APMA 4007 | Sep 5, 2024 | Lecture 2

Outline

1. Review
   - $Ax=b$
   - dot product
2. Dot product appriation
3. Dot product is special case of matrix.mult etc
4. Uniqueness

Review

1. A system of $\boxed{m}$ linear equations in $\boxed{n}$ unknowns can be written as a vector equation where the lhs is a L.C. of $\boxed{n}$ vectors & the rhs is a constant vector w/ $\boxed{m}$ elements.

Example

$$2x_1 - 3x_2 - x_3 = 1$$

$$x_3 - x_1 = 0$$

$$x_2 = 4$$

Take 1 minute & write in the form stated above, LC = const. vector

What are m & n?

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

the scalars are the unknowns

In general

$$x_1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + \cdots + x_n \begin{bmatrix} | \\ a_n \\ | \end{bmatrix} = \left.\begin{bmatrix} | \\ b \\ | \end{bmatrix}\right\} m$$

OR

$$\underset{m\times n}{A}\;\underset{n\times 1}{x} = \underset{m\times 1}{b} \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ | \\ b_m \end{bmatrix}$$

2. Ax is matrix-vector multiplication and

Ax is/means L.C. of the columns of A.

$Ax=b$ states that (for some x), b is a L.C. of the columns of A

The statement that $Ax=b$ has no sol$^{\underline{n}}$ means that b is not a L.C. of the columns of A

Example

$$A=\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}, \quad b=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$Ax=b$ has no sol$^{\underline{n}}$.

Matrix-vector mult. has some properties of ordinary mult. but not all.

$A(x+y) = Ax + Ay$ b/c

$$A(x+y) = (x_1+y_1)\underset{|}{\overset{|}{a_1}} + \dots + (x_n+y_n)\underset{|}{\overset{|}{a_n}}$$

using properties of scaling

$$= x_1\underset{|}{\overset{|}{a_1}} + \dots + x_n\underset{|}{\overset{|}{a_n}} + y_1\underset{|}{\overset{|}{a_1}} + \dots + y_n\underset{|}{\overset{|}{a_n}}$$

$$= Ax + Ay$$

$Ax \neq xA$, $\quad \underset{m\times n}{A}\ \underset{m\times 1}{b} \leftarrow$ does not compute!

3. The dot product of $\vec{x}$ & $\vec{y}$ (same siz) is

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

$\vec{x} \cdot \vec{y} = 0$ means that $\vec{x} \perp \vec{y}$

$\|x\|^2$ = length squared of $\vec{x} = \vec{x} \cdot \vec{x}$

NOTE

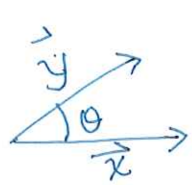
Length of $\vec{x} = \|x\|$ $\qquad$ $\text{len}(x) = n$

For $\vec{x} \neq \vec{0}$, $\dfrac{\vec{x}}{\|x\|}$ is a unit vector

b/c it has length $= 1$

Check

length squared $\dfrac{\vec{x}}{\|x\|} = \dfrac{\vec{x}}{\|x\|} \cdot \dfrac{\vec{x}}{\|x\|} = \dfrac{\vec{x} \cdot \vec{x}}{\|x\|^2} = \dfrac{\|x\|^2}{\|x\|^2} = 1$



$\vec{x} \cdot \vec{y} = \|x\| \, \|y\| \cos\theta$

true but not used so often in this class

Vector notation is compact & concise compared
Matrix
to summation.

Example: statistics (will also provide a geometrical picture)

$$\vec{x} = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$ a vector containing n samples (measurements)

mean $$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\vec{1}\circ\vec{x} \quad \text{where } \vec{1} = \begin{bmatrix} 1 \\ | \\ 1 \end{bmatrix}$$

sample

Variance $$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)^2 = \frac{1}{n-1}\|\vec{x}-\mu_x\vec{1}\|^2$$

if $\mu_x = 0$, variance of $x = \frac{\|x\|^2}{n-1}$ ∴ proportional to length squared.

Covariance

$$\sigma_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y)$$

$$= \frac{1}{n-1}(\vec{x}-\mu_x\vec{1})(\vec{y}-\mu_y\vec{1})$$

if $\mu_x = \mu_y = 0$, $\sigma_{xy} = \frac{\vec{x}\cdot\vec{y}}{n-1}$ & $\sigma_{xy} = 0$ iff $\vec{x}\perp\vec{y}$

independence ~ orthogonality.

correlation

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x\sigma_y}, \quad \text{if } \mu_x = \mu_y = 0 \quad \rho_{xy} = \frac{\vec{X}\cdot\vec{y}}{\|\vec{X}\|\,\|\vec{y}\|} = \underbrace{\cos\theta}_{\text{geometry}}$$

Let us not pretend that matrix mult & the dot product are two different things.

$$\vec{x} \circ \vec{y} = \sum_{i=1}^{n} x_i y_i$$

$$\begin{array}{cccc} 1 \times n & n \times 1 & & \text{ans. is } 1 \times 1 \end{array}$$

$$= \begin{bmatrix} x_1 & \text{---} & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ | \\ y_n \end{bmatrix} \overset{\text{OR}}{=} \begin{bmatrix} y_1 & \text{---} & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$

a matrix w/ one row and n columns

The notation for $[x_1 \text{---} x_n]$ is x^T where "T" means transpose and that means switch rows & columns.

- if x is $n \times 1$, x^T is $1 \times n$
- $(x^T)^T = x$

We will write $x^T y$ (or $y^T x$) and think $\vec{x} \circ \vec{y}$.

$$\|x\|^2 = x^T x$$

$$x^T y = 0 \quad \text{iff} \quad \vec{x} \perp \vec{y}$$

etc

Let us not pretend that matrix mult. & the dot product are two different things (II).

Example

linear combination of columns of A

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cancel{\oplus} - 6\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 0 + 2 \\ 0 - 18 + 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -14 \end{bmatrix}$$

look like dot products

$$= \begin{bmatrix} 5 \cdot 1 - 6 \cdot 0 + 1 \cdot 2 \\ 5 \cdot 0 - 6 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$$

"By the same logic"

The i-th element of Ax denoted $[Ax]_i$ is the i-th row of A dotted with x.

$$[Ax]_i = \sum_{j=1}^{n} \underbrace{A_{ij}}_{\text{row } i \text{, column } j \text{ of } A} x_j = A_{i\,j} \cdot x$$

i-th element of A

i is fixed so i-th row

row x column

m x n

Using the same formula

$$[Ax] = \sum_{j=1}^{n} \underbrace{A_{i\,j}}_{\text{the } j\text{-th column of } A} \overbrace{x_j}^{\text{scalar}} = \text{L.C. of the columns of } A$$

all the elements

2 views of Ax:

1. L.C. of the columns of A
2. Dotting rows of A in x.

Both are useful

<u>Practice/computation</u>

- if x has many zeros #1 is good $A\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} = ?$ 1st column of A
- If we only want 1 element of A, #2 is nice.

<u>theory</u>

- $Ax=b$ has sol$^{\underline{n}}$ iff b is L.C. of columns of A comes from #1. <u>Existence</u>
- #2 is handy for statements about <u>uniqueness</u>

Suppose that there are 2 solutions $\vec{x}$ & $\vec{y}$ (and $\vec{x} \neq \vec{y}$) so that $Ax=b$ & $Ay=b$.

By math

$$A\underbrace{(x-y)}_{\vec{z} \neq \vec{0}} = Ax - Ay = b - b = \vec{0}$$

$\boxed{\text{If}}$ the solⁿ is not unique, $\boxed{\text{there}}$ is a vector $\vec{z} \neq \vec{0}$ such that $Az = \vec{0}$.

[this is unlike scalar mult. where $a\cancel{x}z = 0$ means that eithere $a=0$ or $z=0$]

Example

$$\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$ neither vector is zero but they are $\perp$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{1}$

← each ~~other~~ row of A sums to zero

not zero

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\#$ not unique!

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The logic if solⁿ not unique, then $\exists z$ such that $Az = \vec{0}$, goes both ways.

Suppose $A\vec{z}=0$ and $\vec{z}\neq\vec{0}$. Then a sol$\underline{^{n}}$ of

$Ax=b$ is not unique. why?

$A(x+z) = Az + Ax = b$

two sol$\underline{^{n}}$'s x & $x+z$ and they differ b/c $z\neq\vec{0}$.

Actually more than $\underline{2}$

$$A(\vec{x}+\underset{\substack{\uparrow\\ \text{scalar}}}{c}\vec{z}) = Ax + cAz = b$$

for any scalar c, so an ∞ # of sol$\underline{^{n}}$s.

As we saw in the 2D example the possibilities are

$0, 1, \infty$ # of sol$\underline{^{n}}$'s.

If a sol$\underline{^{n}}$ exists, it is either unique or there are an infinite # of sol$\underline{^{n}}$'s

A statement about uniqueness.

A sol$\underline{^{n}}$ of $Ax=b$ is unique iff the only solution of $Az=0$ is $z=0$.

Note the uniqueness statement does not involve b.

(A does not express a view on existence)

Recall the 2nd view of Ax — dotting rows of A into x

$A\vec{z}=\vec{0}$ means that z is $\perp$ to all the rows of A.

Uniqueness of sol$\underline{n}$ of $Ax=b$ depends on whether there is a non-zero vector z $\perp$ to all the rows of A.

At this point we have theory statements about the existence & uniqueness of sol$\underline{n}$'s of $Ax=b$. But no method for solving.

Sometimes the sol$\underline{n}$ is easy

$$A=\begin{bmatrix} 2 & 0 & 6 & 1 \\ -8 & 4 & 2 & 0 \\ 3 & -3 & 9 & 1 \\ 4 & 6 & 4 & 0 \end{bmatrix}, \quad b=\begin{bmatrix} 0 \\ 4 \\ -3 \\ 6 \end{bmatrix}$$

what is the sol$\underline{n}$ of $Ax=b$?

Since b is the 2nd column of A

$$x=\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$ Next time a ~~[illegible]~~ general method.