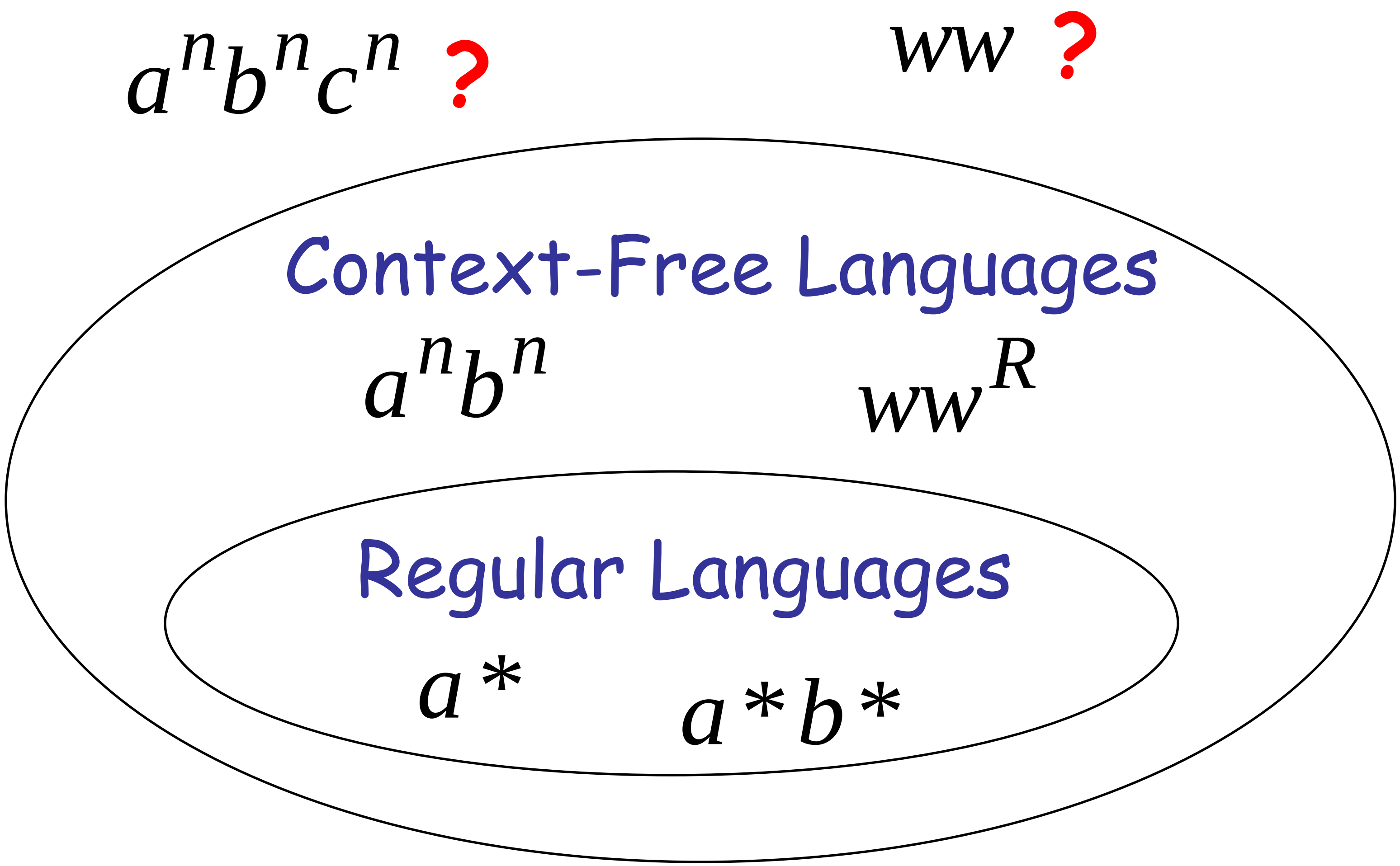
Turing Machines

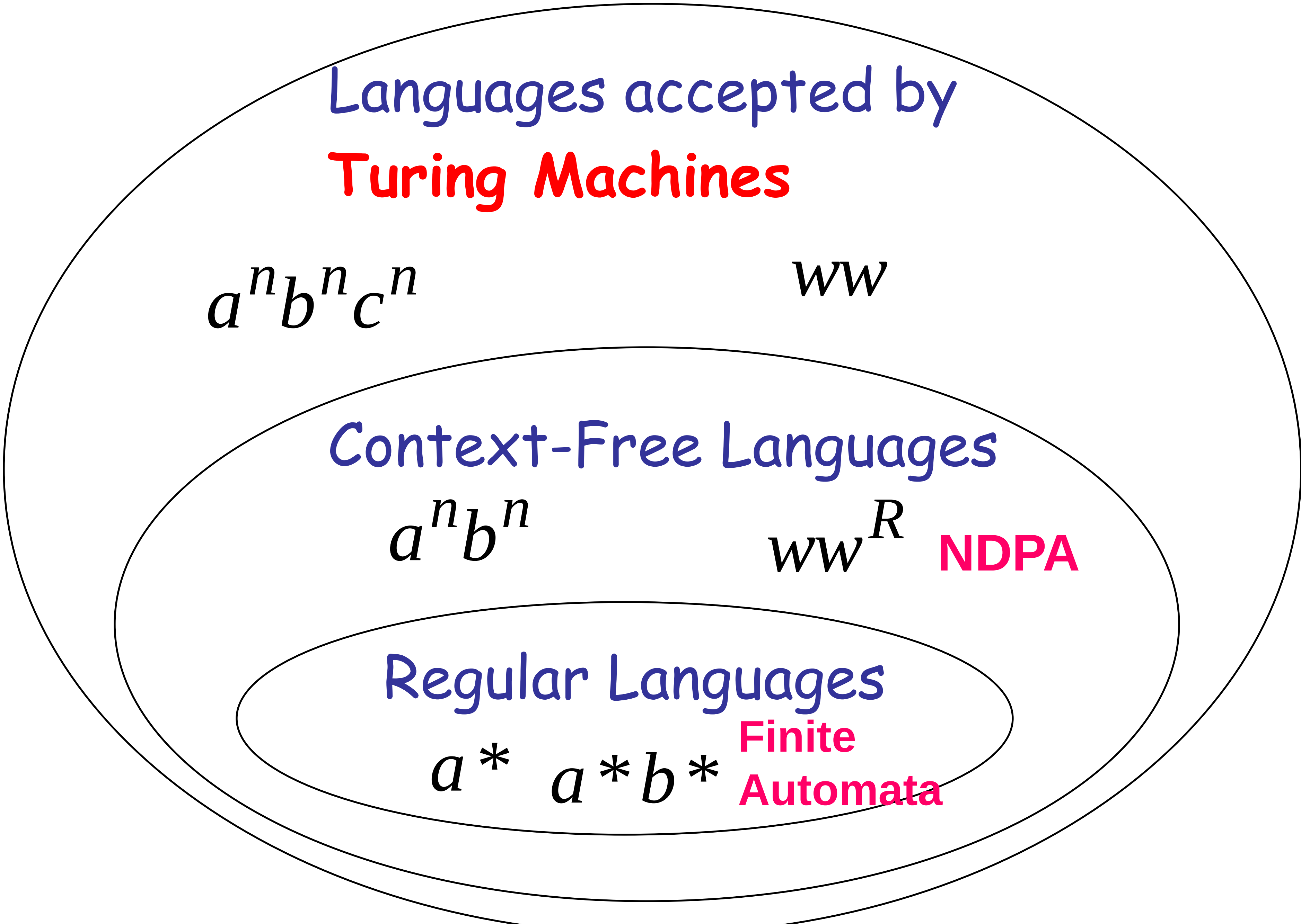
Invented by Alan Turing in 1936.

A simple mathematical model of a general purpose computer.

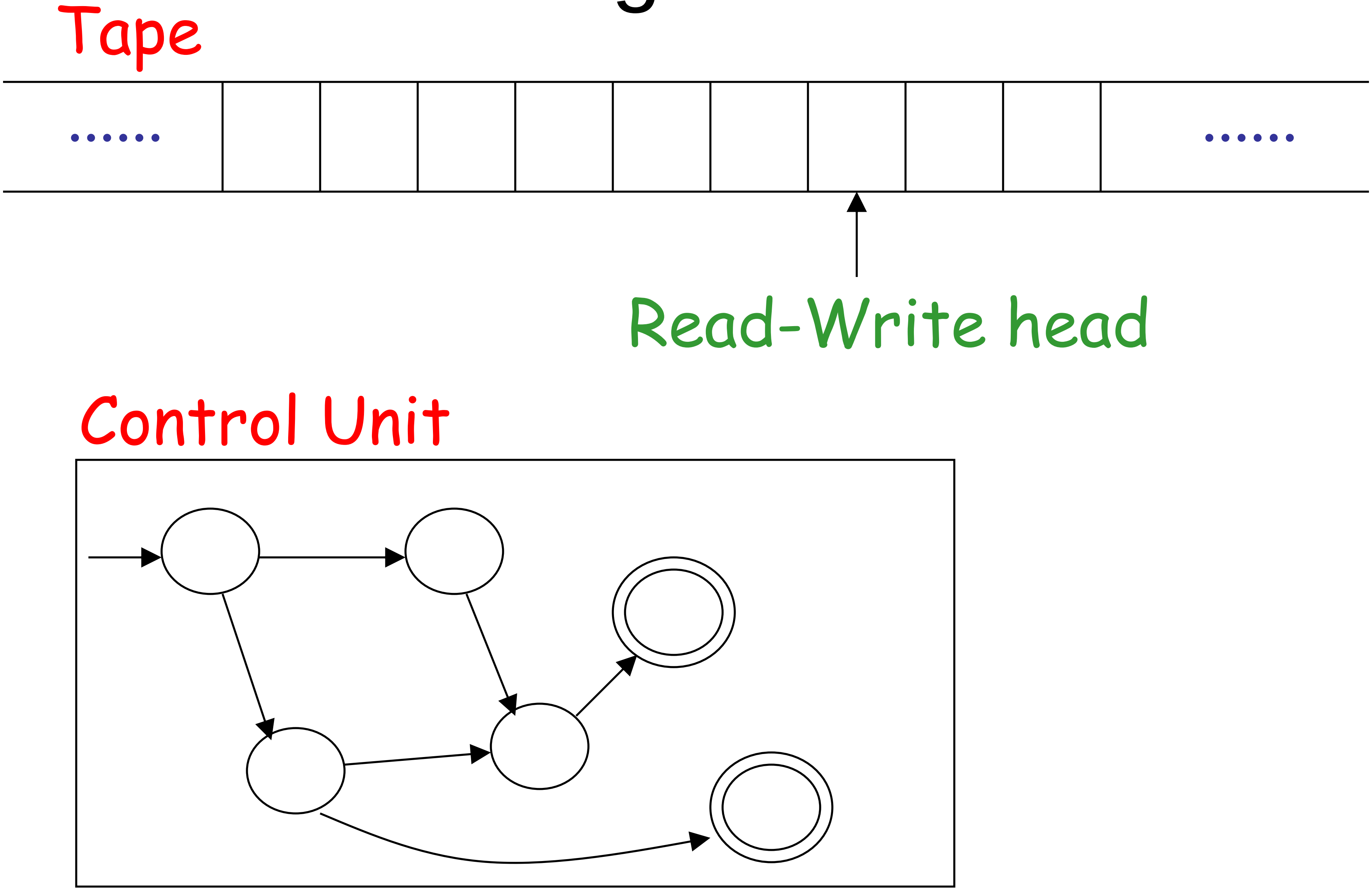
It is capable of performing any calculation which can be performed by any computing machine.

The Language Hierarchy



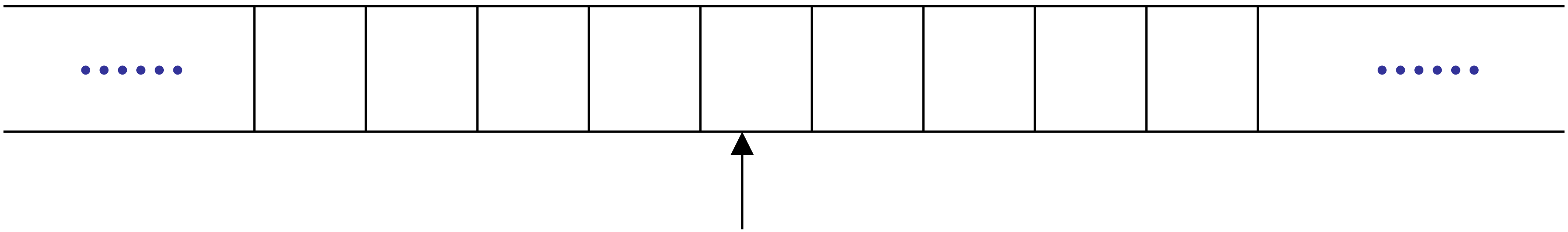

Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
NDPA
Regular Languages
a^*
a^*b^*
Finite
Automata

A Turing Machine



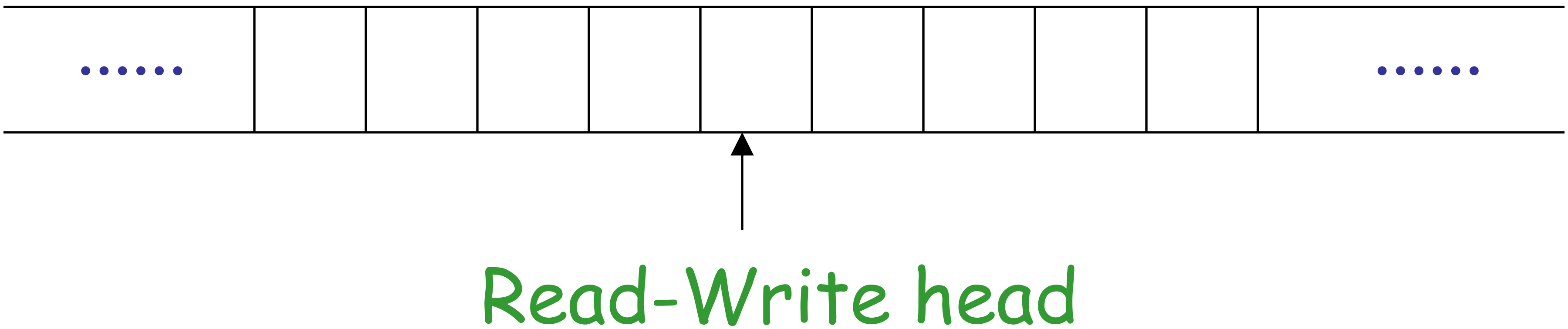
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right

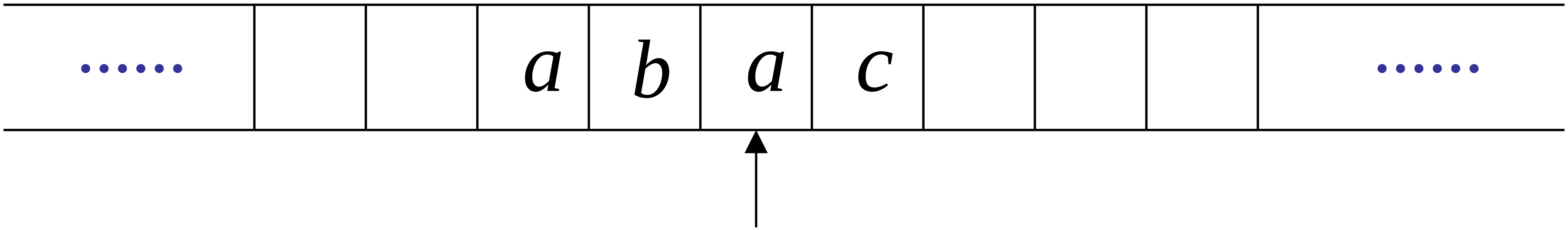


The head at each time step:

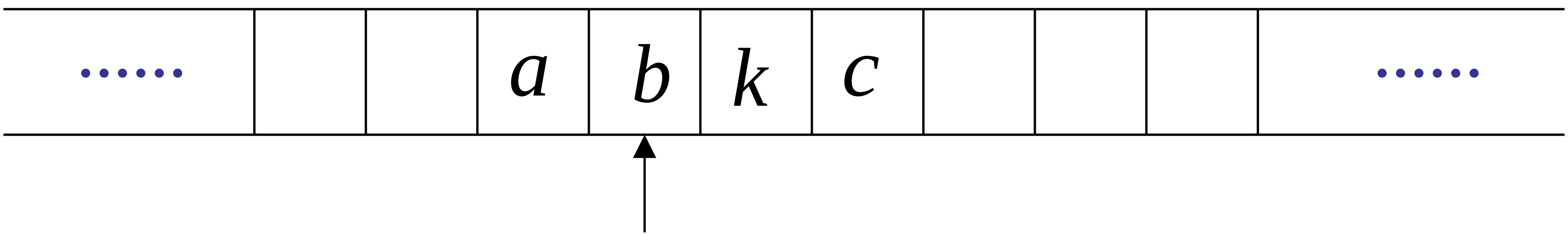
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

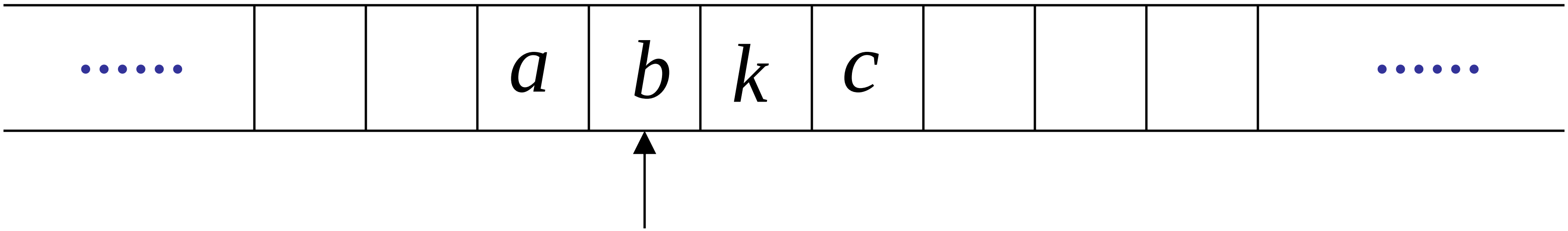


Time 1

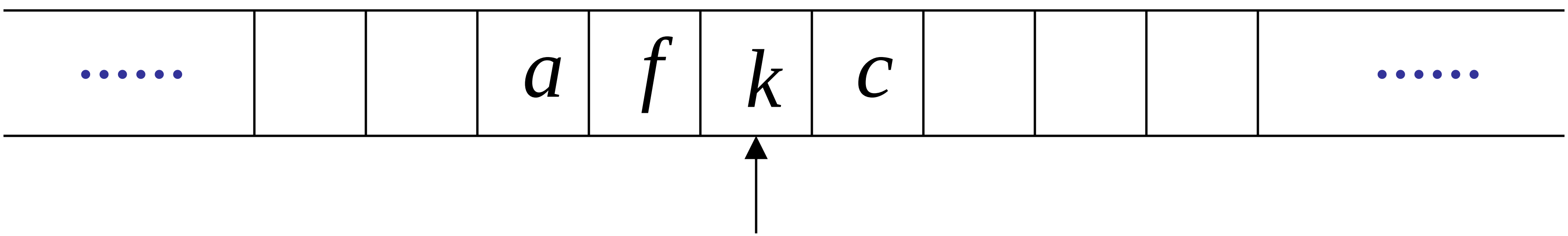


1. Reads a
2. Writes k
3. Moves Left

Time 1

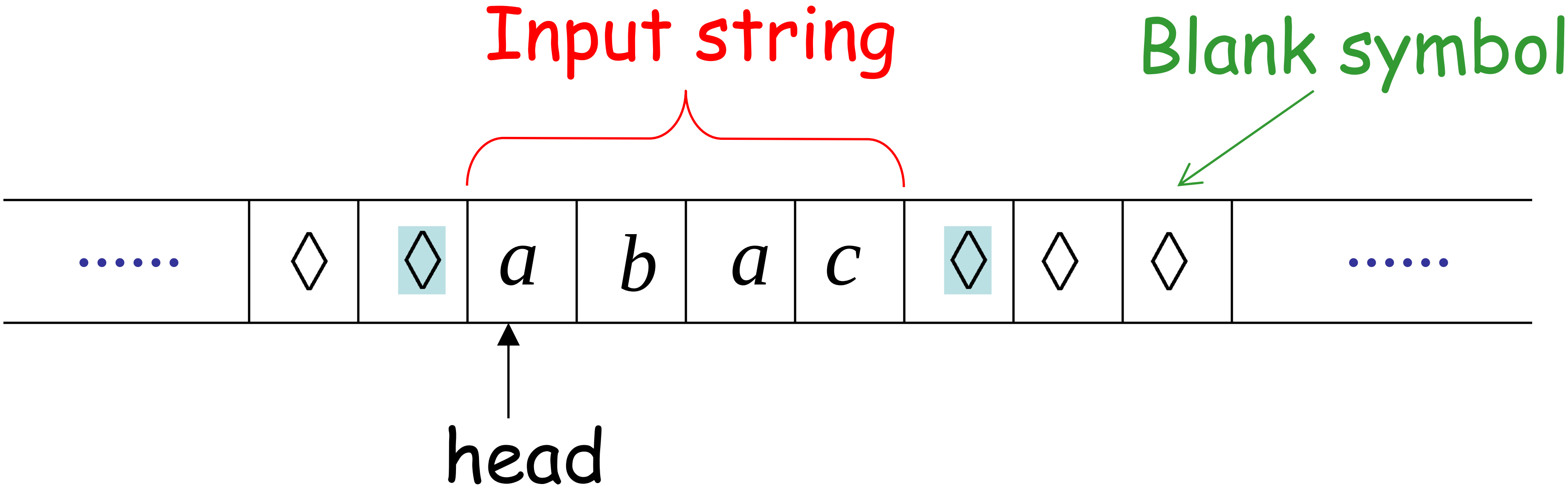


Time 2



1. Reads b
2. Writes f
3. Moves Right

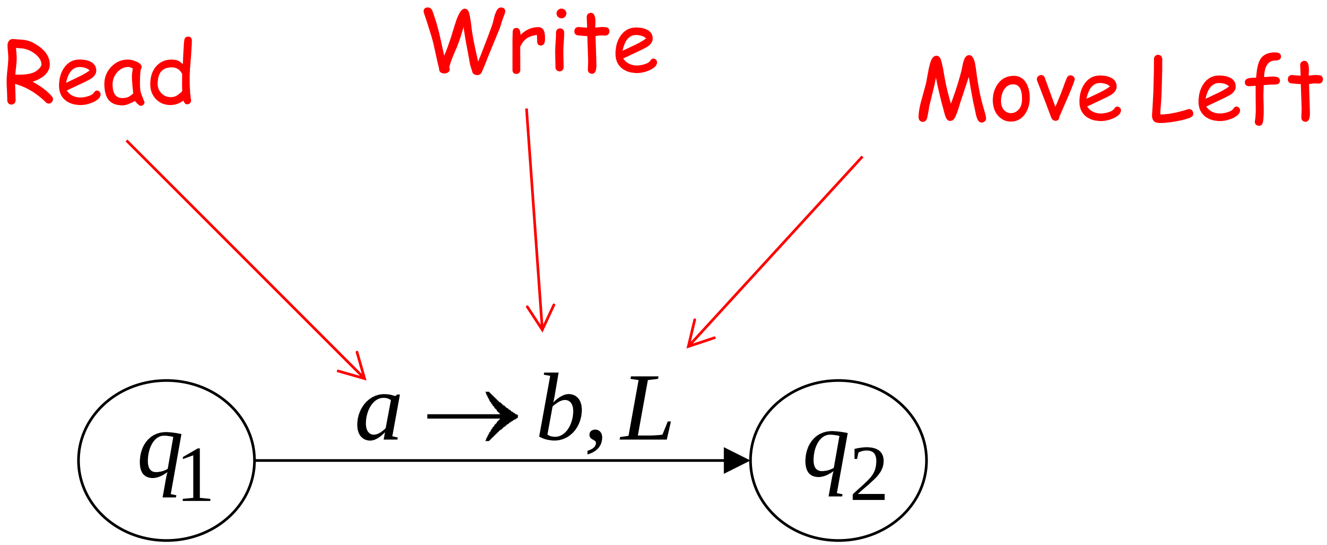
The Input String

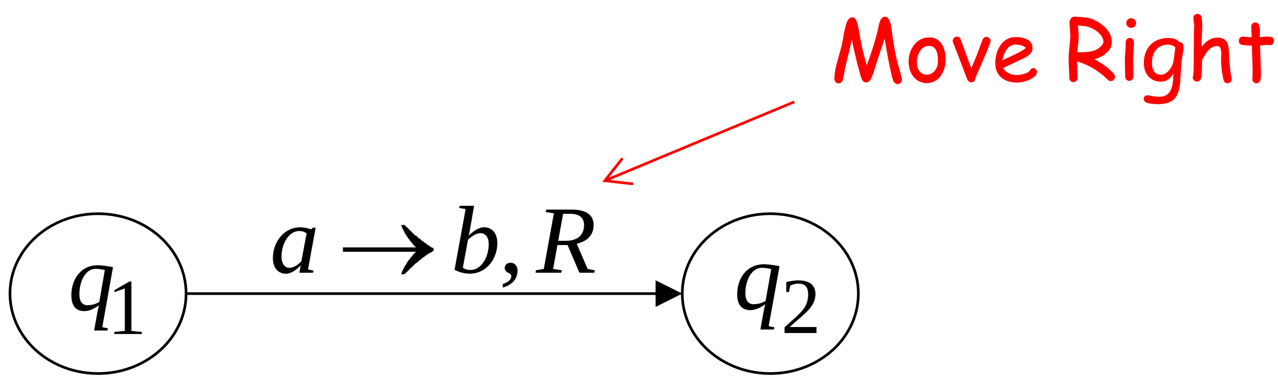


Head starts at the leftmost position of the input string

◊ **Are treated as left and right brackets for the input written on the tape.**

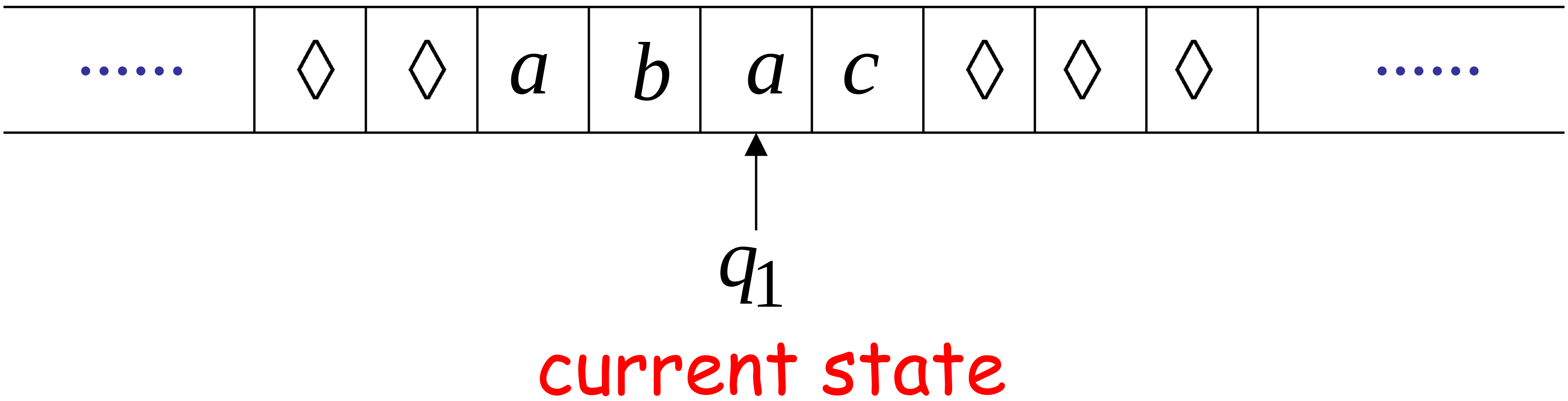
States & Transitions

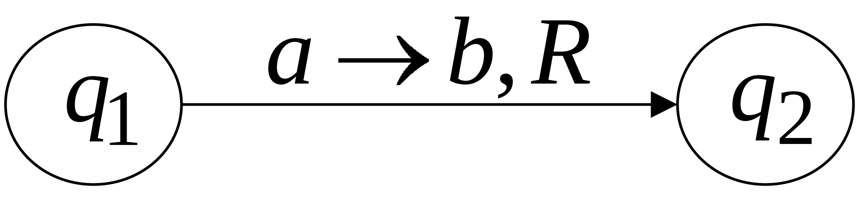


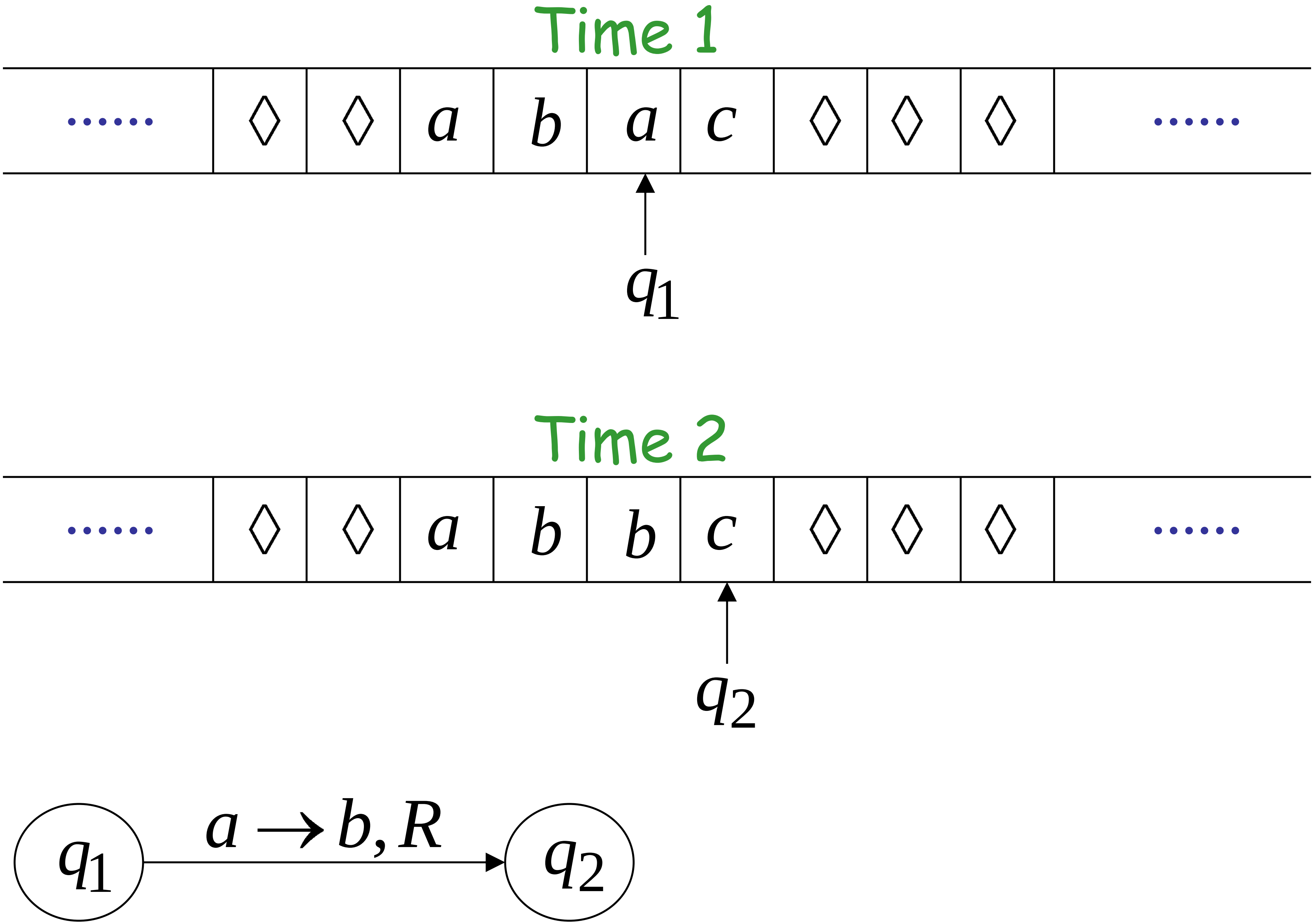


Example:

Time 1

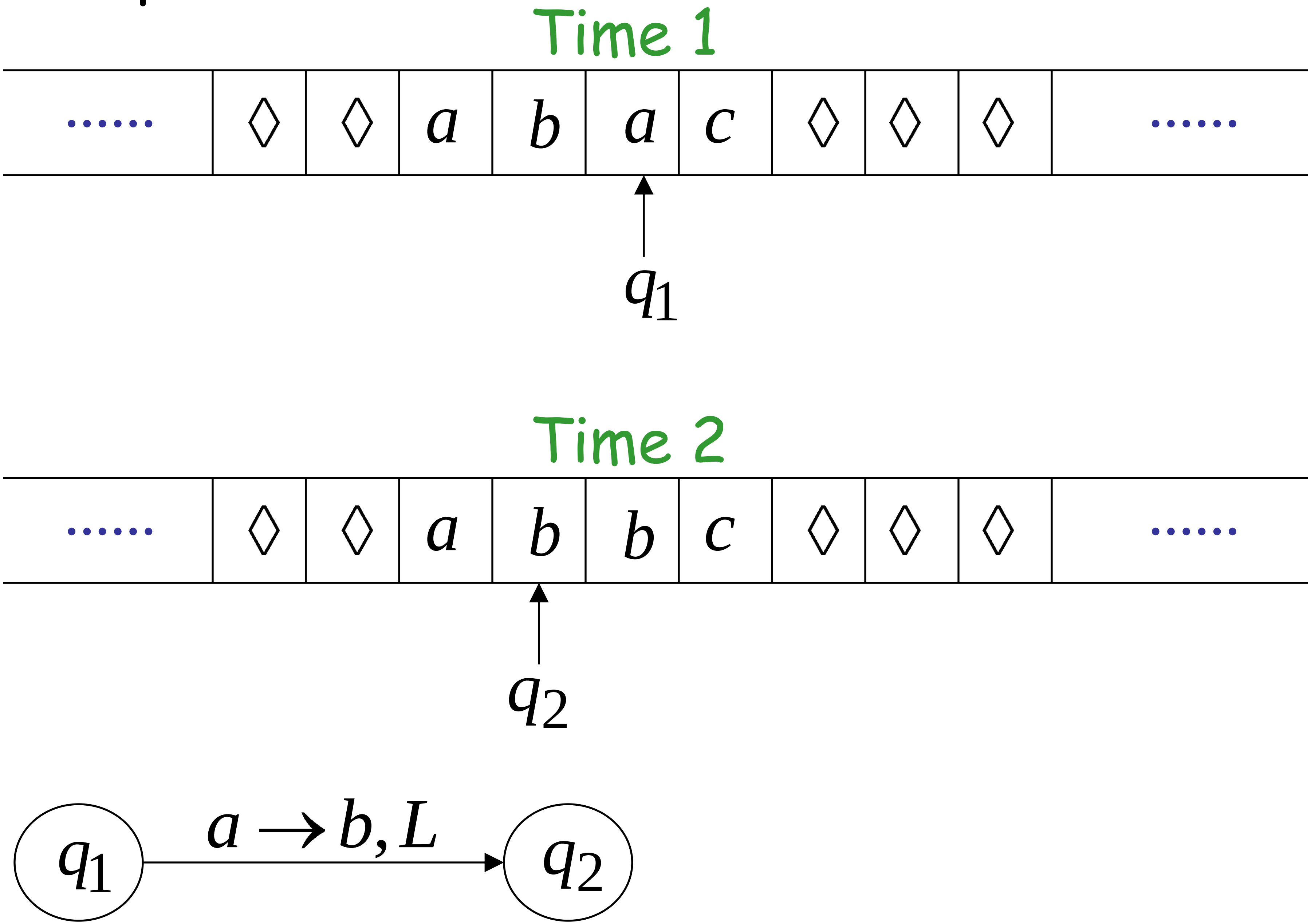




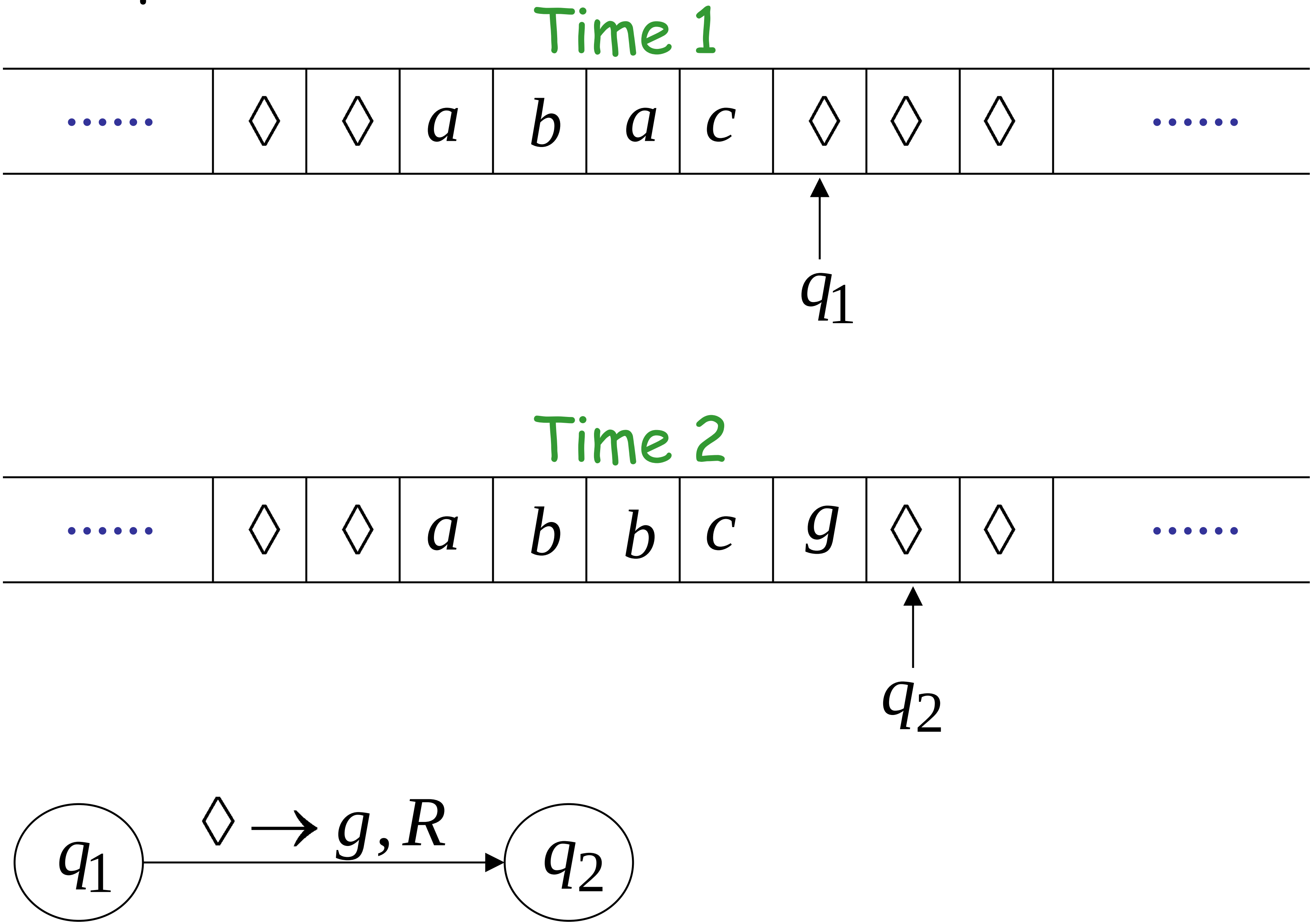


Time 1
a
b
a
c
q1
Time 2
a
b
b
c
q2
q1
a → b, R
q2

Example:



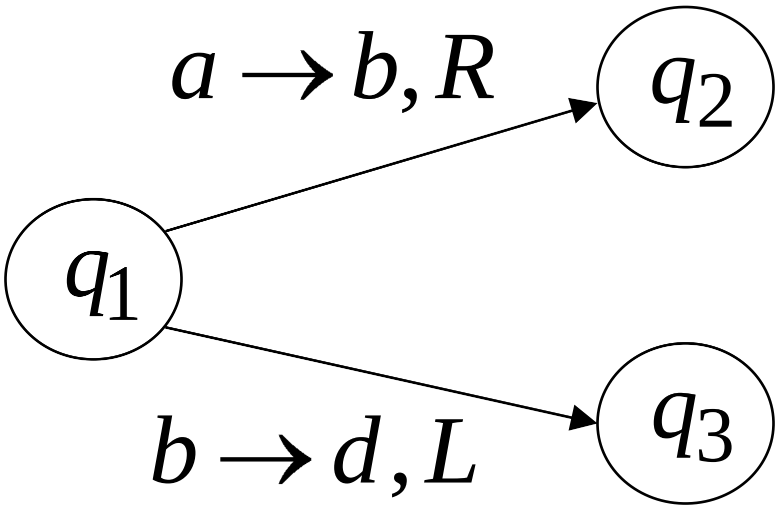
Example:



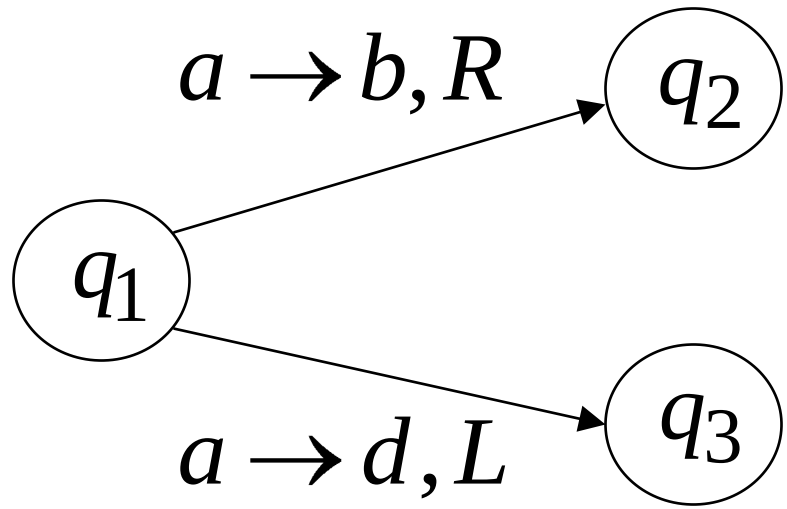
Determinism

Turing Machines are deterministic

Allowed



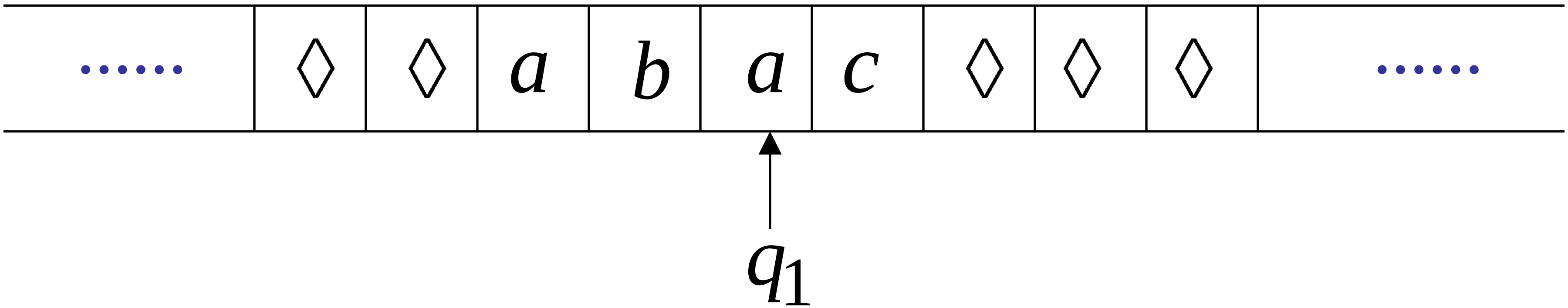
Not Allowed

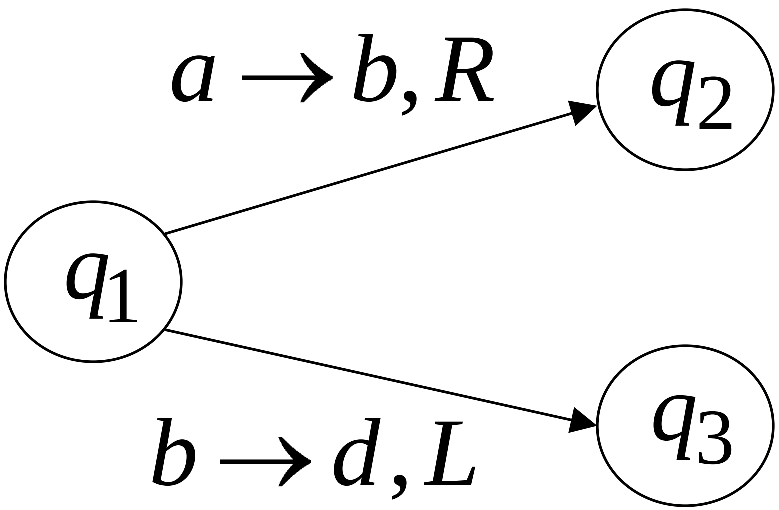


No lambda transitions allowed

Partial Transition Function

Example:





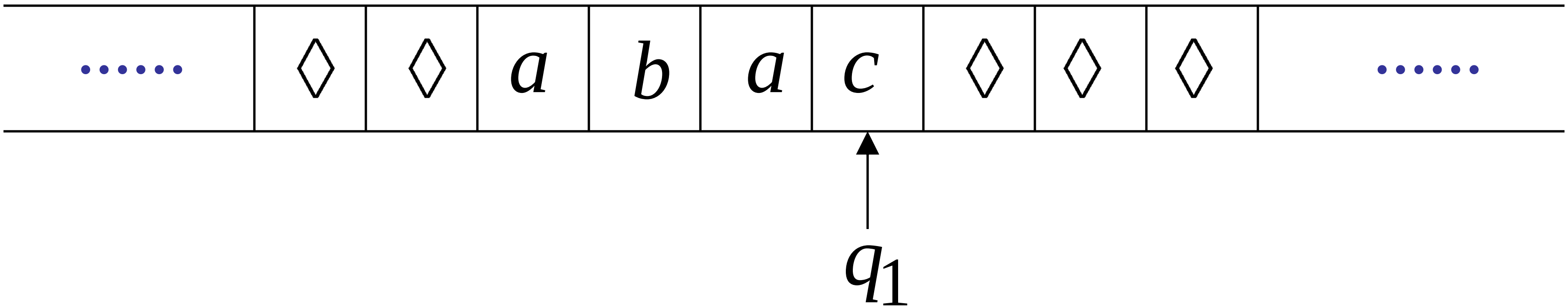
Allowed:

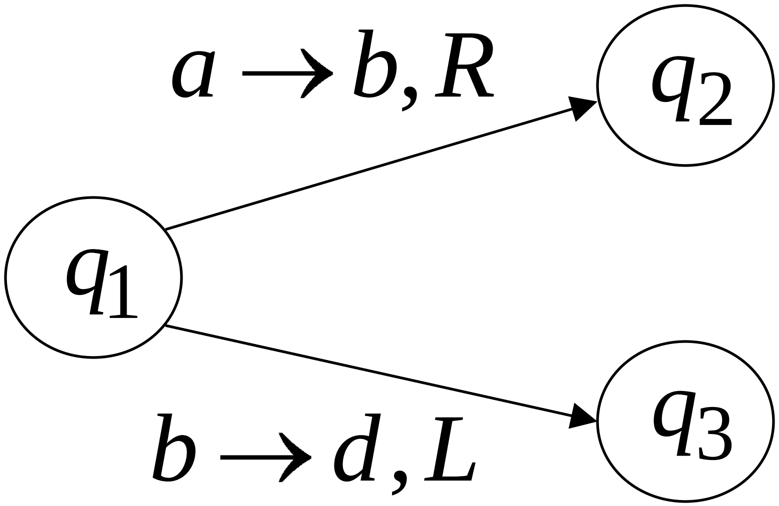
No transition for input symbol c

Halting

The machine ***halts*** if there are no possible transitions to follow

Example:

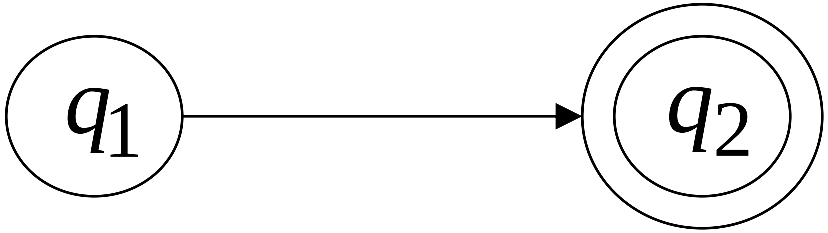




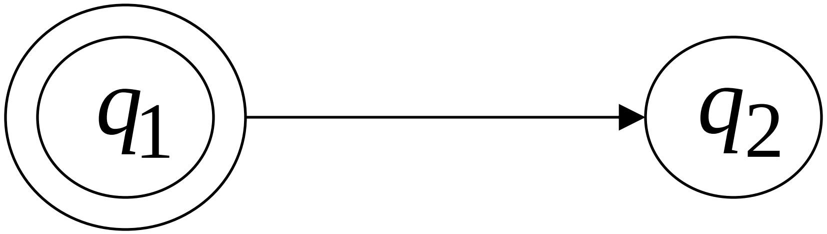
No possible transition

HALT!!!

Final States



Allowed



Not Allowed

- Final states have no outgoing transitions

- In a final state the machine halts

Acceptance

Accept Input ⇒

If machine halts
in a final state

Reject Input ⇒

If machine halts
in a non-final state

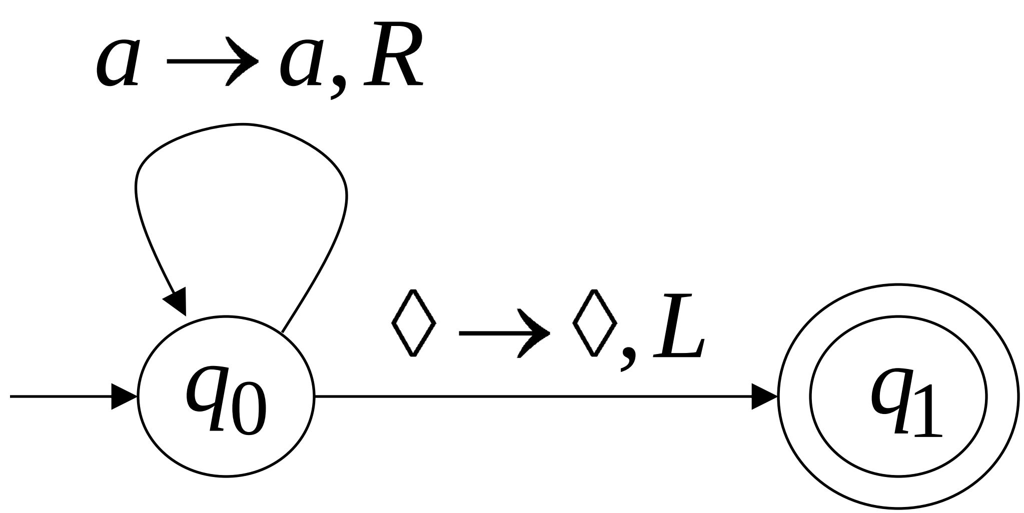
or

If machine enters
an *infinite loop*

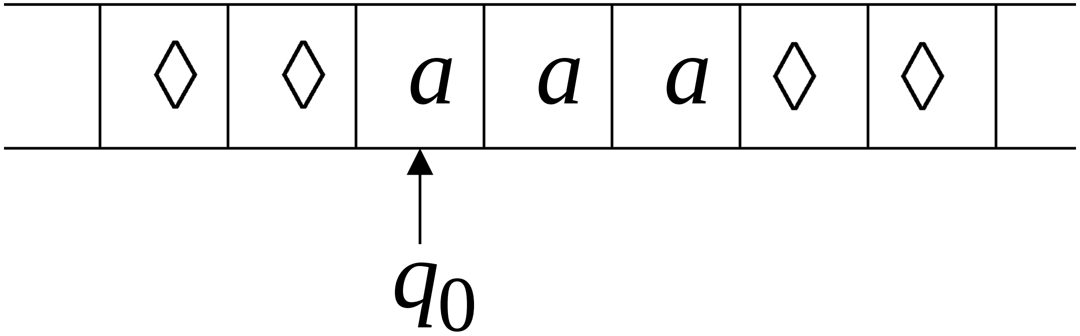
Turing Machine Example

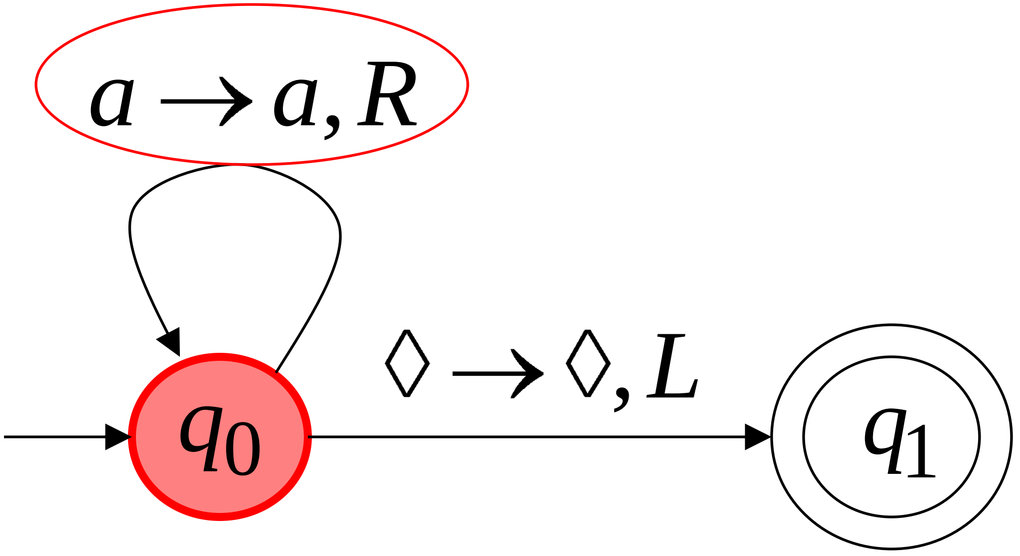
A Turing machine that accepts the language:

$$aa^*$$

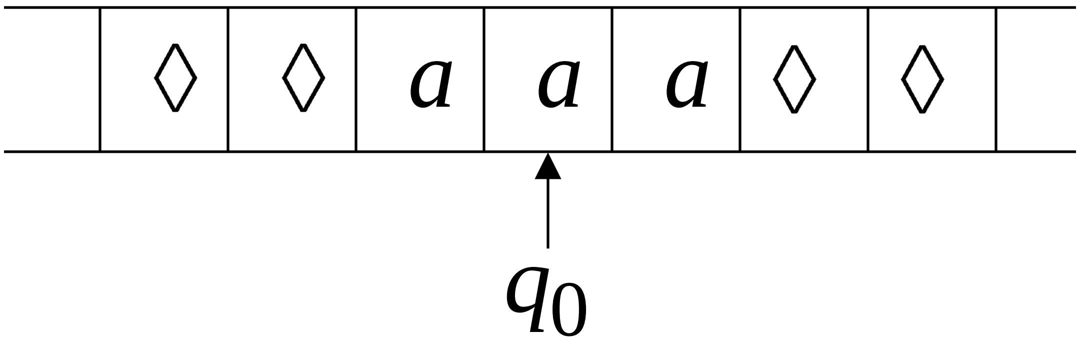


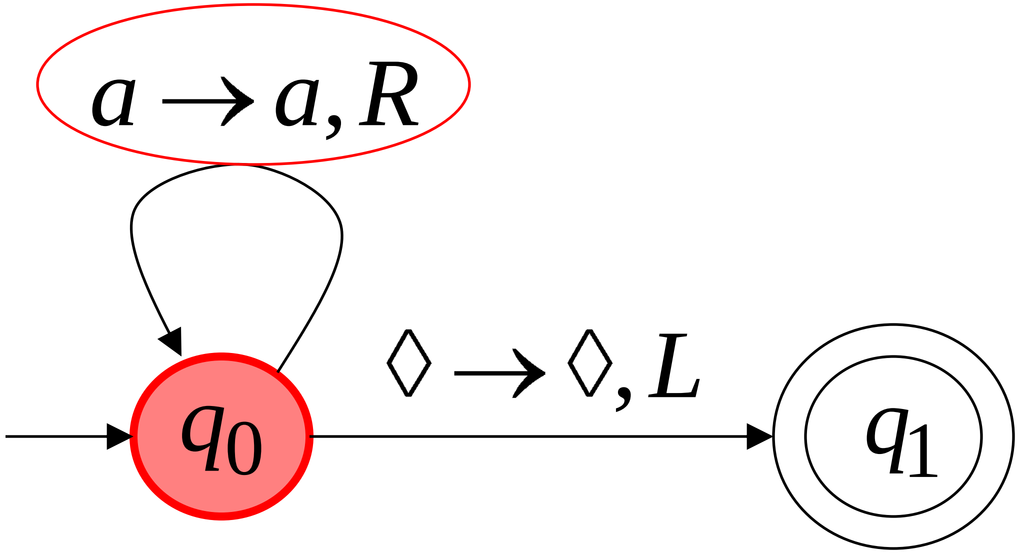
Time 0



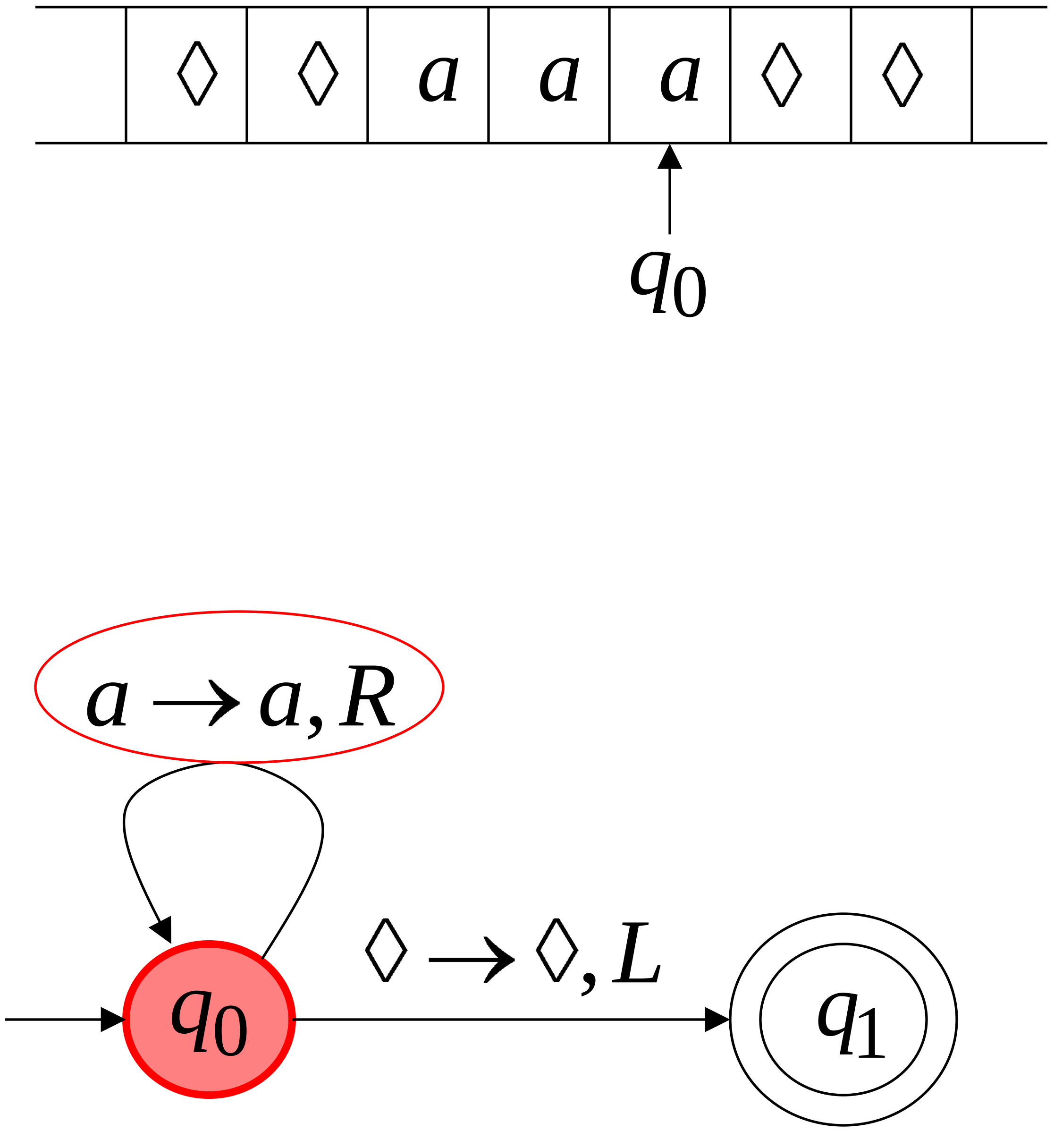


Time 1

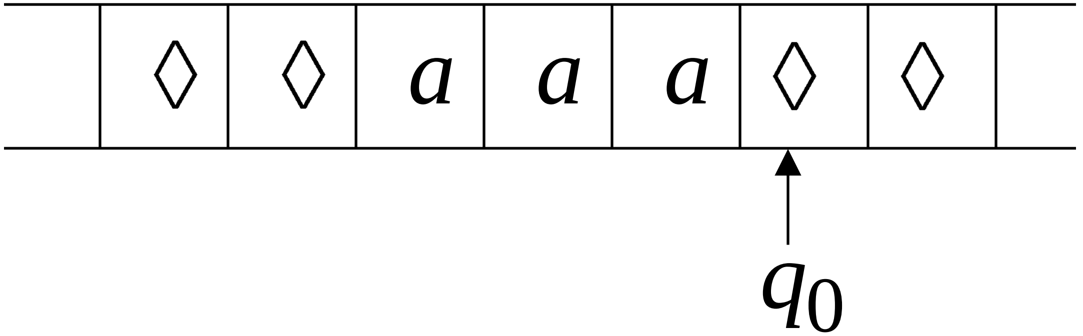


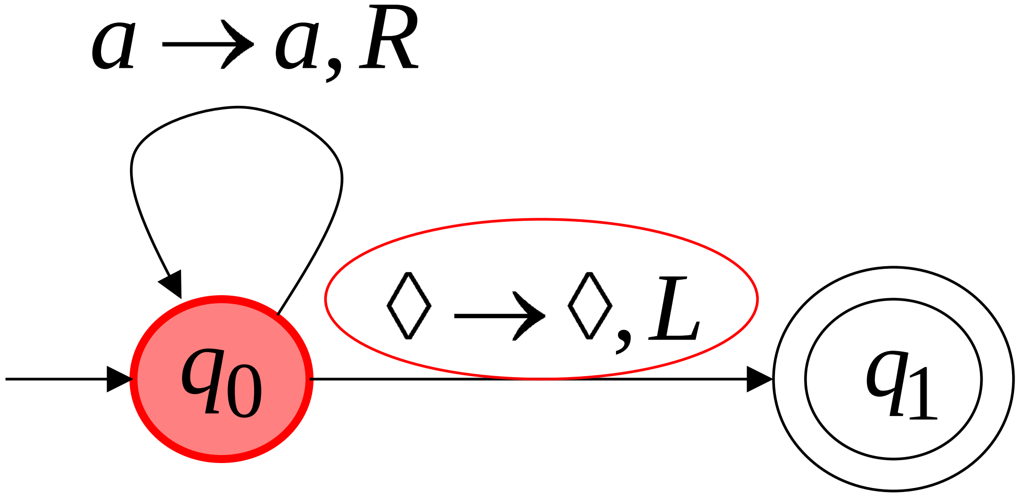


Time 2

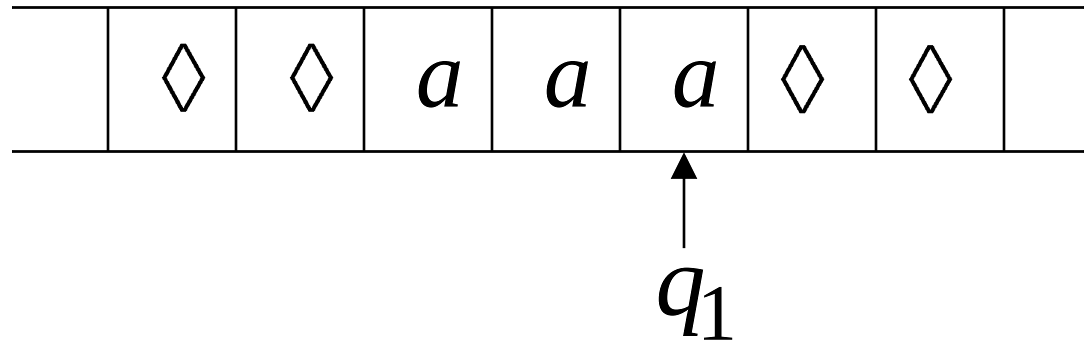


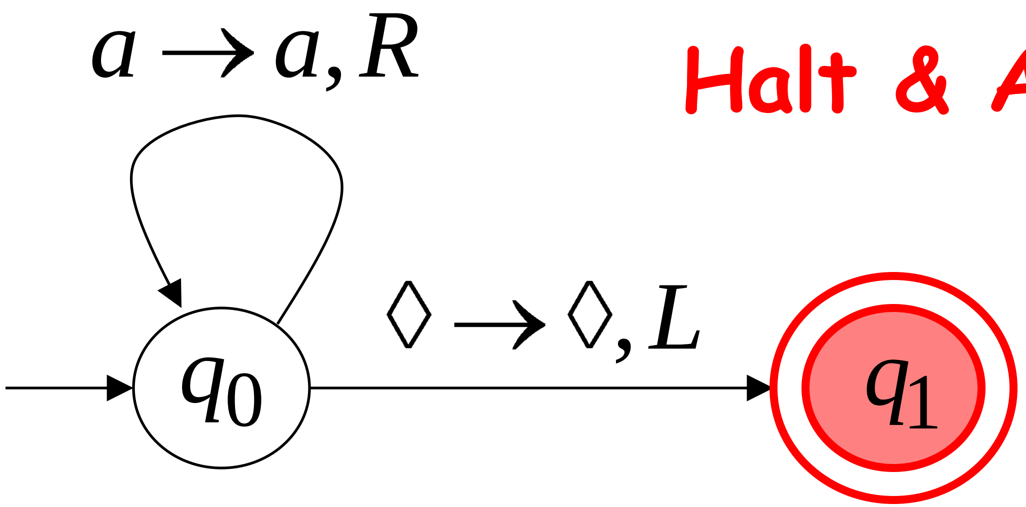
Time 3





Time 4

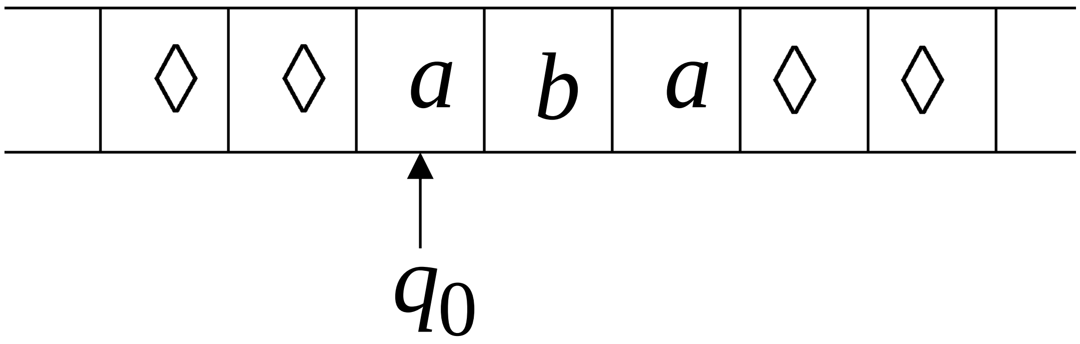


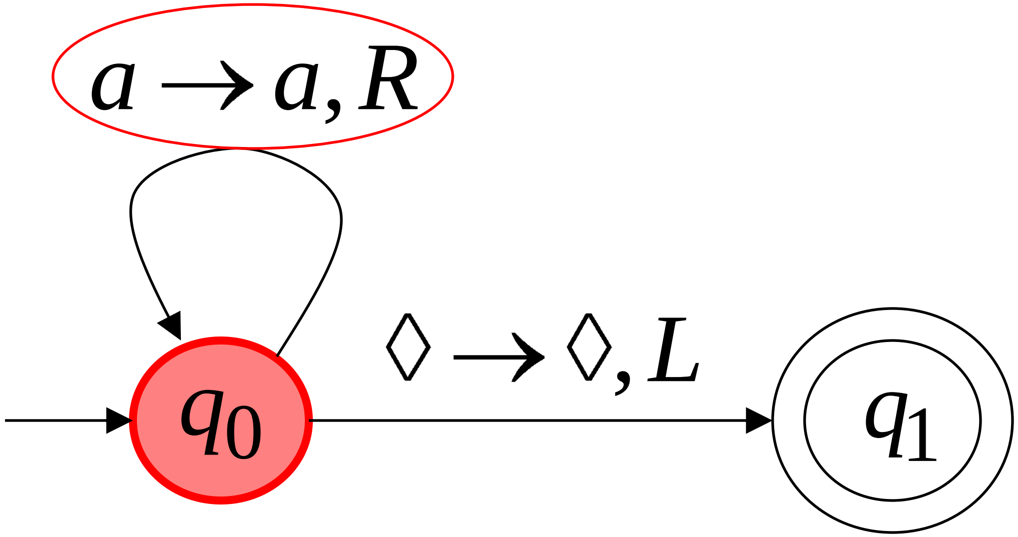


Halt & Accept

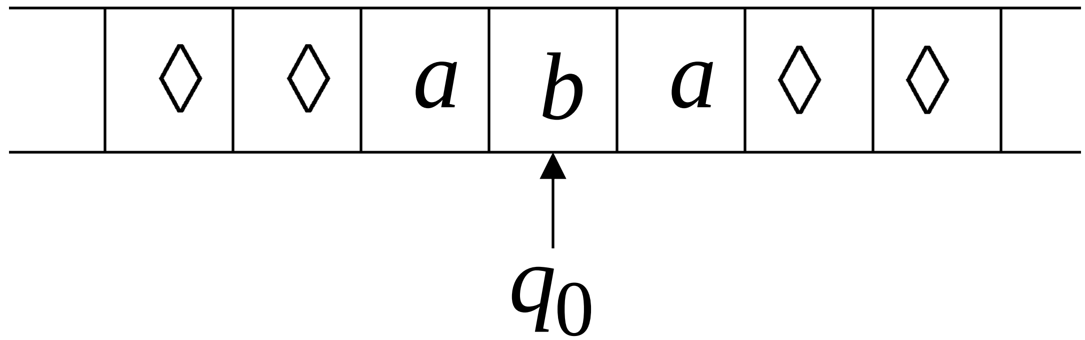
Rejection Example

Time 0



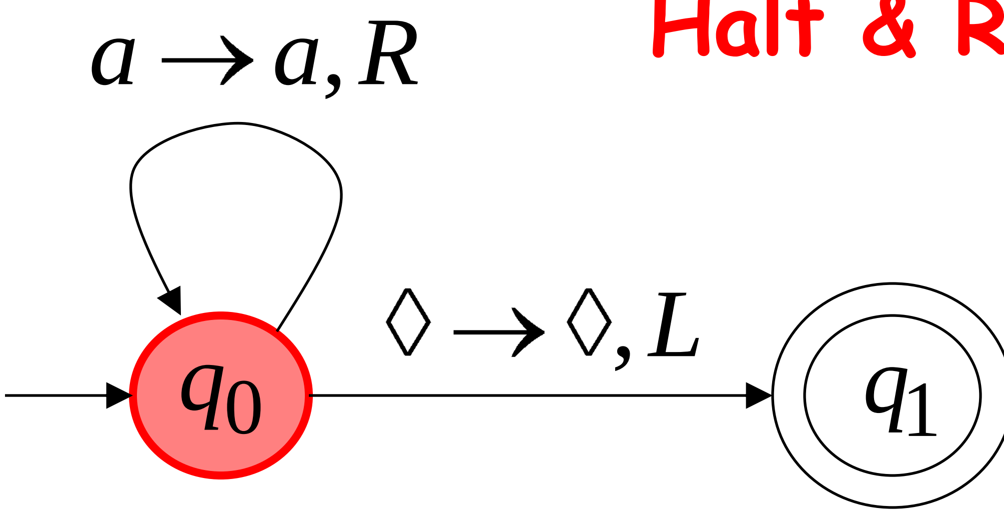


Time 1

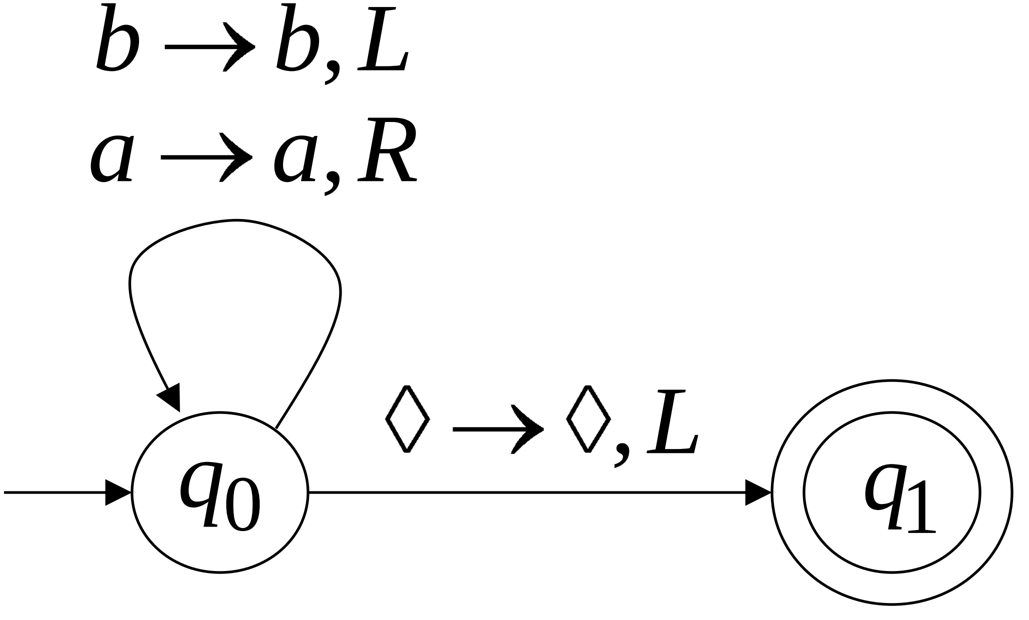


No possible Transition

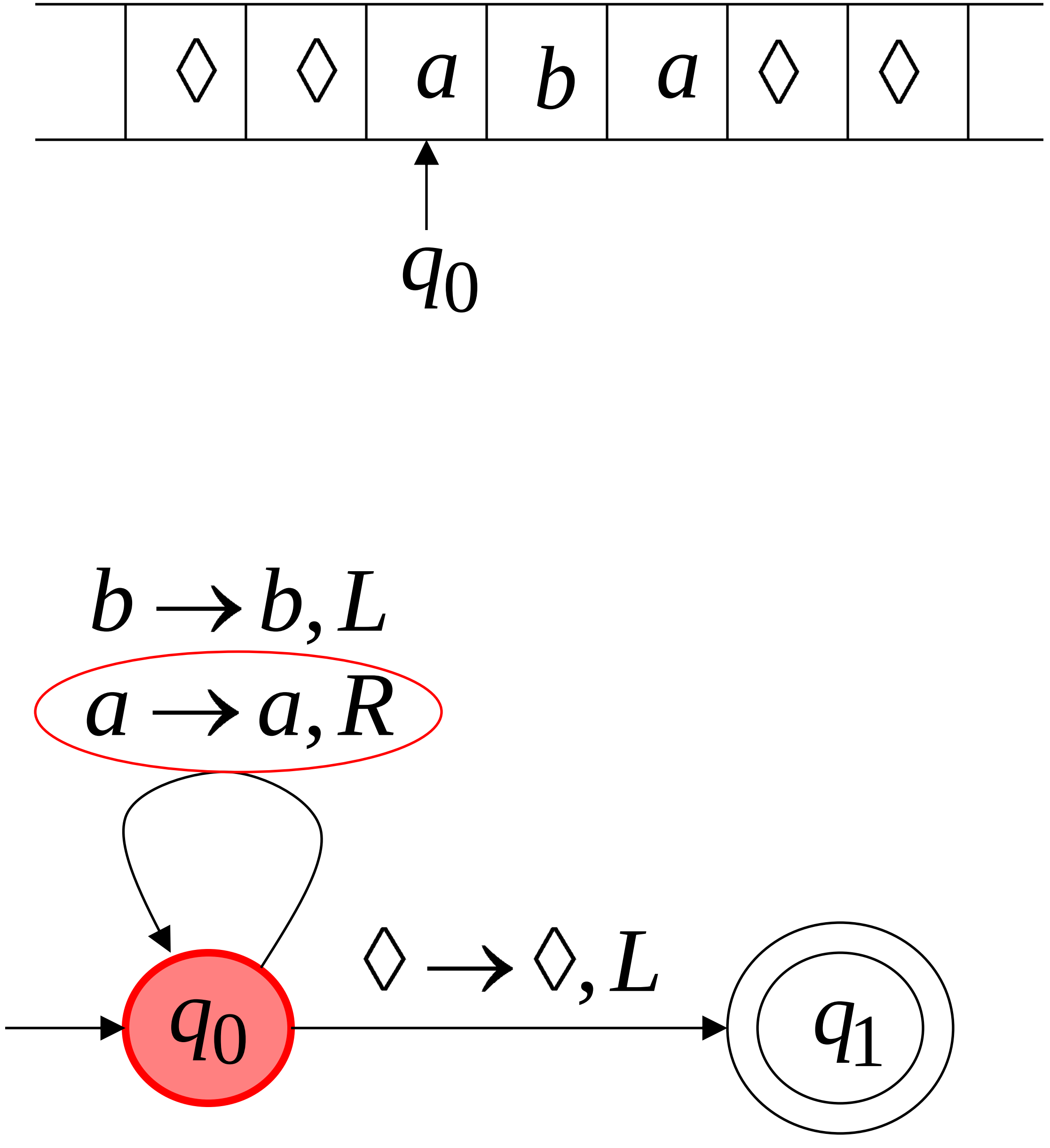
Halt & Reject



Infinite Loop Example

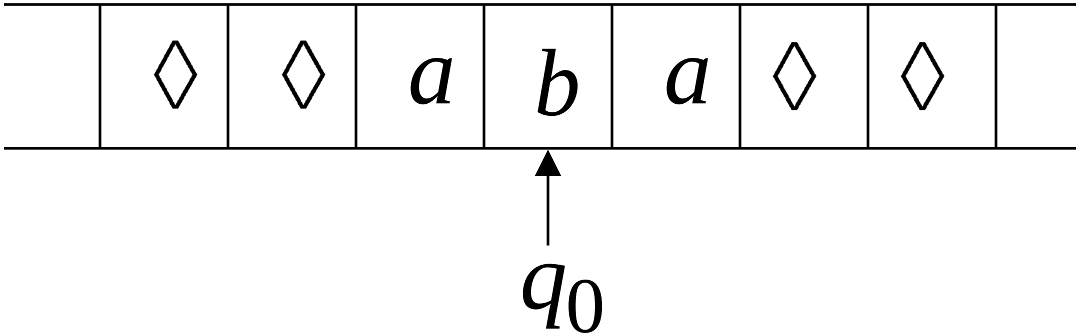


Time 0

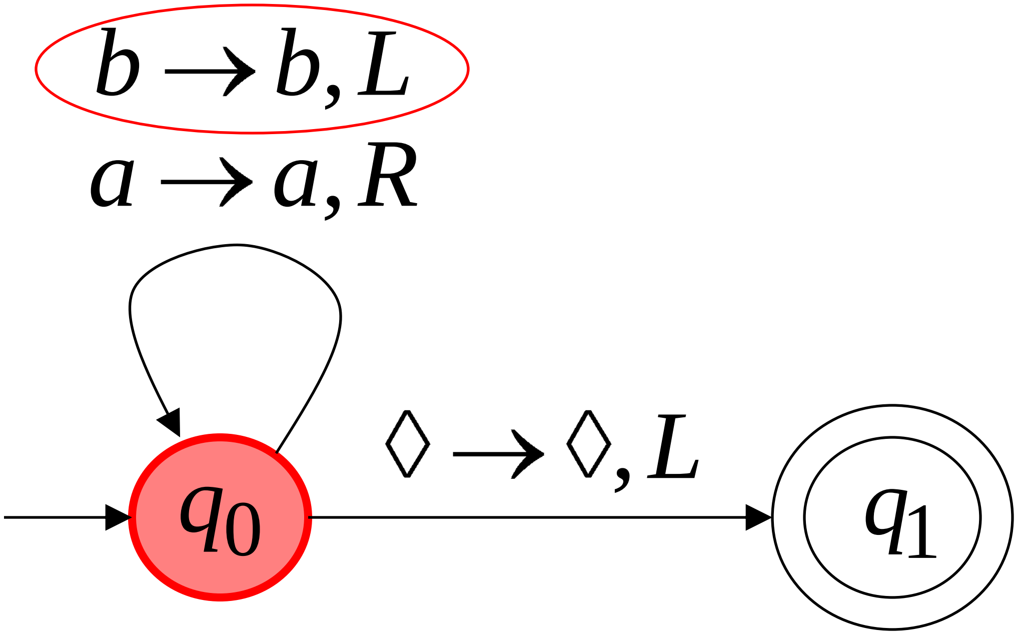


◊
◊
a
b
a
◊
◊
q_0
$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

Time 1

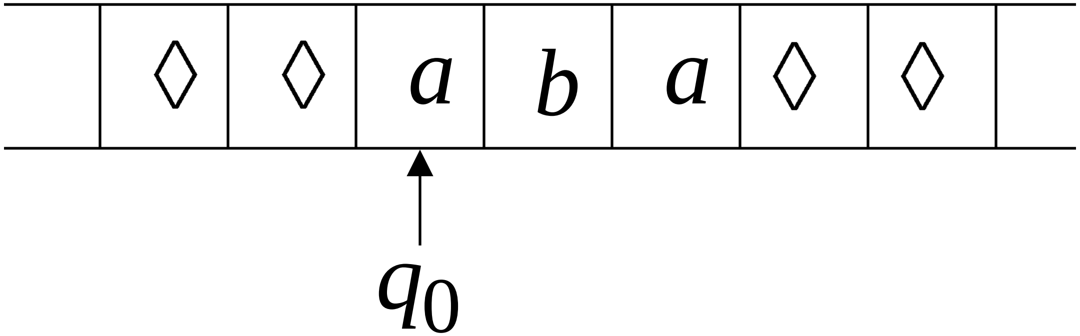


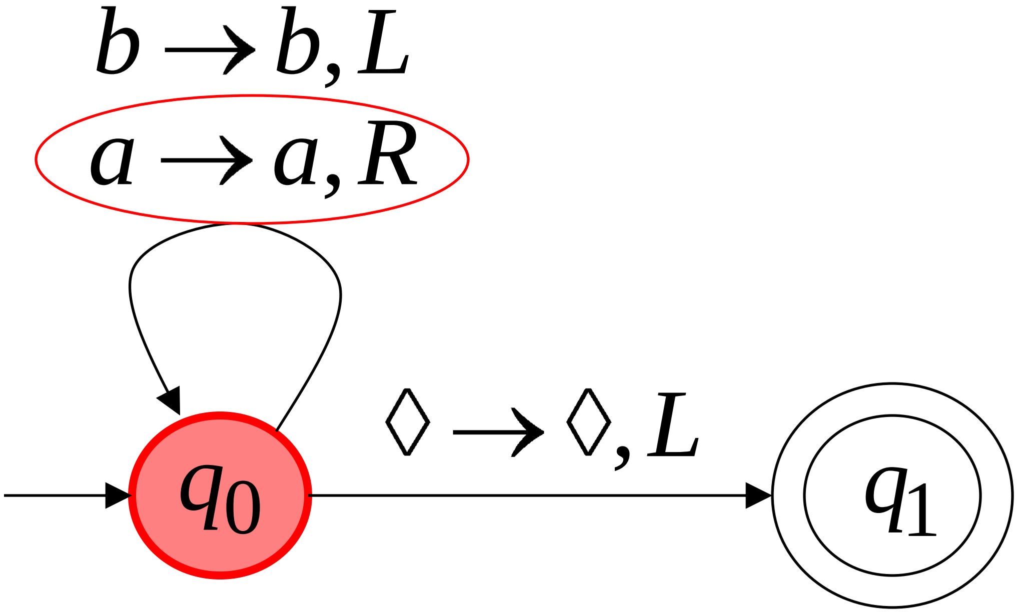
◊ ◊ a b a ◊ ◊
q_0



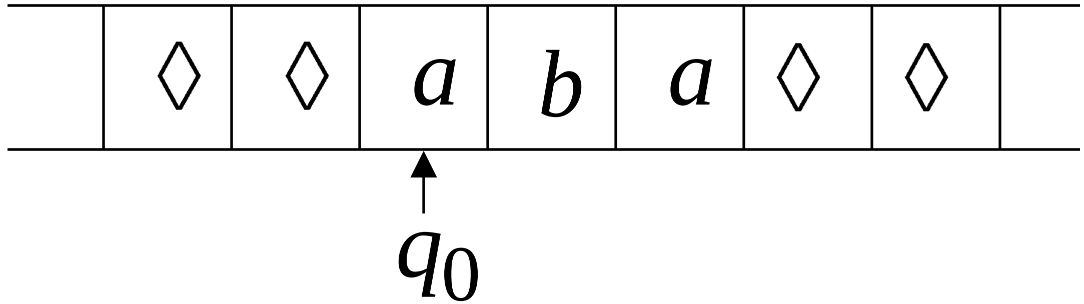
$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

Time 2

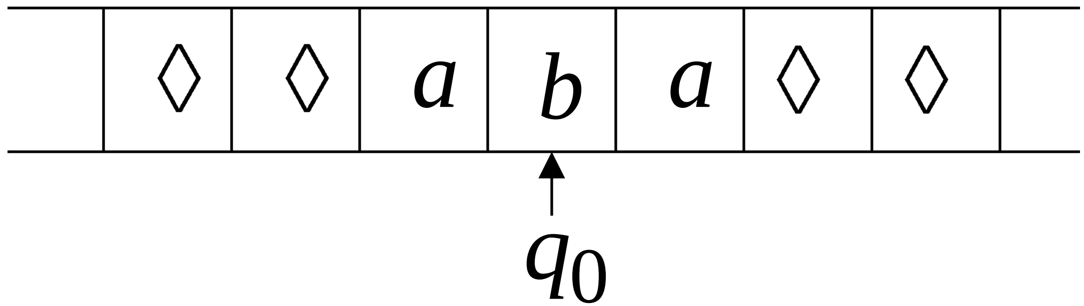




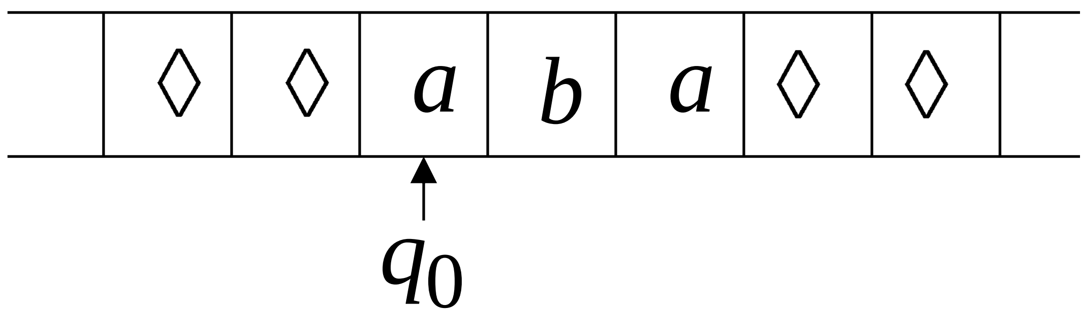
Time 2



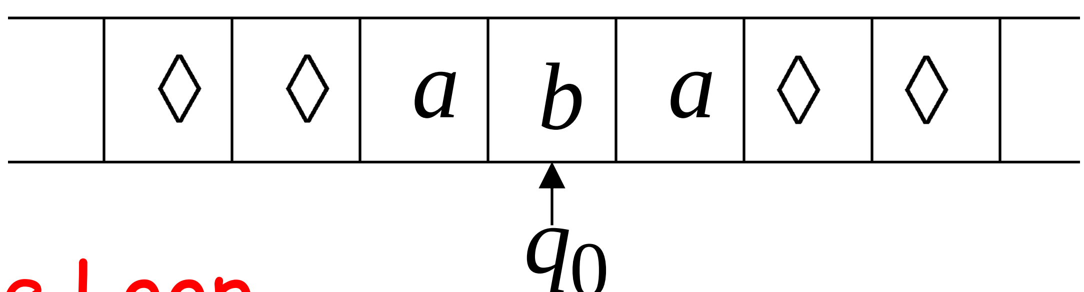
Time 3



Time 4



Time 5



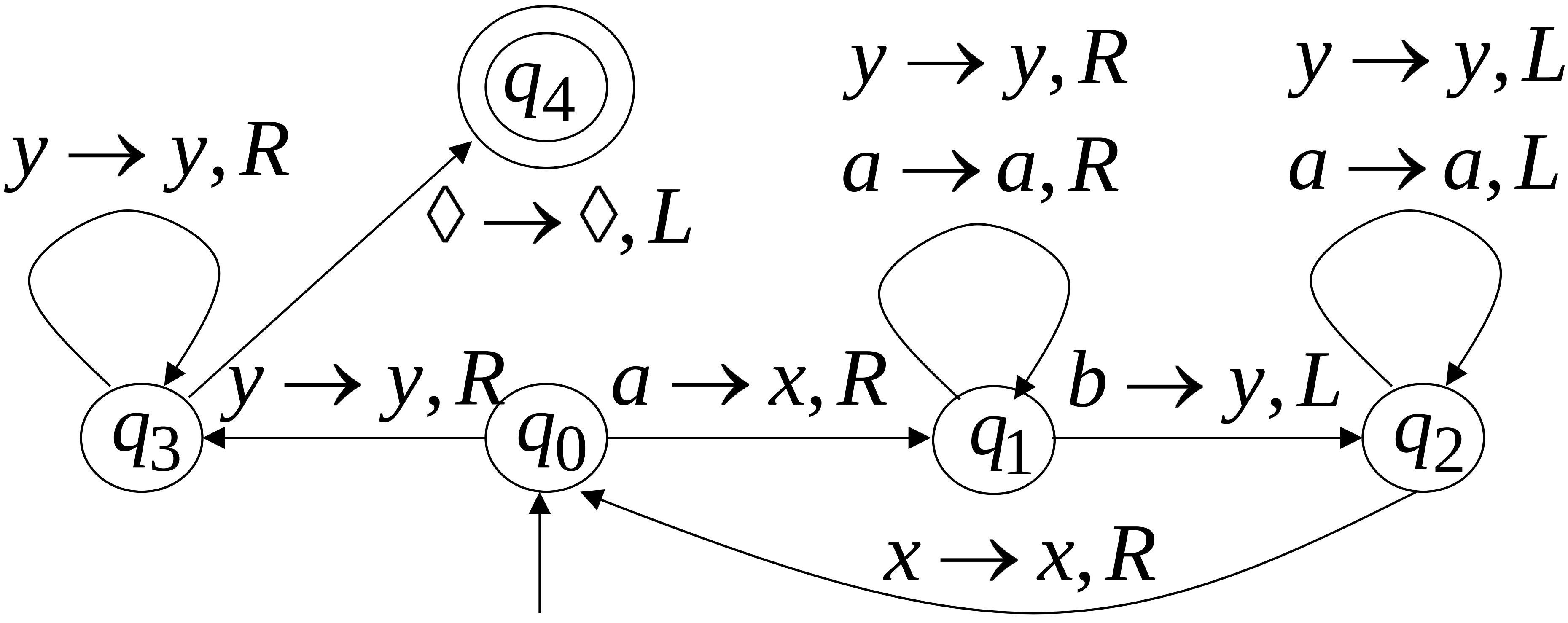
... Infinite Loop

Because of the infinite loop:

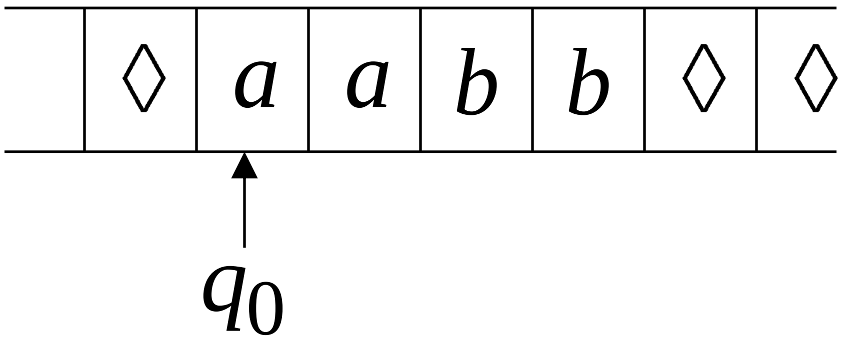
- The final state cannot be reached
- The machine never halts
- The input is not accepted

Another Turing Machine Example

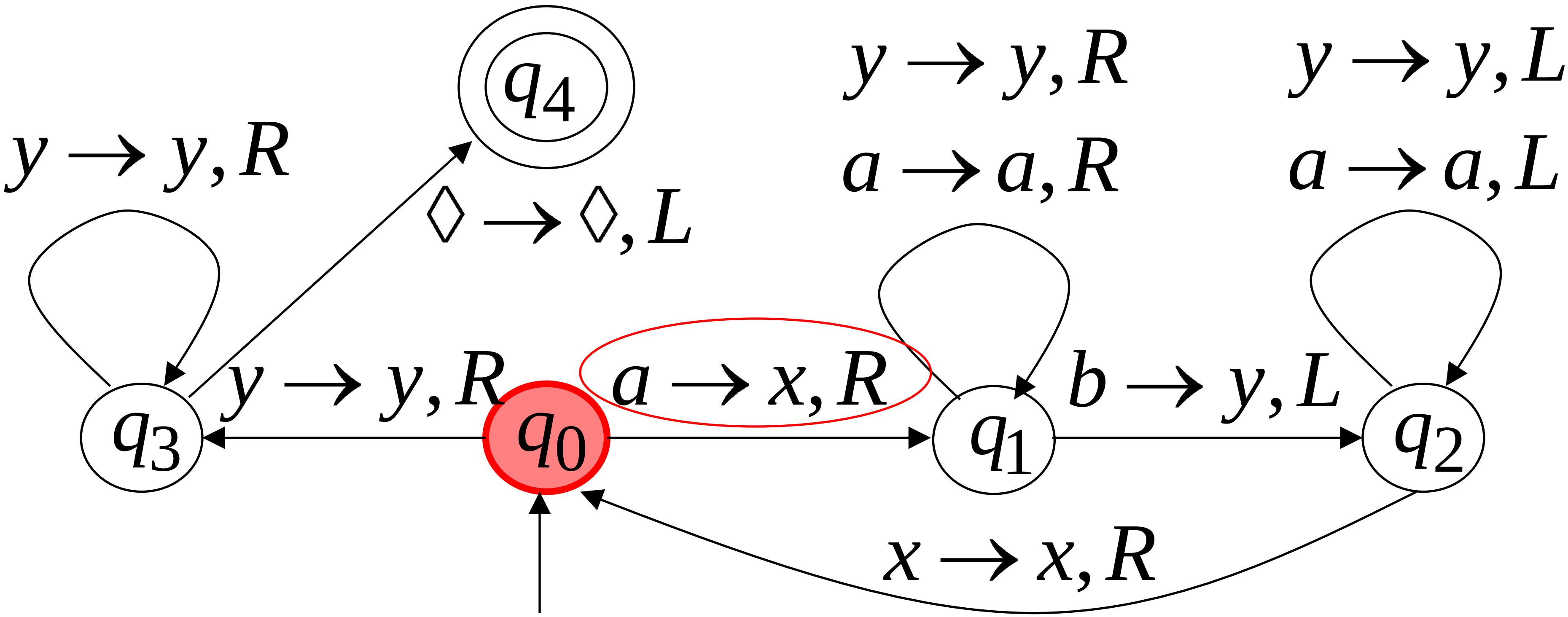
Turing machine for the language $\{a^n b^n\}$



Time 0

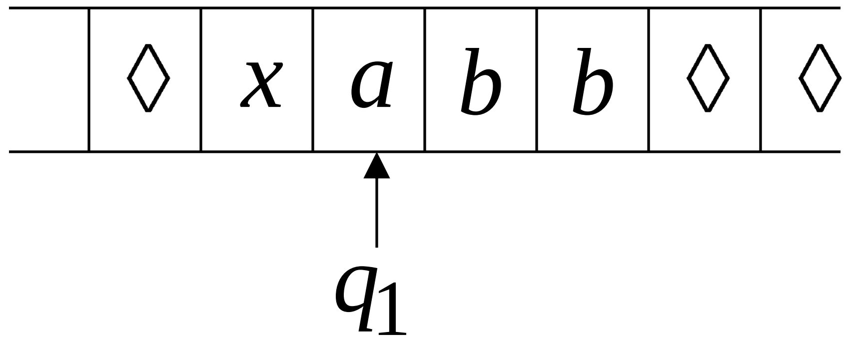


◊
a
a
b
b
◊
◊
q_0

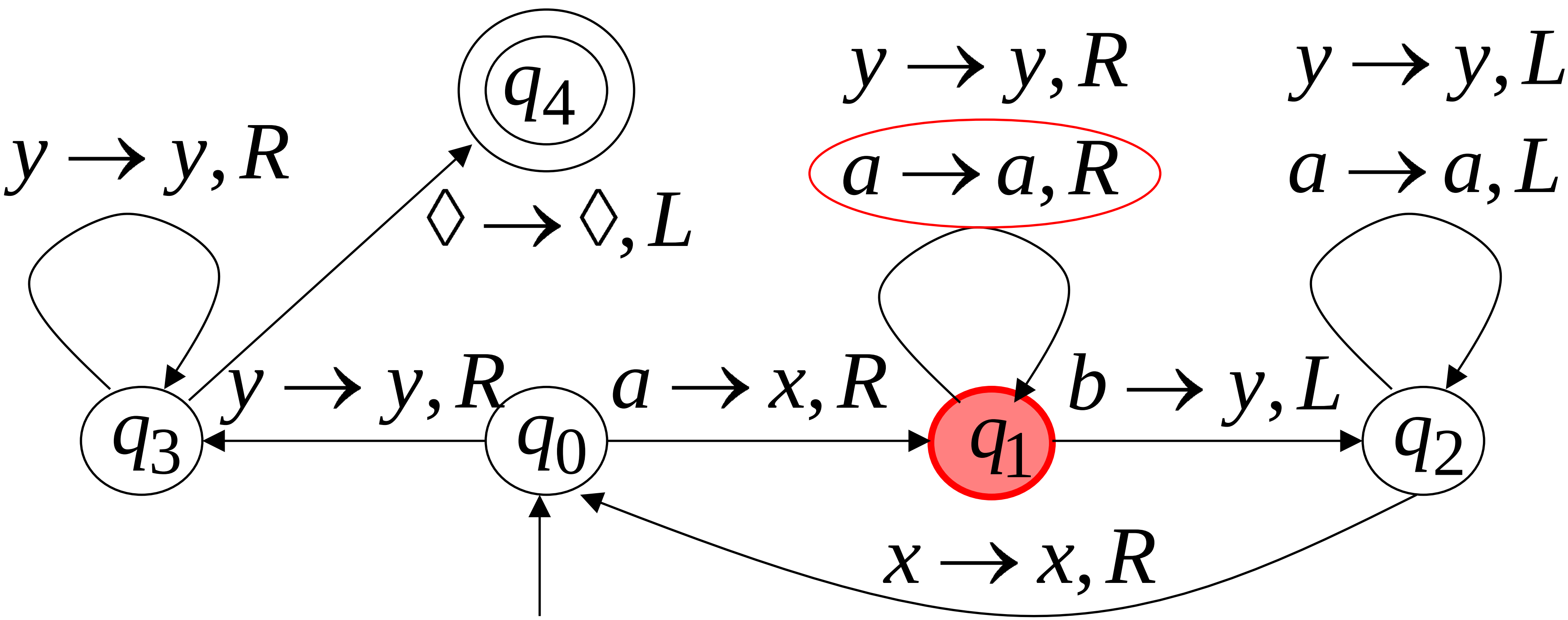


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 1

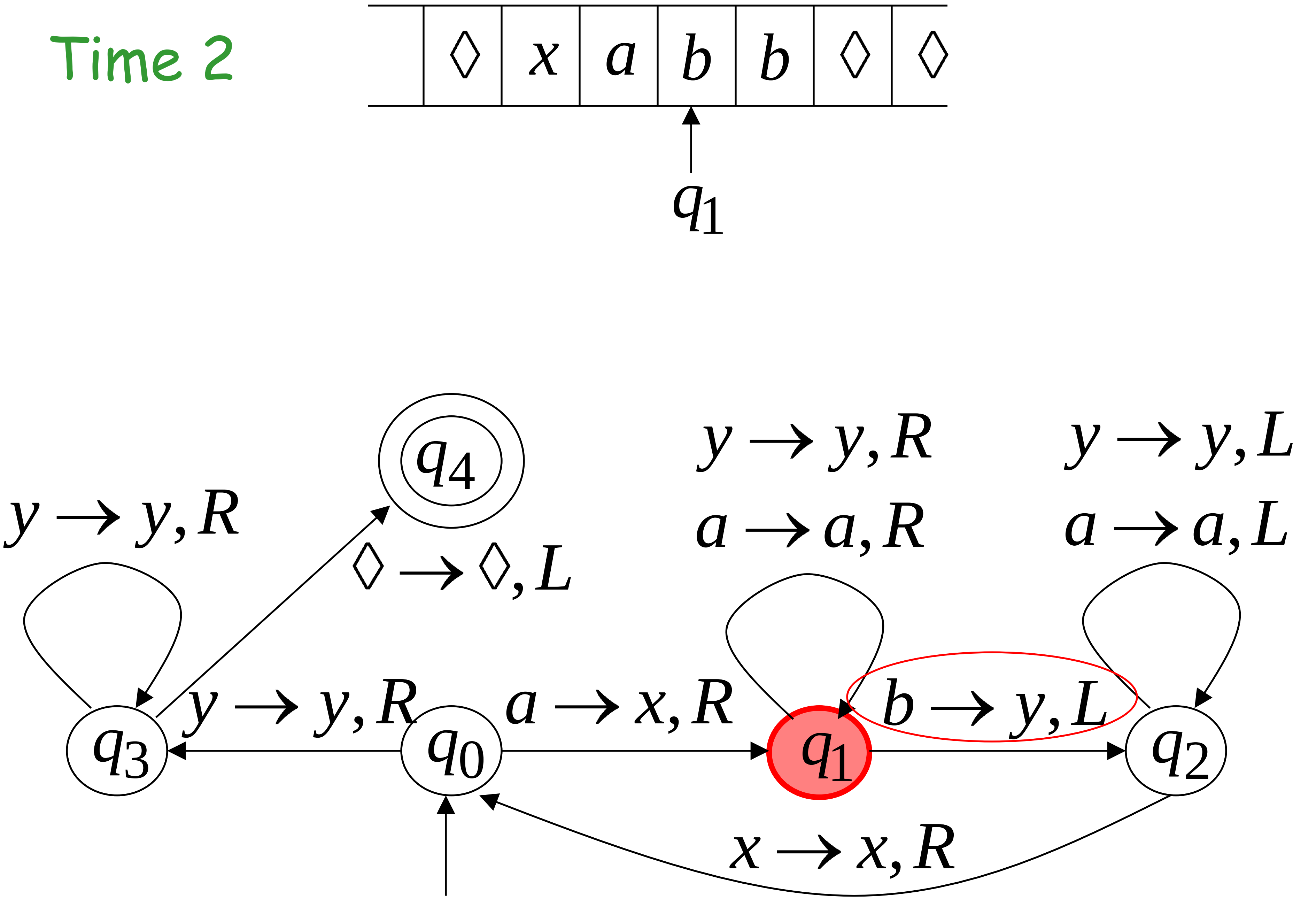


◊
x
a
b
b
◊
◊
q_1

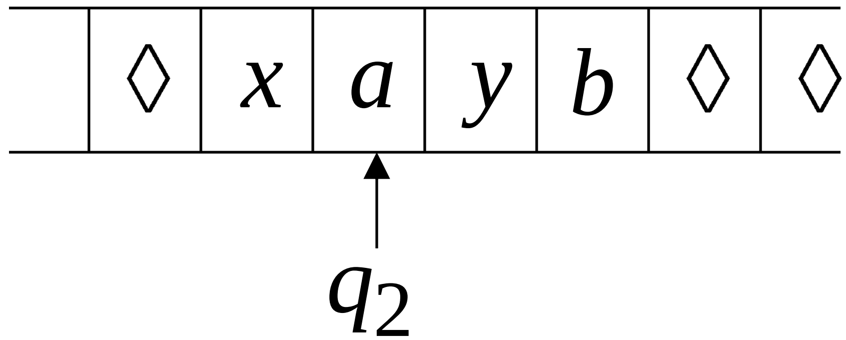


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

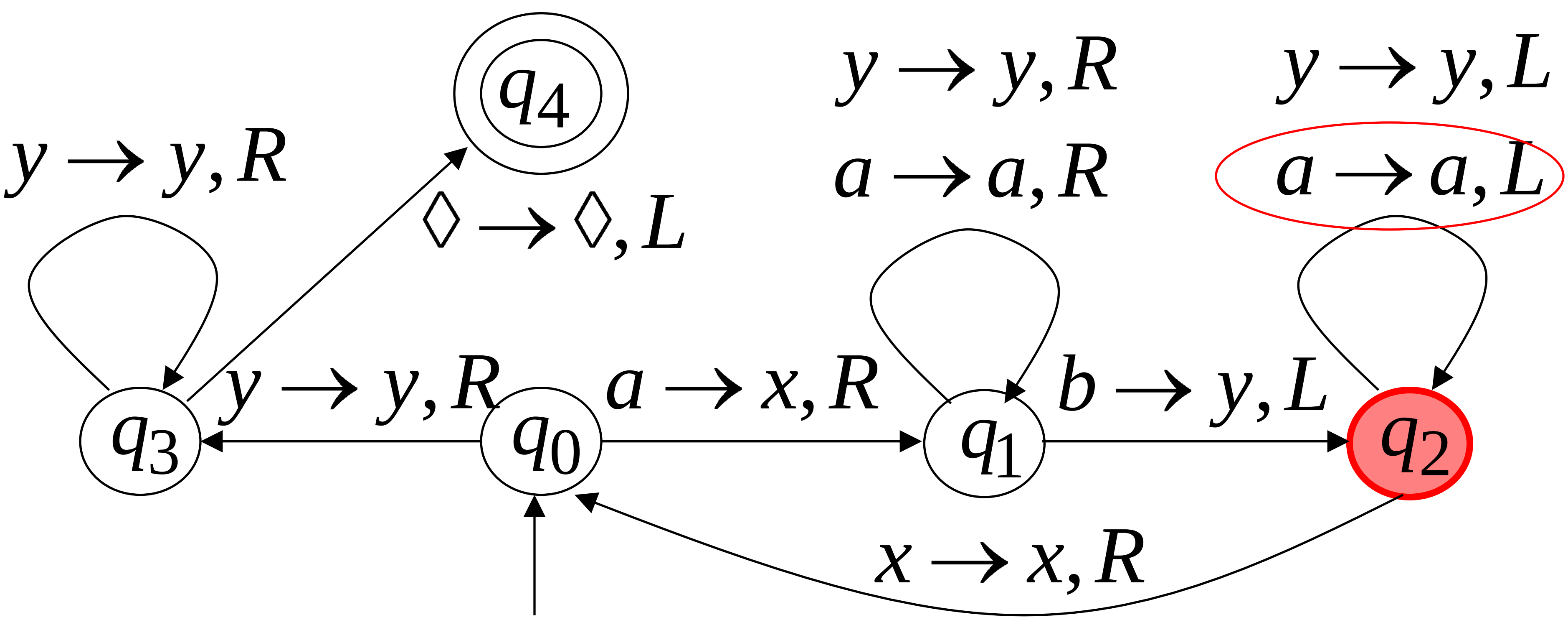
Time 2


◊ x a b b ◊ ◊
q_1
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 3

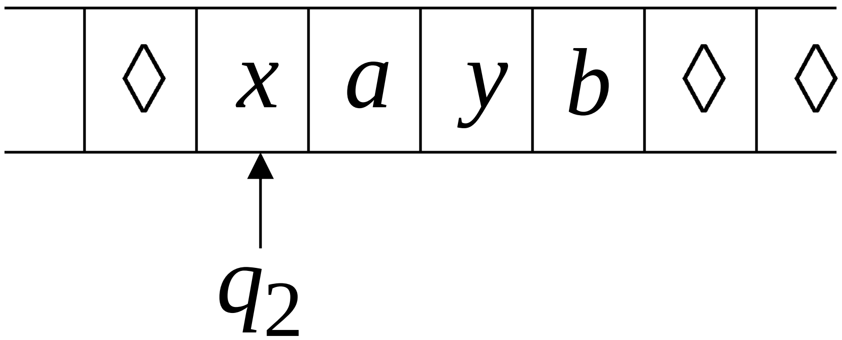


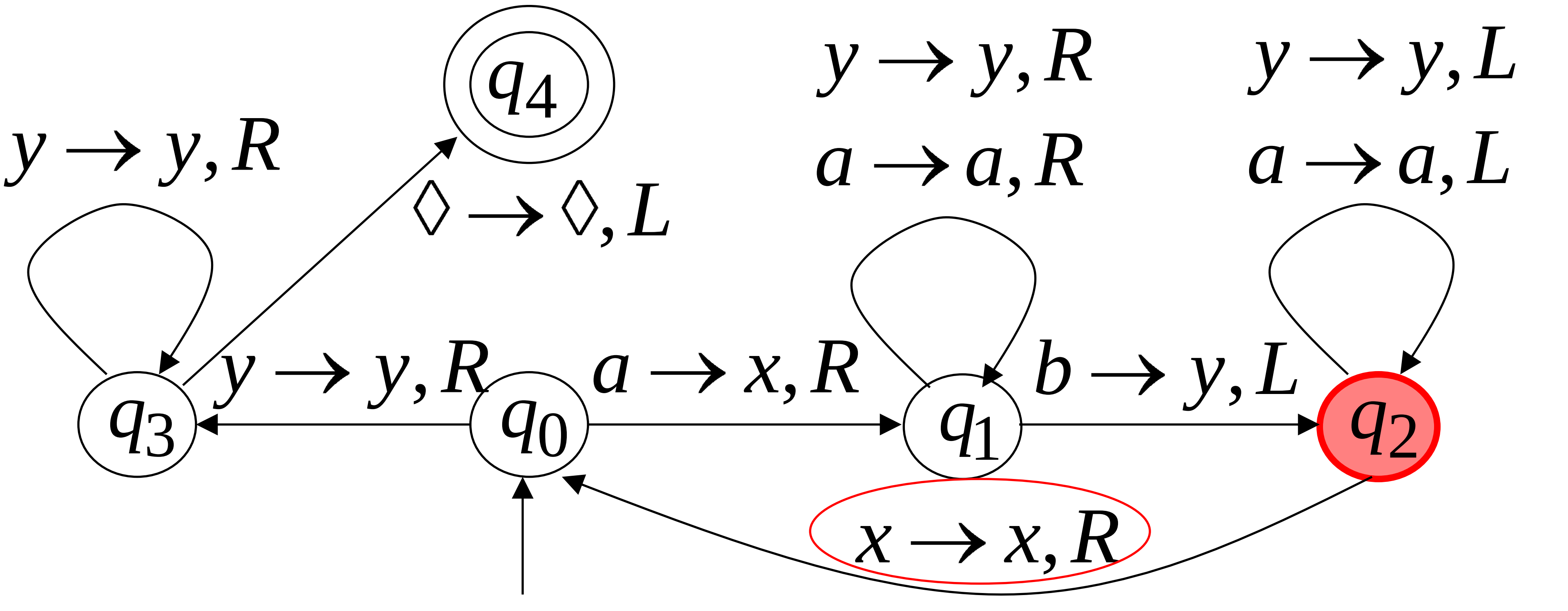
◊
x
a
y
b
◊
◊
q_2



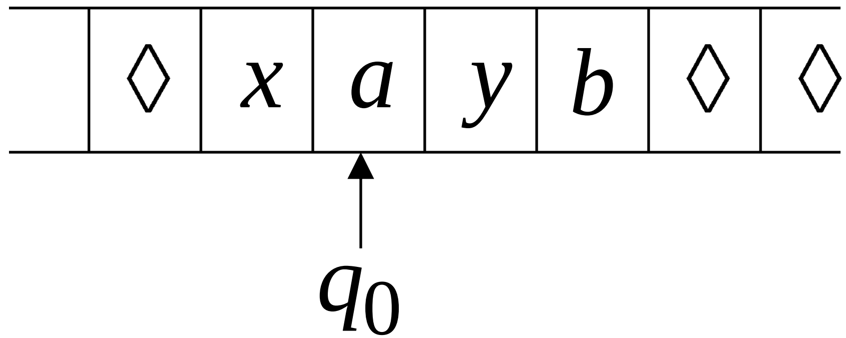
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

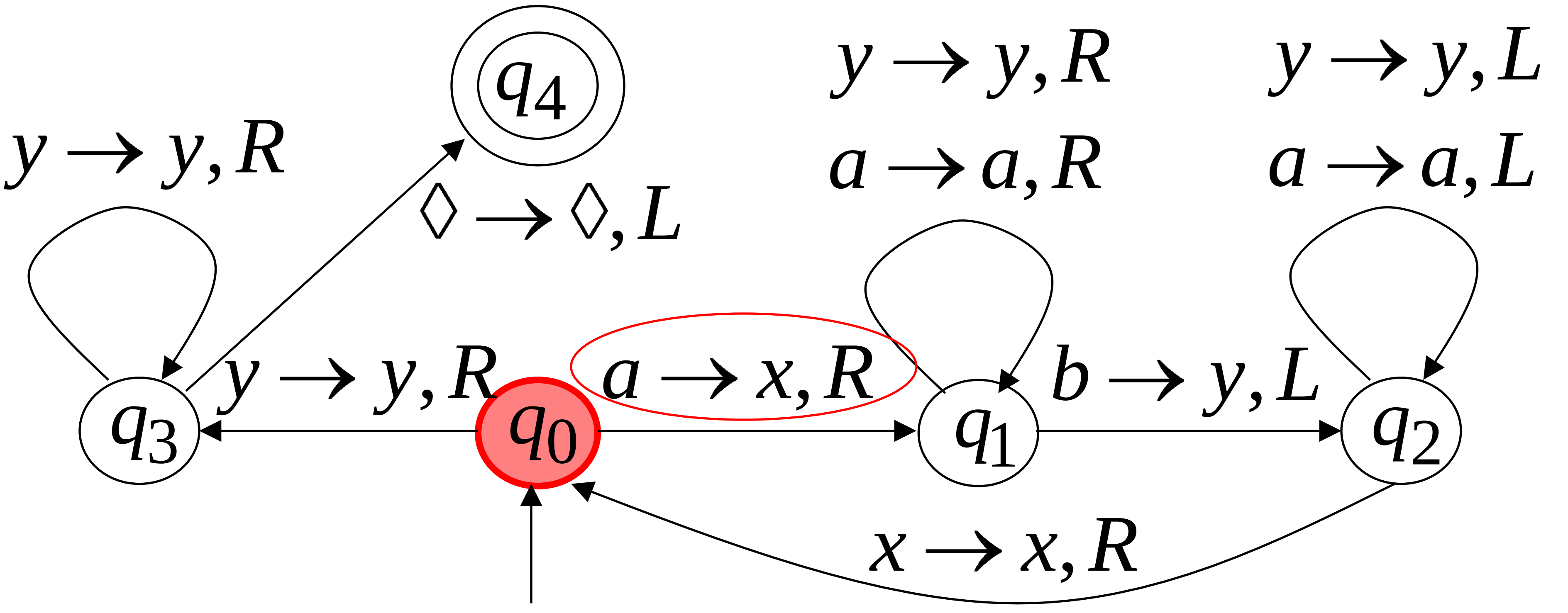
Time 4


◊
x
a
y
b
◊
◊
q_2

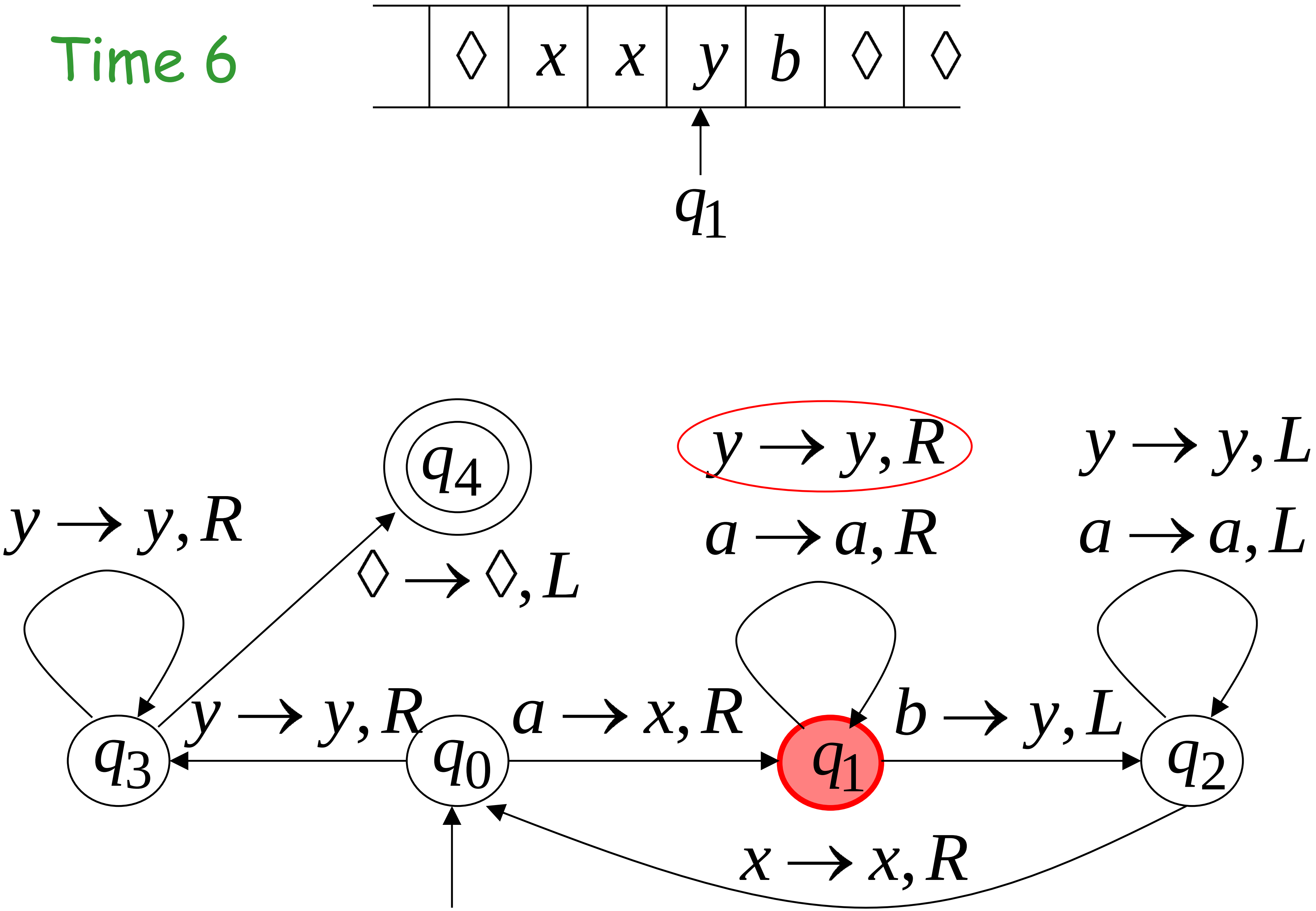

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 5

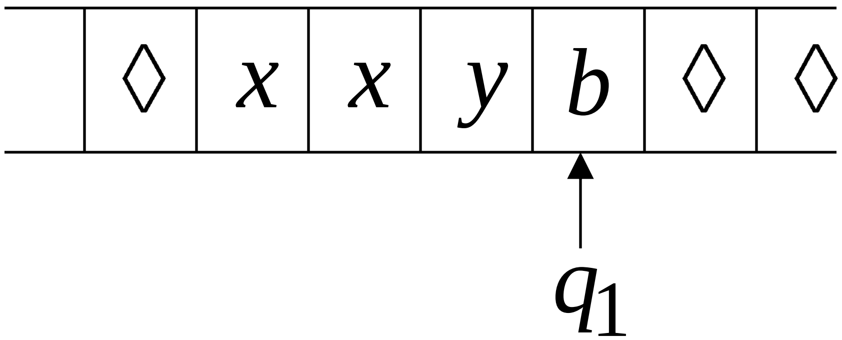

◊
x
a
y
b
◊
◊
q_0


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

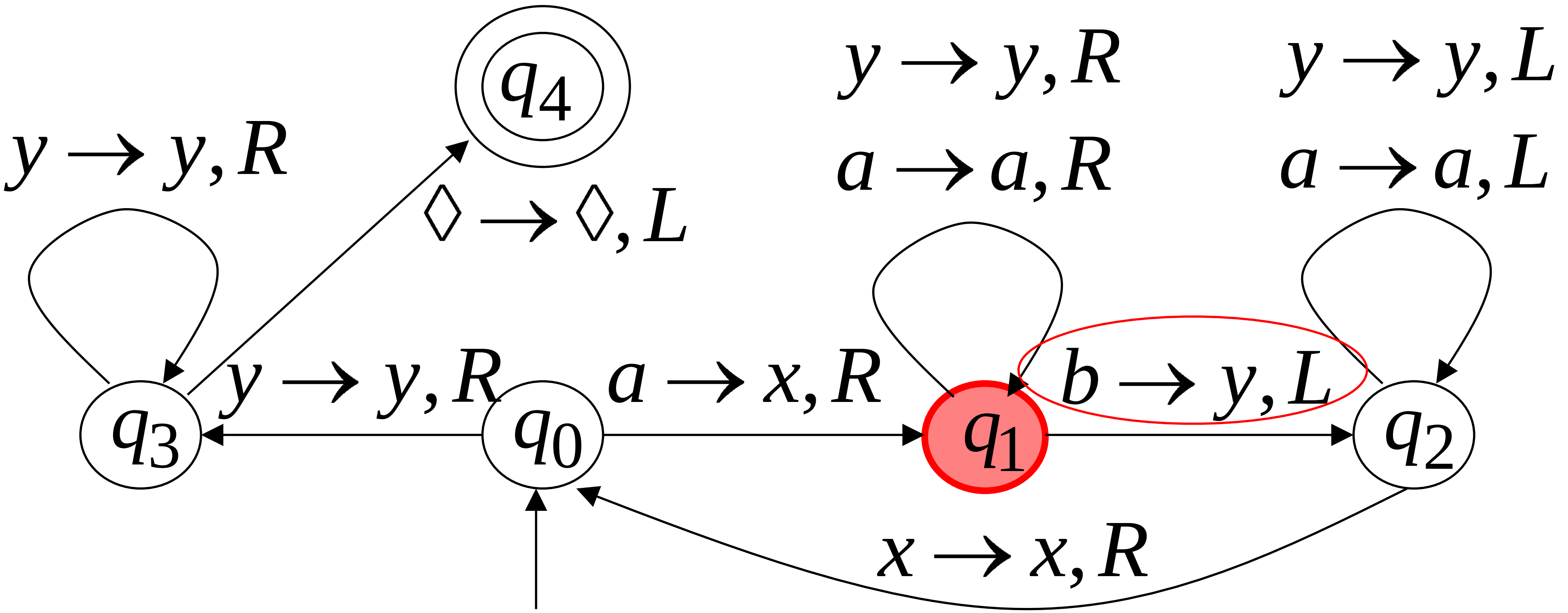
Time 6


◊
x
x
y
b
◊
◊
q_1
q_4
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 7

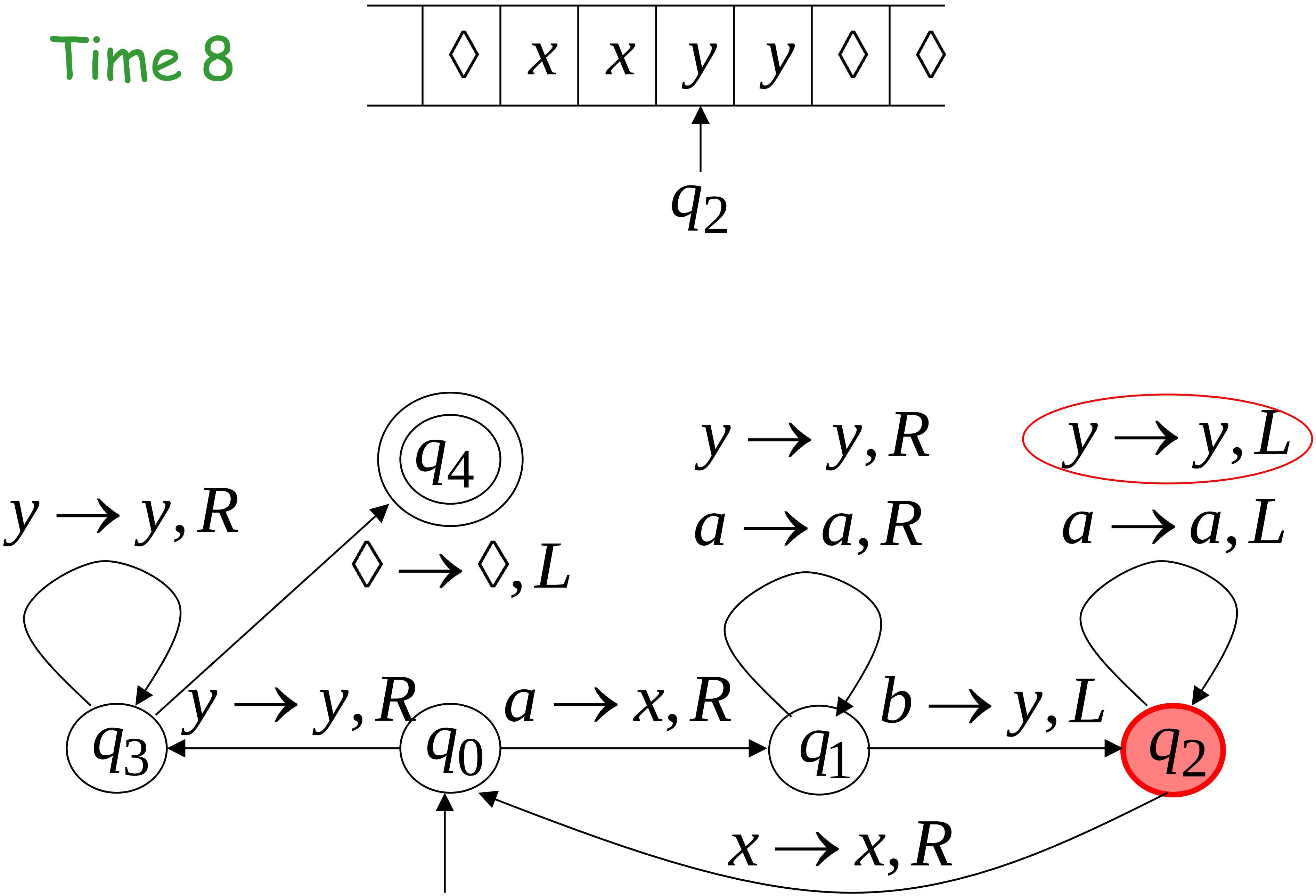


◊
x
x
y
b
◊
◊
q_1



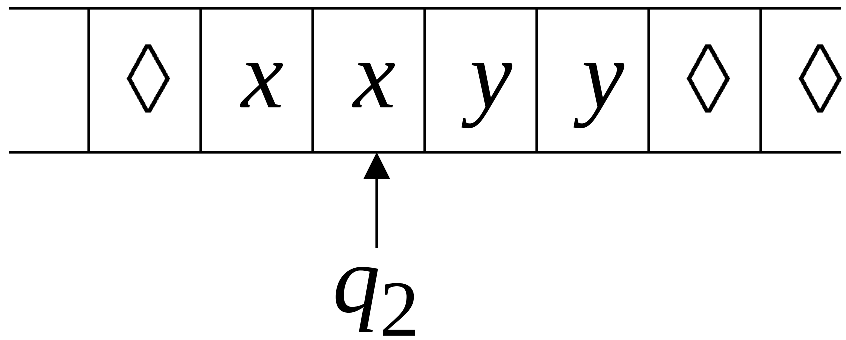
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 8

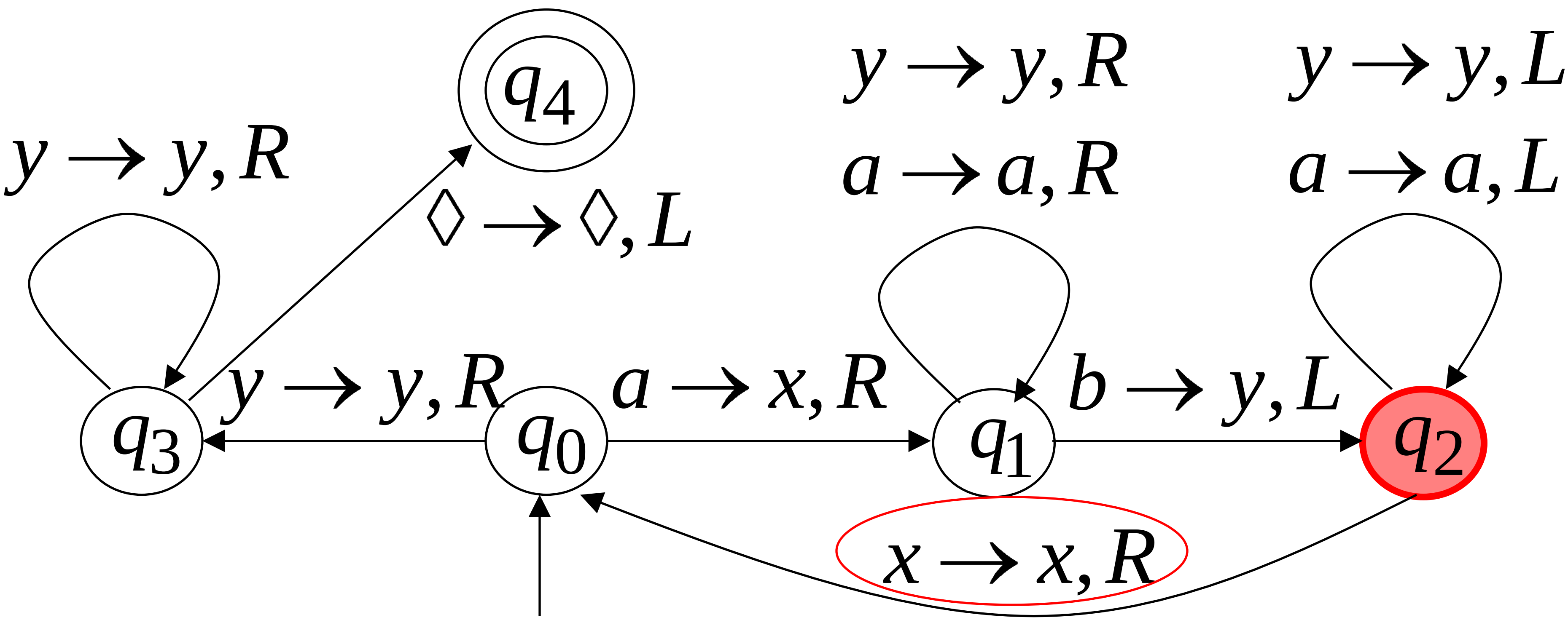


◊ x x y y ◊ ◊
q_2
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
q_4
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 9

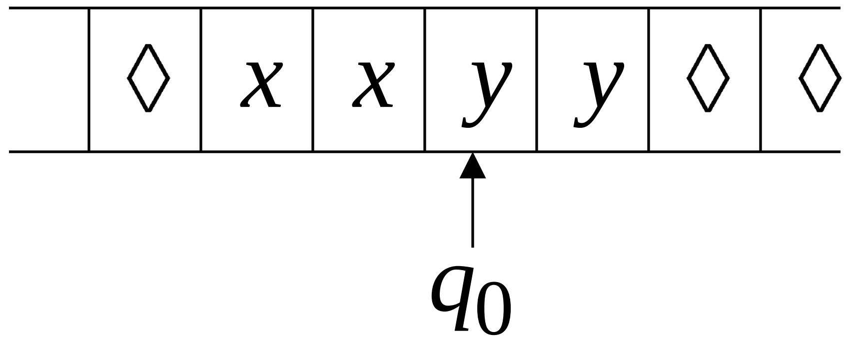


◊ x x y y ◊ ◊
q_2

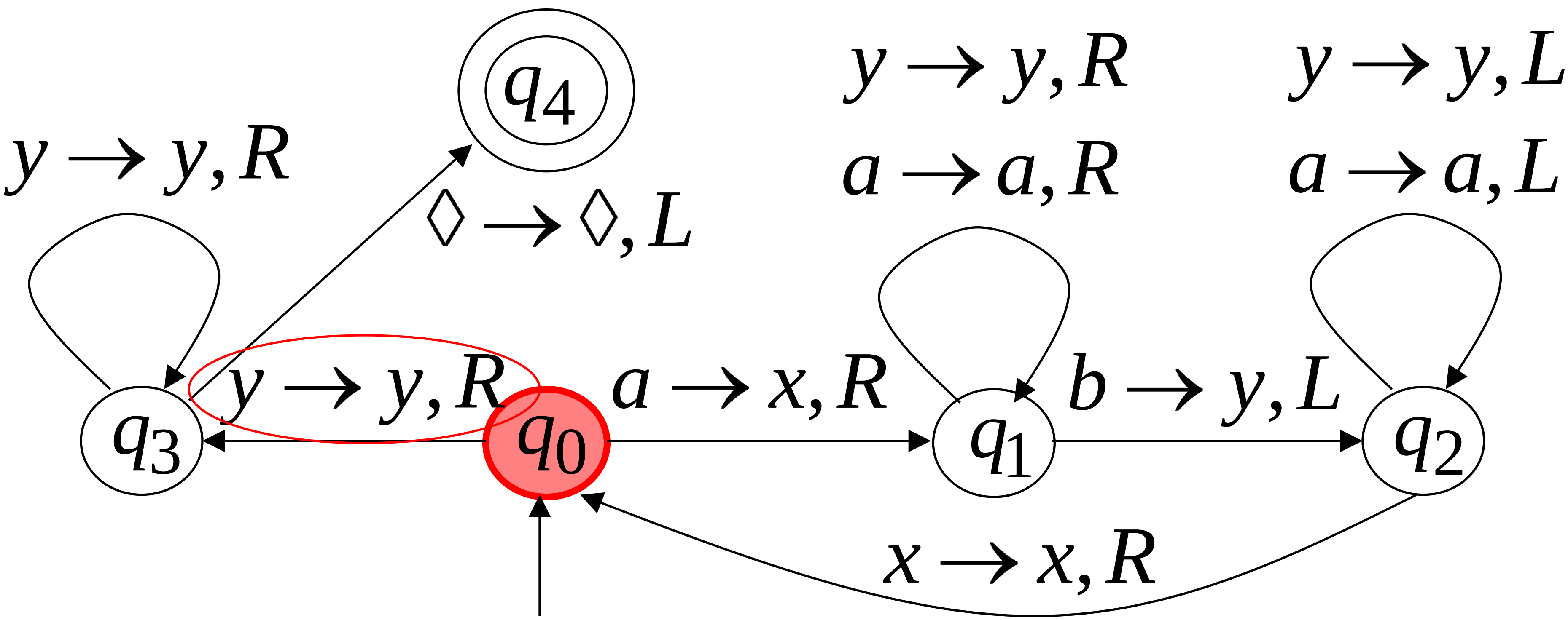


q_4
$y \to y, R$
◊ $\to$ ◊$, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 10

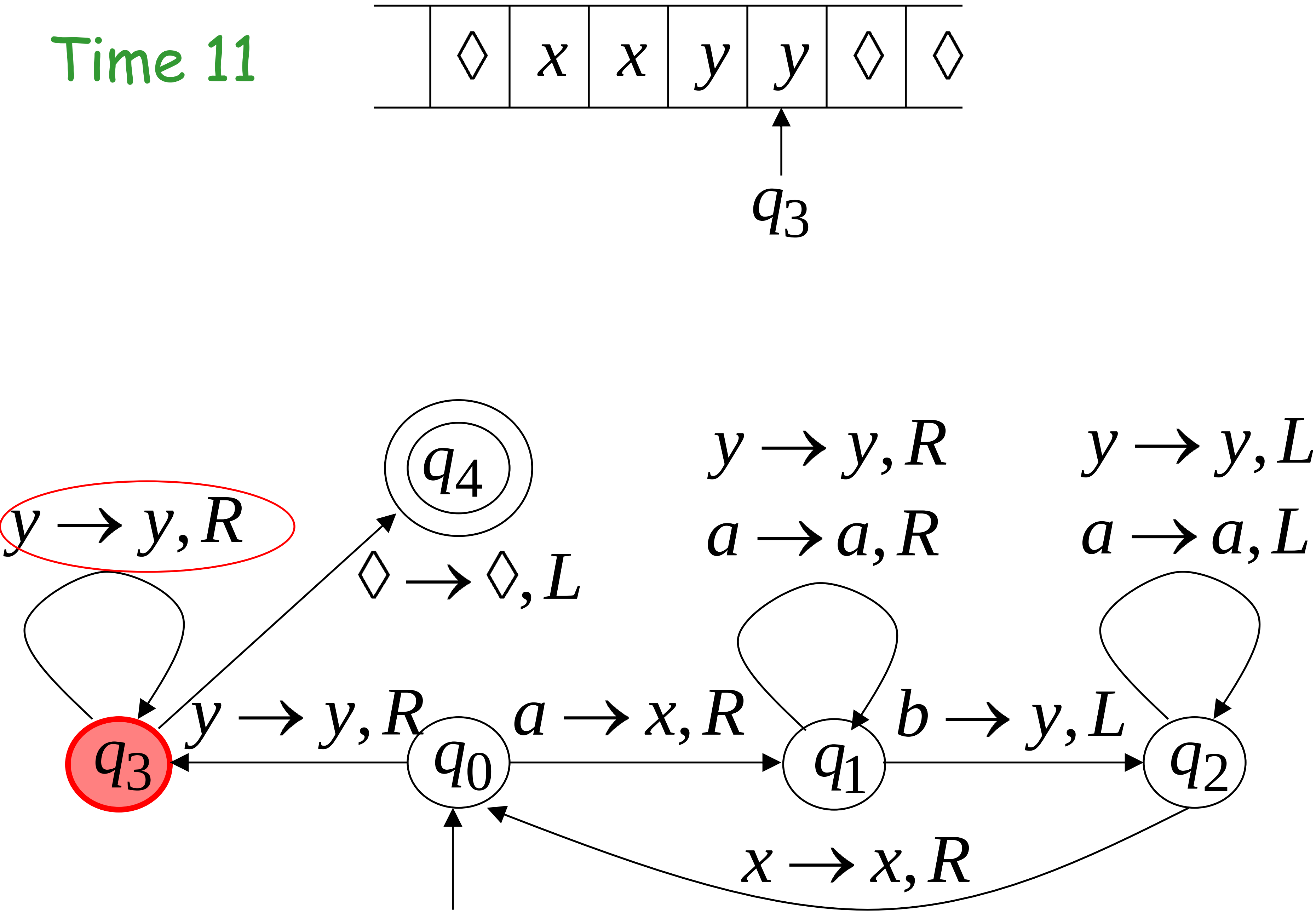


◊
x
x
y
y
◊
◊
q_0

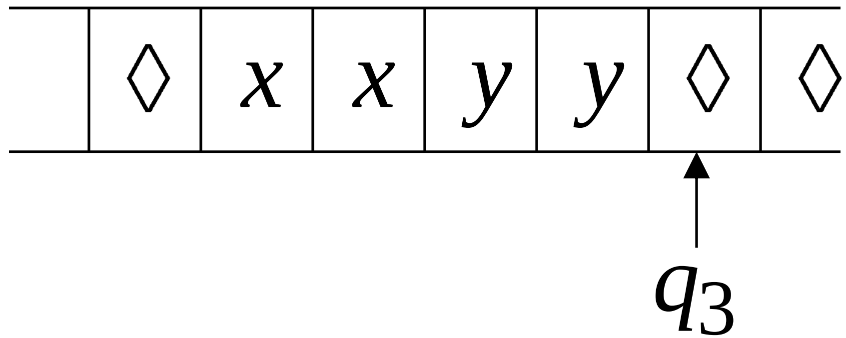


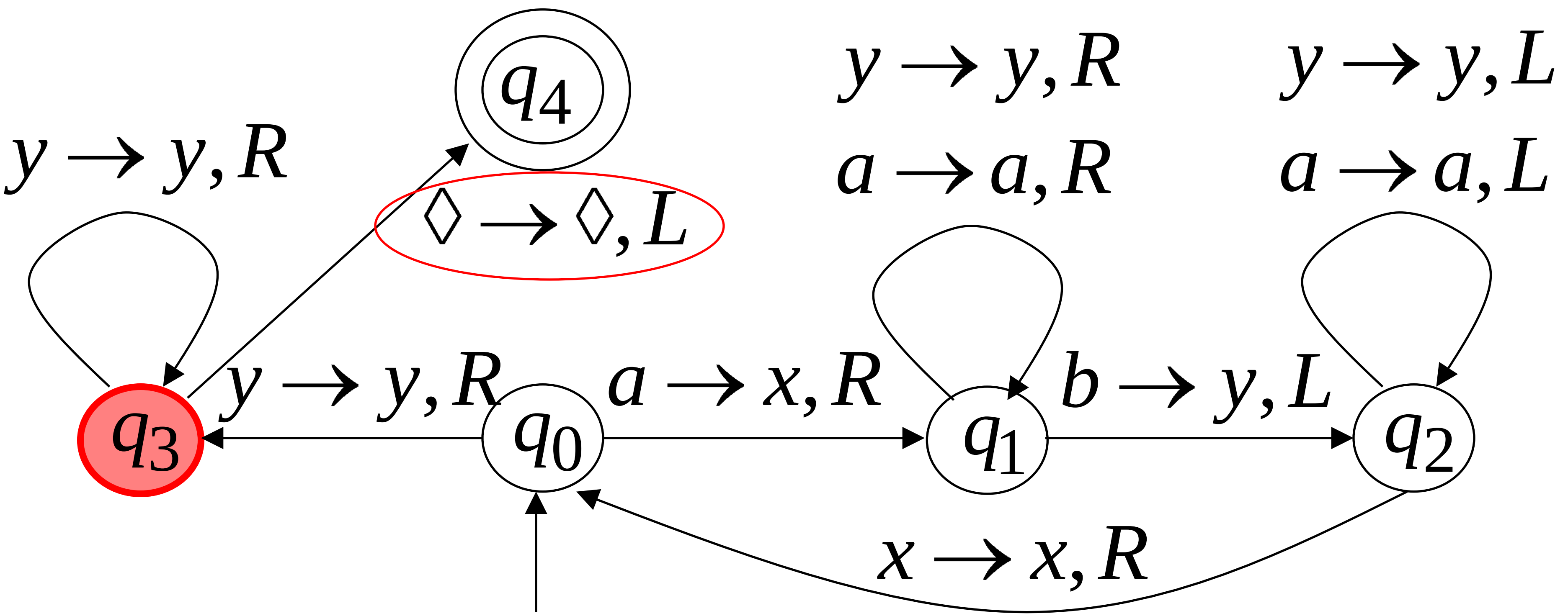
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 11

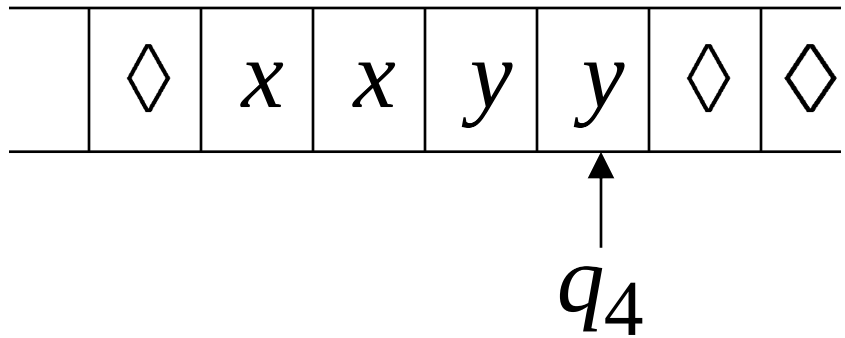


Time 12

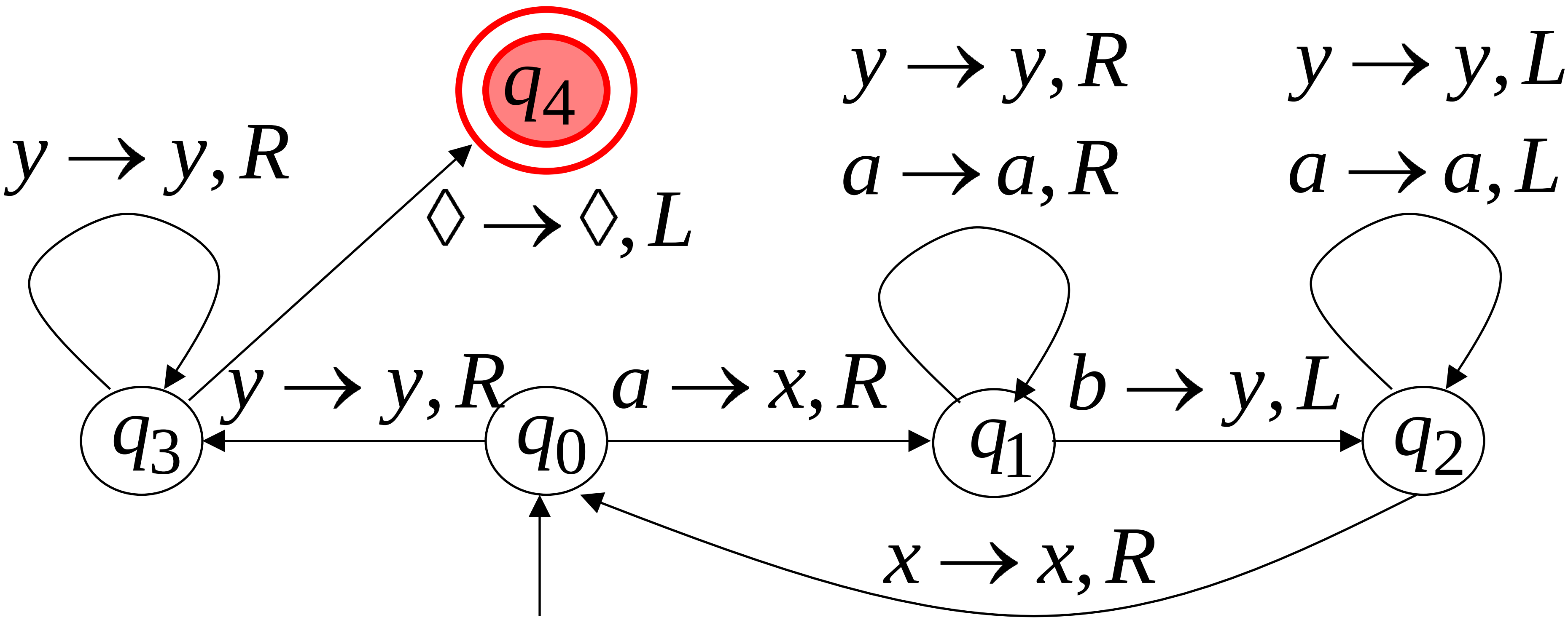

◊ x x y y ◊ ◊
q_3


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 13



Halt & Accept



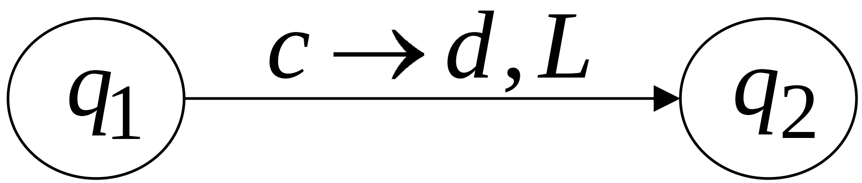
Observation:

If we modify the machine for the language $\{a^n b^n\}$

we can easily construct a machine for the language $\{a^n b^n c^n\}$

Formal Definitions for Turing Machines

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

States

Input alphabet

Tape alphabet

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

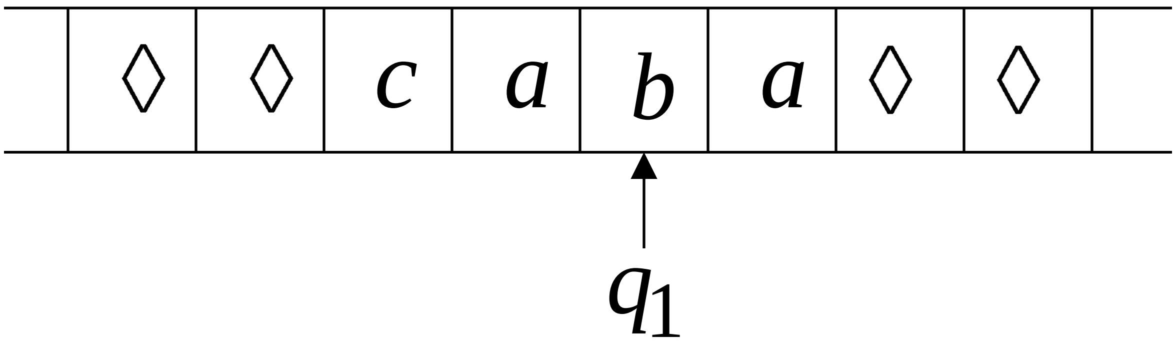
A partial Transition function

Initial state

Blank : a special symbol Of Γ

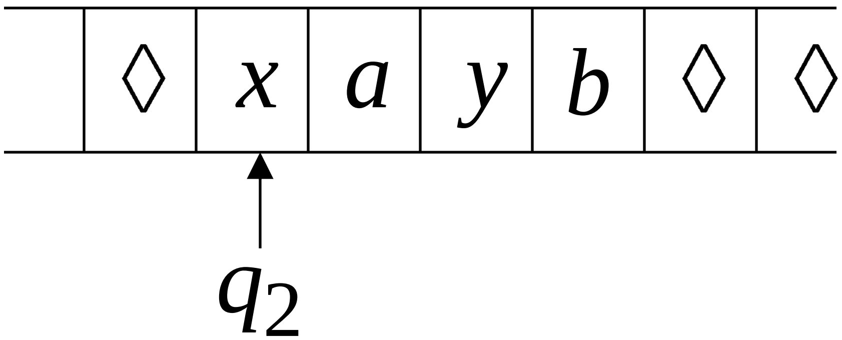
Final states

Configuration

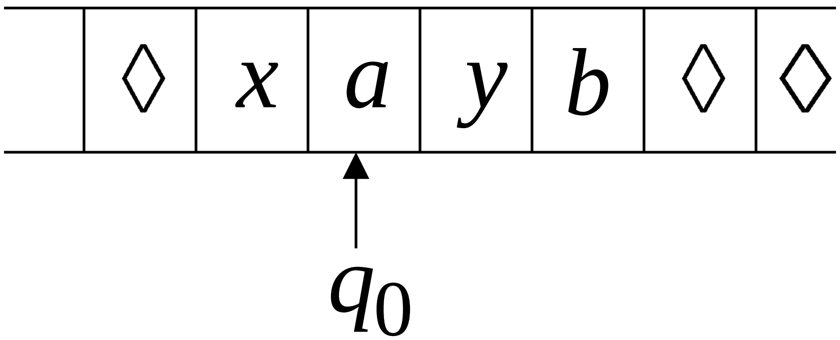


Instantaneous description: $ca\ q_1\ ba$

Time 4

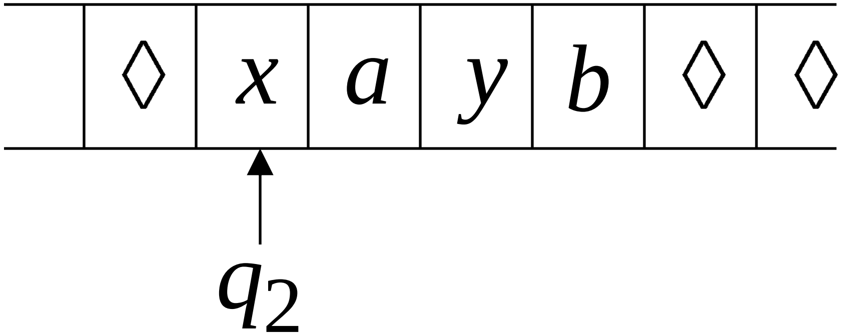


Time 5

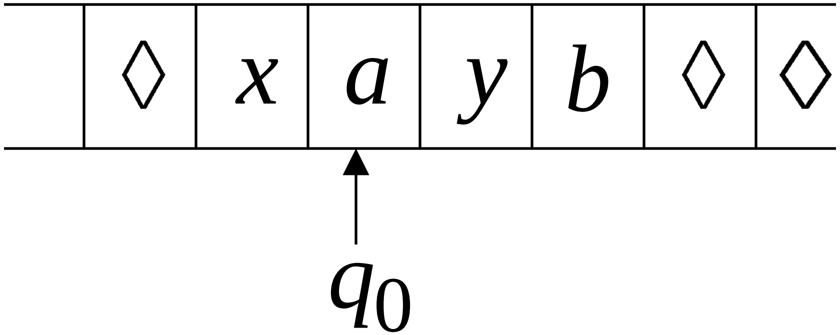


A Move: $q_2\ xayb \succ x\ q_0\ ayb$

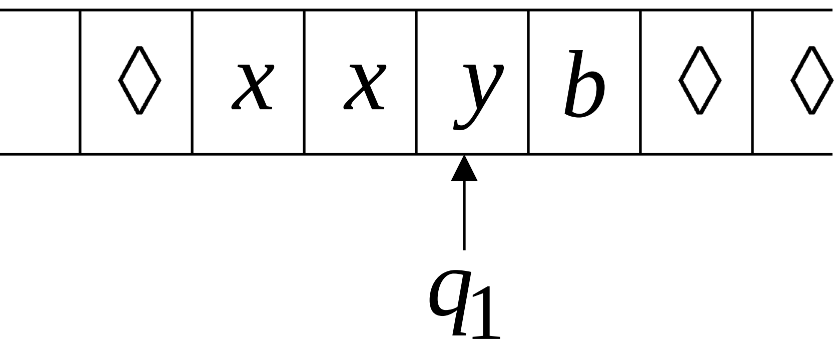
Time 4



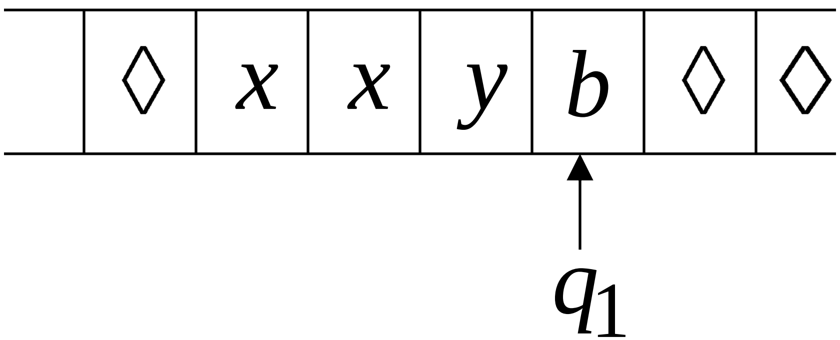
Time 5



Time 6



Time 7

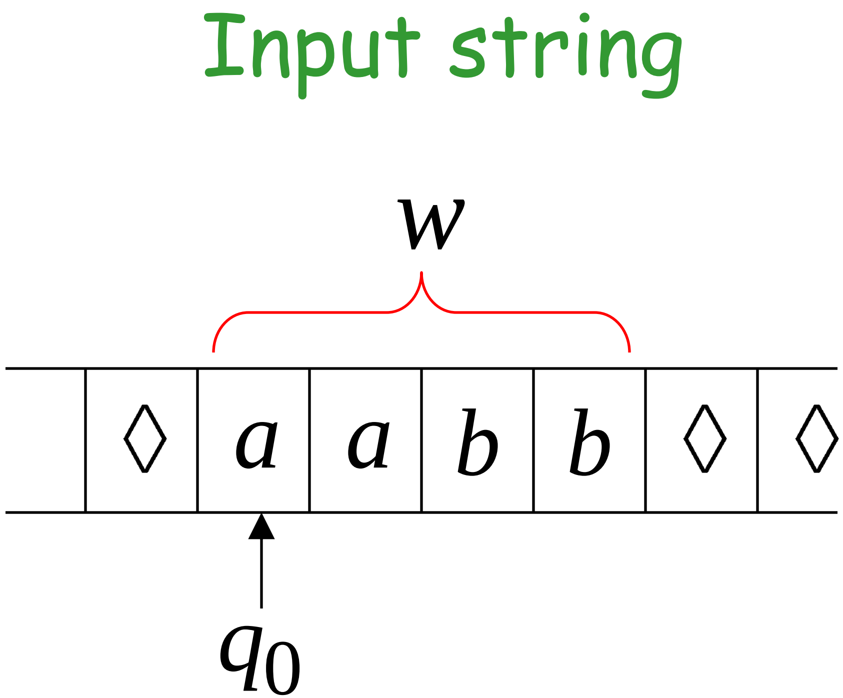


$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\ xayb \succ x\, q_0\ ayb \succ xx\, q_1\ yb \succ xxy\, q_1\ b$$

Equivalent notation: $$q_2\ xayb \overset{*}{\succ} xxy\, q_1\ b$$

Initial configuration: $q_0 \; w$



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

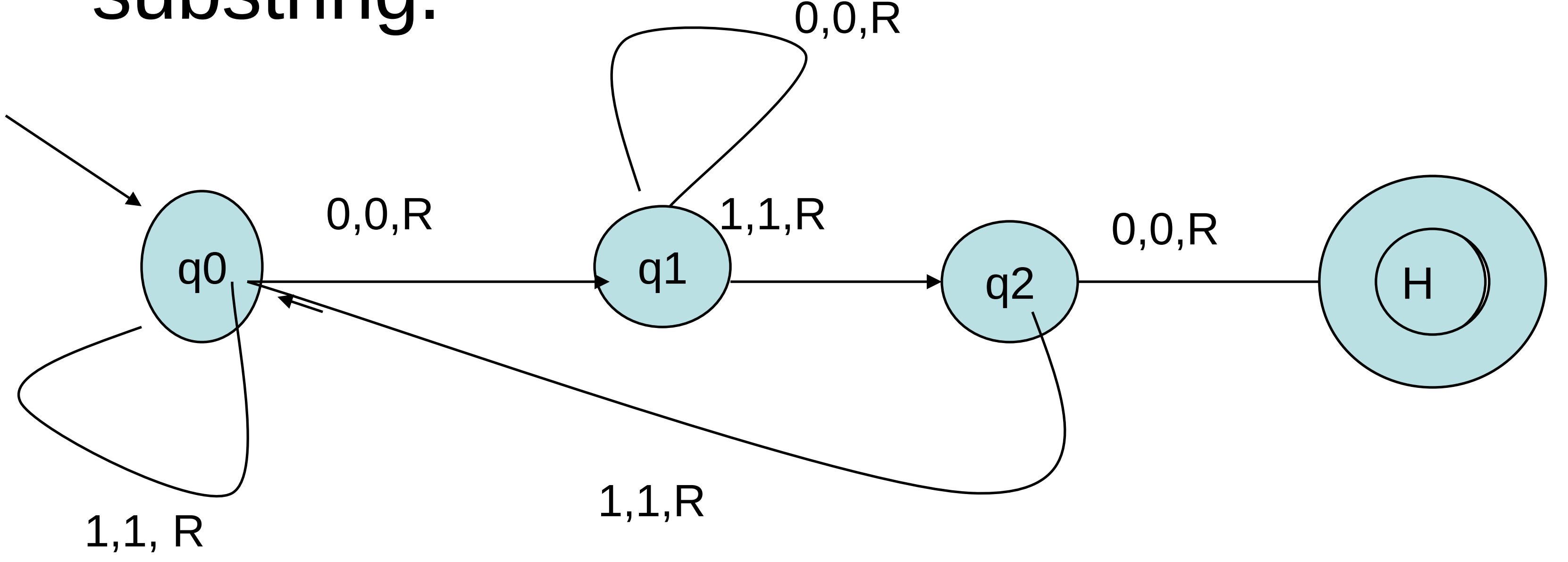
Final state

Standard Turing Machine

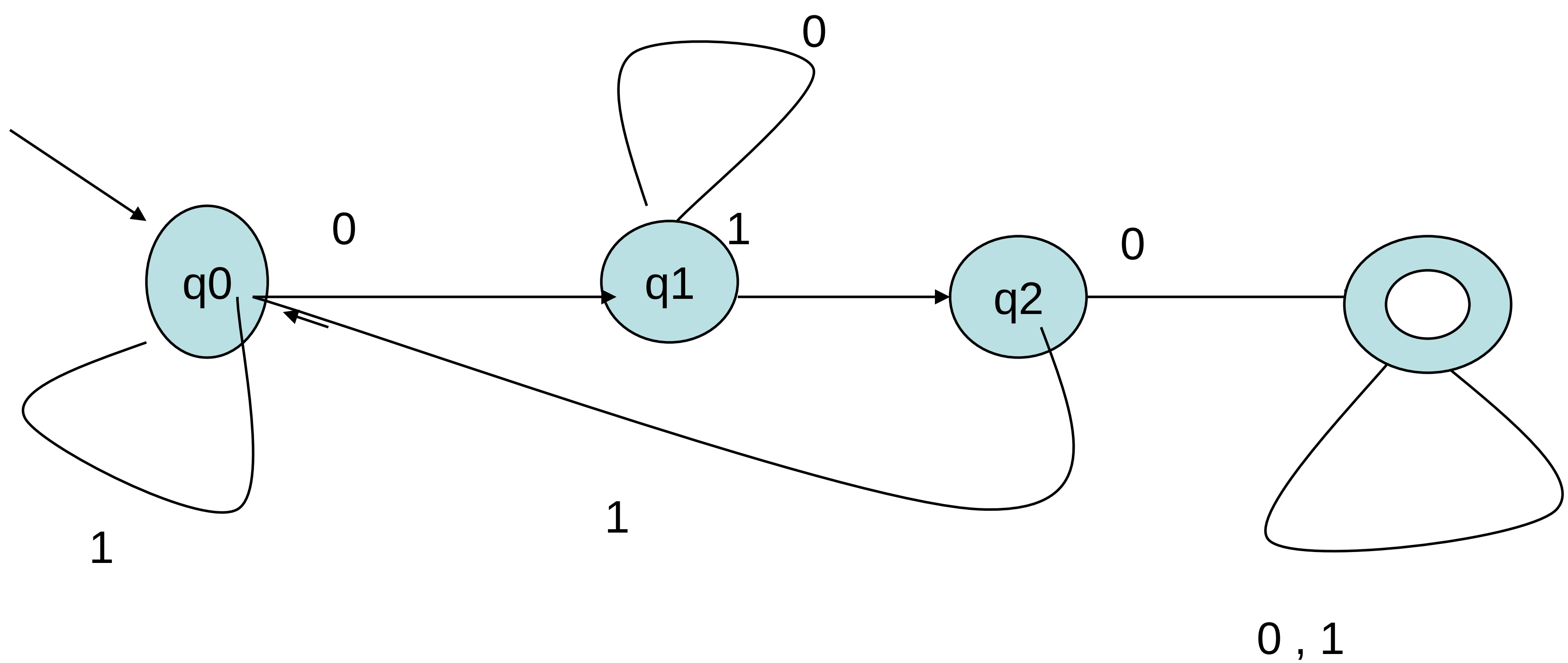
The machine we described is the standard:

- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

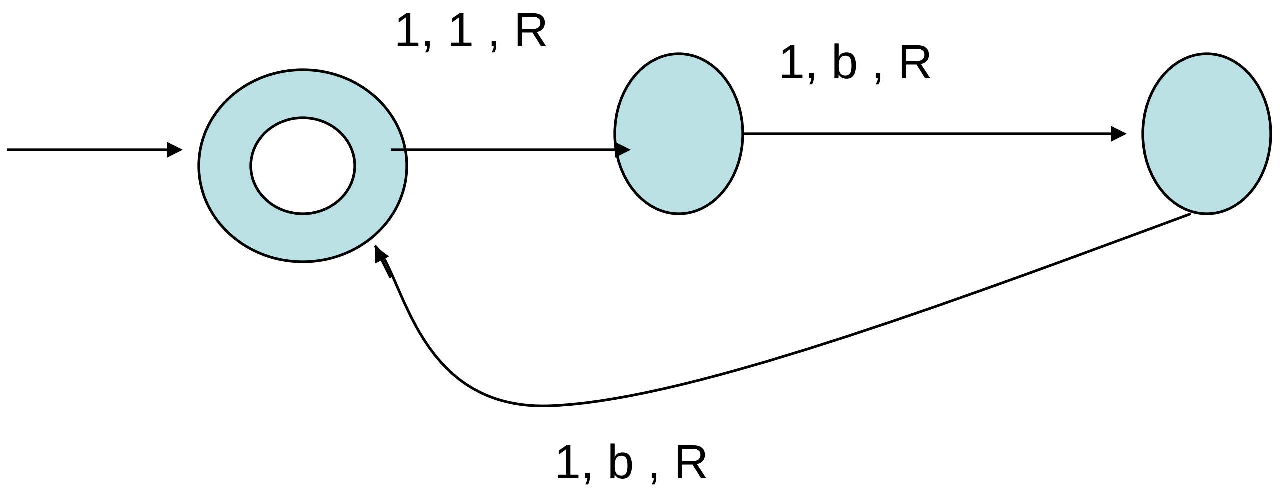
Design a Turing machine to recognize all strings in which 010 is present as a substring.



DFA for the previous language



Turing machine for odd no of 1’s



Recursively Enumerable
and
Recursive

Languages

Definition:

A language is **recursively enumerable** if some Turing machine accepts it

Let L be a recursively enumerable language

and M the Turing Machine that accepts it

For string w :

if $w \in L$ then M halts in a final state

if $w \notin L$ then M halts in a non-final state

or loops forever

Definition:

A language is **recursive**
if some Turing machine accepts it
and halts on any input string

In other words:

A language is recursive if there is
a membership algorithm for it

Let L be a recursive language

and M the Turing Machine that accepts it

For string w :

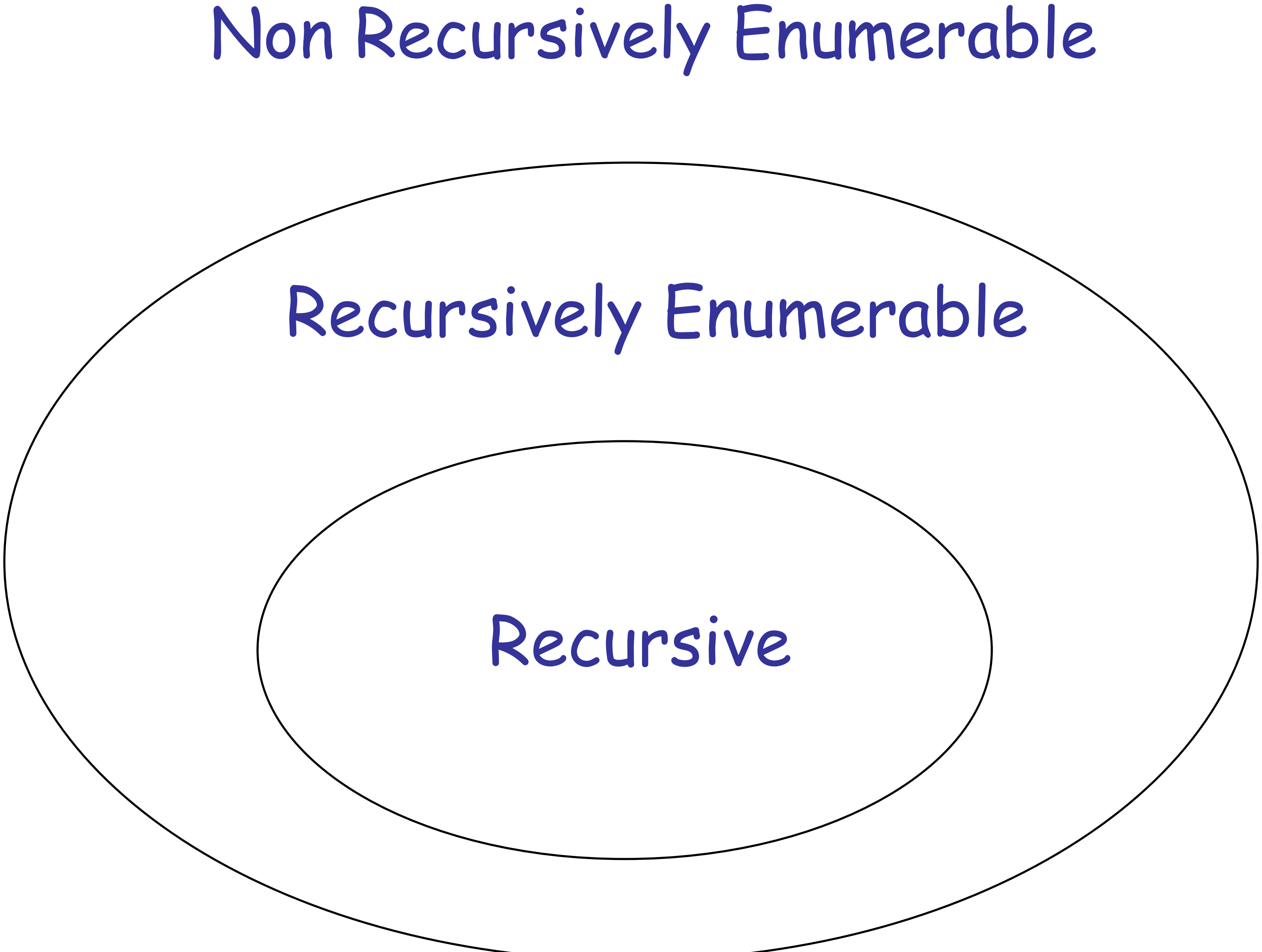
if $w \in L$ then M halts in a final state

if $w \notin L$ then M halts in a non-final state

We will prove:

1. There is a specific language
   which is not recursively enumerable
   (not accepted by any Turing Machine)

2. There is a specific language
   which is recursively enumerable
   but not recursive


Non Recursively Enumerable
Recursively Enumerable
Recursive

We will first prove:

- If a language is recursive then
  there is an enumeration procedure for it

- A language is recursively enumerable
  if and only if
  there is an enumeration procedure for it

The Chomsky Hierarchy

Unrestricted Grammars:

Productions

$$u \rightarrow v$$

String of variables and terminals

String of variables and terminals

Example unrestricted grammar:

$$S \to aBc$$

$$aB \to cA$$

$$Ac \to d$$

Theorem:

A language L is recursively enumerable if and only if L is generated by an unrestricted grammar

Context-Sensitive Grammars:

Productions

$$u \to v$$

String of variables and terminals

String of variables and terminals

and: $|u| \leq |v|$

The language $\{a^n b^n c^n\}$

is context-sensitive:

$$S \rightarrow abc \mid aAbc$$

$$Ab \rightarrow bA$$

$$Ac \rightarrow Bbcc$$

$$bB \rightarrow Bb$$

$$aB \rightarrow aa \mid aaA$$

The language $\{a^n b^n c^n\}$

is context-sensitive:

$$S \to abc \mid aAbc$$

$$Ab \to bA$$

$$Ac \to Bbcc$$

$$bB \to Bb$$

$$aB \to aa \mid aaA$$

Theorem:

A language L is context sensistive
if and only if
L is accepted by a Linear-Bounded automaton

Observation:

There is a language which is context-sensitive but not recursive

The Chomsky Hierarchy