Test Flight Problem Set Solutions

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1. False. If $n \geq 2$, then for any $m, 3m+5n \geq 13$, so we need only show that there is no m such that $3m+5=12$, i.e. no m such that $3m=7$. This is immediate.

2. True. Let $n, n+1, n+2, n+3, n+4$ be any five consecutive integers. Then
$n+(n+1)+(n+2)+(n+3)+(n+4)=5n+1+2+3+4=5n+10=5(n+2)$
which proves the result.

3. True. Consider the two case n even and n odd separately. If n is even, say $n=2k$, then

   $n^2+n+1=4k^2+2k+1=2(2k^2+k)+1$
   which is odd. If n is odd, say $n=2k+1$, then
   $n^2+n+1=(2k+1)^2+(2k+1)+1=4k^2+4k+1+2k+1+1=4k^2+6k+2+1=2(2k^2+3k+1)+1$
   which is odd. In both cases, n^2+n+1 is odd.

4. Let m be a natural number. By the Division Theorem, there are unique numbers n, r such that $m=4n+r$, where $0 \leq r < 4$. Thus m is one of $4n, 4n+1, 4n+2, 4n+3$. Since $4n$ and $4n+2$ are even, if m is odd, the only possibilities are $4n+1$ and $4n+3$.

5. By the Division Theorem, n can be expressed in one of the forms $3q, 3q+1, 3q+2$, for some q. In the first case, n is divisible by 3. In the second case $n+2=3q+3=3(q+1)$, so $n+2$ is divisible by 3. In the third case $n+4=3q+6=3(q+2)$, so $n+4$ is divisible by 3.

6. Consider any three numbers of the form n, $n+2$, $n+4$, where $n>3$. By the answer to the previous question, one of these numbers is divisible by 3, and hence is not prime.

7. Let $S=2+2^2+2^3+...+2^n$. Then $2S=2^2+2^3+2^4+...+2^n+2^{n+1}$. Subtracting the first identity from the second gives $2S-S=2^{n+1}-2$. But $2S-S=S$, so this establishes the stated identity.

8. Let ϵ > 0 be given. By the assumption, we can find an N such that
$n \geq N \Rightarrow |a_n - L| < \epsilon / M$
Then,

$n \geq N \Rightarrow |Ma_n - ML| = M.|a_n - L| < M.\epsilon/M = \epsilon$
which shows that $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML

9. Let $A_n=(0,1/n)$. Clearly, $\bigcap_{n=1}^{\infty} A_n \subseteq A_1=(0,1)$. Hence any element of the intersection must be a member of $(0,1)$. But if $x \in (0,1)$, we can find a natural number n such that $1/n<x$. Then $x \notin A_n$, so $x \notin \bigcap_{n=1}^{\infty} A_n$. Thus $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

10. Let $A_n = [0, 1/n)$. Clearly, $0 \in \bigcap_{n=1}^{\infty} A_n$. But the same argument as above shows that no other number is in the intersection. Hence $\bigcap_{n=1}^{\infty} A_n = \{0\}$.