APMA 4007

Lecture 10

Oct 3, 2024

Outline

1. The really revealing R
   - 4 cases
2. Complete solⁿ
3. Special solⁿ of $A^T y = 0$

upper triangular

row reduced echelon form
really revealing R_0

<u>Review</u> : $A \longrightarrow U \longrightarrow R$

zero above pivots
and divide by pivots

expose pivots
and zero below

What does this give us?

1. The pivots & their location
   - The r pivot columns of A span $C(A)$
   - The rank r which is the number of pivots
   - The $n-r$ special solutions that span $N(A)$

     ~~Existence~~ Uniqueness
   - The $m-r$ rows of zeros

     <u>Existence</u>

—— ?

A useful cartoon for R is the block matrix form

$$\overset{m\times n}{R} = \begin{array}{c} \\ \left[\begin{array}{cc} \overset{r\times r}{I} & \overset{r\times(n-r)}{F} \\ \overset{(m-r)\times r}{0} & \overset{(m-r)\times(n-r)}{0} \end{array}\right] \\ \underbrace{\hphantom{xxxxxxxxxxxxxx}}_{n} \\ \underbrace{\hphantom{xxxxxx}}_{r}\underbrace{\hphantom{xxxxxx}}_{(n-r)} \end{array} \begin{array}{c} r \\ m-r \end{array} \Bigg\} m$$

It is a cartoon in the sense that it assumes the pivot columns come first.

$$\text{if } R = \begin{array}{c} \begin{array}{cccc} p & f & p & f \end{array} \\ \left[\begin{array}{cccc} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

not first

mult. from right by

which is the same as reordering the x's

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ to get } \left[\begin{array}{cc|cc} \boxed{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}} & & \boxed{\begin{array}{cc} 4 & 2 \\ 0 & 5 \end{array}} & \\ \boxed{\begin{array}{cc} 0 & 0 \end{array}} & & \boxed{\begin{array}{cc} 0 & 0 \end{array}} & \end{array}\right]$$

Let's consider 4 special cases

1. $m = n = r$. A is square and full-rank

$[I]$

rank $= n$ (n pivots) A^{-1} exists

solution of $Ax = b$ exists & is unique

$C(A) = \mathbb{R}^n$, $N(A) = \{\vec{0}\}$

Example

$$A = \begin{bmatrix} \boxed{1} & -2 & -1 \\ \enclose{circle}{-2} & 5 & -1 \\ \enclose{circle}{-1} & 2 & 14 \end{bmatrix} \rightarrow \overset{U}{\begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{13} \end{bmatrix}} \rightarrow \overset{R}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$R = [\overset{r \times r}{I}]$ only the upper-left corner, no row of zeros
no special sol$^{\underline{n}}$

~~full column rank~~

2. $\overline{m = r} < n$ A is wide

$[I \ F]$

There is a pivot in each row (full row rank)

No rows of zeros. $C(A) = \mathbb{R}^m$. Always exist

$n - r > 0$ special sol$^{\underline{n}}$s. Never unique.

Example

$$\overset{[A \qquad b]}{\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -3 & 4 & 7 & 1 \end{array}\right]} \rightarrow \overset{[U | ?]}{\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 4 & 4 \end{array}\right]} \rightarrow \overset{[R \mid ??]}{\left[\begin{array}{c|c|c} \underset{I}{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}} & \underset{F}{\boxed{\begin{matrix} 3 \\ 4 \end{matrix}}} & \begin{matrix} 5 \\ 4 \end{matrix} \end{array}\right]}$$

top of cartoon

A particular solution $x_{particular}$ is found by setting the free variables to zero and reading off the rhs for the pivot variables

$$x_{particular} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

not unique b/c special sol$^{\underline{n}}$

What are the special sol$^{\underline{n}}$ s

$$\underset{\text{pivot}}{x_1} = -3\underset{\text{free}}{x_3}$$

$$x_2 = -4x_3$$

$$x_{\text{special}} = \begin{bmatrix} -3x_3 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \overset{-F}{\boxed{\begin{matrix} -3 \\ -4 \end{matrix}}} \\ 1 \end{bmatrix} \leftarrow I$$

so $N(A) = \left\{ c \begin{bmatrix} -\cancel{4}3 \\ -4 \\ 1 \end{bmatrix} \right\} =$ line in $\mathbb{R}^3$ through origin

$$\vec{X} = \underset{\substack{\uparrow \\ \text{particular}}}{\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}} + \cancel{x_3} \underset{\substack{\uparrow \\ \text{special}}}{\begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}} x_3 =$$

is called the complete solution. It is a line (not through origin) in this example

I claim that every solution of $Ax=b$ has this form

$$x = x_p + x_N$$

where $Ax_p = b$ & $Ax_N = \vec{0}$.

Proof

Suppose $Ay = b$. Consider $y - x_p$.

$$A(y-x_p) = Ay - Ap = b - b = \vec{0} \Rightarrow \underset{x_N}{(y-x_p)} \in N(A)$$

$$\Rightarrow y = x_p + \underset{\hookleftarrow \in N(A)}{(y - x_p)}$$

indeed complete.

3. $n = r < m$ A is tall. (full column rank)

$\begin{bmatrix} I \\ 0 \end{bmatrix}$

A has a pivot in each column

There are no special sol$\underline{^n}$s b/c no free variables. Unique

There are $m - r > 0$ rows of zero, so for some b's, $Ax = b$ has no sol$\underline{^n}$.

Example

$$\begin{array}{c} \begin{array}{cccc} A & & b_1 & b_2 \end{array} \\ \left[\begin{array}{cc|cc} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 3 \\ 1 & 2 & 2 & 4 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 1 \\ 0 & \overset{1}{2} & \overset{-1}{-2} & \overset{1}{2} \\ 0 & 3 & 1 & 3 \end{array}\right]$$

$$\overset{U}{\rightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 4 & 0 \\ p & p & \uparrow & \uparrow \end{array}\right]} \rightarrow \overset{R}{\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]}$$

no sol$\underline{^n}$ $0 = 4$

sol$\underline{^n}$ exists

no free variables ↙

$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the unique solution $Ax = \vec{b}_2$

rank = 2

$C(A)$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\dot{\text{c}}$ $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$N(A) = \{\vec{0}\}$

4. $r < m$, $r < n$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

<u>If</u> a solution exists, it is not unique b/c $n-r>0$ special solutions. A solⁿ might not exist b/c $m-r>0$ rows of zeros.

Possible # of solutions: $0, \infty$.

Recall block matrix mult.

$$\underset{\vdash\, r \,\dashv\vdash\, n-r \,\dashv}{\begin{bmatrix} \overset{r\times r}{I} & \overset{r\times(n-r)}{F} \\ \overset{(m-r)\times r}{0} & 0 \end{bmatrix}} \cdot \begin{bmatrix} \overset{r\times(n-r)}{-F} \\ \overset{(n\times r)\times(n-r)}{I} \end{bmatrix}\begin{matrix} r \\ n-r \end{matrix}\Bigg\}\, n = \overset{m\times(n-r)}{\begin{bmatrix} -I_rF + FI_{(n-r)} \\ -F+F \end{bmatrix}} = 0$$

n

cuts in rows match cuts in columns

This means that the $(n-r)$ columns of $\begin{bmatrix} \textcircled{-}F \\ I \end{bmatrix}$ are the special solⁿ (of $Ax=0$)

moving to other side

a 1 in exactly one free variable position

We have assumed that pivot colums come first. So

$$A = \begin{bmatrix} \underset{r \text{ columns}}{A_{\text{pivot}}} & \underset{\substack{\text{columns}\\ n-r}}{A_{\text{free}}} \end{bmatrix}$$

$$A\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} A_{\text{pivot}} & A_{\text{free}} \end{bmatrix}\begin{bmatrix} -F \\ I \end{bmatrix} = -A_{\text{pivot}}F + A_{\text{free}} = 0$$

$\Rightarrow A_{\text{free}} = A_{\text{pivot}}F \Rightarrow$ free columns are L.C. of pivot columns

<u>Moreover</u>

$$A = \begin{bmatrix} A_{\text{pivot}} & A_{\text{free}} \end{bmatrix} = \begin{bmatrix} A_{\text{pivot}} & A_{\text{pivot}}F \end{bmatrix} = A_{\text{pivot}}\overset{\text{pivot rows of } R}{\underbrace{\begin{bmatrix} I & F \end{bmatrix}}_{\text{top part of } R}}$$

$= \begin{bmatrix} r \text{ columns} \end{bmatrix}\begin{bmatrix} r \text{ rows} \end{bmatrix}$

w/o the zero rows

Recalling rows & columns of matrix-matrix mult,

The columns of A are L.C. of the pivot columns. $\underline{\text{span}}$ $C(A)$

The rows of A are L.C. of the rows of $[I \; F]$. $C(A^T)$

$$\left[\begin{array}{cc|c|c} \boxed{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}} & & \boxed{\begin{matrix}3\\4\end{matrix}} & \begin{matrix}1\\1\end{matrix} \\ \boxed{\begin{matrix}0 & 0\end{matrix}} & & \boxed{0} & 0 \end{array}\right]$$

Example

$$[A|b] = \left[\begin{array}{ccc|c} 2 & 0 & 6 & 2 \\ -2 & 5 & 14 & 3 \\ 2 & -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 6 & 2 \\ 0 & \overset{1}{5} & \overset{4}{20} & \overset{1}{5} \\ 0 & -2 & -8 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & 6 & 2 \\ 0 & 5 & 20 & 5 \\ 0 & 0 & \boxed{0} & \boxed{0} \end{array}\right]$$

Sol^{n} exists despite row of zeros

$r = 2$ $\quad$ $C(A)$ spanned by $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$

particular sol^{n} = set free to zero
read pivots

$x_p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, special sol$^{\underline{n}}$ $\quad$ $x_1 = -3x_3$ $\quad$ $x_2 = -4x_3$ $\quad$ $x_N = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -F \\ I \end{bmatrix}$

complete sol^{n} is

any scalar

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} x_3$$

In applications, there might be some additional considerations that cause us to prefer one sol$^{\underline{n}}$. For instance, solve $Ax = b$

AND make $\|x\|^2$ small.

ridge regression / prior on x

$\|x\|^2$ is a "cost"

Another example

$$[A|b] = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & -3 \\ \textcircled{1} & 2 & 9 & -3 \\ \textcircled{-1} & -2 & 10 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 2 & 2 & -3 \\ 0 & 0 & \boxed{7} & 0 \\ 0 & 0 & \textcircled{12} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & -3 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \begin{array}{c} \begin{array}{ccc|c} p & f & p & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & \underline{\underline{0}} \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

pivot columns/variables NOT first.

rank $r = 2$

particular sol$^{\underline{n}}$ = zero free variables & read pivot variables

$$x_p = \begin{bmatrix} -3 \\ 0 \\ \underline{0} \end{bmatrix} \leftarrow \text{free}$$

special
$$\overset{\text{pivot}}{x_1} = \overset{\text{free}}{-2x_2} \qquad x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$
$$x_3 = 0$$

complete

$$x = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2$$

Bonus info

$$A \xrightarrow[\wedge]{} R$$

row operations = mult. by an elimination matrix E

So $\overset{m\times m}{E}\ \overset{m\times n}{A} = \overset{m\times n}{R}$

↑ ~~sp~~ square and invertible

To compute E we could do elimination (Gauss-Jordan style)

on $[A \mid I] \longrightarrow [R \mid E]$

Why? b/c $E[A \mid I] = [R \mid E]$

(if A is square and invertible, $R = I$ & $E = A^{-1}$)

[pivot variables first]

$EA = \begin{bmatrix} I & F \\ \boxed{0 \quad 0} \end{bmatrix}$ says that a L.C. of the rows is zero.

Write $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{matrix} \updownarrow r \\ \updownarrow m-r \end{matrix}$ then $EA = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} A = \begin{bmatrix} E_1 A \\ E_2 A \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

therefor $\overset{(m-r)\times m}{E_2}\ \overset{m\times n}{A} = \overset{(m-r)\times n}{0}$

equivalently recall the bottom m−r rows are compute by dotting the bottom m−r rows of E into A.

$\underset{\equiv}{\text{or}}$ recalling transpose $A^T E_2 = 0$

the $(m-r)$ rows of E_2 [the bottom m−r rows of E] contain the special solutions of $A^T y = 0$.

Knowing such a y is handy when the existence of sol^n of $Ax = b$ is uncertain b/c $\overbrace{y^T A}^{0} x = \overbrace{y^T b}^{\text{check if zero}}$

Since / if

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}, \quad R^T = \begin{bmatrix} I & 0 \\ F & 0 \end{bmatrix}.$$

(with the correct dims on the zero matrices.)

zero below the pivots of R^T to get $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

The rank of R = rank R^T

"It turns out" as we will see later that

rank A = rank A^T.