CHAPTER 11
AN INTRODUCTION TO GRAPH THEORY

Section 11.1

1. (a) To represent the air routes traveled among a certain set of cities by a particular airline.
(b) To represent an electrical network. Here the vertices can represent switches, transistors, etc., and an edge (x, y) indicates the existence of a wire connecting x to y.
(c) Let the vertices represent a set of job applicants and a set of open positions in a corporation. Draw an edge (A,b) to denote that applicant A is qualified for position b. Then all open positions can be filled if the resulting graph provides a matching between the applicants and open positions.

2. (a) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,b\},\{b,c\},\{c,d\}$
(b) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,d\}$
(c) $\{b,e\},\{e,d\}$
(d) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,b\}$
(e) $\{b,e\},\{e,f\},\{f,g\},\{g,e\},\{e,d\},\{d,c\},\{c,b\}$
(f) $\{b,a\},\{a,c\},\{c,b\}$

3. 6

4. We claim that $\kappa(G) = 2$. To verify this consider the following:
(1) Let C_1 be the set of all vertices $v \in V$ where the binary label of v has an even number of 1s. This includes the vertex z whose binary label is the n-tuple of all 0s. For any $v_0 \in C_1$, where $v_0 \neq z$, we can find a path from v_0 to z as follows. Suppose that the binary label for v_0 has $2m$ 1s, where $2 \leq 2m \leq n$. Change the first two 1s in the binary label for v_0 to 0s and call the resulting vertex v_1. Then $v_1 \in C_1$ and $\{v_0, v_1\} \in E$. Now change the first two 1s in the binary label for v_1 to 0s and call the resulting vertex v_2. Once again $v_2 \in C_1$ and $\{v_0, v_1\} \in E$. Continuing this process we reach the vertex $v_m = z$ and find that $\{v_{m-1}, v_m\} \in E$, with $v_{m-1} \in C_1$. Hence each of the vertices in $C_1 - \{z\}$ is connected to z.

(2) Now let C_2 be the set of all vertices $w \in V$ where the binary label for w has an odd number of 1s. Let $z^* \in C_2$ where the binary label for z^* consists of a 1 followed by $n-1$ 0s. For each $w_0 \in C_2$, $w_0 \neq z^*$, one of two possibilities can occur:

(i) There are $2m+1$ 1s in the binary label for w_0, with $3 \leq 2m+1 \leq n$, and the first entry in the label for w_0 is 1. Here we change the next two 1s in the binary label for w_0 to 0s and obtain the vertex $w_1 \in C_2$ with $\{w_0, w_1\} \in E$. Now the first entry in the binary label

for w_1 is a 1 and upon changing the second and third 1s in this label to 0s we obtain the vertex $w_2 \in C_2$ with $\{w_1, w_2\} \in E$. Continuing this process we reach the vertex $w_m = z^*$ with $w_{m-1} \in C_2$ and $\{w_{m-1}, w_m\} \in E$. Consequently each vertex in $C_2 - \{z^*\}$ whose binary label starts with 1 is connected to z^*.
(ii) There are $2m+1$ 1s in the binary label for w_0, with $3 \leq 2m+1 \leq n$, and the first entry in the label for w_0 is 0. Change the first entry in the binary label for w_0 to 1 and the first 1 in the binary label for w_0 to 0. This results in the vertex $w_1 \in C_2$ with $\{w_0, w_1\} \in E$. Upon changing the second and third 1s in the binary label for w_1 to 0s we obtain the vertex $w_2 \in C_2$ with $\{w_1, w_2\} \in E$. Continuing this process we reach the vertex $w_{m+1} = z^*$ with $\{w_m, w_{m+1}\} \in E$. This shows that each vertex in C_2 whose binary label starts with 0 is also connected to z^*.

(3) We claim that the components of G are the graphs determined by C_1 and C_2. Can there exist an edge $\{x, y\} \in E$ where $x \in C_1$, $y \in C_2$? Here the binary label for x has an even number of 1s while the label for y has an odd number of 1s. This contradicts the definition of E – for if $\{a, b\} \in E$ then the total number of 1s in the binary labels for a, b is even.

5. Each path from a to h must include the edge $\{b, g\}$. There are three paths (in G) from a to b and three paths (in G) from g to h. Consequently, there are nine paths from a to h in G.

There is only one path of length 3, two of length 4, three of length 5, two of length 6, and one of length 7.

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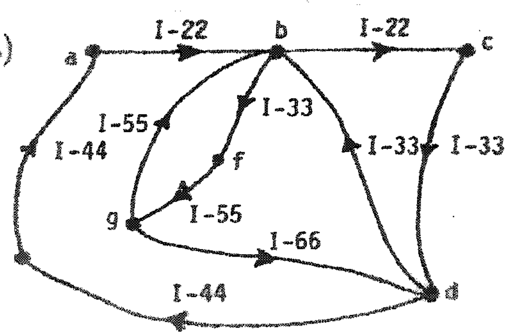
c: 1	e: 1	f: 1	g: 2	h: 3
i: 4	j: 3	k: 2	l: 3	m: 3

7. (a)



(b) $\{(g,d),(d,e),(e,a)\}$;
$\{(g,b),(b,c),(c,d),(d,e),(e,a)\}$.
(c) Two: One of $\{(b,c),(c,d)\}$ and one of $\{(b,f),(f,g),(g,d)\}$.
(d) No
(e) Yes: Travel the path $\{(c,d),(d,e),(e,a),(a,b),(b,f),(f,g)\}$.
(f) Yes: Travel the path $\{(g,b),(b,f),(f,g),(g,d),(d,b),(b,c),(c,d),(d,e),(e,a),(a,b)\}$.

8. The smallest number of guards needed is 3 - e.g., at vertices a, g, i.

9. If $\{a, b\}$ is not part of a cycle, then its removal disconnects a and b (and G). If not, there is a path P from a to b and P, together with $\{a, b\}$, provides a cycle containing $\{a, b\}$.
Conversely, if the removal of $\{a, b\}$ from G disconnects G then there exist $x, y \in V$ such that the only path P from x to y contains $e = \{a, b\}$. If e were part of a cycle

C, then the edges in $(P - \{e\}) \cup (C - \{e\})$ would provide a second path connecting x to y.

10. Any path. **11.** (a) Yes (b) No (c) $n-1$

12. (a) In a loop-free undirected graph (that is not a multigraph) the maximum number of edges is $\binom{v}{2}$. Hence $e \leq \binom{v}{2} = v(v-1)/2$, so $2e \leq v^2 - v$.

(b) In a loop-free directed graph (that is not a multigraph), $e \leq v^2 - v$.

13. This relation is reflexive, symmetric and transitive, so it is an equivalence relation. The partition of V induced by $\mathcal{R}$ yields the (connected) components of G.

14. (a) There are three cycles of length 4 in W_3, five cycles of length 4 in W_4, and five such cycles in W_5.
(b) Denote the consecutive cycle (rim) vertices of W_n by $v_1, v_2, \ldots, v_n$ and the additional (central) vertex by v_{n+1}.

(i) For $n \neq 4$, there are n cycles of length 4:
(1) $v_1 \to v_2 \to v_3 \to v_{n+1} \to v_1$;
(2) $v_2 \to v_3 \to v_4 \to v_{n+1} \to v_2$;
$\ldots$;
$(n-1)$ $v_{n-1} \to v_n \to v_1 \to v_{n+1} \to v_{n-1}$; and
(n) $v_n \to v_1 \to v_2 \to v_{n+1} \to v_n$.

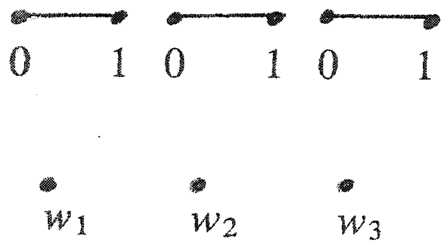
When $n = 4$ the vertices v_1, v_2, v_3, v_4 provide a cycle. The other four cycles of length 4 consist of vertex v_5 and three of the four vertices v_1, v_2, v_3, v_4.

(ii) There are $n+1$ cycles of length n in W_n:
(1) $v_1 \to v_2 \to v_3 \to \ldots \to v_{n-1} \to v_n \to v_1$;
(2) $v_1 \to v_{n+1} \to v_3 \to v_4 \to \ldots \to v_{n-1} \to v_n \to v_1$;
(3) $v_2 \to v_{n+1} \to v_4 \to v_5 \to \ldots \to v_{n-1} \to v_n \to v_1 \to v_2$;
$\ldots$;
(n) $v_{n-1} \to v_{n+1} \to v_1 \to v_2 \to \ldots \to v_{n-3} \to v_{n-2} \to v_{n-1}$; and
$(n+1)$ $v_n \to v_{n+1} \to v_2 \to v_3 \to \ldots \to v_{n-3} \to v_{n-1} \to v_n$.

15. For $n \geq 1$, let a_n count the number of closed $v-v$ walks of length n (where, in this case, we allow such a walk to contain or consist of one or more loops). Here $a_1 = 1$ and $a_2 = 2$. For $n \geq 3$ there are a_{n-1} $v-v$ walks where the last edge is the loop $\{v, v\}$ and a_{n-2} $v-v$ walks where the last two edges are both $\{v, w\}$. Since these two cases are exhaustive and have nothing in common we have $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, $a_1 = 1$, $a_2 = 2$.

We find that $a_n = F_{n+1}$, the $(n+1)$st Fibonacci number.

16. a) There are two other unit-interval graphs for three unit intervals.



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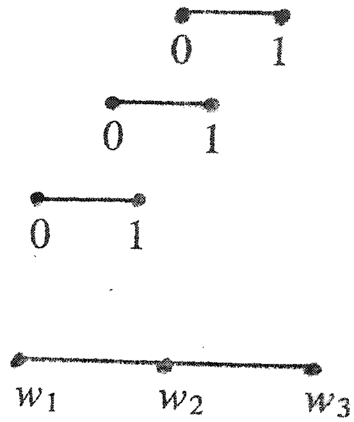

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b) For four unit intervals there are 14 unit-inteval graphs.

c) For $n \geq 1$, there are $b_n = \frac{1}{n+1}\binom{2n}{n}$ unit-interval graphs for n unit intervals. Here b_n is the nth Catalan number. The binary representations set up a one-to-one correspondence with the situations in Example 1.40 – in particular, change 0 to 1 and 1 to 0 in part (b) of Example 1.40 to obtain the binary representations of the 14 unit-interval graphs on four unit intervals.

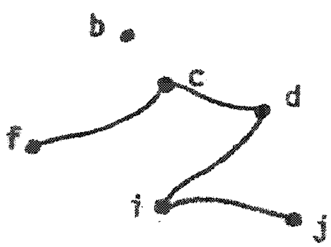
Section 11.2

1. (a) Three: (1) $\{b, a\}, \{a, c\}, \{c, d\}, \{d, a\}$
   (2) $\{f, c\}, \{c, a\}, \{a, d\}, \{d, c\}$
   (3) $\{i, d\}, \{d, c\}, \{c, a\}, \{a, d\}$

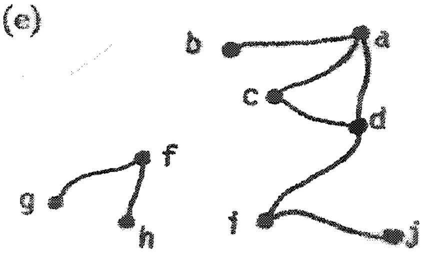
   (b) G_1 is the subgraph induced by $U = \{a, b, d, f, g, h, i, j\}$
   $G_1 = G - \{c\}$

   (c) G_2 is the subgraph induced by $W = \{b, c, d, f, g, i, j\}$
   $G_2 = G - \{a, h\}$

   (d)



   (e)



2. (a) G_1 is *not* an induced subgraph of G if there exists an edge $\{a, b\}$ in E such that

$a, b \in V$, but $\{a, b\} \notin E_1$.

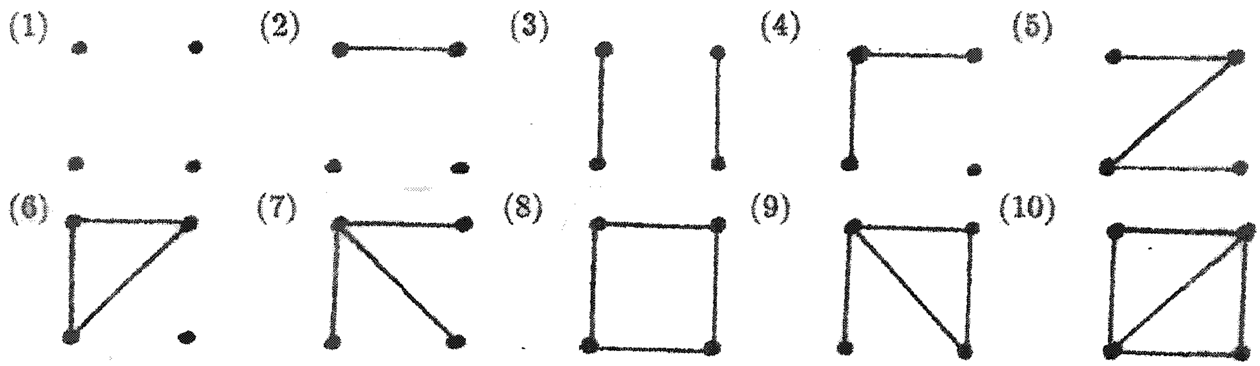
(b) Let $e = \{a, d\}$. Then $G - e$ is a subgraph of G but it is not an induced subgraph.

3. (a) There are $2^9 = 512$ spanning subgraphs.
(b) Four of the spanning subgraphs in part (a) are connected.
(c) 2^6

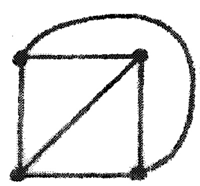
4. There is only one – the graph G itself.

5. G is (or is isomorphic to) the complete graph K_n, where $n = |V|$.

6. There are 11 loop-free nonisomorphic undirected graphs with four vertices.

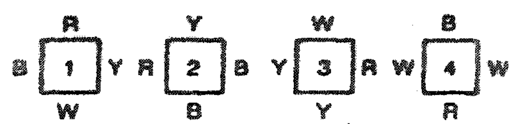


(11)



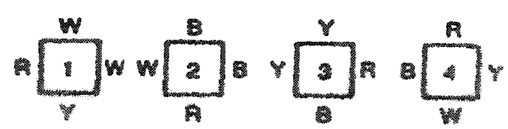
Six of these graphs are connected.

7. (a)



(b) No solution.

(c)



8. (a) There are $(1/2)(7)(6)(5)(4)(3) = 1260$ paths of length 4 in K_7.
(b) The number of paths of length m in K_n, for $0 < m < n$, is $(1/2)(n)(n-1)(n-2)\cdots(n-m)$.

9. (a) Each graph has four vertices that are incident with three edges. In the second graph these vertices (w,x,y,z) form a cycle. This is not so for the corresponding vertices (a,b,g,h) in the first graph. Hence the graphs are *not* isomorphic.

(b) In the first graph the vertex d is incident with four edges. No vertex in the second graph has this property, so the graphs are *not* isomorphic.

10. If G has v vertices and e edges, then by the definition of $\overline{G}$, there are $\binom{v}{2} - e$ edges in $\overline{G}$ since there are $\binom{v}{2}$ edges in K_v.

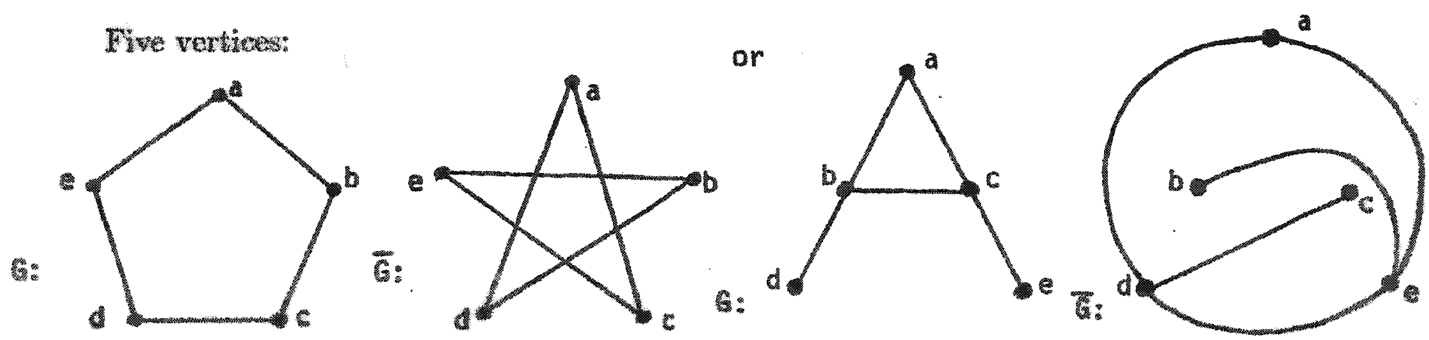
11. (a) If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, then there is a function $f : V_1 \longrightarrow V_2$ that is one-to-one and onto and preserves adjacencies. If $x, y \in V_1$ and $\{x, y\} \notin E_1$, then $\{f(x), f(y)\} \notin E_2$. Hence the same function f preserves adjacencies for $\overline{G}_1, \overline{G}_2$ and can be used to define an isomorphism for $\overline{G}_1, \overline{G}_2$. The converse follows in a similar way.

(b) They are not isomorphic. The complement of the graph containing vertex a is a cycle of length 8. The complement of the other graph is the disjoint union of two cycles of length 4.

12. (a) Let e_1 be the number of edges in G and e_2 the number in $\overline{G}$. For any (loop-free) undirected graph G, $e_1 + e_2 = \binom{n}{2}$, the number of edges in K_n. Since G is self-complementary, $e_1 = e_2$, so $e_1 = (1/2)\binom{n}{2} = n(n-1)/4$.

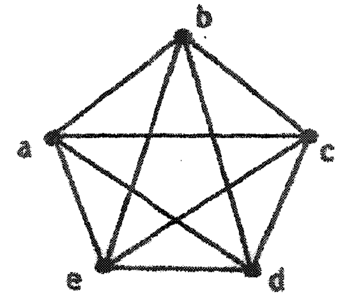
(b) Four vertices:

Five vertices:



(c) From part (a), $4 | n(n-1)$. One of n and $n-1$ is even and the other factor odd. If n is even, then $4 | n$ and $n = 4k$, for some $k \in \mathbf{Z}^+$. If $n-1$ is even, then $4 | (n-1)$ and $n - 1 = 4k$, or $n = 4k + 1$, for some $k \in \mathbf{Z}^+$.

13. If G is the cycle with edges $\{a,b\},\{b,c\},\{c,d\},\{d,e\}$ and $\{e,a\}$, then $\overline{G}$ is the cycle with edges $\{a,c\}$, $\{c,e\}$, $\{e,b\}$, $\{b,d\}$, $\{d,a\}$. Hence, G and $\overline{G}$ are isomorphic. Conversely, if G is a cycle on n vertices and $G,\overline{G}$ are isomorphic, then $n=(1/2)\binom{n}{2}$, or $n=(1/4)(n)(n-1)$, and $n=5$.



14. (a) All of the examples in Exercise 12 above satisfy these conditions.
(b) Since G is not connected, there exist vertices x,y and no path in G connecting these vertices. Hence $\{x,y\}$ is an edge in $\overline{G}$. For each vertex a in G, $a\neq x,y$, either $\{a,x\}$ or $\{a,y\}$ is in $\overline{G}$. If not, both $\{a,x\},\{a,y\}$ are in G and $\{x,a\},\{a,y\}$ provide a path in G connecting x and y. Let $b,c\in V$. If $\{b,x\},\{c,x\}$ are both in $\overline{G}$, there is a path connecting b,c: namely, $\{b,x\},\{x,c\}$. The same is true if $\{b,y\},\{c,y\}$ both occur in $\overline{G}$. If neither of these situations occurs we have $\{b,x\},\{c,y\}$ in $\overline{G}$ (or $\{b,y\},\{c,x\}$) and then the edges $\{b,x\},\{x,y\},\{y,c\}$ provide a path connecting b and c.

15. (a) Here f must also maintain directions. So if $(a,b)\in E_1$, then $(f(a),f(b))\in E_2$.

(b) They are not isomorphic. Consider vertex a in the first graph. It is incident to one vertex and incident from two other vertices. No vertex in the other graph has this property.

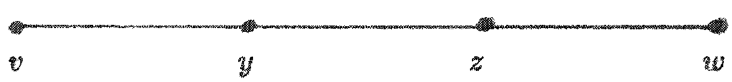
16. (a) $\binom{6}{3}(2^3)=\binom{6}{3}(2^{\binom{3}{2}})$ (b) $\binom{6}{4}(2^{\binom{4}{2}})$

(c) $\sum_{k=1}^{6}\binom{6}{k}(2^{\binom{k}{2}})$ (d) $\sum_{k=1}^{n}\binom{n}{k}(2^{\binom{k}{2}})$

17. There are two cases to consider:

Case 1:

Case 2:



Here there are $n-2$ choices for y – namely, any vertex other than v,w – and there are $n-2$ choices for z – namely, any vertex other than w or the vertex selected for z.

Consequently, there are $(n-1)+(n-2)^2=n^2-3n+3$ walks of length 3 from v to w.

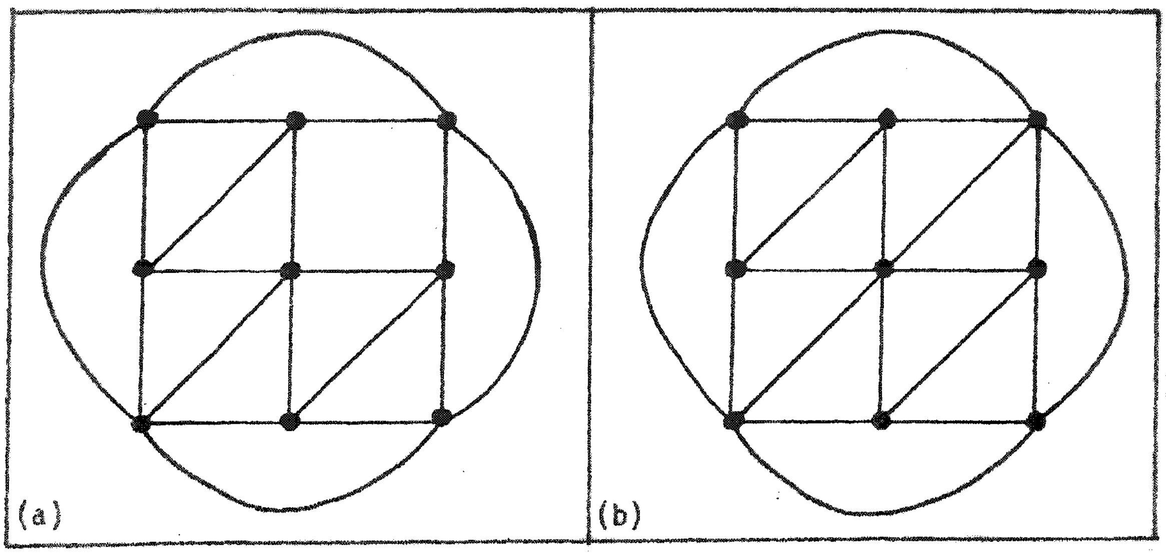
Section 11.3

1. (a) $|V|=6$
(b) $|V|=1$ or 2 or 3 or 5 or 6 or 10 or 15 or 30. [In the first four cases G must be a multigraph; when $|V|=30$, G is disconnected.]

(c) $|V| = 6$

2. $2|E| = 2(17) = 34 = \sum_{v \in V} \deg(v) \geq 3|V|$, so the maximum value of $|V|$ is 11.

3. Since $38 = 2|E| = \sum_{v \in V} \deg(v) \geq 4|V|$, the largest possible value for $|V|$ is 9. We can have (i) seven vertices of degree 4 and two of degree 5; or (ii) eight vertices of degree 4 and one of degree 6. The graph in part (a) of the figure is an example for case (i); an example for case (ii) is provided in part (b) of the figure.



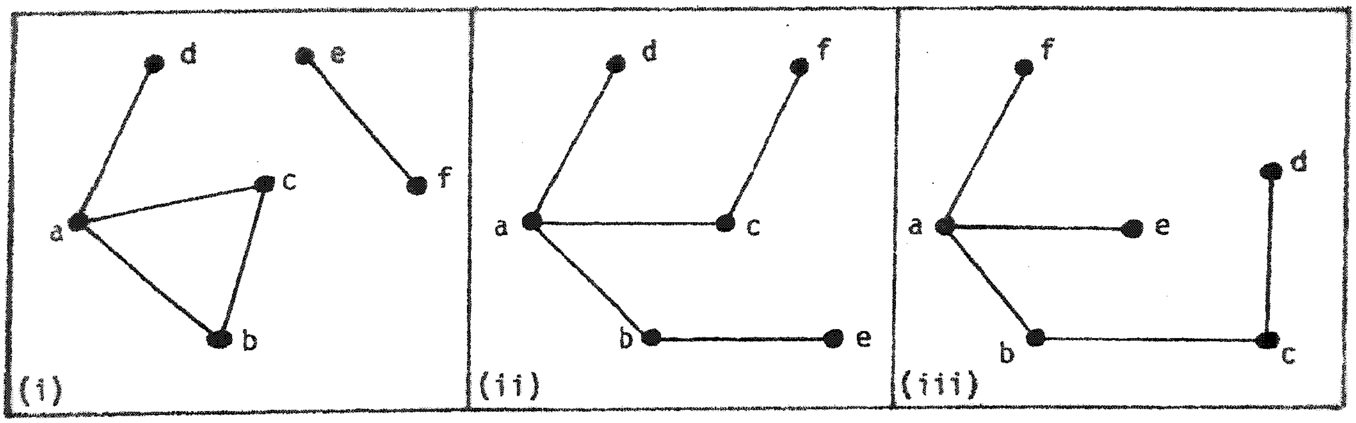
4. a) We must note here that G need *not* be connected. Up to isomorphism G is either a cycle on six vertices or (a disjoint union of) two cycles, each on three vertices.
b) Here G is either a cycle on seven vertices or (a disjoint union of) two cycles — one on three vertices and the other on four.
c) For such a graph G_1, $\overline{G_1}$ is one of the graphs in part (a). Hence there are two such graphs G_1.
d) Here $\overline{G_1}$ is one of the graphs in part (b). There are two such graphs G_1 (up to isomorphism).
e) Let $G_1 = (V_1, E_1)$ be a loop-free undirected $(n-3)$-regular graph with $|V| = n$. Up to isomorphism the number of such graphs G_1 is the number of partitions of n into summands that exceed 2.

5. (a) $|V_1| = 8 = |V_2|$; $|E_1| = 14 = |E_2|$.
(b) For V_1 we find that $\deg(a) = 3$, $\deg(b) = 4$, $\deg(c) = 4$, $\deg(d) = 3$, $\deg(e) = 3$, $\deg(f) = 4$, $\deg(g) = 4$, and $\deg(h) = 3$. For V_2 we have $\deg(s) = 3$, $\deg(t) = 4$, $\deg(u) = 4$, $\deg(v) = 3$, $\deg(w) = 4$, $\deg(x) = 3$, $\deg(y) = 3$, and $\deg(z) = 4$. Hence each of the two graphs has four vertices of degree 3 and four of degree 4.
(c) Despite the results in parts (a) and (b) the graphs G_1 and G_2 are *not* isomorphic.

In the graph G_2 the four vertices of degree 4 — namely, t, u, w, and z — are on a cycle of length 4. For the graph G_1 the vertices b, c, f, and g — each of degree 4 — do not lie on a cycle of length 4.

A second way to observe that G_1 and G_2 are not isomorphic is to consider once again the vertices of degree 4 in each graph. In G_1 these vertices induce a disconnected subgraph consisting of the two edges $\{b,c\}$ and $\{f,g\}$. The four vertices of degree 4 in graph G_2 induce a connected subgraph that has five edges — every possible edge except $\{u,z\}$.

6.



7. a) 19 b) $\sum_{i=1}^{n}\binom{d_i}{2}$ [Note: No assumption about connectedness is made here.]

8. a) There are $8 \cdot 2^7 = 1024$ edges in Q_8.
b) The maximum distance between pairs of vertices is 8. For example, the distance between 00000000 and 11111111 is 8.
c) A longest path in Q_8 contains all of the vertices in Q_8. Such a path has length $2^8 - 1 = 255$.

9. a) $n \cdot 2^{n-1} = 524,288 \Rightarrow n = 16$
b) $n \cdot 2^{n-1} = 4,980,736 \Rightarrow n = 19$, so there are $2^{19} = 524,288$ vertices in this hypercube.

10. The typical path of length 2 uses two edges of the form $\{a,b\}$, $\{b,c\}$. We can select the vertex b as any vertex of Q_n, so there are 2^n choices for b. The vertex b (labeled by a binary n-tuple) is adjacent to n other vertices in Q_n and we can choose two of these in $\binom{n}{2}$ ways. Consequently, there are $\binom{n}{2}2^n$ paths of length 2 in Q_n.

11. The number of edges in K_n is $\binom{n}{2} = n(n-1)/2$. If the edges of K_n can be partitioned into such cycles of length 4, then 4 divides $\binom{n}{2}$ and $\binom{n}{2} = 4t$ for some $t \in \mathbf{Z}^+$. For each vertex v that appears in a cycle, there are two edges (of K_n) incident to v. Consequently, each vertex v of K_n has even degree, so n is odd. Therefore, $n-1$ is even and as $4t = \binom{n}{2} = n(n-1)/2$, it follows that $8t = n(n-1)$. So 8 divides $n(n-1)$, and since n is odd, it follows (from the Fundamental Theorem of Arithmetic) that 8 divides $n-1$. Hence $n-1 = 8k$, or $n = 8k+1$, for some $k \in \mathbf{Z}^+$.

12. a) Let $v \in V$. Then $v\mathcal{R}v$ since v and itself have the same bit in position k and the same bit in position ℓ — hence, $\mathcal{R}$ is reflexive. If $v, w \in V$ and $v\mathcal{R}w$ then v, w have the same bit in position k and the same bit in position ℓ. Hence w, v have the same bit in position k and the same bit in position ℓ. So $w\mathcal{R}v$ and $\mathcal{R}$ is symmetric. Finally, suppose that $v, w, x \in V$ with $v\mathcal{R}w$ and $w\mathcal{R}x$. Then v, w have the same bit in position k and the same bit in position ℓ, and w, x have the same bit in position k and the same bit in position ℓ. Consequently, v, x have the same bit in position k and the same bit in position ℓ, so $v\mathcal{R}x$ — and $\mathcal{R}$ is transitive. In so much as $\mathcal{R}$ is reflexive, symmetric and transitive, it follows that $\mathcal{R}$ is an equivalence relation.

There are four blocks for (the partition induced by) this equivalence relation. Each block contains 2^{n-2} vertices; the vertices in each such block induce a subgraph isomorphic to Q_{n-2}.

(b) For $n \geq 1$ let V denote the vertices in Q_n. For $1 \leq k_1 < k_2 < \ldots < k_t \leq n$ and $w, x \in V$ define the relation $\mathcal{R}$ on V by $w\mathcal{R}x$ if w, x have the same bit in position k_1, the same bit in position $k_2, \ldots$, and the same bit in position k_t. Then $\mathcal{R}$ is an equivalence relation for V and it partitions V into 2^t blocks. Each block contains 2^{n-t} vertices and the vertices in each such block induce a subgraph of Q_n isomorphic to Q_{n-t}.

13. $\delta|V| \leq \sum_{v \in V} \deg(v) \leq \Delta|V|$. Since $2|E| = \sum_{v \in V} \deg(v)$, it follows that $\delta|V| \leq 2|E| \leq \Delta|V|$ so $\delta \leq 2(e/n) \leq \Delta$.

14. (a) f^{-1} is one-to-one and onto. Let $x, y \in V'$ and $\{x, y\} \in E'$. Then f one-to-one and onto $\Longrightarrow$ there exist unique $a, b \in V$ with $f(a) = x$, $f(b) = y$. If $\{a, b\} \notin E$, then $\{f(a), f(b)\} \notin E'$.

(b) If $\deg(a) = n$, then there exist $x_1, x_2, \ldots, x_n \in V$ and $\{a, x_i\} \in E, 1 \leq i \leq n$. Hence, the edge $\{f(a), f(x_i)\} \in E'$ for all $1 \leq i \leq n$, so $\deg(f(a)) \geq n$. If $\deg(f(a)) > n$, let $y \in V'$ such that $y \neq f(x_i)$ for all $1 \leq i \leq n$, and $y = f(x)$. Since f^{-1} is an isomorphism by part (a), $\{a, x\} \in E$ and $\deg(a) > n$. Hence $\deg f(a) = n$.

15. Proof: Start with a cycle $v_1 \to v_2 \to v_3 \to \ldots \to v_{2k-1} \to v_{2k} \to v_1$. Then draw the k edges $\{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \ldots, \{v_i, v_{i+k}\}, \ldots, \{v_k, v_{2k}\}$. The resulting graph has $2k$ vertices each of degree 3.

16. Proof: (By the Alternative Form of the Principle of Mathematical Induction)
The result is true for $n = 1$ (for the complete graph K_2) and for $n = 2$ (for the path on four vertices). So let us assume the result for all $1 \leq n \leq k$, and consider the case for $n = k+1$. Let G' be a graph for $n = k-1$, and add to this graph two isolated vertices x and y. Now introduce two other vertices a and b and the edge $\{a, b\}$. Draw an edge between a and x, and between a and $k-1$ of the vertices (one of each of the degrees $1, 2, \ldots, k-1$) in G'. Now draw an edge between b and y, and between b and the other $k-1$ vertices in G' (the vertices not adjacent to vertex a). The resulting graph has $2(k+1)$ vertices where exactly two vertices have degree i for all $1 \leq i \leq k+1$.
Consequently, the result follows for all $n \in \mathbf{Z}^+$ by the Alternative Form of the Principle of

Mathematical Induction.

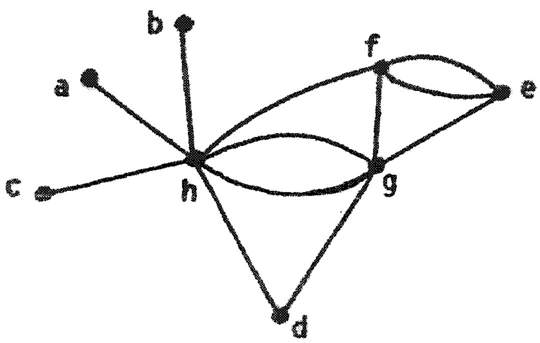
17. (Corollary 11.1) Let $V = V_1 \cup V_2$ where V_1 (V_2) contains all vertices of odd (even) degree. Then $2|E| - \sum_{v \in V_2} \deg(v) = \sum_{v \in V_1} \deg(v)$ is an even integer. For $|V_1|$ odd, $\sum_{v \in V_1} \deg(v)$ is odd.

(Corollary 11.2) For the converse let $G = (V, E)$ have an Euler trail with a, b as the starting and terminating vertices, respectively. Add the edge $\{a, b\}$ to G to form the graph $G' = (V, E')$, where G' has an Euler circuit. Hence G' is connected and each vertex has even degree. Removing edge $\{a, b\}$ the vertices in G will have the same even degree except for a, b. $\deg_G(a) = \deg_{G'}(a) - 1, \deg_G(b) = \deg_{G'}(b) - 1$, so the vertices a, b have odd degree in G. Also, since the edges in G form an Euler trail, G is connected.

18. Select $v_1, v_2 \in V$ where $\{v_1, v_2\} \in E$. Such an edge must exist since $V \neq \emptyset$ and $\deg(v) \geq k \geq 1$ for all $v \in V$. If $k = 1$ the result follows. If $k > 1$, suppose that we have selected $v_1, v_2, \ldots, v_k \in V$ with $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\} \in E$. Since $\deg(v_k) \geq k$, there exists $v_{k+1} \in V_1$ where $v_{k+1} \neq v_i$ for $1 \leq i \leq k-1$, and $\{v_k, v_{k+1}\} \in E$. Then $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}$ provides a path of length k.

19. (a) Let $a, b, c, x, y \in V$ with $\deg(a) = \deg(b) = \deg(c) = 1$, $\deg(x) = 5$, and $\deg(y) = 7$. Since $\deg(y) = 7$, y is adjacent to all of the other (seven) vertices in V. Therefore vertex x is not adjacent to any of the vertices a, b, and c. Since x cannot be adjacent to itself, unless we have loops, it follows that $\deg(x) \leq 4$, and we cannot draw a graph for the given conditions.

(b)



20. (a) $a \to b \to c \to g \to k \to j \to g \to b \to f \to j \to i \to f \to e \to i \to h \to d \to e \to b \to d \to a$

(b) $d \to a \to b \to d \to h \to i \to e \to f \to i \to j \to f \to b \to c \to g \to k \to j \to g \to b \to e$

21. n odd: $n = 2$

22. 1; Any single bridge.

23. Yes. Model the situation with a graph where there is a vertex for each room and the surrounding corridor. Draw an edge between two vertices if there is a door common to both rooms, or a room and the surrounding corridor. The resulting multigraph is connected with every vertex of even degree.

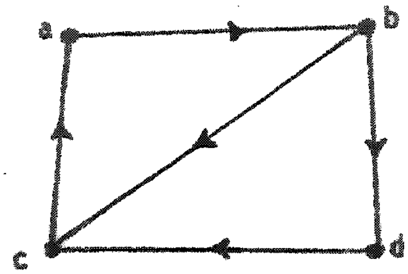
24. We find that $\sum_{v\in V} \mathrm{id}(v) = e = \sum_{v\in V} \mathrm{od}(v)$.

25. (a) (i) Let the vertices of K_6 be $v_1, v_2, v_3, v_4, v_5, v_6$, where $\deg(v_i) = 5$ for all $1 \le i \le 6$. Consider the subgraph S of K_6 obtained (from K_6) by deleting the edges $\{v_2, v_5\}$ and $\{v_3, v_6\}$. Then S is connected with $\deg(v_1) = \deg(v_4) = 5$, and $\deg(v_i) = 4$ for $i \in \{2, 3, 5, 6\}$. Hence S has an Euler trail that starts at v_1 (or v_4) and terminates at v_4 (or v_1). This Euler trail in S is then a trail of maximum length in K_6, and its length is $\binom{6}{2} - (1/2)[6-2] = 15 - 2 = 13$.
(ii) $\binom{8}{2} - (1/2)[8-2] = 28 - 3 = 25$
(iii) $\binom{10}{2} - (1/2)[10-2] = 45 - 4 = 41$
(iv) $\binom{2n}{2} - (1/2)[2n-2] = n(2n-1) - (n-1) = 2n^2 - 2n + 1$.

(b) (i) Label the vertices of K_6 as in section (i) of part (a) above. Now consider the subgraph T of K_6 obtained (from K_6) by deleting the edges $\{v_1, v_4\}$, $\{v_2, v_5\}$, and $\{v_3, v_6\}$. Then T is connected with $\deg(v_i) = 4$ for all $1 \le i \le n$. Hence T has an Euler circuit and this Euler circuit for T is then a circuit of maximum length in K_6. The length of the circuit is $\binom{6}{2} - (1/2)(6) = 15 - 3 = 12$.
(ii) $\binom{8}{2} - (1/2)(8) = 28 - 4 = 24$
(iii) $\binom{10}{2} - (1/2)(10) = 45 - 5 = 40$
(iv) $\binom{2n}{2} - (1/2)(2n) = n(2n-1) - n = 2n^2 - 2n = 2n(n-1)$.

26. (a) If $G = (V, E)$ has a directed Euler circuit, then for all $x, y \in V$ there is a directed trail from x to y (that part of the directed Euler circuit from x to y). This results in a directed path from x to y, as well as one from y to x. Hence G is connected (in fact, G is strongly connected as defined in part (b) of this exercise). Let s be the starting vertex (and terminal vertex) of the directed Euler circuit. For every $v \in V, v \ne s$, each time the circuit comes upon vertex v it must also leave the vertex, so $\mathrm{od}(v) = \mathrm{id}(v)$. In the case of s the last edge of the circuit is different from the first edge and $\mathrm{od}(s) = \mathrm{id}(s)$.

Conversely, if G satisifies the stated conditons, we shall prove by induction on $|E|$ that G has a directed Euler circuit. For $|E| = 1$ the result is true (and the graph consists of a (directed) loop on one vertex). We assume the result for all such graphs with $|E|$ edges where $1 \le |E| < n$. Now consider a directed graph $G = (V, E)$ where G satisfies the given conditions and $|E| = n$. Let $a \in V$. There exists a circuit in G that contains a. If the loop $(a, a) \notin E$, then there is an edge $(a, b) \in E$ for $b \ne a$. If not, a is isolated and this contradicts G being connected. If $(b, a) \in E$ we have the circuit $\{(a, b), (b, a)\}$ containing a. If $(b, a) \notin E$, then there is an edge of the form (b, c), $c \ne b$, $c \ne a$, because $\mathrm{od}(b) = \mathrm{id}(b)$. Continuing this process, since $\mathrm{od}(a) = \mathrm{id}(a)$ and G is finite, we obtain a directed circuit C containing a. If $C = G$ we are finished. If not, remove the edges of C from G, along with any vertex that becomes isolated. The resulting subgraph $H = (V_1, E_1)$ is such that (in H) $\mathrm{od}(v) = \mathrm{id}(v)$ for all $v \in V_1$. However, H is not necessarily connected. But each component of H is connected with $\text{od}(v) = \text{id}(v)$ for each vertex in a component. Consequently, by the induction hypothesis, each component of H has a directed Euler circuit, and each component has a vertex on the circuit C (from above). Hence, starting at vertex a we travel on C until we encounter a vertex v_1 on the directed Euler circuit of the component C_1 of H. Traversing C_1 we return to v_1 and continue on C to vertex v_2 on component C_2 of H. Continuing the process, with G finite we obtain a directed Euler circuit for G.

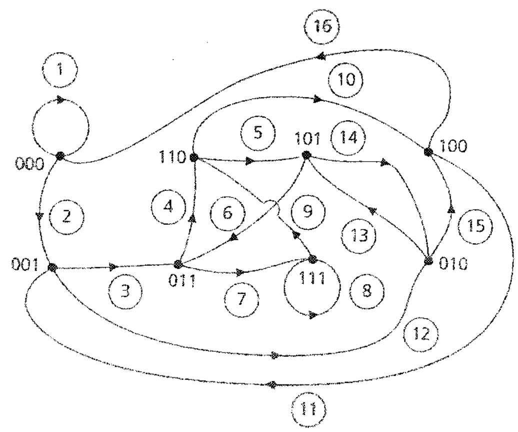


(b) If $G = (V, E)$ is a directed graph with a directed Euler circuit then for all $x, y \in V$, $x \neq y$, there is a directed path from x to y, and one from y to x, so the graph is strongly connected. The converse, however, is false. The directed graph shown here is strongly connected. However, since $\text{od}(b) \neq \text{id}(b)$ the graph does not have a directed Euler circuit.

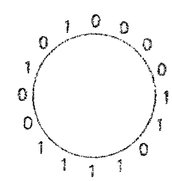
27. From Exercise 24 we see that $\sum_{v \in V}[\text{od}(v) - \text{id}(v)] = 0$. For each $v \in V$, $\text{od}(v) + \text{id}(v) = n - 1$, so $0 = (n-1) \cdot 0 = \sum_{v \in V}(n-1)[\text{od}(v) - \text{id}(v)] = \sum_{v \in V}[\text{od}(v) + \text{id}(v)][\text{od}(v) - \text{id}(v)] = \sum_{v \in V}[(\text{od}(v))^2 - (\text{id}(v))^2]$, and the result follows.

28. Let G be a directed graph satisfying the three conditions. Add the edge (x, y). Then by part (a) of Exercise 26 the resulting graph has a directed Euler circuit C. Removing (x, y) from C yields a directed Euler trail for the given graph G. (This trail starts at y and terminates at x.) In a similar manner we find that if a directed graph G has a directed Euler trail then it satisfies the three conditions.

29. (a) and (b)



(c)



30. 3; 3

31. Let $|V| = n \geq 2$. Since G is loop-free and connected, for all $x \in V$ we have $1 \leq \deg(x) \leq n - 1$. Apply the pigeonhole principle with the n vertices as the pigeons and the $n - 1$ possible degrees as the pigeonholes.

32. (a)

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$I = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) If there is a walk of length two between v_i and v_j, denote this by $\{v_i, v_k\}, \{v_k, v_j\}$. Then $a_{ik} = a_{kj} = 1$ in A and the (i,j)-entry in A^2 is 1. Conversely, if the (i,j)-entry of A^2 is 1 then there is at least one value of k, $1 \leq k \leq n$, such that $a_{ik} = a_{kj} = 1$, and this indicates the existence of a walk $\{v_i, v_k\}, \{v_k, v_j\}$ between the ith and jth vertices of V.

(c) For all $1 \leq i, j \leq n$, the (i,j)-entry of A^2 counts the number of distinct walks of length two between the ith and jth vertices of V.

(d) For v at the top of the column, the column sum is the degree of v, if there is no loop at v. Otherwise, $\deg(v) =$ [(column sum for v) $- 1] + 2$ (number of loops at v).

(e) For each column of I the column sum is 1 for a loop and 2 for an edge that is not a loop.

33. (a) Label the rows and columns of the first matrix with a, b, c. Then the graph for this adjacency matrix is a path of two edges where $\deg(a) = \deg(b) = 1$ and $\deg(c) = 2$.

Now label the rows and columns of the second matrix with x, y, z. The graph for this adjacency matrix is a path of two edges where $\deg(y) = \deg(z) = 1$ and $\deg(x) = 2$.

Define $f : \{a, b, c\} \to \{x, y, z\}$ by $f(a) = y$, $f(b) = z$, $f(c) = x$. This function provides an isomorphism for these two graphs.

Alternatively, if we start with the first matrix and interchange rows 1 and 3 and then interchange columns 1 and 3 (on the resulting matrix), we obtain the second matrix. This also shows us that the graphs (corresponding to these adjacency matrices) are isomorphic.

(b) Yes
(c) No

34. (a) Here each graph is a cycle on three vertices – so they are isomorphic.

(b) The graphs here are not isomorphic. The graph for the first incidence matrix is a cycle of length 3 with the fourth (remaining) edge incident with one of the cycle vertices. The second graph is a cycle on four vertices.
(c) Yes

35. No. Let each person represent a vertex for a graph. If v, w represent two of these people, draw the edge $\{v, w\}$ if the two shake hands. If the situation were possible, then we would have a graph with 15 vertices, each of degree 3. So the sum of the degrees of the vertices would be 45, an odd integer. This contradicts Theorem 11.2.

36. Define the function f from the domain $A \times B$ (or the set of processors of the grid) to the codomain of corresponding vertices of Q_5 as follows:

$$f((ab, cde)) = abcde, \text{ where } ab \in A,\ cde \in B, \text{ and } a, b, c, d, e \in \{0, 1\}.$$

If $f((ab, cde)) = f(a_1b_1, c_1d_1e_1))$, then $abcde = a_1b_1c_1d_1e_1$, so $a = a_1$, $b = b_1$, $c = c_1$, $d = d_1$, $e = e_1$, and $(ab, cde) = (a_1b_1, c_1d_1e_1)$, making f one-to-one. Since $|A \times B| = 15 =$ the number of vertices (of Q_5) in the codomain of f, it follows from Theorem 5.11 that f is also onto.

Now let $\{(ab, cde), (vw, xyz)\}$ be an edge in the 3×5 grid. Then either $ab = vw$ and cde, xyz differ in (exactly) one component or $cde = xyz$ and ab, vw differ in (exactly) one component. Suppose that $ab = vw$ (so $a = v, b = w$) and $c = x$, $d = y$, but $e \neq z$. Then $\{abcde, vwxyz\}$ is an edge in Q_5. [The other four cases follow in a similar way.] Conversely, suppose that $\{f(a_1b_1, c_1d_1e_1), f(v_1w_1, x_1y_1z_1)\}$ is an edge in the subgraph of Q_5 induced by the codomain of f. Then $a_1b_1c_1d_1e_1$ and $v_1w_1x_1y_1z_1$ differ in (exactly) one component – say the last. Then in the 3×5 grid, there is an edge for the vertices (a_1b_1, c_1d_10), $(a_1b_1c_1d_11)$. [Similar arguments can be given for any of the other first four components.] Consequently, f provides an isomorphism between the 3×5 grid and a subgraph of Q_5.

[Note that the 3×5 grid has 22 edges while Q_5 has $5 \cdot 2^4 = 80$ edges.]

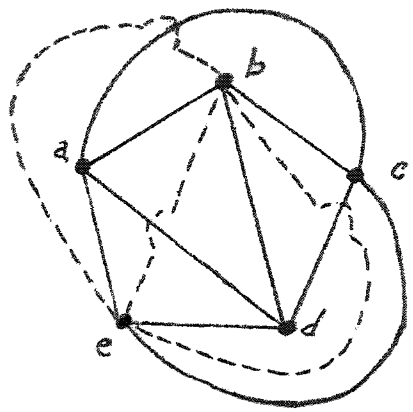
37. Assign the Gray code $\{00, 01, 11, 10\}$ to the four horizontal levels: top – 00; second (from the top) – 01; second from the bottom – 11; bottom – 10. Likewise, assign the same code to the four vertical levels: left (or, first) – 00; second – 01; third – 11; right (or, fourth) – 10. This provides the labels for $p_1, p_2, \ldots, p_{16}$, where, for instance, p_1 has the label $(00, 00)$, p_2 has the label $(01, 00), \ldots, p_7$ has the label $(11, 01), \ldots, p_{11}$ has the label $(11, 10)$, and p_{16} has the label $(10, 10)$.

Define the function f from the set of 16 vertices of this grid to the vertices of Q_4 by $f((ab, cd)) = abcd$. Here $f((ab, cd)) = f((a_1b_1, c_1d_1)) \Rightarrow abcd = a_1b_1c_1d_1 \Rightarrow a = a_1, b = b_1, c = c_1, d = d_1 \Rightarrow (ab, cd) = (a_1b_1, c_1d_1) \Rightarrow f$ is one-to-one. Since the domain and codomain of f both contain 16 vertices, it follows from Theorem 5.11 that f is also onto. Finally, let $\{(ab, cd), (wx, yz)\}$ be an edge in the grid. Then either $ab = wx$ and cd, yz differ in one component or $cd = yz$ and ab, wx differ in one component. Suppose that $ab = wx$ and $c = y$, but $d \neq z$. Then $\{abcd, wxyz\}$ is an edge in Q_4. The other cases follow in a similar way. Conversely, suppose that $\{f((a_1b_1, c_1d_1)), f((w_1x_1, y_1z_1))\}$ is an edge in Q_4. Then $a_1b_1c_1d_1, w_1x_1y_1z_1$ differ in exactly one component – say the first. Then in the grid, there is an edge for the vertices $(0b_1, c_1d_1)$, $(1b_1, c_1d_1)$. The arguments are similar for the other three components. Consequently, f establishes an isomorphism between the three-by-three grid and a subgraph of Q_4.

[Note: The three-by-three grid has 24 edges while Q_4 has 32 edges.]

Section 11.4

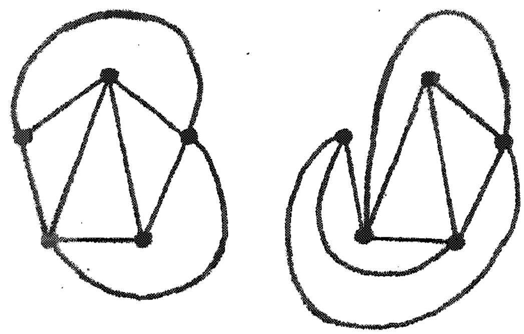
1.



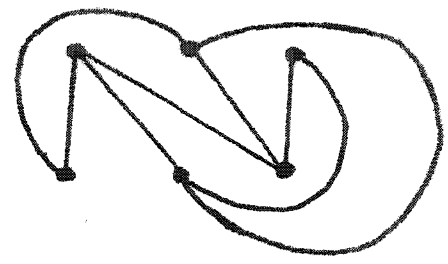
In this situation vertex b is in the region formed by the edges {a,d}, {d,c}, {c,a} and vertex e is outside of this region. Consequently the edge {b,e} will cross one of the edges {a,d}, {d,c}, {a,c} (as shown).

2. From the symmetry in these graphs the following demonstrate the situations we must consider

K_5 :



$K_{3,3}$:



3. (a)

Graph	Number of vertices	Number of edges
$K_{4,7}$	11	28
$K_{7,11}$	18	77
$K_{m,n}$	$m+n$	mn

(b) $m = 6$

4. Let $G = (V, E)$ be bipartite with V partitioned as $V_1 \cup V_2$, so that each edge in E is of the form $\{a, b\}$ where $a \in V_1$, $b \in V_2$. If H is a subgraph of G let W denote the set of vertices for H. Then $W = W \cap V = W \cap (V_1 \cup V_2) = (W \cap V_1) \cup (W \cap V_2)$, where $(W \cap V_1) \cap (W \cap V_2) = \emptyset$. If $\{x, y\}$ is an edge in H then $\{x, y\}$ is an edge in G — where, say, $x \in V_1$ and $y \in V_2$. Hence $x \in W_1$, $y \in W_2$ and H is a bipartite graph.

5. (a) Let $V_1 = \{a, d, e, h\}$ and $V_2 = \{b, c, f, g\}$. Then every vertex of G is in $V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Also every edge in G may be written as $\{x, y\}$ where $x \in V_1$ and $y \in V_2$. Consequently, the graph G in part (a) of the figure is bipartite.
(b) Let $V_1' = \{a, b, g, h\}$ and $V_2' = \{c, d, e, f\}$. Then every vertex of G' is in $V_1' \cup V_2'$ and $V_1' \cap V_2' = \emptyset$. Since every edge of G' may be written as $\{x, y\}$, with $x \in V_1'$ and $y \in V_2'$, it follows that this graph is bipartite. In fact G' is (isomorphic to) the complete bipartite graph $K_{4,4}$.
(c) This graph is *not* bipartite. If $G'' = (V'', E'')$ were bipartite, let the vertices of G'' be partitioned as $V_1'' \cup V_2''$, where each edge in G'' is of the form $\{x, y\}$ with $x \in V_1''$ and $y \in V_2''$. We assume vertex a is in V_1''. Now consider the vertices b, c, d, and e. Since $\{a, b\}$ and $\{a, c\}$ are edges of G'' we must have b, c in V_2''. Also, $\{b, d\}$ is an edge in the graph, so d is in V_1''. But then $\{d, e\} \in E'' \Rightarrow e \in V_2''$, while $\{c, e\} \in E'' \Rightarrow e \in V_1''$.

6. There are four vertices in $K_{1,3}$ and we can select four vertices from those of K_n in $\binom{n}{4}$ ways. Since each of the four vertices (in each of the $\binom{n}{4}$ selections) can be the unique vertex of degree 3 in $K_{1,3}$, there are $4\binom{n}{4}$ subgraphs of K_n that are isomorphic to $K_{1,3}$.

Alternately, select the vertex of degree 3 in $K_{1,3}$ — this can be done in n ways. Then select the remaining pendant vertices — this can be done in $\binom{n-1}{3}$ ways. Hence the number of subgraphs of K_n that are isomorphic to $K_{1,3}$ is

$$n\binom{n-1}{3} = (n)(n-1)(n-2)(n-3)/6 = (4)[(n)(n-1)(n-2)(n-3)/24] = 4\binom{n}{4}.$$

7. The vertices in $K_{m,n}$ may be partitioned as $V_1 \cup V_2$ where $|V_1| = m$, $|V_2| = n$, and each edge of the graph has the form $\{x, y\}$ where $x \in V_1$ and $y \in V_2$.

(a) In order to obtain a cycle of length four we need to select two vertices from each of V_1 and V_2. This can be done in $\binom{m}{2}\binom{n}{2}$ ways — each resulting in a distinct cycle of length four.
[Note: Say we select vertices a, b from V_1 and vertices c, d from V_2. We do *not* distinguish the cycles $a \to c \to b \to d \to a$ and $a \to d \to b \to c \to a$.]
(b) For a path of length two there is one vertex of (path) degree 2 and two vertices of (path) degree 1. If the vertex of (path) degree 2 is in V_1 then there are $m\binom{n}{2}$ such paths. There are $n\binom{m}{2}$ such paths when the vertex of (path) degree 2 is in V_2. Hence there are $m\binom{n}{2} + n\binom{m}{2} = (1/2)(mn)[m + n - 2]$ paths of length 2 in $K_{m,n}$.
(c) Here a path of length 3 has the form $a \to b \to c \to d$ where $a, c \in V_1$ and $b, d \in V_2$. By the rule of product there are $(m)(n)(m-1)(n-1) = 4\binom{m}{2}\binom{n}{2}$ such paths in $K_{m,n}$.

8. (a) 2 (b) 6 $(= 2(3))$ (c) 14 $(= 2(7))$ (d) $2m$

9. (a) 6 (b) $(1/2)(7)(3)(6)(2)(5)(1)(4) = 2520$ (c) 50,295,168,000
(d) $(1/2)(n)(m)(n-1)(m-1)(n-2)\cdots(2)(n-(m+1))(1)(n-m)$

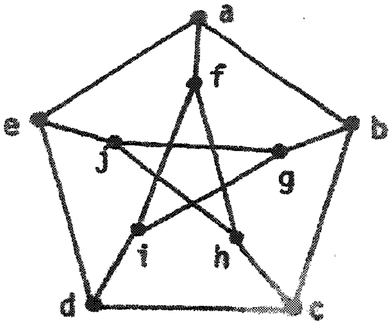
10. Let $G = (V, E)$ be bipartite with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$. If G has a cycle of odd length then there is an edge in the cycle of the form $\{x, y\}$ with $x, y \in V_1$ (or $x, y \in V_2$). This contradicts the definition of a bipartite graph.

11. Partition V as $V_1 \cup V_2$ with $|V_1| = m$, $|V_2| = v - m$. If G is bipartite, then the maximum number of edges that G can have is $m(v-m) = -[m-(v/2)]^2 + (v/2)^2$, a function of m. For a given value of v, when v is even, $m = v/2$ maximizes $m(v-m) = (v/2)[v-(v/2)] = (v/2)^2$. For v odd, $m = (v-1)/2$ or $m = (v+1)/2$ maximizes $m(v-m) = [(v-1)/2][v-((v-1)/2)] = [(v-1)/2][(v+1)/2] = [(v+1)/2][v-((v+1)/2)] = (v^2-1)/4 = \lfloor (v/2)^2 \rfloor < (v/2)^2$. Hence if $|E| > (v/2)^2$, G cannot be bipartite.

12. (a) There are 3: (i) $K_{1,5}$; (ii) $K_{2,4}$; and (iii) $K_{3,3}$.
(b) $\lfloor n/2 \rfloor$ for $n \in \mathbf{Z}^+$, $n \geq 2$.

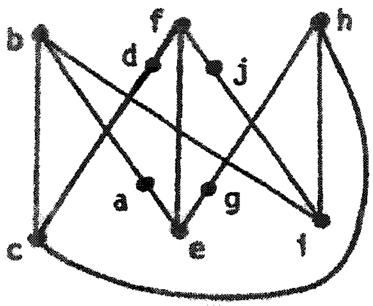
13. (a)

a: {1,2}	f: {4,5}
b: {3,4}	g: {2,5}
c: {1,5}	h: {2,3}
d: {2,4}	i: {1,3}
e: {3,5}	j: {1,4}

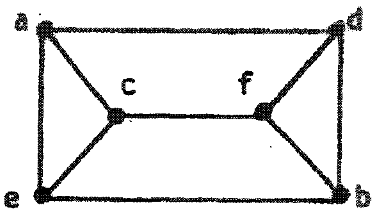


(b) G is (isomorphic to) the Petersen graph. (See Fig. 11.52(a)).

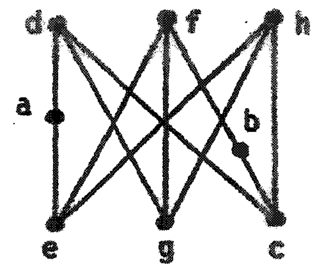
14. (1)



(2)

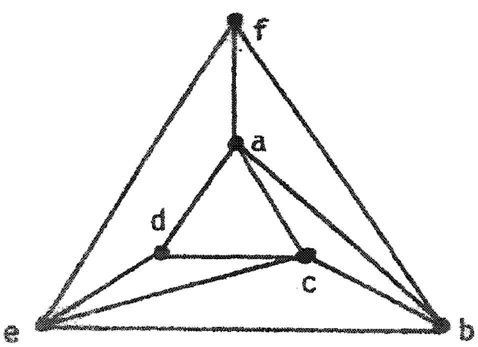


(3)

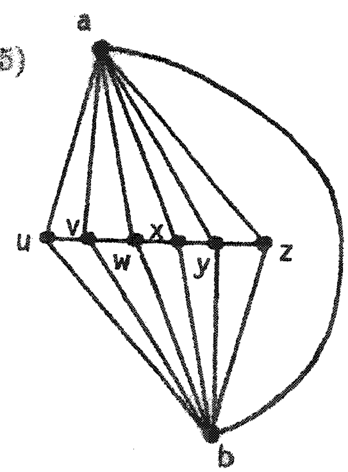


Graph (1) shows that the first graph contains a subgraph homeomorphic to $K_{3,3}$, so it is not planar. The second graph is planar and isomorphic to the second graph of the exercise. The third graph provides a subgraph homeomorphic to $K_{3,3}$ so the third graph given here is not planar. Graph (6) is not planar because it contains a subgraph homeomorphic to K_5.

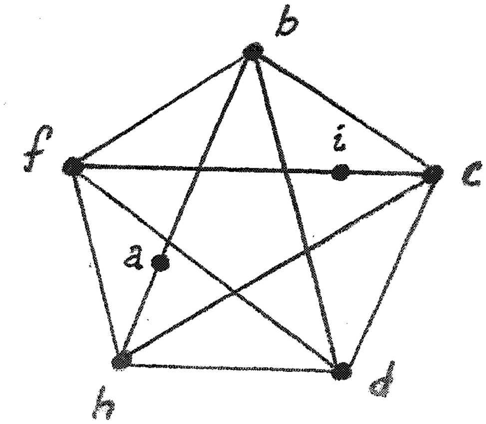
(4)



(5)



(6)



15. The result follows if and only if mn is even (that is, at least one of m, n is even).

Suppose, without loss of generality, that m is even — say, $m = 2t$. Let V denote the vertex set of $K_{m,n}$ where $V = V_1 \cup V_2$ and $V_1 = \{v_1, v_2, \ldots, v_t, v_{t+1}, \ldots, v_m\}$, $V_2 = \{w_1, w_1, \ldots, w_n\}$. The mn edges in $K_{m,n}$ are of the form $\{v_i, w_j\}$ where $1 \leq i \leq m$, $1 \leq j \leq n$. Now consider the subgraphs G_1, G_2 of $K_{m,n}$ where G_1 is induced by $\{v_1, v_2, \ldots, v_t\} \cup V_2$ and G_2 is induced by $\{v_{t+1}, v_{t+2}, \ldots, v_m\} \cup V_2$. Each of G_1, G_2 is isomorphic to $K_{t,n}$, and every edge in $K_{m,n}$ is in exactly one of G_1, G_2.

If both m, n are odd, then $K_{m,n}$ has an odd number of edges and cannot be decomposed into two isomorphic subgraphs — since each such subgraph has the same number of edges as the other.

16. Consider how the vertices of the Petersen graph are labeled in Fig. 11.52(a). The following correspondence of vertices provides an isomorphism for the two graphs:

$$\begin{array}{ccccc} a \to s & b \to v & c \to z & d \to y & e \to t \\ f \to u & g \to r & h \to w & i \to x & j \to q \end{array}$$

17. (a) There are 17 vertices, 34 edges and 19 regions and $v - e + r = 17 - 34 + 19 = 2$.
(b) Here we find 10 vertices, 24 edges and 16 regions and $v - e + r = 10 - 24 + 16 = 2$.

18. Proof: Since each region has at least five edges in its boundary, $2|E| > 5(53)$, or $|E| \geq (1/2)(5)(53)$. And from Theorem 11.6 we have $|V| = |E| - 53 + 2 = |E| - 51 \geq (1/2)(5)(53) - 51 = (265/2) - 51 = 81\frac{1}{2}$. Hence $|V| \geq 82$.

19. 10

20. (a) For each component $C_i = (V_i, E_i)$, $1 \leq i \leq n$, of G, if $e_i = |E_i|$ and $v_i = |V_i|$ then $e_i - v_i + 2 = r_i$. Summing as i goes from 1 to n we have $e - v + 2n = r + (n-1)$ because the infinite region is counted $n = \kappa(G)$ times. Hence $e - v + n + 1 = r = e - v + [\kappa(G) + 1]$.

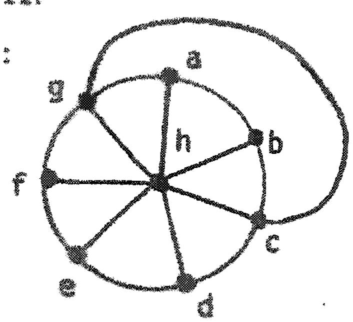
(b) Using the same notation as in part (a) we have $3r_i \leq 2e_i$, $1 \leq i \leq n$, so $3r \leq \sum_{i=1}^{n}(3r_i) \leq \sum_{i=1}^{n} 2e_i = 2e$. Also, $e_i \leq 3v_i - 6$, $1 \leq i \leq n$, so $e = \sum_{i=1}^{n} e_i \leq \sum_{i=1}^{n}(3v_i - 6) = 3v - 6n \leq 3v - 6$.

21. If not, $\deg(v) \geq 6$ for all $v \in V$. Then $2e = \sum_{v \in V} \deg(v) \geq 6|V|$, so $e \geq 3|V|$, contradicting $e \leq 3|V| - 6$ (Corollary 11.3.)

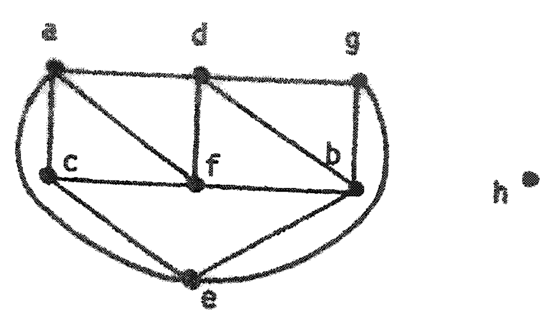
22. (a) Suppose that $G = (V, E)$ with $|V| = 11$. Then $\overline{G} = (V, E_1)$ where $\{a, b\} \in E_1$ iff $\{a, b\} \notin E$. Let $e = |E|$, $e_1 = |E_1|$. If both G and $\overline{G}$ are planar, then by Corollary 11.3 (and part (b) of Exercise 20, if necessary), $e \leq 3|V| - 6 = 33 - 6 = 27$ and $e_1 \leq 3|V| - 6 = 27$. But with $|V| = 11$, there are $\binom{11}{2} = 55$ edges in K_{11}, so $|E| + |E_1| = 55$ and either $e \geq 28$ or $e_1 \geq 28$. Hence, one of G, $\overline{G}$ must be planar.

If $G = (V, E)$ and $|V| > 11$, consider an induced subgraph of G on $V' \subset V$ where $|V'| = 11$.

(b) G :



$\overline{G}$:



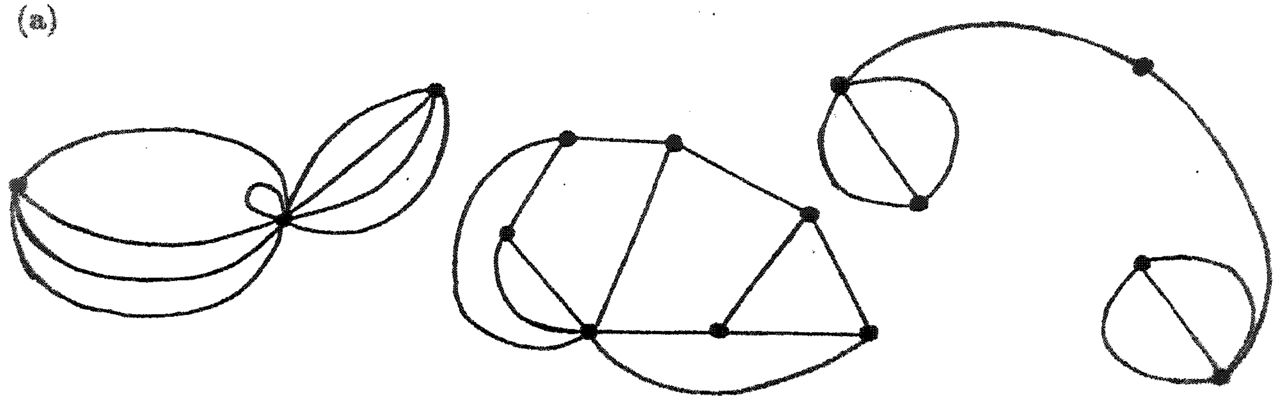
23. (a) $2e \geq kr = k(2+e-v) \Longrightarrow (2-k)e \geq k(2-v) \Longrightarrow e \leq [k/(k-2)](v-2)$.

(b) 4

(c) In $K_{3,3}$, $e = 9$, $v = 6$. $[k/(k-2)](v-2) = (4/2)(4) = 8 < 9 = e$. Since $K_{3,3}$ is connected, it must be nonplanar.

(d) Here $k = 5$, $v = 10$, $e = 15$ and $[k/(k-2)](v-2) = (5/3)(8) = (40/3) < 15 = e$. Since the Petersen graph is connected, it must be nonplanar.

24. (a)

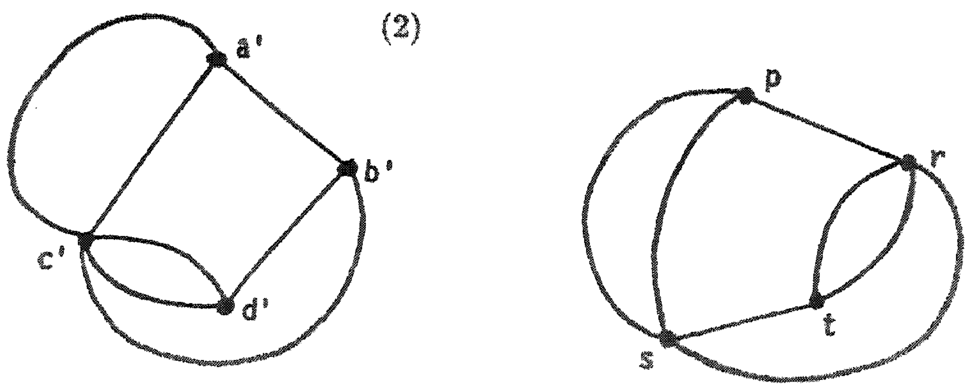


(b) There are no pendant vertices. But this does not contradict the condition mentioned because the loops contain other vertices and edges of the graph.

25. (a) The dual for the tetrahedron (Fig. 11.59(b)) is the graph itself. For the graph (cube) in Fig. 11.59(d) the dual is the octahedron, and vice versa. Likewise, the dual of the dodecahedron is the icosahedron, and vice versa.
(b) For $n \in \mathbf{Z}^+$, $n \geq 3$, the dual of the wheel graph W_n is W_n itself.

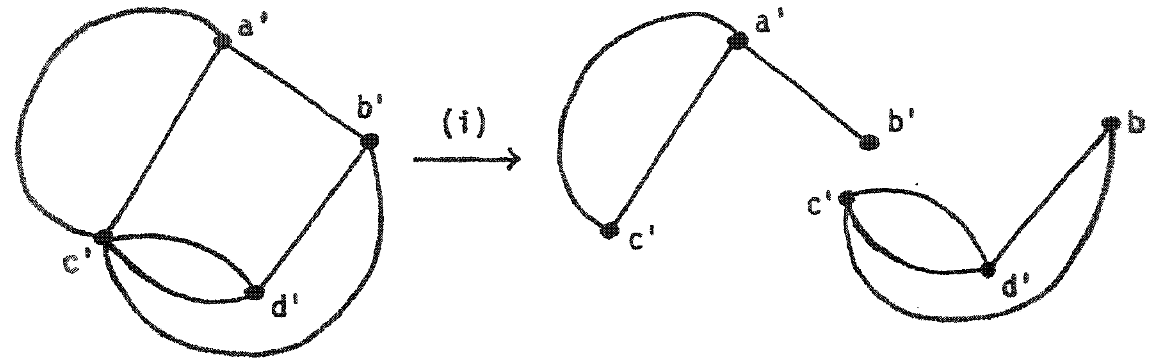
26. (a) The correspondence $a \to v$, $b \to w$, $c \to y$, $d \to z$, $e \to x$ provides an isomorphism.

(b) (1) (2)



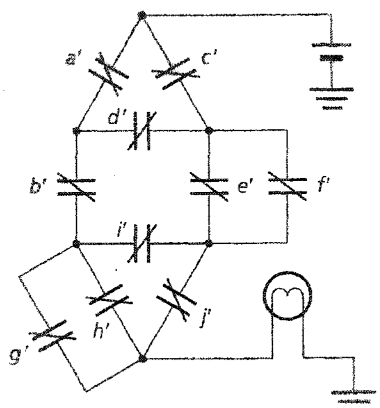
(c) In the first graph in part (b) vertex c' has degree 5. Since no vertex has degree 5 in the second graph, the two graphs cannot be isomorphic.

(d)



(e) $\{\{a', c'\}, \{c', b'\}, \{b', a'\}\}$; $\{\{p, r\}, \{r, t\}, \{r, t\}, \{r, s\}\}$.

27.



28. The number of vertices in G^d, the dual of G, is r, the number of regions in a planar depiction of G. Since G is isomorphic to G^d it follows that $r = n$. Consequently, $|V| - |E| + r = 2 \Rightarrow n - |E| + n = 2 \Rightarrow |E| = 2n - 2$.

29. Proof:

a) As we mentioned in the remark following Example 11.18, when G_1, G_2 are homeomorphic graphs then they may be regarded as isomorphic except, possibly, for vertices of degree 2. Consequently, two such graphs will have the same number of vertices of odd degree.

b) Now if G_1 has an Euler trail, then G_1 (is connected and) has all vertices of even degree – except two, those being the vertices at the beginning and end of the Euler trail. From part (a) G_2 is likewise connected with all vertices of even degree, except for two of odd degree. Consequently, G_2 has an Euler trail. [The converse follows in a similar way.]

c) If G_1 has an Euler circuit, then G_1 (is connected and) has all vertices of even degree. From part (a) G_2 is likewise connected with all vertices of even degree, so G_2 has an Euler circuit. [The converse follows in a similar manner.]

Section 11.5

1.

(a) (b) (c) (d)

2. The graph is a path (cycle).

3. (a) Hamilton cycle: $a \to g \to k \to i \to h \to b \to c \to d \to j \to f \to e \to a$
(b) Hamilton cycle: $a \to d \to b \to e \to g \to j \to i \to f \to h \to c \to a$
(c) Hamilton cycle: $a \to h \to e \to f \to g \to i \to d \to c \to b \to a$
(d) The edges $\{a,c\}$, $\{c,d\}$, $\{d,b\}$, $\{b,e\}$, $\{e,f\}$, $\{f,g\}$ provide a Hamilton path for the given graph. However, there is no Hamilton cycle, for such a cycle would have to include the edges $\{b,d\}$, $\{b,e\}$, $\{a,c\}$, $\{a,e\}$, $\{g,f\}$, and $\{g,e\}$ – and, consequently, the vertex e will have degree greater than 2.
(e) The path $a \to b \to c \to d \to e \to j \to i \to h \to g \to f \to k \to l \to m \to n \to o$ is one possible Hamilton path for this graph. Another possibility is the path $a \to b \to c \to d \to i \to h \to g \to f \to k \to l \to m \to n \to o \to j \to e$. However, there is no Hamilton cycle. For if we try to construct a Hamilton cycle we must include the edges $\{a,b\}$, $\{a,f\}$, $\{f,k\}$, $\{k,l\}$, $\{d,e\}$, $\{e,j\}$, $\{j,o\}$ and $\{n,o\}$. This then forces us to eliminate the edges $\{f,g\}$ and $\{i,j\}$ from further consideration. Now consider the vertex i. If we use edges $\{d,i\}$ and $\{i,n\}$, then we have a cycle on the vertices d,e,j,o,n and i – and we cannot get a Hamilton cycle for the given graph. Hence we must use only one of the edges $\{d,i\}$ and $\{i,n\}$. Because of the symmetry in this graph let us select edge $\{d,i\}$ – and then edge $\{h,i\}$ so that vertex i will have degree 2 in the Hamilton cycle we are trying to construct. Since edges $\{d,i\}$ and $\{d,e\}$ are now being used, we eliminate edge $\{c,d\}$ and this then forces us to include edges $\{b,c\}$ and $\{c,h\}$ in our construction. Also we must include the edge $\{m,n\}$ since we eliminated edge $\{i,n\}$ from consideration. Next we eliminate edges $\{h,m\}$, $\{h,g\}$ and $\{b,g\}$. Finally we must include edge $\{m,l\}$ and then eliminate edge $\{l,g\}$. But now we have eliminated the four edges $\{b,g\}$, $\{f,g\}$, $\{h,g\}$ and $\{l,g\}$ and g is consequently isolated.
(f) For this graph we find the Hamilton cycle $a \to b \to c \to d \to e \to j \to i \to h \to g \to l \to m \to n \to o \to t \to s \to r \to q \to p \to k \to f \to a$.

4. (a) Consider the graph as shown in Fig. 11.52(a). We demonstrate one case. Start at vertex a and consider the partial path $a \to f \to i \to d$. These choices require the removal of edges $\{f,h\}$ and $\{g,i\}$ from further consideration since each vertex of the graph will be incident with exactly two edges in the Hamilton cycle. At vertex d we can

go to either vertex c or vertex e. (i) If we go to vertex c we eliminate edge $\{e,d\}$ from consideration, but we must now include edges $\{e,j\}$ and $\{e,a\}$, and this forces the elimination of edge $\{a,b\}$. Now we must consider vertex b, for by eliminating edge $\{a,b\}$ we are now required to include edges $\{b,g\}$ and $\{b,c\}$ in the cycle. This forces us to remove edge $\{c,h\}$ from further consideration. But we have now removed edges $\{f,h\}$ and $\{c,h\}$ and there is only one other edge that is incident with h, so no Hamilton cycle can be obtained. (ii) Selecting vertex e after d, we remove edge $\{d,c\}$ and include $\{c,h\}$ and $\{b,c\}$. Having removed $\{g,i\}$ we must include $\{g,b\}$ and $\{g,j\}$. This forces the elimination of $\{a,b\}$, the inclusion of $\{a,e\}$ (and the elimination of $\{e,j\}$). We now have a cycle containing a,f,i,d,e, hence this method has also failed.

However, this graph does have a Hamilton path: $a \to b \to c \to d \to e \to j \to h \to f \to i \to g$.

(b) For example, remove vertex j and the edges $\{e,j\},\{g,j\},\{h,j\}$. Then $e \to a \to f \to h \to c \to b \to g \to i \to d \to e$ provides a Hamilton cycle for this subgraph.

5. (a) If we remove any one of the vertices a, b or g, the resulting subgraph has a Hamilton cycle. For example, upon removing vertex a, we find the Hamilton cycle $b \to d \to c \to f \to g \to e \to b$.
(b) The following Hamilton cycle exists if we remove vertex g : $a \to b \to c \to d \to e \to j \to o \to n \to i \to h \to m \to l \to k \to f \to a$. A symmetric situation results upon removing vertex i.

6. Let the vertices on the cycle (rim) of W_n be consecutively denoted by $v_1, v_2, \ldots, v_n$, and let v_{n+1} denote the additional (central) vertex of W_n. Then the following cycles provide n Hamilton cycles for the wheel graph W_n.
(1) $v_1 \to v_{n+1} \to v_2 \to v_3 \to v_4 \to \ldots \to v_{n-1} \to v_n \to v_1$;
(2) $v_1 \to v_2 \to v_{n+1} \to v_3 \to v_4 \to \ldots \to v_{n-1} \to v_n \to v_1$;
(3) $v_1 \to v_2 \to v_3 \to v_{n+1} \to v_4 \to \ldots \to v_{n-1} \to v_n \to v_1$;
$\vdots$
$(n-1)$ $v_1 \to v_2 \to v_3 \to v_4 \to \ldots \to v_{n-1} \to v_{n+1} \to v_n \to v_1$; and
(n) $v_1 \to v_2 \to v_3 \to v_4 \to \ldots \to v_{n-1} \to v_n \to v_{n+1} \to v_1$.

7. (a) $(1/2)(n-1)!$ (b) 10 (c) 9

8. (a) Partition the vertices of $K_{n,n}$ as $X \cup Y$ where $|X| = |Y| = n$. Write $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$; each edge of $K_{n,n}$ is of the form $\{x_i, y_i\}$ where $1 \leq i,j \leq n$. Since x_1 is on every Hamilton cycle of $K_{n,n}$, start with x_1. There are then n choices for y_j where $\{x_1, y_j\}$ is on the cycle. From y_j we can return to X in $n-1$ ways (we cannot use x_1 again), forming the second edge $\{y_j, x_i\}$, where $2 \leq i \leq n$. Continuing in this manner there are $(n-1)!n!$ results. Since directions are not assigned to any of the edges we get a total of $(1/2)(n-1)!n!$ Hamilton cycles for $K_{n,n}$.

(b) In this case the "starting" vertex of a Hamilton path is in one of X or Y (as described in part (a)) and the "terminating" vertex is then in the other set. The number of such paths is $(n!)^2$. (Note: $n = 1$ makes sense in this part but not for the formula in part (a).)

9. Let $G = (V, E)$ be a loop-free undirected graph with no odd cycles. We assume that G is connected – otherwise, we work with the components of G. Select any vertex x in V and let $V_1 = \{v \in V | d(x, v)$, the length of a shortest path between x and v, is odd$\}$ and $V_2 = \{w \in V | d(x, w)$, the length of a shortest path between x and w, is even$\}$. Note that (i) $x \in V_2$; (ii) $V = V_1 \cup V_2$; and (iii) $V_1 \cap V_2 = \emptyset$. We claim that each edge $\{a, b\}$ in E has one vertex in V_1 and the other vertex in V_2.

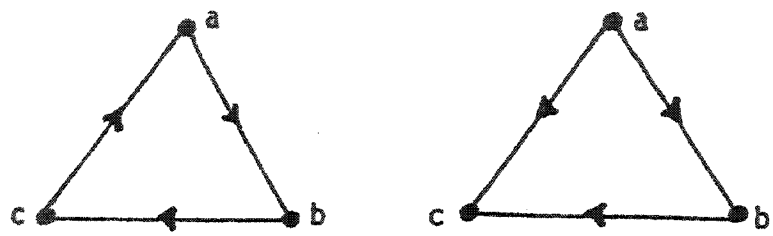
For suppose that $e = \{a, b\} \in E$ with $a, b \in V_1$. (The proof for $a, b \in V_2$ is similar.) Let $E_a = \{\{a, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, x\}\}$ be the m edges in a shortest path from a to x, and let $E_b = \{\{b, v'_1\}, \{v'_1, v'_2\}, \ldots, \{v'_{n-1}, x\}\}$ be the n edges in a shortest path from b to x. Note that m, n are both odd. If $\{v_1, v_2, \ldots, v_{m-1}\} \cap \{v'_1, v'_2, \ldots, v'_{n-1}\} = \emptyset$, then the set of edges $E' = \{\{a, b\}\} \cup E_a \cup E_b$ provides an odd cycle in G. Otherwise, let $w (\neq x)$ be the first vertex where the paths come together, and let $E'' = \{\{a, b\}\} \cup \{\{a, v_1\}, \{v_1, v_2\}, \ldots, \{v_i, w\}\} \cup \{\{b, v'_1\}, \{v'_1, v'_2\}, \ldots, \{v'_j, w\}\}$, for some $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. Then either E'' provides an odd cycle for G or $E' - E''$ contains an odd cycle for G.

10. (a) Suppose that G has a Hamilton cycle C. Then C contains $|V|$ edges and the vertices on C must alternate between vertices in V_1 and those in V_2 because G is bipartite. This forces $|V|$ to be even and $|V_1| = |V_2|$.

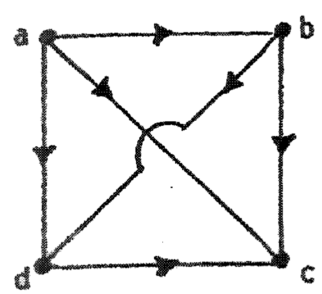
(b) In a similar way, if G has a Hamilton path P, then P has $|V| - 1$ edges and the vertices on P must alternate between the vertices in V_1 and those in V_2. Since $|V_1| \neq |V_2|$, it follows that $|V_1| - |V_2| = \pm 1$.

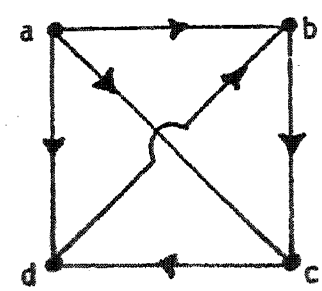
(c) Let $V = \{a, b, c, d, e\}$ with $V_1 = \{a, b\}$, $V_2 = \{c, d, e\}$ and $E = \{\{a, c\}, \{a, d\}, \{a, e\}, \{b, c\}\}$.

11. (a)



(b)





$\text{od}(a) = 3$ $\text{id}(a) = 0$
$\text{od}(b) = 2$ $\text{id}(b) = 1$
$\text{od}(c) = 0$ $\text{id}(c) = 3$
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$\text{od}(a) = 3$ $\text{id}(a) = 0$
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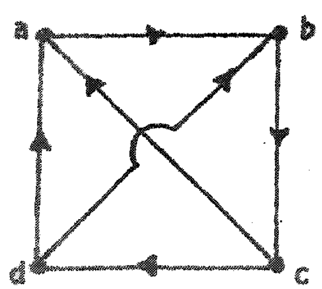

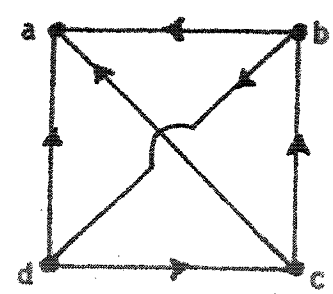

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$\text{od}(b) = 1$ $\text{id}(b) = 2$
$\text{od}(c) = 2$ $\text{id}(c) = 1$
$\text{od}(d) = 2$ $\text{id}(d) = 1$

$\text{od}(a) = 0$ $\text{id}(a) = 3$
$\text{od}(b) = 2$ $\text{id}(b) = 1$
$\text{od}(c) = 2$ $\text{id}(c) = 1$
$\text{od}(d) = 2$ $\text{id}(d) = 1$

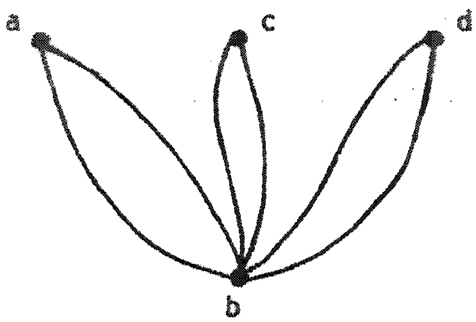
12. Proof: From Example 11.26 we know the result is true for $n = 2$. Assume that Q_n has a Hamilton cycle for some arbitrary (but fixed) $n \geq 2$. Now consider Q_{n+1}. From Example 11.12 we know that Q_{n+1} can be constructed from two copies of Q_n – one copy, $Q_{n,0}$, induced by the vertices of Q_{n+1} that start with 0, the other copy, $Q_{n,1}$, induced by the vertices of Q_{n+1} that start with 1. Each of $Q_{n,0}$, $Q_{n,1}$ has a Hamilton cycle – each may have more than one but we agree to pick the same cycle in each. [The only difference in the cycles is the first bit in the vertices of an edge – that is, if $\{0x, 0y\}$ is an edge in the Hamilton cycle for $Q_{n,0}$ (where x, y are binary strings of length n that differ in only one position), then $\{1x, 1y\}$ is the corresponding edge in the Hamilton cycle for $Q_{n,1}$.] Select edges $\{0v, 0w\}$ and $\{1v, 1w\}$ from the Hamilton cycles for $Q_{n,0}$ and $Q_{n,1}$, respectively. Remove these edges and replace them with the edges $\{0v, 1v\}$, $\{0w, 1w\}$ (in Q_{n+1}). The result is a Hamilton cycle for Q_{n+1}.

It now follows from the Principle of Mathematical Induction that Q_n has a Hamilton cycle for all $n \geq 2$.

13. Proof: If not, there exists a vertex x such that $(v, x) \notin E$ and, for all $y \in V$, $y \neq v, x$, if $(v, y) \in E$ then $(y, x) \notin E$. Since $(v, x) \notin E$, we have $(x, v) \in E$, as T is a tournament. Also, for each y mentioned earlier, we also have $(x, y) \in E$. Consequently, $od(x) \geq od(v)+1$ – contradicting $od(v)$ being a maximum!

14. Let G be any path with more than three vertices.

15.



For the multigraph in the given figure, $|V| = 4$ and $\deg(a) = \deg(c) = \deg(d) = 2$ and $\deg(b) = 6$. Hence $\deg(x) + \deg(y) \geq 4 > 3 = 4 - 1$ for all nonadjacent $x, y \in V$, but the multigraph has no Hamilton path.

16. Corollary 11.4: Proof: For all $x, y \in V$, $\deg(x) + \deg(y) \geq 2[(n-1)/2] = n - 1$, so the result follows from Theorem 11.8.

Corollary 11.5: Proof: Let $a, b \in V$ where $\{a, b\} \notin E$. Then $\deg(a) + \deg(b) \geq (n/2) + (n/2) = n$, so the result follows from Theorem 11.9.

17. For $n \geq 5$ let $C_n = (V, E)$ denote the cycle on n vertices. Then C_n has (actually is) a Hamilton cycle, but for all $v \in V$, $\deg(v) = 2 < n/2$.

18. Construct a graph with 12 vertices, one for each person. If two people know each other, draw an edge connecting their corresponding vertices. By Theorem 11.9 this graph has a Hamilton cycle and this cycle provides such a seating arrangement.

19. This follows from Theorem 11.9, since for all (nonadjacent) $x, y \in V$, $\deg(x) + \deg(y) = 12 > 11 = |V|$.

20. Proof: Let $x, y \in V$ with $\{x, y\} \in E$. Consequently, x, y are nonadjacent in $\overline{G}$. In $\overline{G}$ we find that $\deg_{\overline{G}}(x) = \deg_{\overline{G}}(y) \geq 2n+2-n = n+2$, so $\deg_{\overline{G}}(x)+\deg_{\overline{G}}(y) = 2n+4 > 2n+2 = |V|$. Therefore, by virtue of Theorem 11.9, the graph $\overline{G}$ has a Hamilton cycle.

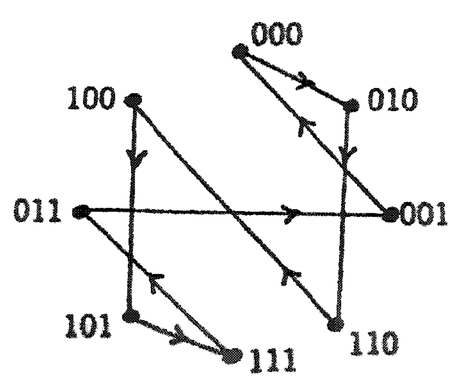
21. When $n = 5$ the graphs C_5 and $\overline{C}_5$ are isomorphic, and both are Hamilton cycles on five vertices.

For $n \geq 6$, let u, v denote nonadjacent vertices in $\overline{C}_n$. Since $\deg(u) = \deg(v) = n - 3$ we find that $\deg(u) + \deg(v) = 2n - 6$. Also, $2n - 6 \geq n \Longleftrightarrow n \geq 6$, so it follows from Theorem 11.9 that the cocycle $\overline{C}_n$ contains a Hamilton cycle when $n \geq 6$.

22. (a) If $x \neq v$ and $y \neq v$, then $\deg(x) = \deg(y) = n - 2$, and $\deg(x) + \deg(y) = 2n - 4 \geq n$, for $n \geq 4$.
If one of x, y is v, say x, then $\deg(x) = 2$ and $\deg(y) = n - 2$, and $\deg(x) + \deg(y) = n$.
(b) From part (a) it follows that $\deg(x) + \deg(y) \geq n$ for all nonadjacent x, y in V. Therefore G_n has a Hamilton cycle — by virtue of Theorem 11.9.
(c) Here $|E| = \binom{n-1}{2} - 1 + 2$, where we subtract 1 for the edge $\{v_1, v_2\}$, and add 2 for the pair of edges $\{v_1, v\}$ and $\{v, v_2\}$. Consequently, $|E| = \binom{n-1}{2} + 1$.
(d) The results in parts (b) and (c) do not contradict Corollary 11.6. They show that the converse of this corollary is false — as is its inverse.

23. (a) The path $v \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \ldots \rightarrow v_{n-1}$ provides a Hamilton path for H_n. Since $\deg(v) = 1$ the graph cannot have a Hamilton cycle.
(b) Here $|E| = \binom{n-1}{2} + 1$. (So the number of edges required in Corollary 11.6 cannot be decreased.)

24. (a)



Since the given graph has a Hamilton path we use this path to provide the following Gray code for $1, 2, 3, \ldots, 8$.

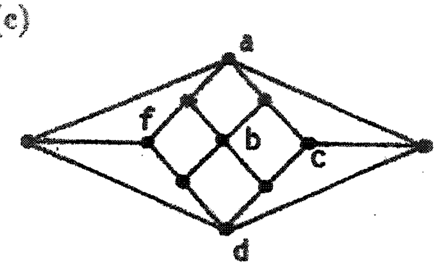
1: 000	2: 010	3: 110	4: 100
5: 101	6: 111	7: 011	8: 001

(b)

1: 0000	2: 0001	3: 0011	4: 0111
5: 1111	6: 1110	7: 1100	8: 1000
9: 1010	10: 1011	11: 1001	12: 1101
13: 0101	14: 0100	15: 0110	16: 0010

25. (a) (i) $\{a,c,f,h\},\{a,g\}$; (ii) $\{z\},\{u,w,y\}$
(b) (i) $\beta(G)=4$; (ii) $\beta(G)=3$
(c) (i) 3 (ii) 3 (iii) 3 (iv) 4 (v) 6
(vi) The maximum of m and n.
(d) The complete graph on $|I|$ vertices.

26. (a) If not, there is an edge $\{a,b\}$ in E where $a,b \in I$. This contradicts the independence of I.
(b) A Hamilton cycle on v vertices must have v edges.
(c)



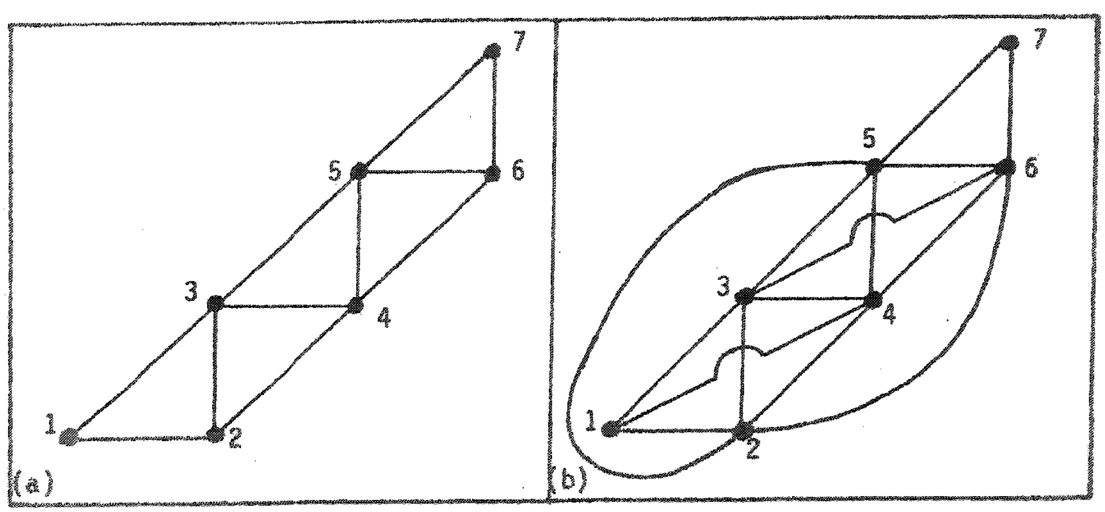
Let $I=\{a,b,c,d,f\}$, as shown in the figure. Here $v=11, e=18$, and $e-\sum_{v\in I}\deg(v)+2|I|=18-(4+4+3+4+3)+2(5)=10<11$, so by part (b), the Herschel graph has no Hamilton cycle.

Section 11.6

1. Draw a vertex for each species of fish. If two species x,y must be kept in separate aquaria, draw the edge $\{x,y\}$. The smallest number of aquaria needed is then the chromatic number of the resulting graph.

2. Draw a vertex for each committee. If someone serves on two committees c_i, c_j draw the edge joining the vertices for c_i and c_j. Then the least number of meeting times is the chromatic number of the graph.

3. We can model this problem with graphs. For either part of the problem draw the undirected graph $G=(V,E)$ where $V=\{1,2,3,4,5,6,7\}$ and $\{i,j\}\in E$ when chemicals i and j require separate storage compartments. For part (a), the graph (in part (a) of the figure) has chromatic number 3, so here Jeannette will need three separate storage compartments to safely store these seven chemicals.



Now consider the graph in part (b) of the figure. Note here that the subgraph induced by the vertices 2,3,4,5,6 is (isomorphic to) K_5. Consequently, with these additional conditions Jeannette will need five separate storage compartments to store these seven chemicals safely.

4. Let G be a cycle on n vertices where n is odd and $n \geq 5$.

5. (a) $P(G, \lambda) = \lambda(\lambda - 1)^3$
(b) For $G = K_{1,n}$ we find that $P(G, \lambda) = \lambda(\lambda - 1)^n$.
$\chi(K_{1,n}) = 2$.

6. (a) (i) Here we have λ choices for vertex a, 1 choice for vertex b (the same choice as that for vertex a), and $\lambda - 1$ choices for each of vertices x, y, z. Consequently, there are $\lambda(\lambda - 1)^3$ proper colorings of $K_{2,3}$ where vertices a and b are colored the same.
(ii) Now we have λ choices for vertex a, $\lambda - 1$ choices for vertex b, and $\lambda - 2$ choices for each of the vertices x, y, and z. And here there are $\lambda(\lambda - 1)(\lambda - 2)^3$ proper colorings.

(b) Since the two cases in part (a) are exhaustive and mutually exclusive, the chromatic polynomial for $K_{2,3}$ is

$$\lambda(\lambda - 1)^3 + \lambda(\lambda - 1)(\lambda - 2)^3 = \lambda(\lambda - 1)(\lambda^3 - 5\lambda^2 + 10\lambda - 7).$$

$\chi(K_{2,3}) = 2$.

(c) $P(K_{2,n}, \lambda) = \lambda(\lambda - 1)^n + \lambda(\lambda - 1)(\lambda - 2)^n$

$\chi(K_{2,n}) = 2$.

7. (a) 2 (b) 2 (n even); 3 (n odd)

(c) Figure 11.59(d): 2; Fig. 11.62(a): 3; Fig. 11.85(i); 2; Fig. 11.85(ii): 3 (d) 2

8. If $G = (V, E)$ is bipartite, then $V = V_1 \cup V_2$ where $V_1 \cap V_2 = \emptyset$ and each edge is of the form $\{x, y\}$ where $x \in V_1, y \in V_2$. Color all the vertices in V_1 with one color and those in V_2 with a second color. Then $\chi(G) = 2$.
Conversely, if $\chi(G) = 2$, let V_1 be the set of all vertices with one color and V_2 the set of vertices with the second color. Then $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and each edge of G has one vertex in V_1 and the other in V_2, so G is bipartite.

9. (a) (1) $\lambda(\lambda - 1)^2(\lambda - 2)^2$; (2) $\lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 2\lambda + 2)$;
(3) $\lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$

(b) (1) 3: (2) 3; (3) 3

(c) (1) 720; (2) 1020; (3) 420

10. (a) These graphs are not isomorphic. The first graph has two vertices of degree 4 – namely, f and k. The second graph has three vertices of degree 4 – namely u,w,z.

(b) For the first graph there are two cases to consider.

Case (i): Vertices f and k have the same color: Here there are $\lambda(\lambda-1)^2(\lambda-2)^2$ ways to properly color the vertices.

Case (ii): Vertices f and k are colored with different colors: Here the vertices can be properly colored in $\lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^2$ ways.

By the rule of sum, $P(G,\lambda) = \lambda(\lambda-1)^2(\lambda-2)^2 + \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)^2 =$ $\lambda(\lambda-1)^2(\lambda-2)^2(\lambda^2-5\lambda+8)$.

Using the same type of argument, with the two cases for vertices u and z, the chromatic polynomial for the second graph is also found to be $\lambda(\lambda-1)^2(\lambda-2)^2(\lambda^2-5\lambda+8)$.

(c) If G_1, G_2 are two graphs with $P(G_1,\lambda) = P(G_2,\lambda)$, it need not be the case that G_1 and G_2 are isomorphic.

11. Let $e = \{v,w\}$ be the deleted edge. There are $\lambda(1)(\lambda-1)(\lambda-2)\cdots(\lambda-(n-2))$ proper colorings of G_n where v, w share the same color and $\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(n-1))$ proper colorings where v, w are colored with different colors. In total there are $P(G_n,\lambda) =$ $\lambda(\lambda-1)\cdots(\lambda-n+2) + \lambda(\lambda-1)\cdots(\lambda-n+1) = \lambda(\lambda-1)\cdots(\lambda-n+3)(\lambda-n+2)^2$ proper colorings for G_n.

Here $\chi(G_n) = n-1$.

12. a) Here $\binom{r}{2}+\binom{g}{2} = \binom{6}{2}+\binom{3}{2} = 15+3 = 18$, and $\binom{r+g}{2} = \binom{9}{2} = 36$. So there are 18 edges that are red or green, and 18 blue edges.

b) $\binom{r}{2}+\binom{g}{2} = (1/2)\binom{r+g}{2} \Leftrightarrow (1/2)r(r-1)+(1/2)g(g-1) = (1/4)(r+g)(r+g-1) \Leftrightarrow$ $2r(r-1)+2g(g-1) = (r+g)(r+g-1) \Leftrightarrow r^2-r+g^2-g = 2rg \Leftrightarrow (r-g)^2 = r+g$. Let $r = g+k$, $k \geq 0$. Then $[(r-g)^2 = k^2 = r+g = 2g+k] \Leftrightarrow [g = (1/2)(k^2-k) = (1/2)k(k-1) = t_{k-1}$ and $r = g+k = (1/2)k(k-1)+k = (1/2)k[(k-1)+2k] = (1/2)k(k+1) = t_k] \Leftrightarrow r, g$ are two consecutive triangular numbers.

13. (a) $|V| = 2n$; $|E| = (1/2)\sum_{v\in V}\deg(v) = (1/2)[4(2)+(2n-4)(3)] = (1/2)[8+6n-12] =$ $3n-2$, $n \geq 1$.

(b) For $n=1$, we find that $G = K_2$ and $P(G,\lambda) = \lambda(\lambda-1) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{1-1}$ so the result is true in this first case. For $n=2$, we have $G = C_4$, the cycle of length 4, and here $P(G,\lambda) = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{2-1}$. So the result follows for $n=2$. Assuming the result true for an arbitrary (but fixed) $n \geq 1$, consider the situation for $n+1$. Write $G = G_1 \cup G_2$, where G_1 is C_4 and G_2 is the ladder graph for n rungs. Then $G_1 \cap G_2 = K_2$, so from Theorem 11.14 we have $P(G,\lambda) = P(G_1,\lambda)\cdot P(G_2,\lambda)/P(K_2,\lambda) =$ $[(\lambda)(\lambda-1)(\lambda^2-3\lambda+3)][(\lambda)(\lambda-1)(\lambda^2-3\lambda+3)^{n-1}]/(\lambda)(\lambda-1) = (\lambda)(\lambda-1)(\lambda^2-3\lambda+3)^n$. Consequently, the result is true for all $n \geq 1$, by the Principle of Mathematical Induction.

14. (a) Select a vertex $v \in V$ and color it with one of the $\Delta+1$ available colors. If $w \in V$ and w has not been colored, since $\deg(w) \leq \Delta$ we can color w, *not* using any of the colors used on the vertices adjacent to w. This procedure is repeated until all of the vertices in V have been (properly) colored.

(b) For $n \in \mathbf{Z}^+$, $n \geq 3$, $\chi(K_n) = n = \Delta + 1$.

15. (a) $\lambda(\lambda-1)(\lambda-2)$ (b) Follows from Theorem 11.10

(c) Follows by the rule of product.

(d)
$$\begin{aligned} P(C_n,\lambda) &= P(P_{n-1},\lambda) - P(C_{n-1},\lambda) = \lambda(\lambda-1)^{n-1} - P(C_{n-1},\lambda) \\ &= [(\lambda-1)+1](\lambda-1)^{n-1} - P(C_{n-1},\lambda) \\ &= (\lambda-1)^n + (\lambda-1)^{n-1} - P(C_{n-1},\lambda) \Longrightarrow \\ & P(C_n,\lambda) - (\lambda-1)^n = (\lambda-1)^{n-1} - P(C_{n-1},\lambda). \end{aligned}$$

Replacing n by $n-1$ yields

$$P(C_{n-1},\lambda) - (\lambda-1)^{n-1} = (\lambda-1)^{n-2} - P(C_{n-2},\lambda) = (-1)[P(C_{n-2},\lambda) - (\lambda-1)^{n-2}].$$

Hence

$$P(C_n,\lambda) - (\lambda-1)^n = P(C_{n-2},\lambda) - (\lambda-1)^{n-2} = (-1)^2[P(C_{n-2},\lambda) - (\lambda-1)^{n-2}].$$

(e) Continuing from part (d),
$$\begin{aligned} P(C_n,\lambda) &= (\lambda-1)^n + (-1)^{n-3}[P(C_3,\lambda) - (\lambda-1)^3] \\ &= (\lambda-1)^n + (-1)^{n-1}[\lambda(\lambda-1)(\lambda-2) - (\lambda-1)^3] \\ &= (\lambda-1)^n + (-1)^n(\lambda-1). \end{aligned}$$

16. (a) $\chi(W_n) = \chi(C_n) + 1$. [$C_n$ has n vertices; W_n has $n+1$ vertices.]

(b) $P(W_n,\lambda) = \lambda P(C_n,\lambda-1) = \lambda[(\lambda-2)^n + (-1)^n(\lambda-2)]$.

(c) (i) and (ii) $P(W_5,\lambda) = \lambda(\lambda-2)^5 + (-1)^5\lambda(\lambda-2)$ – For k colors we have $P(W_5,k) = k(k-2)^5 + (-1)^5 k(k-2) = k(k-2)[(k-2)^4 - 1]$ proper colorings, whenever $k \geq 4$.

17. From Theorem 11.13, the expansion for $P(G,\lambda)$ will contain exactly one occurrence of the chromatic polynomial of K_n. Since no larger graph occurs this term determines the degree as n and the leading coefficient as 1.

18. (a)
$$\begin{array}{lll} |V| = 1: & P(G,\lambda) = \lambda & \\ |V| = 2: & |E| = 0: & P(G,\lambda) = \lambda^2 \\ & |E| = 1: & P(G,\lambda) = \lambda(\lambda-1) = \lambda^2 - \lambda \\ |V| = 3: & |E| = 0: & P(G,\lambda) = \lambda^3 \\ & |E| = 1: & P(G,\lambda) = \lambda^2(\lambda-1) = \lambda^3 - \lambda^2 \\ & |E| = 2: & P(G,\lambda) = \lambda(\lambda-1)^2 = \lambda^3 - 2\lambda^2 + \lambda \\ & |E| = 3: & P(G,\lambda) = \lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda \end{array}$$

(b) Let $G = (V, E)$ be a loop-free undirected graph where $|V| = n \geq 4$ and $|E| = k \geq 1$. (If $k = 0$, $P(G, \lambda = \lambda^n$ and the result is true.) From Theorem 11.10, $P(G, \lambda) = P(G_e, \lambda) - P(G'_e, \lambda)$ where $e = \{a, b\}$ is an edge in G. Since G_e has n vertices but $k - 1$ edges, by the induction hypothesis,

$$P(G_e, \lambda) = \lambda^n - (k-1)\lambda^{n-1} + c_{n-2}\lambda^{n-2} - c_{n-3}\lambda^{n-3} + \ldots + (-1)^{n-1}c_1\lambda,$$

where $k - 1, c_{n-2}, c_{n-3}, \ldots, c_1 \geq 0$. (When a coefficient in this list is zero, all successive coefficients are zero.) Likewise, since G'_e has $n - 1$ vertices, by the induction hypothesis,

$$P(G'_e, \lambda) = \lambda^{n-1} - b_{n-2}\lambda^{n-2} + b_{n-3}\lambda^{n-3} - \ldots + (-1)^{n-2}b_1\lambda,$$

where $b_{n-2}, b_{n-3}, \ldots, b_1 \geq 0$.
Then $P(G, \lambda) = P(G_e, \lambda) - P(G'_e, \lambda) =$

$$\lambda^n - (k)\lambda^{n-1} + (c_{n-2} + b_{n-2})\lambda^{n-2} + \ldots + (-1)^{n-1}(c_1 + b_1)\lambda.$$

(c) This was shown in part (b).

19. (a) For $n \in \mathbf{Z}^+$, $n \geq 3$, let C_n denote the cycle on n vertices.

If n is odd then $\chi(C_n) = 3$. But for each v in C_n, the subgraph $C_n - v$ is a path with $n - 1$ vertices and $\chi(C_n - v) = 2$. So for n odd C_n is color-critical.

However, when n is even we have $\chi(C_n) = 2$, and for each v in C_n, the subgraph $C_n - v$ is still a path with $n - 1$ vertices and $\chi(C_n - v) = 2$. Consequently, cycles with an even number of vertices are not color-critical.

(b) For every complete graph K_n, where $n \geq 2$, we have $\chi(K_n) = n$, and for each vertex v in K_n, $K_n - v$ is (isomorphic to) K_{n-1}, so $\chi(K_n - v) = n - 1$. Consequently, every complete graph with at least one edge is color-critical.

(c) Suppose that G is not connected. Let G_1 be a component of G where $\chi(G_1) = \chi(G)$, and let G_2 be any other component of G. Then $\chi(G_1) \geq \chi(G_2)$ and for all v in G_2 we find that $\chi(G - v) = \chi(G_1) = \chi(G)$, so G is not color-critical.

(d) If not, let $v \in V$ with $\deg(v) \leq k - 2$. Since G is color-critical we have $\chi(G - v) \leq k - 1$, and so we can properly color the vertices in the subgraph $G - v$ with at most $k - 1$ colors. Since $\deg(v) \leq k - 2$, we have used at most $k - 2$ colors to color all vertices in G adjacent to v. Therefore we do not need a new color (beyond those needed to color the subgraph $G - v$) in order to color v and can color all vertices in G with at most $k - 1$ colors. But this contradicts $\chi(G) = k$.

Supplementary Exercises

1. $\binom{n}{2} = 56 + 80 = 136 \Longrightarrow n(n-1) = 272 \Longrightarrow n = 17.$

2. For $n \geq 1$, let c_n count the number of cycles of length four in Q_n. Then $c_1 = 0$ and $c_2 = 1$. Recall the recursive construction of Q_{n+1} from Q_n — given in Section 11.3. Let $V_{n+1}^{(0)}$ denote all the vertices in Q_{n+1} that start with 0, and $V_{n+1}^{(1)}$ those vertices in Q_{n+1} that start with 1. [Each of the subgraphs of Q_{n+1} induced by $V_{n+1}^{(0)}$ and $V_{n+1}^{(1)}$ is isomorphic to Q_n.] Let $v_1 \to v_2 \to v_3 \to v_4 \to v_1$ denote a cycle of length four in Q_{n+1}. There are three cases to consider:

(1) $v_1, v_2, v_3, v_4 \in V_{n+1}^{(0)}$: Here there are c_n such cycles;
(2) $v_1, v_2, v_3, v_4 \in V_{n+1}^{(1)}$: Here there are also c_n such cycles; and,
(3) one edge of the cycle (call it the first) is in $\langle V_{n+1}^{(0)} \rangle$ and another edge (namely, the third) is in $\langle V_{n+1}^{(1)} \rangle$: Here the other two edges are each adjacent to a vertex in $V_{n+1}^{(0)}$ and one in $V_{n+1}^{(1)}$. [Let $\{v_1, v_2\} \in \langle V_{n+1}^{(0)} \rangle$, then $\{v_3, v_4\} \in \langle V_{n+1}^{(1)} \rangle$ and the binary labels on v_1 and v_4 differ only in the first (left-most) position, while the binary labels on v_2 and v_3 also differ only in the first (left-most) position.] Since there are $n2^{n-1}$ possible choices (the number of edges in Q_n) for the so called "first" edge, here we find $n2^{n-1}$ new cycles of length four.

The preceding discussion gives us
$c_{n+1} = 2c_n + n2^{n-1} = 2c_n + (1/2)n2^n \quad n \geq 1,\ c_1 = 0,\ c_2 = 1.$
$c_n^{(h)} = A2^n, \quad c_n^{(p)} = n(B + Cn)2^n$
$(n+1)(B + C(n+1))2^{n+1} = 2n(B + Cn)2^n + n2^{n-1}$
$\Rightarrow [B(n+1) + C(n+1)^2]2^{n+1} = [Bn + Cn^2]2^{n+1} + (n/4)2^{n+1}$
$\Rightarrow 2C = 1/4,\ B + C = 0 \Rightarrow C = 1/8,\ B = -1/8.$
So $c_n^{(p)} = (1/8)(n^2 - n)2^n$.
$0 = c_1 = c_1^{(h)} + c_1^{(p)} = 2A + 0 \Rightarrow A = 0$, so
$c_n = (1/8)(n^2 - n)2^n = \binom{n}{2}2^{n-2}, \quad n \geq 1.$

Alternate Solution: Let $v_1 \to v_2 \to v_3 \to v_4 \to v_1$ be a cycle of length four in Q_n. Say v_1, v_2 differ in position i and v_2, v_4 differ in position j, where $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$. Then v_3 is determined: it differs from v_1 in positions i and j. Starting with v_1 there are 2^n choices. Then for a specific v_1 there are $\binom{n}{2}$ ways to select positions i, j. [Remember that $v_1 \to v_4 \to v_3 \to v_2 \to v_1$ is the same cycle as $v_1 \to v_2 \to v_3 \to v_4 \to v_1$.] So at this point we have $\binom{n}{2}2^n$ cycles. But since each of $v_2 \to v_3 \to v_4 \to v_1 \to v_2$, $v_3 \to v_4 \to v_1 \to v_2 \to v_3$, and $v_4 \to v_1 \to v_2 \to v_3 \to v_4$ is the same cycle as $v_1 \to v_2 \to v_3 \to v_4 \to v_1$, the total number of *distinct* cycles of length four in Q_n is $(1/4)\binom{n}{2}2^n = \binom{n}{2}2^{n-2}, \quad n \geq 1.$

3. (a) Label the vertices of K_6 with $a, b, \ldots, f$. Of the five edges on a at least three have the same color, say red, and let these edges be $\{a, b\}, \{a, c\}, \{a, d\}$. If the edges $\{b, c\}, \{c, d\}, \{b, c\}$ are all blue, the result follows. If not, one of these edges, say $\{c, d\}$, is red and then $\{a, c\}, \{a, d\}, \{c, d\}$ yield a red triangle.

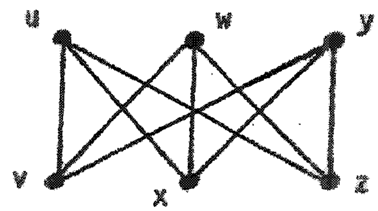
(b) Consider the six people as vertices. If two people are friends (strangers) draw a red (blue) edge connecting their respective vertices. The result then follows from part (a).

4. (a) (i) $|E| = (1/2)\binom{n}{2}$
(ii) For any undirected graph G, if G is not connected then $\overline{G}$ is connected. In this situation $G \cong \overline{G}$, so G is connected.

(b) Proof: When $n = 1$ we have K_1. For $n = 4$ the path on four vertices is an example of a self-complementary graph. The cycle on five vertices provides an example for $n = 5$.

Now suppose we have a self-complementary graph $G = (V, E)$. Construct the graph $G_1 = (V_1, E_1)$ where $V_1 = V \cup \{a, b, c, d\}$ (so none of a, b, c, d is in V) and $E_1 = E \cup \{\{a, b\}, \{b, c\}, \{c, d\}\} \cup \{\{v, a\} | v \in V\} \cup \{\{v, d\} | v \in V\}$. Then G_1 is self-complementary and $|V_1| = |V| + 4$.

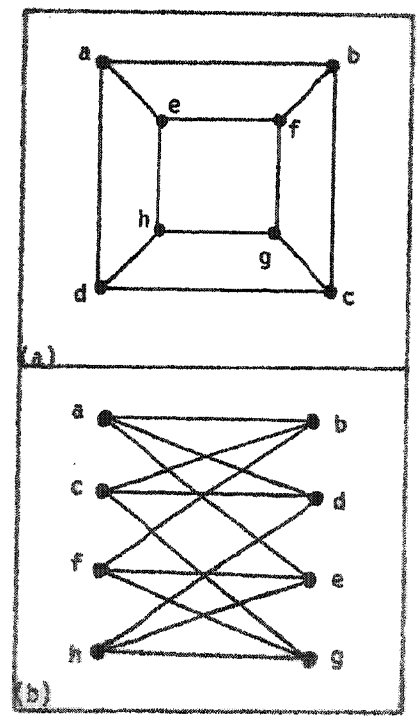
5. (a) We can redraw G_2 as



(b) 72

6.

Only the graph for the cube is bipartite as seen in part (b) of the given figure. In any of the other four graphs (See Fig. 11.59(b) and Fig. 11.60) there are cycles of odd length, so these graphs cannot be bipartite.



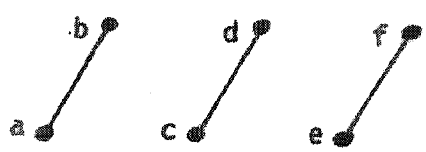
7. (a) Let the vertices of $K_{3,7}$ be partitioned as $V_1 \cup V_2$ where $|V_1| = 3$ and $|V_2| = 7$. Then there are $(3)(7)(2)(6)(1)(5) = 1260$ paths of length 5 where each such path contains all three vertices in V_1.

(b) With V_1, V_2 as in part (a) we find that there are $(1/2)(3)(7)(2)(6)(1)$ paths of length 4 that start and end with a vertex in V_1, and there are also $(1/2)(7)(3)(6)(2)(5)$ paths of length 4 that start and end with a vertex in V_2. Consequently, there are $126 + 630 = 756$ paths of length 4 in $K_{3,7}$.

(c) (Case 1: p is odd, $p = 2k + 1$ for $k \in \mathbf{N}$). Here there are mn paths of length $p = 1$ (when $k = 0$) and $(m)(n)(m-1)(n-1)\cdots(m-k)(n-k)$ paths of length $p = 2k+1 \geq 3$. (Case 2: p is even, $p = 2k$ for $k \in \mathbf{Z}^+$). When $p < 2m$ (i.e., $k < m$) the number of paths of length p is $(1/2)(m)(n)(m-1)(n-1)\cdots(n-(k-1))(m-k) + (1/2)(n)(m)(n-1)(m-1)\cdots(m-(k-1))(n-k)$. For $p = 2m$ we find $(1/2)(n)(m)(n-1)(m-1)\cdots(m-(m-1))(n-m)$ paths of (longest) length $2m$.

8. (a) $(n = 2): X = \{1,2\}$ and G consists of the single vertex v that corresponds to X.
$(n = 3): X = \{1,2,3\}$. Here G is made up of three isolated vertices.
$(n = 4): X = \{1,2,3,4\}$. Now G has six vertices and is drawn as follows:

a: {1,2}	d: {2,4}
b: {3,4}	e: {1,4}
c: {1,3}	f: {2,3}



(b) Let $v(\{a,b\})$ and $w(\{x,y\})$ be two vertices of G. If $\{a,b\} \cap \{x,y\} = \emptyset$, the edge $\{v,w\}$ is in G. If $\{a,b\} \cap \{x,y\} \neq \emptyset$, assume without loss of generality that $a = x$ but $b \neq y$. Hence a, b, y are three distinct elements of X and since $|X| \geq 5$, let $c, d \in X$ with $c \neq d$ and $c, d \notin \{a, b, y\}$. Then there exist edges from $\{a,b\}$ to $\{c,d\}$ and from $\{c,d\}$ to $\{x(=a), y\}$, since $\{a,b\} \cap \{c,d\} = \emptyset = \{c,d\} \cap \{x,y\}$. Hence G is connected.

(c) For $n = 5$ G is (isomorphic to) the Petersen graph, which is nonplanar. For $n \geq 6$ G contains a subgraph isomorphic to the Petersen graph and consequently G is nonplanar.

9. (a) Let I be independent and $\{a,b\} \in E$. If neither a nor b is in $V - I$, then $a, b \in I$, and since they are adjacent, I is not independent. Conversely, if $I \subseteq V$ with $V - I$ a covering of G, then if I is not independent there are vertices $x, y \in I$ with $\{x,y\} \in E$. But $\{x,y\} \in E \Longrightarrow$ either x or y is in $V - I$.

(b) Let I be a largest maximal independent set in G and K a minimal covering. From part (a), $|K| \leq |V - I| = |V| - |I|$ and $|I| \geq |V - K| = |V| - |K|$, or $|K| + |I| \geq |V| \geq |K| + |I|$.

10. (a) Let D be a minimal dominating set for G. If $V - D$ is not dominating, then there is a vertex $x \in D$ such that x is not adjacent to any vertex in $V - D$. Since G has no isolated vertices, x is adjacent to at least one vertex in $D - \{x\}$ and $D - \{x\}$ is a dominating set, contradicting the minimality of D.

(b) Suppose that I is a dominating set. If I is independent but not maximal independent, then there is a vertex $v \in V$ such that v is not in I and is not adjacent to any vertex in I. But this contradicts I being a dominating set. Conversely, if I is maximal independent then every vertex in V is in I or is adjacent to a vertex in I. Hence I is dominating.

(c) $\gamma(G) \leq \beta(G)$ follows from part (b). For the other condition, let $\chi(G) = m$. We can partition the vertices of G into m cells V_i, $1 \leq i \leq m$, where two vertices are in the same cell if they have the same color in G. Each of these cells is an independent set so $|V_i| \leq \beta(G)$, for all $1 \leq i \leq m$. Since $|V| = \sum_{i=1}^{m} |V_i|$, $|V| \leq \sum_{i=1}^{m} \beta(G) = m\beta(G) = \beta(G)\chi(G)$.

11. Since we are selecting n edges and no two have a common vertex, the selection of n edges will include exactly one occurrence of every vertex. We consider two mutually disjoint and exhaustive cases:

(1) The edge $\{x_n, y_n\}$ is in the selection: Then $\{x_{n-1}, x_n\}$ and $\{y_{n-1}, y_n\}$ are not in the selection and we must select the remaining $n-1$ edges from the resulting subgraph (a ladder graph with $n-1$ rungs) in a_{n-1} ways.

(2) The edge $\{x_n, y_n\}$ is not in the selection: Then in order to have x_n and y_n appear in the selection we must include edges $\{x_{n-1}, x_n\}$ and $\{y_{n-1}, y_n\}$. Consequently, we must now select the other $n-2$ edges from the resulting subgraph (a ladder graph with $n-2$ rungs) in a_{n-2} ways.

Hence $a_n = a_{n-1} + a_{n-2}$, $a_0 = 1$, $a_1 = 1$, and $a_n = F_{n+1}$, the $(n+1)$st Fibonacci number.

12. There are two cases to consider:
(1) The vertex y_n is not used. Then there are a_{n-1} independent subsets that contain x_n, and another a_{n-1} such subsets that do not contain x_n.
(2) The vertex y_n is included in the independent subset. Now we cannot use either of the vertices x_n or y_{n-1}. Consequently, there are a_{n-2} such subsets for each of the following situations: (i) x_{n-1} is in the subset; and (ii) x_{n-1} is not in the subset.
These considerations give rise to the recurrence relation

$$a_n = 2a_{n-1} + 2a_{n-2},$$

with initial conditions $a_0 = 1$, $a_1 = 3$. (We used $a_2 = 8$ to determine $a_0 = 1$.)
To solve this recurrence relation let $a_n = Ar^n$, where $A \neq 0$, $r \neq 0$. This leads to the characteristic equation

$$r^2 - 2r - 2 = 0,$$

and the characteristic roots $1 \pm \sqrt{3}$. Consequently, $a_n = A_1(1+\sqrt{3})^n + A_2(1-\sqrt{3})^n$, where A_1, A_2 are constants.

$1 = a_0 = A_1 + A_2$
$3 = a_1 = A_1(1+\sqrt{3}) + A_2(1-\sqrt{3}) = (A_1 + A_2) + \sqrt{3}(A_1 - A_2)$

$= 1 + \sqrt{3}(A_1 - A_2)$, so $2/\sqrt{3} = (A_1 - A_2)$.
Therefore, $A_1 = (\sqrt{3}+2)/2\sqrt{3}$, $A_2 = (\sqrt{3}-2)/2\sqrt{3}$, and
$a_n = [(\sqrt{3}+2)/2\sqrt{3}](1+\sqrt{3})^n + [(\sqrt{3}-2)/2\sqrt{3}](1-\sqrt{3})^n$, $n \geq 0$ (or $n \geq 1$).

13. If the vertex y_n is included in the independent subset then we cannot use any of the vertices y_{n-1}, x_{n-1}, or x_n. There are a_{n-2} such subsets — and another a_{n-2} independent subsets where x_n is included. In addition, there are a_{n-1} independent subsets when both x_n and y_n are excluded. This leads us to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2},$$

with initial conditions $a_1 = 3$, $a_2 = 5$.

To solve this recurrence relation let $a_n = Ar^n$, where $A \neq 0$, $r \neq 0$. This leads to the characteristic equation

$$r^2 - r - 2 = 0,$$

and the characteristic roots -1 and 2. Therefore, $a_n = A_1(-1)^n + A_2(2^n)$, where A_1, A_2 are constants.

$a_1 = 3, a_2 = 5 \Rightarrow 2a_0 = 5 - 3 \Rightarrow a_0 = 1$.
$1 = a_0 = A_1 + A_2$.
$3 = a_1 = -A_1 + 2A_2 = -(1 - A_2) + 2A_2 = -1 + 3A_2$, so $A_2 = 4/3$, and $A_1 = 1 - A_2 = -1/3$.
Consequently, $a_n = (-1/3)(-1)^n + (4/3)(2^n)$, $n \geq 0$ (or $n \geq 1$).

14. $a_0 = a_1 = 0$
For $n \geq 2$, $a_n = \binom{n}{2} = (1/2)n(n-1) > 0$.
$1/(1-x) = 1 + x + x^2 + x^3 + \ldots$
$(d/dx)[1/(1-x)] = 1 + 2x + 3x^2 + 4x^3 + \ldots$
$(d/dx)[1/(1-x)] = (d/dx)[(1-x)^{-1}] = (-1)(1-x)^{-2}(-1) = (1-x)^{-2}$
$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$
$(d/dx)[(1-x)^{-2}] = (-2)(1-x)^{-3}(-1) = 2(1-x)^{-3}$, so $2(1-x)^{-3} = 2 + 3\cdot 2x + 4\cdot 3x^2 + 5\cdot 4x^3 + \ldots$

$$2x^2/(1-x)^3 = 2\cdot 1x^2 + 3\cdot 2x^3 + 4\cdot 3x^4 + 5\cdot 4x^5 + \ldots = \sum_{n=2}^{\infty} n(n-1)x^n = \sum_{n=0}^{\infty} n(n-1)x^n.$$

Hence $f(x) = x^2/(1-x)^3 = \sum_{n=0}^{\infty}[n(n-1)/2]x^n$ is the generating function for the sequence $a_n = \binom{n}{2}$, $n \geq 0$.

15. (a) $\gamma(G) = 2$; $\beta(G) = 3$; $\chi(G) = 4$.

(b) G has neither an Euler trail nor an Euler circuit; G does have a Hamilton cycle.

(c) G is not bipartite but it is planar.

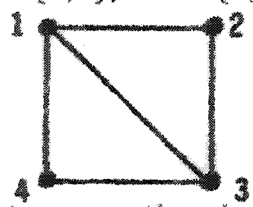
16. (a) (i) $m = 2$, $n = 8$ (ii) $m = n = 4$

(b) (i) $K_{m,n}$, for $m \leq n$, has an Euler circuit but not a Hamilton cycle if m and n are both even and $m \neq n$.

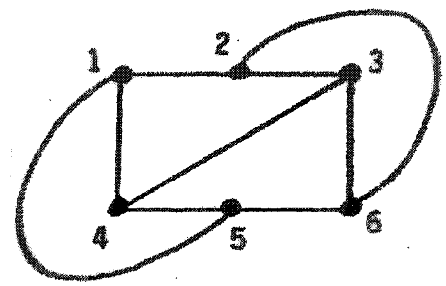
(ii) When m, n are both even and $m = n$, then $K_{m,n}$ has both an Euler circuit and a Hamilton cycle.

17. (a) $\chi(G) \geq \omega(G)$ (b) They are equal.

18. (a) (i) Here vertex 1 is for edge $\{a, c\}$, 2 for $\{a, b\}$, 3 for $\{b, c\}$, and 4 for $\{c, d\}$.

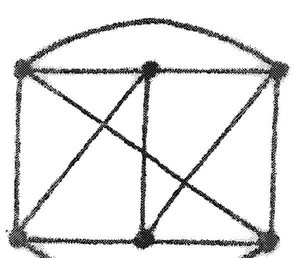


(ii) Here the correspondence between vertices in $L(G)$ and edges in G is given by
$1: \{y, z\}$; $2: \{x, z\}$; $3: \{w, x\}$;
$4: \{w, y\}$; $5: \{u, y\}$; $6: \{u, x\}$



(b) Let $v \in V$ with $\deg(v) = k$. Then there are k edges in G of the form $\{v_i, v\}$, $1 \leq i \leq k$. Any two of these edges are adjacent at v and give rise to an edge in $L(G)$. Hence v brings about $\binom{\deg(v)}{2}$ edges in $L(G)$. In total, $L(G)$ has $\sum_{v \in V} \binom{\deg(v)}{2} = (1/2) \sum_{v \in V} \deg(v)[\deg(v) - 1] = (1/2) \sum_{v \in V} \deg(v)^2 - (1/2) \sum_{v \in V} \deg(v) = (1/2) \sum_{v \in V} \deg(v)^2 - e$ edges.

(c) First we shall prove that $L(G)$ is connected. Let e_1, e_2 be two vertices in $L(G)$ where e_1 arises from edge $\{a, b\}$ and e_2 from edge $\{x, y\}$ in G. Since G is connected there is a path in G from b to x: $b \to v_1 \to v_2 \to \ldots \to v_k \to x$ and a path from a to y: $a \to b \to v_1 \to \ldots \to v_k \to x \to y$. These vertices and edges then determine a path in $L(G)$ from e_1 to e_2, so $L(G)$ is connected. Now for any vertex e in $L(G)$, let $\{a, b\}$ be the edge in G that determines e. Then $\deg(e)$ (in $L(G)$) $= (\deg(a) - 1) + (\deg(b) - 1)$, an even integer, since $\deg(a)$, $\deg(b)$ are both even. Hence by Theorem 11.3, $L(G)$ has an Euler circuit. Furthermore, the ordered list of edges in an Euler circuit for G determine a corresponding Hamilton cycle for the vertices of $L(G)$.



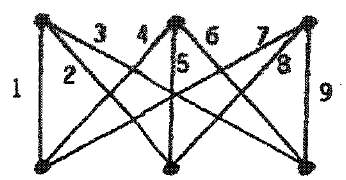
(d) For $G = K_4$, $L(K_4)$ is shown here. This graph has both an Euler circuit and a Hamilton cycle. However, for each vertex v in K_4, $\deg(v) = 3$, so K_4 does not have an Euler circuit.

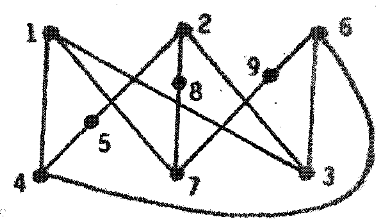
(e) Suppose that $G = (V, E)$ has a Hamilton cycle $v_1 \to v_2 \to v_3 \to \ldots \to v_n \to v_1$ and let $e_i = \{v_i, v_{i+1}\}$, $1 \le i \le n-1$, and $e_n = \{v_n, v_1\}$. Then there is a cycle in $L(G)$ on the vertices e_i, $1 \le i \le n$. If $|E| = n$, then this cycle is a Hamilton cycle. If $|E| > n$, let $e \in E$, where $e \ne e_i$, $1 \le i \le n$, and let $e = \{v_i, v_j\}$, $1 \le i < j \le n$. (This also takes care of the case where G is a multigraph.) In $L(G)$ there are edges $\{e_{i-1}, e\}$, where $e_{i-1} = e_n$ if $i = 1$, and $\{e, e_i\}$, and we can extend the cycle in $L(G)$ by replacing $\{e_{i-1}, e_i\}$ by the edges $\{e_{i-1}, e\}$ and $\{e, e_i\}$. Since $|E|$ is finite, as we continue enlarging our present cycle in this way, we obtain a Hamilton cycle for $L(G)$.

(f) The graph in Fig. 11.99(b) has no Hamilton cycle, but its line graph, as seen in part (a), has a Hamilton cycle.

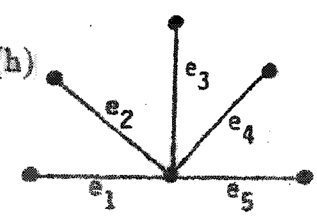
(g) For $G = K_5$, $L(G)$ has 10 vertices and 30 edges. Since G is connected, $L(G)$ is connected. But since $30 > 3(10) - 6$, it follows by Corollary 11.3 that $L(G)$ is nonplanar.

For $G = K_{3,3}$ we number the edges as shown in the first figure. Then in $L(G)$ we find the subraph shown in the second figure, so $L(G)$ is nonplanar.





(h)

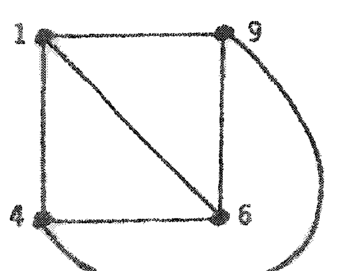


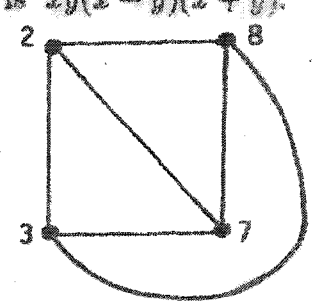
Let G be the graph shown here with six vertices (five pendant and one of degree 5). Then in $L(G)$ there are five vertices each of degree four, and $L(G) = K_5$, a nonplanar graph.

19. (a) The constant term is 3, not 0. This contradicts Theorem 11.11.
(b) The leading coefficient is 3, not 1. This contradicts the result in Exercise 17 of Section 11.6.
(c) The sum of the coefficients in -1, not 0. This contradicts Theorem 11.12.

20. (a) $x^3y - xy^3 = xy(x^2 - y^2) = xy(x - y)(x + y)$
If x or y is even then xy and $xy(x-y)(x+y)$ are both even. When x, y are both odd, then $x - y$ and $x + y$ are both even, as is $xy(x-y)(x+y)$.

(b)





(c) From part (a) $x^3y - xy^3 = xy(x-y)(x+y)$ is always even. If the units digit of either x or y is 0 or 5, then the result follows. Also, if x, y have the same units digit, then $x - y$ is a multiple of 10 and so is $x^3y - xy^3$. In all other cases we have three positive integers x, y, z with distinct units digits in the set $V = \{1, 2, 3, 4, 6, 7, 8, 9\}$. By the pigeonhole principle two of these integers, say x and y, must be in the same component (K_4) of G. Since the component is complete, $\{x, y\}$ is an edge, so either $x + y$ or $x - y$ is divisible by 5. Hence $x^3y - xy^3$ is divisible by 10.

21. (a) $a_1 = 2$, $a_2 = 3$. For $n \geq 3$ label the vertices of P_n as $v_1, v_2, v_3, \ldots, v_n$ where the edges are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$. In constructing an independent subset S from P_n we consider two cases:

(1) $v_n \notin S$: Then S is an independent subset of P_{n-1} and there are a_{n-1} such subsets.

(2) $v_n \in S$: Then $v_{n-1} \notin S$ and $S - \{v_n\}$ is one of the a_{n-2} independent subsets of P_{n-2}.

Hence $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, $a_1 = 2$, $a_2 = 3$, *or* $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 2$. So $a_n = F_{n+2}$, the $(n+2)$nd Fibonacci number.

(b) Consider the subgraph of G_1 induced by the vertices 1,2,3,4. From part (a) we know that this subgraph determines 8 ($= F_6$, the sixth (nonzero) Fibonacci number) independent subsets of {1,2,3,4}. Therefore, the graph G_1 has $1 + F_6$ independent subsets of vertices.

Likewise the graph G_2 has $1 + F_7$ independent subsets (of vertices), and the graph G_n determines $1 + F_{n+2}$ such subsets.

(c) $H_1 : 3 + F_6 = (2^2 - 1) + F_6$
$H_2 : 3 + F_7 = (2^2 - 1) + F_7$
$H_3 : 3 + F_{n+2} = (2^2 - 1) + F_{n+2}$

(d) There are $2^s - 1 + m$ independent subsets of vertices for graph $G' = (V', E')$.

22. Proof: First we prove that G is connected. If not, let C_1, C_2 be two of the components of G and let $v_1, v_2 \in V$ with v_1 a vertex in C_1 and v_2 a vertex in C_2. If C_1 has n_1 vertices and C_2 has n_2 vertices, then $10 = \deg(v_1) + \deg(v_2) \leq (n_1 - 1) + (n_2 - 1) = (n_1 + n_2) - 2 \leq 8$. This contradiction tells us that G is connected.

Here $|E| = (\frac{1}{2})\sum_v \deg(v) = (\frac{1}{2})(50) = 25$. If G were planar, then we would have $25 = |E| \leq 3|V| - 6 = 3(10) - 6 = 24$, according to Corollary 11.3. This contradiction now tells us that G is nonplanar.