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CBSE Class XII
Task 2

43^{rd} *International Mathematical Olympiad*

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Each problem is worth seven points.

1. S is the set of all (h, k) with h, k non-negative integers such that $h+k < n$. Each element of S is colored red or blue, so that if (h, k) is red then (h', k') is also red for all $h' \leq h,\ k' \leq k$.

   A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

2. BC is a diameter of a circle with center O. A is any point on the circle with $\angle AOC > 60°$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

3. Find all pairs of integers $m > 2,\ n > 2$ such that there are infinitely many positive integers k for which $k^m + k^2 - 1 \mid k^n + k - 1$.

4. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \cdots < d_k$, so that $d_1 = 1,\ d_k = n$. Let $d = d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

5. Find all real-valued functions on the reals such that $f(x + y)(f(x) + f(y)) = f(xy) + f(x) + f(y)$ for all real numbers x, y.

6. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \ldots, O_n$. Show that $\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}$.