APMA 4007

Sep 26, 2024

Lecture 8

Outline

1. Vector space
2. Subspace
3. $C(A)$
4. $N(A)$

Let's return to theory.

We have considered vectors in $\mathbb{R}^n$ which is the set of all vectors w/ n elements.

$\mathbb{R}^n$ is a <u>vector space</u> which means that add & scaling (L.C.) vectors is defined and have good properties.

<u>Namely</u>

1. $x+y=y+x$
2. $(x+y)+z = x+(y+z)$
3. $\vec{x}+\vec{0}=\vec{x}$
4. $1\cdot\vec{x}=\vec{x}$
5. $x+(-x)=0$
6. $(c\cdot d)\cdot x = c\cdot(d\cdot x)$
7. $c(x+y) = cx+cy$
8. $(c+d)x = cx+dy$

a set V is a vector space w/ addition & scaling if the above rules are satisfied and if for all x & y in V & scalars c & d

$$cx+dy \in V$$

L.C.s of vectors in V are also in V

Although there are other "kinds" of vectors and definitions of addition & scaling, we will only focus on the usual/familar ones.

In particular, we are interested in vector spaces that ~~are subsets~~ contain vectors from $\mathbb{R}^n$ & $\mathbb{R}^m$ and that related to $Ax=b$. That is to say (vector) subspaces of $\mathbb{R}^n$ & $\mathbb{R}^m$.

S is vector subspace of the vector space V if it satisfies the following rules

1. $S \subseteq V$
2. If $x \in S$, so is cx
3. If x & $y \in S$, so is $x+y$

$\Big\}$ L.C.s stay in S
x & $y \in S \Rightarrow cx + dy \in S$

Subspaces appear to have fewer rules b/c they inherit all the properties of V about addition & scaling.

Two easy consequences are that

1. The origin ($\vec{0}$) is a subspace $\{\vec{0}\}$

   e.g. $\begin{bmatrix}0\\0\\0\end{bmatrix} \subseteq \mathbb{R}^3$ & $c\begin{bmatrix}0\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

2. The origin ($\vec{0}$) is in every subspace

b/c $\underset{\substack{\uparrow \\ \text{scaling by zero}}}{0} \cdot \vec{x} = \vec{0}$.

Is S' a subspace?

Check:

1 Is $S \subseteq V$? (e.g., $\subseteq \mathbb{R}^n$ or $\mathbb{R}^m$)

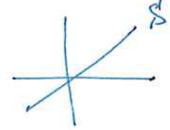
2. Do L.C.s stay in the space?

* Does it contain the origin? If not, NO. If yes, check 2.

Examples

YES

1 A line through the origin. $S \subseteq \mathbb{R}^2$



2. The plane

$3x_1 + 2x_2 - x_3 = 0$

* contains the origin

* L.C.s? Eq. of plane is

$$\underbrace{\begin{bmatrix} 3 & 2 & -1 \end{bmatrix}}_{\text{vector } \perp \text{ to plane}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

if $\vec{x}$ & $\vec{y}$ are in the plane,

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \cdot x = 0 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \cdot y \quad \text{and}$$

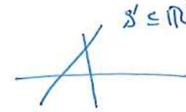
$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \cdot (c\vec{x} + d\vec{y}) = 0 \quad \checkmark$$

3. $\mathbb{R}^2 \subseteq \mathbb{R}^2$ the biggest subspace

4. $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \subseteq \mathbb{R}^2$ " smallest "

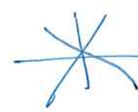
NO

1 A line not through the origin $S' \subseteq \mathbb{R}^2$



2. The plane $x_1 + x_2 + x_3 = 1$. Does not contain $\vec{0}$.

3. 2 lines in $\mathbb{R}^2$ passing through the origin



$$S = \left\{ \begin{matrix} x_2 = x_1 \\ \text{or} \\ x_2 = -x_1 \end{matrix} \right.$$

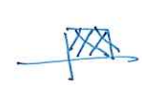
contains $\vec{0}$

but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin S$

4. Half plane

$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_2 \geq 0 \right\}$ why not?

5 box



$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; \begin{matrix} 0 \leq x_1 \leq 1 \\ 0 \leq x_2 \leq 1 \end{matrix} \right\}$$

~~Ex~~ It turns out that examples of subspaces are all there is.

The subspaces of $\mathbb{R}^3$

1. $\vec{0}$
2. A line through origin
3. A plane through origin
4. $\mathbb{R}^3$

Let's start returning to $Ax=b$.

First, ~~write~~ describe a plane through origin (in $\mathbb{R}^3$) using a matrix.

S: $\quad x_1 - 2x_2 + 4x_3 = 0$ ← plane through $\vec{0}$

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$ all points $\perp$ to $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

Since $x_1 = 2x_2 - 4x_3$, $\vec{x} \in S$ if

$$x = \begin{bmatrix} 2x_2 - 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

for some x_2 & x_3. In other words

$S = \{$ all possible L.C. of the columns of A where $A = \begin{bmatrix} 2 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \}$

Recall: $Ax=b$ has sol^n iff b is L.C. of columns of A. So we can answer if

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ has a sol}^n \text{ by}$$

checking if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the plane S, $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 5 \neq 0$

not in S and no sol^n.

Definition: The <u>column space</u> of A denoted $C(A)$ is

$$C(A) = \{ \text{all vectors } v \mid v = \text{L.C. of columns of } A \}$$

$$= \{ \underline{\text{all possible L.C.}} \text{ of columns of } A \}$$

$C(A)$ is the vector space <u>spanned</u> by the columns of A. (The <u>spans</u> of a bridge are the sections that provide support.)

Now we can say $Ax=b$ has sol^n $\Leftrightarrow$ $b \in C(A)$

IS $C(A)$ really a subspace?

1. $C(A) \subseteq \mathbb{R}^m$ (b/c $b \in \mathbb{R}^m$ $m \times 1$)
2. If u & $v \in C(A)$ then $\exists$ x & y such that

$Ax = u$ & $Ay = v$

$cu + dv = cAx + dAy = \cancel{cA}$

$= A(cx + dy)$ ✓

L.C. of columns of A.

Our $Ax=b$ Knowledge can be applied to $C(A)$.

1. $\overset{n\times n}{A}$ & A^{-1} exists. $Ax=b$ has sol$^{\underline{n}}$ $x=A^{-1}b$ for all $b\in\mathbb{R}^n$. Therefore $C(A)=\mathbb{R}^n$ ✓

2. $\overset{m\times n}{A}$ & ~~Ax=b~~ A has m pivots. $Ax=b$ has sol$^{\underline{n}}$ for any b (might not be unique?). $C(A)=\mathbb{R}^m$

3. $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $C(A)=\mathbb{R}^2$ b/c A^{-1} exists.

4. $A=\begin{bmatrix}1&-5\\0&1\end{bmatrix}$, $C(A)=\mathbb{R}^2$, A^{-1} exists, 2 pivots

5. $A=\begin{bmatrix}1&-5&2\\0&1&3\end{bmatrix}$, $C(A)=\mathbb{R}^2$, 2 pivots, adding a column did not make $C(A)$ change.

6. $A=\begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}$, $C(A)=x_1$-x_2 plane

all possible Ax $=\begin{bmatrix}0&1\\1&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_2\\x_1\\0\end{bmatrix}$

7. $A=\begin{bmatrix}0&1&3\\1&0&4\\0&0&0\end{bmatrix}$, $C(A)=\uparrow$ no change

8. $A=\begin{bmatrix}0&1&3\\1&0&2\\0&0&1\end{bmatrix}$, $C(A)=\mathbb{R}^3$ CHANGE

What if $C(A)$ is not obvious?

* Use solving $Ax=b$ skills
  - pivots missing?
- check if $Ax=b$ has sol^n for general b. (one approach)

$$A = \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & b_1 \\ 0 & 4 & 5 & b_2 \\ 1 & -6 & -7 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 4 & 5 & b_2 \\ 0 & -8 & -10 & b_3 - b_1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 4 & 5 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 + 2b_2 \end{array}\right]$$

if $b_3 - b_1 + 2b_2 \neq 0$
NO SOLUTION

$$C(A) = \{ \text{the plane } b_3 - b_1 + 2b_2 = 0 \}$$

$$= \{ \text{all vectors} \perp \text{to} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \}$$

2nd theory statement is that a sol^n of $Ax=b$ is unique $\Leftrightarrow$ the only sol^n of $Ax=0$ is $x=0$.

The <u>null</u> <u>space</u> of A denoted $N(A) = \{ x \mid Ax = \vec{0} \}$

Is $N(A)$ a subspace?

1. $N(A) \subseteq \mathbb{R}^n$ $\quad \overset{n\times 1}{x}$ (where rows of A live)
2. If x & $y \in N(A)$, $Ax = Ay = 0$

$$A(cx+dy) = cAx + dAy = \vec{0} \quad ✓$$

Again putting our $Ax=b$ knowledge to use

1. $\overset{n\times n}{A}$, A^{-1} exists. $N(A) = \{\vec{0}\}$ b/c the unique sol$^{\underline{n}}$ of $Ax=0$ is $\vec{x}=\vec{0}$.

In general, $N(A) = ?$ Solve $Ax=0$ (carefully)

Example

zero do not change unneeded!

① $[A \mid \vec{0}] = \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 2 & 5 & 8 & 0 \\ 2 & 6 & 7 & 4 & 10 & 0 \end{array}\right] \to \begin{bmatrix} \boxed{1} & 3 & 2 & 5 & 8 \\ 0 & 0 & \boxed{3} & -6 & -6 \end{bmatrix}$

elimination below pivots (which are revealed)

② Divide by pivots $\begin{bmatrix} 1 & 3 & 2 & 5 & 8 \\ 0 & 0 & 1 & -2 & -2 \end{bmatrix}$

order of ② & ③ does not matter

③ Zero above pivots $\begin{bmatrix} 1 & 3 & 0 & 9 & 12 \\ 0 & 0 & 1 & -2 & -2 \end{bmatrix}$

echelon ← stairs in French

④ This is called row reduced echelon form

really revealing

⑤ Label columns as pivot (w/ pivot) or free (w/o pivot)

$$\begin{array}{ccccc} p & f & p & f & f \\ \end{array}$$
$$\begin{bmatrix} 1 & 3 & 0 & 9 & 12 \\ 0 & 0 & 1 & -2 & -2 \end{bmatrix} \qquad \begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \end{array} \begin{array}{c} p \\ f \\ p \\ f \\ f \end{array}$$

columns correspond to unknowns

(6). Each equation (row) contains exactly one pivot variable (b/c zero-ed above & below)

* put the pivot variables on the lhs and free on the rhs] always works

$$x_1 = -3x_2 - 9x_4 - 12x_5$$

$$x_3 = 2x_4 + 2x_5$$

* this means if $Ax=0$ then

$$\vec{x} = \begin{bmatrix} -3x_2 - 9x_4 - 12x_5 \\ x_2 \\ 2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{x_2\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -9 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -12 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{special solutions of } Ax=0}$$

$$= \begin{bmatrix} -3 & -9 & -12 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix}$$

every sol^n of $Ax=0$ is L.C. of special sol^n s.

The special solutions span $N(A)$.