APMA 4007

Oct 1, 2024

Lecture 9

Outline

1 Review
   - $C(A)$, $N(A)$

2. $A \to U \to R$
   - pivot columns!

3. Rank.

Last lecture a new idea / vocabular word.

Vector space / vector subspace

- usual rules for adding & scaling vectors
- L.C.s stay in the (sub) space.

We are interested in subspaces related to

$$\overset{m\times n}{A}\,\overset{n\times 1}{x} = \overset{m\times 1}{b} \qquad x \in \mathbb{R}^n \qquad b \in \mathbb{R}^m$$

All possible L.C.s

$C(A) = \{$ the vector space spanned by the columns of $A \}$

$= \{ b \mid Ax=b \text{ has sol}^{\underline{n}} \}$

adding "redundent" columns does not change $C(A)$ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ same $C(A)$

$Ax=b$ has sol$^{\underline{n}}$ iff $b \in C(A)$

$N(A) = \{ x \mid Ax=0 \} = \{$ the space spanned by the special solutions of $Ax=0 \}$

A sol$^{\underline{n}}$ of $Ax=b$ is unique iff $N(A) = \{\vec{0}\}$

If $\overset{n\times n}{A}$ & A^{-1} exists, then $C(A) = \mathbb{R}^n$ & $N(A) = \{\vec{0}\}$

- the question of existence & uniqueness is one for square matrices

In general some work is required to find $C(A)$ & $N(A)$

We saw that

[One approach for $C(A)$ is solve $Ax=b$ w/ letters for b.]

elimination plus ...

N(A) is found by solving $Ax=0$. Today we will see that it also gives C(A)

Example just [A] not [A|0]

$$A=\begin{bmatrix} \boxed{1} & -2 & 4 & 4 & -3 \\ 2 & -4 & 10 & 7 & -6 \\ 3 & -6 & 13 & 12 & -8 \end{bmatrix}$$

Plan

1. Expose pivots & zero below
2. Zero above pivots
3. Divide by pivots

ok

back substitution in the box

$$\to \begin{bmatrix} \boxed{1} & -2 & 4 & 4 & -3 \\ 0 & 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -2 & 4 & 4 & -3 \\ 0 & 0 & \boxed{2} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1/2} & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & -2 & 4 & 0 & -11 \\ 0 & 0 & \boxed{2} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & -15 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = R$$

row-reduced echelon form

or really revealing form

$$\underbrace{[A] \to [U] \to [R]}_{\text{row operations}}$$

if $Rx=0$ then $Ax=0$

b/c math

"plus" Given R find N(A)

1. Label columns of A as pivot or free

has pivot — has no pivot

$$\begin{array}{ccccc} p & f & p & p & f \\ \end{array}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & -15 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

column labels match entries in x

$$\begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \end{array} \begin{array}{c} p \\ f \\ p \\ p \\ f \end{array}$$

2. Take the rows / equations out of the box and add "= 0" that was implicit. Pt.

Each equation contains exactly 1 pivot variable b/c elimination above & below. Put it on the left side & free variables on the right

pivot | free

$$x_1 = 2x_2 + 15x_5$$

$$x_3 = -x_5 \qquad \leftarrow \text{equations for } Ax=0$$

$$x_4 = -2x_5$$

In x, replace pivot variables (lhs) by free (rhs)

Solution for $Ax=0$

$$x_N = \begin{bmatrix} 2x_2+15x_5 \\ x_2 \\ -x_5 \\ -2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 15 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 15 \\ 1 & 0 \\ 0 & -1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$$

special sol^n's

Every sol^n of $Ax=0$ is a L.C. of special sol^{n}s

The special sol^{n}s span N(A).

Recall

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{matrix} p \\ f \\ p \\ p \\ f \end{matrix}$$

Note that each special sol^n contains a 1 in location of one free variable and zeros in the locations of the other free variables (b/c the free variable is the scalar).

And other stuff in the pivot variable locations

With that in mind let's check that the special solutions of $Ax=0$ indeed work.

$$A\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} = \underset{\text{pivot}}{2\times \text{col } 1\ A} + \underset{\text{free}}{1 \text{col } 2\ A} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-2\\-4\\-6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \checkmark$$

$$\checkmark$$

$$A\begin{bmatrix}15\\0\\-1\\-2\\1\end{bmatrix} = \underset{\text{pivot}}{15 \text{ col } 1} \underset{\text{pivot}}{- 1 \text{ col } 3} \underset{\text{pivot}}{- 2 \text{ col } 4} \underset{\text{free}}{+ 1 \text{col } 5} = \begin{bmatrix}15\\30\\45\end{bmatrix} - \begin{bmatrix}4\\10\\13\end{bmatrix} - \begin{bmatrix}8\\14\\24\end{bmatrix} + \begin{bmatrix}-3\\-6\\-8\end{bmatrix}$$

The previous statement ("each special solⁿ has a 1 only one of its free variable slots") means that the L.C = zero ~~has~~ contains exactly one free column and the rest pivot columns. Moving the free column to the other side of the eq gives that each free column of A is a L.C. of pivot columns of A.

$C(A) = \{$ all possible L.C. of the columns of $A\}$

$= \{$ all possible L.C. of the pivot columns of $A\}$

b/c we have demonstrated that the free columns are redundent in regard to $C(A)$)

Here $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\10\\13\end{bmatrix}, \begin{bmatrix}4\\7\\12\end{bmatrix}$ span $C(A)$

1. We did not know where pivot columns were when we started
2. NOT COLUMNS of R. $C(R) \neq C(A)$ b/c row operations scramble

The text (p. 94?) take an equivalent approach with R (called R_0 now) to find the special sol^{n}s.

$$R = \begin{array}{c} \begin{array}{ccccc} p & f & p & f & f \end{array} \\ \begin{bmatrix} 1 & -2 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

1. # of special sol^{n} = # of free variables (here 2)

2. Put a 1 in one free variable slot & the rest zeros

$$\begin{array}{c} \begin{bmatrix} 2 & 1.5 \\ 1 & 0 \\ 0 & -1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} p \\ f \\ p \\ p \\ f \end{array}$$

3. Write the pivot variables for each row/equation (in red above)

It works but seems a little quick.

Summary

1. $[A] \to [U] \to [R]$ row reduced echelon form [called R_0 in 6th edition]

2. $C(A)$ spanned by pivot columns of A

$C(A) \neq C(R)$

3. $N(A)$ is spanned by special solutions

$N(A) = N(U) = N(R)$

Knowing about the pivots is important

- pivot columns span $C(A)$
- $\overset{n\times n}{A}$ is invertible iff A has n pivots.

Def. The rank r of a matrix is the # of pivots it has.

New vocab. word in a sentence

1. $C(A)$ is spanned by the r pivot columns of A.

2. $N(A)$ is spanned by it $n-r$ ~~sep~~ special solutions

\# unknowns $= n$

\# free $=$ total $-$ pivot

\# special sol$^{\underline{n}}$s $=$ # free $= n-r$

The size of A can give clues about its rank.

Suppose $\overset{m\times n}{A}$ is wide $m<n$. more columns than rows

1. A has at most m pivots, one for each row.

   So $r \le m < n$.

2. There are $n-r$ special sol$^{\underline{n}}$s (= # free variables)

   Since $r<n$, $n-r>0 \Rightarrow$ at least one special sol$^{\underline{n}}$. $\Rightarrow$ A sol$^{\underline{n}}$ of $Ax=b$ is not unique.

On the other hand, the size of A is, in general, less informative that its rank.

$$\underset{3\times 3}{A} = \begin{bmatrix} \boxed{1} & 3 & 7 \\ 2 & 6 & 14 \\ 3 & 9 & 21 \end{bmatrix} \rightarrow \begin{array}{ccc} p & f & f \\ \end{array} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A has one pivot. Its rank is 1.

$C(A)$ is spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ NOT $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

There are 2 free variables & 2 special sol$^{\underline{n}}$

$$\overset{p}{x_1} = -3\overset{f}{x_2} - 7\overset{f}{x_3} \qquad x = \begin{bmatrix} -3x_2 - 7x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$$

$A\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$ says (free) column 2 = 3 x col 1

$A\begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$ says free column 3 = 7 x col. 1

this means

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \end{bmatrix}$$

the part of R w/o zeros rows.

pivot column

This confirms our experience that all rank-1 matrices are out products.

Bonus

$C(A^T)$ = the row space of A

= spanned by the rows of A

$C(A^T) = C(R^T)$ b/c R is obtained from A by invertible row operations &

any L.C. of rows of A is L.C. of rows of R

$$(BE)A = BR \qquad BA = (BE^{-1})R$$

The pivot rows of R span $C(R^T)$ b/c the other rows are zeros

and span $C(A^T)$.

A cartoon for R. Suppose

- the pivot variables $x_1, \dots, x_r$ come first followed by the free variables $x_{r+1}, \dots, x_n$
- the zero rows are at the botton of R.

[If this is not the case, the rows & columns can be permuted w/ permutation matrices] $A = [A_{\text{pivot columns}} \; A_{\text{free colums}}]$

$$\begin{array}{c} \text{pivot} \\ \text{rows} \\ \text{zero} \end{array} m \left[\begin{array}{cc} \underset{r\times r}{I} & \underset{r\times(n-r)}{F} \\ \underset{(m-r)\times r}{0} & \underset{(m\times r)\times(n-r)}{0} \end{array}\right]$$

$\longleftarrow n \longrightarrow$

| pivot | free |

columns

zero above & below pivots & divide by pivots = I.

1 $n-r$ = # special sol^n = # free var.

Q: is sol^n unique?

2. $m-r$ = # of zero rows

Q: does sol^n exist?

$0=0$ ok

$0=1$ not ok

only the same question if

$m = n$ = square A.

Recall block - matrix mult.

$$\underbrace{\begin{array}{cc} r\times r & r\times(n-r) \\ \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \end{array}}_{R} \begin{array}{c} r\times(n-r) \\ \begin{bmatrix} -F \\ I \end{bmatrix} \\ (n-r)\times(n-r) \end{array} = -I_r F + F I_{n-r}$$

$$= -F + F = 0$$

this means that the columns of $\begin{bmatrix} -F \\ I \end{bmatrix}$ are

from moving free to other side

a single 1 in each free variable slot.

the special solutions

Since $A\begin{bmatrix} -F \\ I \end{bmatrix} = 0$

and $A = [A_{pivot} \;\; A_{free}]$

$$[A_{pivot} \;\; A_{free}]\begin{bmatrix} -F \\ I \end{bmatrix} = 0 = -A_{pivot}F + A_{free}$$

or

$$A_{free} = \underbrace{A_{pivot}F}_{\text{L.C. of pivot columns}}$$

and

$$A = \begin{bmatrix} A_{pivot} & A_{free} \end{bmatrix} = \begin{bmatrix} A_{pivot} & A_{pivot}F \end{bmatrix}$$

$$= A_{pivot}\underbrace{\begin{bmatrix} I & F \end{bmatrix}}_{\text{the top of } R} = \overset{r}{\text{Columns}} \times \overset{r}{\text{Rows}}$$