Homework 4

Qs 1)
Ans) P, Q - permutation matrix

$$P - Q = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \end{bmatrix}_{n \times n}$$

have some rows 0,
sow rows with +1, 0, -1.

If there is x (non zero) s.t. $(P-Q)x = 0$ then $(P-Q)$ singular.

Take $x = 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\therefore Px = 1$ and $Qx = 1$

$\therefore (P-Q)x = 0$

$\Downarrow$

$(P-Q)$ singular.

Qs 2)

Ans)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}$$

i) $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix}$$

ii) $$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$$E_{32}E_{21}A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{pmatrix} = U$$

$L = E_{21}^{-1} E_{32}^{-1}$

$E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$ $\quad E_{21}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$\therefore \quad A = LU$

Qs3)
Ans)

1) $Lc = b$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}_{c} = \underbrace{\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}}_{b}$$

$c_1 = 4$

$c_1 + c_2 = 5$

$c_1 + c_2 + c_3 = 6$

$\therefore \quad c = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

2) $Ux = c$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 4$

$x_2 + x_3 = 1$

$x_3 = 1$

$$\therefore x = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

3) $\therefore \quad Ax = b$

$A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Q14)

Ans) $A = LU$

1) $$\underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}}$$

$$= \begin{bmatrix} d & e \\ ld & le+f \end{bmatrix}$$

$\therefore$ $d = 0$, $e = 1$

$ld = 2$, $le + f = 3$

$ld = 0$ $\therefore$ impossible.

2) $$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} d & e & g \\ ld & le+f & lg+h \\ md & me+nf & mg+hn+i \end{bmatrix}$$

$d=1$, $e=1$, $g=0$

$ld=1$, $le+f=1$, $lg+h=2$ $\Rightarrow$ $l=1$, $f=0$, $h=2$

$m=1$, $\underline{me+nf=2}$

$\downarrow$

1

(impossible).

As $me+nf=1$ ✗

Q15)
Ans)

1) $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ $|A| = 3$

$A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix} = \frac{1}{|A|} \text{adj } A$

$(A^{-1})^T = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$

$(A^T)^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ $\quad |A| = -c^2$

$A^T = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$

$A^{-1} = \dfrac{1}{-c^2} \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}$

$(A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}$

Q3)

Ans) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$B^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$\therefore \quad (AB)^T = B^T A^T$

$A^T B^T = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} \neq (AB)^T$

$$AA^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$AA^T \neq A^TA$$

$$(AA^T)^T = AA^T$$
$$(A^TA)^T = A^TA$$

$\therefore$ both are symmetric.

Q) 7)

Ans)

a) $((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T$

$= (A^T)^{-1}(B^T)^{-1}$

b) if U is upper triangular,

$\downarrow$

U^{-1} is also upper Δ.

$\downarrow$

$(U^{-1})^T$ is lower triangle matrix.

Q58)
Ans)

1) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = 0$

$\therefore$ take $a=0$, $d=0$, $bc=0$

$\therefore$ take $b=0$.

$\therefore$ $A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ possible if A is not zero matrix.

$A^T = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}$

2) $A^TA = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = 0$

For it to be zero, $a^2+c^2=0$
$b^2+d^2=0$

As $a^2 \geq 0$, all 4 needs to 0.

$\therefore$ only possible for 0 matrix.

Q9)

Ans)

1) P permutation matrix $P^T P = I$

$$x \cdot y = (Px) \cdot (Py).$$

$$x \cdot y = \sum_i x_i y_i$$

P swaps j, k^{th} component $j \leq k$

$$(Px) \cdot (Py) = \sum_{i \neq j,k} x_i y_i + \underbrace{x_k y_k + x_j y_j}_{\text{just order } \cancel{\text{same}} \text{ change}} = \sum_i x_i y_i$$

$$\therefore \quad x \cdot y = (Px) \cdot (Py)$$

2)

$$(Px)^T (Py) = x^T P^T P y = xy$$

$$\therefore \quad P^T P = I$$

3) $x = (1, 2, 3) \quad y = (1, 4, 2)$

$$(Px)\cdot y \neq x\cdot(Py)$$

let ~~$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$~~ $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$(Px)\cdot y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 18$$

$$(x)\cdot(Py) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$= 11$$

So, $(Px)\cdot y$ is not always $x\cdot(Py)$.

Q10)

A) 1) $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

take $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore \; PA = \underset{U}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}}$

2) P_1 - permute row

P_2 - permute column

$$P_1 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \underset{A}{\begin{pmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix}} \underset{P_2}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} = \underset{\underline{L}}{\begin{pmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{pmatrix}}$$

3) P_2 changes <u>column</u> of A.

Qs 11)

Ans)

$$A = \begin{pmatrix} 4 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 4 \end{pmatrix}$$

→ A is a <u>strictly diagnal dominant matrix</u> as

$$|a_{ii}| \geq \sum_{i \neq j} |a_{i,j}| \quad \forall i$$

∴ it is invertible.

<u>Proof</u>) take $Ax = 0$, $x = (x_1, \ldots, x_n)$ (A singular)

let x_k largest element of x.

$$\sum_{j}^{n} a_{kj} x_j = 0$$

$$\Rightarrow \quad a_{kk} x_k = -\sum_{j \neq k} a_{kj} x_j$$

$$\Rightarrow \quad |a_{kk}| = \left| -\sum_{j \neq k} a_{kj} \right| \underbrace{\left| \frac{x_j}{x_k} \right|}_{\leq 1}$$

$$|a_{kk}| \leq \left| \sum_{j \neq k} a_{kj} \right|$$

Contradiction