Homework 7

Qs 1) Harshul Gupta

Ans)

1) null space

2) left null space

→ if A, B rank = 2 then nullity = 1

For AB = 0, all null space dim ≥ 2

∴ it is impossible

Q2)

Ans) 1) 2 dim subspace (L^T is $\perp$ to L)

2) 1 dim subspace ($(L^\perp)^\perp$ is $\perp$ to $L^\perp$)

$$\boxed{(L^\perp)^\perp = L}$$

Q3)

1) if col^n of A are unit vectors -

$$(A^T A)_{ij} = a_i \cdot a_j$$

$$\left.\begin{array}{ll} i=j & a_i \cdot a_j = 1 \\ j \neq i & a_i \cdot a_j = 0 \end{array}\right\}$$

$$\therefore \quad A^T A = I_n$$

Q>4)
Ans)

$$\text{Proj}_a b = \frac{a \cdot b}{a \cdot a} a$$

$$e = b - \text{proj}_a a$$

a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$a \cdot b = 5$

$a \cdot a = 3$

$$\text{proj}_a b = \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$e = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$a \cdot e = 0$ $\therefore$ e perpendicular to a.

b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

$a \cdot b = -11$

$a \cdot a = 11$

$proj_a b = \frac{-11}{11} \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$e = b - proj_a b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a \cdot e = 0$

$\therefore a \perp e$

Q5)
Ans)

$$P = \frac{aa^T}{a^Ta}$$

Proj matrix

a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $a^Ta = 3$

$$P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore P^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = P$$

$$Pb = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

b) $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$

$a^T a = 11$

$$P = \frac{aa^T}{a^T a} = \frac{1}{11}\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}\begin{bmatrix} -1 & -3 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\therefore P^2 = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} = P$$

$$P_b = \begin{bmatrix} 1/11 & 3/11 & 1/11 \\ 3/11 & 9/11 & 3/11 \\ 1/11 & 3/11 & 1/11 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Q6)
Ans)

Solve

1) $(A^T A)\hat{x} = A^T b$

2) $p = A\hat{x}$

$e = b - p$

$A^T e = 0$

a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$A^T b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\therefore$ $\hat{x}_1 = -1$

$\hat{x}_2 = 3$

$$p = A(A^T A)^{-1} A^T b$$

$$p = A\hat{x}$$

$$p = A \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$A^T e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

→ $e \perp$ to col^n of A

b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$

$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}$

$A^T b = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

$x_1 = -2$

$x_2 = 6$

$p = Ax$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

– b is in col^n space of A

Q7)
Ans)

$x - y - 2z = 0$

1) normal vector ⊥ to plane

$$e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

2) $Q = \frac{ee^T}{e^Te}$

$e^Te = 1+1+4 = 6$

$$Q = \frac{1}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Proj

3) $P = I - Q$

$$= \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

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Page

Q08)

A8)

$$\begin{array}{cc} a_1 & a_2 \end{array}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

1) $A^T A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ $1^2+3^2+4^2$

2) $A^T b = \begin{bmatrix} 36 \\ 8+24+80 = 112 \end{bmatrix}$

$A^T A\hat{x} = A^T b$

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$x_1 = 1$

$X_2 = 4$

$\therefore \hat{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

3) $p = A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$

4) $e = b - p = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

$E = \|e\|^2 = 44$

$1^2 + 3^2 + 5^2 + 3^2$

Q.9)

Ans)

$$e = b - p = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}$$

$$p = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix}$$

$$\left.\begin{aligned} e \cdot a_1 &= -1 + 3 - 5 + 3 = 0 \\ e \cdot a_2 &= 0 + 3 - 15 + 12 = 0 \end{aligned}\right\} \text{both col}^n \text{ of } A \perp \text{ to } e$$

Shortest dist $\Rightarrow$ $\|e\| = \sqrt{1+9+25+9}$

$$= \sqrt{44} \approx 6.63$$

Q10)

Ans) parabola

$$b = C + Dt + Et^2$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \underbrace{\begin{bmatrix} C \\ D \\ E \end{bmatrix}}_{x} = b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\therefore A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

Normal eqⁿ

$$\begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

$C = 2$

$D \approx 1.33$

$E \approx 0.67$

* In (a) is fitting 4 points

(b) is a proj in R^4 (geometric proj)

$\Big\}$ Same problem

Q 11)
Ans)

col^n are
$A \rightarrow$ Not independent

$\therefore$ A^TA is not invertible

$\downarrow$

So can't do $P = A(A^TA)^{-1}A$

$\rightarrow$ Take B so that it only keeps the pivot col^n of A.

$$P = B(B^TB)^{-1}B$$

B - full col^n rank matrix

$$A = \begin{bmatrix} 1 & -2 & 2 & -3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B^TB = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

$$|B^TB| = 1$$

$$(B^TB)^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$B(B^TB)^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = B(B^TB)^{-1}\,B^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P \begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\\0 & \cdots & \cdots & 0\end{bmatrix}$$

Ans