 graphs:-

1) equal no of vertices ✓
2) equal no of edges ✓
3) equal no of vertices with a given degree ✓
4) e_1 corresponds to 1
similarly, all edges are same ✓

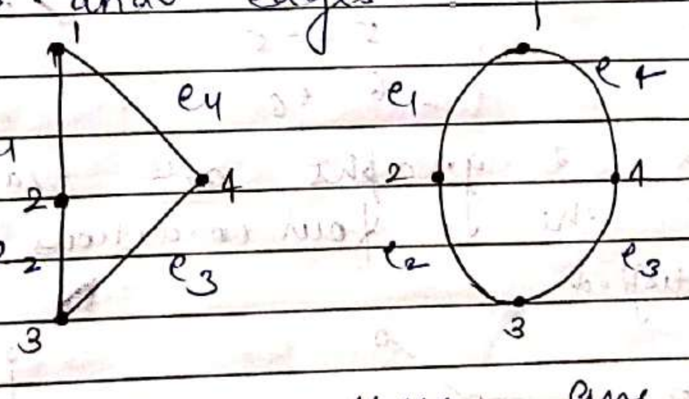


So, G_1 is isomorphic to G_2 and vice versa.

Note

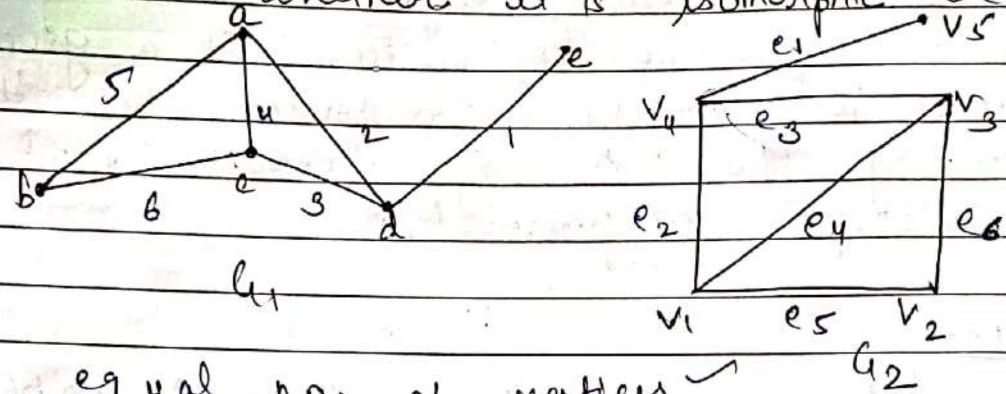
it should be noted that while drawing a graph it is immaterial whether the lines are curved or straight, long or short
The imp- thing is what is incidence b/w vertices and edges.

ie



In geometry, these are different but in graph they both are equivalent (called isomorphism) because no matter that line is curved and straight long or short.

Ques(1) Check whether it is isomorphic or not



1) equal no. of vertices ✓
2) equal no. of edges ✓
3) equal no of vertices with a given degree ✓

G_1 G_2

Degree 1 → e, V_5] in both graph one element of 1 degree

Degree 2 → G_1 b, V_2 G_2] in both graph each has only one element of 2 degree

Degree 3 → a, c, d] G_1 } in both graphs

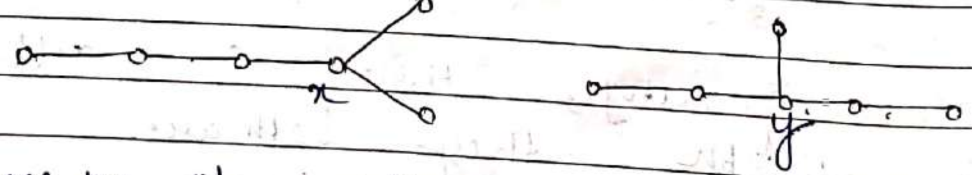
→ V_4, V_1, V_3] G_2 } there are 3 elements of 3-degree

4) ✓

a – V_1	1 – e_1
b – V_2	2 – e_2
c – V_3	3 – e_3
d – V_4	4 – e_4
e – V_5	5 – e_5
	6 – e_6

So these 2 graphs are exactly same as all the four conditions are being satisfied

(2)



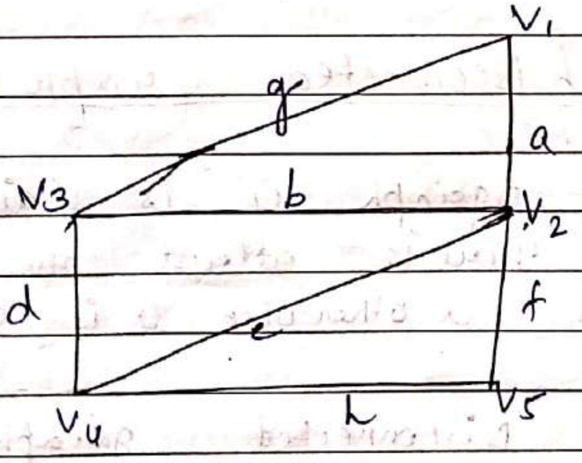
1) Same no of vertices
2) same no of edges.
3) equal vertices with given degree
4) ~~in this gra~~

in graph G_1, x must corresponds to y

in graph G_1, 2 pendant are connected to 3 degree. but in graph G_2 only one pendant is connected to 3 degree.
So, one to one correspondence is not there
So, not a isomorphism.

Walk and Path Circuit

A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices such that no edge appears or travels more than once in a walk, a vertex however may appear more than once. Vertices with which a walk begins and ends are called its terminal vertices. It is possible for walk to begin and at the same vertex. Such a walk is called a closed walk. A walk that is not closed is called an open walk.



$V_1 a V_2 b V_3 c V_3 d V_4 d V_3$ X ∵ edges are repeated

* edge can't be repeated but a vertex can be.

$\boxed{V_1 a V_2 b V_3 c V_3 d V_4 V_5}$ → open walk

$\boxed{V_2 f V_5}$ → open walk

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$\boxed{V_1 a V_2 b V_3 g V_1}$ → closed walk.

Path

→ An open walk in which no vertex appears more than once is called a path or simple path or elementary path

→ The no. of edges in a path is called length of a path.

→ It should be noted that a self loop group can be included in a walk but not in a path.

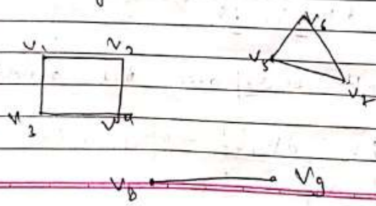
Circuit

A closed walk in which a no vertex (except the initial and final vertex) appears more than once is called a circuit.

Connected & Disconnected Graph And Component

Connected Graph: A graph G is said to be connected if there is atleast one path b/w every pair of vertices in G otherwise it is disconnected.

Disconnected graph is a combination of 2 connected graph.



every edge should traversed exactly once. (all vertex should be of even degree)

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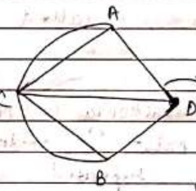
In this disconnected graph we have 3 connected graph. So 3 components of disconnected graph.

Euler's Graph

Graph theory was born in 1736 with Euler's famous paper in which he solved Kӧnigsberg bridge paper. In the same paper, Euler solved a more general problem. In what of graph G is it possible to find a closed walk running through every edge of G exactly once? Such a walk, now called an Euler line and the graph that contain an Euler line is called an Euler graph.

Repetition of vertex allowed

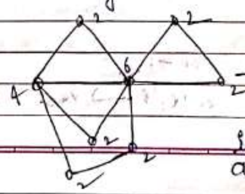
or in other words, you can say, if some closed walk in a graph contains all the edges in a graph then the walk is called an Euler line and the graph is called Euler graph.



→ In this graph all vertices D have are not of even degree so this is not an Euler walk, means a closed walk doesn't exist.

Theorem:- A given connected graph G is an Euler graph if and only if all vertices are of Even degree.

Ex-



→ All the vertices are of even degree so it is an Euler graph and means a closed walk exists.

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every vertex should travel exactly once.

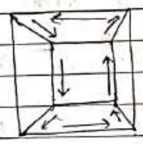
Hamiltonian Circuit and Paths

An Euler line of a connected graph was characterized by the property of being a closed walk (exactly once) that travels every edge of the graph. (exactly once)

A Hamiltonian circuit in a connected graph is defined as a closed walk that travels every vertex of G exactly once except of course the starting vertex at which it also transmits/terminates.

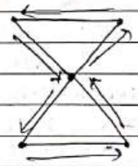
Example:-

(1)



Such a closed walk exist as all vertices are being covered exactly once. So, such a walk is called Hamiltonian circuit.

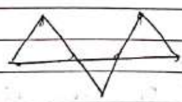
(2)



Hamiltonian's circuit doesn't exist ∵ vertex is being repeated

but this Euler's graph. ∵ every vertex are of even degree and all edges are being also covered.

(3)



Ham → NO

Euler's → Yes (even degree)

Note:- This arises the question: "what is necessary and sufficient condition for a connected graph G to have a Hamiltonian circuit?"

Travelling Salesman Problem

A problem closure related to the Hamiltonian circuit is the travelling salesman problem solved stated as follows:-

Problem: A salesman is required to visit a number of cities during a trip. Given the distances b/w given the cities, in what order should he travel so as he visit every city exactly once and return back to home with min^m millage or (km) travel. representing the cities by vertices and roads b/w them by edges, we get a graph. in this graph, the every edge e_i (e_i) there is associated a real number and the distance in mile. Such a graph is called a weighted graph denoted by $w(e_i)$.

Ques: If ∃ for a Graph G ($f: G \to G$) is given by $f(x) = x^2$ for every x belongs to G ($\forall x \in G$) is a homo-morphism defined as. Prthat G is an abelian?

Sol: $f(a*b) = f(a)*f(b) \to$ Homomorphism condn for $(G,*)$ and $(G,*)$

∵ f is a mapping from G to G

So, $f(a*b) = f(a)*f(b)$

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To prove $a \cdot b = b \cdot a$

$f(a) = a^2$ and $f(b) = b^2$

$f(a \cdot b) = (ab)^2$

Now, $f(ab) = f(ab)^2$ [from funⁿ]

Now means f is a homo- $f(a)\,(f(b))$ should be $=$ to $(ab)^2$

ie, $f(a)\,f(b) = (ab)^2$

$\Rightarrow (a^2)(b^2) = (ab)^2$

$\Rightarrow (a \cdot a)(b \cdot b) = (ab) \cdot (ab)$

$\Rightarrow a(ab)b = a(ba)b$ [Associative law]

$\Rightarrow ab = ba$ [using left & Right cancellation law].

Ques If G be a multiplicative group of 3 cube root of unity ~~Atom~~ $(G = \{1, \omega, \omega^2\})$ where $\omega^3 = 1$ and G' be a additive group of integer modulo 3 $(G' = \{0, 1, 2\}, +_3)$ prove that $f: G \to G'$ is a group isomorphism?

Sol Isomorphism group $\to$ one to one onto. (1st condⁿ) $\hookrightarrow$ homomorphism (2nd condⁿ)

1st condⁿ

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

for one to one onto

we have to prove f is

if $f(x) = f(y)$

then $\boxed{x = y}$

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f image of 1 is 0

i.e. $f(1) = 0$

f image of ω is 1

i.e. $f(\omega) = 1$

and $f(\omega^2) = 2$.

so, this is one to one and onto also.

2nd condⁿ:- $f(a * b) = f$

To prove homo-morphism:-

Table 1:

$f(\omega \cdot \omega^2) = f(\omega^3) = f(1) = \cancel{f}\, 0.$

Table 2:

$f(\omega \cdot \omega^2) = f(\omega) +_3 f(\omega^2)$

$= 1 +_3 2$

$= 0$

So, it is an homomorphism

Now check for (1 and ω)

$f(1 \cdot \omega) = f(\omega) = 1$ | $f(1 \cdot \omega) = f(1) +_3 f(\omega) = 0 +_3 1 = 1$

Also check for $(1, \omega^2)$

$f(1 \cdot \omega^2) = f(\omega^2) = 2$ | $f(1 \cdot \omega^2) = f(1) +_3 f(\omega^2) = 0 +_3 2 = 2$

So, this is a homomorphism

$\therefore$ Both the conditions are being satisfied

So it an isomorphism group.

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