Fundamental concepts of probability

Recently, you learned that **probability** uses math to quantify uncertainty, or to describe the likelihood of something happening. For example, there might be an 80% chance of rain tomorrow, or a 20% chance that a certain candidate wins an election.

In this reading, you'll learn more about fundamental concepts of probability. We'll discuss the concept of a random experiment, how to represent and calculate the probability of an event, and basic probability notation.

Probability fundamentals

Foundational concepts: Random experiment, outcome, event

Let's begin with three concepts at the foundation of probability theory:

- Random experiment
- Outcome
- Event

Probability deals with what statisticians call random experiments, also known as statistical experiments. A **random experiment** is a process whose outcome cannot be predicted with certainty. For example, before tossing a coin or rolling a die, you can't know the result of the toss or the roll. The result of the coin toss might be heads or tails. The result of the die roll might be 3 or 6.

All random experiments have three things in common:

- The experiment can have more than one possible outcome.
- You can represent each possible outcome in advance.
- The outcome of the experiment depends on chance.

In statistics, the result of a random experiment is called an outcome. For example, if you roll a die, there are six possible outcomes: 1, 2, 3, 4, 5, 6.

An event is a set of one or more outcomes. Using the example of rolling a die, an event might be rolling an even number. The event of rolling an even number consists of the outcomes 2, 4, 6. Or, the event of rolling an odd number consists of the outcomes 1, 3, 5.

In a random experiment, an event is assigned a probability. Let’s explore how to represent and calculate the probability of a random event.

The probability of an event

The probability that an event will occur is expressed as a number between 0 and 1. Probability can also be expressed as a percent.

- If the probability of an event equals 0, there is a 0% chance that the event will occur.
- If the probability of an event equals 1, there is a 100% chance that the event will occur.

There are different degrees of probability between 0 and 1. If the probability of an event is close to zero, say 0.05 or 5%, there is a small chance that the event will occur. If the probability of an event is close to 1, say 0.95 or 95%, there is a strong chance that the event will occur. If the probability of an event equals 0.5, there is a 50% chance that the event will occur—or not occur.

Knowing the probability of an event can help you make informed decisions in situations of uncertainty. For example, if the chance of rain tomorrow is 0.1 or 10%, you can feel confident about your plans for an outdoor picnic. However, if the chance of rain is 0.9 or 90%, you may want to think about rescheduling your picnic for another day.

Calculate the probability of an event

To calculate the probability of an event in which all possible outcomes are equally likely, you divide the number of desired outcomes by the total number of possible outcomes. You may recall that this is also the formula for classical probability:

# of desired outcomes ÷ total # of possible outcomes

Let’s explore the coin toss and die roll examples to get a better idea of how to calculate the probability of a single random event.

Example: Coin toss

Tossing a fair coin is a classic example of a random experiment:

- There is more than one possible outcome.
- You can represent each possible outcome in advance: heads or tails.
- The outcome depends on chance. The toss could turn up heads or tails.

Say you want to calculate the probability of getting heads on a single toss. For any given coin toss, the probability of getting heads is one chance out of two. This is 1 ÷ 2 = 0.5, or 50%.

Now imagine that you were to toss a specially designed coin that had heads on both sides. Every time you toss this coin it will turn up heads. In this case, the probability of getting heads is 100%. The probability of getting tails is 0%.

Note that when you say the probability of getting heads is 50%, you aren't claiming that any actual sequence of coin tosses will result in exactly 50% heads. For example, if you toss a fair coin ten times, you may get 4 heads and 6 tails, or 7 heads and 3 tails. However, if you continue to toss the coin, you can expect the long-run frequency of heads to get closer and closer to 50%.

Example: Die roll

Rolling a six-sided die is another classic example of a random experiment:

- There is more than one possible outcome.
- You can represent all possible outcomes in advance: 1, 2, 3, 4, 5, and 6.
- The outcome depends on chance. The roll could turn up any number 1–6.

Say you want to calculate the probability of rolling a 3. For any given die roll, the probability of rolling a 3 is one chance out of six. This is 1 ÷ 6 = 0.1666, or about 16.7%.

Probability notation

It helps to be familiar with probability notation as it's often used to symbolize concepts in educational and technical contexts.

In notation, the letter P indicates the probability of an event. The letters A and B represent individual events.

For example, if you're dealing with two events, you can label one event A and the other event B.

- The probability of event A is written as P(A).
- The probability of event B is written as P(B).
- For any event A, $0 \le P(A) \le 1$. In other words, the probability of any event A is always between 0 and 1.
- If $P(A) > P(B)$, then event A has a higher chance of occurring than event B.
- If $P(A) = P(B)$, then event A and event B are equally likely to occur.

Definitions

Experiment

Any happening whose result is uncertain.

Outcomes

Possible results from an experiment

Sample Space

Set of all possible outcomes

Event

Subset of the sample space. One or more outcomes.

Equally Likely Events

Events which have the same chance of occurring

Probability

Chance that an event will occur. Theoretically for equally likely events, it is the number of ways an event can occur divided by number of outcomes in the sample space. Empirically, the long term relative frequency.

Independent Events

Events in which the occurrence of one event does not change the probability of the occurrence of the other. One does not affect the other.

Dependent Events

Events that are not independent.

Mutually Exclusive Events

Events that can not happen at the same time. Disjoint events.

All Inclusive Events

Events whose union comprises the totality of the sample space.

Complementary Events

Two mutually exclusive events that are all inclusive.

Sample Spaces

The sample space is the set of all the possible outcomes in an experiment and is denoted by a capital letter S.

If you were to roll a single die, then S = { 1, 2, 3, 4, 5, 6 }, the set of all possible outcomes.

If you were to roll two dice and look at the sum of the two dice, then S = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }.

Equally Likely Events

However, not all sample spaces are created equally. In fact, that last example is not. There is only one way that a sum of 2 can be rolled, a 1 on the first die and a 1 on the second die. There are four ways that a sum of five can be rolled: 1-4, 2-3, 3-2, 4-1 (don't be confused here, 1-4 is a 1 on the first die and a 4 on the second and is different than a 4 on the first and a 1 on the second. If it helps, pretend that you're rolling one die and a friend is rolling the other).

We want our sample spaces to be equally likely if at all possible.

Classical / Theoretical Probability

If outcomes are equally likely, then the probability of an event occurring is the number in the event divided by the number in the sample space.

P(E) = n(E) / n(S)

The probability of rolling a six on a single roll of a die is 1/6 because there is only 1 way to roll a six out of 6 ways it could be rolled.

The probability of getting a sum of 5 when rolling two dice is 4/36 = 1/9 because there are 4 ways to get a five and there are 36 ways to roll the dice (Fundamental Counting Principle - 6 ways to roll the first times 6 ways to roll the second).

Do not make the mistake of saying that the probability of rolling a sum of 5 is 1/11 because there is one 5 out of a sample space of 11 sums (2 through 12). When the sample spaces are not equally likely, do not divide by the number in the sample space.

Properties of Probabilities

- All probabilities are between 0 and 1 inclusive.
- A probability of 0 means an event is impossible, it cannot happen.
- A probability of 1 means an event is certain to happen, it must happen.

Addition Rules

When you want to find the probability of one event OR another occurring, you add their probabilities together.

This can lead to problems however, if they have something in common.

The probability of one or both of two events occurring is ...

P(A or B) = P(A) + P(B) - P(A and B)

Mutually Exclusive Events

Mutually Exclusive Events have nothing in common. The intersection of the two events is the empty set. The probability of A and B both occurring is 0 because they can't occur at the same time.

If two events are mutually exclusive, then the probability of one or the other occurring is ...

P(A or B) = P(A) + P(B)

Multiplication Rules

When you want to find the probability of two events both occurring, then you need to apply the Fundamental Counting Principle. This principle can be extended to probabilities.

Independent Events

Independent Events are events where one occurring doesn't change the probability of the other occurring. When events are independent, the probability of them both occurring is ...

P(A and B) = P(A) * P(B)

We don't have time to get into probability very deeply. If we did, we would cover conditional probability - the probability of dependent events.

Complementary Events

The root word in complementary is "complete". Complementary events complete, or make whole. Complementary events are mutually exclusive, but when combined make the entire sample space.

The symbol for the complement of event A is A'. Some books will put a bar over the set to indicate its complement.

Since complementary events are mutually exclusive, we can use the special addition rule to find its probability. Furthermore, complementary events are all inclusive, so they make the sample space when combined, so their probabilities have a sum of 1.

The sum of the probabilities of complementary events is 1.

$P(A) + P(A') = 1$

$P(A') = 1 - P(A)$