APMA 4007 | Sep 10, 2024 | Lecture 3

Outline

1. Elimination + back substitution

   - pivots
   - break down

   $[A|b] \rightarrow [U|c]$

2. Matrix-matrix mult.

   - extension of matrix-vector mult.

We have:

A system of linear equations

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

coeff. unknown const.

Theory

1. $Ax=b$ has a sol$^{\underline{n}}$ iff b is L.C. of columns of A.

2. A sol$^{\underline{n}}$ of $Ax=b$ is unique iff the only sol$^{\underline{n}}$ of $A\vec{z}=\vec{0}$ is $\vec{z}=\vec{0}$.

Missing

1. Method of solving ~~AX~~ $Ax=b$ (also $Az=0$)

2. Method to decide if sol$^{\underline{n}}$ exists.

Elimination + back substitution does both.

Example w/o matrix to warm up

$$2x-y=3$$
$$4x+3y=2$$

Eliminate x from the 2nd eq. by subtracting twice the 1st.

unchanged $2x-y=3$

$$5y=-4 \Rightarrow y=-4/5$$

substitute (back) into the eq. above it

$$2x-(-4/5)=3$$
$$2x = 3-4/5 = 11/5$$
$$x = 11/10$$

check

$$2\left(\tfrac{11}{10}\right)+4/5 = \frac{15}{5} = 3 ✓$$

$$4\left(\tfrac{11}{10}\right)+3\left(-4/5\right) = \frac{22-12}{5} = \frac{10}{5}=2 ✓$$

Now w/ matrices (in the boxes)

$$\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

same elimination step

subtract $2\times$ 1st from 2nd

$$\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

back substitution out of box

$5y = -4$ $\boxed{y = -4/5}$

$2x = -4/5 + 3 = 11/5$

$\boxed{x = 11/10}$

remove redundent notation i.e., "=" & $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 4 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 5 & -4 \end{array}\right]$$

"="

same elimination

$y = -4/5$ same back substitution

In general $[A \mid b]$ "augmented matrix"

Example of a block matrix or partitioned matrix where parts (blocks) of the matrix have names (letters)

Then elimination is accomplished by subtracting (strategically) equations ~~[illegible]~~

Q: Why is add/subtracting rows (equations) allowed?

A larger example

don't forget about

$$2x + y + 3z = 3$$
$$4x + 3y + 8z = 6$$
$$-2x + 3z = 1$$

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 4 & 3 & 8 & 6 \\ -2 & 0 & 3 & 1 \end{array}\right] = [A \mid b]$$

We start at the top.

The pivot (boxed) is the 1st non-zero entry in the row & does the elimination

I use the first equation to eliminate x in the equations below it

I use the pivot (circled) to zero the entries below it.

How? Multiply row 1 by $\dfrac{\text{entry to be zeroed}}{\text{pivot}}$ multiplier

divide by pivot →

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

I put the multipliers in a box for safe keeping in case of audit

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

I put 1's on the diagonal & zeros to right. ↑ to fill the matrix

Now go to the 2nd row, $\boxed{1}$ is the pivot (revealed)

and zero below it

~~diag~~ diagonal = row index = column index

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & \boxed{4} & 4 \end{array}\right]$$

← elimination produces an upper triangular matrix

(zeros below the diagonal)

Back substitution (out of box)

$4z = 4 \qquad z = 1$

$y = -2(1) + 0 = -2$

divide by pivots

$2x = -(-2) - 3(1) + 3$

$2x = 2 \qquad x = 1$

Check in original

$2(1) + (-2) + 3(1) = 3$ ✓

$4(1) + 3(-2) + 8(1) = 6$ ✓

$-2(1) + 3(1) = 1$ ✓

Could anything have gone wrong in this procedure?

Not if all the pivots are there

Another example: Take 3 minutes

$$\left[\begin{array}{ccc|c} \boxed{4} & -2 & 1 & -3 \\ 8 & 2 & 0 & 4 \\ 16 & 10 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{4} & -2 & 1 & -3 \\ 0 & \boxed{16} & -2 & 10 \\ 0 & 18 & -5 & 13 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{4} & -2 & 1 & -3 \\ 0 & \boxed{6} & -2 & 10 \\ 0 & 0 & \boxed{1} & -17 \end{array}\right]$$

$$[A \mid b] \xrightarrow[\text{row operations}]{\text{elimination goes down}} [U \mid d]$$

subtracting/adding L.C. of rows

upper triangular

new constant

multipliers

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

lower triangular

back substitution goes up

$\boxed{x_3 = -17}$

$6x_2 = 2(-17) + 10 = -24$

$\boxed{x_2 = -4}$

$4x_1 = 2(-4) - (-17) - 3 = -8 + 17 - 3$

$4x_1 = 6 \qquad \boxed{x_1 = 6/4 = 3/2}$

Check $\frac{3}{2}\begin{bmatrix} 4 \\ 8 \\ 16 \end{bmatrix} + (-4)\begin{bmatrix} -2 \\ 2 \\ 10 \end{bmatrix} - 17\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 + 8 - 17 \\ 12 - 8 \\ 24 - 40 + 17 \end{bmatrix} = \begin{bmatrix} -3 \\ +4 \\ 1 \end{bmatrix}$ ✓

What can go wrong?

$$\underbrace{\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 4 & 3 & 8 & 6 \\ 6 & 4 & 11 & 9 \end{array}\right]}_{[A|b]} \rightarrow \left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \underbrace{\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{[U|d]}$$

3rd pivot is <u>missing</u> (pivots cannot be zero)

Elimination breaks down.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

In terms of back substitution $0x_3 = 0$ which is true for any ~~x~~ value of x_3. x_3 is <u>free</u> to be whatever it wants (not unique!)

Suppose we start with a different b^*.

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 4 & 3 & 8 & 6 \\ 6 & 4 & 11 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \rightarrow \underbrace{\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{[U|d]}$$

In terms of back substitution $0 \cdot x_3 = 1$ which has no sol^n for any x_3.

Now we can clearly see that $\vec{d}$ is not a L.C. of the columns of U $\Rightarrow$ $U\vec{x} = d$ has no sol^n (which was not obvious w/ $Ax=b$ or $[A|b]$)

Is there something smart to be done when solving $Ax=b$ for multiple rhs?

Sometimes a pivot appears to be missing but isn't

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 4 & 2 & 8 & 6 \\ 6 & 4 & 10 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & -7 \end{array}\right]$$

the pivot for the 2nd column is in the 3rd row. Switch

Why is switching allowed?

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 1 & -7 \\ 0 & 0 & \boxed{2} & 0 \end{array}\right]$$

3 pivots

Note 1. Switching complicates our multiplier accounting

2. We could have switch in advance. (considerable foresight)

$$\left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 6 & 4 & 10 & 2 \\ 4 & 2 & 8 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{2} & 1 & 3 & 3 \\ 0 & \boxed{1} & 9 & -7 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

In summary, elimination w/o breakdown

column 1 use row 1 to zero below 1st pivot

column 2 use row 2 to zero below 2nd pivot

⋮

column m

if there is a zero in a pivot position, switch w/ a row below it that has a pivot

$[A|b] \longrightarrow [U|d]$, multipliers $l_{ij} = \dfrac{\text{row } i \text{ entry to eliminate}}{\text{column } j \text{ pivot}}$

subtrack $l_{ij} \cdot$ row j from row i

[Know the procedure, not so much the words?]

[Post some info on the history of elimination]

1. Old (Chinese book from 200 BC)
2. Not Gaussian
3. Our framing is due to von Neumann 1947, Turing 1948
   - concerned with numerical issues
     e.g., round off error, "pivoting"

I would like to describe elimination using matrix-matrix multiplication so first I will describe m-m mult.

A & B are matrices (usually upper case)

What is AB? Suppose B has 3 columns

$$B = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \quad \text{then } AB = \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix}$$

1. This only works if $A\vec{b}_1$ etc is possible, that is if # rows in B = # columns in A

$$\underset{\boxed{m} \times n}{A} \quad \underset{n \times \boxed{3}}{B}$$

no condition on the dimensions

same rule that adjacent dimensions must match

the result is indeed $m \times 3$ b/c A has m rows and there are 3 columns in B

2. Since Ab_1 etc is a L.C. of the columns of A, the columns of AB are L.C. of the columns of A.

$$\underset{m \times n}{A} \; \underset{n \times k}{B} = \underset{m \times k}{(AB)}$$

Properties of matrix-matrix mult

1. $(AB)C = A(BC)$

2. $c(AB) = (cA)B = A(cB)$

3. $A(B+C) = AB + AC$

4 $\underset{m\times n\ \ n\times k}{AB} \overset{?}{=} \underset{n\times k\ \ m\times n}{BA}$

- does not compute if $k \neq m$

- is not the same size unless

$(m\times k \ ,\ n\times n)$

$m = n = k$ i.e A & B are <u>square</u>

- even if A & B are square, tend to not be equal

When do matrices commute?