Homework 10

Qs1)

Ans)

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$\boxed{\lambda = 5, -1}$

$\rightarrow \lambda = 5$

$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\rightarrow \lambda = -1$

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$B = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

$B - \lambda I = \begin{bmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{bmatrix}$

$\det(B - \lambda I) = 0$

$8 - 6\lambda + \lambda^2 - 8 = 0$

$\boxed{\lambda = 0, 6}$

$\rightarrow \lambda = 6$

$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow \lambda = 0$

$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

- A+I has same eigenvectors as A
- eigenvalues are increased by 1.

Qs 2)

iv) $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\rightarrow \lambda^2 - \lambda - 2 = 0$

$[\lambda = 2, -1]$

$\rightarrow \lambda = 2$

$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow \lambda = -1$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

$\det(A^{-1} - \lambda I) = 0$

$\rightarrow -\frac{1}{2}\lambda + \lambda^2 - \frac{1}{2} = 0$

$2\lambda^2 - \lambda - 1 = 0$

$(2\lambda - 1)(\lambda + 1) = 0$

$\lambda = 1/2, -1$

$\rightarrow \lambda = 1/2$

$\begin{bmatrix} -1 & 1 \\ +1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow \lambda = -1$

$\begin{bmatrix} 1/2 & 1 \\ 1/2 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\rightarrow$ SAME eigenvectors

$\rightarrow \left[A \rightarrow \lambda_1, \lambda_2 \quad , \quad A^{-1} \rightarrow \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right]$

Q3)
Ans)

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

$\det(A-\lambda I) = \lambda^2+\lambda-6$
$= (\lambda+3)(\lambda-2)$

$\lambda = -3, +2$

$\rightarrow \lambda = -3$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\rightarrow \lambda = 2$

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$$

$(7-\lambda)(6-\lambda) - 6 = 0$

$\lambda^2 - 13\lambda + 36 = 0$

$(\lambda-9)(\lambda-4) = 0$

$\lambda = 4, 9$

$\boxed{\lambda_1^2 + \lambda_2^2 = 13}$

$\rightarrow \lambda = 9,$

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\rightarrow \lambda = 4,$

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\rightarrow$ SAME eigenvectors.

$\rightarrow A \Rightarrow \lambda_1, \lambda_2 \quad | \quad A^2 \Rightarrow \lambda_1^2, \lambda_2^2$

Q9)
Ans)

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\det(Q - \lambda I) = 0$

$\lambda^2 - 2\lambda\cos\theta + \sin^2\theta + \cos^2\theta = 0$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$\boxed{\lambda = \cos\theta \pm i\sin\theta} = e^{i\theta}, e^{-i\theta}$

$\lambda_1 = \cos\theta + i\sin\theta$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\lambda_2 = \cos\theta - i\sin\theta$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Q5)

Ans)

eigenvalues - 0, 3, 5
eigenvectors - u, v, w

a) u basis for null space ($Au = 0u$) $N(A) = span\{u\}$

v, w are basis for column space. $C(A) = span\{v, w\}$

b) $A\left(\underbrace{v/3 + w/5}_{x_p}\right) = \frac{3v}{3} + \frac{5w}{5} = \underbrace{v + w}$

↓
particular soln

cu - nullspace

all solution = $\frac{v}{3} + \frac{w}{5} + cu$

c) then u would be in col space

Q5)
A1) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(3-\lambda) - 0 = 0$

$\lambda = 1, 3$

$\lambda = 1$

$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3$

$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\cancel{X_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$A = X \Lambda X^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{bmatrix}$$

$$\lambda_1 = 0, 4$$

$\lambda = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda = 4$

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}, \quad X^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ +1 & 1 \end{bmatrix}$$

$$A = X \Lambda X^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

b)

$$\boxed{A^3 = X\Lambda^3X^{-1}}$$

$$\boxed{A^{-1} = X\Lambda^{-1}X^{-1}}$$

→ A_1

$$\Lambda^3 = \begin{bmatrix} 1^3 & 0 \\ 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix}$$

$$\Lambda^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$$

→ A_2

$$\Lambda^3 = \begin{bmatrix} 0 & 0 \\ 0 & 64 \end{bmatrix}$$

$$\Lambda^{-1} = \begin{bmatrix} \text{undefined} & 0 \\ 0 & 1/4 \end{bmatrix}$$

Qs 7)

Ans) $A = X \Lambda X^{-1}$

$$\left[A+2I = X(\Lambda+2I)X^{-1} \right]$$

$\rightarrow \Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$\Lambda + 2I = diag(\lambda_1+2, \lambda_2+2, \ldots, \lambda_n+2)$

eigenvectors of $A+2I$ are the same as A.

$$A+2I = X(\Lambda+2I)X^{-1}$$
$$= X\Lambda X^{-1} + X2IX^{-1}$$
$$= A+2I$$

Qs 8)

Ans) A eigenvalues : 2,2,5

a) True (no zero eigenvalues)

b) False (repeated $\lambda = 2$ may have only one line of eigenvectors)

c) False (repeated λ - full set of eigenvectors)

Q7)
A3)

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (3-\lambda)^2 = 0$

$\lambda = 3$, algebraic multiplicity $\rightarrow 2$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

rank$(A - 3I) = 1$ (one pivot column)

Nullity $= n - \text{Rank} = 2 - 1 = 1$

A diagonalizable $\rightarrow$

$$A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ diagonalizable.

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Qs10)

A20)

A_2 is $X\Lambda X^{-1}$

$$\Lambda = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.3 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.8 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}$$

$\lambda_1 = 0.3$

$\lambda_2 = 0.9$

$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$A^2 = X\Lambda X^{-1}$

$(A^2)^{10} = X\Lambda^{10}X^{-1}$

$$\Lambda = \begin{bmatrix} 0.9^{10} & 0 \\ 0 & 0.3^{10} \end{bmatrix}$$

$u_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow (A^2)^{10}u_0 = 0.9^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$u_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \rightarrow (A^2)^{10}u_0 = 0.3^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\rightarrow$ $u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ $\boxed{\begin{bmatrix} 1.05 \\ 0.35 \end{bmatrix} = (A_2)^{10}u_0}$

$u_0 = v_1 + v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$(A^2)^{10} u_0 = 0.7^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0.3^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.72 \times 10^{-5} \\ -5.9 \times 10^{-6} \end{bmatrix}$$

$$\rightarrow \quad u_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$(A^2)^{10}\begin{bmatrix} 6 \\ 0 \end{bmatrix} = 0.7^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0.3^{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.05 \\ 0.35 \end{bmatrix}$$

Qs 11)

Ans)

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad Y = \begin{bmatrix} q & r \\ s & t \end{bmatrix}$$

$$XY = \begin{bmatrix} aq+bs & ar+bt \\ cq+ds & cr+dt \end{bmatrix}$$

$$\text{trace}(XY) = aq+bs+cr+dt.$$

$$YX = \begin{bmatrix} aq+rc & ar+rd \\ sq+tc & sr+td \end{bmatrix}$$

$$\text{trace}(YX) = aq+rc+sr+td.$$

$$\boxed{\text{trace}(XY) = \text{trace}(YX)}$$

$\rightarrow$ $\text{trace}(A) = \underline{\text{trace}(X\Lambda X^{-1})}$

$$\text{trace}(X\Lambda X^{-1}) = \text{trace}(\Lambda X^{-1}X) = \text{trace}(\Lambda)$$

$$= \text{trace}(A)$$

$AB - BA = I$ impossible

$\text{trace}(AB - BA) = \text{trace}(AB) - \text{trace}(BA)$

$= 0$

$\text{trace}(I) = 2$

$\boxed{\text{not equall}}$

$0 \neq 2$

Q 12)
iv)

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ 0 & d-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) \Rightarrow (a-\lambda)(d-\lambda)$$

$$(\lambda - a)(\lambda - d) = 0$$

$$\lambda = a, d$$

∴ Cayley Hamilton Th^m

$$(A - aI)(A - dI) = 0$$

$$A - aI = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}$$

$$A - dI = \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$$

$$(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Homework 10

Qs 1)

Ans)

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$\boxed{\lambda = 5, -1}$

→ $\lambda = 5$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

→ $\lambda = -1$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$B = A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{bmatrix}$$

$\det(B - \lambda I) = 0$

$8 - 6\lambda + \lambda^2 - 8 = 0$

$\boxed{\lambda = 0, 6}$

→ $\lambda = 6$

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

→ $\lambda = 0$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

- $A+I$ has <u>same</u> eigenvectors as A
- eigenvalues are <u>increased</u> by 1.

Qs 2)

iv) $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

$\det(A - \lambda I) = 0$

→ $\lambda^2 - \lambda - 2 = 0$

$[\lambda = 2, -1]$

→ $\lambda = 2$

$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

→ $\lambda = -1$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$

$\det(A^{-1} - \lambda I) = 0$

→ $-\frac{1}{2}\lambda + \lambda^2 - \frac{1}{2} = 0$

$2\lambda^2 - \lambda - 1 = 0$

$(2\lambda - 1)(\lambda + 1) = 0$

$\lambda = 1/2, -1$

→ $\lambda = 1/2$

$\begin{bmatrix} -1 & 1 \\ +1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

→ $\lambda = -1$

$\begin{bmatrix} 1/2 & 1 \\ 1/2 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

→ SAME eigenvectors

→ $\left[A \rightarrow \lambda_1, \lambda_2 \quad , \quad A^{-1} \rightarrow \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right]$

Q3)
Ans)

$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 + \lambda - 6$
$= (\lambda+3)(\lambda-2)$

$\lambda = -3, +2$

$\rightarrow \lambda = -3$

$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\rightarrow \lambda = 2$

$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$

$(7-\lambda)(6-\lambda) - 6 = 0$

$\lambda^2 - 13\lambda + 36 = 0$

$(\lambda-9)(\lambda-4) = 0$

$\lambda = 4, 9$ $\quad \boxed{\lambda_1^2 + \lambda_2^2 = 13}$

$\rightarrow \lambda = 9,$

$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\rightarrow \lambda = 4,$

$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow$ SAME eigenvectors.

$\rightarrow A \Rightarrow \lambda_1, \lambda_2 \quad | \quad A^2 \Rightarrow \lambda_1^2, \lambda_2^2$

Q9)
Ans)

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\det(Q - \lambda I) = 0$

$\lambda^2 - 2\lambda\cos\theta + \sin^2\theta + \cos^2\theta = 0$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\boxed{\lambda = \cos\theta \pm i\sin\theta} = e^{i\theta}, e^{-i\theta}$$

$\lambda_1 = \cos\theta + i\sin\theta$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$\lambda_2 = \cos\theta - i\sin\theta$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x = \begin{bmatrix} 1 \\ i \end{bmatrix}$$