APMA 4007 | Sep 17, 2024 | Lecture 5

Outline

1. Matrix-matrix mult. <u>review</u>

2. m-m mult. for elimination

3. $A = LU$

4. Matrix inverse definition & properties

Review matrix-matrix mult. $\overset{m\times n}{A}\ \overset{n\times K}{B}$ $\overset{m\times K}{\text{ans.}}$

* The columns of AB are L.C. of the columns of A.

Ax e.g., $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$

* The rows of AB are L.C. of the rows of AB

y^TA e.g., $[1\ -1\ 0]\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = [1\ 4\ 7] - [2\ 5\ 8]$

$= [-1\ -1\ -1]$

* $[AB]_{ij}$ = row i of A dotted w/ column j o B

x^Ty, $[1\ -1\ 1]\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 - 1 + 3 = 4$

* AB is the sum of the outer products of outer products (rank-1 matrices) of columns of A and rows of B.

$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}[1\ -1\ 2] = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \\ 3 & -3 & 6 \end{bmatrix}$

all columns mult of $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

all rows mult of $[1\ -1\ 2]$

* block matrix mult. Treat letters in matrices as numbers (except for commuting) as long as the dimensions work

$$
m\left\{\begin{bmatrix} \underset{r\times r}{I} & \underset{r\times(n-r)}{F} \\ \underset{(m-r)\times r}{0} & \underset{(m-r)\times(n-r)}{0} \end{bmatrix}\right. \begin{bmatrix} \underset{r\times(n-r)}{-F} \\ \underset{(n-r)\times(n-r)}{I} \end{bmatrix} = \begin{bmatrix} -IF + FI \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} \underset{r\times(n-r)}{0} \\ \underset{(m-r)\times(n-r)}{0} \end{bmatrix}
$$

$m = n = 3$
$r = 2$

$$
\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} -3 \\ -4 \\ \hline 1 \end{array}\right] = \begin{bmatrix} -\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

not zero — zero — zero

Now let us return to elimination & $Ax = b$

a strategic sequence of row operations (L.C.) design to get things ready for back substitution

These row operations can be written as matrix-matrix mult.

The simplest row operation is exchanging two rows and it is accomplished by a permutation matrix

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}
$$

same rows as I but different order

Example Take 1 minute

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & -2 & 1 \\ 2 & -7 & 0 \\ 4 & -17 & 0 \end{bmatrix} = ?$$

E A What did E do?

Ans. Elimination under the first pivot

$$\begin{bmatrix} \boxed{1} & -2 & 1 \\ 0 & \boxed{+3} & -2 \\ 0 & -9 & -4 \end{bmatrix}$$

what matrix does elimination under the 2nd pivot?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$ do nothing to rows 1 & 2

subtract 3 x row 2 from row 3

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$[A|b]$ augmented matrix also block matrix

Another example (w/ b)

$$\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & 4 \\ 3 & -5 & 4 \\ -1 & 3 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -2 & 4 & 3 \\ 3 & -5 & 4 & 2 \\ -1 & 3 & -3 & -1 \end{array}\right] \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 4 & 3 \\ 0 & \boxed{1} & -8 & -7 \\ 0 & 1 & 1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 3 \\ 0 & 1 & -8 & -7 \\ 0 & 0 & 9 & 9 \end{array}\right]$$

U d ready for back sub.

$[A|b] \longrightarrow [U|d]$ now $E_2 E_1 [A|b] = [U|d]$

row operations

Q: How can we undo elimination? Supposing that
A: Add back what was subtracted. we wanted to.

1. To return row 3 to its original state

- add 1x row 2 to row 3
- add (-1) x row 1 to row 3

2. To return row 2 its original state

- add 3 x row 1 to row 2

3. Do nothing to row 1

↑ these row operations can be written as matrix mult.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -5 & 4 \\ -1 & 3 & -3 \end{bmatrix} \checkmark$$

$$L \quad U \quad = \quad A$$

$A = LU$ is our first example of a matrix <u>factorization</u>

Does it ~~all~~ alway work? No, sometimes <u>row switchs</u> are required

↑ sometimes called pivoting

$PA = LU$

↑ row switches

Returning to our example

$$E_2 E_1 = \text{the elimination} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

matrix that does both elimination steps

note the entries here are not obvious from the multipliers

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$ the matrix that undoes it

just the multipliers

Check

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I \leftarrow$ the do-nothing matrix

What a matrix does cannot always be undone

$$\underset{1\times 3}{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}} \underset{3\times K}{B} = \; ? \; = \text{row 1 of } B - \text{row 3 of } B$$

$$= \underset{1 \times K}{\begin{bmatrix} & & \end{bmatrix}}$$

Given this, there is no way to recover B.

* no info about row 2
* impossible to separate rows 1 & 3

Eggs cannot be unscrambled

Paint colors cannot be unmixed

? even if you know exact what was done

"a loss" of information?

if A has fewer rows than B? if A is not square?

For numbers, undoing multiplication is called <u>division</u>, and all numbers except a zero have an inverse $\frac{1}{a}$ with $a \frac{1}{a} = 1$

← identity element under mult.

To solve $ax = b$, (as in the first step of back sub.), mult both sides by $\frac{1}{a}$

$$\frac{1}{a} a x = \frac{1}{a} b$$

$$x = \frac{b}{a}$$

<u>If</u> A has an inverse A^{-1} then the sol^{n} of $Ax = b$ is $x = A^{-1} b$ also b/c mult both sides by A^{-1}

[There is often a theory / practice split on whether knowing A^{-1} is a good idea]

only square matrices have inverses

1st rule of inverse = square

$n \times n$

<u>Definition</u>: A <u>square</u> matrix A is <u>invertible</u> (has an inverse) if there is a matrix A^{-1} such that

$$\underset{n\times n}{A}\,\underset{n\times n}{A^{-1}} = \underset{n\times n}{A^{-1}}\,\underset{n\times n}{A} = \underset{n\times n}{I}$$

A^{-1} is square

<u>vocab.</u>

not invertible = singular

invertible = non singular

Since the sol$^{\underline{n}}$ of $Ax = b$ is $x = A^{-1}b$ when A is invertible, there are many connections between A^{-1} facts & $Ax = b$ facts.

$\underline{A^{-1}}$ <u>facts</u>

0. A must be square (first rule of inverse)

1. A^{-1} exists iff A has n pivots. Otherwise A is singular. Elimination can fail only if pivots are missing [not an explicit proof really, better proof next Lecture?]

2. If A^{-1} exists, it is unique.

* Proof by parenthesis & contradictions

Suppose that ~~A~~ B & C are inverses of A & $B \neq C$.

$$BAC = BAC$$

$$(BA)C = B(AC)$$

we defined inverses as working on both sides

$$IC = BI$$

$$C = B \quad ⋇ \text{ contradiction}$$

3. ~~B~~ If A^{-1} exists $x = A^{-1}b$ <u>is</u> the <u>unique</u> sol$^{\underline{n}}$ of $Ax = b$.

* First it is a sol$^{\underline{n}}$ b/c

$$A(\overset{x}{A^{-1}b}) = AA^{-1}b = b \checkmark$$

* 2nd if there is <u>another</u> sol$^{\underline{n}}$ $Ay = b$ & $y \neq x$

$$A^{-1}(Ay) = A^{-1}b \Rightarrow y = x \quad ⋇$$

4. If there is a vector $x \neq 0$ such that $Ax = 0$, then A is singular.

proof

Suppose $Ax = 0$ & A^{-1} exists.

$$Ax = 0$$

$$A^{-1}A x = A^{-1}\vec{0} = \vec{0}$$

$$\Rightarrow \quad x = 0 \quad \text{※}$$

5. The inverse of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Check $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

6. The inverse of a square diagonal matrix w/ non-zero diagonal elements is

$$\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$$

no inverse if zero on diagonal of diagonal matrix

* scaling by zero cannot be undone

7. The product of invertible matrices A & B is invertible and the inverse is

$$(AB)^{-1} = B^{-1}A^{-1}$$

<u>check</u>

$$(AB)^{-1}AB = B^{-1}\underbrace{A^{-1}A}_{I}B = B^{-1}B = I \checkmark$$

$$AB(AB)^{-1} = A\underbrace{BB^{-1}}_{I}A^{-1} = AA^{-1} = I \checkmark$$

why is the order reversed?

$$B^{-1}A^{-1}ABx$$

sock of, shoes off, shoes on, socks on, feet

<u>Moral</u>: Take off shoes ~~socks~~ before sock. [or pause to think about the order]

<u>Note</u> our ability to solve $Ax=b$ (elimination + back sub.) can be used to compute A^{-1}.

Fill in the blank 1. $A\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ is the ____ of A

2. ~~The~~ If A^{-1} exists, the sol^{n} of $Ax=b$ is ______

3. The sol^{n} of $Ax=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ is ______ of A^{-1}