APMA 4007

Sep 19, 2024

Lecture 6

Outline

1. Review inverse
2. Gauss-Jordan to find A^{-1}
3. Diagonal dominance

APMA 4007

Sep 19, 2024

Lecture 6

Outline

1. Review inverse
2. Gauss-Jordan to find A^{-1}
3. Diagonal dominance

Last lecture defined the matrix inverse A^{-1} of a square matrix A.

$$AA^{-1} = A^{-1}A = I \quad \leftarrow A^{-1} \text{ is also square}$$

Inverse facts

0. A must be square

1. A^{-1} exists $\Leftrightarrow$ A has n pivots (proof later)

   - A has row of zeros, no inverse
   - A has column of zeros, no inverse
   - check by elimination

2. The inverse is unique. If there is a right one and a left one, they are the same.

3. If A^{-1} exists, the sol$^{\underline{n}}$ of $Ax=b$ exists & is unique

$$x = A^{-1}b$$

4. A^{-1} exists $\Leftrightarrow$ $\vec{x} = \vec{0}$ is the unique sol$^{\underline{n}}$ of $Ax=0$.

   - If A has two columns that are the same, no A^{-1}

   e.g., $A \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{bmatrix} = \vec{0}$

   - if the columns of A sum to zero, no A^{-1}

$$A\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \vec{0}$$

5. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6. $\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_n \end{bmatrix}$

7. $(AB)^{-1} = B^{-1}A^{-1}$

8. The sol^n of $A x = \vec{e}_i$, where e_i is the i-th column of I, is the i-th column of A^{-1}.

- b/c $A^{-1} A x = A^{-1}\vec{e}_i$ $\quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i-th position is 1, rest zero

$x = A^{-1} e_i = i$-th column of A^{-1}

Example: Find $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$

1 solve $Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for 1st column of A^{-1}

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & \boxed{-3} & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2/3 \end{array}\right]$$

back sub. in the box

zero above pivots

$$\to \left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{array}\right], \quad \text{1st column of } A^{-1} \text{ is } \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}$$

2. 2nd column of A^{-1}

Note exactly the same row op.

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 2 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-3} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/3 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & -1/3 \end{array}\right], \quad \text{2nd column of } A^{-1} \text{ is } \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$$

so $A^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

check $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

also formula $\dfrac{1}{1-4}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ ✓

Note

1 Same row operations each time, so more efficient

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -2/3 & 1/3 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right]$$

I $\quad$ A^{-1}

This is the same as solving

$A\underline{X} = I$

$\uparrow$ a matrix

More generally do this if solving $Ax=b$ for multiple b's

Gauss-Jordan method of finding A^{-1}

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

row operations
1. Elimination (zero below pivots)
2. Divide by pivots
3. Zero above pivots

~~which~~ row operations can be written as matrix mult. (from the left). What matrix mult?

$$\boxed{A^{-1}}[A \mid I] = [I \mid A^{-1}]$$

aside a matrix equality = many vector equalities

so

$$B[A \mid I] = [I \mid A^{-1}]$$

$\Longleftrightarrow \quad BA = I \quad$; $\quad B = A^{-1}$

Example

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

square ✓
3 pivots ✓
(already upper triangular)

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 3 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 0 & -1 \\ 0 & 4 & 0 & 0 & 1 & -2/3 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 4 & 0 & 0 & 1 & -2/3 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -3/4 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & -1/6 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/4 & -1/6 \\ 0 & 0 & 1/3 \end{bmatrix}$$

check

when to divide by pivots is a choice

$$\begin{bmatrix} 2 & 6 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/4 & -1/6 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Note: If A is upper triangular, so is A^{-1}

why? the i-th column of A^{-1} is the sol$^{\underline{n}}$ of

$A x = \vec{e}_i$ ← n-th row say $a_{nn} x_n = 0$ $\quad i < n$

$n-1$ row says $a_{n-1,n-1} x_{n-1} + a_{n-1,n} \underset{=0}{\underset{\uparrow}{x_n}} = 0$

$\Rightarrow x_{n-1} = 0$

and so on until we get to the i-th row which has one on the rhs.

Nice fact to know in advance

G-J cartoon

$\overset{n\times n}{A}$

1. $[A \mid I] \to [U \mid ?]$

zero below pivots & check that there are n

2. $[U \mid ?] \to [D \mid ??]$

zero above pivots = back substitution in the matrix

3. $[D \mid ??] \to [I \mid A^{-1}]$

divide by pivots.

G-J shows how to solve $Ax=b$ for multiple known rhs's

$A\vec{x} = \vec{b}_1$, $A\vec{y} = \vec{b}_2$

$[A \mid b_1\ b_2] \to [I \mid x\ y]$

Let's revisit the statement that n pivots $\Leftrightarrow$ A^{-1} exists

$\Rightarrow$ n pivots $\Rightarrow$ we can solve $A\vec{c}_i = \vec{e}_i$ $i = 1 \ldots n$

$\Rightarrow AC = I$ $\quad C = [\vec{c}_1 \ldots \vec{c}_n]$

A has a left inverse

G-J says $[A \mid I] \to [I \mid A^{-1}]$ by row ops.

$B[A \mid I] = [I \mid A^{-1}]$

$\Rightarrow BA = I$

and ~~which~~ row ops are m-m mult from the right

So there is a right inverse

privously showed must be the same $(BA)C = B(AC)$

Now $\Leftarrow$ by contrapositive. not having n pivots $\Rightarrow$ singular

missing pivot $\Rightarrow$ $EA = \begin{bmatrix} \text{matrix} \\ \text{with row of zeros} \end{bmatrix}$

note elimination is invertible

Suppose A^{-1} exists. Then $EAA^{-1} = \begin{bmatrix} \text{matrix} \\ \text{w/ row zeros} \end{bmatrix} A^{-1}$

$$E = \begin{bmatrix} \text{another matrix} \\ \text{w/ row of zeros} \end{bmatrix}$$

but a matrix w/ a row of zero is not invertible

b/c if the row i is all zeros $E x = \vec{e}_i$

$0 = 1$ has no sol$^{\underline{n}}$

Deciding if a matrix is invertible

0. Is it square
1. n pivots?
   - elimination
   - diagonal ← look
   - triangular (upper or lower, if lower just switch rows)
   - <u>also</u> <u>anti-triangular</u>

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 5 & 6 \\ 2 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix} \rightarrow \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}}$$

$$\rightarrow \begin{bmatrix} 6 & 5 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$$

2. If 2x2 check $ad - bc \neq 0$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3. Check for rows / columns of zeros. Or rows/ columns that are multiples of each other

Suppose row 1 = row 2

$$\begin{bmatrix} 1 & -1 & & 0 \\ 0 & 1 & & \\ & & 1 & \\ 0 & & & \ddots & 1 \end{bmatrix} A = $$ matrix w/ row of zeros

inverse

no inverse

4. A is invertible if it is the product of 2 invertible matrices ~~say~~ ~~and~~ and $(AB)^{-1} = B^{-1} A^{-1}$

Example $\begin{bmatrix} 10 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ invertible? Yes

Example $\begin{bmatrix} 10 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 8 \end{bmatrix}$ invertible? hard to say but it is "close" to an invertible matrix.

Equivalent question

Does $Ax = 0$ have a sol$^{\underline{n}}$ w/ $\vec{x} \neq \vec{0}$?

* Suppose $|x_1| \geq |x_2|$ & $|x_1| \geq |x_3|$ that is x_1 is the largest in absolute value

first row of $Ax=0$ is $10x_1 + x_2 + x_3 = 0$

or $10x_1 = -(x_2 + x_3)$

$$\Rightarrow \quad 10|x_1| = |x_2 + x_3| \leq |x_2| + |x_3|$$

$$10|x_1| \leq 2|x_1| \quad ※$$

unless $x_1 = 0$ in which case $\vec{x} = 0$ b/c $|x_1| \geq |x_2|$ & $|x_3|$

OK if $|x_2|$ is largests

$\Rightarrow$ $9|x_2| \leq 2|x_2|$ ※

and if $|x_3|$ largest $8|x_3| \leq 2|x_3|$

The general case. Suppose that $Ax=0$ & $|x_i|$ is the largest element in abs. value of x.

Look the i-th row of $Ax=0$ (row i of A dotted w/ x)

$$\sum_{j=1}^{n} A_{ij} x_j = 0 \quad \Rightarrow \quad A_{ii} x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} A_{ij} x_j$$

abs. value

$$|A_{ii}|\,|x_i| = \left|\sum_{\substack{j=1 \\ j\neq i}}^{n} A_{ij} x_j\right| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}|\,|x_j|$$

$$|A_{ii}|\,|x_i| \leq \sum_{\substack{j=1 \\ j\neq i}} |A_{ij}|\,|x_i|$$

$$\Rightarrow \quad |A_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}|$$

abs. diagonal element i

sum of abs. off diag in row i

There if $\boxed{|A_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}|}$ for $i=1,\dots,n$

this condition is called diagonally dominant (strict b/c >)

then $Ax=0$ is impossible

One-way

Diag. dom. $\Rightarrow$ invertible

not Diag. dom $\Rightarrow$ unknown

Example

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

this is a tridiagonal matrix

Is it diag dom?

No b/c $|2| > |2|$ not true

Is it invertible yes

~~A^i has positive entries for some $i \le n$.~~

It turns out that if A satisfies a condition called irreducible / strongly connected then

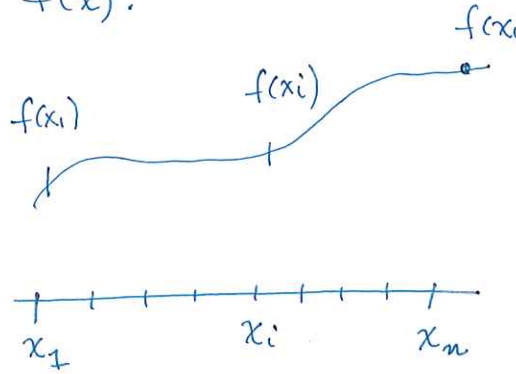
$$|A_{ii}| \boxed{\geq} \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}| \quad \text{for all } i = 1, \dots, n$$

with $>$ (strict) for at least one i the A is invertible.

advanced not easy to check reducible

Where does this tridiagonal matrix come from?

Suppose we wish to approximate numerically the derivative of a function $f(x)$.



then $\dfrac{f(x_{i+1}) - f(x_i)}{h}$ (forward) is a finite difference approximation as

is $\dfrac{f(x_i) - f(x_{i-1})}{h}$ (backward)

$x_{i+1} - x_i = h$ spacing

and likewise a (centered) approximation of $\dfrac{d^2f}{dx^2}$ is their difference

$$\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

Suppose $\frac{d^2 f}{dx^2} = g(x)$ then

$\uparrow$ for unknown $\qquad$ $\uparrow$ given

the i-th row of the finite difference approximation is

$$\frac{1}{h^2}\left(f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right) = g(x_i)$$

which is a linear equation in the values of f

Writing it in matrix form w/ $n=4$

$$\overset{K}{\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ g(x_3) \\ g(x_4) \end{bmatrix}$$

Note that the 1st & last equations (rows) are missing $f(x)_0$ & $f(x)_{n+1}$ respectively. In other word they are taken to be zero. Solving

$$\frac{1}{h^2} K \vec{f} = \vec{g}$$

is an approximation of $f'' = g$ with $f = 0$ at the boundaries (x_0 & x_{n+1})

b/c K is invertible, it has a unique soln. Other BCs lead to other first & last rows and matrices which might not be invertible.