EC303 - Digital Signal Processing

M04L001 - Frequency Analysis of Signal and Systems

Dr. M. C. Patel

Autumn Semester, 2025-26

Information

- Slides are prepared to use in class room purpose, only used as a reference material
- All the slides are prepared based on the reference material.
- Most of the figures/content used in this material are redrawn, some of the figures/pictures are downloaded from the Internet.
- This material is not for commercial purpose.
- This material is prepared based on course name EC303-Digital Signal Processing for Electronic Communication Engineering, of Department of Electronics Engineering, S. V. National Institute of Technology, Surat, Gujarat, INDIA (SVNIT-Surat) syllabus.

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1 The Fourier Series for Discrete-Time Periodic Signals

The Fourier Series for Discrete-Time Periodic Signals

- The Fourier representation of signal maps the signal into frequency domain.
- The Fourier transform provides a different way to interpret signals and systems.
- It is useful for convolution operation in time domain maps into multiplication in frequency domain.
- It gives us information about system or signal characteristic of behavior.

The Fourier Series for Discrete-Time Periodic Signals

- A given periodic sequence $x(n)$ with period N, that is $x(n) = x(n+N)$.
- The Fourier representation of $x(n)$ can be expressed as (n is varying from 0 to N-1),

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}$$

  Where C_k are the coefficients in the series representation. This equation is called the $\underline{Discrete-Time\ Fourier\ Series (DTFS)}$

- The Fourier coefficients $\{C_k\}, k = 1, 2, 3, .., N-1$ provides the description of $x(n)$ in the frequency domain.

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

The Fourier Series for Discrete-Time Periodic Signals

- C_k represents the amplitude and phase associated with the frequency component.

$$S_k = e^{j\frac{2\pi}{N}kn} = e^{j\omega_k n}$$

Where $\omega_k = \frac{2\pi}{N}k$

- The function s_k are periodic N Hence by using synthesis from ,

$$s_k(n) = s_k(n+N)$$

- So, That,

$$C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}k(n+N)} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = C_k$$

Therefore C_k is periodic sequence with fundamental period of N.

- Instead of focusing in periodic range $k = 0, 1, ..., N-1$ in frequency domain $0 < \omega_k = \frac{2\pi}{N}k < 2\pi$ for $0 < k < N-1$, We do in range of $-\pi < \omega_k = \frac{2\pi}{N}k < \pi$, which corresponds to $-\frac{N}{2} < k < \frac{N}{2}$.

The Fourier Series for Discrete-Time Periodic Signals

Example

Determine the spectra of the following signal.

$$x(n) = \cos\left(\frac{\pi n}{3}\right), \qquad \omega_0 = \frac{\pi}{3} \qquad f_0 = \frac{1}{6}$$

Solution:Hence, $x(n)$ is periodic with fundamental period of $N = 6$.

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j\frac{2\pi}{6}kn}, k = 0, 1, 2.., 5$$

$$x(n) = \cos(\frac{\pi n}{3}) = \cos(\frac{2\pi n}{6}) = \frac{1}{2}e^{j\frac{2\pi}{6}n} + \frac{1}{2}e^{-j\frac{2\pi}{6}n}$$

$$e^{j\frac{2\pi}{6}(-1)n} = e^{j\frac{2\pi}{6}(5-6)n} = e^{j\frac{2\pi}{6}5n} \text{ which means } C_{-1} = C_5$$

$$C_k = \frac{1}{6}\sum_{n=0}^{5} \cos(\frac{n\pi}{3})e^{-j\frac{2\pi}{N}kn}, k = 0, 1, 2, .., 5$$

The Fourier Series for Discrete-Time Periodic Signals

Solution(cont.)

$$C_k = \frac{1}{6}\Bigg(1 + \cos(\frac{\pi}{3})e^{-j\frac{2\pi}{6}k} + \cos(\frac{2\pi}{3})e^{-j\frac{2\pi}{6}2k} + \cos(\frac{3\pi}{3})e^{-j\frac{2\pi}{6}3k}$$
$$+ \cos(\frac{4\pi}{3})e^{-j\frac{2\pi}{6}4k} + \cos(\frac{5\pi}{3})e^{-j\frac{2\pi}{6}5k}\Bigg), k = 0, 1, 2, ..5$$

So, $C_0 = 0$, $C_1 = \frac{1}{2}$, $C_2 = 0$, $C_3 = 0$, $C_4 = 0$, $C_5 = \frac{1}{2}$

```
% Find the Fourier Series coefficients of
x(n)=cos(pi n/3)

for k=1:6
  c(k)=0;
  for n=1:6
    c(k)=c(k)+cos(pi*(n-1)/3)
          *exp(-j*2*pi*(k-1)*(n-1)/6);
  end
  c(k)=c(k)/6;
end
for k=1:6;
  rats(c(k))
end
```

The Fourier Series for Discrete-Time Periodic Signals

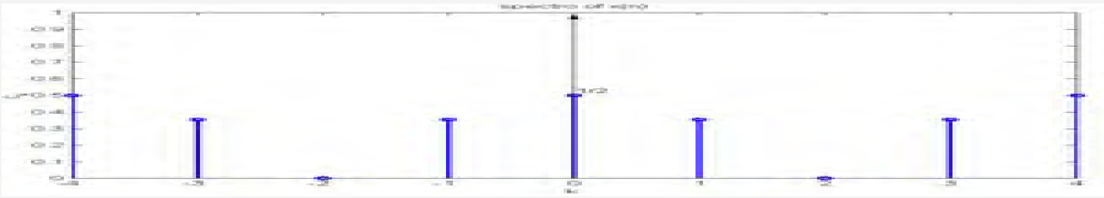
Example:Determine the spectra of the following periodic signal with N =4

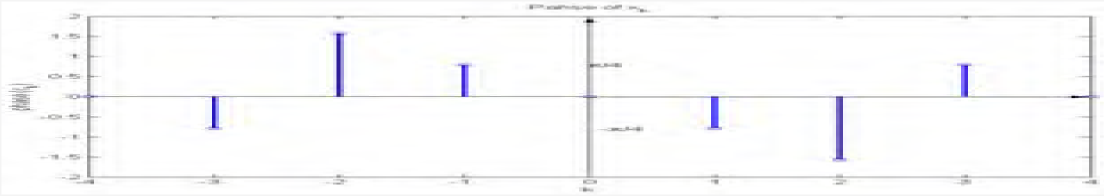
$$x(n) = \{1, 1, 0, 0\}$$

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$
$$= \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j\frac{2\pi}{4}kn}, k = 0, 1, 2, 3$$
$$= \frac{1}{4}(1 + e^{-j\frac{2\pi}{4}k}), k = 0, 1, 2, 3$$

So, $C_1 = \frac{1}{2}$, $C_1 = \frac{1}{4}(1-j)$, $C_2 = 0$, $C_3 = \frac{1}{4}(1+j)$.

so Magnitude for complex, $|C_1| = \frac{\sqrt{2}}{4}$, $\angle C_1 = \frac{-\pi}{4}$, $|C_3| = \frac{\sqrt{2}}{4}$, $\angle C_1 = \frac{\pi}{4}$





The Fourier Series for Discrete-Time Periodic Signals

Magnitude Plot:

Phase Plot:

2 Fourier Transform of Discrete-Time

Fourier Transform of Discrete-Time

The Fourier Transform of a finite-energy discrete time signal $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$X(\omega)$ signal is periodic with 2π.

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega n)}e^{-j(2\pi kn)} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned}$$

The inverse Fourier Transform of discrete-time signal,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

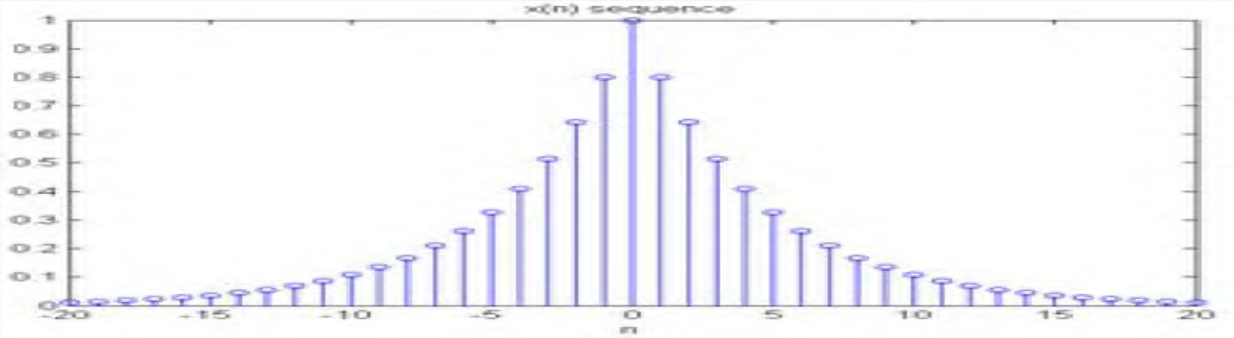
3 Properties of Fourier Transform for DT

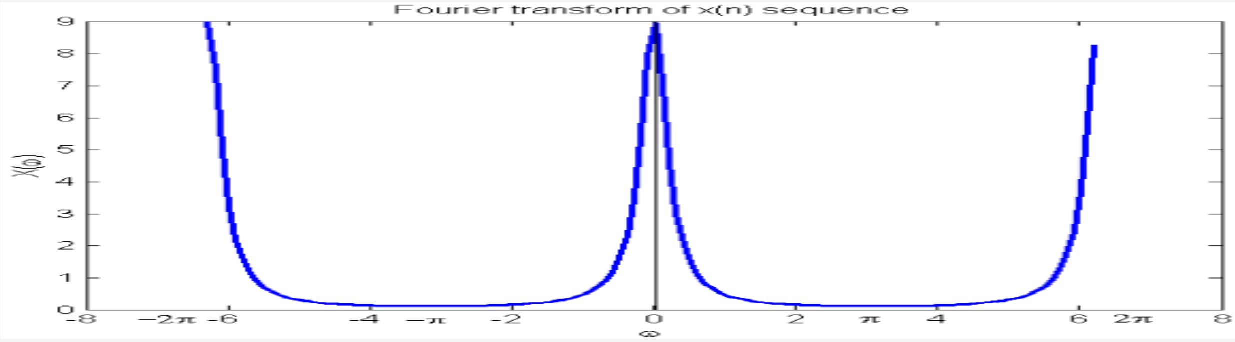
Properties of Fourier Transform for DT

Properties	Time Domain	Frequency Domain
Notation	$x(n)$, $x_1(n)$, $x_2(n)$	$X(\omega)$, $X_1(\omega)$, $X_2(\omega)$
Linearity	$ax_1(n) + bx_2(n)$	$aX_1(\omega) + bX_2(\omega)$
Time Shifting	$x(n-k)$	$e^{-jk\omega}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2} = x_1(l) * x_2(-l)$	$S_{x_1x_2} = X_1(\omega)X_2(-\omega) = X_1(\omega)X_2^*(\omega)$
Frequency Shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x_1(n)\cos(\omega_0 n)$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$

Properties of Fourier Transform for DT

Properties	Time Domain	Frequency Domain
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in frequency Domain	$x^*(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$nx(n)$	$X^*(-\omega)$
Parseval Theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$





4 Examples

Example

Find Frequency Spectra:
$x(n) = 0.8^{|n|}$, $-\infty < n < \infty$. By linearity $x(n) = x_1(n) + x_2(n)$.

$$x_1(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}, \qquad x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

$$\begin{aligned} X_1(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{\infty}(0.8e^{-j\omega})^n \\ &= \frac{1}{1-0.8e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.8} \end{aligned}$$

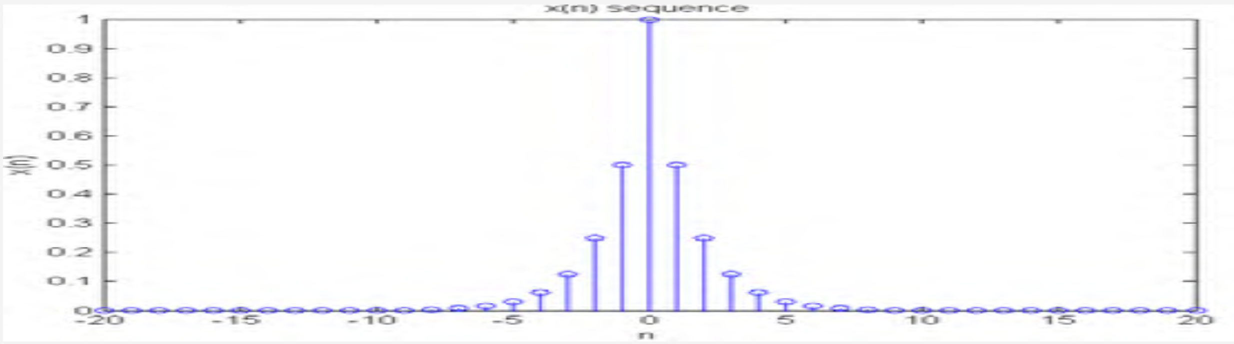
$$\begin{aligned} X_2(\omega) &= \sum_{n=-\inf}^{\infty} x(n)e^{-j\omega n} = \sum_{n=\infty}^{-1}(0.8e^{j\omega})^{-n} \\ &= (0.8e^{j\omega})^n = \frac{0.8e^{j\omega}}{1-0.8e^{j\omega}} \end{aligned}$$

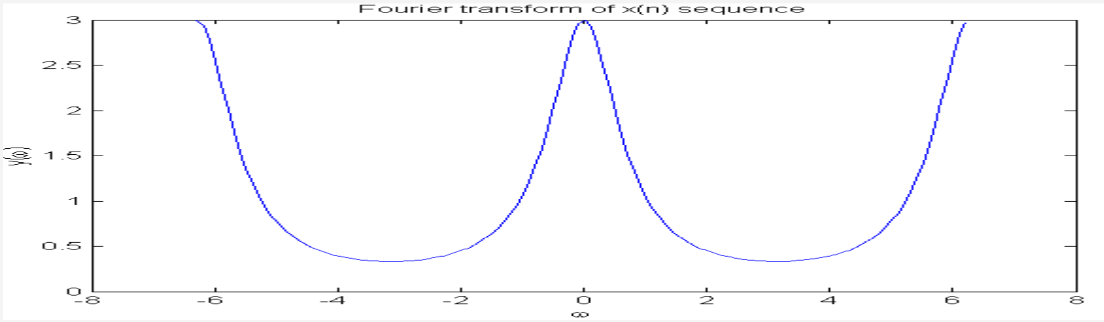
Example

Solution(Cont.):

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{1}{1-0.8e^{-j\omega}} + \frac{0.8e^{j\omega}}{1-0.8e^{j\omega}} = \frac{1-0.8^2}{1-2(0.8)\cos(\omega)+0.8^2}$$

```
n=-20:20;
x=0.8.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')
figure;

w=-2*pi:0.1:2*pi;
y=(1-0.8^2)./
  (1-2*0.8*cos(w)+0.8^2);
plot (w,y);
xlabel ('\omega')
ylabel ('y(\omega)')
title ('Fourier transform of
      x(n) sequence')
```





Example

Example 2:

$$x(n) = 0.5^{|n|}, \qquad -\infty < n < \infty$$

```
n=-20:20;
b=0.5
x=b.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
y=(1-b^2)./(1-2*b*cos(w)+b^2);
plot (w,y);
xlabel ('\omega')
ylabel ('y(\omega)')
title ('Fourier transform of
      x(n) sequence')
```

$$X(\omega) = \frac{1 - 0.5^2}{1 - \cos(\omega) + 0.5^2}$$

Example

Solution(Example-2)

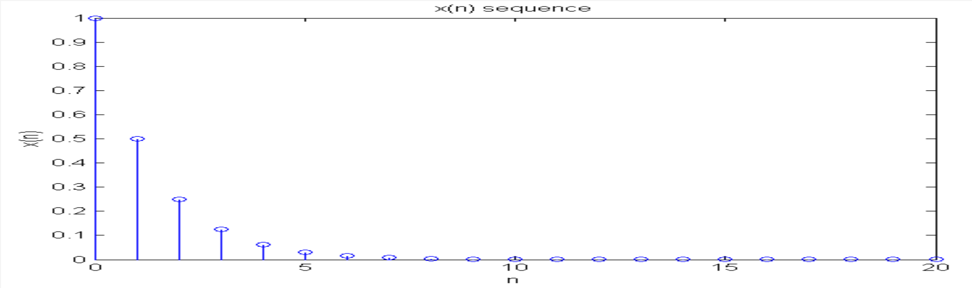
$$X(\omega) = \frac{1 - 0.5^2}{1 - \cos(\omega) + 0.5^2}$$

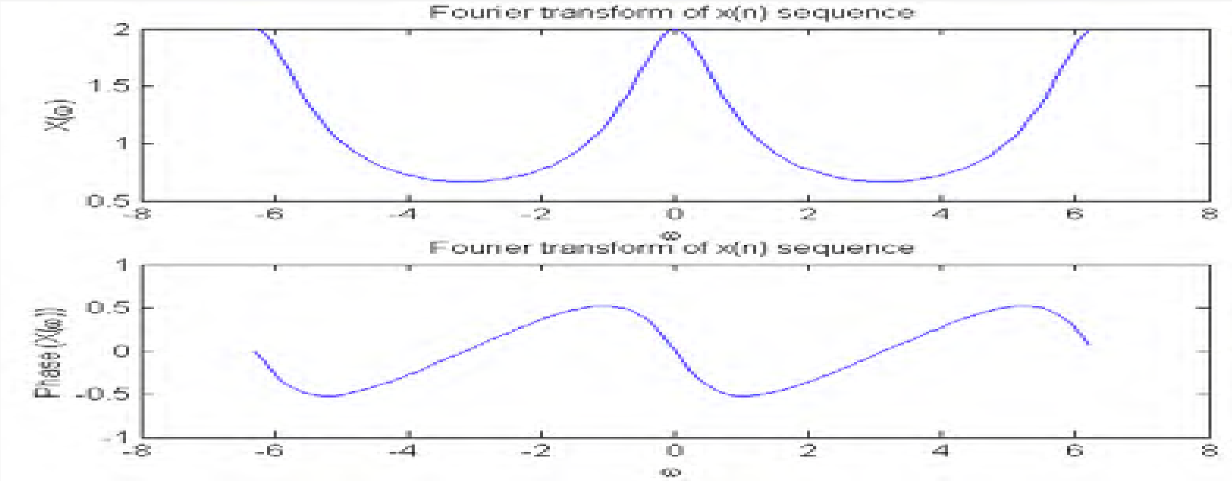
Example

Example 3:

$$x(n) = 0.5^n, \qquad 0 < n < \infty$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{\infty}(0.5e^{-j\omega})^n$$
$$= \frac{1}{1 - 0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$$





Example

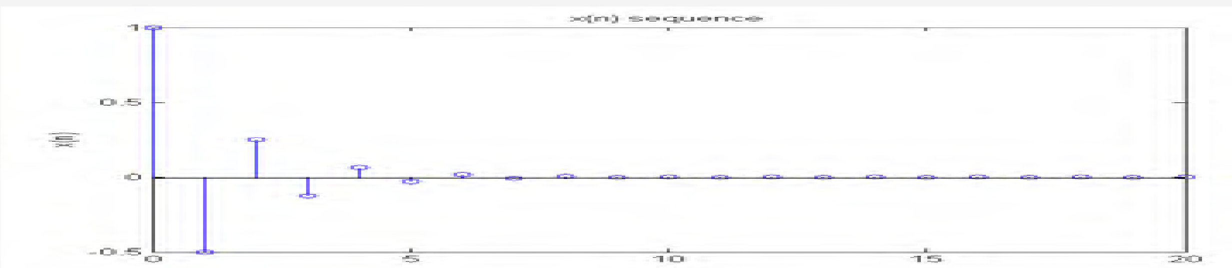
Solution(Example-3)

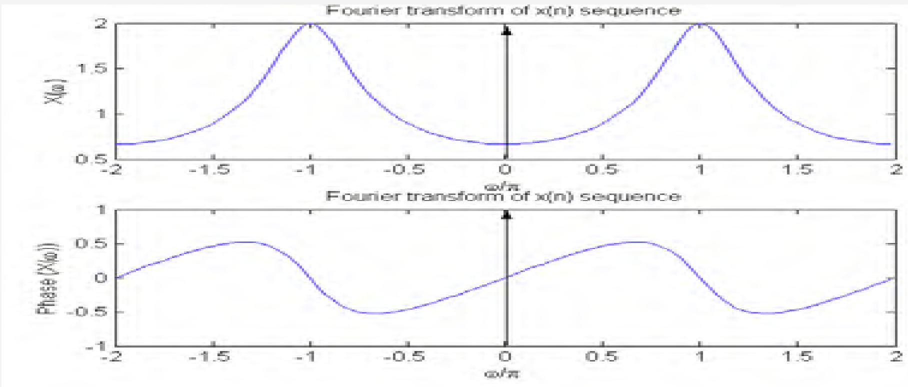
```
n=-0:20;
b=0.5
x=b.^abs(n);
stem (n,x)
xlabel ('n')
ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.1:2*pi;
X=exp(j*w)./(exp(j*w)-b);
subplot (2,1,1)
plot (w,abs(X));
xlabel ('\omega')
ylabel ('X(\omega)')
title ('Fourier transform of
     x(n) sequence')
subplot (2,1,2);
plot (w,phase(X));
xlabel ('\omega')
ylabel ('Phase (X(\omega))')
title ('Fourier transform of
      x(n) sequence')
```

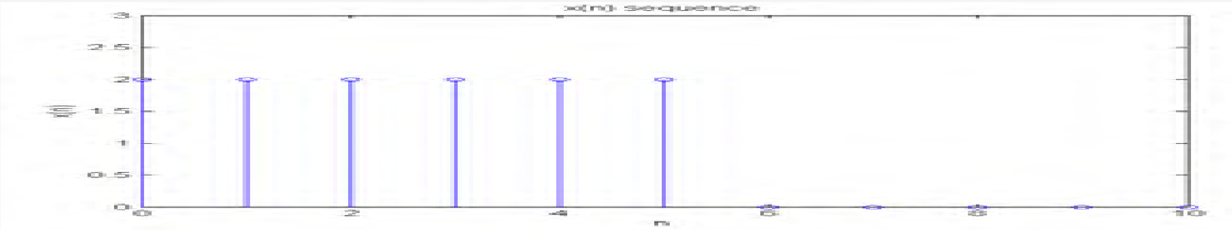
Example

Example 4:

$$x(n) = (-0.5)^n u(n)$$







$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{\infty}(-0.5e^{-j\omega})^n$$
$$= \frac{1}{1+0.5e^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}+0.5}$$

```
n=0:20; b=-0.5; x=b.^n;
stem (n,x);
xlabel ('n'); ylabel ('x(n)');
title ('x(n) sequence');

figure;
w=-2*pi:0.1:2*pi; wpi=w/pi;
X=exp(j*w)./(exp(j*w)-b);
subplot (2,1,1); plot (wpi,abs(X));
xlabel ('\omega/\pi'); ylabel ('X(\omega)')
title ('Fourier transform of x(n)
sequence')

subplot (2,1,2); plot (wpi,phase(X));
xlabel ('\omega/\pi')
ylabel ('Phase (X(\omega))')
title ('Fourier transform of x(n)
sequence')
```

Example

Solution (Example-4)

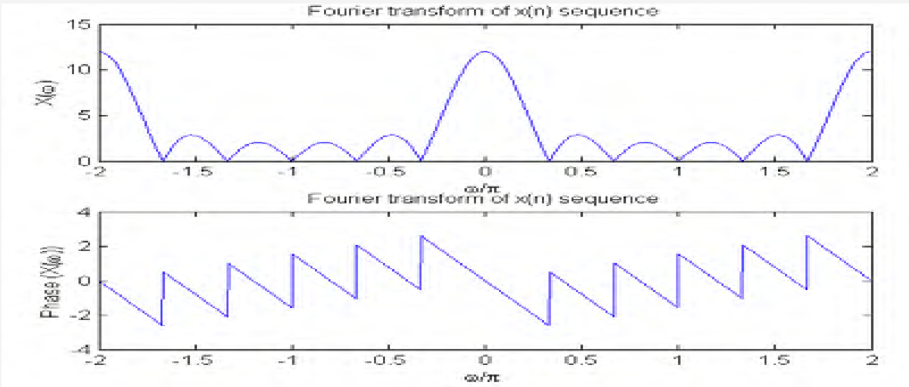
$$X(\omega) = \frac{e^{j\omega}}{e^{j\omega}+0.5}$$

Example

Example 5:

$$x(n) = \begin{cases} 2, & 0 \le n \le 5 \\ 0, & \text{Otherwise} \end{cases}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{5}(2e^{-j\omega})^n$$
$$= 2e^{-j\frac{5}{2}\omega}\,\frac{\sin(3\omega)}{\sin(\omega/2)}$$




```
n=0:10; x=2*(n>=0 & n<=5);
stem (n,x); xlabel ('n'); ylabel ('x(n)')
title ('x(n) sequence')

figure;
w=-2*pi:0.01:2*pi; wpi=w/pi;
X=2*(1-exp(-j*6*w))./(1-exp(-j*w));
subplot (2,1,1); plot (wpi,abs(X));
xlabel ('\omega/\pi'); ylabel ('X(\omega)')
title ('Fourier transform of x(n) sequence')

subplot (2,1,2); plot (wpi,phase(X));
xlabel ('\omega/\pi'); ylabel ('Phase (X(\omega))')
title ('Fourier transform of x(n) sequence')
```

Example

Solution (Example-4)

$$X(\omega) = 2e^{-j\frac{5}{2}\omega}\frac{\sin(3\omega)}{\sin(\omega/2)}$$

More Example is covered in Tutorial Class.
Questions?

EC303 - Digital Signal Processing

M04L002 - Various Convergence Criteria for DTFT and others Transformation

Dr. M. C. Patel

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Table of contents

Contents

So Much of things should be review....
so you are expected to read it for refresh your memory.
We'll focus on a few topics you might not have seen before...

1 Summery of DTFS and DTFT

The Fourier Series for Discrete-Time Periodic Signals

The idea here is the same as FS for CT periodic signals.. just a few details are different

- Let $x[n]$ be a periodic signal with period of N; $x[n+N] = x[n]$.
- We define the fundamental frequency as $2\pi/N$ in unit of rad/sample.
- In exactly the same way as for CT-FS, we can decompose the periodic signal $x[n]$ into sum of sinusoids with frequencies that are integer multiples of the fundamental.
- The key difference, here is that for DT frequencies, we can limit ourselves to the range of 0 to 2π. So the frequencies of interest are: $\frac{2\pi}{N}k$, $k = 0, 1, 2, ..., N-1$.
- FS of DT signal:

$$x[n] = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}, n = 0, 1, 2, ..., N-1$$

  where this $N-1$ though extendable to other n via periodicity.

The Fourier Series for Discrete-Time Periodic Signals

- Similar to how we find the FS coefficient for CT..., The DTFS coefficients are:

$$C_k = \frac{1}{N}\sum_{k=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

- It is easy to verify $C_{k+N} = C_k$.

DTFS		DTFT
$x[n] = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}$ $C_k = \frac{1}{N}\sum_{k=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$		$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$ $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ **Infinite summation.. So, Major issues of convergence & Truncation.**

Summary of Analysis and synthesis Formula

Notes
Periodicity $w/$"period" α in one domain automatically implies discretization with "spacing" of $1/\alpha$ in the other domain and vice versa!

Convergence of DTFT

- The equation for finding the DTFT of a signal is $X^f(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$.
- It involves an infinite sum, so there are issues with its convergence, Recall our study of ROC for ZT and the link between ZT & DTFT!.
- Define $X_N^f(\omega) = \sum_{n=-N}^{N} x[n]e^{-j\omega n}$.
- Mathematicians have many different ways to characterize convergence... the one we consider first is called "Uniform Convergence". We say that, $X_N^f(\omega)$ converge uniformaly to $X^f(\omega)$ when

$$\lim_{N\to\infty}\{sup_\omega|X^f(\omega) - X_N^f(\omega)|\} = 0$$

- **A more general concept of "max". This roughly says that "largest" difference between the two keeps getting smaller.**
- Like we saw in our studies of Z transformation. A sufficient condition is absolutely summability of $x[n]$. The DTFT converges uniformly if,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

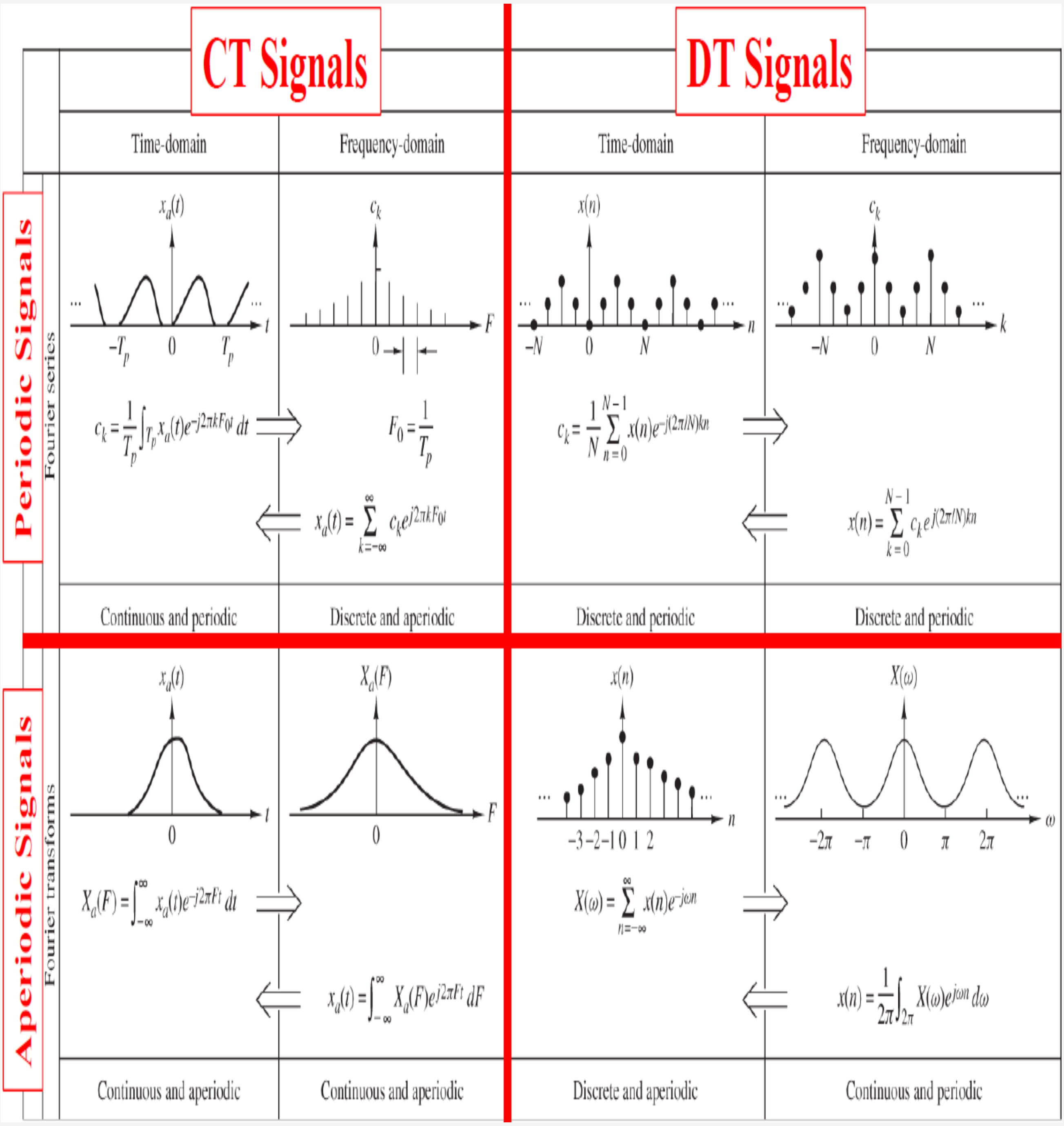
  However, in DSP the class of signal of interest are energy signals ("square summable") and not all of those are absolutely summable.
- To allow us to deal with energy signals we have to relax our expectation on the type of convergence we will accept. We consider "Mean Square Convergence". Defines as,

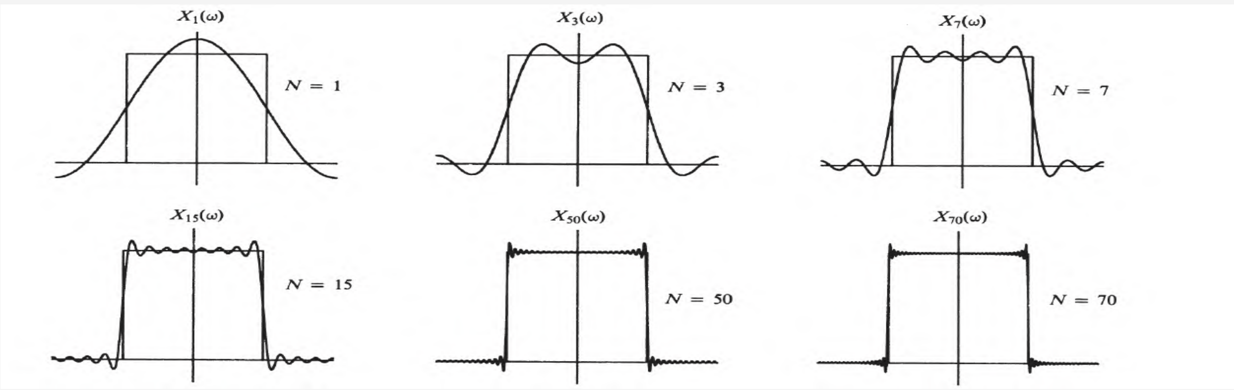
$$\lim_{N\to\infty}\int_{-\pi}^{\pi} |X^f(\omega) - X_N^f(\omega)|^2 d\omega = 0$$

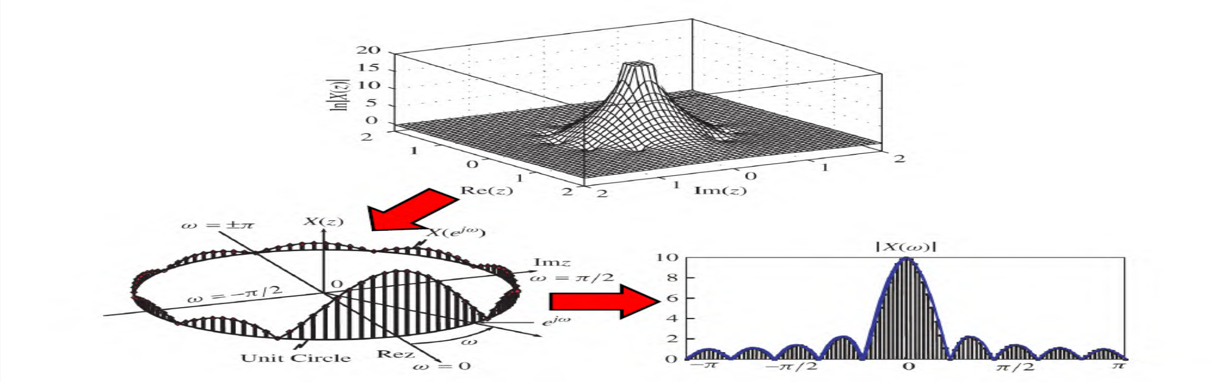
- This says that we were looking for the "area of squared error" to go to zero.

DTFT of SINC function

Example
Peak of ripple stays same area of squared error goes to zero.
However, the area can go to zero even though the "largest difference" between the two does not go to zero.


CT Signals
DT Signals
Time-domain
Frequency-domain
Time-domain
Frequency-domain
Periodic Signals
Fourier series
$c_k = \frac{1}{T_p}\int_{T_p} x_a(t)e^{-j2\pi kF_0 t}\,dt$
$F_0 = \frac{1}{T_p}$
$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$
$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}$
$x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$
Continuous and periodic
Discrete and aperiodic
Discrete and periodic
Discrete and periodic
Aperiodic Signals
Fourier transforms
$X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\,dt$
$x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\,dF$
$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$
$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$
Continuous and aperiodic
Continuous and aperiodic
Discrete and aperiodic
Continuous and periodic


$X_1(\omega)$
$N = 1$
$X_3(\omega)$
$N = 3$
$X_7(\omega)$
$N = 7$
$X_{15}(\omega)$
$N = 15$
$X_{50}(\omega)$
$N = 50$
$X_{70}(\omega)$
$N = 70$



2 Relationship of DTFT of Z-Transformation

Relationship of DTFT of Z-Transformation

We have already discussed this. but we will see it again here.

$$X^z(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \text{ ROC:} R_2 < |z| < R_1$$

If the ROC contains the Unit circle then we can replace z by its values on the Unit circle $z \to e^{j\omega}$.

$$X^z(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \Rightarrow X^z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\Rightarrow X^f(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Viewed as a function of ω yields the DTFT

So, If the ROC contains the Unit Circle then we get the DTFT by evaluating the z-transformation on the Unit Circle.

But. there are some signals with DTFT that do not "come from" a z-transformation. One of those is the sinc function. It has a DTFT but does not have a Z-transformation.

3 DTFT of Signals with poles on Unit Circle

DTFT of Signal with Poles on Unit Circle

Even though these don't satisfy the "Unit Circle in ROC" criteria. Of we allow the DTFT to contain delta functions we get DTFT result for such signals.

Consider the signal $x[n] = u[n]$, whose Z-transform is $X^2(z) = \frac{1}{1-z^{-1}}$.

It has a pole on unit circle the ROC does not contain the unit circle.

But. in Signal and system we said that its DTFT is

$$X^f(\omega) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=\infty}^{\infty} \delta(\omega - 2\pi k)$$

.**Evaluate z-transform on Unit Circle except at poles**.

.**Account for poles at $z = 1$. which is at $\omega = 2\pi k$**

4 Symmetry Properties of the DTFT

Symmetry Properties of the DTFT

In this section, we will allow the signal to be complex valued in general (Obviously, the DTFT is also in general complex values).

$$x[n] = x_R[n] + jx_I[n]$$
$$X^f(\omega) = X_R^f(\omega) + jX_I^f(\omega)$$

$$X^f(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[x_R[n] + jx_I[n]\right]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \Big[x_R[n] + jx_I[n]\Big]\Big[\cos(\omega n) - j\sin(\omega n)\Big]$$

$$= \sum_{n=-\infty}^{\infty} \Big[x_R[n]\cos(\omega n) + x_I[n]\sin(\omega n)\Big] + j\Big[-x_R[n]\sin(\omega n) + x_I[n]\cos(\omega n)\Big]$$

Thus .. $X^f(\omega) = X_R^f(\omega) + jX_I^f(\omega)$ with.
$X_R^f(\omega) = \sum_{n=-\infty}^{\infty} \Big[x_R[n]\cos(\omega n) + x_I[n]\sin(\omega n)\Big]$
$X_I^f(\omega) = -\sum_{n=-\infty}^{\infty} \Big[x_R[n]\sin(\omega n) - x_I[n]\cos(\omega n)\Big]$
Similarly, $x[n] = x_R[n] + jx_I[n]$ with.
$X_R[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[X_R^f(\omega)\cos(\omega n) - X_I^f(\omega)\sin(\omega n)\big]d\omega$
$X_I[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[X_R^f(\omega)\sin(\omega n) - X_I^f(\omega)\cos(\omega n)\big]d\omega$
We can now use these general result to explore several special cases!

Special Case: Real Signals

For real signal $x_I[n] = 0$, and $x_R[n] = x[n]$.
$X_R[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[X_R^f(\omega)\cos(\omega n) - 0\big]d\omega$
$X_I[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[X_R^f(\omega)\sin(\omega n) - 0\big]d\omega$
$X_R^f(-\omega) = X_R^f(\omega)$, as cos is even function so function is also even.
$X_I^f(-\omega) = -X_I^f(\omega)$, as sin is odd function so function is also odd.
$X^{f*} = X^f(-\omega)$,
Magnitude: $|X^f(-\omega)| = |X^f(\omega)|$,
Phase : $\angle X^f(-\omega) = -\angle X^f(\omega)$.
Proof : $X^f(-\omega) = X_R^f(\omega) + jX_I^f(-\omega) = X_R^f(\omega) - jX_I^f(\omega) = X^{f*}(\omega)$
So, $x[n] = x_R[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big[X_R^f(\omega)\cos(\omega n) - X_I^f(\omega)\sin(\omega n)\big]d\omega$
If you look as the term First term consist of Even term x Even Term = Even Term. While Second term is Even = Odd x odd.

and so we need to integrate over half range and then multiply with 2.

$$x[n] = \frac{1}{\pi}\int_0^{\pi} \big[X_R^f(\omega)\cos(\omega n) - X_I^f(\omega)\sin(\omega n)\big]d\omega$$

Special Case: Real and Even Signals

From the fact that $x[n]$ is real we recall that,
$X_R^f(\omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)$ which is Even = Even x Even.
$X_I^f(\omega) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)$ which is Odd = Even x odd.
Overall $X_R^f(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)$ and $X_I^f(\omega) = 0$. So Even and Real function is

$$X^f(\omega) = X_R^f(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)$$

From the fact that $x[n]$ is real we recall that,

$$x[n] = \frac{1}{\pi}\int_0^{\pi} \big[X_R^f(\omega)\cos(\omega n) - X_I^f(\omega)\sin(\omega n)\big]d\omega$$

$$x[n] = \frac{1}{\pi}\int_0^{\pi} X_R^f(\omega)\cos(\omega n)d\omega$$

Special Case: Real and Odd Signals

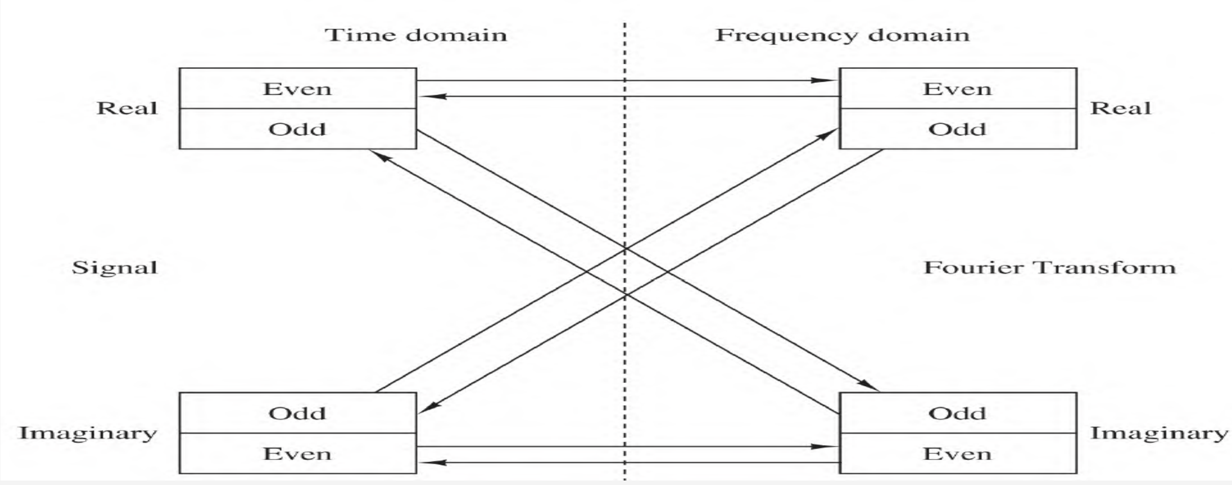
Using a similar arguments,
$X_R^f(\omega) = 0$.
$X_I^f(\omega) = -2\sum_{n=-1}^{\infty} x[n]\sin(\omega n)$.
So Odd and Imaginary function is

$$X^f(\omega) = jX_I^f(\omega) = -2j\sum_{n=1}^{\infty} x[n]\sin(\omega n)$$

From the fact that $x[n]$ is real we recall that,

$$x[n] = \frac{1}{\pi}\int_0^{\pi} \big[X_R^f(\omega)\cos(\omega n) - X_I^f(\omega)\sin(\omega n)\big]d\omega$$

$$x[n] = -\frac{1}{\pi}\int_0^{\pi} X_I^f(\omega)\sin(\omega n)d\omega$$



Special Case: Purely Imaginary Signal

From the fact that $x_R[n] = 0$ and $x_I[n] = x[n]$,
$X_R^f[\omega] = \sum_{n=-\infty}^{\infty} \left[0 + X_I[n]\sin(\omega n)\right]$
$X_I^f[\omega] = \sum_{n=-\infty}^{\infty} \left[0 - X_I[n]\cos(\omega n)\right]$
$X_R^f(\omega) = \sum_{n=-\infty} \infty x[n]\sin(\omega n)$ and $X_I^f(\omega) = \sum_{n=-\infty} \infty x[n]\cos(\omega n)$
$X_R^f(-\omega) = -X_R^f(\omega)$, as sin is odd function so function is also odd.
$X_I^f(-\omega) = X_I^f(\omega)$, as cos is even function so function is also even.
So,

$$x[n] = \frac{1}{\pi}\int_0^{\pi} \left[X_R^f(\omega)\sin(\omega n) - X_I^f(\omega)\cos(\omega n)\right] d\omega$$

Special Case: Purely Imaginary Signal and Even

Here, $X_R^f(\omega) = 0$ and $X_I^f(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)$
So, Even and Imaginary Function is :

$$X^f(\omega) = jX_I^f(\omega) = j\Big[x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)\Big]$$

$$x[n] = \frac{1}{\pi}\int_0^{\pi} X_I^f(\omega)\cos(\omega n) d\omega$$

Special Case: Purely Imaginary Signal and Odd

Here, $X_I^f(\omega) = 0$ and $X_R^f(\omega) = 2\sum_{n=1}^{\infty} x[n]\sin(\omega n)$
So, Odd and Real Function is :

$$X^f(\omega) = X_R^f(\omega) = \Big[2\sum_{n=1}^{\infty} x[n]\sin(\omega n)\Big]$$

$$x[n] = \frac{1}{\pi}\int_0^{\pi} X_R^f(\omega)\sin(\omega n) d\omega$$

Properties of Fourier Transform

Summery of symmetry properties for the Fourier Transform

Questions?

EC303 - Digital Signal Processing

M04L003 - System Analysis by DTFT

Dr. M. C. Patel

Autumn Semester, 2025-26

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1 Ideal Low-Pass Filter

ideal LP filter

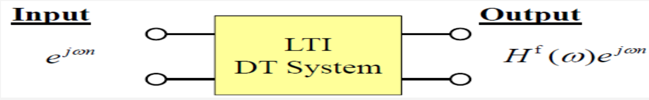
As for a Continuous Time system, hypothesize this:

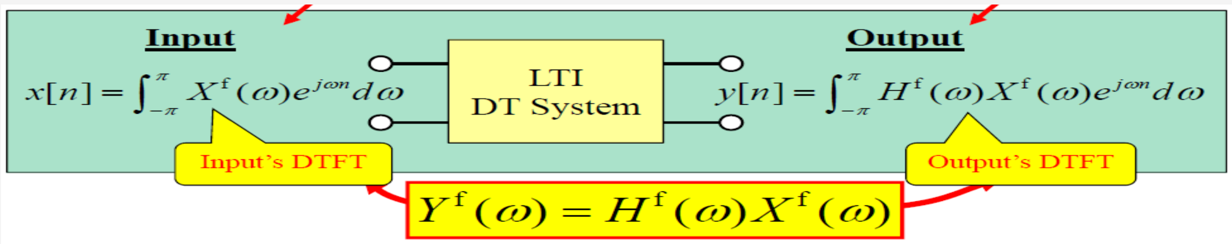
$$x[n] = e^{j\omega n} \rightarrow y[n] = H^f(\omega)e^{j\omega n}$$

Now substitute into this Differential equation the hypothesized input and output:

$$y[n] = a_1 y[n-1] + ... + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

- Sub In: $H^f(\omega)e^{j\omega n} + a_1 H^f(\omega)e^{j\omega(n-1)} + ... + a_N H^f(\omega)e^{j\omega(n-N)} = b_0 e^{j\omega n} + b_1 e^{j\omega(n-1)} + ... + b_M e^{j\omega(n-M)}$
- Algebra: $H^f(\omega)e^{j\omega n}\left[1 + a_1 e^{j\omega(-1)} + ... + a_N e^{j\omega(-N)}\right] = e^{j\omega n}\left[b_0 + b_1 e^{j\omega(-1)} + ... + b_M e^{j\omega(-M)}\right]$
- Algebra: $H^f(\omega) = \frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega M}}$
- **So. can just write $H^f(\omega)$ by inspection of differential equation coefficient**





1.1 LTI System Response to a Sinusoid

System Response to a Sinusoid

We have just shown that

$$x[n] = e^{j\omega n} \to y[n] = H^f(\omega)e^{j\omega n}$$

By using Euler's formula and linearity we can extend this to:

$$x[n] = A\cos(\omega_0 n + \theta)$$

which means

$$y[n] = |H^f(\omega_0)A\cos(\omega_0 n + \theta + \angle H^f(\omega_0))|$$

This tells us that a discrete time LTI system does two things to a sinusoidal input:

- It change its amplitude multiplicatively with a factor of $|H^f(\omega_0)|$.
- It change its phase additively with factor $\angle H^f(\omega_0)$.

.

Alternate way to find the frequency response: Take the DTFT of the difference equation and use the delay property:

$$y[n] = a_1 y[n-1] + ... + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

- DTFT : $DTFT\{y[n] = a_1 y[n-1] + ... + a_N y[n-N]\} = DTFT\{b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]\}$
- Applying Delay Property: $Y^f(\omega)+a_1Y^f(\omega)e^{-j\omega}+...+a_NY^f(\omega)e^{-j\omega N} = b_0X^f(\omega)+b_1X^f(\omega)e^{-j\omega}+...+b_MX^f(\omega)e^{-j\omega M}$
- Algebra: $Y^f(\omega)e^{j\omega n}\left[1 + a_1e^{-j\omega} + ... + a_N e^{-j\omega N}\right] = X^f(\omega)\left[b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}\right]$
- Algebra:
$$Y^f(\omega) = \left[\frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega M}}\right] X^f(\omega)$$
- **Same result as on previous page..2**

2 System Analysis Via DTFT (Graphical)

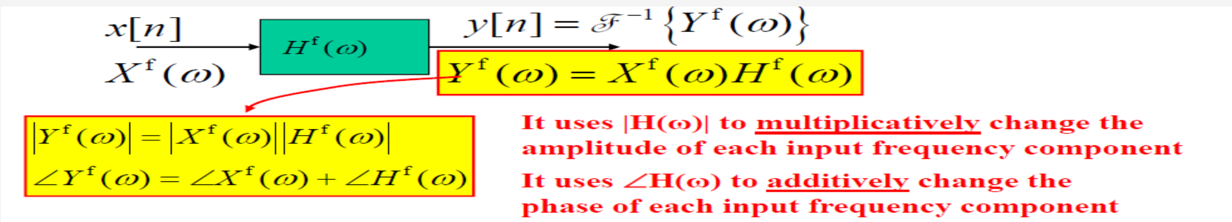
System Analysis via DTFT

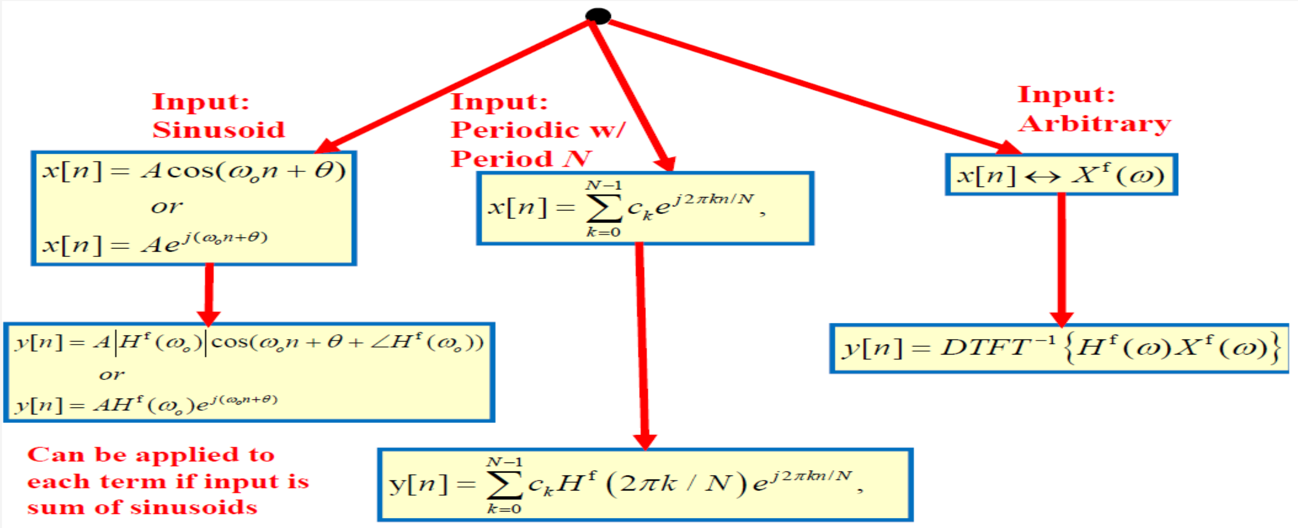
Recall the definition of the frequency response:
Input $x[n]$ is linear combination of sinusoids.. the output is a linear combination.
So we have
So in genral we see that the system frequency response re-shape the input DTFT's magnitude and phase.
$\Rightarrow$ System Can:

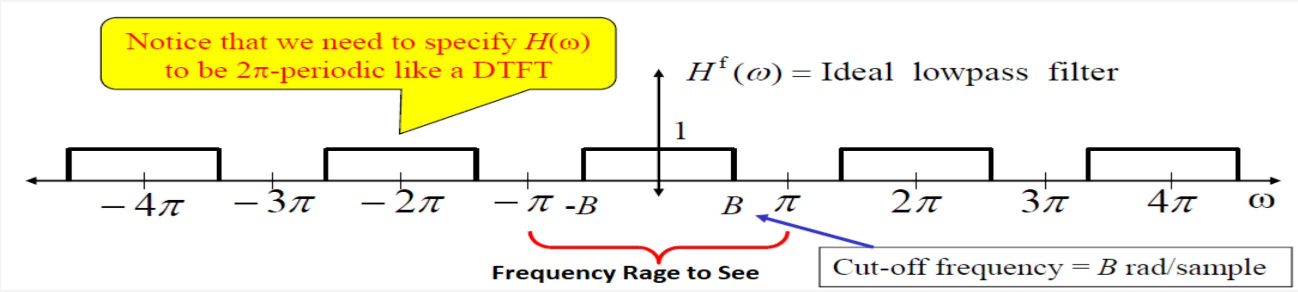
- Emphasize some frequency.
- De-emphasize other frequency.

Actually this is the perfectly parallel to the same ideas for continuous system...!!!.

The above shows how to use DTFT to do genral Discrete Time system analyses.. and it is virtually same as for the Continous time case!!!







Three main ways to Use frequency Response for DT-LTI system

3 Filter Example Using DTFT

Example: Ideal DT lowpass filter (LPF)

We will see later that we can't really build such an "ideal" filter but we can strive to get very close.. As always with DT .... we only need to look the frequency range of $-\pi$ to π

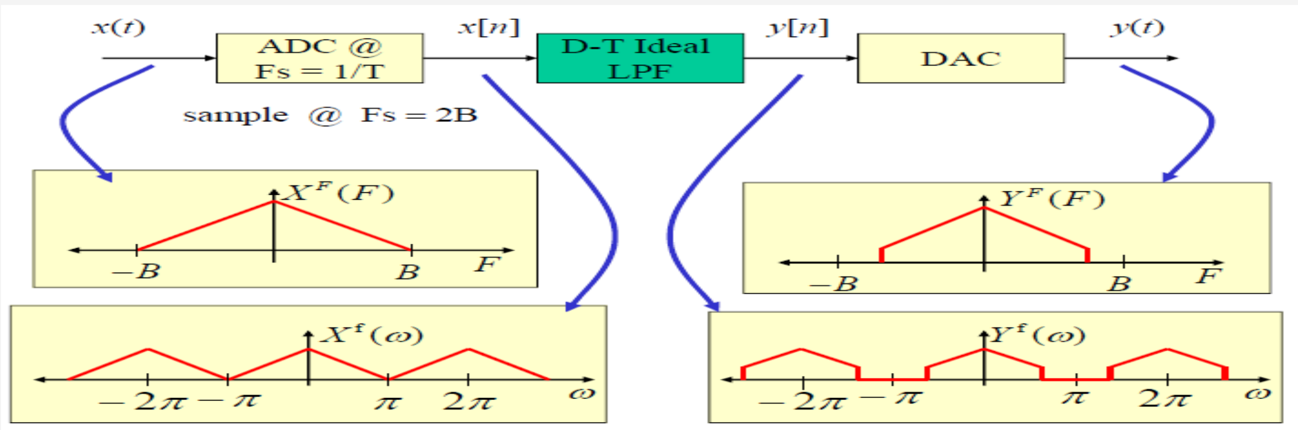
Ideal Filters can't be built in practice

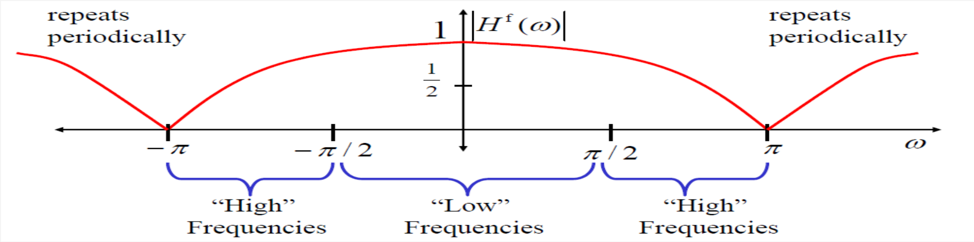
This slide shows how a DT filter might be employed. but ideal filter can't be built in practice. We will see later a few practical DT filter **Whole Sytem (ADC-DT filter - DAC) acts like an equivalent CT system**

4 Example of Simple Filter

Simple Non-recursive filter

Here is very simple, low quality LPF. It difference equation and block diagram are:





$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

The general results for differential equation and frequency response are:

$$y[n] + a_1 y[n-1] + ... + a_N y[n-M] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

and

$$H^f(\omega) = \frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega N}}$$

Note that the given filter has none of the so-called feedback terms. such a filter is called a non-recursive filter. Using the general result for this filter gives.

$$H^f(\omega) = \frac{1}{2}\left[1 + e^{-j\omega}\right]$$

Magnitude Response of Filter

Now, to see what this looks like we find its manitude...

$$\begin{aligned} H^f(\omega) = & \quad \tfrac{1}{2}\left[1 + e^{-j\omega}\right] \\ = & \quad \tfrac{1}{2}\left[(1 + \cos(\omega)) - j\sin(\omega)\right] \end{aligned}$$

Now,

$$\begin{aligned} |H^f(\omega)| = & \quad \sqrt{\left[\tfrac{1}{2}(1+\cos(\omega))\right]^2 + (-\tfrac{1}{2}\sin(\omega))^2} \\ = & \quad \tfrac{1}{2}\sqrt{1 + 2\cos(\omega) + \cos^2(\omega) + \sin^2(\omega)} \\ = & \quad \tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\omega)} \end{aligned}$$

Now, Plot this to see if it is a good LPF !

Magnitude Response Plot

Example

Here is a plot of this filter's frequency response magnitude:

- Well this does attenuate high frequency but doesn't really stop them !.
- It is a low pass filter but not a very good one!.
- How do we make a better LPF???
- We could try longer non-recursive filter having N terms:

$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

Magnitude Response Plot with More N

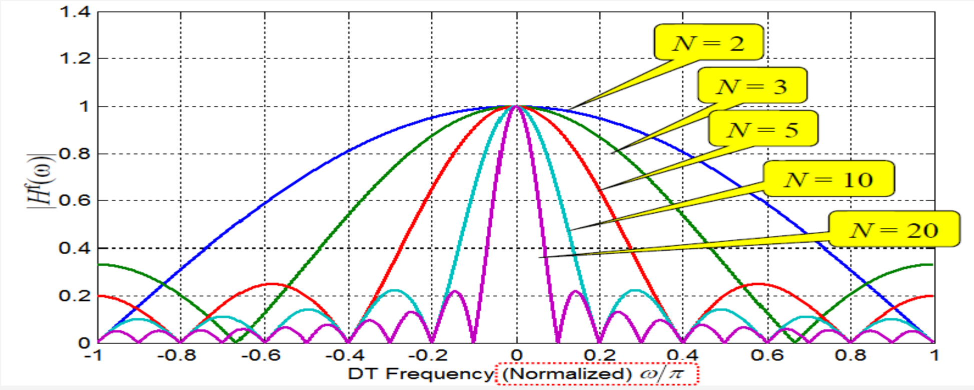
Plot of frequency response for various N values

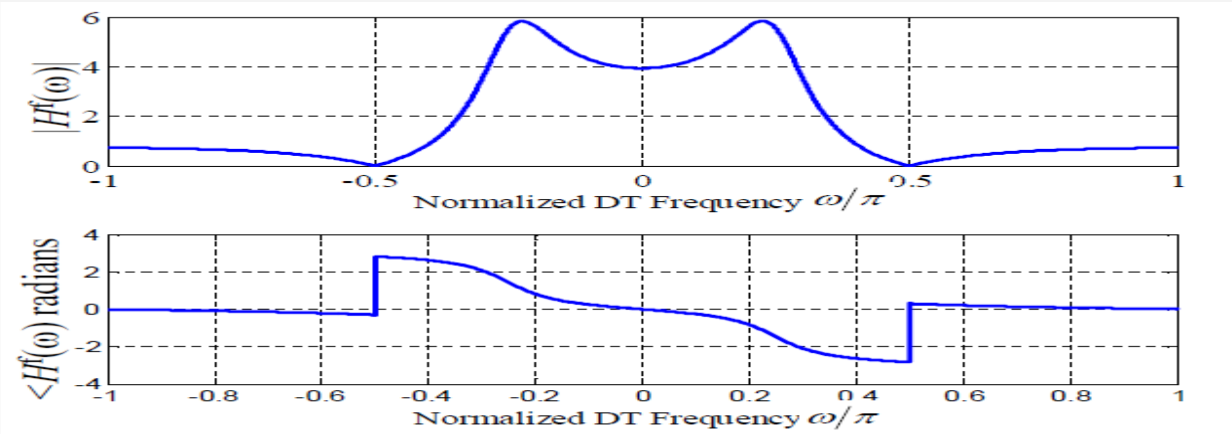
$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

$$H^f(\omega) = \frac{1}{N} + \frac{1}{N}e^{-j\omega} + ... + \frac{1}{N}e^{-j\omega M}$$

Increasing the length cause the passband to get narrower. but the quality of the filter doesn't better. so we generally need other types of filters.

We will see that better filters can be made from this form by allowing the coefficients to be non-uniform!!!!!





5 MATLAB Command to compute the DT Frequency Response

MATLAB command to compute the DT frequency Response

$H = freqz(b, a, w)$ gives frequency response points in vector H at the frequency points in vector w.

$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

$$H^f(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + ... + b_M e^{-j\Omega M}}{1 + a_1 e^{-j\Omega} + ... + a_N e^{-j\Omega N}}$$

The numerator and denominator coefficient from vector b and a is used in $freqz$ command.

Example

Solution

system : $y[n] - 1.1314y[n-1] + 0.64y[n-2] = x[n] + x[n-2]$ So $H^f(\omega) = \frac{1+1e^{-j2\omega}}{1-1.1314e^{-j\omega}+0.64e^{-j2\omega}}$

```
>> w = linspace(-pi,pi,2000);
>> a = [1 -1.1314 0.64];
>> b = [1 0 1];
>> H = freqz(b,a,w);
>> subplot (2,1,1);
>> plot (w/pi,abs(H));
>> subplot (2,1,2);
>> plot (w/pi,angle(H));
```

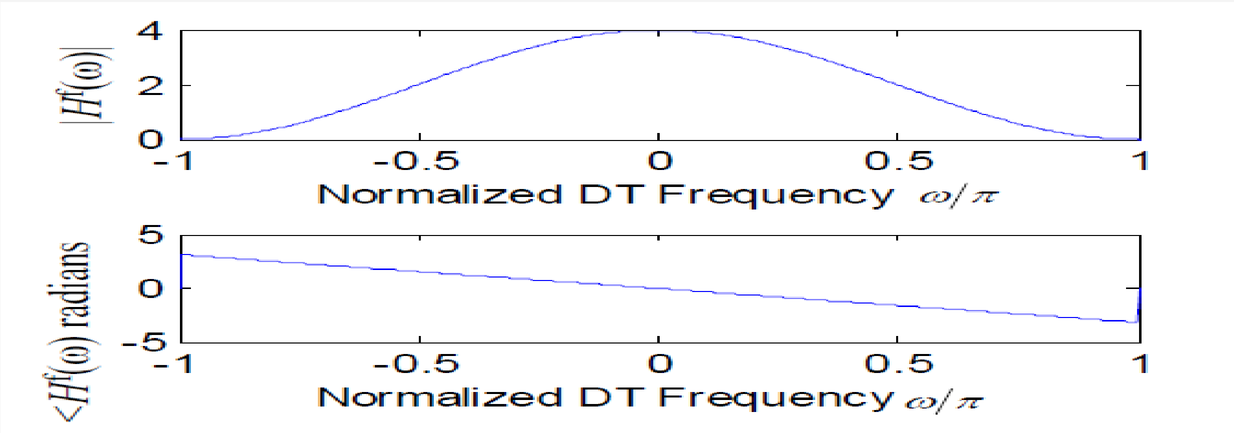
Example

Solution

Non recursive filter have no feedback and system : $y[n] = x[n] + 2x[n-1] + 1x[n-2]$ So $H^f(\omega) = \frac{1+2e^{-j1\omega}+1e^{-j2\omega}}{1}$

```
>> w = linspace(-pi,pi,2000);
>> a = [1];
>> b = [1 2 1];
>> H = freqz(b,a,w);
>> subplot (2,1,1);
>> plot (w/pi,abs(H));
>> subplot (2,1,2);
>> plot (w/pi,angle(H));
```

Questions?



$|H^f(\omega)|$
4
2
0
-1
-0.5
0
0.5
1
Normalized DT Frequency ω/π
$<H^f(\omega)$ radians
5
0
-5
-1
-0.5
0
0.5
1
Normalized DT Frequency ω/π

EC303 - Digital Signal Processing

M04L004 - Simple Filter Design by DTFT

Dr. M. C. Patel

Autumn Semester, 2025-26

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Finding The Frequency Response from Difference Equation

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Now substitute into this Differential equation the hypothesized input and output:

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- Algebra: $H^f(\omega)e^{j\omega n}\left[1 + a_1 e^{j\omega(-1)} + ... + a_N e^{j\omega(-N)}\right] = e^{j\omega n}\left[b_0 + b_1 e^{j\omega(-1)} + ... + b_M e^{j\omega(-M)}\right]$
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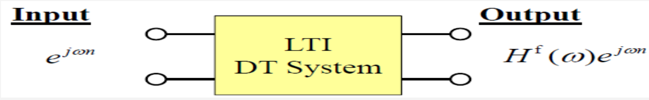
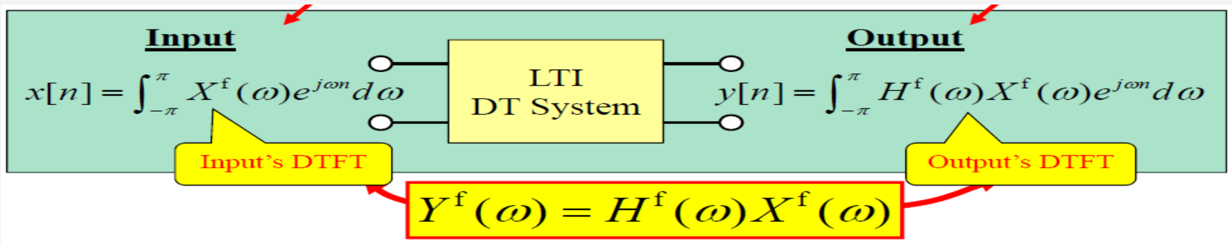
- **So. can just write $H^f(\omega)$ by inspection of differential equation coefficient**





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- DTFT : $DTFT\{y[n] = a_1 y[n-1] + ... + a_N y[n-N]\} = DTFT\{b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]\}$
- Applying Delay Property: $Y^f(\omega) + a_1 Y^f(\omega)e^{-j\omega} + ... + a_N Y^f(\omega)e^{-j\omega N} = b_0 X^f(\omega) + b_1 X^f(\omega)e^{-j\omega} + ... + b_M X^f(\omega)e^{-j\omega M}$
- Algebra: $Y^f(\omega)e^{j\omega n}\left[1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega N}\right] = X^f(\omega)\left[b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}\right]$
- Algebra:
$$Y^f(\omega) = \left[\frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega M}}\right] X^f(\omega)$$
- **Same result as on previous page..2**

2 System Analysis Via DTFT (Graphical)

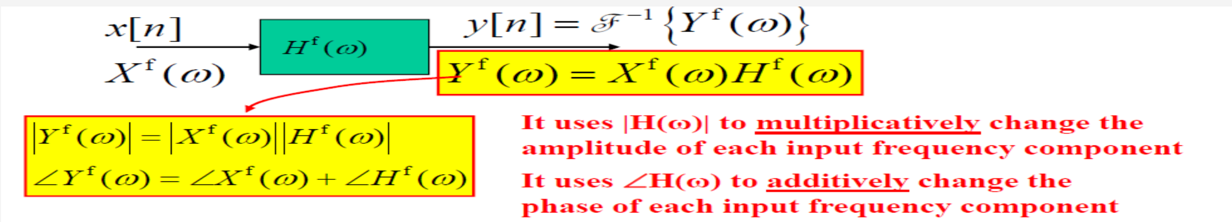
System Analysis via DTFT

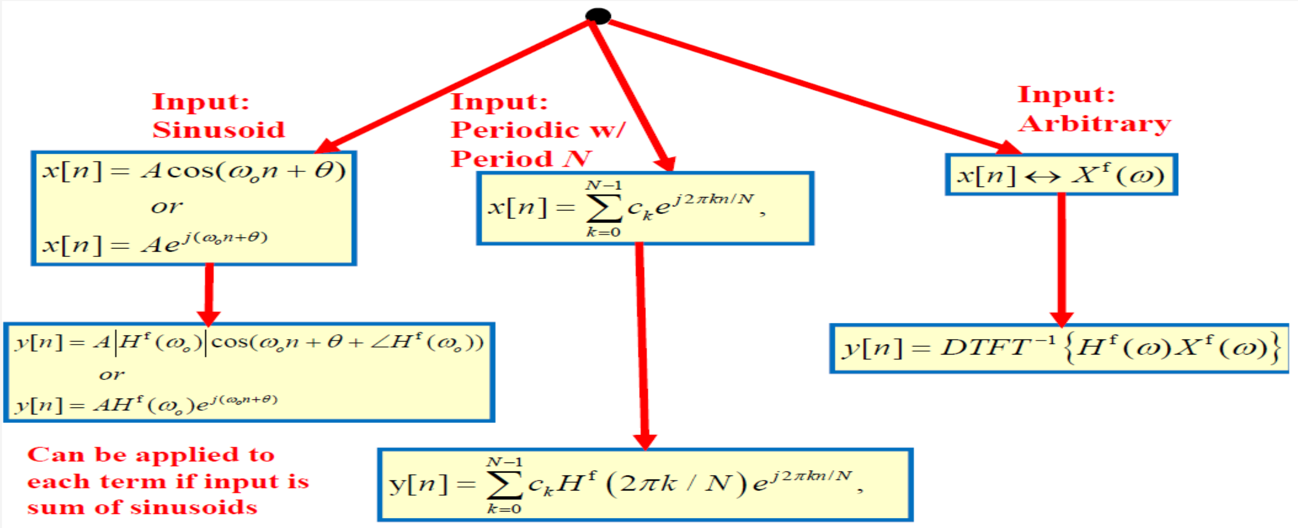
Recall the definition of the frequency response:
Input $x[n]$ is linear combination of sinusoids.. the output is a linear combination.
So we have
So in genral we see that the system frequency response re-shape the input DTFT's magnitude and phase.
$\Rightarrow$ System Can:

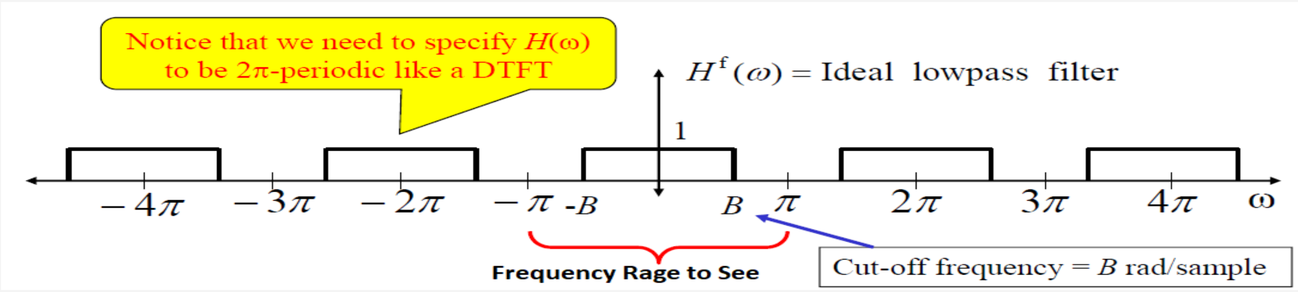
- Emphasize some frequency.
- De-emphasize other frequency.

Actually this is the perfectly parallel to the same ideas for continuous system...!!!.

The above shows how to use DTFT to do genral Discrete Time system analyses.. and it is virtually same as for the Continous time case!!!







Three main ways to Use frequency Response for DT-LTI system

3 Filter Example Using DTFT

Example: Ideal DT lowpass filter (LPF)

We will see later that we can't really build such an "ideal" filter but we can strive to get very close.. As always with DT .... we only need to look the frequency range of $-\pi$ to π

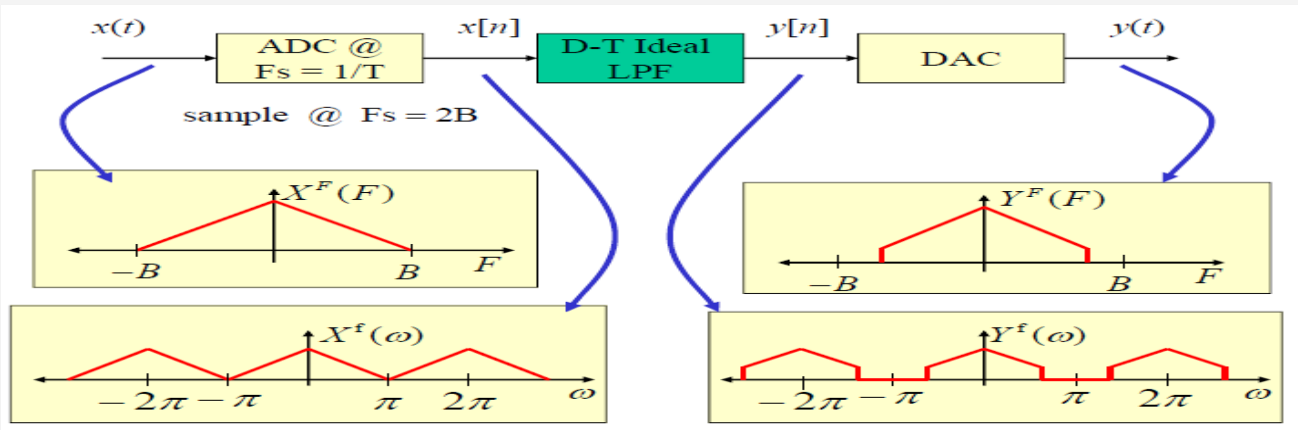
Ideal Filters can't be built in practice

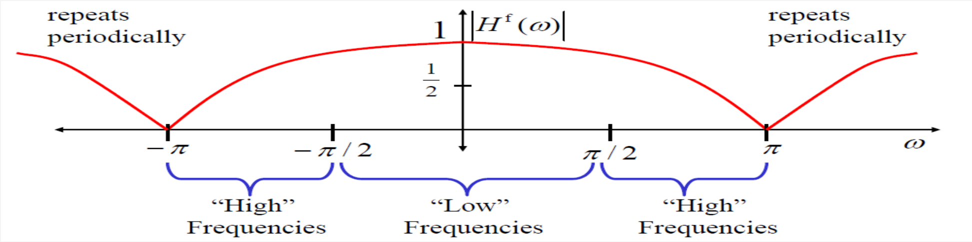
This slide shows how a DT filter might be employed. but ideal filter can't be built in practice. We will see later a few practical DT filter **Whole Sytem (ADC-DT filter - DAC) acts like an equivalent CT system**

4 Example of Simple Filter

Simple Non-recursive filter

Here is very simple, low quality LPF. It difference equation and block diagram are:





$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

The general results for differential equation and frequency response are:

$$y[n] + a_1 y[n-1] + ... + a_N y[n-M] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

and

$$H^f(\omega) = \frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega N}}$$

Note that the given filter has none of the so-called feedback terms. such a filter is called a non-recursive filter. Using the general result for this filter gives.

$$H^f(\omega) = \frac{1}{2}\left[1 + e^{-j\omega}\right]$$

Magnitude Response of Filter

Now, to see what this looks like we find its manitude...

$$\begin{aligned} H^f(\omega) &= \tfrac{1}{2}\left[1 + e^{-j\omega}\right] \\ &= \tfrac{1}{2}\left[(1 + \cos(\omega)) - j\sin(\omega)\right] \end{aligned}$$

Now,

$$\begin{aligned} |H^f(\omega)| &= \sqrt{\left[\tfrac{1}{2}(1+\cos(\omega))\right]^2 + (-\tfrac{1}{2}\sin(\omega))^2} \\ &= \tfrac{1}{2}\sqrt{1 + 2\cos(\omega) + \cos^2(\omega) + \sin^2(\omega)} \\ &= \tfrac{1}{\sqrt{2}}\sqrt{1 + \cos(\omega)} \end{aligned}$$

Now, Plot this to see if it is a good LPF !

Magnitude Response Plot

Example

Here is a plot of this filter's frequency response magnitude:

- Well this does attenuate high frequency but doesn't really stop them !.
- It is a low pass filter but not a very good one!.
- How do we make a better LPF???
- We could try longer non-recursive filter having N terms:

$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

Magnitude Response Plot with More N

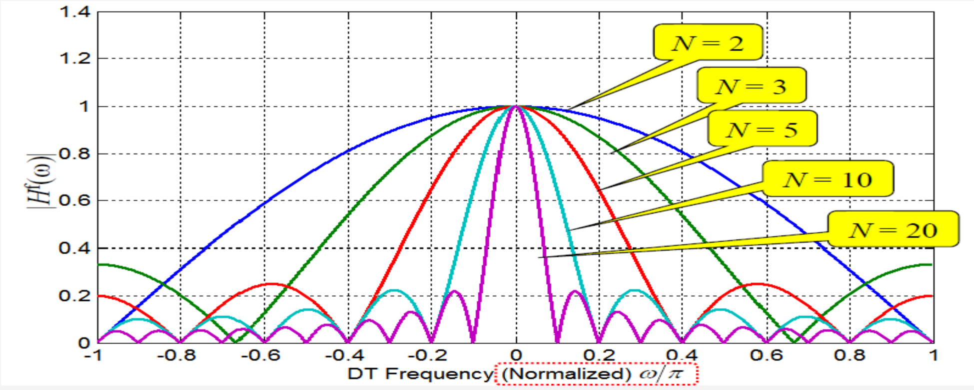
Plot of frequency response for various N values

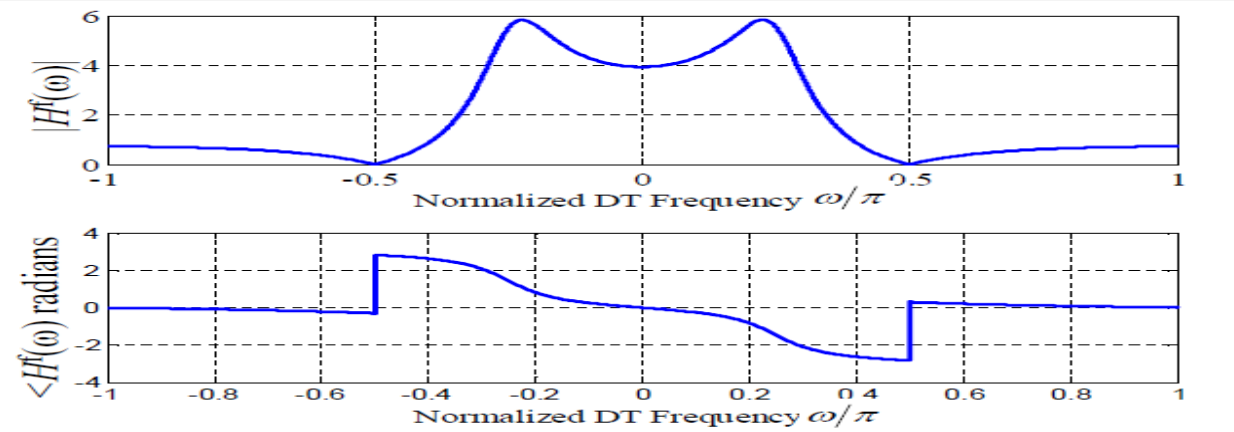
$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

$$H^f(\omega) = \frac{1}{N} + \frac{1}{N}e^{-j\omega} + ... + \frac{1}{N}e^{-j\omega M}$$

Increasing the length cause the passband to get narrower. but the quality of the filter doesn't better. so we generally need other types of filters.

We will see that better filters can be made from this form by allowing the coefficients to be non-uniform!!!!!





5 MATLAB Command to compute the DT Frequency Response

MATLAB command to compute the DT frequency Response

$H = freqz(b, a, w)$ gives frequency response points in vector H at the frequency points in vector w.

$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + ... + \frac{1}{N}x[n-(N-1)]$$

$$H^f(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + ... + b_M e^{-j\Omega M}}{1 + a_1 e^{-j\Omega} + ... + a_N e^{-j\Omega N}}$$

The numerator and denominator coefficient from vector b and a is used in $freqz$ command.

Example

Solution

system : $y[n] - 1.1314y[n-1] + 0.64y[n-2] = x[n] + x[n-2]$ So $H^f(\omega) = \frac{1+1e^{-j2\omega}}{1-1.1314e^{-j\omega}+0.64e^{-j2\omega}}$

```
>> w = linspace(-pi,pi,2000);
>> a = [1 -1.1314 0.64];
>> b = [1 0 1];
>> H = freqz(b,a,w);
>> subplot (2,1,1);
>> plot (w/pi,abs(H));
>> subplot (2,1,2);
>> plot (w/pi,angle(H));
```

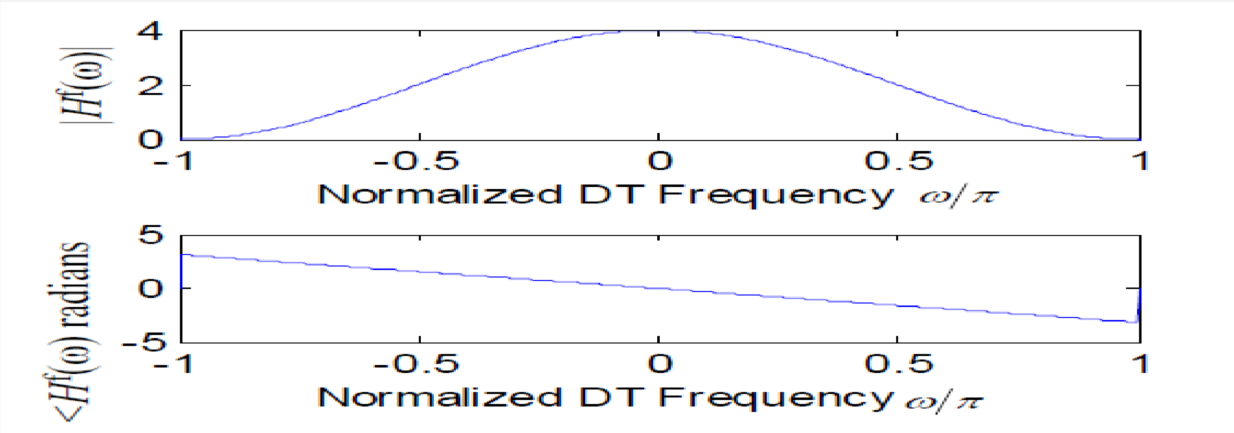
Example

Solution

Non recursive filter have no feedback and system : $y[n] = x[n] + 2x[n-1] + 1x[n-2]$ So $H^f(\omega) = \frac{1+2e^{-j1\omega}+1e^{-j2\omega}}{1}$

```
>> w = linspace(-pi,pi,2000);
>> a = [1];
>> b = [1 2 1];
>> H = freqz(b,a,w);
>> subplot (2,1,1);
>> plot (w/pi,abs(H));
>> subplot (2,1,2);
>> plot (w/pi,angle(H));
```

Questions?


|Hf(ω)|
4
2
0
-1
-0.5
0
0.5
1
Normalized DT Frequency ω/π
<Hf(ω) radians
5
0
-5
-1
-0.5
0
0.5
1
Normalized DT Frequency ω/π

EC303 - Digital Signal Processing

M04L005 - Few More Filter Design by DTFT

Dr. M. C. Patel

Autumn Semester, 2025-26

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1 DT Filter

LTI-Ideal LP Filter

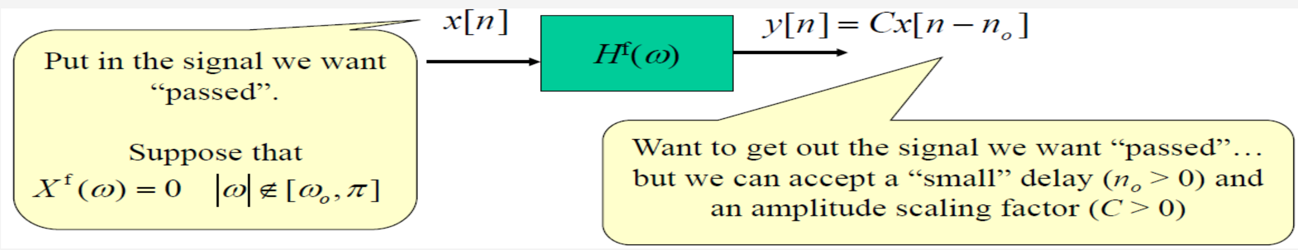
from the time-shift property of the DTFT then we need:

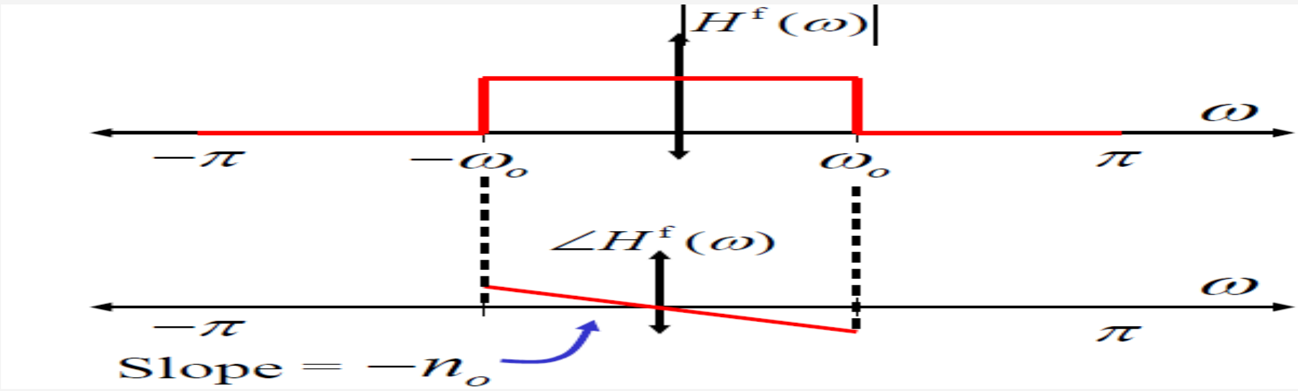
$$Y^f(\omega) = X^f(\omega)Ce^{-j\omega n_0} \tag{1}$$

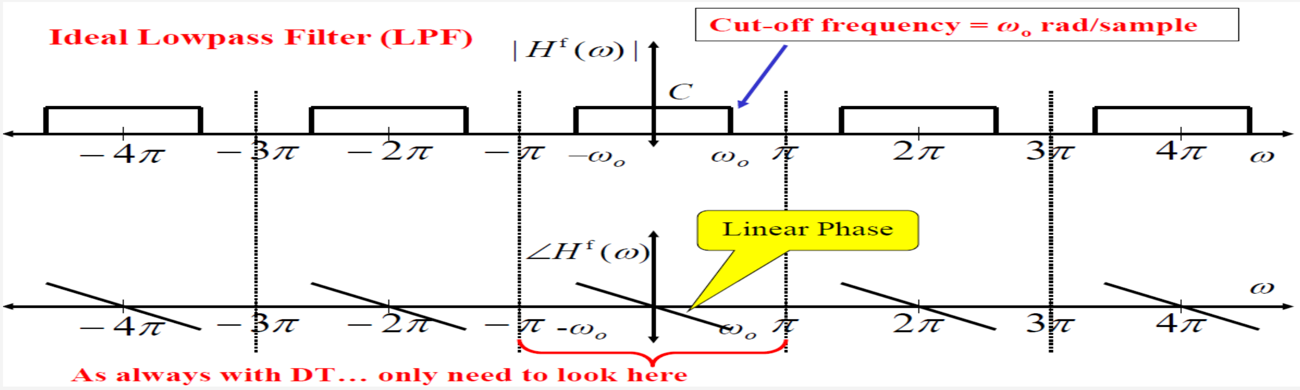
Typically the C is consider as 1.

Thus we should treat this as $H^f(\omega) = \frac{Y^f(\omega)}{X^f(\omega)}$, so we have:

- ▷ Magnitude : $|H^f(\omega)| = |Ce^{-j\omega n_0}| = C$. For ω in the "Pass band" of the filter $\omega \in [-\omega_0, \omega_0]$.
- ▷ Phase : $\angle H^f(\omega) = \angle Ce^{-j\omega n_0} = -\omega n_0$. Line of slope $-n_0$ is "Linear Phase".







So.. for an ideal low-pass filter (LPF we have) Magnitude and Phase response as.

$$H(\omega) = \begin{cases} Ce^{-j\omega n} & , -\omega_0 < \omega < \omega_0 \\ 0 & , \text{ Otherwise.} \end{cases} \tag{2}$$

If you refer to phase in stop-band it is : $H^f(\omega) = 0 = 0e^{j\theta}$. so Calculate $\angle 0 =?$ it is either 0 or $\pi/2$.
i.e. phase is undefined for frequencies outside the ideal pass-band.

Summery of Ideal Filters

Here is final Summery:

1. For Magnitude Response:
   (a) Constant in Passband.
   (b) Zero in Stopband.
2. For Phase Response:
   (a) Linear in Passband. (negative slope = delay)
   (b) Undefined in Stopband.

Ideal Filters

Remember that for DT frequency response is a DTFT so is periodic.

2 Why can't an ideal filter exists in practice??

Why can't an ideal filter exist in practice??

To answer this, we will find the filter's impulse response, which is the IDTFT of the frequency response.
The frequency response of the ideal LPF is:

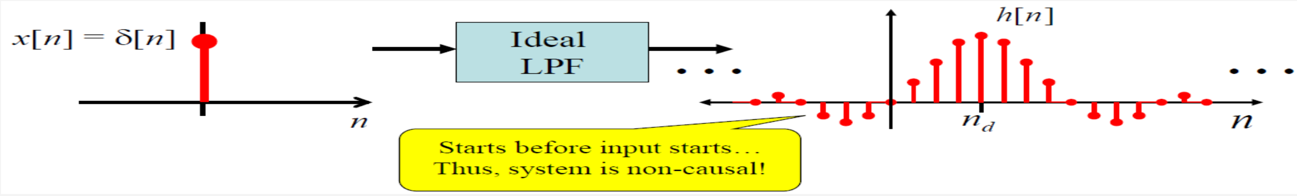
$$H(\omega) = \begin{cases} Ce^{-j\omega n} & , -\omega_0 < \omega < \omega_0 \\ 0 & , \text{ Otherwise.} \end{cases} \tag{3}$$

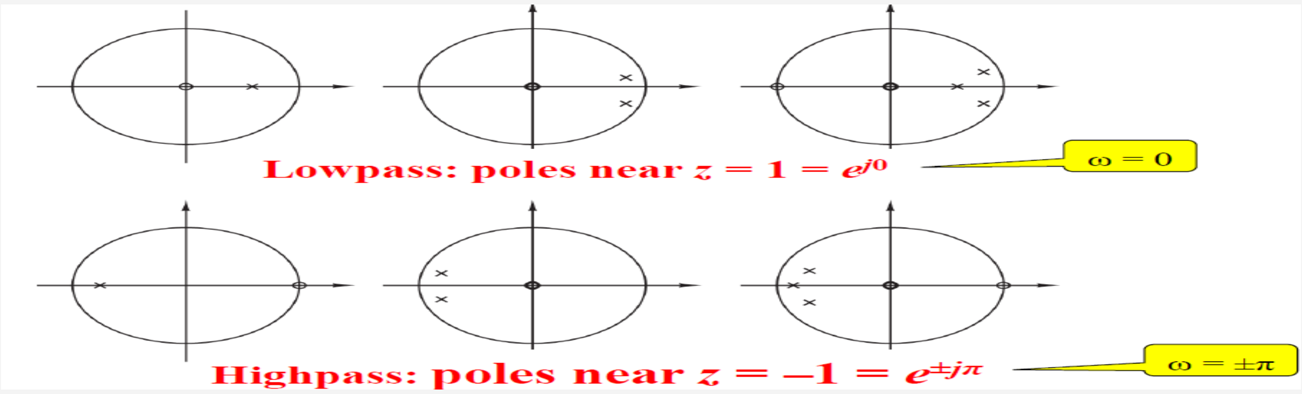
Using the IDTFT of a rectangle together with time-shift properties gives impulse response as:

$$h[n] = (\frac{\omega_0}{\pi}) sinc\big[(\frac{\omega_0}{\pi})(n - n_0)\big] \tag{4}$$

2.1 Type of ideal Filter

Types of Ideal Filter





So far we've limited discussion to ideal lowpass filters. These ideas can be extended to other filter types. To be ideal they need to have.

- Constant Magnitude.
- Constant Phase

in their passband.

Note: Although it is not shown here, all of these repeat periodically outside $[-\pi, \pi]$

3 Poles Zero Placement to Yield Filter Types

Pole-Zero Placement to Yield Filter Types

Although there are high-powered methods of filter design.... it is useful to understand how to achieve some simple filters via proper placement of poles and zeros.

For that it is essential to convert DTFT domain to z-transform... How???

Put $e^{j\omega} = z$ in the system equation:

$$H^f(\omega) = \left[\frac{b_0 + b_1 e^{-j\omega} + ... + b_M e^{-j\omega M}}{1 + a_1 e^{-j\omega} + ... + a_N e^{-j\omega M}}\right] \tag{5}$$

So It is :

$$H^z(z) = \left[\frac{b_0 + b_1 z^{-1} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + ... + a_N z^{-N}}\right] \tag{6}$$

Poles-Zero Placement for LPF

$$H^z(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 \prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \tag{7}$$

Here, b_0 sets overall gain of filter. The location of the poles p_k and zeros z_k impact the shape of the frequency response.

Effect of Poles-Zeros on Frequency Response of DT Filters

4 Example of Simple Low Pass Filter

Simple Low Pass Filter

Case - 1 for Low pass filter

$$H_1(z) = \frac{1-a}{1-az^{-1}} \tag{8}$$

Case - 2 for Low pass filter

$$H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}} \tag{9}$$

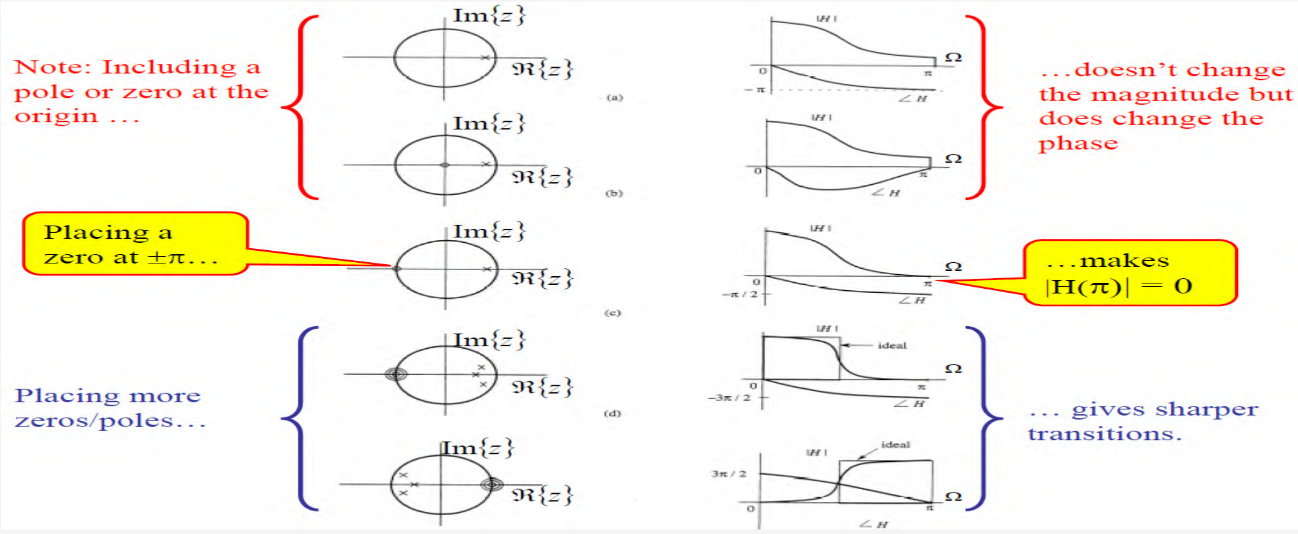


Figure 1: Figure from B. P. Lathi, Signal Processing and Linear System

Magnitude and Phase plot Comparison For Both Cases

The Comparison is carried out at $a = 0.9$. More phase linearity can be acquired by making zeros at farthest point.

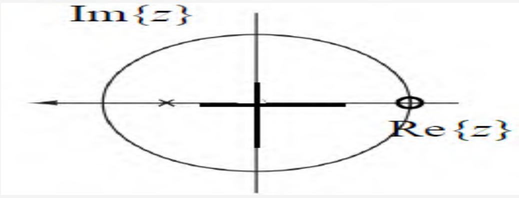


Figure 6: Poles zero Plot

A MATLAB Command:

```
w= linspace(-pi, pi, 2000);
b = 0.1; % Numerator Coefficient
a=[1, -0.9] % Denomintor Coefficient
H=freqz(b,a,w);
```

5 Example of Simple High Pass Filter

Simple High Pass Filter

Looking back at pole-zero plots for HPF and LPF, we see that each LPF can be converted into a HPF by flipping,

$$z \to -z \tag{10}$$

Then

$$LPF : H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}} \tag{11}$$

$$HPF : H_3(z) = H_2(-z) = \frac{1-a}{2}\frac{1-z^{-1}}{1+az^{-1}} \tag{12}$$

Simple High Pass Filter

The Magnitude and Phase Plot

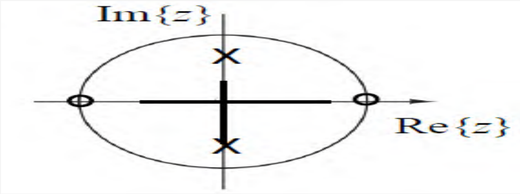


Figure 9: Poles zero Plot

6 A Simple Bandpass Filters

Simple band-pass Filter

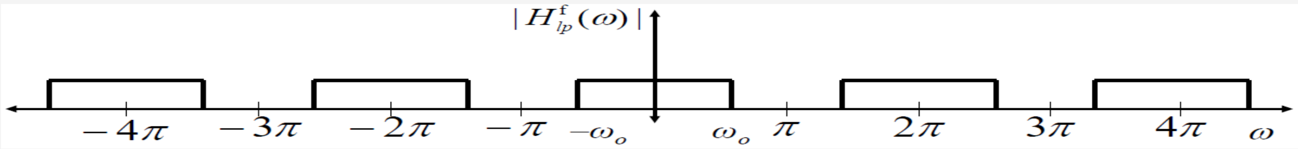
We can get a simple Band-pass filter, if we put poles at $p_{1,2} = re^{\pm j\pi/2}$. and zeros at $z = \pm 1$

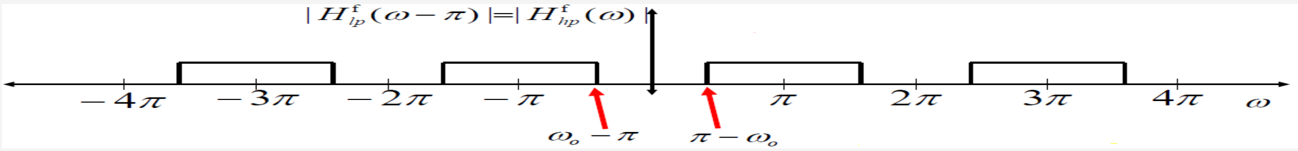
$$H(z) = G\left[\frac{(z-1)(z+1)}{(z-jr)(z+jr)}\right] \tag{13}$$

$$= G\left[\frac{z^2-1}{z^2+r^2}\right] \tag{14}$$

Simple band-pass Filter

The Magnitude and Phase Plot





7 Simple LPF to HPF Transformation

A Simple LPF-to-HPF Transformation

If we have a lowpass filter but want to use it as a way to create a highpass filter that is easily done as follows:We will illustrate the idea using an ideal LPF (even though those don't really exit!): Shift this frequency response by π rad/samples: This finally gives HPF!!!, $H_{hp}^{f}(\omega) = H_{lp}^{f}(\omega - \pi)$

A Simple LPF-to-HPF Transformation

So... This gives us what we want.... bit how do we actually *do* it???

If the frequency response of the LPF is given by:

$$H_{lp}^{f}(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k e^{-j\omega k}} \tag{15}$$

$$\Rightarrow H_{hp}^{f}(\omega) = H_{lp}^{f}(\omega-\pi) = \frac{\sum_{k=0}^{M} b_k e^{-j(\omega k-\pi)}}{1+\sum_{k=1}^{N} a_k e^{-j(\omega k-\pi)k}} \tag{16}$$

$$\Rightarrow H_{hp}^{f}(\omega) = \frac{\sum_{k=0}^{M} b_k e^{j\pi k} e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k e^{j\pi k} e^{-j\omega k}} \tag{17}$$

$$\text{Put } e^{j\pi} = -1 \quad . \tag{18}$$

$$\Rightarrow H_{hp}^{f}(\omega) = \frac{\sum_{k=0}^{M} b_k (-1)^k e^{-j\omega k}}{1+\sum_{k=1}^{N} a_k (-1)^k e^{-j\omega k}} \tag{19}$$

So LPF to HPF can be easily done by multiply $(-1)^k$ with low-pass coefficient.

A Simple LPF-to-HPF Transformation

Now changing the focus to the transfer function.

$$H_{lp}^{z}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=0}^{N} a_k z^{-k}} \tag{20}$$

$$H_{hp}^{z}(z) = \frac{\sum_{k=0}^{M} b_k (-1)^k z^{-k}}{1+\sum_{k=1}^{N} a_k (-z)^{-k}} \tag{21}$$

$$= \frac{\sum_{k=0}^{M} b_k (-z)^{-k}}{1+\sum_{k=1}^{N} a_k (-z)^{-k}} \tag{22}$$

$$= H_{lp}^{z}(-z) \tag{23}$$

So what do you mean by $H_{hp}^{z}(z) = H_{lp}^{z}(-z)$....?

$\Rightarrow$ It is just the flips the poles and zeros with respect to Imaginary and Real Axis.

Let's Take filter with real coefficients. Means it have pole-zero symmetry across the real axis.

$\Rightarrow$ It is just the flips the poles and zeros with respect to Imaginary and Real Axis.

A Simple LPF-to-HPF Transformation

And these results impact the difference equation view:

Let take Difference Equation for LPF as:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \tag{24}$$

and the Derived Difference equation for HPF is:

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k) \tag{25}$$

A Simple LPF-to-HPF Transformation

Suppose you don't have the transfer function, Frequency response function or difference equation... but only impulse response then.....

Applying the modulation (frequency shift) property of DTFT gives,

$$H^f_{hp}(\omega) = H^f_{lp}(\omega - \pi) \quad \Rightarrow h_{hp}[n] = e^{j\pi n} h_{lp}[n] \tag{26}$$

$$\Rightarrow h_{hp}[n] = (-1)^n h_{lp}[n] \tag{27}$$

8 Summary : LPF-to-HPF Transformation

Summery: LPF-to-HPF Transformation

- ⇒ Flip poles/zeros means: $H^z_{hp}(z) = H^z_{lp}(-z)$.
- ⇒ Shift the frequency response by π means:$H^f_{lp}(\omega) = H^f_{lp}(\omega - \pi)$.
- ⇒ Alternating sign change means: $h_{hp}[n] = (-1)^n h_{lp}[n]$.
- ⇒ coefficient value change means, $a_{k,hpf} = (-1)^k a_{k,lpf}$ and $b_{k,hpf} = (-1)^k b_{k,lpf}$

Questions?

EC303 - Digital Signal Processing

M04L005 - Some Useful Filter Design by DTFT

Dr. M. C. Patel

Autumn Semester, 2025-26

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1 Some Useful Filter Design by Pole-Zero Placement

1.1 Digital Oscillator

Digital Resonators

This filter has two complex-conjugate poles placed near the Unit-Circle to create a resonate peak at a desired frequency. Their locations determines characteristics:

- Angle will be approximately at the resonate peak.
- Radius determines how pronounced the peak is.

It has two zeros that can be placed where desired... usually either,

- Both at the origin.
- One at $z = 1$ ($\omega = 0$) and one at $z = -1$ ($\omega = \pm\pi$).

Zeros at Origin

$$\begin{aligned} H^z(z) &= \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} \\ &= \frac{b_0}{1-(2r\cos(\omega_0))z^{-1}+r^2z^{-2}} \end{aligned}$$

Zeros at $z = \pm 1$

$$\begin{aligned} H^z(z) &= G\frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} \\ &= G\frac{(1-z^{-2})}{1-(2r\cos(\omega_0))z^{-1}+r^2z^{-2}} \end{aligned}$$

Resonator with zeros at the origin

$$H^z(z) = \frac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} = \frac{b_0}{1-(2r\cos(\omega_0))z^{-1}+r^2z^{-2}}$$

Book Shows That,

$$\begin{aligned} \omega_r &= \cos^{-1}\left(\tfrac{1+r^2}{2r}\cos(\omega)\right) \\ \Delta &\equiv 2(1-r) \end{aligned}$$

Resonator with zeros at ± 1

$$H^z(z) = G\frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} = G\frac{(1-z^{-2})}{1-(2r\cos(\omega_0))z^{-1}+r^2z^{-2}}$$

1.2 Oscillator

Oscillator

$$\begin{aligned} H^z(z) &= \tfrac{b_0}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} \\ &= \tfrac{b_0}{1-(2r\cos(\omega_0))z^{-1}+r^2z^{-2}} \end{aligned}$$

So,

$$h(n) = \frac{b_0 r^n}{\sin(\omega_0)} \sin\big(\omega_0(n+1)\big)u(n)$$

If we put the pole on the unit circle ($r = 1$) then this impulse response does not decay and the system can be used as an oscillator.

For more details see the Section 5.4.7 of the textbook...!

1.3 Notch Filters

Notch Filters

This simple version has two complex-conjugate zeros placed on the Unit circle to create a null a desired frequency. Their angle will be at the null frequency.

Has two poles that can be placed where desired ... usually either

- Both at the origin (this results in an FIR filter).
- Two complex-conjugate poles at $p_{1,2} = re^{\pm j\omega_0}$

Poles at Origin

$$\begin{aligned} H^z(z) = &\quad b_0(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1}) \\ = &\quad b_0\big(1-2\cos(\omega_0)z^{-1}+z^{-2}\big) \\ = &\quad \tfrac{b_0\left(z^2-2\cos(\omega_0)z+1\right)}{z^2} \end{aligned}$$

Poles at $p_{1,2} = re^{\pm j\omega_0}$

$$\begin{aligned} H^z(z) = &\quad \tfrac{b_0(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} \\ = &\quad \tfrac{b_0\left(1-2\cos(\omega_0)z^{-1}+z^{-2}\right)}{\left(1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}\right)} \\ = &\quad \tfrac{b_0\left(z^2-2\cos(\omega_0)z+1\right)}{\left(z^2-2r\cos(\omega_0)z+r^2\right)} \end{aligned}$$

Notch Filters

Poles at Origin
Poles at $p_{1,2} = re^{\pm j\omega_0}$

1.4 Comb Filters

Comb Filters

These have a variety of uses... When you have harmonics that either need to be passed and/or stopped.

The names comes from that fact that these filters have a Finite response magnitude that looks like a comb -many "teeth".

Simplest form is an FIR filter with "uniform weights":

$$y(n) = \frac{1}{M+1}\sum_{k=0}^{M} x(n-k), \qquad \Rightarrow \text{ M Even}$$

Transfer Function

$$\begin{aligned} H^z(z) = &\quad \tfrac{1}{M+1}\textstyle\sum_{k=0}^{M} z^{-k} \\ = &\quad \tfrac{1}{M+1}\left[\tfrac{1-z^{-(M+1)}}{1-z^{-1}}\right] \end{aligned}$$

Here, zeros are at $z = e^{j2\pi k/(M+1)}$, and $k = 0, 1, ..., M$. and Poles are at $z = 1$ which is canceled by zero at $z = 1$.

Impulse Response

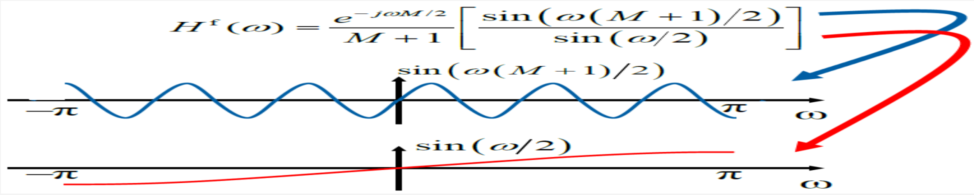
$$h(n) = \frac{1}{M+1}\Big[\underset{\Uparrow}{1}, 1, 1, ...1\Big]$$

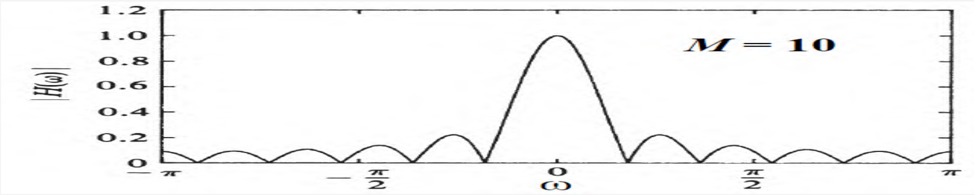
It is Rectangle function starting at $n = 0$.

Frequency Response

$$H^f(\omega) = \frac{e^{-j\omega M/2}}{M+1}\left[\frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}\right]$$

This can be found by taking DTFT of rectangle starting at 0 (For this use time-shift property of DTFT).





Magnitude Response

Taking a look at the frequency response over the $[\pi, \pi]$ range:
Numerator is zero when $\omega(M+1)/2 = k\pi \Rightarrow \omega = k2\pi/(M+1)$.

1.4.1 More General Approach For Design Filter

More General Approach

- Start with some FIR Filter $H^z(z) = \sum_{k=0}^{M} h(k)z^{-k}$.
- Replace z by z^L where L is positive integer: $H_L^z(z) = \sum_{k=0}^{M} h(k)z^{-kL}$.
- The resulting frequency response is;

$$H_L^f(\omega) = \sum_{k=0}^{M} h(k)e^{-jkL\omega} = H^f(L\omega)$$

Means the "Scrunches" by factor of L... e.g. when $\omega = \pi/L$, we get the original Frequency Response point at π.

For illustration consider the Frequency response with Triangle response. Not a real FIR's shape! For easiness to see it.

Let's see what $z \to z^L$ does from an impulse response and bloc diagram viewpoint.

$$\begin{aligned} H_L^z(z) = & \quad \sum_{k=0}^{M} h(k)z^{-kL} = h(0) + h(1)z^{-L} + h(2)z^{-2L} + ... + h(M)z^{-ML} \\ = & \quad h(0) + 0z^{-1} + ... + 0z^{-(L-1)} + h(1)z^{-L} + 0z^{-(L+1)} + ... + 0z^{-(2L-1)} + h(M)z^{-ML} \\ h_L(n) = & \quad \left[h(0), 0...0, h(1), 0...0, h(2), 0...0, h(3), 0...0, ..., ...h(M)\right] \end{aligned}$$

Applying this idea to the uniform weight FIR filter we get,

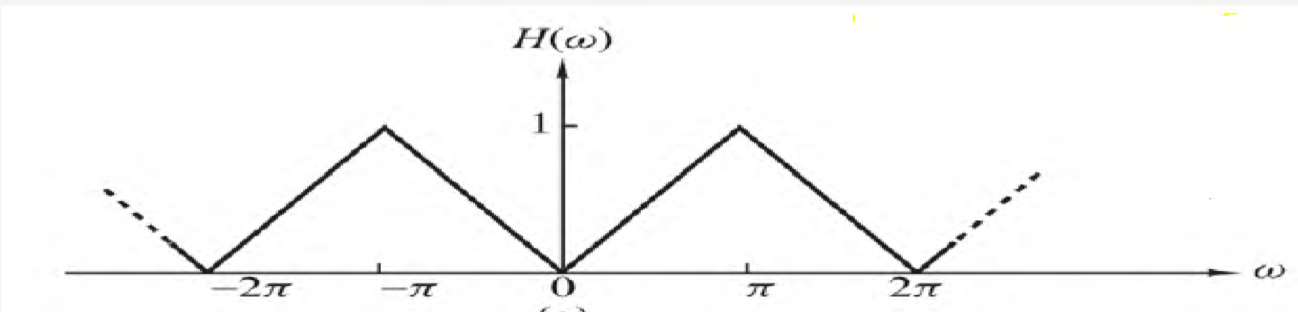
Transfer Function

$$H^z(z) = \frac{1}{M+1}\left[\frac{1 - z^{-L(M+1)}}{1 - z^{-L}}\right]$$

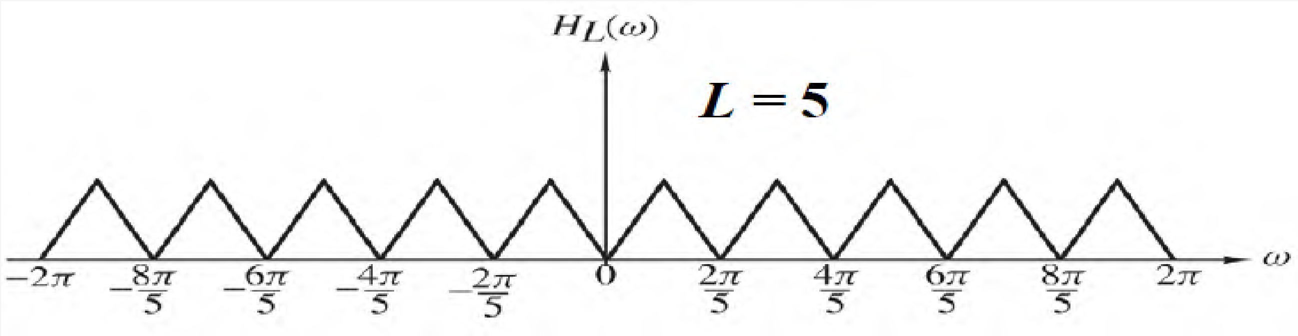
Frequency Response

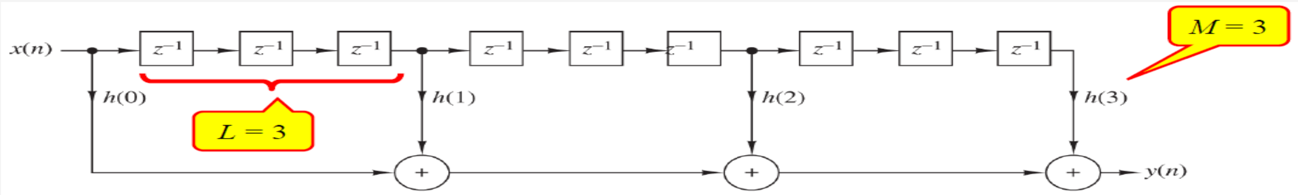
$$H_L^f(\omega) = \sum_{k=0}^{M} h(k)e^{-jkL\omega} = H^f(L\omega)$$

See book's discussion of the use of such a comb filter to separate solar harmonics from lunar harmonics in ionospheric measurements!



(a)





1.5 All-Pass Filter

All-Pass Filters

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2 Summary : LPF-to-HPF Transformation

Summery: LPF-to-HPF Transformation

⇒ Flip poles/zeros means: $H^z_{hp}(z) = H^z_{lp}(-z)$.

⇒ Shift the frequency response by π means:$H^f_{lp}(\omega) = H^f_{lp}(\omega - \pi)$.

⇒ Alternating sign change means: $h_{hp}[n] = (-1)^n h_{lp}[n]$.

⇒ coefficient value change means, $a_{k,hpf} = (-1)^k a_{k,lpf}$ and $b_{k,hpf} = (-1)^k b_{k,lpf}$

Questions?