Homework - 6

Name: Harshul Gupta

Q1)
Ans)

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

1) $$[A|b] = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 2 & 5 & 7 & 6 & 3 \\ 2 & 3 & 5 & 2 & 6 \end{array}\right]$$

$$\Big\downarrow \; R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & +1 \end{array}\right]$$

$$\Big\downarrow \; R_3 \to R_3 + R_2$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$U \qquad c$

2) $b_1 \to b_1$

$b_2 \to b_2 - b_1$

$$b_3 - b_1 + b_2 - b_1 = 0$$

$$b_3 + b_2 - 2b_1 = 0$$

3)
$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Col space = span $\left\{ \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \right\}$

CS contain all vector $b_3 + b_2 - 2b_1 = 0$

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4)

$$U = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\Rightarrow \quad x_1 + 2x_2 + 3x_3 + 2x_4 = 0$$

$$x_2 + x_3 + 2x_4 = 0$$

$$x_2 = -x_3 - 2x_4$$

$$x_1 + x_3 - 2x_4 = 0$$

$$x_1 = +2x_4 - x_3$$

Special solⁿ

$$\left(\begin{matrix} x_3 = 0, & x_4 = 1 \\ x_3 = 1, & x_4 = 0 \end{matrix} \right)$$

Null space $\Rightarrow$ span $\left[\left\{ \begin{matrix} -1 \\ -1 \\ 1 \\ 0 \end{matrix} \right\}, \left\{ \begin{matrix} 2 \\ -2 \\ 0 \\ 1 \end{matrix} \right\} \right]$

$$x_n = c_1 s_1 + c_2 s_2$$

5)

$$[U\ c] \xrightarrow[R_1 \to R_1 - 2R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$R_0 \quad d$

6) $x_{particular} \Rightarrow$ $x_1 = 4$, $x_2 = -1$, $x_3 = 0$, $x_4 = 0$

$$x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_{complete} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Q2)

Ans)

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad x + 3y + z + 2t = 1$

$2z + 4t = 1 \Rightarrow z = \frac{1-4t}{2}$

$\therefore x = \frac{1}{2} - 3y$

$x = \frac{1}{2} - 3y$ | $z = \frac{1}{2} - 2t$

$y = y$ | $t = t$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_n}$$

$\therefore$ Complete ~~Particular~~ sol^n

(x_c)

x complete

Q3)
Ans)

A (1X3) system

$$\underset{A}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = b$$

$$X_p = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \qquad X_n = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

→ eq^n above has only 1 eq^n & 3 variables.

∴ 2 free variables

$Rank(A) \leq 1$

$2a_1 + 4a_2 + 0a_3 = b$

$a_1 + a_2 + a_3 = 0$ (then row is LD)

∴ can't be possible.

Q4)
d m)

1) x_5 is a free variable.

2) if $Ax=b$ has a sol^n, then it has infinitely many solutions.

Q5)

Ans) 1) Largest Rank - 3

2) pivot in every row.

3) Solution always exist / not unique.

4) Col space - R^3

5) Ex of A -

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix}$$

Rank - 3

Q6)

Ans)

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{rank}(A) = 3$

$\det(A) = 1$

v_1, v_2, v_3 are linearly independent.

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\Big\downarrow R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{Rank}(A_2) = 3 < 4$

($\because$ 1 free variable)

$\therefore$ its linear dependent.

$Ac = 0$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 + c_3 + 2c_4 &= 0 \\ \cancel{c_2+c_3+c_4}\ c_2 + c_3 + c_4 &= 0 \\ c_3 + 4c_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \\ c_3 &= -4 \\ c_4 &= 1 \end{aligned}$$

Q) 7)
Ans) $v_i \in R^3$, $i \in [1,4]$

a)   4 vectors $A(3 \times 4)$ R^3

$\downarrow$

max Rank $3 < 4$

$\downarrow$

one free variable $\rightarrow$ dependent

b) v_1, v_2 dependent if

$\boxed{v_1 = c v_2}$ or

$[v_1 \ v_2]$ has rank ≤ 1

c) v_1, $(0,0,0)$ k dependent as

$[v_1 \ 0]$ has rank ≤ 1

Q08)

Ans)

a) Vector = $\begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$

basis = $\{(1,1,1,1)\}$

b) $v = (a, b, c, -a-b-c) \Rightarrow a(1,0,0,-1)$

$\therefore$ basis = $\{(1,0,0,-1),$
$(0,1,0,-1),$
$(0,0,1,-1) \ldots\}$

c) $\perp$ to $(1,1,0,0)$, $(1,0,1,1)$
(a,b,c,d)

$a+b=0$
$a+c+d=0$

(dot product = 0)

$\therefore v = (a, -a, c, -a-c)$

basis = $\{(1,-1,0,-1), (0,0,1,-1)\}$

d) $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Basis (col space) = $\{(1,0,0,0),$
$(0,1,0,0),$
$(0,0,1,0),$
$(0,0,0,1)\}$

Basis (null space) = Empty set

Qs9)
Ans)

$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ | $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

1) Column space -

$C_2 = C_3 + C_1$ ($\therefore$ Linear dependent) | $C_2 = C_3 + C_1$

$C(A) = \{(1,0,1), (3,1,3)\}$ | $C(U) = \{(1,0,0), (3,1,0)\}$

2) Row space -

$R_3 = R_1$ (LD) | $R_3 = 0$

$R(A) = \{(1,3,2), (0,1,1)\}$ | $R(A) = \{(1,3,2), (0,1,1)\}$

3) Null space -

$Ax = 0$ | $Ux = 0$

$x_2 + x_3 = 0$
$x_1 + 3x_2 + 2x_3 = 0$
$x_1 + x_2 = 0$

| $x_1 + 3x_2 + 2x_3 = 0$
$x_2 + x_3 = 0$

$x = (1, -1, 1)$ | $x = (1, -1, 1)$

Null space, Row space same for A, U

- Col^n space changed for A, U.
  during elimination.

Q10)

a) A — 7 X 9 matrix.

Rank(A) = 5.

dim(Col(A)) = 5

dim(Null(A)) = 9-5 = 4

dim(Row(A)) = 5

dim(left Null space(A)) = 7-5 = 2.
N(A^T)

sum ⇒ 5+5+4+2 = 16 = m+n

b)

3X4 matrix

Rank(3)

Colspace (A) = 3 (R^3)

dim (left Null space) = 3-3 = 0

Q11)
Ans)

a) rank $(r) \leq \min(m, n)$

no sol^n so,

$[r < m]$

b) $A^T y = 0$

dim (Left Null space) $= m - r$

if $m > r$
$m - r > 0$
$\therefore$ (non trivial)

$\therefore$ So has sol^n (non zero)

Q) 12)
Ans)

$$[A|b] \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

$\therefore b_3 - 2b_2 + b_1 = 0$

$\underline{R_3 - 2R_2 + R_1 = 0}$ (Rows give 0)

Vectors in Null space of A^T,

$A^T y = 0$

(combination of Row give 0)

$$\therefore \underline{y} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Vector in Null space of A -

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_3 = 1, \; x_2 = -2, \; x_1 = 1$$

$$\therefore \; x = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ null space of A