Data Driven Model Predictive Control using Gaussian Processes

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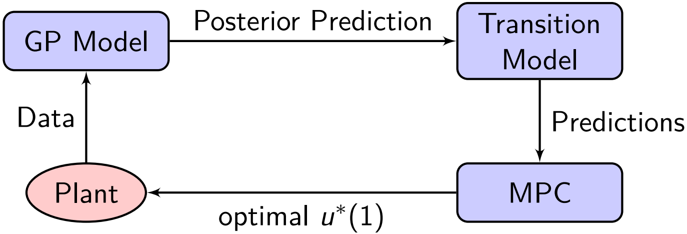
Motivation

- Learning Non-linear systems
- System identification requires Persistent Excitation condition
- Drawbacks of data-driven approaches
  - learning process is slow
  - requires an large number of interactions
  - Data inefficiency makes control learning of robotic systems and other complex systems impractical
- State and input constraints

Approach

- Model Predictive Control(MPC)
- Model-Based Learning
- Gaussian process
- A transition model is proposed for long-term prediction

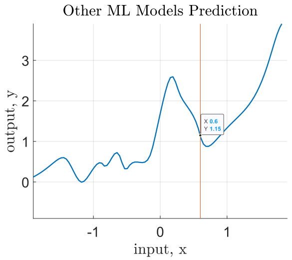


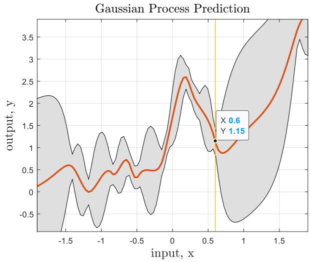
Contents

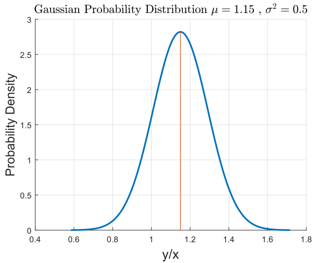
Gaussian Process [1]

Definition : A Gaussian process (GP) is a collection of random variables, any finite number of which have consistent joint Gaussian distributions.

- The parameter space is denoted by $\mathcal{X}$, for example, $\mathcal{X} \subset \mathbb{R}^D$.
- Evaluation points $x^* \in \mathcal{X}$ and $f^* = f(x^*)$.
- Gaussian multivariate distribution: $f^* \sim \mathcal{N}(\mu, \Sigma)$







Gaussian Process Regression

- Consider the parameter space $\mathcal{X} = [0, 5]$ and sample points $x^* = [0; 1]^T$, $f(x^*) = [0; 0]$, So $\mu = [0; 0]^T$
- squared exponential (SE) kernel

$$k(x, x') = \sigma^2 \exp\left(-\frac{(x - x')^2}{2\ell^2}\right)$$

- assume $\sigma^2 = 1, \ell = 1$, SE covariance function

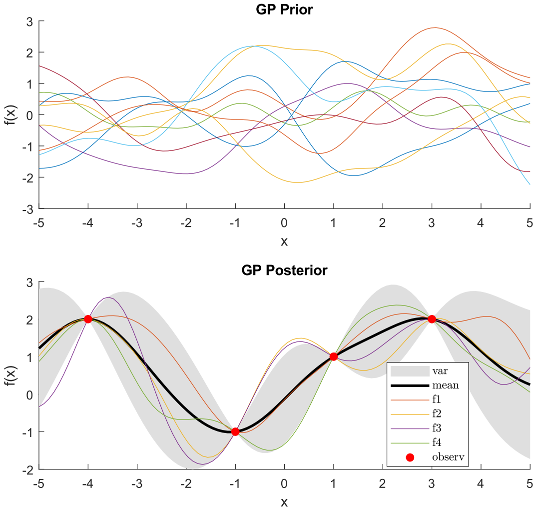
$$\Sigma = \begin{pmatrix} k(x_1, x_1) & k(x_1, x_2) \\ k(x_2, x_1) & k(x_2, x_2) \end{pmatrix} = \begin{pmatrix} 1 & 0.607 \\ 0.607 & 1 \end{pmatrix}$$

- Gaussian Prior $\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.607 \\ 0.607 & 1 \end{bmatrix}\right)$

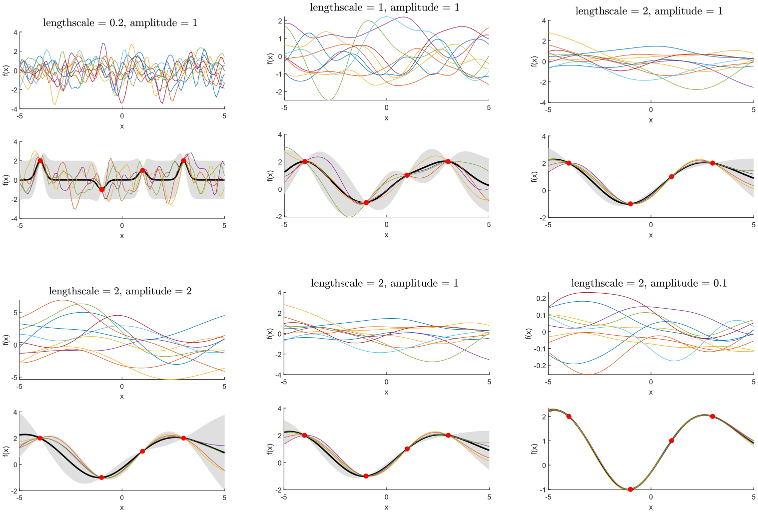
Observation

- Training points x and observations $f(x)$.
- Noisy observations are given by:$y = f(x) + \omega$ where ω is independent Gaussian noise with variance σ_n^2.
- The covariance of y is: $\text{cov}(y) = k(X, X') + \sigma_n^2 I$
- The joint distribution $\begin{bmatrix} y \\ f^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu \\ \mu^* \end{bmatrix}, \begin{bmatrix} K(X,X) + \sigma_n^2 I & K(X,X^*) \\ K(X^*,X) & K(X^*,X^*) \end{bmatrix}\right)$
- $f^*|X^*, X, y \sim \mathcal{N}(\mu^* + K(X,X^*)(K(X,X) + \sigma_n^2 I)^{-1}(y - \mu),$
  $\text{K}(\text{X}^*, X^*) - K(X^*,X)(K(X,X) + \sigma_n^2 I)^{-1}K(X,X^*))$

Graphical Interpretation

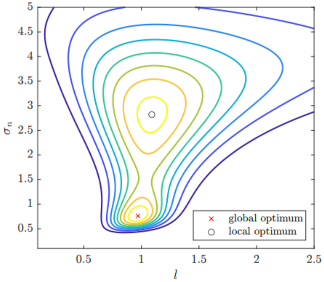


Hyper parameters



Optimization of the Hyperparameters[2]

- Necessary to the use of computationally efficient optimization methods.
- Two such methods are maximizing the marginal likelihood and cross-validation.
- In marginal liklihood, the gradient-based algorithm is used.

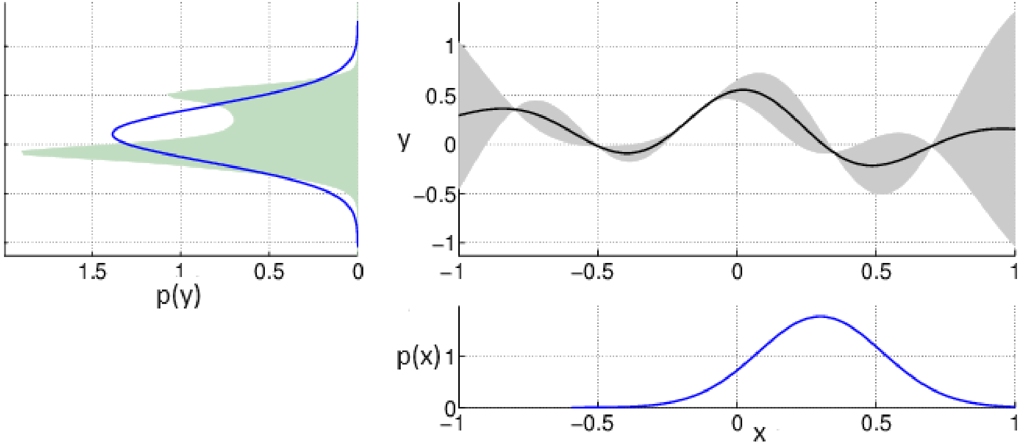


Long-term Prediction

$$p(f^*|\mu_x^*, \Sigma_x^*, \mathbf{D}) = \int p(f^*|X^*, \mathbf{D}) \cdot p(X^*|\Sigma_x^*, \mathbf{D})\, dx^* \tag{1}$$

- The integral is analytically intractable.
- Approximate the integral numerically using Monte-Carlo methods.
- To approximate the posterior distribution as a Gaussian by calculating its mean and variance (Moment Matching).

Moment Matching Approximation [3]



Model Predicitve control

- Model Predictive Control (MPC) is an advanced control method that uses a model of the system to predict and optimize the control actions over a future time horizon.
- MPC operates on a receding horizon principle
- directly address State and input constraints
- the high computational cost of solving the OCP

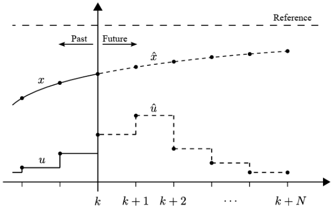
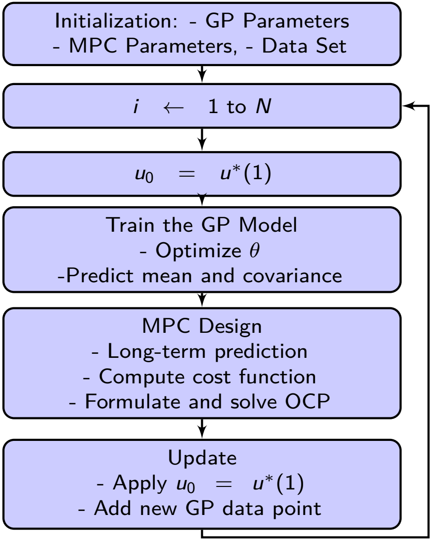


Figure: MPC Illustration[4]

GP-MPC framework



Simulation Results

- Linear system - DC motor
- Non-Linear system - Van der Pol oscillator
- Validation of GP Model and the Moment Matching model
- Modeling without uncertainty
- Modeling with uncertainty

DC Motor[5]

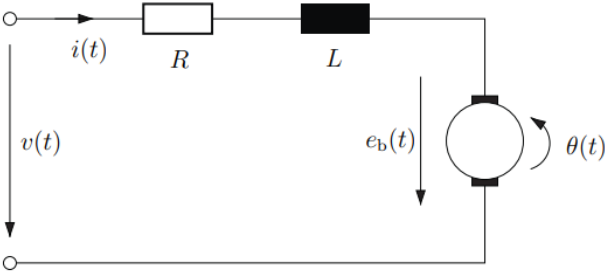


Figure: DC Motor

$$V(t) = L\frac{di}{dt} + R_m i(t) + K_e \omega(t)$$

$$G(s) = \frac{21}{s(1.1s+1)}$$

Validation of DC Motor Learning

- x_1 is the angular velocity ($\dot{\theta}$)
- x_2 is the angle (θ)
- control input is voltage (v)

$$\dot{x} = \begin{bmatrix} -0.9091 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u$$

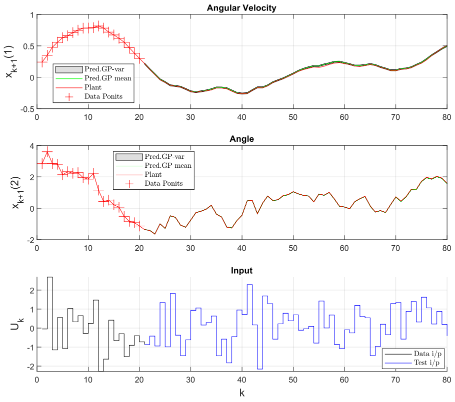
$$y = \begin{bmatrix} 0 & 4.7727 \end{bmatrix} x$$

- Zoh discretization with $T_s = 0.1$.
- initial states are $x_1 = 0$ and $x_2 = \pi$

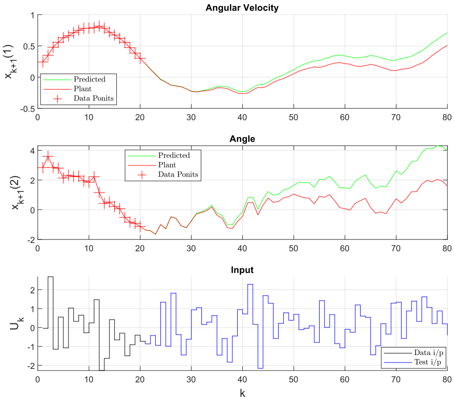
$$x_{k+1} = \begin{bmatrix} 0.9131 & 0 \\ 0.0956 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.3824 \\ 0.0194 \end{bmatrix} u_k$$

$$yk = \begin{bmatrix} 0 & 4.7727 \end{bmatrix} x_k$$

Validation of DC Motor Learning

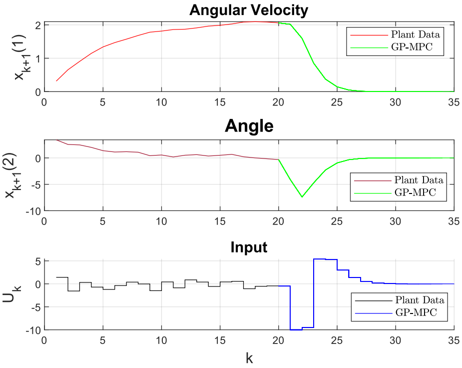


(a) GP model validation

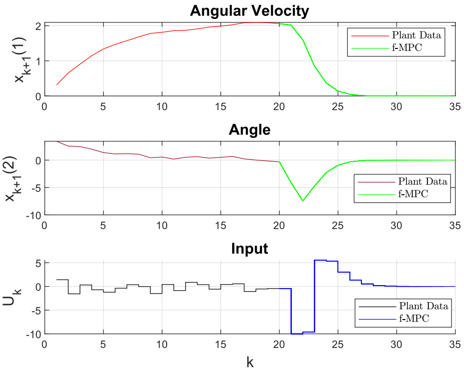


(b) Moment Matching model Validation

DC Motor Without Uncertainty(Disturbances)



(c) GP-MPC controller $t_s = 8$



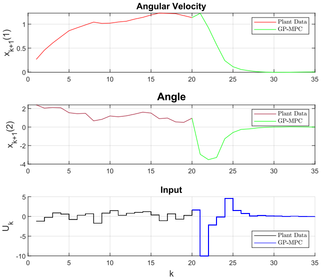
(d) f-MPC controller $t_s = 7$

DC Motor With Uncertainty(Disturbances)

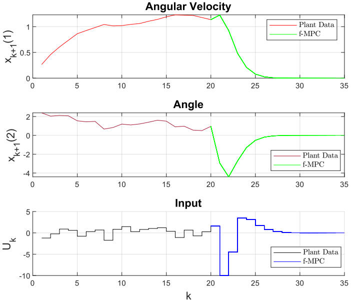
- Uncertainty Model

$$x_{k+1} = A_d x_k + B_d u_k + \epsilon_d$$

$$A_d = A + \lambda_a A, \qquad B_d = B + \lambda_b B$$



(e) GP-MPC controller $t_s = 11$



(f) f-MPC controller $t_s = 9$

Van der Pol Oscillator [6]

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

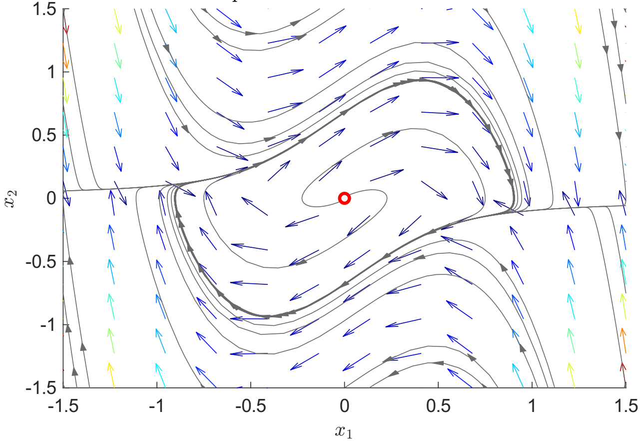
- Exhibits limit cycle
- μ represented the damping term $-10x_2(1 - x_2^2)$, indicating the nonlinearity and damping strength.
- unstable fixed point at the origin and a stable limit cycle around the origin.
- fourth-order Runge-Kutta (RK4) method with $T_s = 0.2$

$$\dot{x}_1 = 2x_2$$

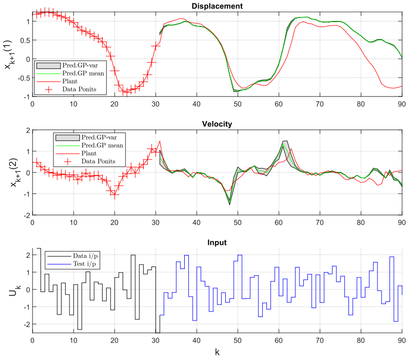
$$\dot{x}_2 = -0.8x_1 + 2x_2 - 10x_1^2x_2 + u$$

Phase Portrait

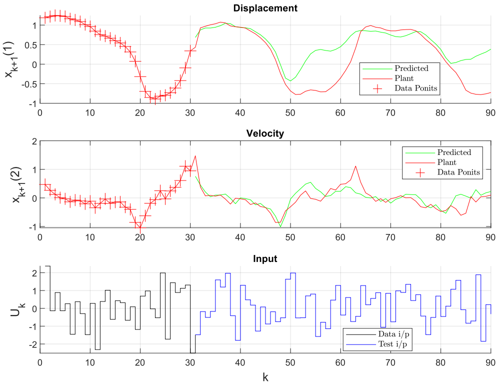
Phase portrait of Van der Pol oscillator



Validation of Van der Pol Oscillator Learning

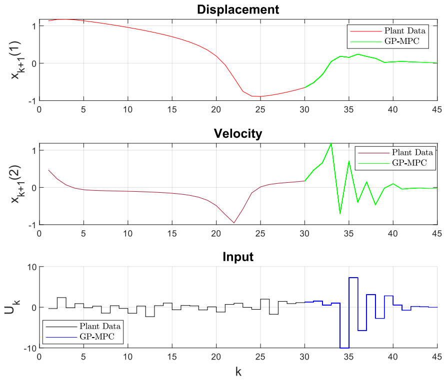


(g) GP model validation

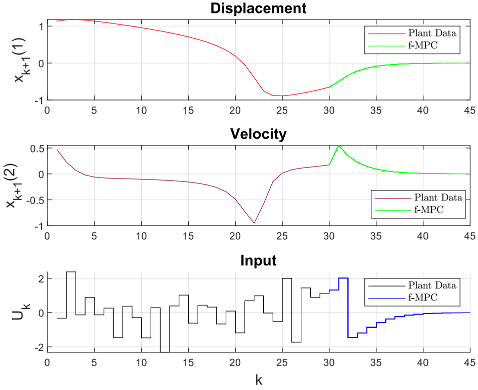


(h) Moment Matching model validation

Van der Pol oscillator Without Uncertainty



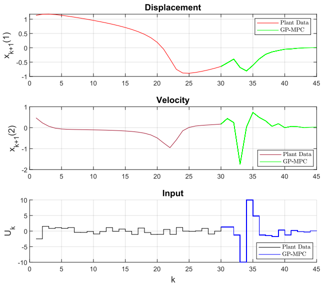
(i) GP-MPC controller $t_s = 13$



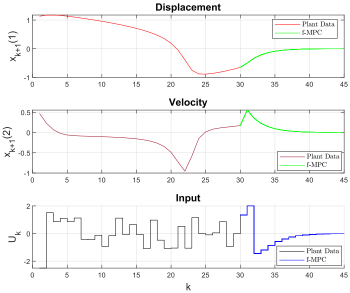
(j) f-MPC controller $t_s = 10$

Van der Pol oscillator With Uncertainty

$$\dot{x}_1 = (2 + 2\lambda)x_2$$
$$\dot{x}_2 = -(0.8 + 0.8\alpha)x_1 + (2 + 2\lambda)x_2 - (10 + 10\gamma)x_1^2 x_2 + u$$

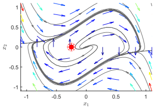


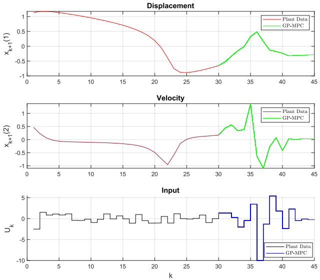
(k) GP-MPC controller $t_s = 13$



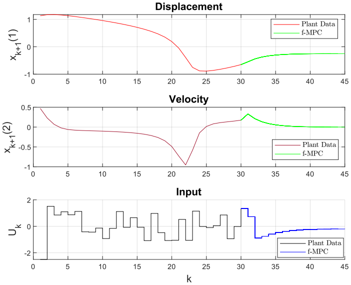
(l) f-MPC controller $t_s = 9$

Reference Tracking $x_f = [-0.25, 0]$



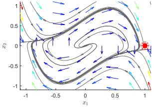


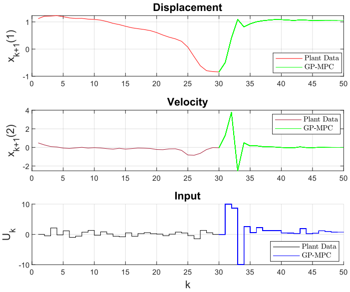
(n) GP-MPC controller $t_s = 13$



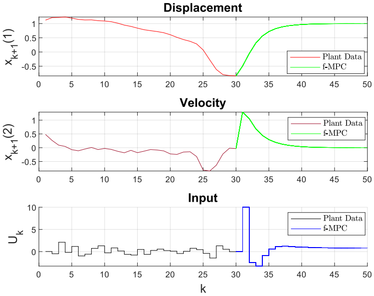
(o) f-MPC controller $t_s = 9$

Reference Tracking $x_f = [1, 0]$





(q) GP-MPC controller $t_s = 14$



(r) f-MPC controller $t_s = 12$

Conclusion

- Validated GP and Moment Matching Prediction Model - model errors in horizon end
- DC Motor Model- GP-MPC controller closely resembles f-MPC controller
- Precise and faster control in noise-free scenarios of DC motor system
- Van der Pol Oscillator- initial struggle to control, satisfactory performance later
- Reference tracking of the non linear system to desired point
- GP-MPC controller is faster(within $t_s = 15$)
- Non-linear model based learning with non linear controller
- GP-MPC controller is data efficient (requires 20-50)
- Single data-driven controller framework for variety of systems with good performance

Outlook

- Optimize the Moment Matching model for speed and accuracy using advanced modeling techniques and algorithmic improvements
- Explore the application of GP-MPC in Multi-Input Multi-Output (MIMO) systems
- Implementation GP-MPC framework to real-world systems, including both linear and nonlinear systems

The End

Thank you very much for your attention!

[1] J. Quiñonero-Candela, C. E. Rasmussen, and C. K. I. Williams, "Approximation methods for gaussian process regression," *Applied Games, Microsoft Research Ltd.*, May 2007.

[2] J. O. Lübsen, "Bayesian optimization for the control parameters of the optical synchronization system at european xfel," M.S. thesis, TUHH, 2022.

[3] M. P. Deisenroth and C. E. Rasmussen, "Pilco: A model-based and data-efficient approach to policy search," in *Proceedings of the International Conference on Machine Learning*, 2011.

[4] M. Maiworm, "Gaussian processes in control: Model predictive control with guarantees and control of scanning quantum dot microscopy," Ph.D. Dissertation, Jun. 2021. [Online]. Available: `https://doi.org/10.25673/38665`.

[5] H. Werner, *Introduction to control systems: Lecture notes*, Lecture notes, TUHH.

[6] M. Korda and I. Mezić, *Optimal construction of koopman eigenfunctions for prediction and control*, 2020. arXiv: `1810.08733 [math.OC]`.

GP prior

assumes a prior that function values behave according to

$$p(f|x_1, x_2, \ldots, x_n) = \mathcal{N}(0, K(\mathbf{x_1}, \mathbf{x_2})), \tag{2}$$

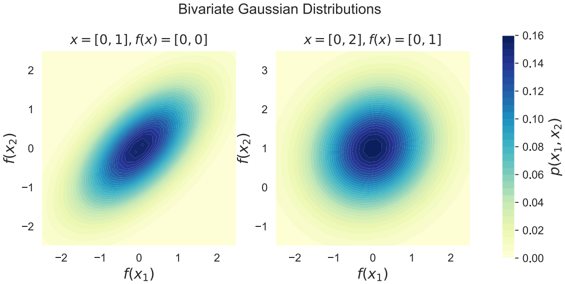
where $f = [f_1, f_2, \ldots, f_n]^\top$ is a vector of latent function values, $f_i = f(\mathbf{x_i})$, and $K(\mathbf{x_1}, \mathbf{x_2})$ is a covariance matrix, whose entries are given by the covariance function, $K(\mathbf{x_1}, \mathbf{x_2})_{ij} = k(\mathbf{x_i}, \mathbf{x_j})$.

$$K_{ij} = k(\mathbf{x_i}, \mathbf{x_j}) = \sigma_f^2 \exp\left(-\frac{(\mathbf{x_i} - \mathbf{x_j})^2}{2l^2}\right), \tag{3}$$

where σ_f^2 controls the prior variance, and l is an isotropic lengthscale parameter that controls the rate of decay of the covariance

Gaussian Process Regression

- Ex 1, $x^* = [0,1]^T$, $f(x^*) = [0,0]$, $\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0.607\\0.607 & 1\end{bmatrix}\right)$
- Ex 2, $x^* = [0,2]^T$, $f(x^*) = [0,1]$, $\mathcal{N}\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1 & 0.135\\0.135 & 1\end{bmatrix}\right)$



Maximizing Marginal likelihood Method

Following the GP assumption, the distribution of the training outputs is given as

$$p(y|X,\theta) = \mathcal{N}(0, \Sigma_\theta), \tag{4}$$

where $\Sigma_\theta = K + \sigma_n^2 I$ and θ is the collection of the unknown hyperparameters. Therefore, the negative log marginal likelihood (nlml) is

$$L(\theta) = -\log p(y|X,\theta) = \frac{1}{2} y^T \Sigma_\theta^{-1} y + \frac{1}{2} \log \det \Sigma_\theta + \frac{n}{2} \log 2\pi, \tag{5}$$

and the partial derivatives of nlml with respect to the hyperparameters are given by

$$\frac{\partial}{\partial \theta_i} L(\theta) = \frac{1}{2} \mathrm{tr} \left(\Sigma_\theta^{-1} \frac{\partial \Sigma_\theta}{\partial \theta_i} \right) - \frac{1}{2} y^T \Sigma_\theta^{-1} \frac{\partial \Sigma_\theta}{\partial \theta_i} \Sigma_\theta^{-1} y. \tag{6}$$

Moment Matching Approximation

- Alternatively, the posterior distribution $p(f|x, \mathbf{X}, \mathbf{D})$ can be approximated as a Gaussian by calculating its mean and variance.
- Several methods exist for this Gaussian approximation, such as Moment Matching and linearization of the posterior GP mean function.
- Moment Matching computes the first two moments of the predictive distribution exactly, whereas linearization provides a computationally efficient approximation by explicitly linearizing the posterior GP.
- Moment Matching is better than linearization, we will focus on the Moment Matching Gaussian approximation.

Mean Prediction

Following the law of iterated expectations, for target dimensions $a = 1, ..., D$ we obtain the predictive mean:

$$\begin{aligned}\mu_t^a &= \mathbb{E}_{\tilde{\boldsymbol{x}}_{t-1}}\left[\mathbb{E}_{f_a}\left[f_a\left(\tilde{\boldsymbol{x}}_{t-1}\right) \mid \tilde{\boldsymbol{x}}_{t-1}\right]\right] = \mathbb{E}_{\tilde{\boldsymbol{x}}_{t-1}}\left[m_{f_a}\left(\tilde{\boldsymbol{x}}_{t-1}\right)\right] \\ &= \int m_{f_a}\left(\tilde{\boldsymbol{x}}_{t-1}\right) \mathcal{N}\left(\tilde{\boldsymbol{x}}_{t-1} \mid \tilde{\boldsymbol{\mu}}_{t-1}, \tilde{\boldsymbol{\Sigma}}_{t-1}\right) d\tilde{\boldsymbol{x}}_{t-1} \\ &= \boldsymbol{\beta}_a^T \boldsymbol{q}_a\end{aligned} \tag{7}$$

where, $\quad \boldsymbol{\beta}_a = \left(\boldsymbol{K}_a + \sigma_{w_a}^2\right)^{-1} \boldsymbol{y}_a, \qquad \boldsymbol{q}_a = \left[q_{a_1}, \ldots, q_{a_n}\right]^T.$

$$\begin{aligned}q_{a_i} &= \int k_a\left(\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{x}}_{t-1}\right) \mathcal{N}\left(\tilde{\boldsymbol{x}}_{t-1} \mid \tilde{\boldsymbol{\mu}}_{t-1}, \tilde{\boldsymbol{\Sigma}}_{t-1}\right) d\tilde{\boldsymbol{x}}_{t-1} \\ &= \sigma_{f_a}^2 \left|\tilde{\boldsymbol{\Sigma}}_{t-1} \boldsymbol{\Lambda}_a^{-1} + \boldsymbol{I}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \boldsymbol{\nu}_i^T \left(\tilde{\boldsymbol{\Sigma}}_{t-1} + \boldsymbol{\Lambda}_a\right)^{-1} \boldsymbol{\nu}_i\right),\end{aligned} \tag{8}$$

where we define

$$\nu_i := \left(\tilde{\boldsymbol{x}}_i - \tilde{\boldsymbol{\mu}}_{t-1}\right) \tag{9}$$

Covariance Matrix Prediction

The Predictive covariance matrix $\boldsymbol{\Sigma}_\Delta \in \mathbb{R}^{D \times D}$ is given by

$$\sum(:) = \begin{bmatrix} \sigma_{aa}^2 & \sigma_{ab}^2 & \cdots\cdots \\ \sigma_{ab}^2 & \sigma_{bb}^2 & \cdots\cdots \\ . & . & \cdots\cdots \\ . & . & \cdots\cdots \end{bmatrix}_{DxD} \tag{10}$$

where,

$$\sigma_{aa}^2 = \mathbb{E}_{\overline{\boldsymbol{x}}_t}\left[\operatorname{var}_f\left[\Delta_a \mid \tilde{\boldsymbol{x}}_t\right]\right] + \mathbb{E}_{f,\overline{\boldsymbol{x}}_t}\left[\Delta_a^2\right] - \left(\boldsymbol{\mu}_\Delta^a\right)^2, \tag{11}$$

$$\sigma_{ab}^2 = \mathbb{E}_{f,\overline{\boldsymbol{x}}_t}\left[\Delta_a \Delta_b\right] - \boldsymbol{\mu}_\Delta^a \boldsymbol{\mu}_\Delta^b, \quad a \neq b, \tag{12}$$

$$\begin{aligned} \mathbb{E}_{f,\overline{\boldsymbol{x}}_t}\left[\Delta_a \Delta_b\right] &= \mathbb{E}_{\overline{\boldsymbol{x}}_t}\left[\mathbb{E}_f\left[\Delta_a \mid \tilde{\boldsymbol{x}}_t\right] \mathbb{E}_f\left[\Delta_b \mid \tilde{\boldsymbol{x}}_t\right]\right] \\ &= \int m_f^a\left(\tilde{\boldsymbol{x}}_t\right) m_f^b\left(\tilde{\boldsymbol{x}}_t\right) p\left(\tilde{\boldsymbol{x}}_t\right) \mathrm{d}\tilde{\boldsymbol{x}}_t \\ &= \beta_a^\top \boldsymbol{Q} \beta_b, \end{aligned} \tag{13}$$

Covariance Matrix Prediction

$$\boldsymbol{Q} := \int k_a\left(\tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{X}}\right)^\top k_b\left(\tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{X}}\right) p\left(\tilde{\boldsymbol{x}}_t\right) \mathrm{d}\tilde{\boldsymbol{x}}_t. \tag{14}$$

Using standard results from Gaussian multiplications and integration, we obtain the entries Q_{ij} of $Q \in \mathbb{R}^{n \times n}$

$$Q_{ij} = |\boldsymbol{R}|^{-\frac{1}{2}} k_a\left(\tilde{\boldsymbol{x}}_i, \tilde{\boldsymbol{\mu}}_t\right) k_b\left(\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{\mu}}_t\right) \exp\left(\frac{1}{2}\boldsymbol{z}_{ij}^\top \boldsymbol{T}^{-1} \boldsymbol{z}_{ij}\right) \tag{15}$$

where we define $\boldsymbol{R} := \tilde{\boldsymbol{\Sigma}}_t\left(\boldsymbol{\Lambda}_a^{-1} + \boldsymbol{\Lambda}_b^{-1}\right) + \boldsymbol{I}, \qquad \boldsymbol{T} := \boldsymbol{\Lambda}_a^{-1} + \boldsymbol{\Lambda}_b^{-1} + \tilde{\boldsymbol{\Sigma}}_t^{-1},$

$\boldsymbol{z}_{ij} := \boldsymbol{\Lambda}_a^{-1}\boldsymbol{\nu}_i + \boldsymbol{\Lambda}_b^{-1}\boldsymbol{\nu}_j,$

From (11), we see that the diagonal entries contain the additional term

$$\mathbf{E}_{\tilde{\boldsymbol{x}}_t}\left[\mathrm{var}_f\left[\Delta_a \mid \tilde{\boldsymbol{x}}_t\right]\right] = \sigma_{f_a}^2 - \mathrm{tr}\left(\left(\boldsymbol{K}_a + \sigma_{w_a}^2 \boldsymbol{I}\right)^{-1} \boldsymbol{Q}\right) + \sigma_{w_a}^2 \tag{16}$$

$\sigma_{w_a}^2$ being the system noise variance of the ath target dimension.