Multiple Linear Regression

Multiple linear regression attempts to model the relationship between two or more explanatory variables and a response variable by fitting a linear equation to observed data. Every value of the independent variable x is associated with a value of the dependent variable y. The population regression line for p explanatory variables $x_1, x_2, \ldots, x_p$ is defined to be $\mu_y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p$. This line describes how the mean response μ_y changes with the explanatory variables. The observed values for y vary about their means μ_y and are assumed to have the same standard deviation σ. The fitted values $b_0, b_1, \ldots, b_p$ estimate the parameters $\beta_0, \beta_1, \ldots, \beta_p$ of the population regression line.

Since the observed values for y vary about their means μ_y, the multiple regression model includes a term for this variation. In words, the model is expressed as DATA = FIT + RESIDUAL, where the "FIT" term represents the expression $\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_p x_p$. The "RESIDUAL" term represents the deviations of the observed values y from their means μ_y, which are normally distributed with mean 0 and variance σ. The notation for the model deviations is ε.

Formally, the model for multiple linear regression, given *n* observations, is
$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots \beta_p x_{ip} + \varepsilon_i$ **for** $i = 1,2, \ldots n$.

In the least-squares model, the best-fitting line for the observed data is calculated by minimizing the sum of the squares of the vertical deviations from each data point to the line (if a point lies on the fitted line exactly, then its vertical deviation is 0). Because the deviations are first squared, then summed, there are no cancellations between positive and negative values. The least-squares estimates $b_0, b_1, \ldots b_p$ are usually computed by statistical software.

The values fit by the equation $b_0 + b_1 x_{i1} + \ldots + b_p x_{ip}$ are denoted $\hat{y}_i$, and the residuals e_i are equal to $y_i - \hat{y}_i$, the difference between the observed and fitted values. The sum of the residuals is equal to zero.

The variance σ^2 may be estimated by $s^2 = \dfrac{\sum e_i^2}{n - p - 1}$, also known as the mean-squared error (or MSE).

The estimate of the standard error s is the square root of the MSE.

Example

The dataset "Healthy Breakfast" contains, among other variables, the *Consumer Reports* ratings of 77 cereals and the number of grams of sugar contained in each serving. (*Data source: Free publication available in many grocery stores. Dataset available through the Statlib Data and Story Library (DASL).*)

A simple linear regression model considering "Sugars" as the explanatory variable and "Rating" as the response variable produced the regression line
Rating = 59.3 - 2.40 Sugars, with the square of the correlation $r^2 = 0.577$ (see Inference in Linear Regression for more details on this regression).

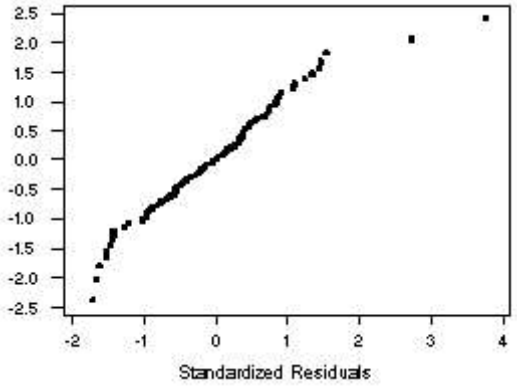
The "Healthy Breakfast" dataset includes several other variables, including grams of fat per serving and grams of dietary fiber per serving. Is the model significantly improved when these variables are included?

Suppose we are first interested in adding the "Fat" variable. The correlation between "Fat" and "Rating" is equal to -0.409, while the correlation between "Sugars" and "Fat" is equal to 0.271. Since "Fat" and "Sugar" are not highly correlated, the addition of the "Fat" variable may significantly improve the model.

The MINITAB "Regress" command produced the following results:

```
Regression Analysis

The regression equation is
Rating = 61.1 - 3.07 Fat - 2.21 Sugars
```



After fitting the regression line, it is important to investigate the residuals to determine whether or not they appear to fit the assumption of a normal distribution. A normal quantile plot of the standardized residuals $y - \hat{y}$ is shown to the left. Despite two large values which may be outliers in the data, the residuals do not seem to deviate from a random sample from a normal distribution in any systematic manner.

The MINITAB output provides a great deal of information. Under the equation for the regression line, the output provides the least-squares estimates for each parameter, listed in the "Coef" column next to the variable to which it corresponds. The calculated standard deviations are provided in the second column.

Predictor	Coef	StDev	T	P
Constant	61.089	1.953	31.28	0.000
Fat	-3.066	1.036	-2.96	0.004
Sugars	-2.2128	0.2347	-9.43	0.000

S = 8.755 R-Sq = 62.2% R-Sq(adj) = 61.2%

Significance Tests

The third column "T" of the MINITAB "REGRESS" output provides test statistics. As in linear regression, one wishes to test the significance of the parameters included. For any of the variables x_j included in a multiple regression model, the null hypothesis states that the coefficient β_j is equal to 0. The alternative hypothesis may be one-sided or two-sided, stating that β_j is either less than 0, greater than 0, or simply not equal to 0.

The test statistic t is equal to b_j/s_{bj}, the parameter estimate divided by its standard deviation. This value follows a *t(n-p-1)* distribution when *p* variables are included in the model.

In the example above, the parameter estimate for the "Fat" variable is -3.066 with standard deviation 1.036 The test statistic is t = -3.066/1.036 = -2.96, provided in the "T" column of the MINITAB output. For a two-sided test, the probability of interest is $2P(T \geq |-2.96|)$ for the $t(77-2-1) = t(74)$ distribution, which is about 0.004. The "P" column of the MINITAB output provides the *P*-value associated with the two-sided test. Since the *P*-values for both "Fat" and "Sugar" are highly significant, both variables may be included in the model.

Condidence Intervals for Regression Parameters

A level C confidence interval for the parameter β_j may be computed from the estimate b_j using the computed standard deviations and the appropriate critical value t^* *from the t(n-p-1) distribution.* *The confidence interval for β_j takes the form $b_j \pm t^* s_{bj}$.*

Continuing with the "Healthy Breakfast" example, suppose we choose to add the "Fiber" variable to our model. The MINITAB results are the following:

```
Regression Analysis

The regression equation is
Rating = 53.4 - 3.48 Fat + 2.95 Fiber - 1.96 Sugars

Predictor       Coef       StDev          T        P
Constant      53.437       1.342      39.82    0.000
Fat          -3.4802      0.6209      -5.61    0.000
Fiber         2.9503      0.2549      11.57    0.000
Sugars       -1.9640      0.1420     -13.83    0.000

S = 5.235       R-Sq = 86.7%     R-Sq(adj) = 86.1%
```

The squared multiple correlation R^2 is now equal to 0.861, and all of the variables are significant by the t tests. Examination of the residuals indicates no unusual patterns. The inclusion of the "Fat," "Fiber," and "Sugars" variables explains 86.7% of the variability of the data, a significant improvement over the smaller models.

For additional tests and a continuation of this example, see ANOVA for Multiple Linear Regression.

RETURN TO MAIN PAGE.