APMA 4007 | Sep 3, 2024 | Lecture 1

Outline

1. System of linear equations
   - row view

2. Vectors, L.C.

3. Matrix-vector product $Ax=b$ existence

4. Dot product

Course details on courseworks (canvas)

- Syllabus
- HW + midterm + final

HW ↑ weekly; no late HW; 2 drops

midterm ↑ check the date

final ↑ projected date

Target audience — no linear algebra experience

Classical applications of linear algebra

1. Numerical soln of differential equations
   - systems of linear equations w/ (many) unknowns
     numerical methods
     another class
   - unknowns appear linearly

2. Data analysis / statistics
   - curve (line) fitting via least squares
   - covariance matrices e.g., PCA, eigenvalue problems.

Soul of linear algebra =

systems	S
of	O
linear	L
equations	E

Example

$$\boxed{\begin{matrix} 2x & + & y \\ x & - & y \end{matrix}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

lhs unknows w/ coefficients

rhs constant

Take 2 minutes to solve for x & y.

Q: 1. What is the method?

2. Is there a useful picture? What questions does it answer?

wait 2 minutes

What is the answer? $x = 2$
$y = 1$

I added the 2nd equation to the first to get $3x = 6$ ← y is eliminated

Divided by 3 to get $x = 2$

I substituted (back) into the 2nd to get

$2 - y = 1 \Rightarrow y = 1$. I checked the soln.

$2 \cdot 2 + 1 = 5$ ✓

$2 - 1 = 1$ ✓

<u>Goal:</u> A systematic <u>method</u> (could be coded) of addition/subtraction of equations that gives an equation w/ a <u>single</u> unknown (the others have been <u>eliminated</u>), which is then solved and whose sol<u>n</u> is then substituted back into the other equations.

Very practical.

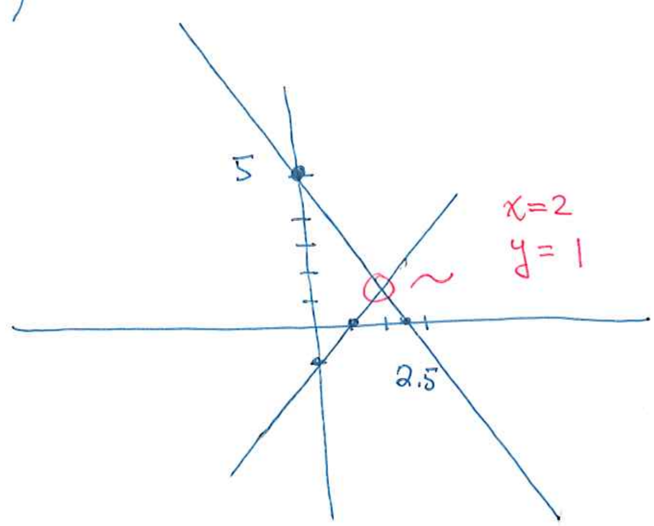
Theoretical concerns? [which are important in practice]

1. Is there a sol<u>n</u>? <u>Existence</u>

2. Is there more than one sol<u>n</u>? <u>Uniqueness</u>

The picture is useful for this)

$y = x - 1$ a line w/ slope 1 passing through $x=0$, $y=-1$; $x=1$, $y=0$



$y = -2x + 5$ a line w/ slope -2 passing through $x=0$, $y=5$; $x=2.5$, $y=0$

- The 2 equations = 2 lines
- A sol<u>n</u> <u>exists</u> if the lines cross (not parallel)
- if the lines are parallel
  * no sol<u>n</u> if the lines are different
  * infinite # of sol<u>n</u> if they are the same line

<u>How many sol<u>n</u>'s? 0, 1, ∞</u>

This is the row picture.

Each equation (a row) is a line. ← 2 unknowns
2 equations

In 3D, each row (equation) is a plane. Harder to draw. Harder to think about the ways that 3 plane can intersect.

is the equation of a plane

Another viewpoint (picture) uses columns, which brings use to vectors.

Vectors — ordered list of numbers
— column in a spreadsheet
— a 1D array shape(x) = (n,1)

by default columns

[all arrays are 1D b/c x.flatten()]

Example

sometimes

$$\vec{x} = \begin{bmatrix} 13 \\ 22 \\ 31 \end{bmatrix}$$

bold font

Key property is its size/ = 3 elements

NOTE len(x) = 3 but we will use length for something else

italic not bold

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

vector $\vec{x}$ w/ n elements

bold

(literacy reading equations)

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

bold

some abuse of notation b/c limited # of good letters

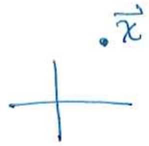
not bold

$$\vec{x} = \begin{array}{c} | \\ x \\ | \end{array}$$

visual representation that x is a column vector

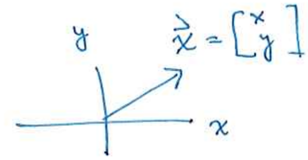
Example

1. $\vec{x} \in \mathbb{R}^n$



point in $\mathbb{R}^n$

2. $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$



vector in 2D

3. $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0.6 \\ 0.6 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

salmon   red

4. x_i is a measurement at t_i

$\vec{x} = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$ is a time series (date)

Central Park temp

stock price

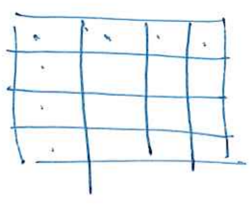
heart rate

5. $\vec{S} = \begin{bmatrix} S_1 \\ | \\ S_n \end{bmatrix}$ is a portfolio w/ S_i = # of shares (or \$) of company i

positive = long

negative = short

6. $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{16} \end{bmatrix}$



4x4 grayscale image (flattened)

with some convention

"C" along rows

"F" along columns

What can be done w/ vectors?

1. Check equality $\vec{x} = \vec{y}$ iff $\Leftrightarrow$ $x_i = y_i \quad i = 1, \dots, n$

must be the same size.

A concise way of writing n equations

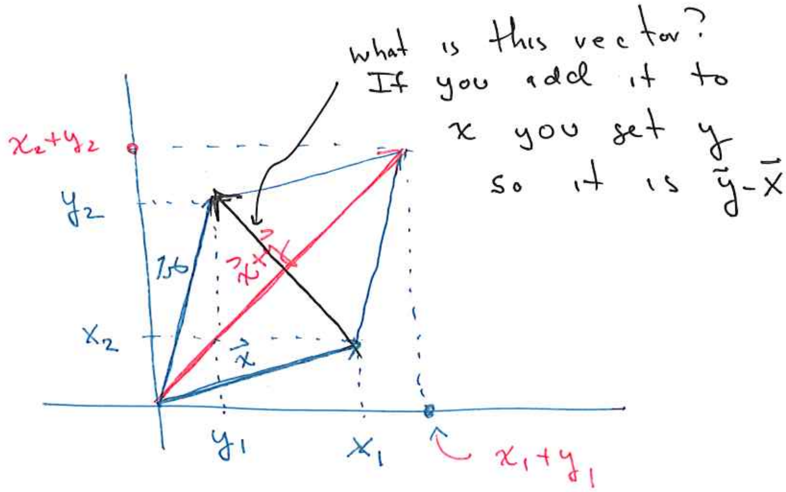
2. Add (elementwise) $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ | \\ x_n + y_n \end{bmatrix}$

must be the same size

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

The picture

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$



Graphical addition

- Put the start of y at then end of x to get $x + y$.

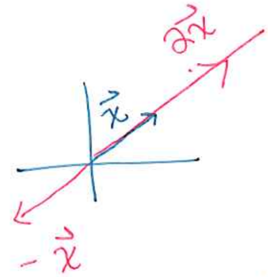
- Since vector addition is built upon elementwise additition, it has similar properties (easily checked)

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

3. <u>Scale</u> a vector. Make it longer or shorter w/o changing direction.

$$c\vec{x} = \begin{bmatrix} cx_1 \\ | \\ cx_n \end{bmatrix}$$ each element scaled



c ← a number, a <u>scalar</u>

negative scalar flips

but I think that is the same direction (2-way, like streets)

<u>Properties</u>

* $(c + d)\vec{x} = c\vec{x} + d\vec{x}$ — c, d: scalars

* $c\vec{x} = \vec{x}c$

* $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$

check using element wise definition — not mysterious

4. Linear combination = adding and scaling

$c\vec{x} + d\vec{y}$ — $\vec{x}$, $\vec{y}$: vectors; c, d: scalars

$\vec{x}$ & $\vec{y}$ are the ~~same~~ same size or addition does not work.

<u>Example</u> n = # of students $\vec{x}$ = HW grade average w/ 2 lowest dropped

$\vec{y}$ = midterm grade

$\vec{z}$ = final grade

course grades = $0.2\boxed{\vec{x}} + 0.4\vec{y} + 0.4\vec{z}$

Let us connect systems of linear equations w/ vectors.

$$x - y = 1$$
$$2x + y = 5$$

A system of linear equation can be written as a (single) vector equation

$$\begin{bmatrix} x-y \\ 2x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

n eq. $\rightarrow$ 1 vector eq

unaddition

$$\begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad (*)$$

unscale

$$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Take 2 minutes

1. Check that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is still a sol$^{\underline{n}}$

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \checkmark$$

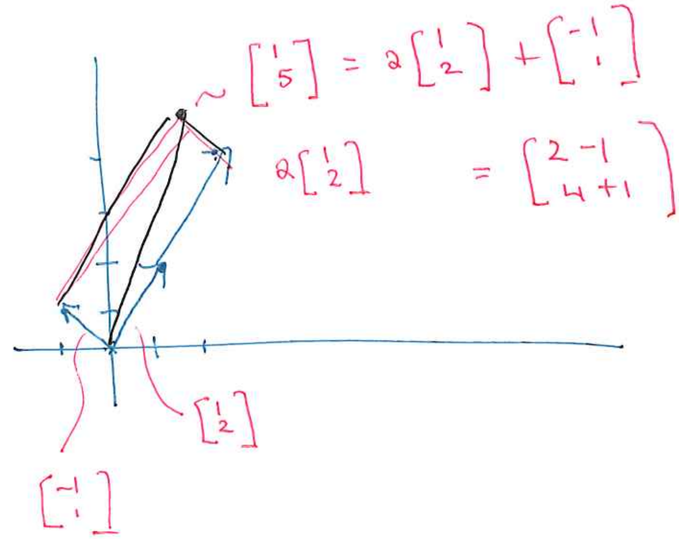
2. Use your new vocab to express $(*)$ as a sentence.

= means "is" $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is a L.C. of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3. Draw the picture.

The column picture

sol$^{\underline{n}}$ exists b/c there is a way to write $\begin{bmatrix}1\\5\end{bmatrix}$ as L.C. of $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$



It is fair to say that the questions of existence and uniqueness are less obvious (for now)

A notation for L.C.

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix}$$

- put columns in a box $\begin{array}{cc} x & y \\ \end{array}$ $\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}$

- put the scalars that mult. them above

- actually put them on the side

$$\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

BUT remember they go w/ columns NOT rows

Now the SOLE is

$$\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

Tradition uses these letters

$$A\vec{x} = \vec{b}$$

matrix of coefficients

vector of

constant vector

matrix-vector product (multiplication)

$\boxed{Ax} = b$

of <u>rows</u> of A = # of equations = $\underline{m}$

of columns of A = # of unknowns = n

b/c each unknown multiplies a column

$$\text{size } A = \underset{\text{rows}}{m} \times \underset{\text{columns}}{n}$$

size x = $n \times 1$

size b = $m \times 1$ ← each element of b is the 'constant' in the m equations.

must match

$$\underbrace{\overset{m\times n}{A}\ \overset{n\times 1}{x}}_{\text{size of result } m\times 1} = \overset{m\times 1}{b}$$

must match rhs

$$\begin{array}{c} \begin{array}{ccc} x_1 & & x_n \end{array} \\ \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \end{array}$$

A w/ n columns

each column has m elements

Some properties of matrix-vector mult are familar.

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$

BUT $A\vec{x} \neq \vec{x}A$ b/c does not compute

A nice theory statement about $Ax=b$.

$Ax=b$ has a sol$^{\underline{n}}$ <u>iff</u> b is a L.C. of the columns of A.

There is another product the dot product

$\vec{u} \cdot \vec{v}$ = a number

$= u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$

Properties

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

3 $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

Length in 2D $\|u\|^2$ = length square

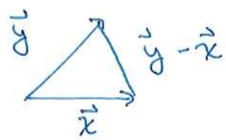
$= u_1^2 + u_2^2 = \vec{u} \cdot \vec{u}$

and so on in n dimensions

$$\|u\|^2 = \sum_{i=1}^{n} u_i^2 = \vec{u} \cdot \vec{u}$$

Properties

1. $\|c\vec{x}\| = |c| \, \|x\|$

2. Triangle inequality 

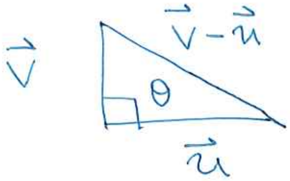
$\|y - x\| \leq \|x\| + \|y\|$ or replacing x by $-x$

$\|x + y\| \leq \|x\| + \|y\|$

3. $\|x\| \geq 0$

4. $\|x\| = 0$ iff $\vec{x} = \vec{0}$ (the zero vector)

The only vector w/ length zero is the zero vector.



if $\theta = 90°$ then

$$\|u\|^2 + \|v\|^2 = \|u-v\|^2$$

in general $\|u-v\|^2 = (u-v)\cdot(u-v)$

$$= u\cdot u + v\cdot v - 2u\cdot v$$

$$= \|u\|^2 + \|v\|^2 - 2u\cdot v$$

so if $\theta = 90°$ $u\cdot v = 0$ $u \perp v$

dot product is zero if $\perp$

The law of cosines says

$$\|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos\theta = \|u-v\|^2$$

$$\Rightarrow \quad 2\|u\|\,\|v\|\cos\theta = 2u\cdot v$$

$$\Rightarrow \quad \frac{u\cdot v}{\|u\|\,\|v\|} = \cos\theta$$

since $|\cos\theta| \le 1$

$$|u\cdot v| \le \|u\|\,\|v\|$$

Cauchy - Schwartz inequality