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CBSE Class XII
Task 2

44^{th} *International Mathematical Olympiad*

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Each problem is worth seven points.

1. S is the set $\{1, 2, 3, \dots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S such that the sets $\{a + x_i \mid a \in A\}$ are all pairwise disjoint.

2. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2m^2-n^2+1}$ is a positive integer.

3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Show that all the hexagon's angles are equal.

4. $ABCD$ is cyclic. The feet of the perpendiculars from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of $\angle ABC$ and $\angle CDA$ meet on the line AC if and only if $RP = RQ$.

5. Given $n > 2$ and reals $x_1 \leq x_2 \leq \cdots \leq x_n$, show that $\sum_{i<j} |x_i - x_j|^2 \leq \frac{n^2-1}{3} \sum_{i<j} (x_i - x_j)^2$. Show that equality holds if and only if the sequence is an arithmetic progression.

6. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.