Functions : plain and one-to-one :-

<u>Definition</u> : For nonempty sets A, B, a function, or mapping, f from A to B, denoted $f: A \to B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

<u>Note</u>: We often write $f(a) = b$ when (a,b) is an ordered pair in the function f. For $(a,b) \in f$, b is called the image of a under f, whereas a is a preimage of b. In addition, the definition suggests that f is a method for associating with each $a \in A$ the unique element $f(a) = b \in B$. Consequently, $(a,b), (a,c) \in f$ implies $b = c$.

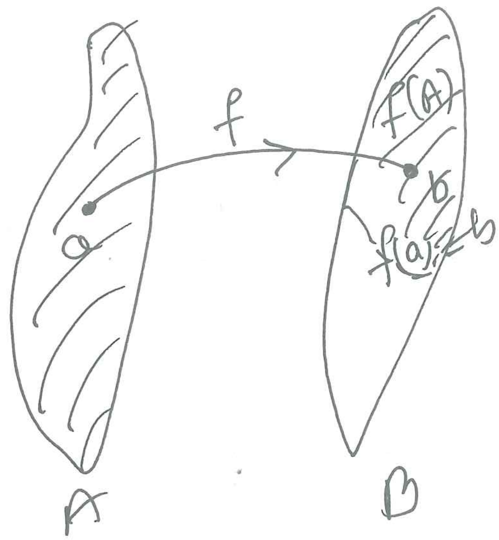
<u>Ex</u>: For $A = \{1,2,3\}$ and $B = \{w, x, y, z\}$, $f = \{(1,w), (2,x), (3,x)\}$ is a function, and consequently a relation, from A to B. $R_1 = \{(1,w), (2,x)\}$ and $R_2 = \{(1,w), (2,w), (2,x), (3,z)\}$ are relations, but not functions, from A to B.

<u>Definition</u>: For the function $f: A \to B$, A is called the domain of f and B is called the codomain of f. The subset of B consisting of those elements that appear as second components in the ordered pairs of f is called the range of f and is denoted by $f(A)$ because it is the set of images under f.

Ex: For $A=\{1,2,3\}$ and $B=\{w,x,y,z\}$,
$f=\{(1,w),(2,x),(3,x)\}$

The domain is $A=\{1,2,3\}$, the co-domain is $B=\{w,x,y,z\}$ and the range of $f = f(A)=\{w,x\}$

Pictorial representations of these ideas is shown below.



Functions in computer science:

a) A common function encountered is the <u>greatest integer function</u>, or <u>floor function</u>.

This fn $f: R \to Z$, is given by
$f(x) = \lfloor x \rfloor =$ the greatest integer less than or equal to x.

Ex (1) $\lfloor 3.8 \rfloor = 3$, $\lfloor 3 \rfloor = 3$, $\lfloor -3.8 \rfloor = -4$ $\lfloor -3 \rfloor = -3$

$\lfloor 7.1+8.2 \rfloor = \lfloor 15.3 \rfloor = 15 = 7+8 = \lfloor 7.1 \rfloor + \lfloor 8.2 \rfloor$

b) <u>Ceiling function</u>. This function $g: R \to Z$ is defined by $g(x) = \lceil x \rceil =$ least integer that is ~~not~~ greater than or equal to x.

Ex: $\lceil 3 \rceil = 3$, $\lceil 3.01 \rceil = \lceil 3.7 \rceil = \lceil 4 \rceil = 4$.
$\lceil -3 \rceil = -3$, $\lceil -3.01 \rceil = -3$, $\lceil 3.3+4.2 \rceil = \lceil 7.5 \rceil = 8$.

Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$, here $|A| = 3$ & $|B| = 4$.
$\therefore$ $A \times B$ has $3 \times 4 = 12$ pairs, hence there are 2^{12} ~~functions~~ Relations are possible from A to B. In a similer way, How many functions are there from A to B?

In general, let A and B be nonempty sets with $|A| = m$, $|B| = n$. Consequently, if $A = \{a_1, a_2, a_3, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$, then a typical function $f : A \to B$ can be described by $\{(a_1, x_1), (a_2, x_2), (a_3, x_3) \ldots (a_m, x_m)\}$. We can select any of the n elements of B for x_1, and then do the same for x_2. We continue this selection process untill one of the n elements of B is finally selected for x_m. In this way using the rule of product, there are $n^m = |B|^{|A|}$ functions from A to B.

$\therefore$ In the above example, for A, B there are $4^3 = 64$ functions from A to B and $3^4 = |A|^{|B|} = 81$ functions from B to A. In general we do not expect $|A|^{|B|}$ to equal to $|B|^{|A|}$. Unlike the situation for relations, we cannot always obtain a function from B to A by simply interchanging the components in the ordered pairs of a function from A to B or (vice versa).

Definition: A function $f: A \to B$ is called one-to-one, or injective, if each element of B appears at most once as the image of an element of A.

~~Ex~~ $f: A \to B$ is one-to-one, with A, B, finite, we must have $|A| \le |B|$. For arbitrary sets A, B, $f: A \to B$ is one-to-one iff for all $a_1, a_2 \in A$ $f(a_1) = f(a_2) = a_1 = a_2$

Ex: $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x+7$ for all $x \in \mathbb{R}$. then for any $x_1, x_2 \in \mathbb{R}$ we find that

$f(x_1) = f(x_2) = 3x_1 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \therefore x_1 = x_2$

Ex: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. The function $f = \{(1,1), (2,3), (3,4)\}$ is a one-to-one fun from A to B.

$g = \{(1,1), (2,3), (3,3)\}$ is a function from A to B, but it fails to be one-to-one because $g(2) = g(3)$ but $2 \ne 3$.

Note: For A, B in above ex, there are 2^{15} relations from A to B and 5^3 of these are functions from A to B. The next question we want to answer is how many functions $f: A \to B$ are one-to-one?

when $A = \{a_1, a_2, a_3, \dots, a_m\}$, $B = \{b_1, b_2, b_3, \dots, b_n\}$, and $m \le n$, a one-to-one function $f : A \to B$ has the form $\{(a_1, x_1), (a_2, x_2), (a_3, x_3), \dots, (a_m, x_m)\}$, where there are n choices for x_1 (i-e any element of B), $n-1$ choices for x_2 (i-e any element of B except the one choosen for x_1), $n-2$ choices for x_3, and so on, finishing with $n-(m-1)$ $= n-m+1$ choices for x_m. By the rule of product, the number of one-to-one functions from A to B is

$$n(n-1)(n-2)\dots(n-m+1) = \frac{n!}{(n-m)!} = P(n,m) = P(|B|, |A|).$$

Ex $A = \{1, 2, 3\}$ & $B = \{1, 2, 3, 4, 5\}$, there are $5 \cdot 4 \cdot 3 = 60$ One-to-one functions $f : A \to B$.

Definition : If $f : A \to B$ and $A_1 \subseteq A$, then

$f(A_1) = \{b \mid b = f(a),\ b \in B$ and some $a \in A\}$

and $f(A_1)$ is called the image of A_1 under f.

Ex: For $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$; let $f : A \to B$ be given by $f = \{(1, w), (2, x), (3, x), (4, y), (5, y)\}$. Then for $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2, 3\}$, $A_4 = \{2, 3\}$ and $A_5 = \{2, 3, 4, 5\}$, we find the following images under f.

$f(A_1) = \{w\}$

$f(A_2) = \{w, x\}$

$f(A_3) = \{w, x\}$

$f(A_5) = \{x, y\}$

Onto functions:

Definition: A function $f : A \to B$ is called onto, or surjective, if $f(A) = B$. i.e, if for all $b \in B$ there is at least one $a \in A$ with $f(a) = b$.

Ex: The function $f : R \to R$ defined by $f(x) = x^3$ is an onto function. For here we find that if r is any real number in the codomain of f, then the real number $\sqrt[3]{r}$ is in the domain of f and $f(\sqrt[3]{r}) = (\sqrt[3]{r})^3 = r$. Hence the codomain and the function of $f = R =$ the range of f, and f is onto.

The function $g : R \to R$, where $g(x) = x^2$ for
each real number x, is not onto function.
In this case no negative real numbers appears
in the range of g. For ex, for -9 to be in the
range of g, we would have to be able to find
a real number r with $g(r) = r^2 = -9$. But this
is not possible.

Ex: If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then
$f_1 = \{(1,z), (2,y), (3,x), (4,y)\}$ and
$f_2 = \{(1,x), (2,x), (3,y), (4,z)\}$ are both onto functions
from A onto B. However, the function $g = \{(1,x), (2,x), (3,y), (4,y)\}$ is not onto, because $g(A) = \{x, y\} \subset B$.

Counting number of onto functions from $A \to B$.

If $A = \{x, y, z\}$ and $B = \{1, 2\}$, then all functions $f: A \to B$ are onto except $f_1 = \{(x,1), (y,1), (z,1)\}$ and $f_2 = \{(x,2), (y,2), (z,2)\}$, the constant functions. So there are $|B|^{|A|} - 2 = 2^3 - 2 = 6$ onto functions from A to B.

In general, if $|A| = m \geq 2$ and $|B| = 2$, then there are $2^m - 2$ onto functions from A to B.

when $m = 1$, $2^1 - 2 = 0$ onto functions from A to B.

Ex: For $A = \{w, x, y, z\}$ and $B = \{1, 2, 3\}$, there are 3^4 functions from A to B. Considering subset of B of size 2, there are 2^4 functions from A to $\{1, 2\}$, 2^4 fn from A to $\{1, 3\}$, and 2^4 fn from A to $\{2, 3\}$. $\therefore$ we have $3 \cdot (2^4)$ function from A to B that are definitely not onto.

$\therefore 3^4 - \binom{3}{2} 2^4$ number of onto functions. But out $\binom{3}{2} 2^4$ functions are not distinct.

For example, if we consider all functions from A to $\{1, 2\}$, out of 2^4 fn we are removing $\{(w,2), (x,2), (y,2), (z,2)\}$. Then, considering the fn from A to $\{2, 3\}$, we remove the same fn.

$\therefore$ In the result $3^4 - \binom{3}{2} 2^4$ we removed twice each of the constant fn $f: A \to B$ where $f(A)$ is one of the sets $\{1\}$, $\{2\}$ or $\{3\}$.

$\therefore 3^4 - \binom{3}{2} 2^4 + 3 = \binom{3}{3} 3^4 - \binom{3}{2} 2^4 + \binom{3}{1} 1^4 = 36$ onto fn

Keeping $B = \{1,2,3\}$, for any set A with $|A| = m \geq 3$ there are $\binom{3}{3}3^m - \binom{3}{2}2^m + \binom{3}{1}1^m$ onto functions from A to B.

if $m=1$ & $m=2$ this yields -2 & -2 respectively (impossible).

<u>General result</u>:

For finite sets A, B with $|A| = m$ and $|B| = n$, there are

$$\binom{n}{n}n^m - \binom{n}{n-1}(n-1)^m + \binom{n}{n-2}(n-2)^m - \cdots + (-1)^{n-2}\binom{n}{2}2^m + (-1)^{n-1}\binom{n}{1}1^m$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n}{n-k}(n-k)^m$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{n-k}(n-k)^m$$ onto functions from A to B.

<u>Ex</u>: Let $A = \{1,2,3,4,5,6,7\}$ and $B = \{w, x, y, z\}$. Applying the general formula with $m=7$ & $n=4$, we get

$$\binom{4}{4}4^7 - \binom{4}{3}3^{\cancel{4}7} + \binom{4}{2}2^7 - \binom{4}{1}1^7$$

$$= \sum_{k=0}^{4} (-1)^k \binom{4}{4-k}(4-k)^7 = 4^7 - \frac{4!}{3!}(3^7) + \frac{4!}{2!\,2!}(2^7) - \frac{4!}{3!}(1^7)$$

$$= 4^7 - 4(3^7) + 6(2^7) - 4(1^7)$$

$$= 16{,}384 - 8748 + 786 - 4 = 8400$$
or
8418

Onto functions from A to B.

Ex: If $A=\{a,b,c,d\}$ and $B=\{1,2,3\}$, then there are 36 onto functions from A to B or, equivalently, 36 ways to distribute four distinct objects into three distinguishable containers, with no container empty. Among these 36 distributions we find the following collection of six

1) $\{a,b\}_1\ \{c\}_2\ \{d\}_3$ 2) $\{a,b\}_1\ \{d\}_2\ \{c\}_3$
3) $\{c\}_1\ \{a,b\}_2\ \{d\}_3$ 4) $\{c\}_1\ \{d\}_2\ \{a,b\}_3$
5) $\{d\}_1\ \{a,b\}_2\ \{c\}_3$ 6) $\{d\}_1\ \{c\}_2\ \{a,b\}_3$

where, for example, the notation $\{c\}_2$ means that c is in the second container. Now if we no longer distinguish the containers, these $6=3!$ distributions become identical, so there are $36/3!=6$ ways to distribute the distinct objects a, b, c, d among three identical containers, leaving no container empty.

General result:

For $m \geq n$ there are $\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$ ways to distribute m distinct objects into n numbered containers with no container left empty. Removing the numbers on the containers, so that they are now identical in appearance, we find that one distribution into these n (nonempty) identical containers corresponds with n! such distributions into the numbered containers. So the number of ways in which it is possible to distribute the m distinct objects into n identical containers, with no containers left empty is

$$\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m.$$

This will be denoted by $S(m,n)$ and is called as the Stirlling number of the second kind. Note that $|A|=m \geq n=|B|$, there are $n! \cdot S(m,n)$ onto functions from A to B.

Table below lists some Stirling numbers of second kind.

m \ n	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

Definition: If $f: A \to B$, then f is said to be bijective, or to be a one-to-one correspondence, if f is both one-to-one and onto.

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{w, x, y, z\}$, then $f = \{(1, w), (2, x), (3, y), (4, z)\}$ is a one-to-one correspondance from A to B, and $g = \{(w, 1), (x, 2), (y, 3), (z, 4)\}$ is a one-to-one correspondance from B to A.

Definition: The function $I_A: A \to A$, denoted by $I_A(a) = a$ for all $a \in A$, is called the identity function for A.

Definition: If $g, f: A \to B$, we say that f and g are equal and we write $f = g$, if $f(a) = g(a)$ for all $a \in A$

Definition:
If $f: A \to B$ and $g: B \to C$, we define the composition function, which is denoted by $g \circ f: A \to C$ by $(g \circ f)(a) = g(f(a))$, for each $a \in A$.

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ with $f: A \to B$ and $g: B \to C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$, for each element of A we find:

$$(g \circ f)(1) = g(f(1)) = g(a) = x$$
$$(g \circ f)(2) = g(f(2)) = g(a) = x$$
$$(g \circ f)(3) = g(f(3)) = g(b) = y$$
$$(g \circ f)(4) = g(f(4)) = g(c) = z$$

So, $g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$

Example: Let $f: R \to R$, $g: R \to R$ be defined by $f(x) = x^2$, $g(x) = x+5$, then

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2+5$$

whereas

$$(f \circ g)(x) = f(g(x)) = f(x+5) = (x+5)^2 = x^2+10x+25.$$

If $x=1$

$$(f \circ g)(1) = 1^2+10+25 = 36$$

$$(g \circ f)(1) = 1^2+5 = 6$$

Therefore $(f \circ g) \neq (g \circ f)$ i.e composition of functions is not commutative.

Theorem: Let $f: A \to B$ and $g: B \to C$

a) If f, g are one-to-one, then $g \circ f$ is one-to-one.

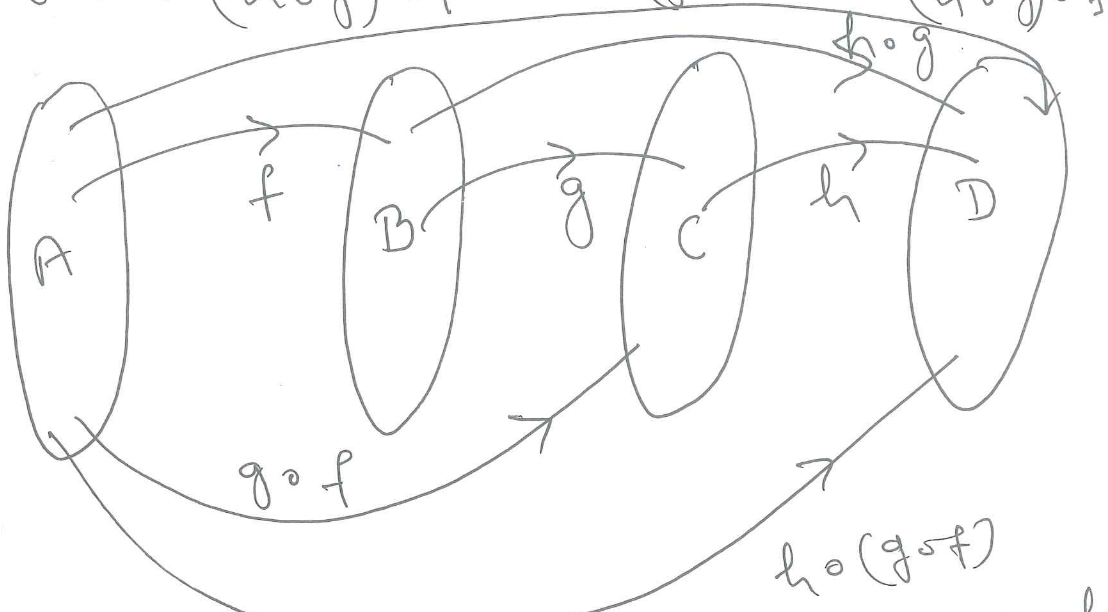
b) If f, g are onto, then $g \circ f$ is onto.

Proof: (a) $g \circ f: A \to C$ is one-to-one, let $a_1, a_2 \in A$ with $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then $(g \circ f)(a_1) = g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2)$ because g is one-to-one. Also, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ because f is one-to-one. Consequently, $g \circ f$ is one-to-one.

(b) For $g \circ f: A \to C$, let $z \in C$, since g is onto there exists $y \in B$ with $g(y) = z$. With f is onto there exists $x \in A$ with $f(x) = y$. Hence $z = g(y) = g(f(x)) = (g \circ f)(x)$, so the range of $(g \circ f) = C =$ the co-domain of $g \circ f$, and hence $g \circ f$ is onto.

Function composition is associative :-

If $f: A \to B$, $g: B \to C$, and $h: C \to D$ then $(h \circ g) \circ f = h \circ (g \circ f)$.



Proof: Since both the functions have the same domain A, and co-domain D, the result will follows by showing that for every $x \in A$, see the fig above.

$$((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$$

$$(h \circ g)(f(x)) = (h \circ (g \circ f)(x))$$

$$h(g(f(x))) = h(g(f(x)))$$

Therefore, composition of functions is associative.

By virtue of the associative property for function composition, we can write $h \circ g \circ f$, $(h \circ g) \circ f$ or $h \circ (g \circ f)$ without any ambiguity.

Definition: If $f: A \to A$, we define $f^1 = f$ and for $n \in \mathbb{Z}^+$, $f^{n+1} = f \circ (f^n)$

Example: With $A=\{1,2,3,4\}$ and $f:A\to A$ defined by $f=\{(1,2),(2,2),(3,1),(4,3)\}$, we have

$f^2=f\circ f=\{(1,2)(2,2),(3,2),(4,1)\}$ and

$f^3=f\circ f^2=\{(1,2),(2,2),(3,2),(4,2)\}$

similerly find f^4 & f^5

$f^4=f\circ f^3=\{(1,2),(2,2),(3,2),(4,2)\}$

$f^5=f\circ f^4=\{(1,2),(2,2),(3,2),(4,2)\}$

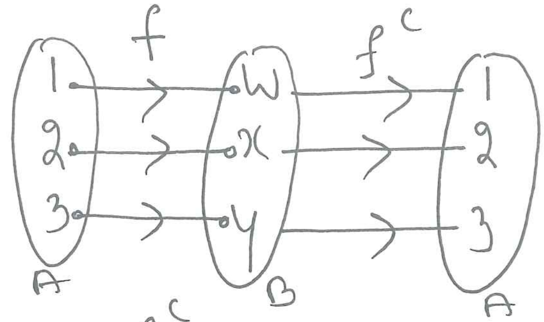
Invertible functions and its properties:

Definition: For sets $A, B\subseteq \mathcal{U}$, if R is a relation from A to B, then the Converse of R, denoted R^c, is the relation from B to A defined by R^c $=\{(b,a)\mid (a,b)\in R\}$

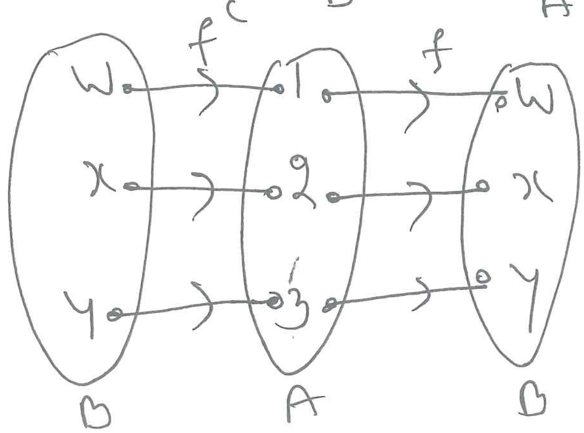
To get R^c from R, we simply interchange the components of each ordered pair in R. So if $A=\{1,2,3,4,5\}$, $B=\{w,x,y\}$, and $R=\{(1,w),(2,w),(3,x)\}$ then, $R^c=\{(w,1),(w,2),(x,3)\}$, a relation from B to A.

For the same sets A, B, let $f: A\to B$ where $f=\{(1,w),(2,x),(3,y),(4,x)\}$. Then $f^c=\{(w,1),(x,2),(y,3),(x,4)\}$, a relation, but not a function from B to A.

Ex: For $A=\{1,2,3\}$ and $B=\{w,x,y\}$, let $f: A\to B$ be given by $f=\{(1,w),(2,x),(3,y)\}$. Then $f^c=\{(w,1),(x,2),(y,3)\}$ is a function from $B\to A$, and we observe that $f^c\circ f=I_A$ and $f\circ f^c=I_B$



$\therefore \quad f^c\circ f = I_A$



$f\circ f^c = I_B$

Definition: If $f: A\to B$, then f is said to be invertible if there is a function $g: B\to A$ such that $g\circ f=I_A$ and $f\circ g=I_B$

Note: g is also invertible.

Ex: Let $f, g: R\to R$ be defined by $f(x)=2x+5$, $g(x)=(1/2)(x-5)$. Then

$(g\circ f)(x) = g(f(x)) = g(2x+5) = \frac{1}{2}[(2x+5)-5] = x$

$(f\circ g)(x) = f(g(x)) = f(\frac{1}{2}(x-5)) = 2(\frac{1}{2}(x-5))+5 = x\cancel{-5}\cancel{+5}=x$

so $f\circ g = I_R$ and $g\circ f = I_R$. Consequently, f and g are invertible.

Theorem :- If a function $f: A \to B$ is invertible and a function $g: B \to A$ satisfying $g \circ f = I_A$ and $f \circ g = I_B$, then this function g is unique.

Proof : If g not unique, let h is a fn from $B \to A$ such that $h \circ f = I_A$ & $f \circ h = I_B$. Consequently,

$$h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g.$$

NOTE :- Let us call this function g the inverse of f and shall adopt the notation $g = f^{-1}$.
Also $f^{-1} = f^c$. We also see that if f is a invertible function f^{-1}. $\therefore (f^{-1})^{-1} = f$, by the uniqueness.

Theorem :- A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof : Assuming that $f: A \to B$ is invertible, we have a unique function $g: B \to A$ with $g \circ f = I_A$, $f \circ g = I_B$.
If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$ or $(g \circ f)(a_1) = (g \circ f)(a_2)$. With $g \circ f = I_A$ it follows that $a_1 = a_2$, so f is one-to-one. For the onto property, let $b \in B$. Then $g(b) \in A$, so we can talk about $f(g(b))$. Since $f \circ g = I_B$, we have $b = I_B(b)$ $= (f \circ g)(b) = f(g(b))$, so f is onto.

Ex : The function $f_1: \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x^2$ is not invertible, but $f_2: [0, +\infty] \to [0, +\infty]$ defined by $f_2(x) = x^2$ is invertible with $f_2^{-1} = \sqrt{x}$.

Theorem :- If $f: A \to B$, $g: B \to C$ are invertible functions ... $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

The Pigeonhole Principle :-

If m pigeons occupy n pigeonholes and $m > n$ then at least one pigeonhole has two or more pigeons roosting in it.

Ex (1) : An office employs 13 file clerks, so at least two of them must have birthdays during the same month.

$\Rightarrow$ Here 13 = (m) pigeons (clerks) and 12 = (n) pigeonholes (months)

Ex (2) : Mr. John operates a computer with a magnetic tape drive. One day he is given with a tape that contains 500,000 words of four or fewer lowercase letters (consequitive words separated by blank), Can it be that the 500,000 words are all distinct.

$\Rightarrow$ From the rules of sum and product, the total number of different words possible using four or fewer letters are

$$26^4 + 26^3 + 26^2 + 26 = 475,254$$

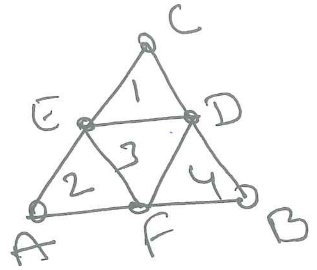
with this 475,254 as pigeonholes and 500,000 words on tape as pigeons, atleast one word is repeated on the tape.

Ex (3) : Any subset of size 6 from the set $S = \{1, 2, 3, \ldots, 9\}$ must contain two elements whose sum is 10.

$\Rightarrow$ Here, pigeons are all subsets of length - 6 and the pigeon holes are the $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$. When six pigeons go to the respective pigeonholes, they must fill at least one of the two element subsets whose members sums to 10.

Ex (4): Triangle ABC is equilateral with AC = 1. If five points are selected from the interior of the triangle, there are at least two whose distance apart is less than 1/2.

$\Rightarrow$



For the $\triangle$le shown, the four smaller triangles are congruent equilateral triangles and $AE = 1/2$. We break $\triangle$le ABC into the following four regions which are mutually disjoint in pairs.

Since there are 4 regions (pigeonholes) and 5 points are choosen. Therefore, according to pigeon hole principle, at least two points lies on the same region.

Computational complexity:

Definition: Let $f, g : Z^+ \to R$. We say that g dominates f (or f dominates g) if there exist constants $m \in R^+$ and $K \in Z^+$ such that $|f(n)| \leq m|g(n)|$ for all $n \in Z^+$, where $n \geq K$.

Note that as we consider the values of $f(1), g(1), f(2), g(2), \ldots$ there is a point (namely, K) after which the size of $f(n)$ is bounded above by a positive multiple (m) of the size of $g(n)$. Also, when g dominates f, then $|f(n)/g(n)| \leq m$ i.e, the size of the quotient $f(n)/g(n)$ is bounded by m, for those $n \in Z^+$ where $n \geq K$ and $g(n) \neq 0$.

When f is dominated by g we say that f is of order (at most) g and we use what is called "big-oh" notation to designate this. We write $f \in O(g)$, where $O(g)$ is order of g or big-oh of g.

Ex (1): Let $f, g : Z^+ \to R$ be given by $f(n) = 5n$, $g(n) = n^2$, for $n \in Z^+$. If we compute $f(n)$ and $g(n)$ for $1 \leq n \leq 4$, we find that $f(1) = 5$, $g(1) = 1$, $f(2) = 10$, $g(2) = 4$, $f(3) = 15$, $g(3) = 9$, & $f(4) = 20$, $g(4) = 16$. However, $n \geq 5 \Rightarrow n^2 \geq 5n$ and we have $|f(n)| = 5n \leq n^2 = |g(n)|$. So with $m=1$ & $k=5$, we find that for $n \geq k$, $f(n) \leq m\,|g(n)|$. Consequently g dominates f and $f \in O(g)$.

Ex (2): Let $f, g : Z^+ \to R$ with $f(n) = 5n^2 + 3n + 1$ and $g(n) = n^2$. Then $|f(n)| = |5n^2 + 3n + 1| = 5n^2 + 3n + 1 \leq 5n^2 + 3n^2 + n^2 = 9n^2 = 9\,|g(n)|$.

$\therefore |f(n)| \leq m\,|g(n)|$ for $m \geq 9$, hence $f \in O(g)$

$\boxed{O(g) = O(f)}$ is also holds good.

Similarly, if $f(n) = 3n^3 + 7n^2 - 4n + 2$ &
$g(n) = n^3$

here $|f(n)| = |3n^3 + 7n^2 - 4n + 2| \leq |3n^3| + |7n^2| + |-4n| + |2|$
$\leq 3n^3 + 7n^3 + 4n^3 + 2n^3 = 16n^3 = 16\,|g(n)|$ for all $n \geq 1$. So with $m = 16$ & $K = 1$, we find that f is dominated by g. & $f \in O(g)$ or $f \in O(n^3)$

$\boxed{O(g) = O(f)}$