ID2204: Constraint Programming

Introduction & Overview

Lecture 01, 2018-03-19

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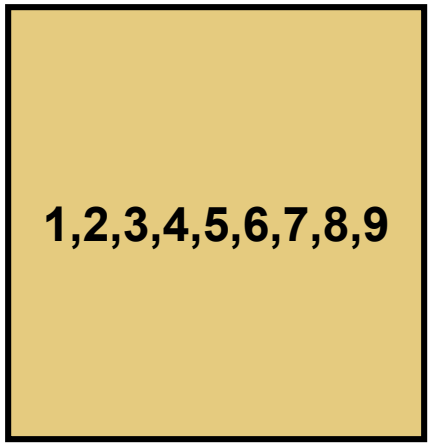
Sudoku

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Assign blank fields digits such that:
  digits distinct per rows, columns, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			



- Prune digits from fields such that:
  digits distinct per rows, columns, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

1,3,5,6,7,8

- Prune digits from fields such that:
  digits distinct per **rows**, columns, blocks

Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
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- Prune digits from fields such that:
  digits distinct per rows, **columns**, blocks

Propagation

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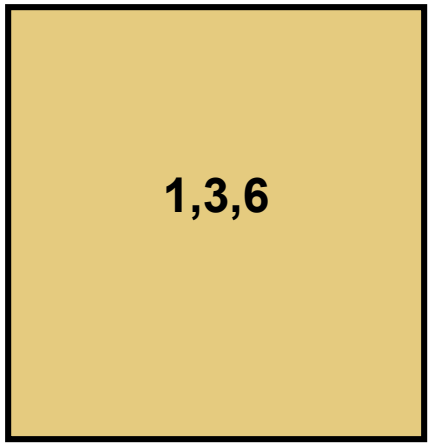


- Prune digits from fields such that:
  digits distinct per rows, columns, **blocks**

Iterated Propagation

			2		5			
	9					7	3	
		2			9		6	
2						4		9
				7				
6		9						1
	8		4			1		
	6	3					8	
			6		8			

- Iterate propagation for rows, columns, blocks
- What if no assignment: search... later

Running Example: SMM

- Find distinct digits for letters, such that

```
  SEND
+ MORE
------
= MONEY
```

Constraint Model for SMM

- Variables:

  $S,E,N,D,M,O,R,Y \in \{0,\ldots,9\}$

- Constraints:

  $$\text{distinct}(S,E,N,D,M,O,R,Y)$$

  $$\begin{array}{rr} & 1000\times S+100\times E+10\times N+D \\ + & 1000\times M+100\times O+10\times R+E \\ = & 10000\times M+1000\times O+100\times N+10\times E+Y \end{array}$$

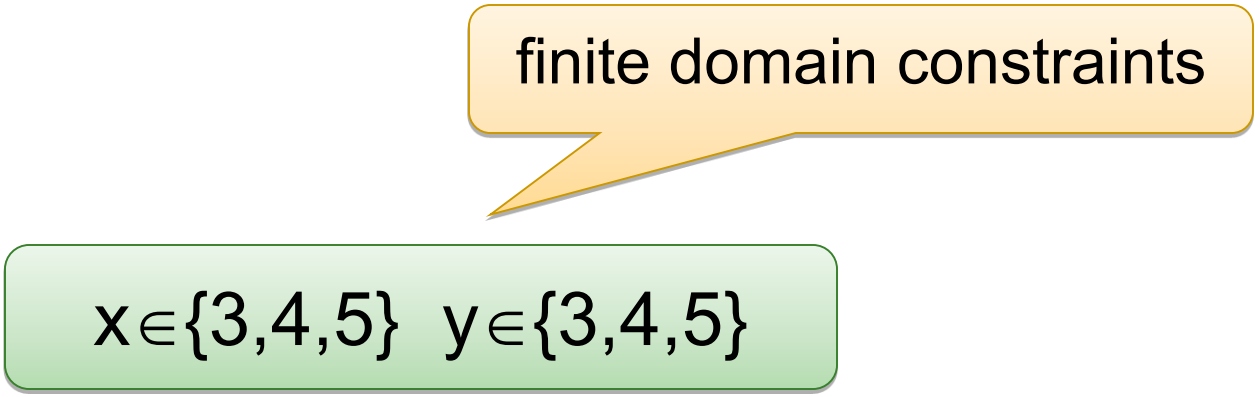
  $$S\neq 0 \qquad M\neq 0$$

Finding a Solution

- Compute with possible values
  - rather than enumerating assignments
- Prune inconsistent values
  - constraint propagation
- Search
  - branch: define search tree
  - explore: explore search tree for solution

Constraint Propagation

Constraint Store



- Maps variables to possible values
- Others: finite sets, intervals, trees, ...

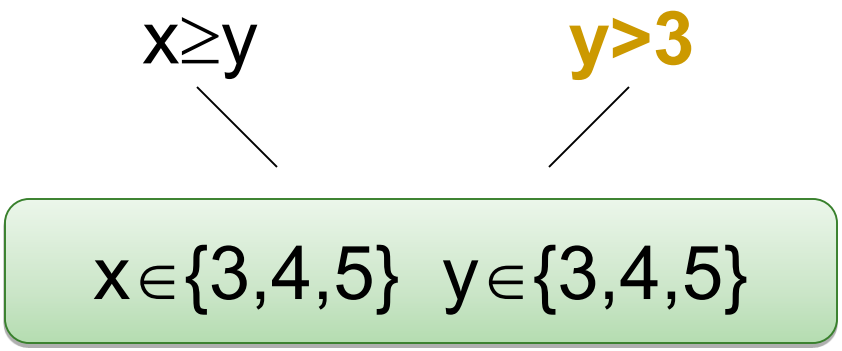
Propagators

- Implement (non-basic) constraints

  $\texttt{distinct}(x_1,\ldots,x_n)$

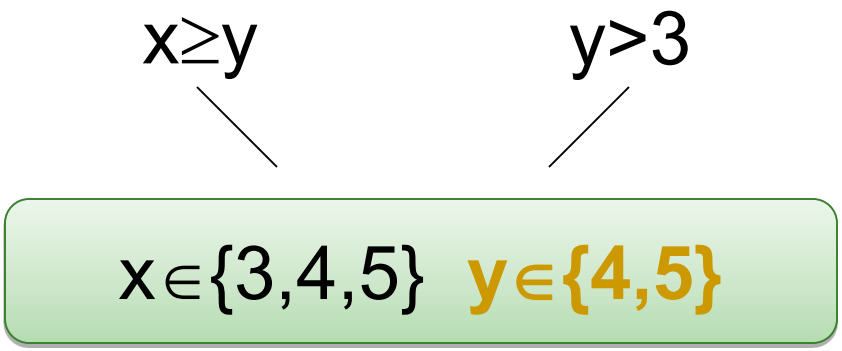
  `x + 2*y = z`

Propagators



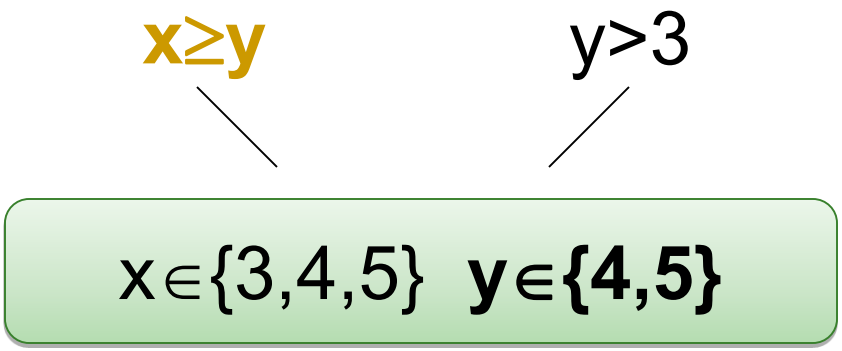
- Amplify store by constraint propagation

Propagators



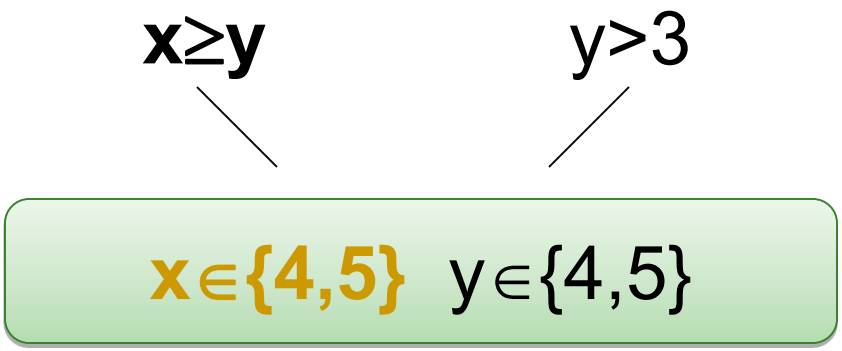
- Amplify store by constraint propagation

Propagators



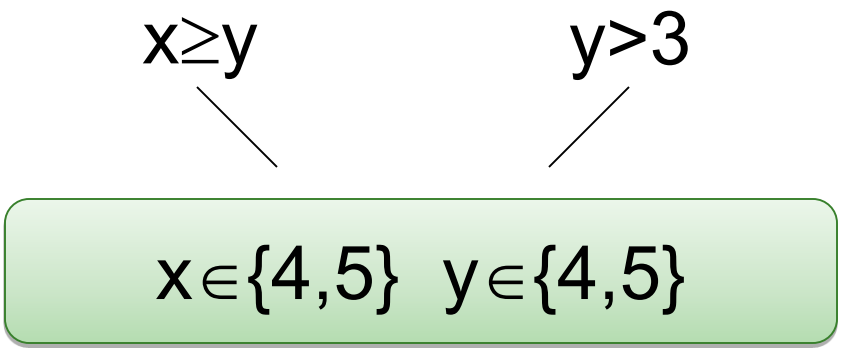
- Amplify store by constraint propagation

Propagators



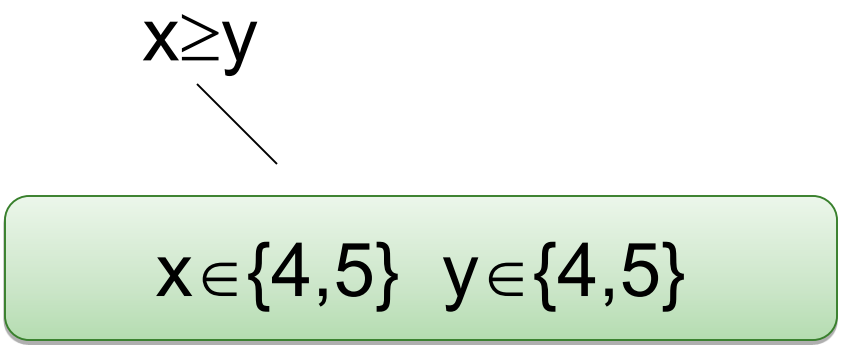
- Amplify store by constraint propagation

Propagators



- Amplify store by constraint propagation
- Disappear when done (subsumed, entailed)
  - no more propagation possible

Propagators



- Amplify store by constraint propagation
- Disappear when done (subsumed, entailed)
  - no more propagation possible

Propagation for SMM

- Results in store

```
S∈{9}  E∈{4,…,7}  N∈{5,…,8} D∈{2,…,8}
M∈{1}  O∈{0}      R∈{2,…,8} Y∈{2,…,8}
```

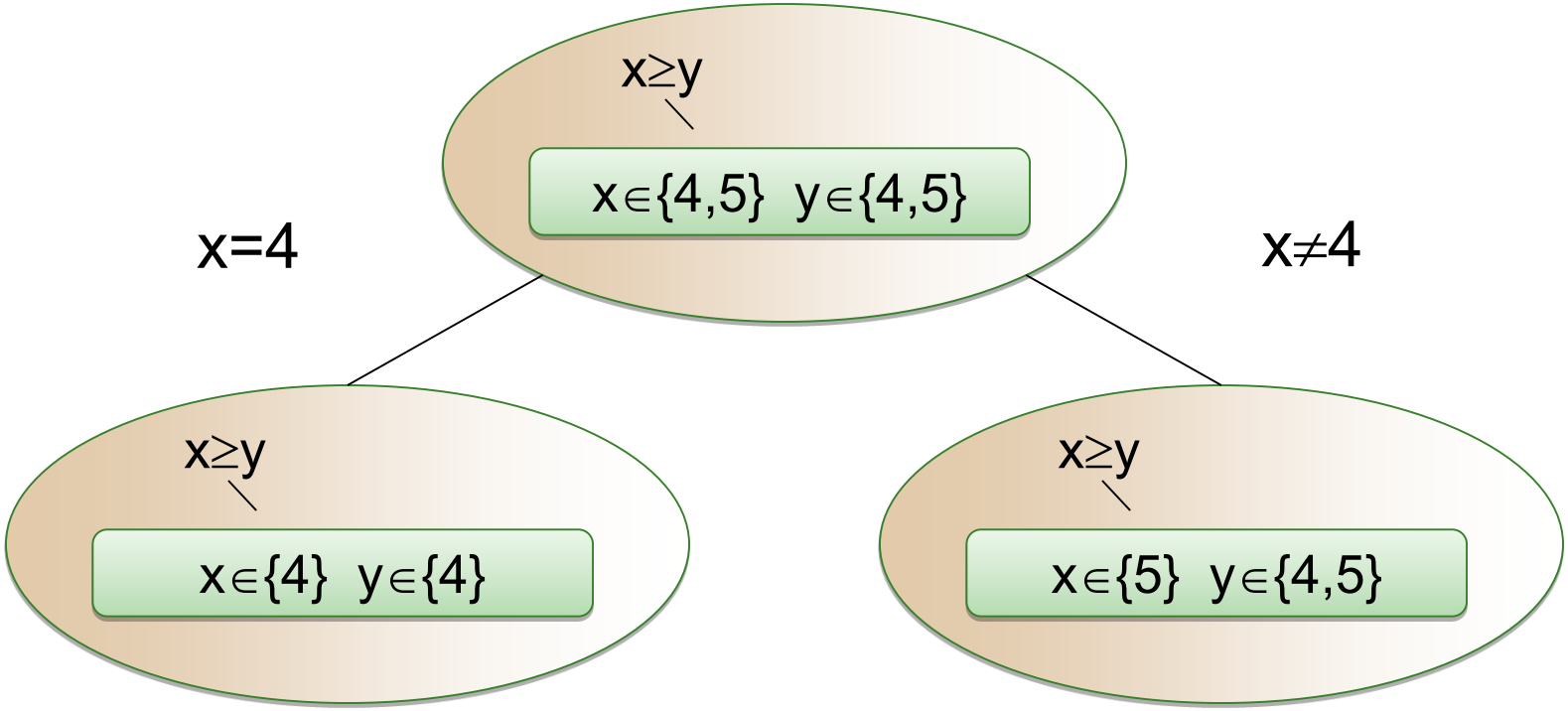
- Propagation **alone** not sufficient!
  - create simpler sub-problems
  - branching

Constraints and Propagators

- Constraints state relations among variables
  - which value combinations satisfy constraint
- Propagators implement constraints
  - prune values in conflict with constraint
- Constraint propagation drives propagators for several constraints

Search

Search: Branching



- Create subproblems with additional information
  - enable further constraint propagation

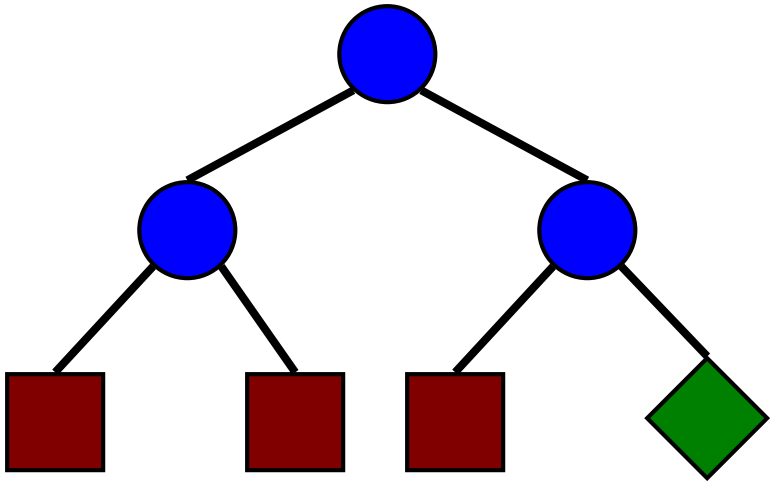
Example Branching Strategy

- Pick variable x with at least two values
- Pick value n from domain of x
- Branch with

  $x=n$ and $x \neq n$

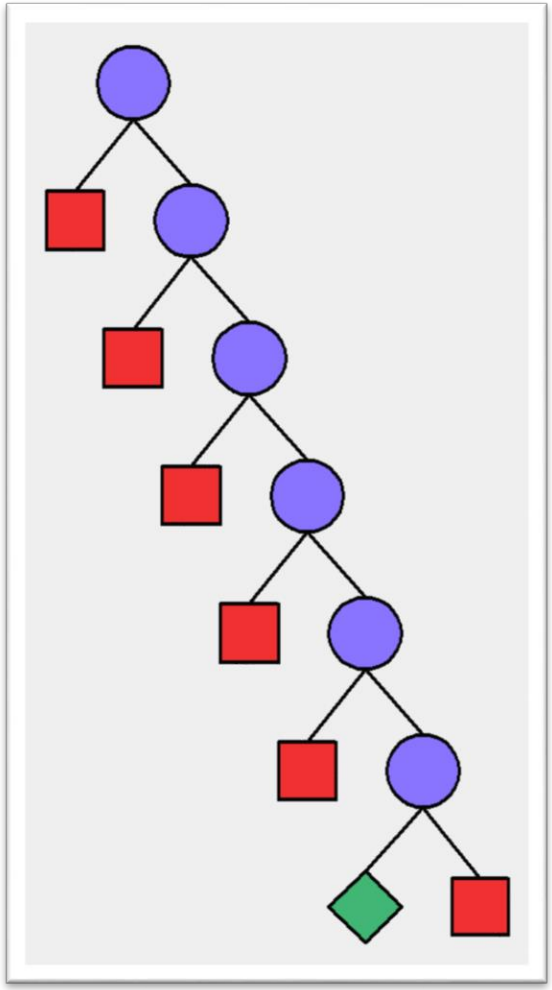
- Part of model

Search: Exploration



- Iterate propagation and branching
- Orthogonal: branching ⇆ exploration
- Nodes:
  - **Unsolved** • **Failed** • **Succeeded**

SMM: Unique Solution



```
   SEND
+  MORE
-------
= MONEY
```

```
   9567
+  1085
-------
= 10652
```

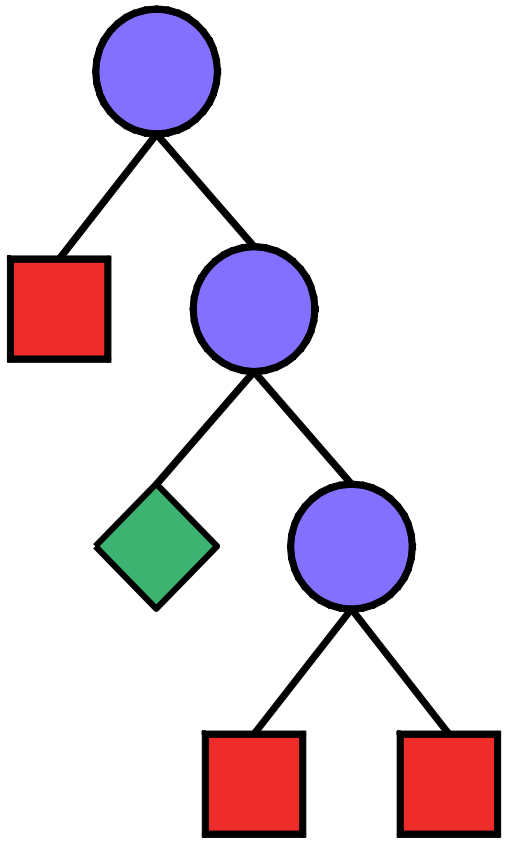
Heuristics for Branching

- Which variable
  - least possible values (first-fail)
  - application dependent heuristic
- Which value
  - minimum, median, maximum

    $x=m$ or $x\neq m$
  - split with median m

    $x<m$ or $x\geq m$
- Problem specific

SMM: Solution With First-fail



```
   SEND
+  MORE
-------
= MONEY
```

```
   9567
+  1085
-------
= 10652
```

Send Most Money (SMM++)

- Find distinct digits for letters, such that

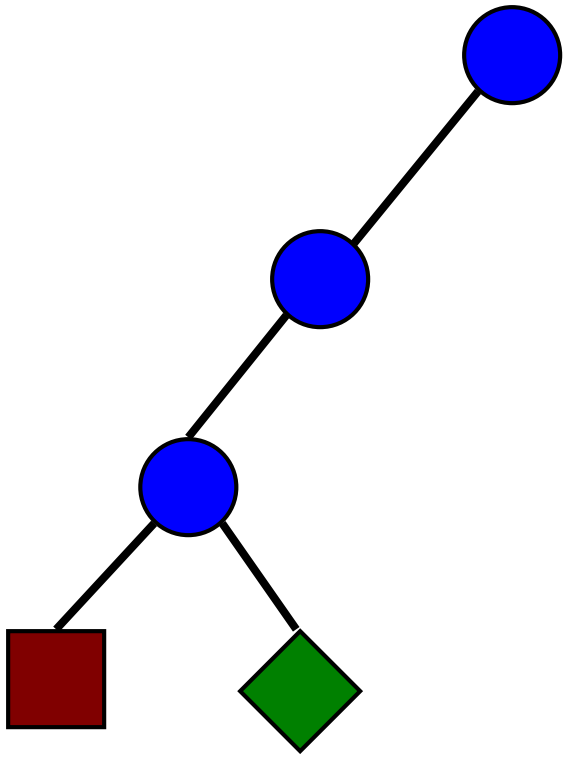
```
  SEND
+ MOST
------
= MONEY
```

and `MONEY` maximal

Best Solution Search

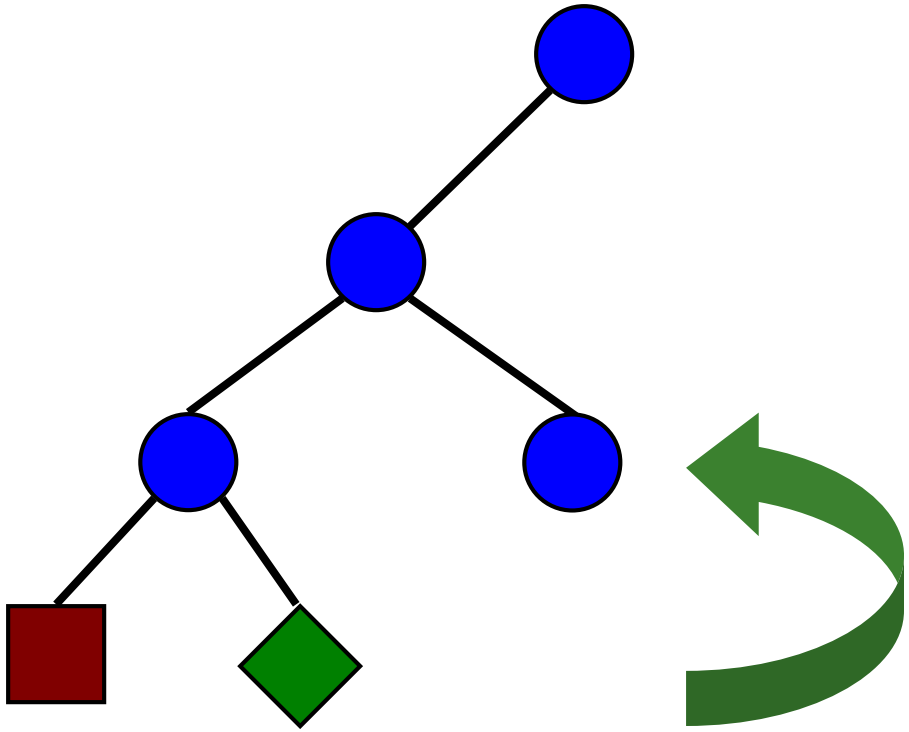
- Naïve approach:
  - compute all solutions
  - choose best
- Branch-and-bound approach:
  - compute first solution
  - add “betterness” constraint to open nodes
  - next solution will be “better”
  - prunes search space

Branch-and-bound Search



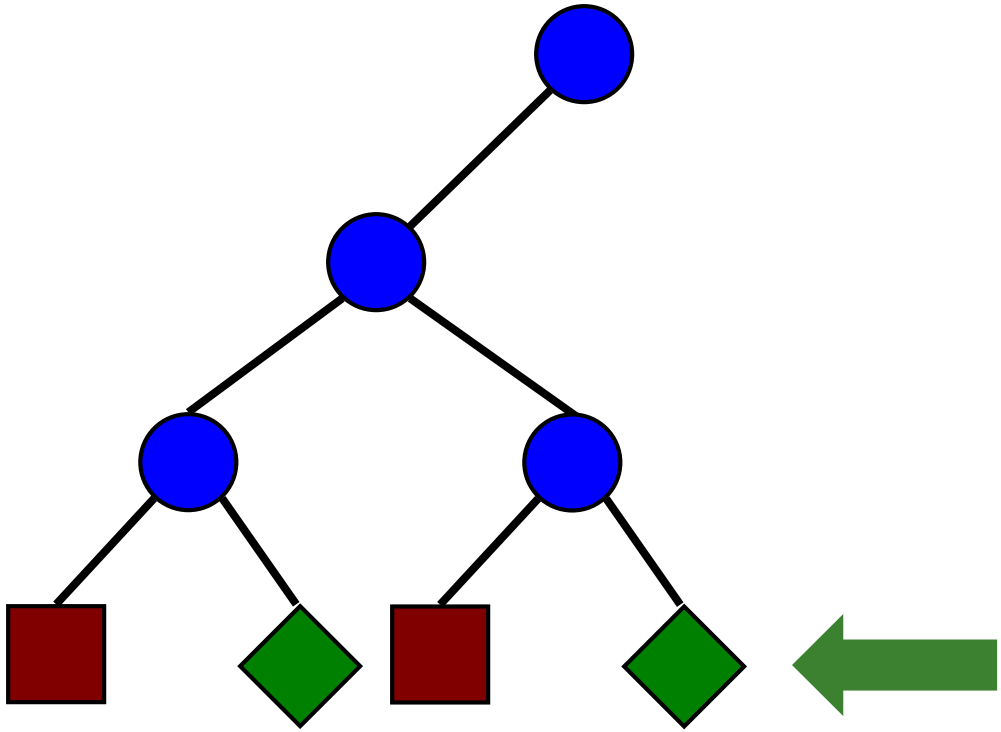
- Find first solution

Branch-and-bound Search



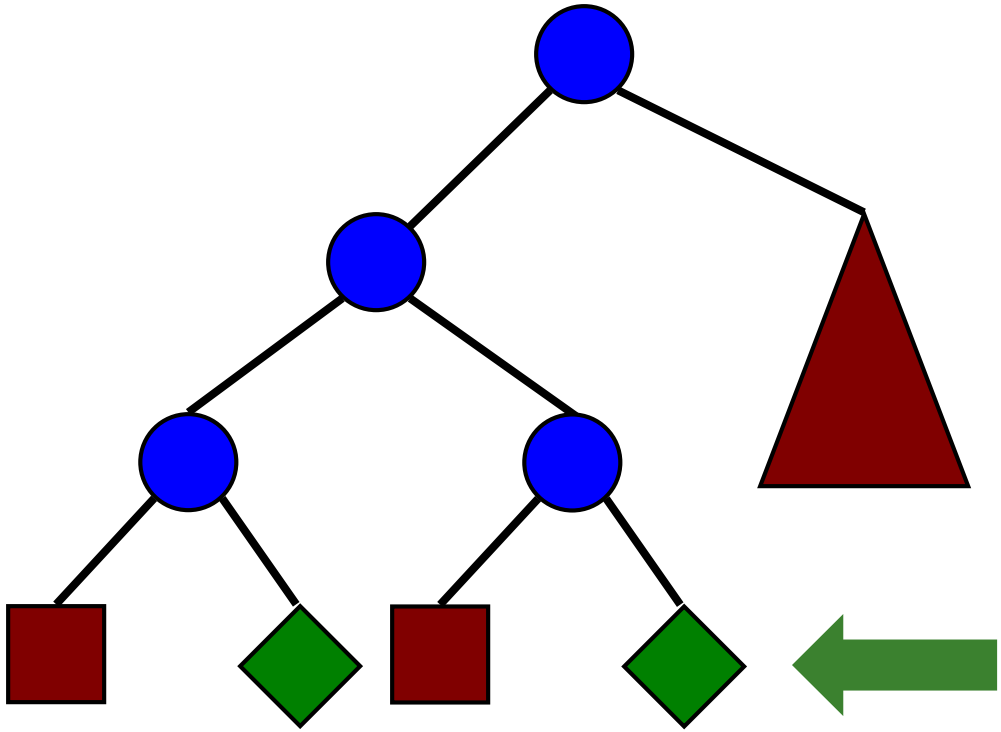
- Explore with additional constraint

Branch-and-bound Search



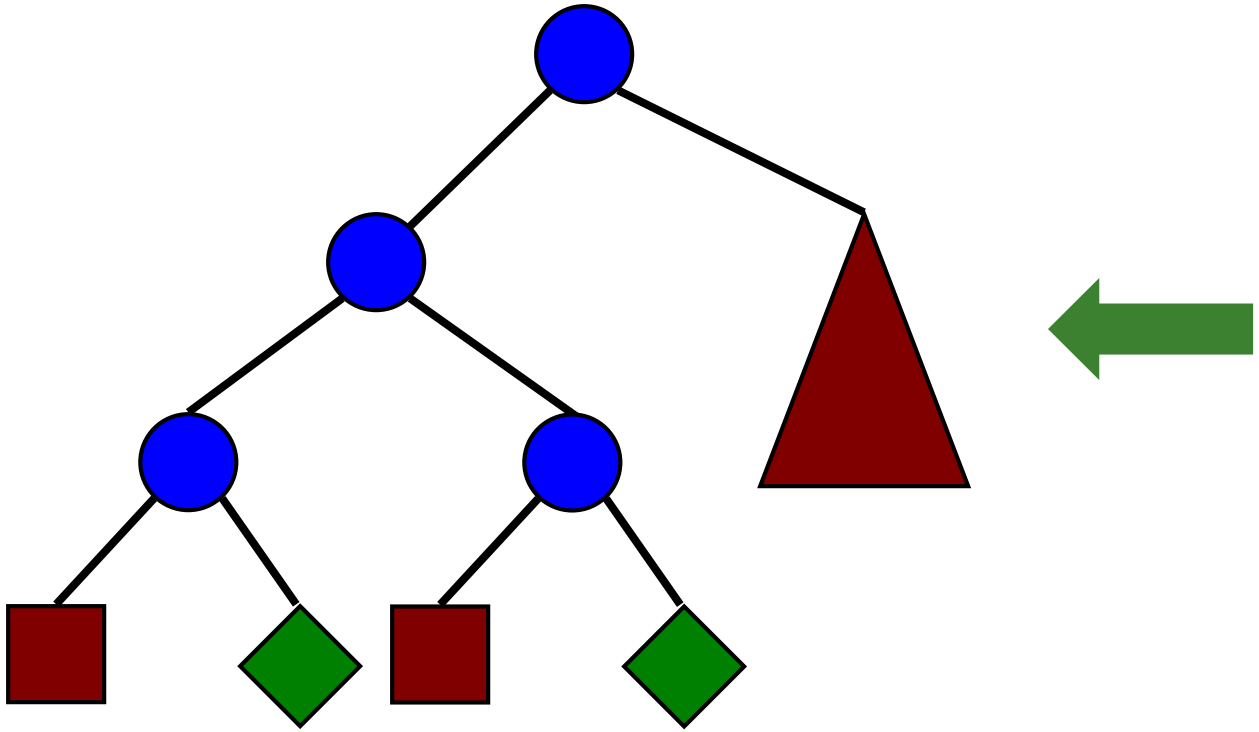
- Guarantees better solutions

Branch-and-bound Search



- Last solution best

Branch-and-bound Search



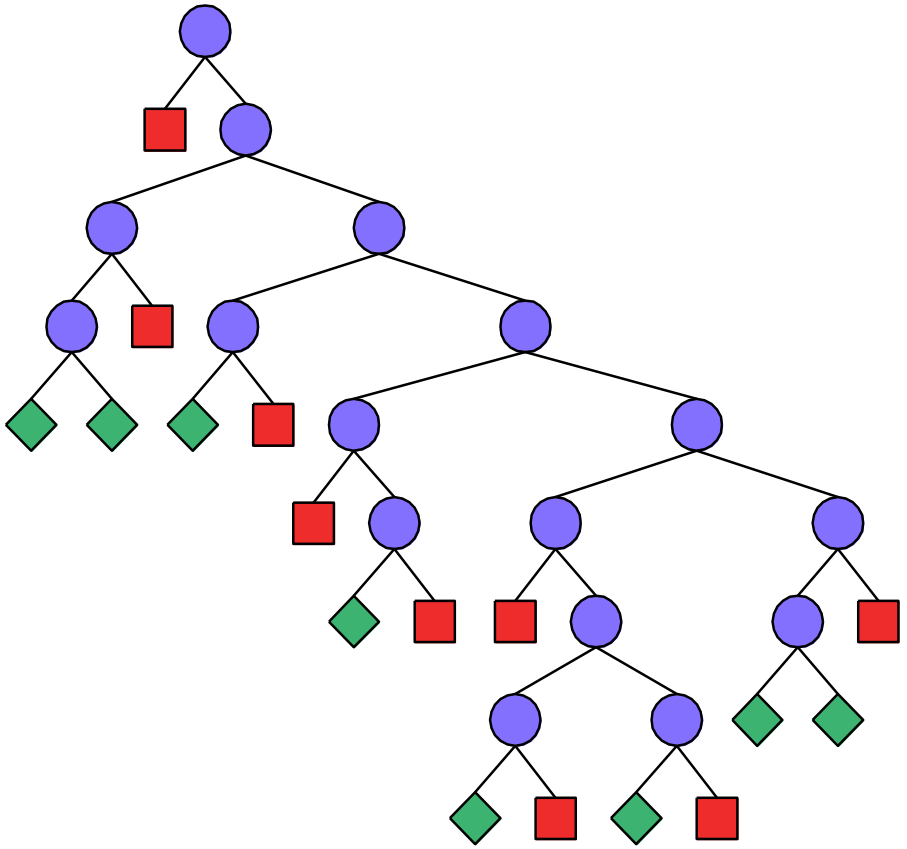
- Proof of optimality

Modelling SMM++

- Constraints and branching as before
- Order among solutions with constraints
  - so-far-best solution `S,E,N,D,M,O,T,Y`
  - current node `S,E,N,D,M,O,T,Y`
  - constraint added

$$10000 \times M + 1000 \times O + 100 \times N + 10 \times E + Y$$
$$<$$
$$10000 \times M + 1000 \times O + 100 \times N + 10 \times E + Y$$

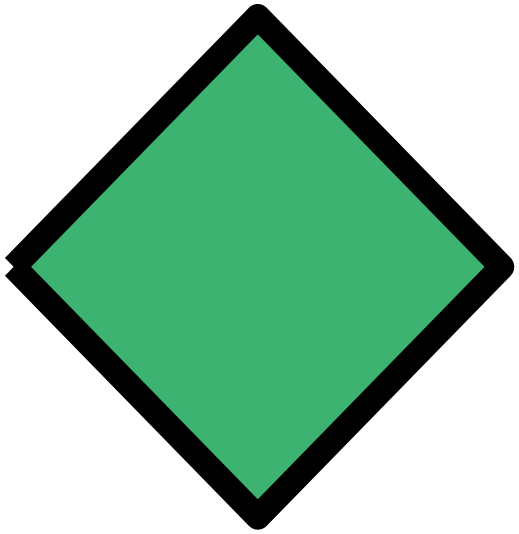
SMM++: Branch-and-bound



```
   SEND
+  MOST
------
= MONEY

   9782
+  1094
------
= 10876
```

SMM: Strong Propagation



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    9567
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Acknowledgments

- I am grateful to Pierre Flener for helpful comments and bugreports on these slides