er-5: Relations 3

Definition-1: For sets $A, B \subseteq \mathcal{U}$, the cartesian product, or cross product, of A and B is denoted by $A \times B$ and equals $\{(a,b) \mid a \in A,\ b \in B\}$.

We say that the elements of $A \times B$ are ordered pairs. For $(a,b), (c,d) \in A \times B$, we have $(a,b) = (c,d)$ iff $a = c$ & $b = d$.

If A and B are finite, it follows from the rule of product that $|A \times B| = |A| \cdot |B|$.

In general we will not have $A \times B = B \times A$, but we will have $|A \times B| = |B \times A|$.

Even though $A, B \subseteq \mathcal{U}$, but $A \times B$ need not be $\subseteq \mathcal{U}$.

$A_1 \times A_2 \times \cdots \times A_n$ equals $\{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \le i \le n\}$.

Ex: Let $\mathcal{U} = \{1, 2, 3, \ldots, 7\}$, $A = \{2, 3, 4\}$ & $B = \{4, 5\}$, Then

a) $A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}$.

b) $B \times A = \{(4,2), (4,3), (4,4), (5,2), (5,3), (5,4)\}$.

c) $B^2 = B \times B = \{(4,4), (4,5), (5,4), (5,5)\}$.

d) $B^3 = B \times B \times B = \{(a,b,c) \mid a, b, c \in B\}$; for example $(4,5,5) \in B^3$

Definition-2: For sets $A, B \subseteq \mathcal{U}$, any subset of $A \times B$ is called a relation from A to B. Any subset of $A \times A$ is called as binary relation on A.

Ex: A, B & $\mathcal{U}$ are as in ex(1).

a) ϕ b) $\{(2,4)\}$ c) $\{(2,4), (2,5)\}$

d) $\{(2,4), (3,4), (4,4)\}$ e) $\{(2,4), (3,4), (4,5)\}$ f) $A \times B$

Since $|A \times B| = 6$, it follows form definition (2) that there are 2^6 possible relations from B to A or A to B.

In general, for finite sets A, B with $|A| = m$ and $|B| = n$, there are 2^{mn} relations from A to B, including the empty relation as well as the relation $A \times B$ itself. There are 2^{nm} $(= 2^{mn})$ relations from B to A, one of which is also ϕ and $B \times A$ itself. The reason we get the same number of relations from B to A as we have from A to B is that any relation R_1 from B to A can be obtained from a unique relation R_2 from A to B by simply reversing the components of each ordered pair in R_2 (and vice-versa).

Ex (3) Let $B = \{1, 2\} \subseteq N$, $\mathcal{U} = P(B)$ and $A = \mathcal{U}$ $= \{\phi, \{1\}, \{2\}, \{1, 2\}\}$. The following is an example of a binary relation on A: $R = \{(\phi, \phi), (\phi, \{1\}), (\phi, \{2\}), (\phi, \{1,2\}), (\{1\}, \{1\}), (\{1\}, \{1,2\}), (\{2\}, \{2\}), (\{2\}, \{1,2\}), (\{1,2\}, \{1,2\})\}$. We say that the relation R is the subset relation where $(C, D) \in R$ iff $C, D \subseteq B$ and $C \subseteq D$.

<u>Theorem 1</u>: For any sets $A, B, C \subseteq \mathcal{U}$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

<u>proof</u>: For $a, b \in \mathcal{U}$, $(a, b) \in A \times (B \cap C)$

$\Leftrightarrow a \in A$ and $b \in (B \cap C)$

$\Leftrightarrow a \in A$ and $b \in B, C$

$\Leftrightarrow a \in A, b \in B$, and $a \in A, b \in C$

$\Leftrightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C$

$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$.

Similarly, the followings are correct

1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Definition - 3 : For nonempty sets A, B, a function, or mapping, f from A to B, denoted $f: A \to B$ is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

Ex : Let $A = \{1,2,3\}$
$B = \{w, x, y, z\}$
$f = \{(1,w), (2,x), (3,x)\}$ is a function from A to B.
$R_1 = \{(1,w), (2,x)\}$ and
$R_2 = \{(1,w), (2,w), (2,x), (3,z)\}$ are relations, but not functions, from A to B.

Representations of relations.

I Set notation
II Matrix form.

Consider the finite sets $A = \{a_1, a_2, \ldots, a_m\}$ & $B = \{b_1, b_2, \ldots, b_n\}$ of orders m and n respectively then $A \times B$ consists of all ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$. Let R be a relation from A to B then R is a subset of $A \times B$. Define $m \times n$ matrix, m_{ij} as follows

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

Ex : $A = \{0,1,2\}$, $B = \{p, q\}$ & $R = \{(0,p), (1,q), (2,p)\}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a, b, c \in A$. Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$.

Example: The relations R_3, R_5 above are reflexive since they both contain all pairs of the form (a,a) namely, $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. The other relations are not reflexive since they do not contain all of these ordered pairs.

The relations R_2, and R_3 are symmetric, because in each case (b,a) belongs to the relation whenever (a,b) does. For R_2, the only thing to check is that both $(1,2)$ and $(2,1)$ are in the relation. For R_3, it is necessary to check that both $(1,2)$ and $(2,1)$ belongs to the relation, and $(1,4)$ and $(4,1)$ belongs to the relation.

R_4, R_5 and R_6 are all antisymmetric.

R_4, R_5 and R_6 are all transitive.
R_4 is transitive because $(3,2)$ and $(2,1)$, $(4,2)$ and $(2,1)$, $(4,3)$ and $(3,1)$, and $(4,3)$ and $(3,2)$ are the only such sets of pairs, and $(3,1)$, $(4,1)$ & $(4,2)$ belong to R_4.
R_1 is not transitive since $(3,4)$ and $(4,1)$ belongs to R_1, but $(3,1)$ does not. R_2 is not transitive since $(2,1)$ & $(1,2)$ belong to R_2, but $(2,2)$ does not. R_3 is not transitive since $(4,1)$ and $(1,2)$ belongs to R_3, but $(4,2)$ does not.

Problem: Consider a relation "divides" on set of all integers. Is this relation reflexive, symmetric, antisymetric & transitive?

Soln: Since $a|a$ whenever a is a +ve integer, the divides relation is reflexive.
Suppose that a divides b, ~~and~~ then $b \nmid a$, for ep $1|2$ but $2 \nmid 1$, hence it is not symmetric, but it is antisymmetric, because if $a|b$ and $b|a$ iff $a = b$.
Suppose that $a|b$ and $b|c$. Then there are positive integers k & l such that $b = ak$ and $c = bl = a(kl)$. Hence a divides c. It follows that the relation "divides" is a transitive relation.

Operations on relations:-

Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Ex: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

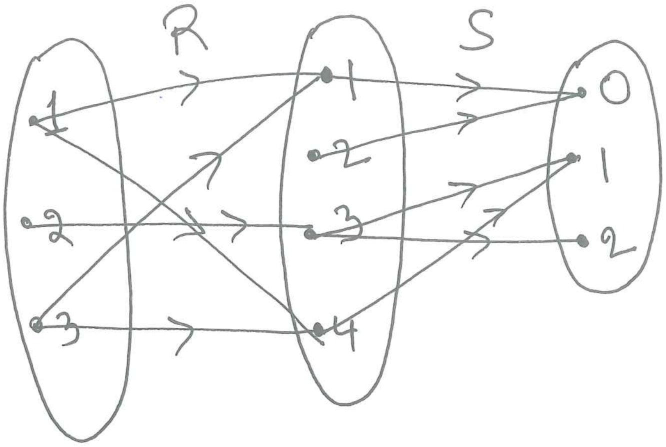
$R_1 \triangle R_2$ or $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1) = \{(1,2),(1,3),(1,4),(2,2),(3,3)\}$.

Composition of relations:-

Let R be a relation from a set A to a set B and S be a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. This is denoted by $S \circ R$

Ex (1): What is the composite of the relations R & S where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R=\{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with $S=\{(1,0),(2,0),(3,1),(3,2),(4,1)\}$?

Soln:



$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

Ex (2): Composing the parent relation with itself.

Let R be the relation on the set of all people such that $(a,b) \in R$ if person a is a parent of person b. Then $(a,c) \in R \circ R$ iff there is a person b such that $(a,b) \in R$ and $(b,c) \in R$, i.e. iff there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a,c) \in R \circ R$ iff a is a grandparent of c.

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R$$

$$\therefore R^2 = R \circ R,\ R^3 = R^2 \circ R = (R \circ R) \circ R, \text{ and so on.}$$

Ex (3): Let $R=\{(1,1),(2,1),(3,2),(4,3)\}$. Find R^2, R^3.

Soln: $R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$.

$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$.

Similarly $R^4 = R^5 = \cdots = R^n = R^3$.

Closures of Relations :-

The relation $R = \{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A = \{1,2,3\}$ is not reflexive. How to produce a reflexive relation containing R that is as small as possible? This can be done by adding (2,2) and (3,3) to R, since these are the only pairs of the form (a, a) that are not in R, clearly this new relation contains R.

Any relation R' which is a reflexive relation and it contains the relation R is called as the <u>Reflexive closure</u>.

$\therefore \quad R' = R \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$ is the diagonal relation.

<u>Ex</u>: what is the reflexive closure of the relation $R = \{(a,b) \mid a < b\}$ on the set of integers?

<u>Soln</u>: $R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in Z\}$

$= \{(a,b) \mid a \leq b\}$.

The relation $\{(1,1), (1,2), (2,3), (2,2), (3,1), (3,2)\}$ on $\{1,2,3\}$ is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only to add (2,1) and (1,3), since these are the only pairs of the form (b, a) with $(a,b) \in R$ that are not in R. This new relation is symmetric and contains R. furthermore, any symmetric relation that contains R must contain (2,1) and (1,3). This new relation is called the <u>symmetric closure</u> of R.

R^{-1} : inverse relation of $R = \{(a,b) \mid (b,a) \in R\}$

<u>Ex</u>: what is the symmetric closure of the relation $R = \{(a,b) \mid a > b\}$ on the set of positive integers.

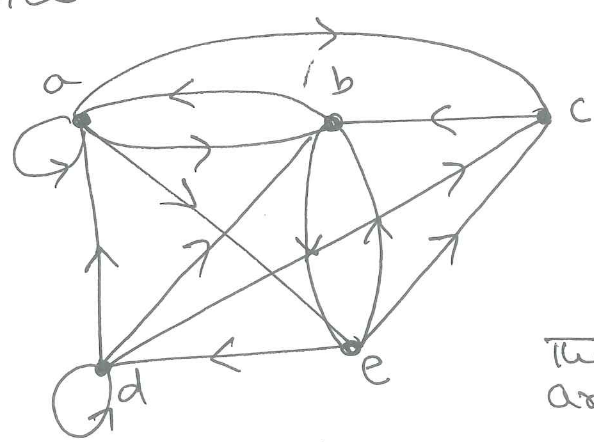
$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\} = \{(a,b) \mid a \neq b\}$

paths in directed graphs :-

Definition: A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ in G, where n is a non-negative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \dots, x_{n-1}, x_n$ and has length n. Empty set of edges ~~from~~ can be viewed as a path from a to a. A path of length ≥ 1 that has its begin and end at the same vertex is called a circuit or cycle.

A path in a graph can pass through a vertex more than once and the edge can appear more than once.

Ex:



a, b, e, d : is a path of length 3

a, e, c, d, b : is not a path.

b, a, c, b, a, a, b : is a path of length 6.

c, b, a : is a path of length 2.

e, b, a, b, a, b, e : is a path of length 6.

The paths b, a, e, b, d, b & e, b, a, b, a, b, e are circuits.

The term path also applies to relations. Carrying over the definition from directed graphs to relations, there is a path from a to b in R if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \dots, (x_{n-1}, b) \in R$.

Theorem: Let R be a relation on set A. There is a path of length n, where n is a positive integer, from a to b iff $(a, b) \in R^n$.

Transitive closures:

Definition: Let R be a relation on a set A. The connectivity relation R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

Since R^n consists of paths (a,b) of length n from a to b, it follows that R^* is the union of all R^n.

$$\boxed{R^* = \bigcup_{i=1}^{\infty} R^i}$$

$$R^1 \cup R^2 \cup R^3 \cup R^4 \cup \cdots \cup R^{\infty} = R^*$$

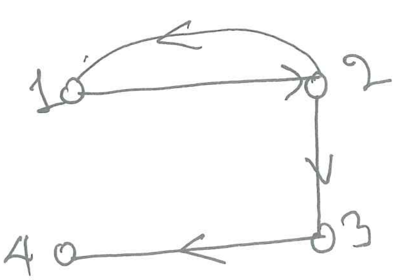
The transitive closure of the relation R equals the connectivity relation R^*.

Ex: Let $A = \{1,2,3,4\}$ and $R = \{(1,2),(2,3),(3,4),(2,1)\}$

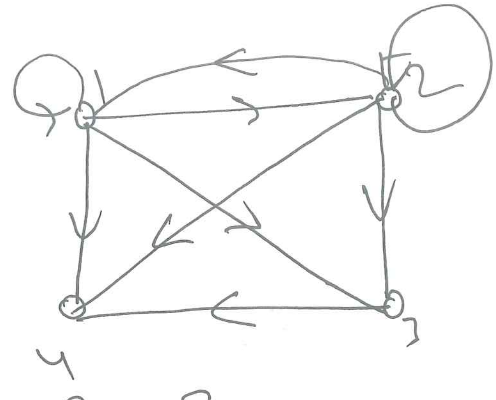
find R^*.

$R^* = \{(1,2),(2,3),(1,3),(3,4),(2,4),(1,4),(2,1),(1,1),(2,2)\}$

Digraph of R:



Digraph of R^*:



Note: $R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$ where $n = |A|$

verify this with above example :-

$R^* = R \cup R^2 \cup R^3 \cup R^4$

$A = \{1,2,3,4\}$ $\quad |A| = 4$

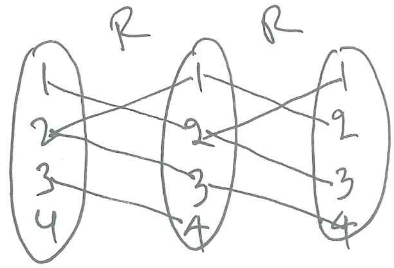
$R = \{(1,2),(2,3),(3,4),(2,1)\}$

$R^2 = R \circ R = \{(1,3),(2,4),(1,1),(2,2)\}$

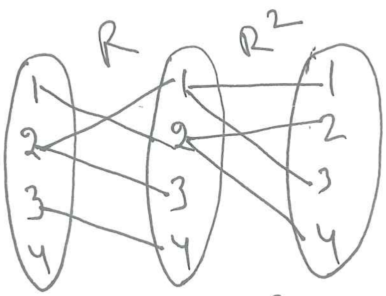
$R^3 = R \circ R^2 = \{(1,4),(2,3),(2,1),(1,2)\}$

$R^4 = R \circ R^3 = \{(1,3),(1,1),(2,4),(2,2)\}$

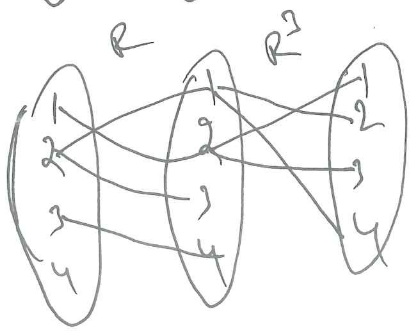
$R^* = R \cup R^2 \cup R^3 \cup R^4 = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$



$= R \circ R = R^2 = \{(1,1),(1,3),(2,2),(2,4)\}$



$= R \circ R^2 = R^3 = \{(1,2),(2,4),(2,1),(2,3)\}$



$= R \circ R^3 = R^4 = \{(1,1),(1,3),(2,2),(2,4)\}$

Warshal's algorithm to compute transitive closure:

Let R be a relation on A.
Represent R by a relation matrix M_R.
Compute Matrix M_{R^*} by filling the cells such that
if $M_{ij} = 1$ if $M_{R_{ij}} = 1$ or $M_{R_{ik}} = 1$ & $M_{R_{kj}} = 1$
else $M_{ij} = 0$.

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for i ← 1 to n do
  for j ← 1 to n do
    for k ← 1 to n do
      M_ij = M_ij ∨ (M_ik ∧ M_kj)
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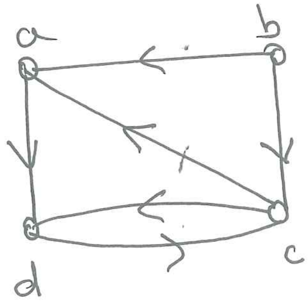
Ex(1):



$R = \{(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)\}$

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find $M_{R^*} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ How ?

I find R^2, R^3, R^4 and their matrices and take union (or) of all these matrices to get M_{R^*}.

II List all paths of length 2, 3, 4 and their corresponding matrices and finally or ed them.

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Equivalence Relations:

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation are called *equivalent*.

Ex(1): Suppose that R is the relation on the set of strings of English letters such that aRb iff $l(a) = l(b)$, where $l(x)$ denotes the length of the string x. Is R an equivalence relation.

Soln: Since $l(a) = l(a)$, it follows tha aRa whenever a is a string, so that R is *reflexive*.
Suppose that aRb, so that $l(a) = l(b)$. Then bRa, since $l(b) = l(a)$. Hence, R is symmetric.
Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(c) = l(a)$, so that aRc. Consequently, R is transitive.
It follows that R is an equivalence relation.

Ex(2): Let R be the relation on the set of real numbers such that aRb if and only if $a-b$ is an integer. Is R an equivalence relation?

Ex(3): Congruence Modulo m. Let m be a positive integer with $m > 1$. S.T. the relation $R = \{(a,b) \mid a \equiv b \pmod{m}\}$ is an equivalence relation.

Ex(4): which of these are equivalence relations on $\{0,1,2,3\}$

a) $\{(0,0), (1,1), (2,2), (3,3)\}$

b) $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$

c) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

d) $\{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

e) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

Equivalence classes and partition of set :-

If R is a equivalence relation on a set A and $a \in A$, then the R-relative set of a, namely R(a) is defined as $[a] = R(a) = \{x \in A \mid aRx\}$, is called as equivalence ~~relation~~ class, determined by a w.r.t. R.

What is the maximum number of equivalence classes of R on A ?

If $|A| = n$ ∴ There are n equivalence classes.

Theorem-1: Let R be an equivalence relation on a set A, and let $a, b \in A$. Then aRb iff $R(a) = R(b)$.

⇒ Suppose aRb. Take any $x \in R(a)$, then aRx. Since R is symmetric, it follows that xRa. Then since R is transitive, we have xRa & aRb. Since, R is symmetric, it follows that xRb. Since R is symmetric, it follows that bRx ∴ $x \in R(b)$. Hence, $R(a) \subseteq R(b)$. Similarly, we find $R(b) \subseteq R(a)$. Therefore $R(a) = R(b)$.

Partition of a set :-

Let A be a non-empty set, and P be a family of non-empty subsets of A such that

i) Every element of A belongs to one of the sets in P.

ii) Any two distinct sets belongs to P are mutually distinct.

Then we call P as the partition, decomposition or quotient set of A. The sets in P are called as blocks or cells.

A partition of a set consisting of 6 blocks is shown in fig.

Ex: Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and its subsets
$A_1 = \{1, 3, 5, 7\}$, $A_2 = \{2, 4\}$, $A_3 = \{6, 8\}$,
$A_4 = \{1, 3, 5\}$, $A_5 = \{5, 6, 8\}$.

Let $P = \{A_1, A_2, A_3\}$ is partition of A because Every element of A belongs one of the subsets A_1, A_2, A_3 and all these are disjoint.

Suppose if $P = \{A_2, A_3, A_4\}$ then P is not a partition of A because 7 not belongs to any of these three sets, even though they are all disjoint.

If $P = \{A_1, A_2, A_5\}$ is not a partion because 5 belongs to A_1 & A_5 hence these two are not disjoint.

Theorem: Let R be an equivalence relation on set A, and let P be the collection of all distinct R-relative sets in A. Then P is the partition of A, and R is the equivalence relation determined by P.

$\Rightarrow$ Here P is the collection of all distinct relative sets $R(a)$, $a \in A$.
We note that, for all $a \in A$, we have aRa, that is, $a \in R(a)$, because R is reflexive. Thus, every element of A belongs to one of the sets in P.
If $R(a)$ and $R(b)$ are distinct sets belongs to P, then $R(a) \cap R(b) = \phi$. This fact prove that P is a partition of the set A. This partition determines the relation R in the sense that aRb iff a and b [illegible] of the partition and hence

Ex: Let $A = \{a, b, c, d\}$, and $P = \{\{a,b,c\},\{d\}\}$
Find the equivalence relation R induced by P.

⇒ Since a, b, c belongs to one block we have
$\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$.
& d belongs to another block ∴ $(d,d) \in R$.

∴ $R = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d)\}$

Ex: Let $A = \{1,2,3,4,5,6,7\}$, and R be the relation on A defined by aRb whenever $a-b$ is divisible by 3. S.T R is an equivalence relation. Determine the partition of A induced by R.

⇒ For any $a \in A$, we have $a - a = 0$ which is divisible by 3, Therefore, aRa for $a \in A$. Accordingly, R is reflexive.
Next, suppose aRb, for $a, b \in A$. Then $a-b$ is divisible by 3. This implies that $b-a$ is divisible by 3, so that bRa, hence R is symmetric.
Lastly, suppose aRb and bRc, $a,b,c \in A$. Then $a-b$ is divisible by 3 and $b-c$ is divisible by 3, so that aRc, hence R is transitive. ∴ R is equivalence relation.

$R = \{(1,1),(1,4),(1,7),(2,2),(2,5),(3,3),(3,6),(4,1),(4,4),$
$(4,7),(5,2),(5,5),(6,3),(6,6),(7,1),(7,4),(7,7)\}$

$R(1) = \{1,4,7\} = R(4) = R(7)$.
$R(2) = \{2,5\} = R(5)$
$R(3) = \{3,6\} = R(6)$

∴ The partion of set A is $P = \{\{1,4,7\},\{2,5\},\{3,6\}\}$.

Partial Ordering :-

Definition: A relation R on a set S is called a
partial ordering or partial order if it is reflexive,
antisymmetric, and transitive. A set S together
with a partial ordering R is called a partially
ordered set, or poset, and is denoted by (S, R).

Ex(1): Show that the "greater than or equal" relation
$(\geq)$ is a partial ordering on the set of integers.

$\Rightarrow$ Since $a \geq a$ for every integer a, $\geq$ is reflexive.
If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymme
-tric. Finally, $\geq$ is transitive since $a \geq b$ and $b \geq c$
imply that $a \geq c$. It follows that $\geq$ is a partial
ordering on the set of integers and $(Z, \geq)$ is a poset.

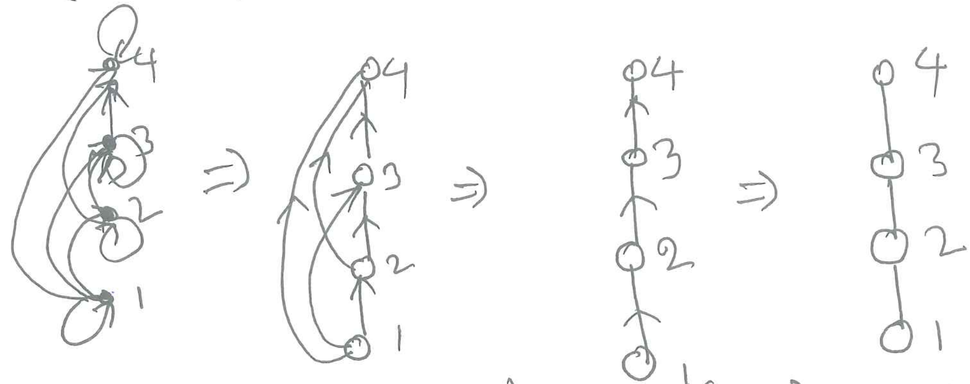
Ex(2): The relation "divides" on the set of positive
integers is a partial ordering as we saw earlier
that it is reflexive, antisymmetric and transitive
Hence, $(Z^{+}, |)$ is a poset.

Ex(3): S.T the inclusion relation $\subseteq$ is a partial
ordering on the power set of a set S.

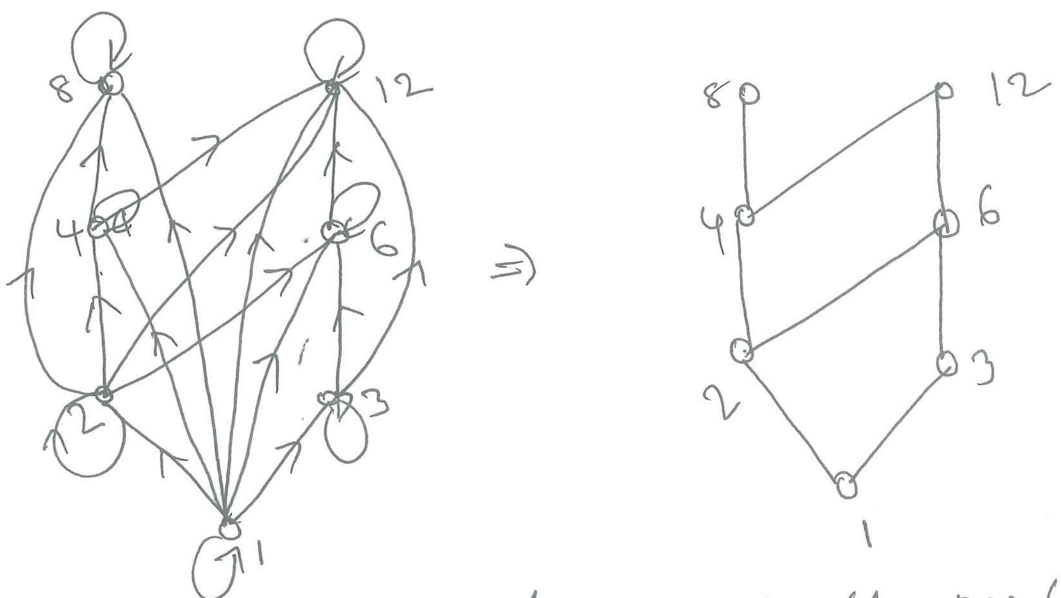
$\Rightarrow$ Since $A \subseteq A$ whenever A is a subset of S,
$\subseteq$ is reflexive. It is antisymmetric since $A \subseteq B$
and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is
transitive, since $A \subseteq B$ and $B \subseteq C$ imply that
$A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$,
and $(P(S), \subseteq)$ is a poset.

HASSE DIAGRAMS:
Many edges in the directed graph for a finite
poset do not have to be shown since they must
be present. i.e self loops and transitive edges need
not be shown in the graph. Moreover if will assume
all the edges are directed upwords then we can
[illegible] the arrows or direction. This modified

EX (1): Let $A = \{1, 2, 3, 4\}$ and the relation is $\leq$. Then $(A, \leq)$ is a poset. Its directed graph is

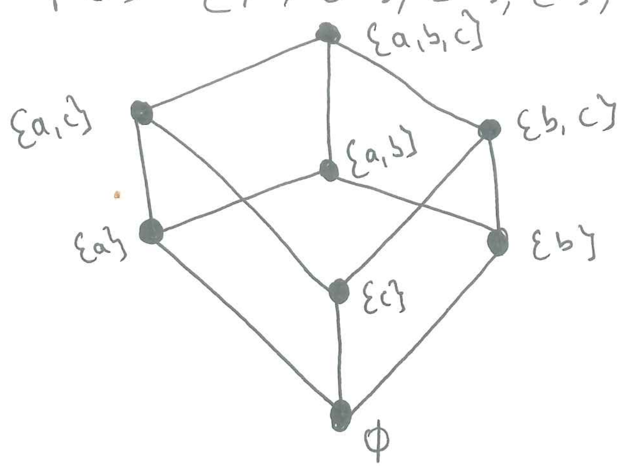


EX (2) Let $A = \{1, 2, 3, 4, 6, 8, 12\}$. Is $(A, |)$ is poset, if yes give its Hasse diagram.



EX (3): Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



MAXIMAL AND MINIMAL ELEMENTS :-

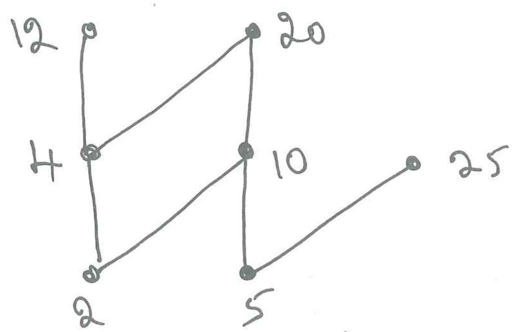
Elements of poset that have certain extermal properties are important for many applications.

An element of a poset is called maximal if it is not less than any element of the poset. i-e if a, is maximal in the poset (S, R) if there is no $b \in S$ such that aRb.

Similerly, an element of a poset is called minimal if it not greater than any element of the poset. i-e if a is said to be minimal if there is no $b \in S$ such that b R a.

There are the elements at top and bottom of a Hasse diagram of the poset.

Ex(1): which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are maximal, and which are minimal.



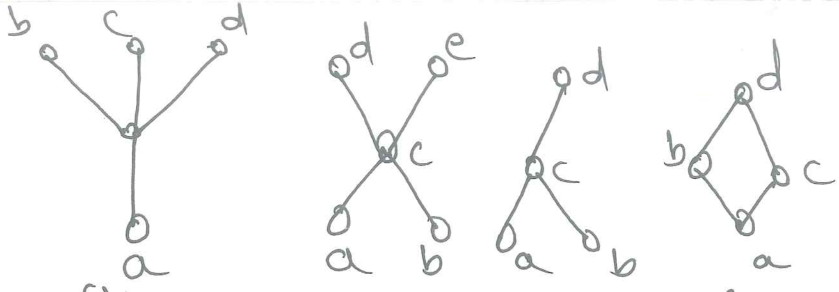
Soln: Maximal elements are 12, 20, 25

Minimal elements are 2, 5.

There fore, a poset can have more than one maximal and minimal elements.

Greatest and Least element of a poset :-

If there is an element in a poset that is greater than every other element. Such an element is called the greatest element. i-e if a is greatest element of the poset (S, R) if b R a, for all $b \in S$. The greatest element is unique if it exists. Likewise, an element is called the least element if it is less than all the

Ex(2): 

(i) (ii) (iii) (iv)

Determine whether the posets given above have a greatest and least element.

Soln: (i) Has least element i.e a
This poset has no greatest element.
(ii) This poset has neither least nor greatest element.
(iii) This has the greatest element i.e d but no least element.
(iv) This has both greatest and least elements
greatest element is d
least element is a

Ex(3): Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

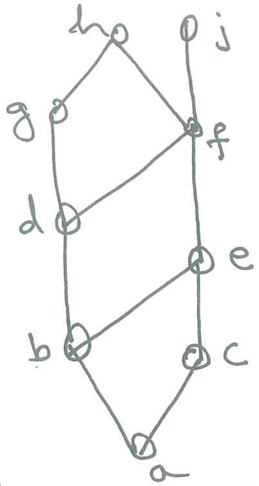
Soln: The least element is ϕ (null set). $\because \phi \subseteq$ all sets
The greatest element is the set S itself. $\because$ all subsets are subsets to S.

Ex(4): Is there a greatest element and a least element in the poset $(Z^+, |)$.

Soln: The integer 1 is the least element since $1|n$ whenever n is a positive integer. Since there is no integer that is divisible by all positive integers, there is no greatest element.

Sometimes it is possible to find an element that is greater than all the elements in a subset A of a poset (S, R). If u is an element of S such that aRu for all $a \in A$ then u is called as upper bound of A.
Likewise, there may be an element less than all the elements in A. If l is an element of S such that lRa

Ex(5) :



Find the lower bounds and upper bounds of the subsets {a, b, c}, {j, h}, and {a, c, d, f}.

Soln: {a, b, c} upper bounds are e, f, h, j
lower bounds are only a

{h, j} upper bounds are nil
lower bounds are a, b, c, d, e, f

{a, c, d, f} upper bounds are h, j
lower bounds are a only.

The element x is called the least upper bound of the subset A of x is an upper bound that is less than every other upper bound of A. Since there is only one such element, if it exists, it makes sense to call this element the least upper bound.

Similarly, the element y is called the greatest lower bound of A if y is a lower bound of A and zRy whenever z is a lower bound of A.

The least upper bound and the greatest lower bound is unique if it exists.

Ex(6) : Find glb and lub of {b, d, g}, if they exist in the poset shown in Ex(5).

Soln : The upper bounds of {b, d, g} are g & h
since g lower than h, g is the lub.
The lower bounds of {b, d, g} are {a, b}
Since b is upper than a ∴ glb{b, d, g} is b.

Ex(7) : (2+1) lub{3, 9, 12} = 36 = lcm(3, 9, 12)