5 Nov 24

# Homework 8

(Qs 1)

(a)

$$v_1 = (2, 2, -1)$$
$$v_2 = (-1, 2, 2)$$

$$\|v_1\| = 3$$
$$\|v_2\| = 3$$

$$q_1 = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) \quad \Big| \quad q_2 = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$$

(orthogonal matrix)

$$Q^TQ = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$QQ^T = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$$

Q2)

Ans) plane $x+y+2z=0$

1) $z=0$, $x=1, y=-1$

$v_1 = (1,-1,0)$

2) $x=1, y=1, z=-1$

$v_2 = (1,1,-1)$

} Orthogonal vectors

$v_1 \cdot v_2 = 1-1+0 = 0$

$\|v_1\| = \sqrt{2}$

$\|v_2\| = \sqrt{3}$

$q_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$q_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

} Orthonormal vectors.

Q3)

Ans)

a) $a = (1, 3, 4, 5, 7)$

$b = (-6, 6, 8, 0, 8)$

$\|a\| = \sqrt{1^2+3^2+4^2+5^2+7^2}$

$= \sqrt{100} = 10$

$\|b\| = \sqrt{6^2+6^2+8^2+8^2} = 10\sqrt{2}$

$q_1 = \frac{1}{10}(1, 3, 4, 5, 7)$

$proj_{q_1} b = (b \cdot q_1)\, q_1$

$b \cdot q_1 = \frac{1}{10}(-6 \times 1 + 6 \times 3 + 8 \times 4 + 0 \times 5 + 8 \times 7)$

$= 10$

$proj_{q_1} b = (1, 3, 4, 5, 7)$

$v_1 = b - \text{proj}_{q_1} b = (-7, 3, 9, -5, 1)$

$\|v_1\| = \sqrt{7^2 + 3^2 + 9^2 + 5^2 + 1^2}$

$= 10$

$\therefore q_2 = \frac{1}{10}(-7, 3, 9, -5, 1)$

$q_1, q_2$ are orthonormal vectors.

b) $v = (1,0,0,0,0)$

$\text{proj}_{\text{plane}}\, v = (v \cdot q_1)\, q_1 + (v \cdot q_2)\, q_2$

$v \cdot q_1 = 1 \times 0.1 + 0 + 0 + 0 + 0$

$= 0.1$

$v \cdot q_2 = -0.7$

$\text{proj}_{\text{plane}}\, v = (0.1)\, q_1 + -(0.7)\, q_2$

$= \frac{1}{100}(1,3,4,5,7) - \frac{7}{100}(-7,3,4,5,1)$

$= \frac{1}{100}(50, -18, -24, 40, 0)$

$= (0.5, -0.18, -0.24, 0.4, 0)$

Q4)

Ans)

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

a)

$$q_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \qquad q_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\|q_1\| = 3$$

$$q_1 = \frac{q_1}{\|q_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$q_2 \cdot q_1 = -3$$

$$\text{proj}_{q_1} q_2 = |q_2 \cdot q_1| \, q_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$v_1 = q_2 - \text{proj}_{q_1} q_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|v_1\| = 3$$

$$q_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$q_3 = q_1 \times q_2$$

$$= \begin{vmatrix} i & j & k \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{vmatrix}$$

$$q_3 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

b) Nullspace $A^T$ containing $q_3$.

c) $Ax = (1, 2, 7)$ by least square.

$$\vec{x} = (A^T A)^{-1} A^T b$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -7 \\ -7 & 18 \end{bmatrix} \begin{bmatrix} -9 \\ 27 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Q5)

Ans)

$a = (4, 5, 2, 2)$

$b = (1, 2, 0, 0)$

$$\text{proj}_a b = \frac{a \cdot b}{a \cdot a}\, a$$

$a \cdot b = 4 + 10 + 0 + 0 = 14$

$a \cdot a = 4^2 + 5^2 + 2^2 + 2^2 = 16 + 25 + 4 + 4 = 49$

$$p = \text{proj}_a b = \frac{14}{49} a = \frac{2}{7} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{2}{7} a$$

$a = (4, 5, 2, 2)$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{7}\begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$

$$B = b - p = b - \frac{2}{7}a$$

$$= \frac{1}{2}(-1, 1, -1, -1)$$

$\|B\| = 1$

$\therefore B = q_2$

$q_1, q_2$ orthonormal

Q06)

Ans)

$a = (1,-1,0,0)$

$b = (0,1,-1,0)$

$c = (0,0,1,-1)$

$d = (1,1,1,1)$

1) $A = a = (1,-1,0,0)$

2) $\text{proj}_A b = \left(\frac{b \cdot A}{A \cdot A}\right) A$

$= \left(\frac{-1}{2}\right) (1,-1,0,0)$

$B = b - \text{proj}_A b = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$

3) $\text{proj}_A c = \left(\frac{c \cdot A}{A \cdot A}\right) A = 0$

$c \cdot A = 0$

$$\text{proj}_B C = \left(\frac{C \cdot B}{B \cdot B}\right) B$$

$$= \frac{-1}{3/2} B$$

$$= \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \\ 0 \end{bmatrix}$$

$$C = C - \text{proj}_A C - \text{proj}_B C$$

$$= \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

Orthogonal A, B, C

$d = (1,1,1,1)$

all of A, B, C are $\perp$ to d.

$A \cdot d = 0$

$B \cdot d = 0$

$C \cdot d = 0$

&

$a \cdot d = 0$

$b \cdot d = 0$

$c \cdot d = 0$