	P	q	$P \to q$	$\neg P \vee q$
1	F	F	T	T
2	F	T	T	T
3	T	F	F	F
4	T	T	T	T

Semantic entailment holds from ①, ② and ④.

Step 1 To show

$$\models (P \to q) \to (\neg P \vee q)$$

$(P \to q) \to (\neg P \vee q)$

This can evaluate F if $(P \to q)$ is T and $(\neg P \vee q)$ is F.

Step 2 if $\models \eta$ holds, then $\vdash \eta$ is valid. In other words if η is a tautology, then η is a theorem.

Let ϕ be a formula such that $P_1, P_2 \ldots P_n$ are its only propositional atoms. Let l be any line number in ϕ's truth table. For all $1 \leq i \leq n$ let $\hat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\hat{p}_i$ is $\neg P_i$

1) $\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n \vdash \phi$ is provable if the entry for ϕ in line l is T.

2) $\hat{P}_1, \hat{P}_2, \ldots \hat{P}_n \vdash \neg\phi$ is provable if the entry for ϕ in line l is F.

ϕ is of the form $\phi_1 \to \phi_2$ where

$\phi_1 = P \to q$

$\phi_2 = \neg P \vee q$

Let $q_1, \ldots, q_k$ be propositional atoms of ϕ_1

Let $r_1, \ldots, r_k$ be propositional atoms of ϕ_2

Then we have $\{q_1, q_2 \ldots q_l\} \cup \{r_1, \ldots r_k\} = \{p_1, \ldots, p_n\}$. Therefore, whenever $\hat{q}_1, \hat{q}_2 \ldots \hat{q}_l \vdash \psi_1$ and $\hat{r}_1, \hat{r}_2 \ldots \hat{r}_k \vdash \psi_2$ ~~and~~ are valid, then we have $\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n \vdash \psi_1 \wedge \psi_2$.

ϕ evaluates to F. then we know ϕ_1 evaluates to T and ϕ_2 evaluates to F.

Then we know $\hat{q}_1, \hat{q}_2 \ldots \hat{q}_l \vdash \phi_1$ and $\hat{r}_1, \hat{r}_2 \ldots \hat{r}_k \vdash \neg\phi_2$

So $\hat{p}_1, \hat{p}_2 \ldots \hat{p}_n \vdash \phi_1 \wedge \neg\phi_2$.

This amounts to -

$$\phi_1 \wedge \neg\phi_2 \vdash \neg(\phi_1 \to \phi_2)$$

$$(p \to q) \wedge \neg(\neg p \vee q) \vdash \neg((p \to q) \to (\neg p \vee q))$$

1. $(p \to q) \wedge \neg(\neg p \vee q)$ premise
2. $(p \to q) \to (\neg p \vee q)$ assumption
3. $(p \to q)$ $\wedge e_1$ 1
4. $\neg p \vee q$ $\to e$ 2,3
5. $\neg(\neg p \vee q)$ $\wedge e_2$ 1
6. $\bot$ $\neg e$ 4,5
7. $\neg((p \to q) \to (\neg p \vee q))$ PBC 2-6

If ϕ evaluates to T, we have three cases.

i) ϕ_1 evaluates to T and ϕ_2 evaluates to T

$\hat{q}_1, \hat{q}_2 \ldots \hat{q}_l \vdash \phi_1$ and $\hat{r}_1, \hat{r}_2, \ldots \hat{r}_k \vdash \phi_2$

$\therefore \hat{p}_1, \hat{p}_2 \ldots \hat{p}_n \vdash \phi_1 \wedge \phi_2$

$$\phi_1 \wedge \phi_2 \vdash \phi_1 \to \phi_2$$

$$(p \to q) \wedge (\neg p \vee q) \vdash (p \to q) \to (\neg p \vee q)$$

1. $(p \to q) \wedge (\neg p \vee q)$ premise
2. $(p \to q)$ assumption
3. $\neg p \vee q$ $\wedge e_2$ 1
4. $(p \to q) \to (\neg p \vee q)$ $\to i$ 2-3

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ii) If ϕ_1 evaluates to F and ϕ_2 evaluates to F.

$\neg\phi_1 \land \neg\phi_2 \vdash \phi_1 \to \phi_2$

$\neg(p\to q) \land \neg(\neg p \lor q) \vdash (p\to q) \to (\neg p \lor q)$

1.	$\neg(p\to q) \land \neg(\neg p \lor q)$	premise
2.	$(p\to q)$	assumption
3.	$\neg(p\to q)$	$\land e_1$ 1
4.	$\bot$	$\neg e$ 2,3
5.	$(\neg p \lor q)$	$\bot e$ 4
6.	$(p\to q) \to (\neg p \lor q)$	$\to i$ 2-5

(iii) $\phi_1 = F$, $\phi_2 = T$

$\neg\phi_1 \land \phi_2 \vdash \phi_1 \to \phi_2$

$\neg(p\to q) \land (\neg p \lor q) \vdash (p\to q) \to (\neg p \lor q)$

1.	$\neg(p\to q) \land (\neg p \lor q)$	premise
2.	$(p\to q)$	assumption
2.	$\neg(p\to q)$	$\land e_1$ 1
4.	$\bot$	$\neg e$ 2,3
5.	$\neg p \lor q$	$\bot e$ 4
6.	$(p\to q) \to (\neg p \lor q)$	$\to i$ 2-5

$\therefore$ It is true for 2^n ~~lines of~~ truth table.

Hence $\vdash (p\to q) \to (\neg p \lor q)$

Taking $p\to q$ as premice and applying $\to e$

$\therefore$ $(p\to q) \vdash \neg p \lor q$

Hence, Completeness has been established.