2.5.1 Chromatic Polynomials

Given a connected graph G and λ number of different colors, let us take up the problem of finding the number of different ways of properly coloring G with these λ colors.

First, consider the null graph N_n with n vertices. In this graph, no two vertices are adjacent. Therefore, a proper coloring of this graph can be done by assigning a single color to all the vertices. Thus, if there are λ number of colors, each vertex of the graph has λ possible choices of colors assigned to it, and as such the graph can be properly colored in λ^n different ways.

Next, consider the complete graph K_n. In this graph, every two vertices are adjacent, and as such there must be at least n colors for a proper coloring of the graph. If the number of different colors available is λ, then the number of ways of properly coloring K_n is (i) zero if $\lambda < n$, (ii) one if $\lambda = n$, and (iii) greater than 1 if $\lambda > n$. Let $v_1, v_2, v_3, \ldots v_n$ be the vertices of K_n and suppose $\lambda > n$. For a proper coloring of K_n, the vertex v_1 can be assigned any of the λ colors, the vertex v_2 can be assigned any of the remaining $\lambda - 1$ colors, the vertex v_3 can be assigned any of the remaining $\lambda - 2$ colors and finally the vertex v_n can be assigned any of the $\lambda - n + 1$ colors. Thus, K_n can be properly colored in $\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$ different ways if $\lambda > n$.

Lastly, consider the graph L_n which is a path consisting of n vertices $v_1, v_2, \ldots, v_n$ shown below:

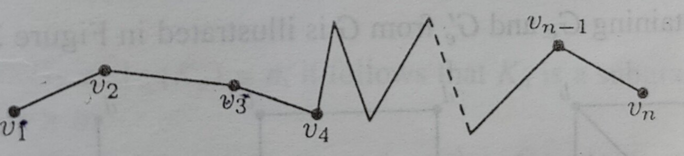

Figure 2.44

This graph cannot be properly colored with one color, but can be properly colored with 2 colors – by assigning one color to $v_1, v_3, v_5, \ldots$ and another color to $v_2, v_4, v_6, \ldots$. Suppose there are $\lambda \geq 2$ number of colors available. Then, for a proper coloring of the graph, the vertex v_1 can be assigned any one of the λ colors and each of the remaining vertices can be assigned any one of $\lambda - 1$ colors. (Bear in mind that alternative vertices can have the same color). Thus, the graph L_n can be properly colored in $\lambda(\lambda-1)^{n-1}$ different ways.

The number of different ways of properly coloring a graph G with λ number of colors is denoted by $P(G, \lambda)$. Thus, from what is seen in the above three illustrate examples, we note that

(i) $P(N_n, \lambda) = \lambda^n$,

(ii) $P(K_n, \lambda) = 0$ if $\lambda < n$,

? $P(K_n, n) = 1$ if $\lambda = n$, and
$P(K_n, \lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$ if $\lambda > n$,

(iii) $P(L_n, \lambda) = \lambda(\lambda - 1)^{n-1}$ if $\lambda \geq 2$.

We observe that in each of the above cases, $P(G, \lambda)$ is a *polynomial*. Motivated by these cases, we take that $P(G, \lambda)$ is a polynomial for all connected graphs G. This polynomial is called the ***Chromatic Polynomial***.

It follows that if a graph G is made up of n parts $G_1, G_2, \ldots G_n$, then $P(G, \lambda)$ is given by the following

Product Rule:

$$P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda) \cdot \cdots \cdot P(G_n, \lambda).$$

In particular, if G is made up of two parts G_1 and G_2, then we have $P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda)$ so that

$$P(G_2, \lambda) = P(G, \lambda)/P(G_1, \lambda).$$

Decomposition Theorem

Let G be a graph and $e = \{a, b\}$ be an edge of G. Let $G_e = G - e$ be that subgraph of G which is obtained by deleting e from G without deleting vertices a and b*. Suppose we construct a new graph G'_e by coalescing (identifying/merging) the vertices a and b in G_e. Then G'_e is a subgraph of G_e as well as G.

The process of obtaining G_e and G'_e from G is illustrated in Figure 2.45.

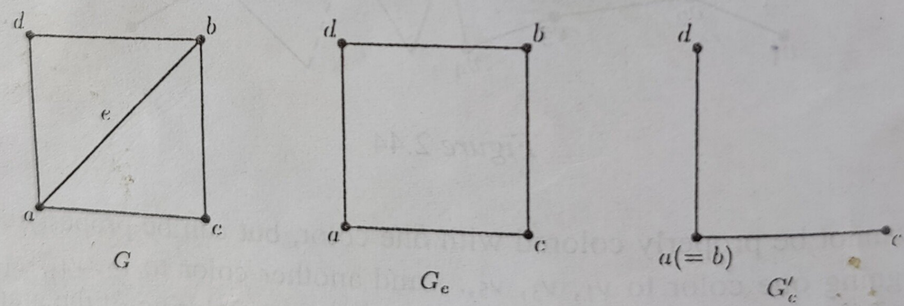

Figure 2.45

The following theorem called the *Decomposition theorem for chromatic polynomials* gives an expression for $P(G, \lambda)$ in terms of $P(G_e, \lambda)$ and $P(G'_e, \lambda)$ for a connected graph G.

Theorem 1 *If G is a connected graph and $e = \{a, b\}$ is an edge of G, then*

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda)$$

Proof: In a proper coloring of G_e, the vertices a and b can have the same color or different colors. In every proper coloring of G, the vertices a and b have different colors and in every

*See Section 1.5.

proper coloring of G'_e these vertices have the same color. Therefore, the number of proper colorings of G_e is the sum of the number of proper colorings of G and the number of proper colorings of G'_e. That is,

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda)$$

This completes the proof of the theorem.

Multiplication Theorem

The following theorem gives an expression for $P(G, \lambda)$ for a special class of graphs.

Theorem 2 *If a graph G has subgraphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_n$ for some positive integer n, then*

$$P(G, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}$$

where

$$\lambda^{(n)} = \lambda(\lambda - 1)(\lambda - 2)\ldots(\lambda - n + 1).$$

Proof: Since $K_n = G_1 \cap G_2$ and $\chi(K_n) = n$, it follows that K_n is a subgraph of both G_1 and G_2 and $\chi(G_1) \geq n$ and $\chi(G_2) \geq n$.

Given $\lambda > n$ number of different colors, there are $\lambda^{(n)} = \lambda(\lambda-1)(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$ number of proper colorings of K_n*. For each of these $\lambda^{(n)}$ proper colorings of K_n, the product rule yields $P(G_1, \lambda)/\lambda^{(n)}$ ways of properly coloring the remaining vertices of G_1. Similarly, there are $P(G_2, \lambda)/\lambda^{(n)}$ ways of properly coloring the remaining vertices of G_2. As such,

$$\begin{aligned}
P(G, \lambda) &= P(K_n, \lambda) \cdot \frac{P(G_1, \lambda)}{\lambda^{(n)}} \cdot \frac{P(G_2, \lambda)}{\lambda^{(n)}} \\
&= \lambda^{(n)} \cdot \frac{P(G_1, \lambda)}{\lambda^{(n)}} \cdot \frac{P(G_2, \lambda)}{\lambda^{(n)}} \\
&= \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}
\end{aligned}$$

This completes the proof of the theorem.

Example 1 *Find the chromatic polynomial for the graph shown in Figure 2.46. What is its chromatic number?*

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*Recall that $P(K_n, n) = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - n + 1)$ for $\lambda > n$.

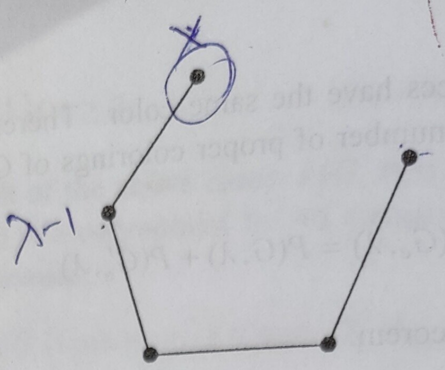


Figure 2.46

► We observe that the given graph G is a path of length $n = 5$, namely L_5. Therefore, its chromatic polynomial is

$$P(G, \lambda) = \lambda(\lambda - 1)^{n-1} = \lambda(\lambda - 1)^4.$$

Next, we note that the chomatic number of the graph is $\chi(G) = 2$. (Because, the graph cannot be properly colored with one color but can be properly colord with 2 colors by assigning two colors to the alternative vertices). ∎

Example 2 *Find the chromatic number and the chromatic polynomial for the graph $K_{1,n}$*

► We note that $K_{1,n}$ is the complete bipartite graph wherein one bipartite of the vertex set has only one vertex, say v, and the other bipartite has n vertices, say $v_1, v_2, \ldots, v_n$. A proper coloring of this graph cannot be done with just one color and but can be done with two colors - by assigning one color to v and another color to all of $v_1, v_2, \ldots, v_n$. Thus, the chromatic number of this graph is 2.

If λ colors are available, then the vertex v can be colored in λ ways and each of the vertices $v_1, v_2, \ldots, v_n$ can be colored in $\lambda - 1$ ways. Therefore, the number of ways of properly coloring the graph is $\lambda(\lambda - 1)^n$. This is the chromatic polynomial for the graph. ∎

Example 3 (a) *Consider the graph $K_{2,3}$ shown in Figure 2.47. Let λ denote the number of colors available to properly color the vertices of this graph. Find:*

(i) *how many proper colorings of the graph have vertices a, b colored the same colors.*

(ii) *how many proper colorings of the graph have vertices a, b colored with different colors.*

(iii) *the chromatic polynomial of the graph.*

(b) *For the graph $K_{2,n}$ what is the chromatic polynomial?*

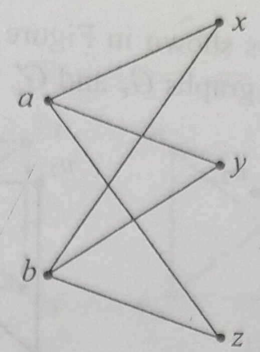


Figure 2.47

► (a) (i) If the vertices a and b are to have the same color, then there are λ choices for coloring the vertex a and only one choice for the vertex b (or vice versa). Consequently, there are $\lambda - 1$ choices for each of the vertices x, y, z. Hence , the number of proper colorings (in this case) is $\lambda(\lambda-1)^3$.

(ii) If the vertices a and b are to have different colors, then there are λ choices for coloring the vertex a and $(\lambda - 1)$ choices for the vertex b (or vice versa). Consequently, there are $(\lambda - 2)$ choices for each of the vertices x, y, z. Hence the number of proper colorings (in this case) is $\lambda(\lambda-1)(\lambda-2)^3$.

(iii) Since the two cases of the vertices a and b having the same color or different colors are exhaustive and mutually exclusive, the chromatic polynomial of the graph is

$$P(K_{2,3}, \lambda) = \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)^3$$

(b) Let $V_1 = \{a, b\}$ and $V_2 = \{x_1, x_2, x_3, \ldots, x_n\}$ be the two bipartites of $K_{2,n}$. Then, if a and b are to have the same color, the number of proper colorings of $K_{2,n}$ is $\lambda(\lambda-1)^n$, as in case (i) above. If a and b are to have different colors, the number of proper colorings is $\lambda(\lambda-1)(\lambda-2)^n$, as in case (ii) above. Consequently, the chromatic polynomial for $K_{2,n}$ is

$$P(K_{2,n}, \lambda) = \lambda(\lambda-1)^n + \lambda(\lambda-1)(\lambda-2)^n.$$ ■

Example 4 *Find the chromatic polynomial for the cycle C_4 of length 4.*

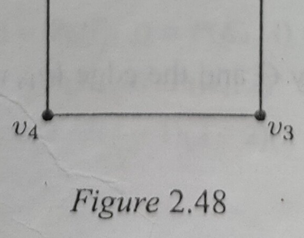


Figure 2.48

▶ A cycle of length 4, namely C_4, is shown in Figure 2.48. Let us redesignate it as G and denote the edge $\{v_2, v_3\}$ as e. Then the graphs G_e and G'_e would be as shown below.

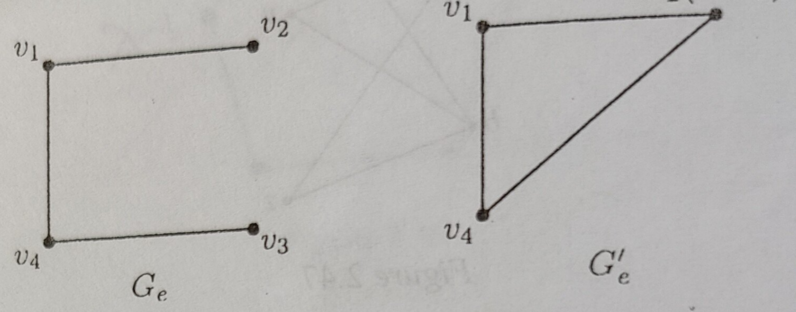


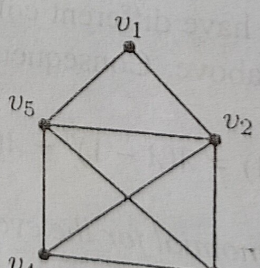
Figure 2.49

We note that the graph G_e is a path with 4 vertices. Therefore, $P(G_e, \lambda) = \lambda(\lambda-1)^3$. Also, the graph G'_e is the graph K_3. Therefore, $P(G'_e, \lambda) = \lambda(\lambda-1)(\lambda-2)$. Accordingly, using the decomposition theorem, we find that

$$\begin{aligned} P(C_4, \lambda) \equiv P(G, \lambda) &= P(G_e, \lambda) - P(G'_e, \lambda) \\ &= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda. \end{aligned}$$

This is the chromatic polynomial for the given cycle. ∎

Example 5 *Find the chromatic polynomial for the graph shown below. If 5 colors are available, in how many ways can the vertices of this graph be properly colored?*



G

Figure 2.50

▶ Let us denote the given graph by G and the edge $\{v_1, v_2\}$ by e. Then the graphs G_e and G'_e would be as shown in Figure 2.51.

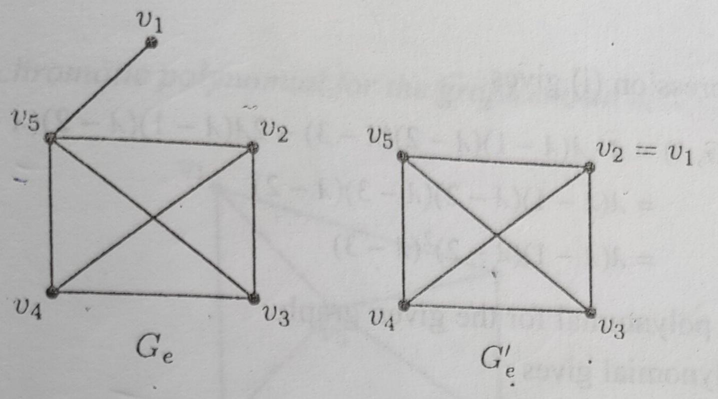


Figure 2.51

Let us redesignate the graph G_e as H and denote the edge $\{v_1, v_5\}$ as f. Then the graphs H_f and H'_f would appear as shown below:

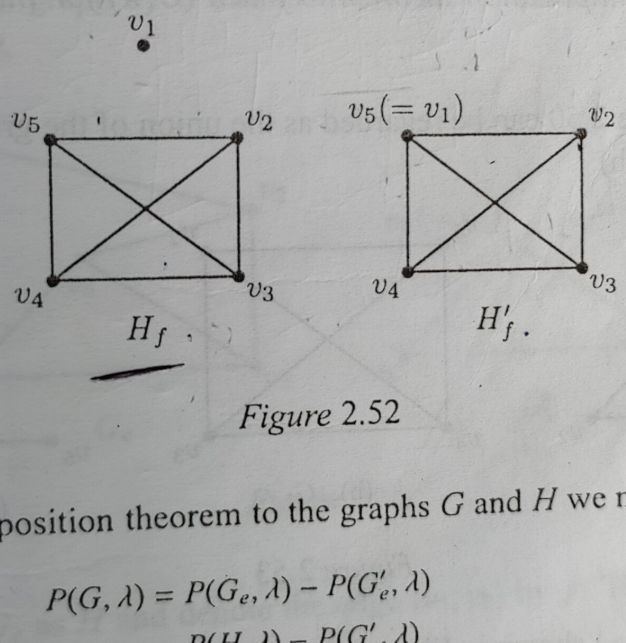


Figure 2.52

Applying the decomposition theorem to the graphs G and H we note that

$$\begin{aligned} P(G, \lambda) &= P(G_e, \lambda) - P(G'_e, \lambda) \\ &= P(H, \lambda) - P(G'_e, \lambda) \\ &= \{P(H_f, \lambda) - P(H'_f, \lambda)\} - P(G'_e, \lambda) \end{aligned} \tag{i}$$

We observe that both of the graphs G'_e and H'_f are the graph K_4 and the graph H_f is a disconnected graph having N_1 (–null graph of order 1 consisting of the single vertex v_1) and K_4 as components. Accordingly,

$$P(G'_e, \lambda) = P(H'_f, \lambda) = P(K_4, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

and

$$\begin{aligned} P(H_f, \lambda) &= P(N_1, \lambda) \cdot P(K_4, \lambda)^* \\ &= \lambda \cdot \lambda(\lambda-1)(\lambda-2)(\lambda-3) \end{aligned}$$

*by the Product Rule.

Consequently, expression (i) gives

$$\begin{aligned}P(G,\lambda) &= \lambda\cdot\lambda(\lambda-1)(\lambda-2)(\lambda-3) - 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)\\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-2)\\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)\end{aligned}$$

This is the chromatic polynomial for the given graph.

For $\lambda = 5$, this polynomial gives

$$P(G,5) = 5\times 4\times 3^2\times 2 = 360.$$

This means that if 5 colors are available, the vertices of the graph can be properly colored in 360 different ways.

Example 6 *Use the multiplication theorem to find $P(G,\lambda)$ for the graph shown in Figure 2.50.*

▶ The graph G in Figure 2.50 can be regarded as the union of the graphs G_1 and G_2 shown in Figures 2.53(a) and 2.53(b).

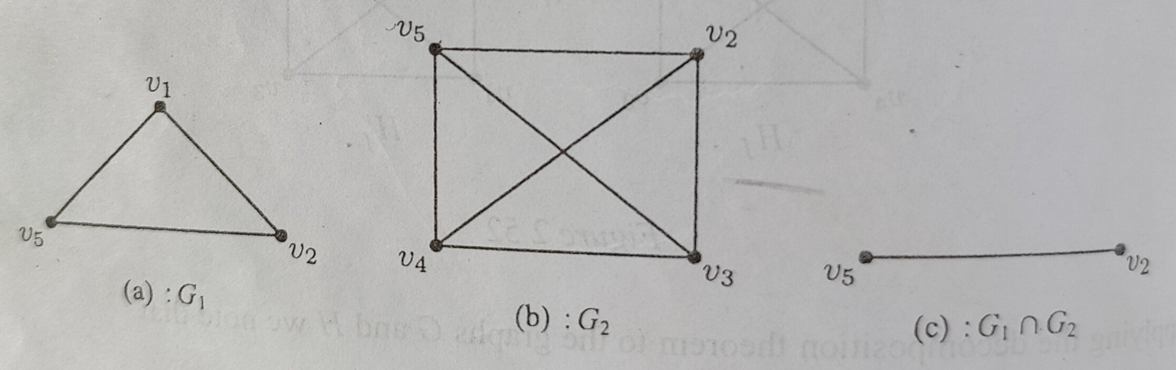


Figure 2.53

Then $G_1 \cap G_2 = \{v_5, v_2\}$ shown in Figure 2.53(c).

We note that G_1 is the same as K_3, G_2 is the same as K_4 and $G_1 \cap G_2$ is the same as K_2. Hence, using the multiplication theorem (Theorem 2), we get

$$\begin{aligned}P(G,\lambda) &= \frac{P(G_1,\lambda)\cdot P(G_2,\lambda)}{\lambda^{(2)}}\\ &= \frac{P(K_3,\lambda)\cdot P(K_4,\lambda)}{\lambda^{(2)}}\\ &= \frac{\lambda(\lambda-1)(\lambda-2)\cdot\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\lambda(\lambda-1)}\\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)\end{aligned}$$

as the chromatic polynomial for the given G. (This result agrees with the result proved in Example 5).

Example 7 *Find the chromatic polynomial for the graph shown below:*

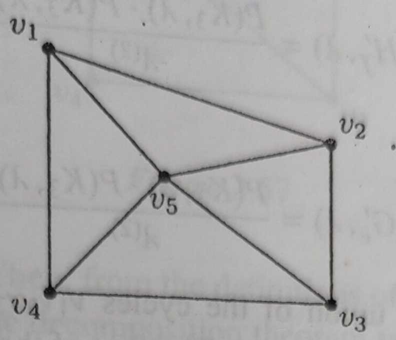


Figure 2.54

▶ Let us denote the given graph by G and the edge $\{v_1, v_5\}$ as e. Then the graphs G_e and G'_e would be as shown below.

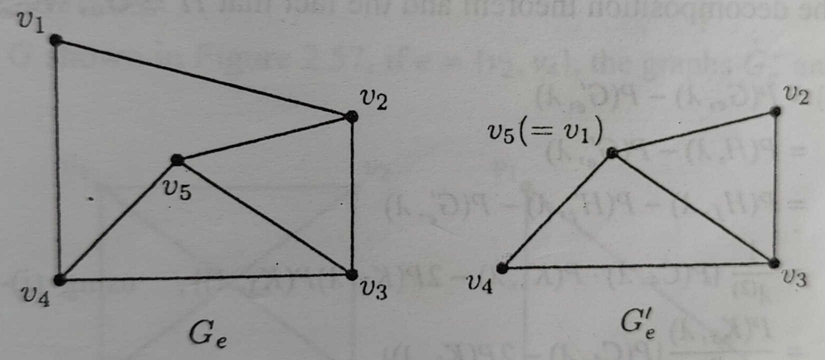


Figure 2.55

Let us redesignate G_e as H and denote the edge $\{v_5, v_2\}$ by f. Then the graphs H_f and H'_f are as shown below.

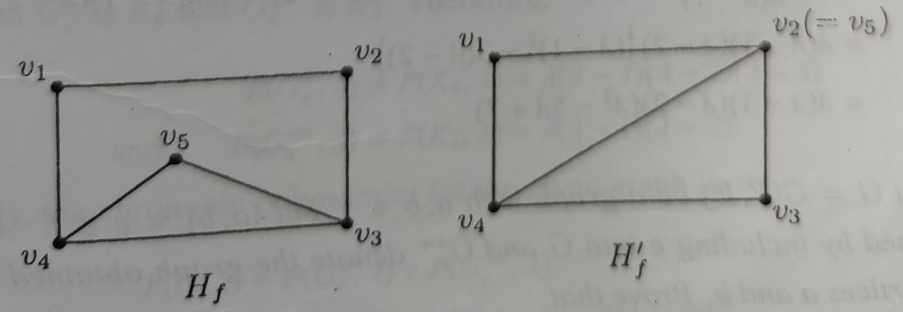


Figure 2.56

Now, we note that H'_f is the union of the cycles $v_1v_4v_2v_1$ and $v_2v_3v_4v_2$ each of which is the same as K_3, and that the intersection of these cycles is the edge $\{v_4, v_2\}$ which is the same as

K_2. Therefore, by the multiplication theorem, we have

$$P(H'_f, \lambda) = \frac{P(K_3, \lambda) \cdot P(K_3, \lambda)}{\lambda^{(2)}} \tag{i}$$

Similarly,

$$P(G'_e, \lambda) = \frac{P(K_3, \lambda) \cdot P(K_3, \lambda)}{\lambda^{(2)}} \tag{ii}$$

Next, we note that H_f is the union of the cycles $v_1v_2v_3v_4v_1$ and $v_5v_3v_4v_5$ and that the intersection of these cycles is the edge $\{v_4, v_3\}$. The first of these cycles is C_4, the second cycle is K_3 and the edge $\{v_4, v_3\}$ is K_2. Therefore, by the multiplication theorem, we have

$$P(H_f, \lambda) = \frac{P(C_4, \lambda) \cdot P(K_3, \lambda)}{\lambda^{(2)}} \tag{iii}$$

Now, by using the decomposition theorem and the fact that $H \equiv G_e$, we get

$$\begin{aligned}
P(G, \lambda) &= P(G_e, \lambda) - P(G'_e, \lambda) \\
&= P(H, \lambda) - P(G'_e, \lambda) \\
&= P(H_f, \lambda) - P(H'_f, \lambda) - P(G'_e, \lambda) \\
&= \frac{1}{\lambda^{(2)}} \{P(C_4, \lambda) \cdot P(K_3, \lambda) - 2P(K_3, \lambda)P(K_3, \lambda)\}, \quad \text{using (i)-(iii)} \\
&= \frac{P(K_3, \lambda)}{\lambda^{(2)}} \{P(C_4, \lambda) - 2P(K_3, \lambda)\}
\end{aligned}$$

Using the result of Example 4 and the expressions for $P(K_3, \lambda)$ and $\lambda^{(2)}$, this becomes

$$\begin{aligned}
P(G, \lambda) &= \frac{\lambda(\lambda-1)(\lambda-2)}{\lambda(\lambda-1)} \left\{\lambda\left\{(\lambda-1)^3 - (\lambda-1)(\lambda-2)\right\} - 2\lambda(\lambda-1)(\lambda-2)\right\}. \\
&= \lambda(\lambda-1)(\lambda-2)\left\{(\lambda-1)^2 - 3(\lambda-2)\right\} \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 5\lambda + 7)
\end{aligned}$$

Example 8 *Let $G = G(V, E)$ be a graph with $a, b \in V$ but $\{a, b\} = e \notin E$. Let G_e^+ denote the graph obtained by including e into G and G_e^{++} denote the graph obtained by coalescing (merging) the vertices a and b. Prove that*

$$P(G, \lambda) = P(G_e^+, \lambda) + P(G_e^{++}, \lambda)$$

Hence find the chromatic polynomial for the graph shown in Figure 2.57.

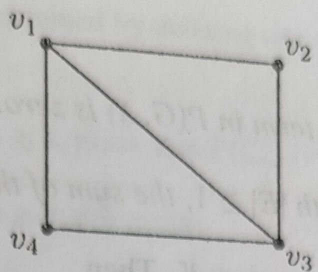


Figure 2.57

► Let us redesignate G_e^+ as H. Then, from the definitions of G_e^+ and G_e^{++}, we find that $H_e = G$ and $H_e' = G_e^{++}$. Now, applying the decomposition theorem to H, we get

$$P(H_e, \lambda) = P(H, \lambda) + P(H_e', \lambda)$$

This is the same as

$$P(G, \lambda) = P(G_e^+, \lambda) + P(G_e^{++}, \lambda)$$

which is the required result.

For the graph G shown in Figure 2.57, if $e = \{v_2, v_4\}$, the graphs G_e^+ and G_e^{++} are as shown below:

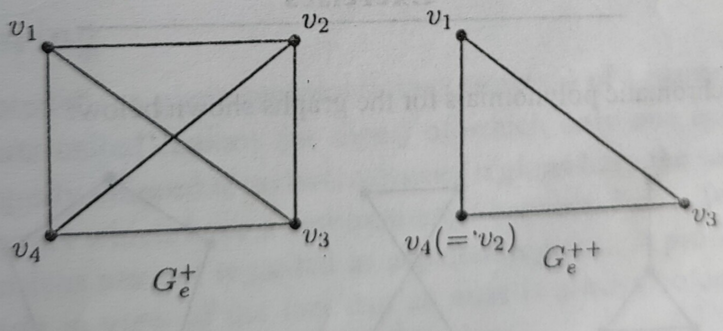


Figure 2.58

We note that G_e^+ is K_4 and G_e^{++} is K_3. Therefore,

$$P(G_e^+, \lambda) = P(K_4, \lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$$

and $$P(G_e^{++}, \lambda) = P(K_3, \lambda) = \lambda(\lambda-1)(\lambda-2).$$

Accordingly, the chromatic polynomial for the given graph is

$$\begin{aligned} P(G, \lambda) &= P(G_e^+, \lambda) + P(G_e^{++}, \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)(\lambda-2)^2 \end{aligned}$$

■

Example 9 *Prove the following:*

(a) *For any graph G, the constant term in $P(G, \lambda)$ is zero.*

(b) *For any graph $G = G(V, E)$ with $|E| \geq 1$, the sum of the coefficients in $P(G, \lambda)$ is zero.*

▶ Let $P(G, \lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_r\lambda^r$. Then

$$P(G, 0) = a_0 \text{ and } P(G, 1) = a_0 + a_1 + a_2 + \cdots + a_r.$$

(a) For any graph G, $P(G, 0)$ represents the number of ways of properly coloring G with zero number of colors. Since a graph cannot be colored with no color on hand, it follows that $P(G, 0) = 0$; that is $a_0 = 0$.

(b) For any graph G, $P(G, 1)$ represents the number of ways of properly coloring G with 1 color. If G has at least one edge, G cannot be properly colored with 1 color. This means that, for $G = G(V, E)$ with $|E| \geq 1$, we have

$$P(G, 1) = 0; \quad \text{that is,} \quad a_0 + a_1 + \cdots + a_r = 0. \quad ■$$

Exercises

1. Determine the chromatic polynomials for the graphs shown below:

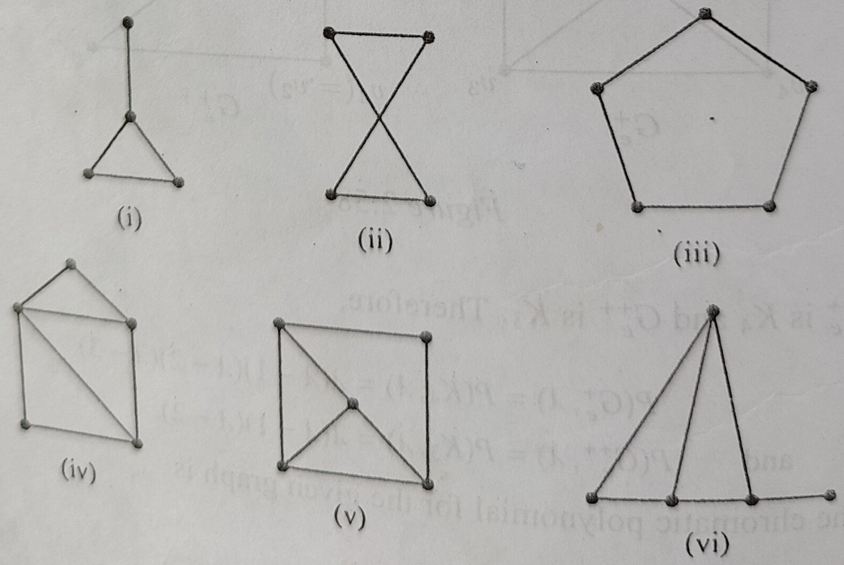


Figure 2.59

2. If 4 colors are available, in how many different ways can the vertices of each graph in Figure 2.59 be properly colored?

3. For $n \geq 3$, Let G_n be the graph obtained by deleting one edge from K_n. Determine $P(G_n, \lambda)$ and $\chi(G_n)$.

4. If C_n denotes a cycle of length $n \geq 3$, prove that $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$.

5. If C_n denotes a cycle of length $n \geq 4$, prove that

$$P(C_n, \lambda) + P(C_{n-1}, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

Answers

1. (i) $\lambda(\lambda-1)^2(\lambda-2)$, (ii) $\lambda(\lambda-1)^2(\lambda-2)^2$, (iii) $\lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$, (iv) $\lambda(\lambda-1)(\lambda-2)^3$, (v) $\lambda(\lambda-1)(\lambda-2)(2\lambda-5)$, (vi) $\lambda(\lambda-1)^2(\lambda-2)^2$.

2. (i) 72 (ii) 144 (iii) 240 (iv) 96 (v) 72 (vi) 144

3. $P(G_n, \lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+3)(\lambda-n+2)^2, \chi(G_n) = n-1$

2.6 Map Coloring

In Section 2.2, we noted that a plane representation (drawing) of a planar graph divides a plane into a number of parts called regions (or faces) of which only one is exterior. We say that these regions are properly colored if no two *adjacent regions* have the same color. By adjacent regions we mean regions which have a common edge between them. Two regions having one or more common vertices are *not* regarded as adjacent regions. A proper coloring of regions is called ***map coloring*** in view of the fact that an atlas is always colored in such a way that countries with common boundaries have different colors.

The following Figure illustrates a proper coloring of regions of a planar graph.