APMA 4007 | Lecture 11 | Oct 8, 2024

Outline

1. Review
2. Independent / dependent
3. Span (review)
4. Basis
5. Dimension
6. The big picture.

Review

$$Ax=b \qquad [A \mid b] \longrightarrow [U \mid ?] \longrightarrow [R \mid d]$$

What elimination tells us about $Ax=b$?

- rank r (# pivots)
- $m-r$ rows of zeros (existence)
- $n-r$ free variables (uniqueness)
- pivot columns of A span $C(A)$

$n-r$

- special sol$^{\underline{n}}$s ($d=0$)
- particular sol$^{\underline{n}}$
  * free variables $=0$
  * pivot variables read.

if $b \in C(A)$ complete sol$^{\underline{n}}$ $= x_{particular} + x_N$ $\in N(A)$

$$Ax_p = b, \quad Ax_N = 0$$

Cartoon (pivot variables come first) $x = \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix}$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}, \quad \text{special sol}^{\underline{n}} \begin{bmatrix} -F \\ I \end{bmatrix}$$

~~new~~ bonus

Thursday? If E is the matrix that does all the row ops $A \to R$

$$\underset{m\times m}{E}\ \underset{m\times n}{A} = \underset{m\times n}{R}$$

① $E[A \mid I] = [R \mid E]$ (like Gauss-Jordan)

② $EA = \underbrace{\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}}_{R}$, write $\overset{m\times m}{E} = \begin{array}{c} \\ \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \end{array}\begin{array}{l} \updownarrow r \\ \updownarrow m-r \end{array}$ ← square and invertible

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} A = \begin{bmatrix} E_1 A \\ \boxed{E_2 A} \end{bmatrix} = \begin{bmatrix} I & F \\ \boxed{0 \quad 0} \end{bmatrix}$$

$\Rightarrow E_2 A = 0$ or

$A^T E_2^T = 0$

$A^T E_2^T = 0$ means that the bottom $m-r$ rows of E contain the special sol$\underline{n}$'s of $A^T y = 0$

Also if $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$, $R^T = \begin{bmatrix} I & 0 \\ F^T & 0 \end{bmatrix}$

zeroing under the pivots of R^T gives $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

so the rank of R^T = rank R.

It turns out that the same is tre for A. rank A = rank A^T

? skip

Vocabulary (to talk about our results)

Independent. The columns of A are linearly independent iff the only sol$^{\underline{n}}$ of $Ax=0$ is $x=0$.

Dependent: (The opposite of L.I.) $Ax=0$ for some $\vec{x}\neq\vec{0}$.

why this word?

$$A=\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \quad \& \quad Ax=0 \text{ for } x\neq 0$$

That is $x_1 a_1 + \ldots + x_n a_n = 0$

Suppose $x_1\neq 0$ then $a_1 = -\frac{1}{x_1}\left[x_2 a_2 + \ldots + x_n a_n\right]$

a_1 is a L.C. of the other ~~colu~~ columns.

a_1 depends linearly on the other columns

Example

$$A=\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

columns of A are dependent

(free column is L.C. of pivot columns)

L. I. facts.

1. A set of L.I vectors does not include the zero vector.

$$A = \begin{bmatrix} | & \\ 0 & \cdots \\ | & \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

$$\uparrow \neq \vec{0}$$

2. Every sub set of L.I vectors is L.I.

Suppose $\{ \vec{a}_1 \cdots \vec{a}_n \}$ are L.I and $\{ \vec{a}_1 \cdots \vec{a}_k \}$ $k \le n$ are not. That is $\begin{bmatrix} | & & | \\ a_1 & \cdots & a_k \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_k \end{bmatrix} = \vec{0}$

$$\Rightarrow \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0} \quad \# \quad \text{b/c } a_1, \ldots, a_n \text{ L.I}$$

Taking away a vector from a L.I set does not make the subset dependent <u>BUT</u>

<u>adding</u> a vector to a set of L.I vectors can. make it dependent. (e.g., add a repeat)

3. How do we check for L.I? Put the vectors in A & check for special sol^{n}s.

4. Any set of columns of I are L.I.

I has a pivot in every column, subsets are L.I.

in fact a subset might look like $\begin{bmatrix} I \\ 0 \end{bmatrix}$ Keeping the first few columns

5. The pivot columns of R are L.I. b/c columns of I.

6. A sol^{n} of $Ax=b$ is unique iff the columns of A are L.I.

7. The rank of $\overset{m\times n}{A}$ is n iff the columns of A are L.I. ($r=n$, full column rank)

8. If $n>m$, n vectors in $\mathbb{R}^m$ cannot be L.I. b/c $\overset{m\times n}{A}$ is wide $r \leq m < n$ means $n-r>0$ special sol^{n}s.

9. The r pivot columns of A are L.I b/c if we put in another matrix (w/o free columns), the elimination steps will be the same b/c they only depend on pivot column values <u>but</u> no free columns $\Leftrightarrow$ L.I.

10 The special sol^{n}s are L. I

<u>cartoon</u> <u>version</u> $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ special sol^{n} $\begin{bmatrix} -F \\ I \end{bmatrix}$ switch rows

$\longleftrightarrow \begin{bmatrix} I \\ -F \end{bmatrix} \to \begin{bmatrix} I \\ 0 \end{bmatrix}$ zero below pivots

L. I

if the pivot variables/columns do not come first, the rows of $\begin{bmatrix} -F \\ I \end{bmatrix}$ are reordered which does not change the rank.

[each special sol^{n} has exactly one 1 in a free variable slot & the ~~reset~~ rest zeros in free variable slots]

11. The pivot rows of R are L.I. (which span $C(A^T)$) b/c the rest are zero rows).

they look like $[I\ F]$ possibly w/ columns reordered

put the rows in a matrix $\begin{bmatrix} I \\ F^T \end{bmatrix} \to \begin{bmatrix} I \\ 0 \end{bmatrix}$

L.I.

12. The columns of any (square) invertible matrix are L.I. b/c no free variables / special sol^{n}s

13. The ~~row~~ m-r bottom ~~row~~ rows of E (E_2) are L.I skip?

<u>Span</u> (review): A set of vectors <u>span</u> a space if ~~those~~ their L.C.s "fill" the space. Each element of the space is a L.C. of the vectors.

More use in a sentence

1. The columns of A span $C(A)$
   - the r pivot columns of A span $C(A)$

2. The special sol$^{\underline{n}}$s span $N(A)$.

3. The columns of an invertible matrix span $\mathbb{R}^n$. (e.g. I).

4. The rows of A span $C(A^T)$.
   - the r pivot rows of R span $C(A^T)$

<u>Basis</u> (span + LI): A basis for a vector space is a set of ~~the~~ L.I. vectors that span it.

<u>Sentences</u>

1. The pivot columns of A are a basis for $C(A)$
2. The special sol^{n}'s are a basis for $N(A)$.
3. The columns of any invertible matrix are a basis for $\mathbb{R}^n$
4. The pivot rows of R are a basis for $C(A^T)$.
5. The $m-r$ bottom rows of E (E_2) are a basis for $N(A^T)$ ← skip.

Remarks

1. Given vectors that span a space, how to find a basis? Put them in a matrix & find the pivot columns (which are L.I & span the space)

2. If $a_1, \dots a_n$ is a basis for V & $b \in V$ then b is a unique L.C. of those vectors

$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$, $b \in C(A) = V$, $Ax = b$ has unique sol^{n} b/c no free variables. (every column is a pivot column)

L.I. $\Leftrightarrow$ $Ax = 0$ $\Leftrightarrow$ $x = 0$

Dimension: The dimension of a vector space is the # of vectors in its basis. * Gap: is this well-defined? Next lecture.

1. $\dim C(A) = r$ [pivot columns] $\quad C(A) \subseteq \mathbb{R}^m$
2. $\dim N(A) = n - r$ [special sol^{n}] $\quad N(A) \subseteq \mathbb{R}^n$

* 3. $\dim C(A^T) = r$ [pivot rows of R] $\quad$ rank A = rank A^T $= r$ ✓

- since the pivot "columns" of A^T are also a basis, there are r so the dim matches

4. $\dim N(A^T) = m - r$ [$m-r$ rows of E]

E invertible, rows L.I.

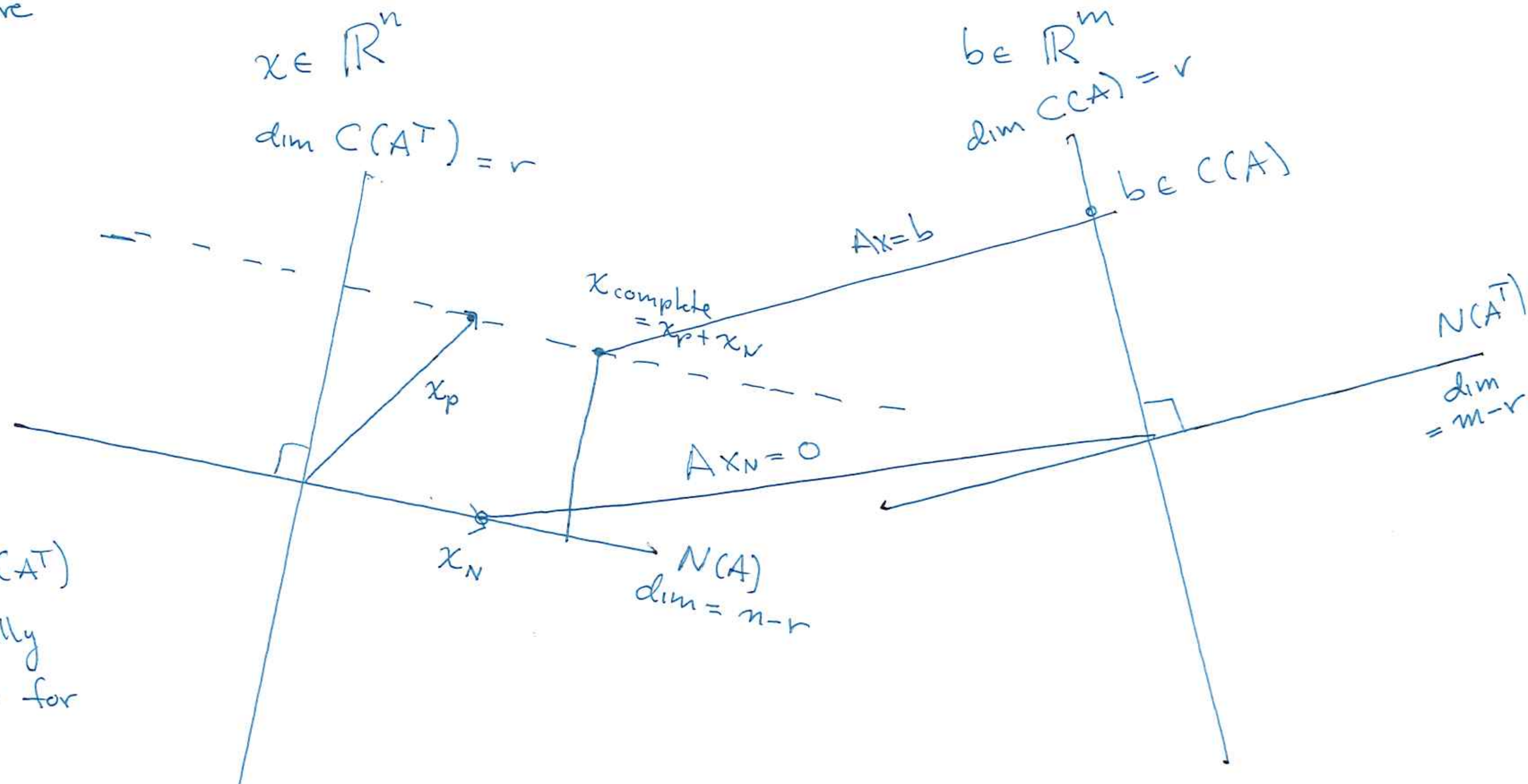
* rule 2 dim of null space = # columns − rank $= m - r$

special sol^{n} of $A^T y = 0$

$\Leftrightarrow$ $\underbrace{y^T A = 0}$

"left null space of A"

(σ) the big picture



(2) I have drawn $x_p \notin C(A^T)$ & that is generally true. The basis for

$C(A^T)$ is $\begin{bmatrix} I \\ F^T \end{bmatrix}$ (pivot rows of R) and does not (in general) have zeros in the free variable slots but $x_{particular}$ does. Example

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_p = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

basis $C(A^T)$

$$\xrightarrow{x_p} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 2 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -9 \end{array}\right] \#$$

(1) I have drawn the subspaces $\perp$ to each other & they are.

If $x_N \in N(A)$ & $y \in C(A^T)$ then

$Ax_N = 0$ $\quad y = A^T v$ (L.C. of rows of A)

$$y^T x_N = v^T \underbrace{\boxed{A x_N}}_{0} = 0$$

$x_N \perp y$. ✓ Every vector in $C(A^T)$ is $\perp$ to every vector in $N(A)$

Likewise if $b \in C(A)$ the $b = Ax$ for some x

if $u \in N(A^T)$, $A^T u = 0$.

$$b^T u = (Ax)^T u = x^T \underbrace{A^T u}_{\overset{\shortparallel}{0}} = 0$$

Every vector in $C(A)$ is $\perp$ to every vector in $N(A^T)$.

—

ok

vectors in $N(A^T)$ are handy to checking if $Ax = b$ has soln. If $u \in N(A^T)$ the $u^T A = 0$ and we assume $Ax = b$ (soln exists)

$$\underset{\overset{\shortparallel}{0}}{u^T A x} = \underset{\overset{\shortparallel}{?}}{u^T b}$$

if $u^T b \neq 0$

※ no soln

—

Sometimes this is rearranged to say

Either

$Ax = b$ has soln

OR

$A^T u = 0$ has soln w/ $u^T b \neq 0$

(b not $\perp$ to $N(A^T)$, i.e. not in $C(A)$)

Fredholm alternative.