APMA 4007 | Lecture 12 | Oct 10, 2024

Outline

1. A basis always has the same # vectors
2. Basis for $N(A^T)$
3. The right # of L.I vectors is a basis
4. $x_p \notin C(A^T)$ generally
5. Orthogonal subspaces

Review of new vocab. (Reverse order)

Dimension of a subspace : # of vectors in its basis

Basis of a subspace : a set of L.I. vectors that span it

Linearly independent vectors : No (non-zero) linear combination is zero. The columns of A are L.I means that the only solⁿ of $Ax=0$ is $x=0$.

Span : A set of vectors span a subspace if every element is a L.C. of them.

A basis of a vector space V contains exactly the right vectors (and number).

1. If you take away one the space is no ~~xxx~~ longer spanned

2. If you add ~~one~~ any vector from V they are no longer linearly independent

Proofs Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V

1. If you take away $\vec{v}_n$, $\{\vec{v}_1, \dots, \vec{v}_{n-1}\}$ no longer span V b/c

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_{n-1} \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ | \\ x_{n-1} \end{bmatrix} = \begin{matrix} | \\ v_n \\ | \end{matrix}$$ has no sol$^{\underline{n}}$

and $v_n \in V$. ~~there~~ The reason it has no sol$^{\underline{n}}$ is that if it did, then

$$\begin{bmatrix} | & & | & | \\ v_1 & \cdots & v_{n-1} & v_n \\ | & & | & | \end{bmatrix}\begin{bmatrix} x_1 \\ | \\ x_{n-1} \\ -1 \end{bmatrix} = \vec{0} \quad \#$$ b/c L.I.

2. Add any $v_{n+1} \in V$ to the basis vectors.

Since $v_{n+1} \in V$, it is a L.C. of basis vectors

$$v_{n+1} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$

which means

$$\begin{bmatrix} | & & | & | \\ v_1 & \cdots & v_n & v_{n+1} \\ | & & | & | \end{bmatrix}\begin{bmatrix} x_1 \\ \\ x_n \\ -1 \end{bmatrix} = \vec{0}$$

not L.I., not a basis.

I noted last lecture that for any $b \in V$, b is a unique L.C. of basis vectors. b/c the sol$^{\underline{n}}$ of

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} x = b \quad \text{is unique}$$

when the columns are L.I. (no special sol$^{\underline{n}}$s)

On the other hand, sets of basis vectors are not unique. For instance the columns of any $n \times n$ invertible matrix are a basis for $\mathbb{R}^n$

To get a better idea of the different basis sets that a vector space can have, let's ~~cons~~ fill a gap regarding the definition of dimension, namely that all basis sets of a vector space have the same number of vectors. (i.e., dimension is a well-defined quantity).

Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ & $\{\vec{w}_1, \ldots, \vec{w}_m\}$ are sets of basis vectors for the same vector space and that $m < n$.

Since the $\vec{w}_i$'s are a basis each $\vec{v}_i$ is a L. C. of $\vec{w}_i$s. That is,

$$\begin{bmatrix} | & & | \\ w_1 & \cdots & w_m \\ | & & | \end{bmatrix} \begin{bmatrix} | \\ a_i \\ | \end{bmatrix} = \vec{v}_i$$

$\overline{W}$ ↗ (the matrix) ↑ the vector of L.C. scalars

true for each i

$\overline{W}\,\vec{a}_i = \vec{v}_i \qquad i = 1, \ldots, n$

turning the n vector equalities into a matrix equality gives

$$\overline{W} \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

↑ A ↑ $\overline{V}$

$$\overline{W} \cdot A = \overline{V}$$

what is the size of A?

- A has n columns
- A has m rows, one for each column of $\overline{W}$

$\therefore A$ is $m \times n$. w/ $m < n$ $\therefore$ A is <u>wide</u>

rank $A \leq m < n$ $\Rightarrow$ at least on special sol$^{\underline{n}}$ x w/ $Ax = 0$ & $x \neq \vec{0}$.

$$V x = W \underbrace{A x}_{=0}$$ (mult. both sides by x)

$Vx = 0$ & $x \neq 0$ $\Rightarrow$ columns L.I

$\Rightarrow$ not a basis # $m < n$ is impossible

likewise (switching the roles of V & W), $n < m$ is impossible, $\Rightarrow$ $m = n$.

This means that ~~matrix~~ A is square and that the only sol$^{\underline{n}}$ of $Ax = 0$ is $x = 0$ (otherwise same argument leads to $Vx = 0$)

$\Rightarrow A$ is invertible.

Any two sets of basis vectors are related by an invertible matrix A.

$$V = W A$$

The fundamental theorem of linear algebra (I) is about dimensions

	basis
1 $\dim C(A) = r$	r pivot columns of A
2 $\dim C(A^T) = r$	r pivot rows of R
3 $\dim N(A) = n - r$	$n-r$ special sol$^{\underline{n}}$s
4. $\dim N(A^T) = m - r$	?

3. gives the rule that dim of null space of a matrix = # of columns − rank and that gives $\dim N(A^T) = m - r$.

But what is a basis for $N(A^T)$?

$y \in N(A^T) \iff A^T y = 0$ or $y^T A = 0$

which says a L.C. of the rows of A is a zero row.

$$\text{Recall } R = \underbrace{\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}}_{\text{cartoon}} = \underbrace{\begin{bmatrix} R_{pivot} \\ 0 \end{bmatrix}}_{\text{true in general}}$$

has zero rows that appear by row operations (zero above/below divide by pivots)

Let's encode those row ops in the matrix E

$$\overset{m\times m}{E}\,\overset{m\times n}{A} = \overset{m\times n}{R}$$

* how to find E? G-J.

E is square and invertible b/c all row ops can be undone.

Divide the rows in E to match those in R

$$E = \begin{bmatrix} \overset{r\times m}{E_1} \\ \overset{(m-r)\times m}{E_2} \end{bmatrix}, \text{ then } EA = \begin{bmatrix} E_1 A \\ E_2 A \end{bmatrix} = \begin{bmatrix} R_{pivot} \\ 0 \end{bmatrix}$$

~~this is two sets of equations.~~

~~1 $E_1 A = R_{pivot}$ which says the pivot rows of R are L.C. of rows of A~~

This says that $E_2 A = 0$

which means that the $m-r$ rows of E_2 are in $N(A^T)$

And how do you find E? Gauss-Jordan
$E[A|I] = [EA|E] = [R|E]$

$$\begin{bmatrix} — & e_{r+1}^T & — \\ & & \\ — & e_m^T & — \end{bmatrix} A = 0$$

e_i^T is the i-th row of E

These rows are L.I b/c the rows of an invertible matrix.

Q: Are they a basis for $N(A^T)$? In particular do they span $N(A^T)$.

A: Yes. If you have the right number of L.I vectors, they are a basis.

Proof

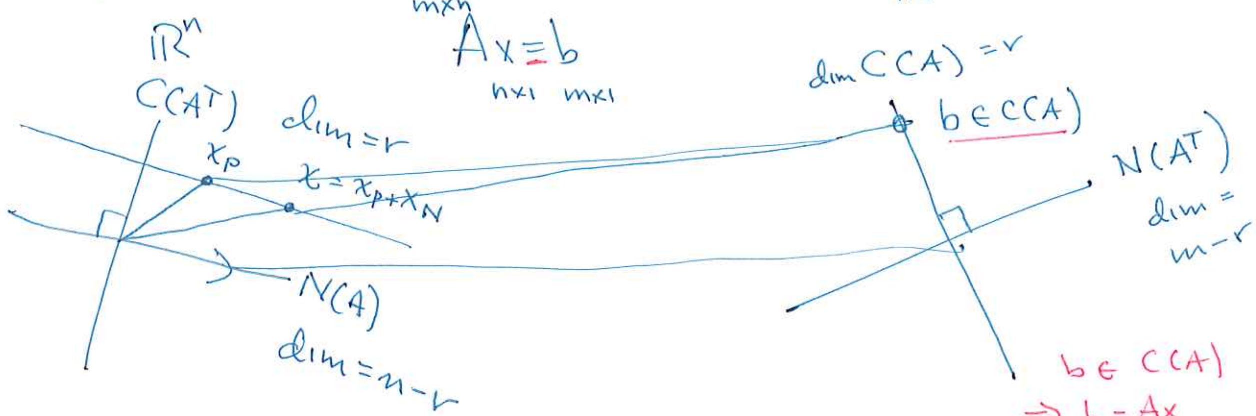
SKIP in class

Suppose $v \in N(A^T)$ & v is NOT a L.C. of the $e_{r+1}, \ldots e_m$. In that case, $\{e_{r+1}, \ldots, e_m, v\}$ is a set of $m-r+1$ L.I vectors but $m-r+1$ L.I vectors in a vector space of dimension $m-r+1$ leads to a contraction

wide so has a special sol$^{\underline{n}}$. #

$$\underset{n \times (m-r+1)}{\begin{bmatrix} | & & | & | \\ e_{r+1}, & \ldots & e_m & v \\ | & & | & | \end{bmatrix}} = \underset{n \times (m-r)}{\begin{bmatrix} | & & | \\ w_1, & \ldots & w_{m-r} \\ | & & | \end{bmatrix}} \underset{(m-r) \times (m-r+1)}{A}$$

The big picture again



$y \in C(A^T)$
$\Rightarrow y = A^T v$

$x \in N(A) \Rightarrow Av = 0$

$y^T x = v^T A x = 0$ ✓

$b \in C(A)$
$\Rightarrow b = Ax$
$y \in N(A^T) \Rightarrow$
$A^T y = 0$
$b^T y =$
$x^T A^T y = 0$ ✓

Complete $\text{sol}^{\underline{n}} = x = x_p + \underbrace{x_N}_{\in N(A)}$

x_p generally not in $C(A^T)$

Example

$$\begin{array}{c} \begin{array}{ccc|c} & A & & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} p & p & f \end{array} \end{array} \qquad x_p = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \begin{array}{c} p \\ p \\ f \end{array}$$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$x_p = \begin{bmatrix} \text{stuff} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ not $\perp$

$x_N = \begin{bmatrix} -F \\ I \end{bmatrix}$

$$x_N = \begin{bmatrix} -4 \\ -5 \\ 0 \end{bmatrix} x_3$$

is $x_p \in C(A^T)$?

also $\begin{bmatrix} I \\ F^T \end{bmatrix}$ basis for $C(A^T)$

No b/c $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 4 & 5 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -23 \end{array}\right]$$ no $\text{sol}^{\underline{n}}$

OR

~~recall that A~~ just check if x_p is $\perp$ to x_N

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -5 \\ 0 \end{bmatrix} = -23 \quad \text{NO.}$$

Now we can define what is means for subspaces to be orthogonal ($\perp$) as drawn

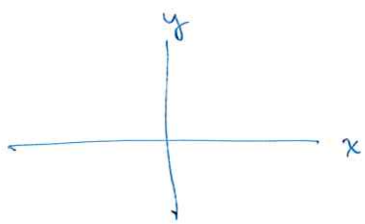
The subspaces V & W are orthogonal means that for all $v \in V$ & $w \in W$ $v^T w = 0$,

And we write $V \perp W$.

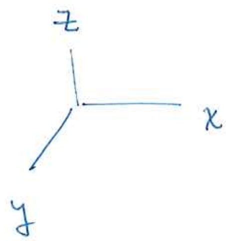
Examples

- $N(A) \perp C(A^T)$
- $N(A^T) \perp C(A)$

other examples

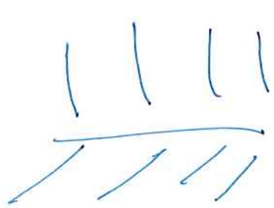


x-axis $\perp$ y-axis



z-axis $\perp$ xy-plane

not an exampl



wall xz-plane

floor xy-plane

NOT $\perp$ subspaces b/c x-axis is in both e.g. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and it cannot be $\perp$ to itself

2 $\perp$ subspaces only intersect at $\vec{0}$ (origin)