IIITB COMET FOUNDATION

Harshita N Kumar
ID: COMETFWC052

CBSE Class XII
Task–2 : IMO 2003 Problems

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1. Let

   $S = \{1, 2, 3, \ldots, 1000000\}$.

   Show that for any subset A of S with 101 elements, we can find 100 distinct elements x_i of S such that the sets

   $\{a + x_i \mid a \in A\}$

   are all pairwise disjoint.

2. Find all pairs (m, n) of positive integers such that

   $$\frac{m^2}{2mn^2 - n^3 + 1}$$

   is a positive integer.

3. A convex hexagon has the property that for any pair of opposite sides, the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Show that all the hexagon's angles are equal.

4. $ABCD$ is cyclic. The feet of the perpendiculars from D to the lines AB, BC, CA are P, Q, R respectively.

   Show that the angle bisectors of $\angle ABC$ and $\angle CDA$ meet on the line AC if and only if

   $RP = RQ$.

5. Given $n > 2$ and real numbers

   $x_1 \leq x_2 \leq \cdots \leq x_n$,

   show that

   $$\left(\sum_{i<j} |x_i - x_j|\right)^2 \leq \frac{2}{3}(n^2 - 1)\sum_{i<j}(x_i - x_j)^2.$$

   Show that equality holds if and only if the sequence is an arithmetic progression.

6. Show that for each prime p, there exists a prime q such that

   $n^p - p$

   is not divisible by q for any positive integer n.