

(1)

Joint Probability Mass Function

Let S be a sample space and X and Y be two discrete random variables defined on S .

Let $X = f(S)$, $Y = g(S)$, $s \in S$, where f and g are real-valued functions defined on S , and

$$f(S) = \{x_1, x_2, x_3, \dots\} \text{ and } g(S) = \{y_1, y_2, y_3, \dots\}$$

be the image sets of S under f and g respectively, where x_1, x_2, \dots are the values of X and y_1, y_2, \dots are the values of Y .

The probability that $X = x_i$, denoted by $P(x_i)$, is

$$\text{defined by } P(X=x_i) = P(x_i) = P\{s \in S | f(s)=x_i\}.$$

The probability that $Y = y_j$, denoted by $P(y_j)$, is

$$\text{defined by } P(Y=y_j) = P(y_j) = P\{s \in S | g(s)=y_j\}.$$

Then their joint probability mass function is given

$$\text{by } p_{ij} = P(x_i, y_j) = P\{s \in S | f(s)=x_i \text{ and } g(s)=y_j\}.$$

such that (i) $p_{ij} \geq 0$, for $i=1, 2, 3, \dots, j=1, 2, 3, \dots$

$$\text{(ii)} \quad \sum_i \sum_j p_{ij} = 1$$

The values of X , Y and $P(X, Y)$ can be displayed in the form of the following table, called the Probability Contingency Table or Joint Probability Table.

$X \setminus Y$	y_1	y_2	y_3	\dots
x_1	P_{11}	P_{12}	P_{13}	\dots
x_2	P_{21}	P_{22}	P_{23}	\dots
x_3	P_{31}	P_{32}	P_{33}	\dots
\vdots				

Marginal probability distributions

$$\text{Let } P_i = P_{i1} + P_{i2} + P_{i3} + \dots = \sum_j P_{ij} \quad (\text{fixed } i)$$

$$Q_j = P_{j1} + P_{j2} + P_{j3} + \dots = \sum_i P_{ij} \quad (\text{fixed } j)$$

The sets $\{P_i\}$, $i=1, 2, 3, \dots$, and $\{Q_j\}$, $j=1, 2, 3, \dots$ are called the marginal probability distributions

of X and Y respectively. These P_i and Q_j are identical with the individual probability distributions of X and Y (respectively), so that

$$P_i = P(x_i) = P(X=x_i) \quad \text{and} \quad Q_j = P(y_j) = P(Y=y_j)$$

Stochastic Independence

The random variables X and Y , defined on the same sample space S , are said to be stochastically independent or statistically independent.

$$P_{ij} = P_i Q_j$$

Expectations

(2)

Consider a function $\phi(x, y)$ of x and y .
 The function $E[\phi(x, y)]$ defined by the formula

$$E[\phi(x, y)] = \sum_i \sum_j p_{ij} \phi(x_i, y_j)$$

is called the mathematical expectation (or expectation) of $\phi(x, y)$ in the joint distribution of X and Y .

Suppose $\phi(x, y) = (x - \mu_x)^r (y - \mu_y)^s$,

where $\mu_x = E(X)$ and $\mu_y = E(Y)$ are the means of the distributions of X and Y respectively, r and s are non-negative integers. Then the above formula yields.

$$E[(x - \mu_x)^r (y - \mu_y)^s] = \sum_i \sum_j p_{ij} (x_i - \mu_x)^r (y_j - \mu_y)^s$$

The function $E[(x - \mu_x)^r (y - \mu_y)^s]$ is called the $(r, s)^{th}$ moment about the mean or the $(r, s)^{th}$ central moment of the joint distribution of X, Y and is denoted by μ_{rs} .

In particular, μ_{11} is called the covariance of X and Y and is denoted by $\text{Cov}(X, Y)$.

$$\text{Thus, } \text{Cov}(X, Y) = \mu_{11} = E[(x - \mu_x)(y - \mu_y)]$$

$$= \sum_i \sum_j p_{ij} (x_i - \mu_x)(y_j - \mu_y)$$

Note: $\text{Cov}(X, Y)$ is the expectation of the function $\phi(x, y) = (x - \mu_x)(y - \mu_y)$

* If $y = x$, then

$$\text{Cov}(x, x) = E[(x - \mu_x)^2] = \text{Var}(x)$$

* $\text{Cov}(x, y) = E[xy] - E[x]E[y]$

* Let σ_x and σ_y be the standard deviations of the distributions of x and y respectively.

Then $r(x, y)$ is defined by

$$r(x, y) = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y},$$

which is called the correlation coefficient of the joint distribution of x, y .

* $\text{Cov}(x, y) = \sum_i \sum_j p_{ij} (x_i - \mu_x)(y_j - \mu_y)$

$$E[xy] = \sum_i \sum_j p_{ij} x_i y_j$$

$$\mu_x = E[x] = \sum_i x_i p_i = \sum_i \sum_j p_{ij} x_i$$

$$\mu_y = E[y] = \sum_j y_j p_j = \sum_i \sum_j p_{ij} y_j$$

Also $\sum_i \sum_j p_{ij} = 1$

$$\begin{aligned} \therefore \text{Cov}(x, y) &= \sum_i \sum_j p_{ij} (x_i y_j - \mu_y \sum_i p_{ij} x_i - \mu_x \sum_j p_{ij} y_j) \\ &= E[xy] - \mu_y \mu_x - \mu_x \mu_y + \mu_x \mu_y \cdot 1 \\ &= E[xy] - E[x]E[y] \end{aligned}$$

The joint probability distribution of two random variables X and Y is given by the following table: (3)

$X \backslash Y$	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine the individual (marginal) distributions of X and Y . Also, verify that X and Y are stochastically independent.

Sol)

$X \backslash Y$	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

$$P_1 = 0.3$$

$$P_2 = 0.7$$

$$\theta_1 = 0.2 \quad \theta_2 = 0.5 \quad \theta_3 = 0.3$$

X	1	2
$P(X)$	0.3	0.7

Y	2	3	4
$P(Y)$	0.2	0.5	0.3

$$P_1 \theta_1 = 0.06 = P_{11}, \quad P_1 \theta_2 = 0.15 = P_{12}, \quad P_1 \theta_3 = 0.09 = P_{13}$$

$$P_2 \theta_1 = 0.14 = P_{21}, \quad P_2 \theta_2 = 0.35 = P_{22}, \quad P_2 \theta_3 = 0.21 = P_{23}$$

Thus $P_i \theta_j = P_{ij}$ for all allowable values of i and j . Hence X and Y are stochastically independent.

* A joint distribution of two random variables X and Y is given by the following table:

X\Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

, Are X and Y independent?

Determine (i) The marginal distributions of X and Y.

(ii) $E(X)$ and $E(Y)$, (iii) $E(XY)$

Soln From the given table, we have

i	X	1	5	Y	-4	2	7
P(X)	1/2	1/2	P(Y)	3/8	3/8	2/8	

$P_1 \theta_1 = \frac{3}{16} \neq P_{11}$. Thus $P_i \theta_j \neq P_{ij}$ for $i=1, j=1$,
Therefore, X and Y are not independent.

(ii) $E(X) = \mu_X = x_1 P_1 + x_2 P_2 = 1 \times \frac{1}{2} + 5 \times \frac{1}{2} = 3$

$$E(Y) = \mu_Y = y_1 \theta_1 + y_2 \theta_2 + y_3 \theta_3 = (-4) \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times \frac{2}{8} = 1$$

(iii) $E(XY) = \sum_i \sum_j P_{ij} x_i y_j$

$$= (P_{11} x_1 y_1 + P_{12} x_1 y_2 + P_{13} x_1 y_3) + (P_{21} x_2 y_1 + P_{22} x_2 y_2 + P_{23} x_2 y_3)$$

$$= \frac{1}{8} \times 1 \times (-4) + \frac{1}{4} \times 1 \times 2 + \frac{1}{8} \times 1 \times 7 + \frac{1}{4} \times 5 \times (-4) + \frac{1}{8} \times 5 \times 2 + \frac{1}{8} \times 5 \times 7$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{7}{8} - 5 + \frac{5}{4} + \frac{35}{8}$$

$$= \frac{3}{2}$$

* The joint distribution of two random variables ④
 X and Y is given by the following table:

X\Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find the marginal distribution of X and Y,
 and evaluate $\text{Cov}(X, Y)$ and $P(X, Y)$.

Sol From the given table .

X	2	4	6
$P(X)$	1/4	2/4	1/4

Y	1	3	9
$P(Y)$	3/6	2/6	1/6

$$E[X] = \sum x_i P_i = 2 \times \frac{1}{4} + 4 \times \frac{2}{4} + 6 \times \frac{1}{4} = 4$$

$$E[Y] = \sum y_j P_j = 1 \times \frac{3}{6} + 3 \times \frac{2}{6} + 9 \times \frac{1}{6} = 3$$

$$\begin{aligned} E[XY] &= \sum_i \sum_j P_{ij} x_i y_j = P_{11} x_1 y_1 + P_{12} x_1 y_2 + P_{13} x_1 y_3 \\ &\quad + (P_{21} x_2 y_1 + P_{22} x_2 y_2 + P_{23} x_2 y_3) \\ &\quad + (P_{31} x_3 y_1 + P_{32} x_3 y_2 + P_{33} x_3 y_3) \\ &= (2 \times \frac{1}{8} \times 2 \times 1 + \frac{1}{24} \times 2 \times 3 + \frac{1}{12} \times 2 \times 9) + (\frac{1}{4} \times 4 \times 1 + \frac{1}{4} \times 4 \times 3 + 0 \times 4 \times 9) \\ &\quad + (\frac{1}{8} \times 6 \times 1 + \frac{1}{24} \times 6 \times 3 + \frac{1}{12} \times 6 \times 9) \end{aligned}$$

$$E[XY] = 12$$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 12 - 4 \times 3 = 0$$

$$\text{As } P(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0$$

* The distributions of two stochastically independent random variables X and Y defined on the same sample space are given by the following tables:

X	0	1
$P(X)$	0.2	0.8

Y	1	2	3
$P(Y)$	0.1	0.4	0.5

Find the joint distribution of X and Y .

Also evaluate $\text{Cov}(X, Y)$.

Using $P_{ij} = P_i Q_j \Rightarrow$

$X \setminus Y$	1	2	3
0	0.02	0.08	0.10
1	0.08	0.32	0.40

$$P_i = P(x_i)$$

$$Q_j = P(y_j)$$

$$E[X] = \sum x_i P_i = 0 \times 0.2 + 1 \times 0.8 = 0.8$$

$$E[Y] = \sum y_j Q_j = 1 \times 0.1 + 2 \times 0.4 + 3 \times 0.5 = 2.4$$

$$E[XY] = \sum_i \sum_j P_{ij} x_i y_j = 0.02 \times 0 \times 1 + 0.08 \times 0 \times 2 + 0.10 \times 0 \times 3 \\ + 0.08 \times 1 \times 1 + 0.32 \times 1 \times 2 + 0.40 \times 1 \times 3 \\ = 1.92$$

$$\therefore \text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y]$$

$$= 1.92 - 0.8 \times 2.4$$

$$= 0$$

- * The joint probability function for two discrete random variables X and Y is given by $f(x,y) = c(2x+y)$ where x and y can assume all integral values such that $0 \leq x \leq 2$, $0 \leq y \leq 3$, and $f(x,y) = 0$ otherwise. Find (i) the value of the constant c , (ii) $P(X=2, Y=1)$, (iii) $P(X \geq 1, Y \leq 2)$, (iv) $P(X+Y \leq 1)$ (v) $P(X+Y > 1)$.

$x \backslash y$	0	1	2	3
0	0	c	$2c$	$3c$
1	$2c$	$3c$	$4c$	$5c$
2	$4c$	$5c$	$6c$	$7c$

$$\text{① } \sum_i \sum_j P_{ij} = 1$$

$$\Rightarrow (0 + c + 2c + 3c) + (2c + 3c + 4c + 5c) + (4c + 5c + 6c + 7c) = 1$$

$$\Rightarrow 42c = 1 \Rightarrow c = 1/42$$

$$\text{② } P(X=2, Y=1) = P(x_3, y_2) \quad P_{32} = 5c = 5/42$$

$$\begin{aligned} \text{③ } P(X \geq 1, Y \leq 2) &= P(X=1, Y \leq 2) + P(X=2, Y \leq 2) \\ &= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=2) \\ &\quad + P(X=2, Y=0) + P(X=2, Y=1), P(X=2, Y=2) \end{aligned}$$

$$\begin{aligned} &= P_{21} + P_{22} + P_{23} + P_{31} + P_{32} + P_{33} \\ &= (2c + 3c + 4c + 5c) + (4c + 5c + 6c + 7c) \\ &= 24c = 24/42 = 4/7 \end{aligned}$$

$$\begin{aligned} \text{④ } P(X+Y \leq 1) &= P(X=0, Y=0) + P(X=0, Y=1) + P(X=1, Y=0) \\ &= P_{11} + P_{12} + P_{21} \\ &= 0 + c + 2c \\ &= 3c = 3/42 = 1/14 \end{aligned}$$

$$\begin{aligned} \text{⑤ } P(X+Y > 1) &= 1 - P(X+Y \leq 1) \\ &= 1 - \frac{1}{14} \\ &= 13/14 \end{aligned}$$

* A coin is tossed three times. Let X be equal to 0 or 1 according as a head or a tail occurs on the first toss. Let Y be equal to the total number of heads which occur.

Determine (i) the marginal distributions of X and Y , and (ii) the joint distribution of X and Y ,

(iii) expected values of $X, Y, X+Y$ and XY ,

(iv) σ_X^2 and σ_Y^2 (v) $\text{Cov}(X, Y)$ and $P(X, Y)$.

(vi) σ_X^2 and σ_Y^2

Soln: $S = \{HHH, HHT, HTT, TTT, THH, THT, TTH\}$

i	X	0	1
	$P(X)$	$4/8$	$4/8$

y	0	1	2	3
$P(Y)$	$1/8$	$3/8$	$3/8$	$1/8$

x	0	1	2	3
0	0	$1/8$	$2/8$	$1/8$
1	$1/8$	$2/8$	$1/8$	0

$$(iii) E[X] = \mu_X = \sum x_i P(x_i) \\ = 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$$

$$E[Y] = \mu_Y = \sum y_j P(y_j) \\ = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} \\ = \frac{12}{8}$$

$$E[X+Y] = \sum_i \sum_j p_{ij} (x_i + y_j) \\ = P_{11}(x_1 + y_1) + P_{12}(x_1 + y_2) + P_{13}(x_1 + y_3) \\ + P_{14}(x_1 + y_4) + P_{21}(x_2 + y_1) + P_{22}(x_2 + y_2) \\ + P_{23}(x_2 + y_3) + P_{24}(x_2 + y_4)$$

$$= 0(0+0) + \frac{1}{8}(0+1) + \frac{2}{8}(0+2) \\ + \frac{1}{8}(0+3) + \frac{1}{8}(1+0) + \frac{2}{8}(1+1) \\ + \frac{1}{8}(1+2) + 0(1+3) \\ = \frac{16}{8} = 2$$

$$E[XY] = \sum_i \sum_j p_{ij} x_i y_j \\ = P_{11} x_1 y_1 + P_{12} x_1 y_2 + P_{13} x_1 y_3 + P_{14} x_1 y_4 \\ + P_{21} x_2 y_1 + P_{22} x_2 y_2 + P_{23} x_2 y_3 + P_{24} x_2 y_4 \\ = 0+0+0+0+\frac{2}{8}+\frac{2}{8}+0 \\ = \frac{1}{2}$$

$$(iv) \sigma_X^2 = E(X^2) - \mu_X^2 \\ = \sum x_i^2 P(x_i) - [E(X)]^2 \\ = 0^2 \times \frac{4}{8} + 1^2 \times \frac{4}{8} - \left(\frac{4}{8}\right)^2 = \frac{1}{4}$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2 \\ = \sum y_j^2 P(y_j) - [E(Y)]^2 \\ = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} \\ - \left(\frac{12}{8}\right)^2 = \frac{3}{4}$$

$$\text{Cov}(X, Y) = \frac{1}{4} E(XY) - \mu_X \mu_Y \\ = \frac{1}{2} - \frac{1}{2} \times \frac{3}{2} = -\frac{1}{4}$$

$$P(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{-1/4}{(\sqrt{2})(\sqrt{3})} = -\frac{1}{\sqrt{3}}$$

1. The joint probability distribution of two random variables X and Y is defined by the function $P(X, Y) = \frac{1}{27}(2X+Y)$ where X and Y assume the integer values 0, 1, 2. Find the marginal distributions of X and Y . Are X and Y independent? (6)

X	0	1	2	Y	0	1	2
$P(X)$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{27}$	$P(Y)$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{27}$

X and Y are not independent.

2. Consider an experiment that consists of 2 throws of a fair die. Let X be the number of 4s and Y be the number of 5s obtained in the two throws. Find the joint probability distribution of X and Y . Also evaluate $P(2X+Y < 3)$.

$X \setminus Y$	0	1	2	
0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$	$P(2X+Y < 3)$
1	$\frac{8}{36}$	$\frac{2}{36}$	0	
2	$\frac{1}{36}$	0	0	

3. The joint distribution of two random variables X and Y is as given below:

$X \setminus Y$	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Find the marginal distributions of X and Y . Also determine μ_x, μ_y and the covariance of X and Y .

X	1	2	Y	-2	-1	4	5	$\mu_x = 1.4$
$P(X)$	0.6	0.4	$P(Y)$	0.3	0.3	0.1	0.3	$\mu_y = 1.0$

$$\text{Cov}(x, y) = -0.5$$

