Homework 5

Q2)

Ans) S is subspace of V if

1) $S \subseteq V$
2) if $x \in S$, so is $cx \in S$
3) if $x, y \in S$, so is $x+y \in S$
4) $\vec{0} \in S$

a) vector $\overbrace{(b_1, b_2, b_3) \text{ where } b_1 = b_2}^{S}$ is subset of R^3.

i) $(0,0,0) \in S$

ii) $S \subseteq R^3$

iii) if $(b_1, b_2, b_3) \in S$ & $(c_1, c_2, c_3) \in S$ | $b_1 = b_2$, $c_1 = c_2$

$\Downarrow$

$(b_1 + c_1, b_2 + c_2, b_3 + c_3) \in S$

iv) if $(b_1, b_2, b_3) \in S$ then | $b_1 = b_2$

$\Downarrow$ | $\Downarrow$

C constant | $(cb_1, cb_2, cb_3) \in S$ | $cb_1 = cb_2$

$\therefore$ S is subspace of R^3.

$$\overbrace{\text{plane of vectors with } b_1 = 1}^{S}$$

b) plane of vectors with $b_1 = 1$

(b_1, b_2, b_3)

as $b_1 = 1$

$\therefore$ $\vec{0}$ vector can't exist in S.

$\therefore$ S is not subspace of R^3.

c) vectors with $b_1 b_2 b_3 = 0$ $\}$ S

i) $\vec{0} \in S$ ✓

ii) $(b_1, b_2, b_3) \in S \longrightarrow b_1 b_2 b_3 = 0$

$\Downarrow$

$\therefore (cb_1, cb_2, cb_3) \in S \longrightarrow c^3 b_1 b_2 b_3 = c^3 \cdot 0 = 0$ ✓

iii) $(b_1, b_2, b_3) \in S$ & $(c_1, c_2, c_3) \in S$

$\Downarrow$

$\therefore (b_1+c_1, b_2+c_2, b_3+c_3) \notin S$ (not necessary)

Eg: $(0,1,1)$ & $(1,0,0) \in S$ ✗

but $(1,1,1) \notin S$

$\therefore$ Not subspace

$v = (1,9,0)$ & $w = (2,2,2)$ (Linear combination)

i) $\vec{0} = 0 \cdot v + 0 \cdot w \in S$

ii) $v + w \in S$

iii) $c \cdot v \in S$

$\therefore$ is a subspace

$b_1 + b_2 + b_3 = 0$

i) $\vec{0} \in S$

ii) $(b_1, b_2, b_3) \in S \longrightarrow (cb_1, cb_2, cb_3) \in 0$

iii) $(b_1, b_2, b_3) \in S$ & $(c_1, c_2, c_3) \in S$

$\downarrow$

$(b_1 + c_1, b_2 + c_2, b_3 + c_3) \in S$

$b_1 + b_2 + b_3 = 0$

$c_1 + c_2 + c_3 = 0$

$\therefore$ is a subspace

f) $b_1 \leq b_2 \leq b_3$

i) $\vec{0} \in S$

ii) if (b_1, b_2, b_3) is S. | $b_1 \leq b_2 \leq b_3$

$\Downarrow$ if $c = -1$

$(-b_1, -b_2, -b_3) \notin S$

$\therefore$ Not a subspace.

Q5 2)

Ans)
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Column space = $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 0 \\ 0 \end{bmatrix}$

(cs)

$C(A)$

line in x axis

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$C(B) = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}$ plane in x-y plane, $z=0$ in R^3

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$C(C) = \begin{bmatrix} c_1 \\ 2c_1 \\ 0 \end{bmatrix}$ line through origin, span by single vector.

Q3)

1) a) $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Aug matrix -

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} R_3 \to R_3 + R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

Solvable if -

$b_2 - 2b_1 = 0 \Rightarrow b_2 = 2b_1$

$b_3 + b_1 = 0 \Rightarrow b_3 = -b_1$

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Aug matrix -

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right]$$

$$\Bigg\downarrow \begin{array}{l} R_3 \to R_3 + R_1 \\ R_2 \to R_2 - 2R_1 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{array}\right]$$

$\therefore \quad b_3 + b_1 = 0$ (solvable only if)

$\therefore \quad b_3 = -b_1$

Qs 4)

Ans)

→ unless the new b is a linear combination of columns of A.

$col(A) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

→ if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[A\ b] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ No sol^n

col^n space get larger.

→ if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[A\ b] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Sol^n exist

col^n space same

$\begin{bmatrix} x \\ y \end{bmatrix}$

- if it doesn't get longer.

means b is linear comb^n of A col^n.

$\therefore$ $\boxed{Ax = b}$

Q5)

Ans) $Ax = b$ &
$Ay = b^*$ $\Big\}$ $A(x+y) = b+b^*$

then $Az = b+b^*$ (also solvable)

$\therefore$ $z = x+y$

if $b, b^* \in C(A)$ then $b+b^* \in C(A)$

classmate

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Q26)

Ans) 9×12 sys

$Ax = b$ solvable for every $b \in R^9$

$\therefore$ A can produce all vector in R^9

$\downarrow$

$Col(A) = R^9$

$C(A) = \underline{R^9}$

Q.7)

Ans)

a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

$\downarrow R_2 \to R_2 - R_1$

$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

$\downarrow R_3 \to R_3 - R_2$

$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U(\Delta \text{ echelon form})$

x_1, x_3 - pivot

x_2, x_4, x_5 - free variables.

b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$

$\downarrow R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow$ $R_1 \to R_1/2$, $R_2 \to R_2/4$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U$

pivot $\to$ x_1, x_2

free variables $\to$ x_3

Qs 8)
Ans)

a) False,

eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\downarrow R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

pivot - x_1
free variables - x_2

b) True, invertible matrix all column are LI
(Linear independent)

All column pivot.
No free variable.

True, $m \times n$ matrix

only has n col^n

↳ so max n pivots.

True; $m \times n$ matrix

- each pivot should be in a distinct row

↓

$\therefore$ # pivots $\leq$ m rows

Qs 9)
Ans)

x_4 - <u>free variable</u>

<u>Special</u> soln - $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

X

where $Ax = 0$.

2(a)
Ans)

i) $m \times n$ matrix

· n col^n
· r pivots

$\underline{n-r}$ special sol^n by Counting theorem.

ii) Nullspace contain $x=0$ when $\underline{r=n}$ (no free variable)

iii) $C(A) = R^m$ if $\underline{r=m}$.

(all pivots $= m$)

Q3 11)
Ans) A st $N(A)$ = all multiples of $(4,3,2,1)$

$$Ax = 0 \quad \forall\ x = (4c, 3c, 2c, 1c)^T$$

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ & \vdots & & \\ & \vdots & & \end{bmatrix} \begin{bmatrix} 4c \\ 3c \\ 2c \\ 1c \end{bmatrix} = 0$$

$\therefore\ 4cx_1 + 3cx_2 + 2cx_3 + x_4 c = 0$

Let			
$x_1 = 0$	$x_1 = 1$	$x_1 = 0$	$x_1 = 0$
$x_2 = 0$	$x_2 = 0$	$x_2 = 1$	$x_2 = 0$
$x_3 = 0$	$x_3 = 0$	$x_3 = 1$	$x_3 = 1$
$x_4 = 0$	$x_4 = -4$	$x_4 = -5$	$x_4 = -2$

$$A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ \text{Rank}(A) = 3$

12)

$\dim(\text{col}(A))$ $\dim(\text{null}(A))$

$\text{rank}(A) + \text{nullity}(A) = 2$

$\downarrow$ $\downarrow$

1 1

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\text{Col}(A) = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = 1$
$x_2 = 1$

$$N(A) = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

1/4

1/0

1 1

Q. 13)

Ans) A — 4×4 invertible matrix

$$B = \begin{bmatrix} A & A \end{bmatrix}_{4\times 8}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_1, x_2 \in \mathbb{R}^4$

$Bx = 0$

$$\begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow \quad Ax_1 + Ax_2 = 0$

$\Rightarrow \quad A(x_1 + x_2) = 0$

$\therefore \quad x \in N[B]$ must be $x = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ $\quad x_1 \in \mathbb{R}^4$

$\dim(N(B)) = 4$

→ if A is $n \times n$ matrix,

$-A = (-1)A$

- $A, -A$ have same row space.

  ∴ they make the same reduced row echelon form.