



Finite Automata

Reading: Chapter 2



Finite Automaton (FA)

- Informally, a state diagram that comprehensively captures all possible states and transitions that a machine can take while responding to a stream or sequence of input symbols
- Recognizer for “Regular Languages”
- Deterministic Finite Automata (DFA)
 - The machine can exist in only one state at any given time
- Non-deterministic Finite Automata (NFA)
 - The machine can exist in multiple states at the same time



Deterministic Finite Automata

- Definition

- A Deterministic Finite Automaton (DFA) consists of:
 - $Q \Rightarrow$ a finite set of states
 - $\Sigma \Rightarrow$ a finite set of input symbols (alphabet)
 - $q_0 \Rightarrow$ a start state
 - $F \Rightarrow$ set of accepting states
 - $\delta \Rightarrow$ a transition function, which is a mapping between $Q \times \Sigma \Rightarrow Q$
- A DFA is defined by the 5-tuple:
 - $\{Q, \Sigma, q_0, F, \delta\}$



What does a DFA do on reading an input string?

- Input: a word w in Σ^*
- Question: Is w acceptable by the DFA?
- Steps:
 - Start at the “start state” q_0
 - For every input symbol in the sequence w do
 - Compute the next state from the current state, given the current input symbol in w and the transition function
 - If after all symbols in w are consumed, the current state is one of the accepting states (F) then *accept* w ;
 - Otherwise, *reject* w .



Regular Languages

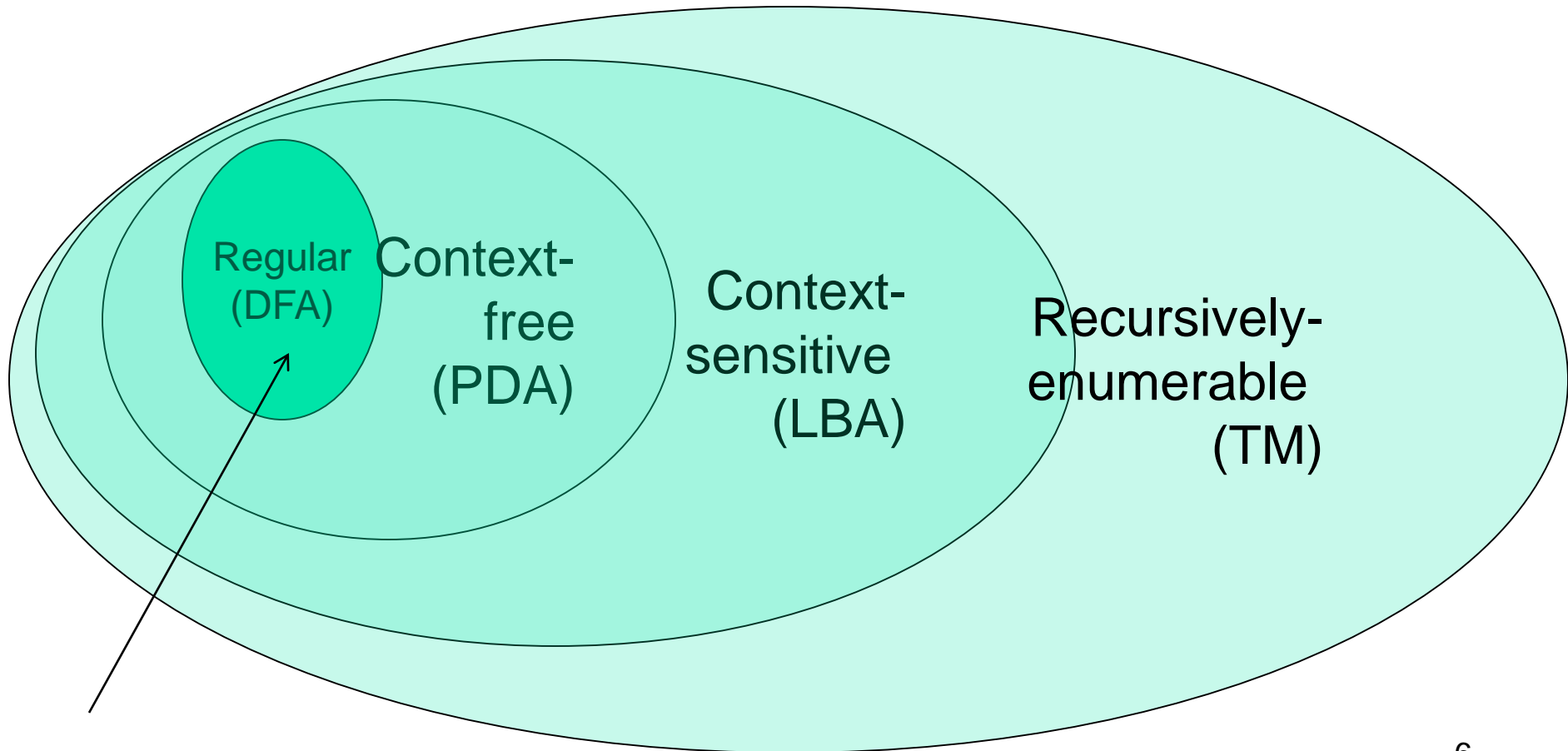
- Let $L(A)$ be a language *recognized* by a DFA A .
 - Then $L(A)$ is called a “*Regular Language*”.
- Locate regular languages in the Chomsky Hierarchy



The Chomsky Hierarchy



- A containment hierarchy of classes of formal languages





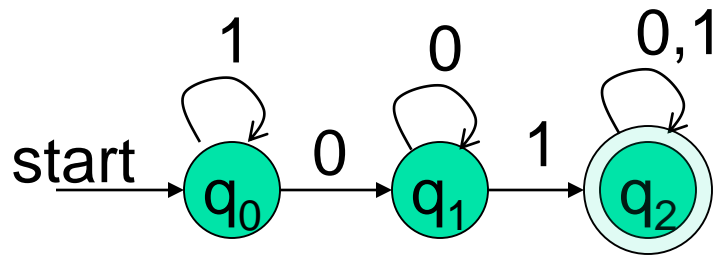
Example #1

- Build a DFA for the following language:
 - $L = \{w \mid w \text{ is a binary string that contains } 01 \text{ as a substring}\}$
- Steps for building a DFA to recognize L:
 - $\Sigma = \{0,1\}$
 - Decide on the states: Q
 - Designate start state and final state(s)
 - δ : Decide on the transitions:
- “Final” states == same as “accepting states”
- Other states == same as “non-accepting states”

Regular expression: $(0+1)^*01(0+1)^*$

DFA for strings containing 01

- What makes this DFA deterministic?



Accepting state

- What if the language allows empty strings?

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{0, 1\}$
- start state = q_0
- $F = \{q_2\}$

Transition table

| | | symbols | |
|----------|--------|---------|-------|
| δ | | 0 | 1 |
| states | q_0 | q_1 | q_0 |
| | q_1 | q_1 | q_2 |
| | $*q_2$ | q_2 | q_2 |



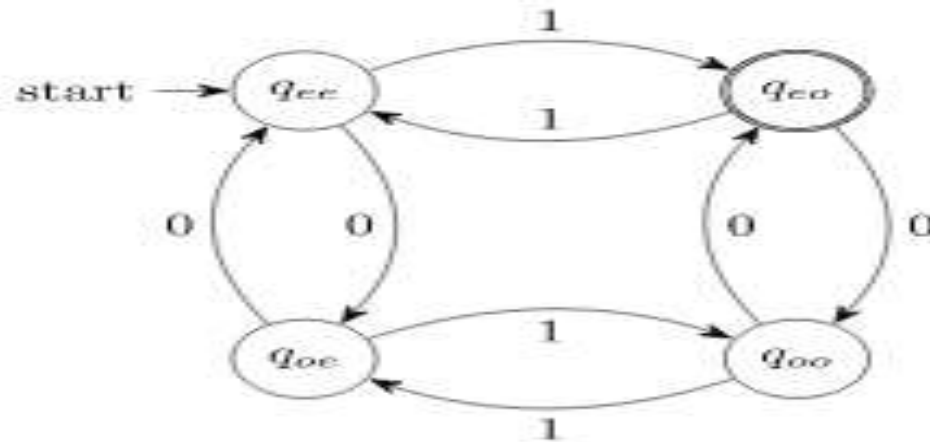
Example #2

Clamping Logic:

- A clamping circuit waits for a "1" input, and turns on forever. However, to avoid clamping on spurious noise, we'll design a DFA that waits for *two consecutive 1s* in a row before clamping on.
- Build a DFA for the following language:
$$L = \{ w \mid w \text{ is a bit string which contains the substring } 11 \}$$
- State Design:
 - q_0 : start state (initially off), also means the most recent input was not a 1
 - q_1 : has never seen 11 but the most recent input was a 1
 - q_2 : has seen 11 at least once

Example #3

- Build a DFA for the following language:
 $L = \{ w \mid w \text{ is a binary string that has even number of 1s and even number of 0s} \}$





Extension of transitions (δ) to Paths ($\hat{\delta}$)

- $\hat{\delta}(q, w) = \text{destination state from state } q \text{ on input string } w$
- $\hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a)$
- Work out example #3 using the input sequence $w=10010$, $a=1$:
 - $\hat{\delta}(q_0, wa) = ?$



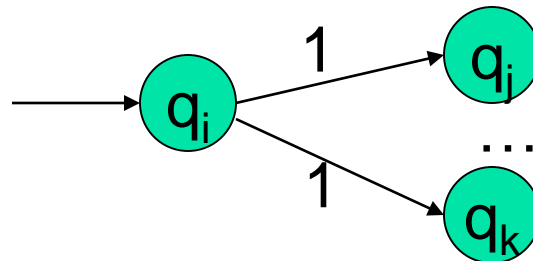
Language of a DFA

A DFA A accepts string w if there is a path from q_0 to an accepting (or final) state that is labeled by w

- *i.e., $L(A) = \{ w \mid \hat{\delta}(q_0, w) \in F \}$*
- *L.e., $L(A) = \text{all strings that lead to an accepting state from } q_0$*

Non-deterministic Finite Automata (NFA)

- A Non-deterministic Finite Automaton (NFA)
 - is of course “non-deterministic”
 - Implying that the machine can exist in more than one state at the same time
 - Transitions could be non-deterministic



- Each transition function therefore maps to a set of states



Non-deterministic Finite Automata (NFA)

- A Non-deterministic Finite Automaton (NFA) consists of:
 - $Q \implies$ a finite set of states
 - $\Sigma \implies$ a finite set of input symbols (alphabet)
 - $q_0 \implies$ a start state
 - $F \implies$ set of accepting states
 - $\delta \implies$ a transition function, which is a mapping between $Q \times \Sigma \implies$ subset of Q
- An NFA is also defined by the 5-tuple:
 - $\{Q, \Sigma, q_0, F, \delta\}$



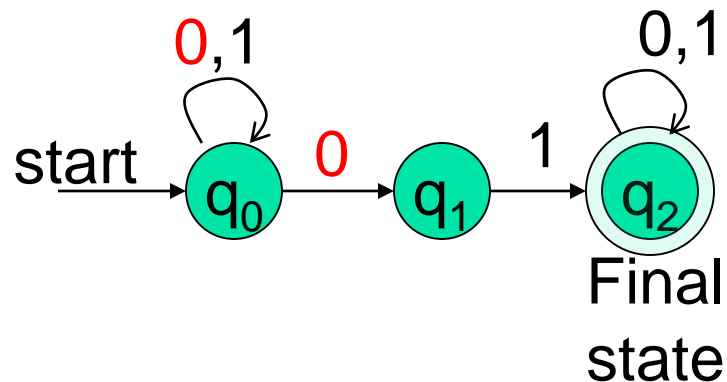
How to use an NFA?

- Input: a word w in Σ^*
- Question: Is w acceptable by the NFA?
- Steps:
 - Start at the “start state” q_0
 - For every input symbol in the sequence w do
 - Determine **all possible next states from all current states**, given the current input symbol in w and the transition function
 - If after all symbols in w are consumed and if at least **one of** the current states is a final state then *accept* w ;
 - Otherwise, *reject* w .

Regular expression: $(0+1)^*01(0+1)^*$

NFA for strings containing 01

Why is this non-deterministic?



What will happen if at state q_1 an input of 0 is received?

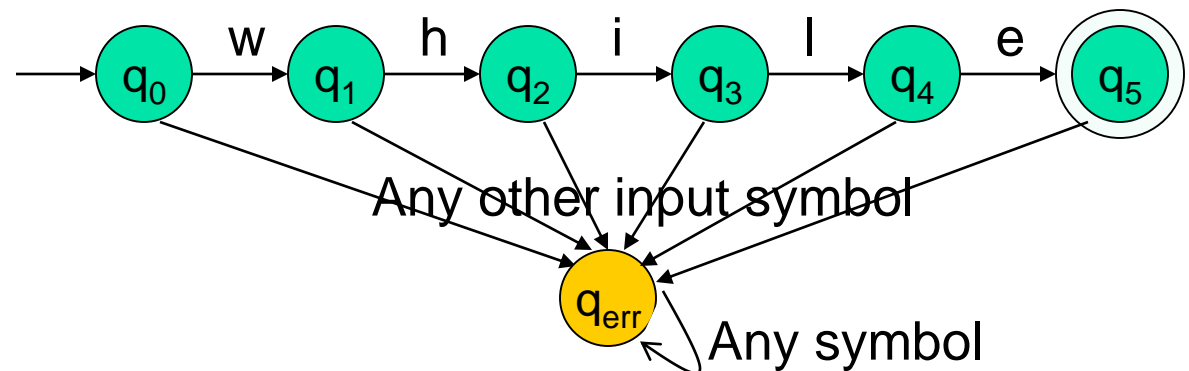
- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{0, 1\}$
- start state = q_0
- $F = \{q_2\}$
- Transition table

| symbols | | |
|--------------------------|----------------|-----------|
| δ | 0 | 1 |
| states $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| q_1 | Φ | $\{q_2\}$ |
| $*q_2$ | $\{q_2\}$ | $\{q_2\}$ |

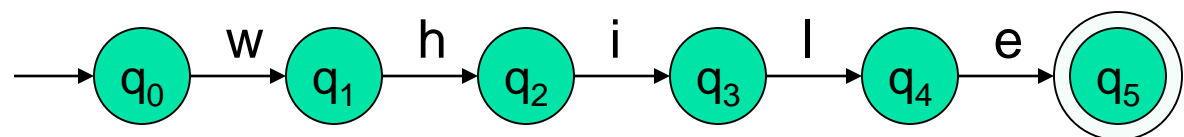
Note: Omitting to explicitly show error states is just a matter of design convenience (one that is generally followed for NFAs), and i.e., this feature should not be confused with the notion of non-determinism.

What is an “error state”?

- A DFA for recognizing the key word “*while*”



- An NFA for the same purpose:



Transitions into a dead state are implicit



Example #2

- Build an NFA for the following language:
$$L = \{ w \mid w \text{ ends in } 01 \}$$
- ?
- Other examples
 - Keyword recognizer (e.g., if, then, else, while, for, include, etc.)
 - Strings where the first symbol is present somewhere later on at least once



Extension of $\hat{\delta}$ to NFA Paths

- Basis: $\hat{\delta}(q, \varepsilon) = \{q\}$
- Induction:
 - Let $\hat{\delta}(q_0, w) = \{p_1, p_2, \dots, p_k\}$
 - $\delta(p_i, a) = S_i$ for $i=1, 2, \dots, k$
 - Then, $\hat{\delta}(q_0, wa) = S_1 \cup S_2 \cup \dots \cup S_k$



Language of an NFA

- An NFA accepts w if *there exists at least one* path from the start state to an accepting (or final) state that is labeled by w
- $L(N) = \{ w \mid \hat{\delta}(q_0, w) \cap F \neq \Phi \}$



Advantages & Caveats for NFA

- Great for modeling regular expressions
 - String processing - e.g., grep, lexical analyzer
- Could a non-deterministic state machine be implemented in practice?
 - Probabilistic models could be viewed as extensions of non-deterministic state machines (e.g., toss of a coin, a roll of dice)
 - They are not the same though
 - A parallel computer could exist in multiple “states” at the same time

But, DFAs and NFAs are equivalent in their power to capture languages !!

Differences: DFA vs. NFA

■ DFA

1. All transitions are deterministic
 - Each transition leads to exactly one state
2. For each state, transition on all possible symbols (alphabet) should be defined
3. Accepts input if the last state visited is in F
4. Sometimes harder to construct because of the number of states
5. Practical implementation is feasible

■ NFA

1. Some transitions could be non-deterministic
 - A transition could lead to a subset of states
2. Not all symbol transitions need to be defined explicitly (if undefined will go to an error state – this is just a design convenience, not to be confused with “non-determinism”)
3. Accepts input if *one of* the last states is in F
4. Generally easier than a DFA to construct
5. Practical implementations limited but emerging (e.g., Micron automata processor)



Equivalence of DFA & NFA

- Theorem:

Should be
true for
any L

- → A language L is accepted by a DFA if and only if it is accepted by an NFA.

- Proof:

1. If part:

- Prove by showing every NFA can be converted to an equivalent DFA (in the next few slides...)

2. Only-if part is trivial:

- Every DFA is a special case of an NFA where each state has exactly one transition for every input symbol. Therefore, if L is accepted by a DFA, it is accepted by a corresponding NFA. □



Proof for the if-part

- If-part: A language L is accepted by a DFA if it is accepted by an NFA
 - rephrasing...
 - Given any NFA N , we can construct a DFA D such that $L(N)=L(D)$
-
- How to convert an NFA into a DFA?
 - Observation: In an NFA, each transition maps to a *subset* of states
 - Idea: Represent:
each “subset of NFA_states” \rightarrow a single “DFA_state”

Subset construction



NFA to DFA by subset construction

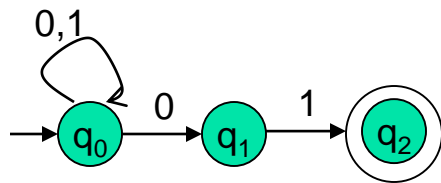
- Let $N = \{Q_N, \Sigma, \delta_N, q_0, F_N\}$
- Goal: Build $D = \{Q_D, \Sigma, \delta_D, \{q_0\}, F_D\}$ s.t.
 $L(D) = L(N)$
- Construction:
 1. Q_D = all subsets of Q_N (i.e., power set)
 2. F_D = set of subsets S of Q_N s.t. $S \cap F_N \neq \emptyset$
 3. δ_D : for each subset S of Q_N and for each input symbol a in Σ :
 - $\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a)$

Idea: To avoid enumerating all of power set, do "lazy creation of states"

NFA to DFA construction: Example

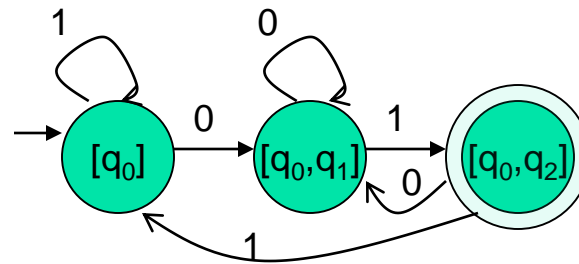
- $L = \{w \mid w \text{ ends in } 01\}$

NFA:



| δ_N | 0 | 1 |
|-------------------|----------------|-------------|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| q_1 | \emptyset | $\{q_2\}$ |
| $*q_2$ | \emptyset | \emptyset |

DFA:



| δ_D | |
|--|--|
| \emptyset | |
| $\rightarrow [q_0]$ | |
| $[q_1]$ | |
| $*[q_2]$ | |
| $[q_0, q_1]$ | |
| $*[q_0, q_2]$ | |
| $*[q_1, q_2]$ | |
| $*[q_0, q_1, q_2]$ | |

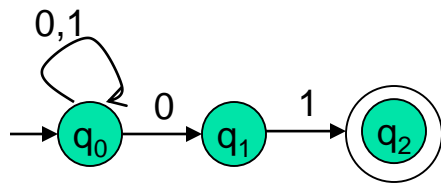
| δ_D | 0 | 1 |
|---------------------|--------------|--------------|
| $\rightarrow [q_0]$ | $[q_0, q_1]$ | $[q_0]$ |
| $[q_0, q_1]$ | $[q_0, q_1]$ | $[q_0, q_2]$ |
| $*[q_0, q_2]$ | $[q_0, q_1]$ | $[q_0]$ |

- Enumerate all possible subsets
- Determine transitions
- Retain only those states reachable from $\{q_0\}$

NFA to DFA: Repeating the example using *LAZY CREATION*

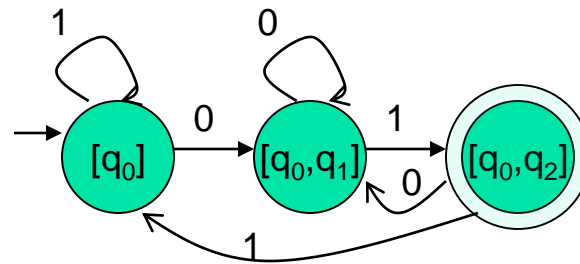
- $L = \{w \mid w \text{ ends in } 01\}$

NFA:



| δ_N | 0 | 1 |
|-------------------|----------------|-------------|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| q_1 | \emptyset | $\{q_2\}$ |
| $*q_2$ | \emptyset | \emptyset |

DFA:



| δ_D | 0 | 1 |
|------------|--------------|---------|
| $[q_0]$ | $[q_0, q_1]$ | $[q_0]$ |

Main Idea:

Introduce states as you go
(on a need basis)

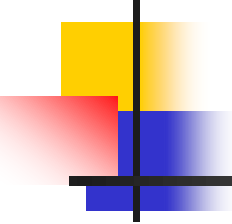


Correctness of subset construction

Theorem: *If D is the DFA constructed from NFA N by subset construction, then $L(D)=L(N)$*

■ Proof:

- Show that $\hat{\delta}_D(\{q_0\}, w) \equiv \hat{\delta}_N(q_0, w)$, for all w
- Using induction on w 's length:
 - Let $w = xa$
 - $\hat{\delta}_D(\{q_0\}, xa) \equiv \hat{\delta}_D(\hat{\delta}_N(q_0, x), a) \equiv \hat{\delta}_N(q_0, w)$



A bad case where $\#states(DFA) \gg \#states(NFA)$

- $L = \{w \mid w \text{ is a binary string s.t., the } k^{\text{th}} \text{ symbol from its end is a } 1\}$
 - NFA has $k+1$ states
 - But an equivalent DFA needs to have at least 2^k states

(Pigeon hole principle)

- m holes and $>m$ pigeons
 - \Rightarrow at least one hole has to contain two or more pigeons



Applications

- Text indexing
 - inverted indexing
 - For each unique word in the database, store all locations that contain it using an NFA or a DFA
- Find pattern P in text T
 - Example: Google querying
- Extensions of this idea:
 - PATRICIA tree, suffix tree



A few subtle properties of DFAs and NFAs

- The machine never really terminates.
 - It is always waiting for the next input symbol or making transitions.
- The machine decides when to consume the next symbol from the input and when to ignore it.
 - (but the machine can never skip a symbol)
- \Rightarrow A transition can happen even *without* really consuming an input symbol (think of consuming ε as a free token) – if this happens, then it becomes an ε -NFA (see next few slides).
- A single transition *cannot* consume more than one (non- ε) symbol.



FA with ε -Transitions

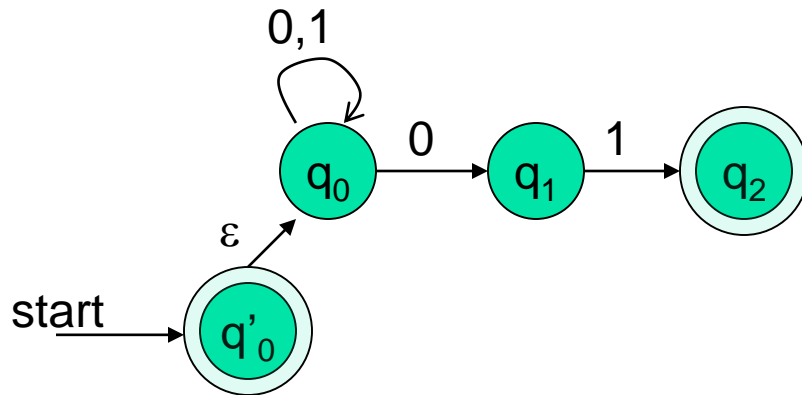
- We can allow explicit ε -transitions in finite automata
 - i.e., a transition from one state to another state without consuming any additional input symbol
 - Explicit ε -transitions between different states introduce non-determinism.
 - Makes it easier sometimes to construct NFAs

Definition: ε -NFAs are those NFAs with at least one explicit ε -transition defined.

- ε -NFAs have one more column in their transition table

Example of an ε -NFA

$L = \{w \mid w \text{ is empty, or if non-empty will end in } 01\}$



- ε -closure of a state q , **ECLOSE(q)**, is the set of all states (including itself) that can be *reached* from q by repeatedly making an arbitrary number of ε -transitions.

| δ_E | 0 | 1 | ε | |
|---------------------|----------------|-------------|-----------------|------------------|
| $\rightarrow *q'_0$ | \emptyset | \emptyset | $\{q'_0, q_0\}$ | ECLOSE(q'_0) |
| q_0 | $\{q_0, q_1\}$ | $\{q_0\}$ | $\{q_0\}$ | ECLOSE(q_0) |
| q_1 | \emptyset | $\{q_2\}$ | $\{q_1\}$ | ECLOSE(q_1) |
| $*q_2$ | \emptyset | \emptyset | $\{q_2\}$ | ECLOSE(q_2) |

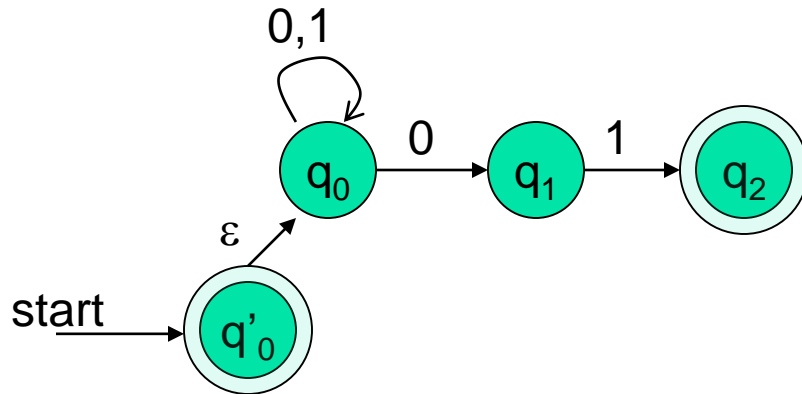
To simulate any transition:

Step 1) Go to all immediate destination states.

Step 2) From there go to all their ε -closure states as well.

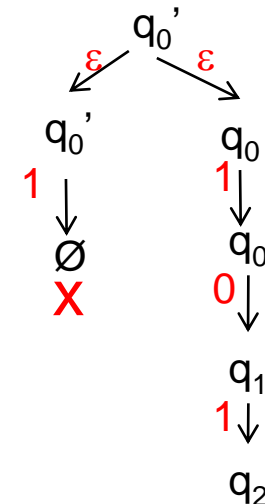
Example of an ε -NFA

$L = \{w \mid w \text{ is empty, or if non-empty will end in } 01\}$



Simulate for $w=101$:

| δ_E | 0 | 1 | ε | |
|---------------------|----------------|-------------|-----------------|------------------|
| $\rightarrow *q'_0$ | \emptyset | \emptyset | $\{q'_0, q_0\}$ | ECLOSE(q'_0) |
| q_0 | $\{q_0, q_1\}$ | $\{q_0\}$ | $\{q_0\}$ | ECLOSE(q_0) |
| q_1 | \emptyset | $\{q_2\}$ | $\{q_1\}$ | |
| $*q_2$ | \emptyset | \emptyset | $\{q_2\}$ | |

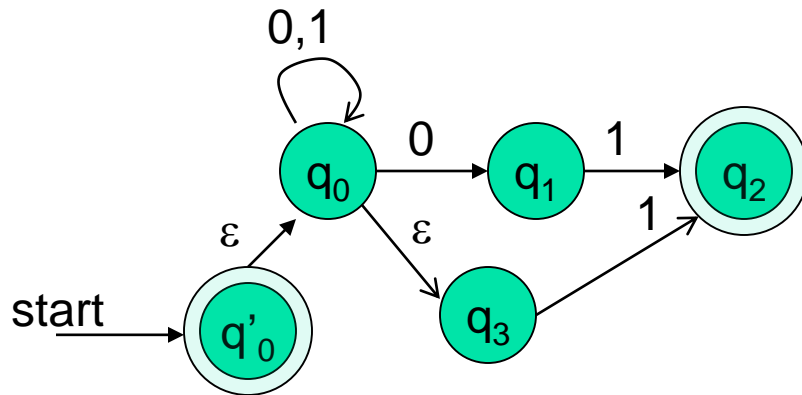


To simulate any transition:

Step 1) Go to all immediate destination states.

Step 2) From there go to all their ε -closure states as well.

Example of another ε -NFA



Simulate for $w=101$:

?

| δ_E | 0 | 1 | ε |
|---------------------|----------------|-------------|----------------------|
| $\rightarrow *q'_0$ | \emptyset | \emptyset | $\{q'_0, q_0, q_3\}$ |
| q_0 | $\{q_0, q_1\}$ | $\{q_0\}$ | $\{q_0, q_3\}$ |
| q_1 | \emptyset | $\{q_2\}$ | $\{q_1\}$ |
| $*q_2$ | \emptyset | \emptyset | $\{q_2\}$ |
| q_3 | \emptyset | $\{q_2\}$ | $\{q_3\}$ |



Equivalency of DFA, NFA, ϵ -NFA

- Theorem: A language L is accepted by some ϵ -NFA if and only if L is accepted by some DFA
- Implication:
 - $\text{DFA} \equiv \text{NFA} \equiv \epsilon\text{-NFA}$
 - (all accept Regular Languages)



Eliminating ε -transitions

Let $E = \{Q_E, \Sigma, \delta_E, q_0, F_E\}$ be an ε -NFA

Goal: To build DFA $D = \{Q_D, \Sigma, \delta_D, \{q_D\}, F_D\}$ s.t. $L(D) = L(E)$

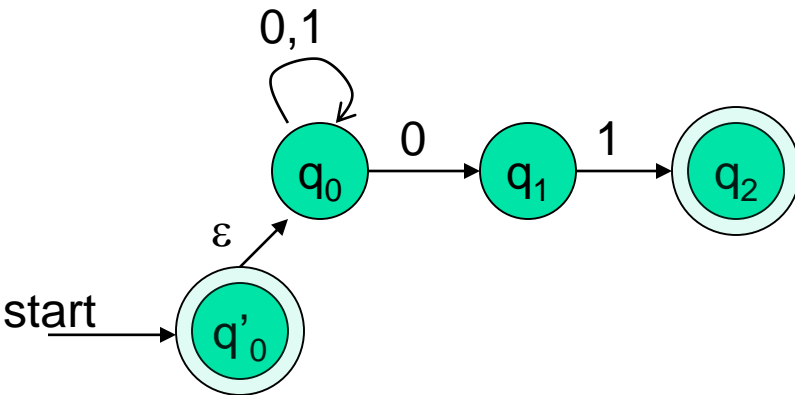
Construction:

1. Q_D = all reachable subsets of Q_E factoring in ε -closures
2. $q_D = \text{ECLOSE}(q_0)$
3. F_D = subsets S in Q_D s.t. $S \cap F_E \neq \emptyset$
4. δ_D : for each subset S of Q_E and for each input symbol $a \in \Sigma$:
 - Let $R = \bigcup_{p \in S} \delta_E(p, a)$ // go to destination states
 - $\delta_D(S, a) = \bigcup_{r \in R} \text{ECLOSE}(r)$ // from there, take a union of all their ε -closures

Reading: Section 2.5.5 in book

Example: ε -NFA \rightarrow DFA

$L = \{w \mid w \text{ is empty, or if non-empty will end in } 01\}$

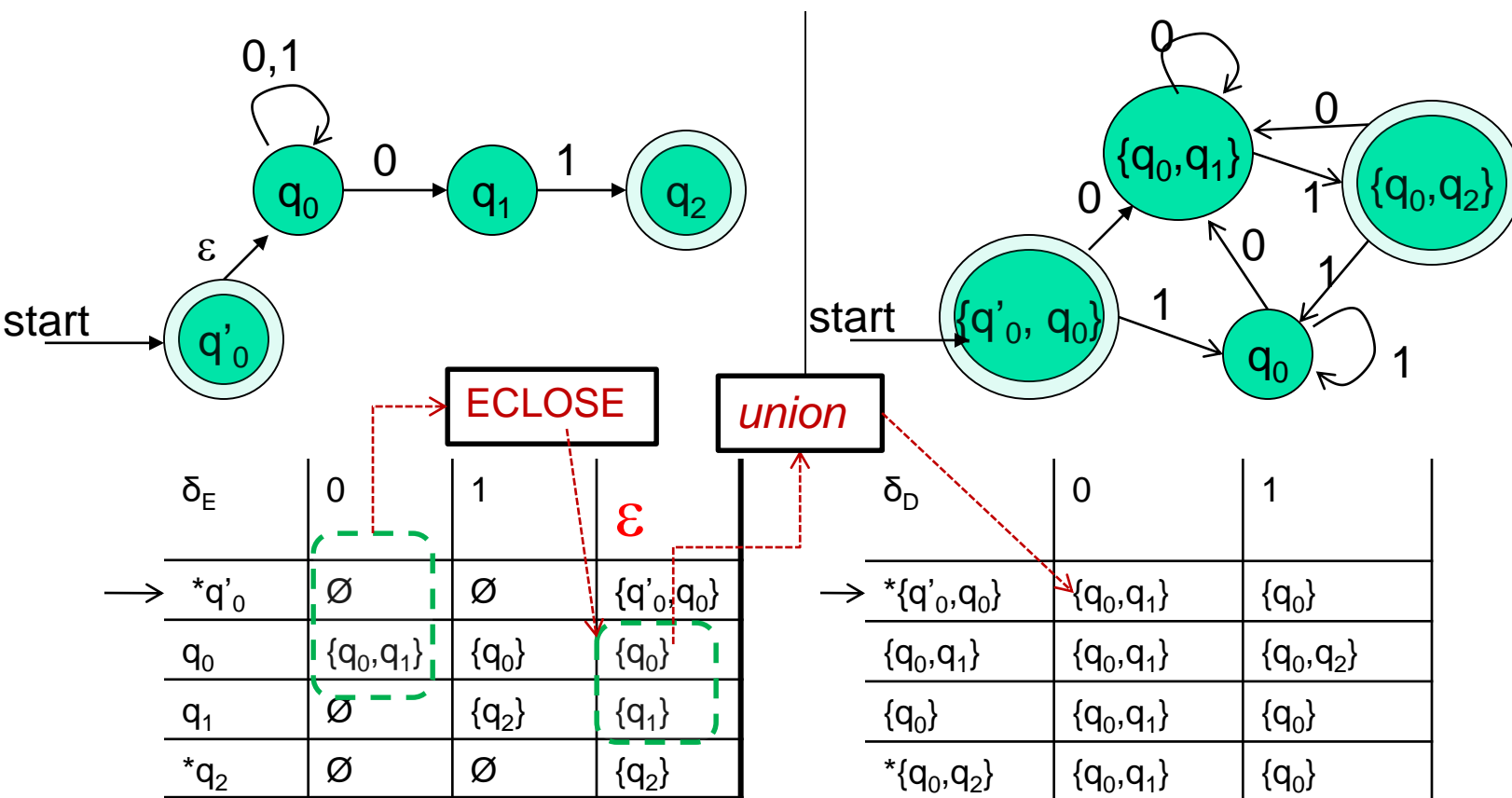


| δ_E | 0 | 1 | ε |
|---------------------|----------------|-------------|-----------------|
| $\rightarrow *q'_0$ | \emptyset | \emptyset | $\{q'_0, q_0\}$ |
| q_0 | $\{q_0, q_1\}$ | $\{q_0\}$ | $\{q_0\}$ |
| q_1 | \emptyset | $\{q_2\}$ | $\{q_1\}$ |
| $*q_2$ | \emptyset | \emptyset | $\{q_2\}$ |

| δ_D | 0 | 1 |
|-------------------------------|---|---|
| $\rightarrow * \{q'_0, q_0\}$ | | |
| ... | | |

Example: ε -NFA \rightarrow DFA

$L = \{w \mid w \text{ is empty, or if non-empty will end in } 01\}$





Summary

- DFA
 - Definition
 - Transition diagrams & tables
- Regular language
- NFA
 - Definition
 - Transition diagrams & tables
- DFA vs. NFA
- NFA to DFA conversion using subset construction
- Equivalency of DFA & NFA
- Removal of redundant states and including dead states
- ϵ -transitions in NFA
- Pigeon hole principles
- Text searching applications