Partial Differential Equation ("PDE")

Def^n ⟹ A differential eq^n which involves partial derivatives is called "PDE".

Ex:-
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ are partial derivatives.

Notation ⟹ let, $z = f(x, y)$

$$\left\{ \begin{aligned} \frac{\partial z}{\partial x} &= p \\ \frac{\partial z}{\partial y} &= q \\ \frac{\partial^2 z}{\partial x^2} &= r \\ \frac{\partial^2 z}{\partial y^2} &= t \\ \frac{\partial^2 z}{\partial x \partial y} &= s \end{aligned} \right.$$

Formation of PDE by Eliminating arbitary constant

① Form PDE, $z = ax + by + ab$ by eliminating arbitary constant.

Sol^n ⟹
$$z = ax + by + ab$$

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$ {differentiating w.r.t x}

$\frac{\partial z}{\partial y} = b \Rightarrow q = b$ {differentiating w.r.t y}

by putting $p = a$ and $q = b$ in given eq^n

$$z = ax + by + ab$$
$$z = px + qy + pq$$

(2) $z = (x^2+a)(y^2+b)$

diffⁿ w.r.t (x) :—

$$\frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\Rightarrow p = 2x(y^2+b) \Rightarrow (y^2+b) = \frac{p}{2x}$$

diffⁿ w.r.t (y) :—

$$\frac{\partial z}{\partial y} = 2y(x^2+a)$$

$$q = 2y(x^2+a) \Rightarrow (x^2+a) = \frac{q}{2y}$$

by putting the values of (x^2+a) & (y^2+b) in given eqⁿ; —

$$z = \frac{q}{2y} \times \frac{p}{2x}$$

$$4xyz = pq$$

(3) $z = (x-a)^2 + (y-b)^2$

diffⁿ w.r.t (x) :—

$$\frac{\partial z}{\partial x} = 2(x-a) \Rightarrow p = 2(x-a)$$

$$\Rightarrow (x-a) = \frac{p}{2}$$

diffⁿ w.r.t (y) :—

$$\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b)$$

$$\Rightarrow (y-b) = \frac{q}{2}$$

by putting the values of $(x-a)$ & $(y-b)$ in given eqⁿ;

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$4z = p^2 + q^2$$

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Formation of PDE by eliminating arbitary function :-

① $z = f\left(\frac{y}{x}\right)$

diffⁿ w.r.t (x) :-

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$p = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-x^2 p}{y} \quad \text{— ①}$$

diffⁿ w.r.t (y) :-

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$$

$$\Rightarrow q = f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{y}{x}\right) = xq \quad \text{— ②}$$

By ~~putting~~ equating the values of eqⁿ ① & ② :-

$$\frac{-x^2 p}{y} = xq \Rightarrow -xp = yq$$

or, $xp + yq = 0$ Ans

② $z = f(x^2 - y^2)$ — ①

diffⁿ w.r.t (x) :-

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)\cdot 2x$$

$$\frac{p}{2x} = f'(x^2 - y^2) \quad \text{— ②}$$

diffⁿ w.r.t (y) :-

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)\cdot(-2y)$$

$$q = f'(x^2 - y^2)\cdot(-2y)$$

$$\frac{-q}{2y} = f'(x^2 - y^2) \quad \text{— ③}$$

By eqⁿ ② & ③ :— (equating)

$$\frac{p}{2x} = \frac{-q}{2y}$$

$$py + qx = 0 \quad \text{Ans}$$

Q → $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

diffⁿ (x)

$$\frac{\partial z}{\partial x} = 2f'\left\{\frac{1}{x} + \log y\right\} \cdot \left(\frac{-1}{x^2}\right)$$

$$\frac{-px^2}{2} = f'\left(\frac{1}{x} + \log y\right) \quad \text{--- ①}$$

diffⁿ (y)

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$$

$$\frac{y}{2}(q - 2y) = f'\left(\frac{1}{x} + \log y\right)$$

$$\frac{yq}{2} - y^2 = f'\left(\frac{1}{x} + \log y\right) \quad \text{--- ②}$$

by equating eqⁿ ① & ② :—

$$\frac{-px^2}{2} = \frac{yq}{2} - y^2$$

$$-px^2 = yq - 2y^2$$

$$x^2p + yq = 2y^2 \quad \text{Ans}$$

II^{nd} form

(1) $z = f(x+at) + x\,g(x+at)$ — (1)

diff^n (x) $\rightarrow$ $\dfrac{\partial z}{\partial x} = f'(x+at) + x\,g'(x+at) + g(x+at)$

$p = f'(x+at) + x\,g'(x+at) + g(x+at)$ — (2)

diff^y (t) $\rightarrow$ $q = \dfrac{\partial z}{\partial t} = a f'(x+at) + a x\,g'(x+at)$ — (3)

diff^y again (x) $\rightarrow$ $r = \dfrac{\partial^2 z}{\partial x^2} = f''(x+at) + x\,g''(x+at) + g'(x+at) + g'(x+at)$ — (4)

again diff (t) $\rightarrow$ $\dfrac{\partial^2 z}{\partial t^2} = t = a^2 f''(x+at) + a^2 x\,g''(x+at)$ — (5)

diff^y eq^n (3) w.r.t (x) $\rightarrow$ $s = \dfrac{\partial^2 z}{\partial x \partial t} = a f''(x+at) + a\{x\,g''(x+at) + g'(x+at)\}$

$s = a f''(x+at) + a x\,g''(x+at) + a g'(x+at)$ — (6)

Here, eq^n (4) (5) (6) $\rightarrow$

$$\begin{cases} r = f''(x+at) + x\,g''(x+at) + 2g'(x+at) & \text{— (7)} \\ t = a^2 f''(x+at) + a^2 x\,g''(x+at) & \text{— (8)} \\ s = a f''(x+at) + a x\,g''(x+at) + a g'(x+at) & \text{— (9)} \end{cases}$$

&

* [$a^2 \times$ eq^n (7), $2a \times$ eq^n (9)], we get

$$\boxed{a^2 r - 2as + t = 0}$$ Ans

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Q $z = f(x+ay) + g(x-ay)$
then P.T. $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol^n

$\Rightarrow p = \frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay)$

$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$ —— (1)

$q = \frac{\partial z}{\partial y} = a f'(x+ay) - a g'(x-ay)$

$t = \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$

$s = \frac{\partial^2 z}{\partial x \partial y} = a f''(x+ay) - a g''(x-ay)$ —— (2)

By eq^n (1) and (2) :-
It is clear that

$$\boxed{a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}}$$ Hence proved

Q $z = y f(x) + x g(y)$ —— 1)

$p = \frac{\partial z}{\partial x} = y f'(x) + g(y)$ —— 2)

$q = \frac{\partial z}{\partial y} = f(x) + x g'(y)$ —— 3)

$r = \frac{\partial^2 z}{\partial x^2} = y f''(x)$ —— 4)

$t = \frac{\partial^2 z}{\partial y^2} = x g''(y)$ —— 5)

$s = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$ —— 6)

By eq^n (6) $\rightarrow$ $s = f'(x) + g'(y)$

By eq^n (2), $f'(x) = \frac{p - g(y)}{y}$

& By eq^n (3) $\rightarrow$ $g'(y) = \frac{q - f(x)}{x}$

Putting the values of $f'(x)$ & $g'(x)$ in s.

$$s = \frac{p - g(y)}{y} + \frac{q - f(x)}{x}$$

$$s = \frac{xp - xg(y) + qy - yf(x)}{xy}$$

$$sxy = xp + qy - (xg(y) + yf(x))$$

$$\boxed{sxy = xp + qy - z}$$ Ans (By (1))

Q $z = f(x) + e^y g(x)$ — 1)

$p = f'(x) + e^y g'(x)$ — 2)

$\frac{\partial z}{\partial y} = q = e^y g(x)$ — 3)

$r = f''(x) + e^y g''(x)$ — 4)

$\frac{\partial^2 z}{\partial y^2}$ $t = e^y g(x)$ — 5)

$s = e^y g'(x)$ — 6)

By eq^n (3) & eq^n (6) :—

$$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$$

or $q = t$ — A

Que If $u = f_1(x^2 + 2yz, y^2 + 2zx)$,

P.T. $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Sol^n $\rightarrow$

$$u = f(x^2 + 2yz, y^2 + 2zx)$$

$u \to f < \begin{matrix} v \\ w \end{matrix} > < \begin{matrix} x \\ y \\ z \end{matrix}$ Here, $u =$ Dependent var.

$x, y, z =$ Independent var.

Let, $u = f(v, w)$

Here, $v = x^2 + 2yz$ & $w = y^2 + 2zx$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\cdot(2x) + \frac{\partial f}{\partial w}(2z) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial v}(2z) + \frac{\partial f}{\partial w}(2y) \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial v}(2y) + \frac{\partial f}{\partial w}(2x) \quad \text{--- (3)}$$

LHS $\rightarrow$ $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$ --- (4)

Putting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial z}$ in (4)

$= 0 =$ RHS.

2 Independent variables

Single Independent variable

Direct Integration Method.

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(1) Solve, $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

By direct Integration Method.

Sol^n $\Rightarrow$ $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

* Integrating w.r.t x {Keeping y fixed}

$$\frac{\partial^2 z}{\partial x \partial y} + 18y^2\left(\frac{x^2}{2}\right) - \frac{\cos(2x-y)}{2} = f(y)$$

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{\cos(2x-y)}{2} = f(y)$$

* Integrating w.r.t x {Keeping y fixed}

$$\frac{\partial z}{\partial y} + 9y^2\left(\frac{x^3}{3}\right) - \frac{\sin(2x-y)}{4} = x f(y) + g(y)$$

* Integrating w.r.t y {keeping x fixed}

$$z + 3x^3\left(\frac{y^3}{3}\right) + \frac{\cos(2x-y)}{-4} = x\int f(y)\,dy + \int g(y)\,dy$$

$$\Rightarrow z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\,\underset{(let)}{u(y)} + \underset{(let)}{v(y)}$$

(2) Solve, $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$ by direct Integration Method.

Sol^n $\Rightarrow$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$

* Integrating w.r.t x :- (keeping y as fixed)

$$\frac{\partial z}{\partial y} = \frac{1}{y}\left(\frac{x^2}{2}\right) + ax + f(y)$$

* Integrating w.r.t y :- (Keeping x as fixed)

$$z = \frac{x^2}{2}\left(\frac{-1}{y^2}\right) + axy + \int f(y)\,dy + f(x)$$

$$z = -\frac{x^2}{2y^2} + axy + \underset{(let)}{u(y)} + f(x)$$

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Q $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \sin y$, for which

$\frac{\partial z}{\partial y} = -2\sin y$, when $x=0$ & $z=0$

when $z=0$ when y is an odd multiple of $\pi/2$.

Soln $\Rightarrow$ $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \sin y$

* Integrating w.r.t x :-

$$\frac{\partial z}{\partial y} = -\cos x \cdot \sin y + f(y) \quad \text{—} (1)$$

* Integrating w.r.t. y :-

$$Z = \cos x \cdot \cos y + \int f(y)\,dy + f(x)$$

$$Z = \cos x \cdot \cos y + \underbrace{u(y)}_{(\text{let})} + f(x) \quad \text{—} (2)$$

By eqn (1) :- $\frac{\partial z}{\partial y} = -\cos x \cdot \sin y + f(y)$

Here, $\boxed{\frac{\partial z}{\partial y} = -2\sin y,\ x=0 \text{ (Given)}}$

$$-2\sin y = -\cos 0 \cdot \sin y + f(y)$$

$$-2\sin y + \sin y = f(y)$$

So, $\boxed{f(y) = -\sin y}$

Put $f(y) = -\sin y$ in eqn (2) :-

$$Z = \cos x \cdot \cos y + \int -\sin y\,dy + f(x)$$

$$Z = \cos x \cdot \cos y + \cos y + f(x) \quad \text{—} (3)$$

$\boxed{\text{put, } Z = 0 \text{, at } y = \pi/2 \text{ (Given)}}$

$$0 = \cos x \cdot \cos\frac{\pi}{2} + \cos\frac{\pi}{2} + f(x)$$

$$\boxed{f(x) = 0}$$

So finally we get, By (3) $Z = \cos x \cdot \cos y + \cos y$ Ans

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Q. Solve, $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given that $x=0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Solⁿ $\Rightarrow$ $\frac{\partial^2 z}{\partial x^2} + z = 0$

$(D^2+1)z = 0$ (Notation $\frac{\partial}{\partial z} = D$)

Auxiliary eqⁿ :— $D^2+1=0$

$D = \pm i$ or $0 \pm i$

$\alpha = 0, \beta = 1$

Solⁿ is :— { * $c_1 \cos x + c_2 \sin x \rightarrow$ C.F. of the diffⁿ eqⁿ }

{ * Formula:- $e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$ }

So, Solⁿ is, $z = f_1(y)\cos x + f_2(y)\sin x$ —— (1)

Arbitary fuⁿ

Given, $x=0$ & $z = e^y$

Put $x=0$ & $z=e^y$ in eqⁿ (1) :—

$e^y = f_1(y)\cos 0 + f_2(y)\sin 0$

$\Rightarrow \boxed{e^y = f_1(y)}$ —— (2)

again, Given, $\frac{\partial z}{\partial x} = 1$, $x=0$:—

diffⁿ eqⁿ (1) w.r.t (1) :—

we get, $\frac{\partial z}{\partial x} = -f_1(y)\sin x + f_2(y)\cos x$

Put $\frac{\partial z}{\partial x} = 1$ & $x=0$:—

$\boxed{1 = f_2(y)}$ —— (3)

Put $f_1(y) = e^y$ & $f_2(y) = 1$ in eqⁿ (1) :—

$\boxed{z = e^y \cos x + \sin x}$ Ans.

(6) H.W. Solve, $\frac{\partial^2 z}{\partial x^2} = a^2 z$, when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

Ans :- $z = \sin y + \sinh ax$

Langrange's Linear PDE.

The PDE of the form,
$Pp + Qq = R$ is the standard form of Linear PDE of order one is called "Langrange's LDE".

Here, P, Q, R are any fn of x, y, z.

Working Rule :- To Solve $Pp + Qq = R$ by Langrage's LDE :-

Step ① $\rightarrow$ $Pp + Qq = R$ (standard form)

Step ② $\rightarrow$ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step ③ $\rightarrow$ equate any 2 Independent terms each other and find the Soln.

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(1) Solve, $xzp + yzq = xy$

Sol^n ⟹ By Langrange's LDE :—

$Pp + Qq = R$

By comparing given eq^n we get,

$P = xz$, $Q = yz$ and $R = xy$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

I II III (members)

Taking Ist two members :—

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

By Integrating, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\log x = \log y + \log c_1$

$x = y c_1$ $\xrightarrow{or}$ $\boxed{c_1 = \frac{x}{y}}$ $\Rightarrow x = c_1 y$

again, by equating last 2 members, $\frac{dy}{\cancel{y}z} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{x}$$

Here, variable x is different from derivatives

So put $\underline{x = c_1 y}$

$$\frac{dy}{z} = \frac{dz}{c_1 y}$$

$\Rightarrow c_1 y\,dy = z\,dz \Rightarrow \int c_1 y\,dy = \int z\,dz$

$$c_1 \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

again put $c_1 = x/y$

$$\frac{x}{y} \times \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{\frac{xy}{2} + \frac{z^2}{2} = c_2}$$

So final Ans is

$$\phi\left\{\frac{xy}{2} + \frac{z^2}{2}, \frac{x}{y}\right\} = 0$$

Ans.

(2) Solve,

$$yzp + zxq = xy$$

Sol^n $\Rightarrow$

$$P = yz, \quad Q = zx, \quad R = xy$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

I II III

By Ist 2 members :-

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x} \Rightarrow \int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \xrightarrow{or} \boxed{c_1 = \frac{x^2}{2} - \frac{y^2}{2}}$$

By last 2 members :-

$$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$\int y\,dy = \int z\,dz \Rightarrow \frac{y^2}{2} = \frac{z^2}{2} - c_2$$

$$\xrightarrow{or} \boxed{c_2 = \frac{y^2}{2} - \frac{z^2}{2}}$$

So,
$$\phi\left\{\frac{x^2}{2} - \frac{y^2}{2}, \frac{y^2}{2} - \frac{z^2}{2}\right\} = 0 \quad \text{Ans}$$

Que $\Rightarrow$ Solve, $y^2p - xyq = x(z-2y)$

Sol^n $\Rightarrow$

$$P = y^2, \quad Q = -xy, \quad R = x(z-2y)$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

I II III

By Ist 2 members $\Rightarrow$ $\frac{dx}{y^2} = \frac{dy}{-xy}$

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$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy$$

By Integration; $-\int x\,dx = \int y\,dy$

$$-\frac{x^2}{2} = \frac{y^2}{2} - C_1$$

$$\boxed{\frac{y^2}{2} + \frac{x^2}{2} = C_1}$$ (our assumption)

By last 2 members;

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$(z-2y)\,dy = -y\,dz$$

$$\underbrace{z\,dy + y\,dz} - 2y\,dy = 0$$

This is an exact Diffⁿ eqⁿ.

$$d(yz) - 2y\,dy = 0$$

By Integration, $yz - 2\frac{y^2}{2} = C_2$

$$\boxed{yz - y^2 = C_2}$$

So, $\phi\left\{\frac{y^2}{2} + \frac{x^2}{2},\ yz - y^2\right\} = 0$ Ans.

Q ⟹ Solve, $x^2p + y^2q = z^2$

Solⁿ ⟹ $P = x^2,\ Q = y^2,\ R = z^2$

$$\underset{\text{I}}{\frac{dx}{x^2}} = \underset{\text{II}}{\frac{dy}{y^2}} = \underset{\text{III}}{\frac{dz}{z^2}}$$

By Ist 2 members;

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} - C_1$$

then, $\boxed{C_1 = \frac{1}{x} - \frac{1}{y}}$

By last 2 membes;

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

By Integration, $\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$

$$-\frac{1}{y} = \frac{-1}{z} - C_2$$

So, $\boxed{C_2 = \frac{1}{y} - \frac{1}{z}}$

then, $\phi\left\{\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right\} = 0$ Ans

Case - III $f(x,y) = x^m y^n$

Que ① $(D^2 - 6DD' + 9D'^2)Z = 12x^2 + 36xy$

Soln $(D^2 - 6DD' + 9D'^2)Z = 0$

A. eqn $D^2 - 6DD' + 9D'^2 = 0$

$m^2 - 6m + 9 = 0$ $\left\{ \begin{array}{l} D \to m \\ D' \to 1 \end{array} \right.$

$m = 3, 3$

C.F. $f_1(y+3x) + x f_2(y+3x)$

P.I. $\frac{1}{f(D,D')} 12x^2 + 36xy$

$\frac{1}{D^2 - 6DD' + 9D'^2} 12x^2 + 36xy$

$\Rightarrow \frac{1}{D^2\left\{1 - 6\frac{D'}{D} + 9\frac{D'^2}{D^2}\right\}} 12x^2 + 36xy$

$\Rightarrow \frac{1}{D^2}\left\{1 - \left(6\frac{D'}{D} - 9\frac{D'^2}{D^2}\right)\right\}^{-1} 12x^2 + 36xy$

$\Rightarrow \frac{1}{D^2}\left[1 + \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right) + \left(6\frac{D'}{D} - \frac{9D'^2}{D^2}\right)^2 + \cdots\right] 12x^2 + 36xy$

** $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots \infty$

$\Rightarrow \frac{1}{D^2}\left[(12x^2 + 36xy) + \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)(12x^2 + 36xy) + \left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)^2 (12x^2 + 36xy) + \cdots\right]$

$\Rightarrow \frac{1}{D^2}\left[(12x^2 + 36xy) + \frac{6}{D} \times D'(12x^2 + 36xy) - \frac{9}{D^2} \times D'^2 (12x^2 + 36xy) + \frac{6^2}{D^2} \times D'^2(12x^2 + 36xy) + \frac{9}{D^4} \times D'^4 (12x^2 + 36xy) - 2\left(\frac{6D'}{D} - \frac{9D'^2}{D^2}\right)(12x^2 + 36xy) \cdots \infty\right.$

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$$\Rightarrow \frac{1}{D^2}\left[12x^2-36xy+\frac{6}{D}[36x]-\frac{9}{D^2}[0]+\frac{6}{D^2}[0]+\frac{9}{D^4}\times 0-\frac{12}{D}\times D'(12x^2+36xy)\right]$$

$$\Rightarrow \frac{1}{D^2}\left[12x^2-36xy+216x\int x\,dx-\frac{12}{D}\times 36x\right]$$

$$\Rightarrow \frac{1}{D^2}\left[12x^2-36xy+\overset{108}{\cancel{216}}\times\frac{x^2}{\cancel{2}}-432\int x\,dx\right]$$

$$\Rightarrow \frac{1}{D^2}\left[12x^2-36xy+108x^2-\overset{216}{\cancel{432}}\times\frac{x^2}{\cancel{2}}\right]$$

$$\Rightarrow \frac{1}{D^2}\left[12x^2-36xy+108x^2-216x^2\right]$$

$$\Rightarrow \frac{1}{D^2}\left[-96x^2-36xy\right]$$

$$\Rightarrow \int\left(\int -96x^2-36xy\;dx\right)dx$$

$$\Rightarrow \int\left[-96\times\frac{x^2}{2}-36y\times\frac{x^2}{2}\right]dx$$

$$\Rightarrow -\overset{16}{\cancel{96}}\times\frac{x^3}{3\times\cancel{2}}-\overset{12\ 6}{\cancel{36}}y\times\frac{x^3}{\cancel{3\times 2}}$$

$$\Rightarrow -16x^3-6x^3y$$

$$z = C.F + P.I.$$

Que $\Rightarrow$ $(D^2 - 2DD' + D'^2)z = x^3$

Sol^n $\Rightarrow$ $(D^2 - 2DD' + D'^2)z = 0$

A.eq^n $\Rightarrow$ $D^2 - 2DD' + D'^2 = 0$

$m^2 - 2m + 1 = 0$ $\left\{ \begin{array}{l} D \to m \\ D' \to 1 \end{array} \right.$

$m = 1, 1$

C.F. $\Rightarrow$ $f_1(y+x) + x f_2(y+x)$

P.I. $\Rightarrow$ $\dfrac{1}{f(D,D')} F(x,y)$

$$\frac{1}{D^2 - 2DD' + D'^2} x^3$$

$$\frac{1}{D^2\left\{1 - \frac{2D'}{D} + \frac{D'^2}{D^2}\right\}} x^3$$

$$\Rightarrow \frac{1}{D^2}\left[1 - \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right)\right]^{-1} x^3$$

$$\Rightarrow \frac{1}{D^2}\left[1 + \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right) + \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right)^2 + \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right)^3 + \cdots\cdots\right] x^3$$

$$\Rightarrow \frac{1}{D^2}\left[1 + \left(\frac{2D'}{D} - \frac{D'^2}{D^2}\right) + \left(\frac{4D'^2}{D^2} + \frac{D'^4}{D^4} - \frac{4D'^3}{D^3}\right) \frac{8D'^3}{D^3} + \cdots\cdots\right] x^3$$

$$\Rightarrow \frac{1}{D^2}\left[x^3 - \frac{2}{D} \times 0 - \frac{1}{D^2} \times 0 + \frac{4}{D^2} \times 0 \cdots\right]$$

$$\Rightarrow \iint x^3\, dx\, dx = \int \frac{x^4}{4} dx$$

$$\Rightarrow \frac{x^5}{20}$$

So, $Z = C.F + P.I$

Ans

Que $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Sol^n $(D^2 + DD' - 6D'^2)z = y\cos x$

A.eq^n $D^2 + DD' - 6D'^2 = 0$

$m^2 + m - 6 = 0$ $\quad D \to m$

$m = -3, 2$ $\quad D' \to 1$

C.F. $f_1(y-3x) + f_2(y+2x)$

P.I. $\frac{1}{D^2 + DD' - 6D'^2} \cdot y\cos x$

By using General method (which is also known as falwike case),

By 1st factor $(D+3D')$

$\int F(x,y)\,dx$ $\quad$ let, $y-3x = c$ then $y = c+3x$

$\int F(x, c+3x)\,dx$

$\int y\cos x\,dx$

$\int \underset{u}{(c+3x)}\underset{v}{\cos x}\,dx$

$(c+3x)\sin x + 3\cos x$ $\quad$ again put, $c = y-3x$

$(y - \cancel{3x} + \cancel{3x})\sin x + 3\cos x$

$y\sin x + 3\cos x$ —— (1)

again, by 2nd factor; $(D-2D')$

$\int F(x,y)\,dx$ $\quad$ put, $y+2x = c$ then $y = c-2x$

$\int F(x, c-2x)\,dx$

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$$\int (y\sin x + 3\cos x)\,dx$$ By eq^n ①

$$\Rightarrow \int \left(\frac{c-2x}{4}\right)\sin x + 3\cos x\,dx$$

$$\Rightarrow \left[(c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x\right]$$

again, put $c = y+2x$

$$\Rightarrow \left[(y+\cancel{2x}-\cancel{2x})(-\cos x) - 2\sin x + 3\sin x\right]$$

$$\Rightarrow \left[-y\cos x + \sin x\right]$$

So, $Z = C.F. + P.I.$

$$Z = f_1(y-3x) + f_2(y+2x) + \left[\sin x - y\cos x\right]$$

Ans

Que $(D^3 - D^2D' + DD'^2 - D'^3)Z = e^x\cos 2y$

Sol^n $D^3 - D^2D' + DD'^2 - D'^3 = 0$

$$m^3 - m^2 + m - 1 = 0 \qquad \left.\begin{matrix} D \to m \\ D' \to 1 \end{matrix}\right\}$$

$$m = -1, -1, 1$$

C.F. $\rightarrow$ $f_1(y-x) + x f_2(y-x) + f_3(y+x)$

P.I. $\rightarrow$ $$\frac{1}{D^3 - D^2D' + DD'^2 - D'^3}\, e^x\cos 2y$$

By Using General Method

By Taking 1st factor, $(D+D')$

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let, Put $c = y-x \Rightarrow y = c+x$

$\int F(x,y)\,dx$

$\Rightarrow \int (x, c+x)\,dx$

$\Rightarrow \int e^x \cos 2y\,dx$

$\Rightarrow \int e^x \cos 2(c+x)\,dx$

$\Rightarrow \left[\frac{e^x}{1+4}\left\{\cos 2(c+x) + 2\sin 2(c+x)\right\}\right]$

$\Rightarrow \left[\frac{e^x}{5}\left\{\cos 2(c+x) + 2\sin 2(c+x)\right\}\right]$

Put, $c = y-x$

$\Rightarrow \frac{e^x}{5}\left[\cos 2(y-x+x) + 2\sin 2(y-x+x)\right]$

$\Rightarrow \frac{e^x}{5}\left[\cos 2y + 2\sin 2y\right]$ ---------- (1)

again, by 2^{nd} factor, $(D+D')$

Put, $c = y-x$
$y = c+x$

$\Rightarrow \int F(x,y)\,dx$

$\Rightarrow \int \frac{e^x}{5}\left\{\cos 2y + 2\sin 2y\right\}].\,dx$

$\Rightarrow \int \frac{e^x}{5}\left\{\cos 2(c+x) + 2\sin 2(c+x)\right\}\,dx$

$\Rightarrow \frac{1}{5}\left[\int e^x \cos 2(c+x)\,dx + \int e^x \sin 2(c+x)\,dx\right]$

$\Rightarrow \frac{1}{5}\left[\frac{e^x}{1+4}\left\{\cos 2(c+x) + 2\sin(c+x)\right\} + \frac{e^x}{1+4}\left\{\sin 2(c+x) - 2\cos(c+x)\right\}\right]$

$\Rightarrow \frac{1}{5}\left[\frac{e^x}{5}\left\{\cos 2(c+x) + 2\sin(c+x)\right\} + \frac{e^x}{5}\left\{\sin 2(c+x) - 2\cos(c+x)\right\}\right]$

$\Rightarrow \frac{1}{25}\left[e^x\left\{\cos 2(y-x+x) + 2\sin(y-x+x)\right\} + e^x\left\{\sin 2(y-x+x) - 2\cos(y-x+x)\right\}\right]$ {again put,

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$$\Rightarrow \frac{e^x}{25}\left[\cos 2y + 2\sin y + \sin 2y - 2\cos y\right]$$

A

$z = C.F + P.I.$

Non-Homogeneous Fuⁿ

P.I. of Non-Hom fuⁿ is same as Hom. fuⁿ.
only C.F. is different.

In the Non-Hom. fuⁿ factor is in the form of $(D - mD' - a)$

m_1, m_2 are roots then, $\xrightarrow{C.F}$ $e^{ax} f_1(y + m_1 x) + e^{ax} f_2(y + m_2 x)$

Que $\Rightarrow$ $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$

Solⁿ $\Rightarrow$ $D^2 - DD' + D' - 1 = 0$

$(D^2 - 1) + D'(-D + 1) = 0$

$(D+1)(D-1) - D'(D-1) = 0$

$(D-1)(D - D' + 1) = 0$

Or, $(D - 0D' - 1)(D - D' + 1) = 0$

Here factors are in the form of $(D - mD' - a)$ so, by comparing

$a = 1$ & $m = 0$ (by 1st factor) | $a = -1$, $m = 1$ (by 2nd factor)

then, $\xrightarrow{C.F.}$ $e^{x}[f_1(y + 0x)] + e^{-x}[f_2(y + x)]$

$\Rightarrow e^{x} f_1(y) + e^{-x} f_2(y + x)$

P.I. $\rightarrow \dfrac{1}{f(D,D')} F(x,y)$

$\dfrac{1}{D^2 - DD' + D' - 1} \cos(x+2y)$

put, $\begin{cases} D^2 \rightarrow -1 \\ DD' \rightarrow -2 \end{cases}$

$\Rightarrow \dfrac{1}{-1+2+D'-1} \cos(x+2y)$

$\Rightarrow \dfrac{1}{D'} \cos(x+2y)$

$\Rightarrow \int \cos(x+2y)\,dy$ Integration w.r.t y

$\Rightarrow \dfrac{\sin(x+2y)}{2}$

So, $Z = C.F + P.I.$ Ans

(2) $[(D-D'-1)(D-D'-2)]z = e^{2x-y}$

Sol^n $\Rightarrow$ By comparing with factors $(D-mD'-a)$

By 1st factor :- $(D-D'-1)$ $m=1, a=1$

and By 2nd factor :- $(D-D'-2)$ $m=1, a=2$

C.F. $\rightarrow e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

P.I. $\rightarrow \dfrac{1}{f(D,D')} F(x,y)$

$\dfrac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$ Put $D \rightarrow 2$, $D' \rightarrow -1$

$\dfrac{1}{(2+1-1)(2+1-2)} e^{2x-y}$

$\dfrac{1}{2} e^{2x-y}$

So, $Z = C.F + P.I.$ Ans

Langrange's Linear PDE.

The PDE of the form,
$Pp + Qq = R$ is the standard form of Linear PDE of order one is called "Langrange's LDE".

Here, P, Q, R are any fn of x, y, z.

Working Rule :- To Solve $Pp + Qq = R$ by Langrage's LDE :-

Step ① → $Pp + Qq = R$ (standard form)

Step ② → $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step ③ → equate any 2 Independent terms each other and find the Soln

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(1) Solve, $xzp + yzq = xy$

Sol^n By Langrange's LDE :-

$Pp + Qq = R$

By comparing Given eq^n we get,

$P = xz$, $Q = yz$ and $R = xy$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

I II III (members)

Taking Ist two members :-

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

By Integrating, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\log x = \log y + \log c_1$.

$x = y c_1$ $\xrightarrow{\text{or}}$ $\boxed{c_1 = \frac{x}{y}} \Rightarrow x = c_1 y$

again, by equating last 2 members, $\frac{dy}{\cancel{y}z} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{x}$$

Here, variable x is different from derivatives

So put $\underline{x = c_1 y}$

$$\frac{dy}{z} = \frac{dz}{c_1 y}$$

$\Rightarrow c_1 y \, dy = z \, dz \Rightarrow \int c_1 y \, dy = \int z \, dz$

$$c_1 \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

again put $c_1 = x/y$

$$\frac{x}{y} \times \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{\frac{xy}{2} + \frac{z^2}{2} = c_2}$$

So final Ans is

$$\phi\left\{\frac{xy}{2} + \frac{z^2}{2}, \frac{x}{y}\right\} = 0$$

A.

(2) Solve,

$$yzp + zxq = xy$$

Sol^n $\Rightarrow$ $P = yz$, $Q = zx$, $R = xy$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

I II III

By Ist 2 members :-

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x} \Rightarrow \int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \xrightarrow{or} \boxed{c_1 = \frac{x^2}{2} - \frac{y^2}{2}}$$

By last 2 members :-

$$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$\int y\,dy = \int z\,dz \Rightarrow \frac{y^2}{2} = \frac{z^2}{2} - c_2$$

$$\xrightarrow{or} \boxed{c_2 = \frac{y^2}{2} - \frac{z^2}{2}}$$

So, $\phi\left\{\frac{x^2}{2} - \frac{y^2}{2}, \frac{y^2}{2} - \frac{z^2}{2}\right\} = 0$ Ans

Solve at last

Que (3) $\Rightarrow$ Solve, $y^2p - xyq = x(z-2y)$

Sol^n $\Rightarrow$ $P = y^2$, $Q = -xy$, $R = x(z-2y)$

different

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

I II III

By Ist 2 members $\Rightarrow$ $\frac{dx}{y^2} = \frac{dy}{-xy}$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy$$

By Integration; $-\int x\,dx = \int y\,dy$

$$-\frac{x^2}{2} = \frac{y^2}{2} - c_1 \rightarrow \text{(our assumption)}$$

$$\boxed{\frac{y^2}{2} + \frac{x^2}{2} = c_1}$$

By last 2 members;

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$(z-2y)\,dy = -y\,dz$$

$$\underbrace{z\,dy + y\,dz}_{} - 2y\,dy = 0$$

This is an exact Diffⁿ eqⁿ.

$$d(yz) - 2y\,dy = 0$$

By Integration, $yz - 2\frac{y^2}{2} = c_2$

$$\boxed{yz - y^2 = c_2}$$

So, $\phi\left\{\frac{y^2}{2} + \frac{x^2}{2},\ yz - y^2\right\} = 0$ Ans.

Q(4) Solve, $x^2p + y^2q = z^2$

Solⁿ ⟹ $P = x^2,\ Q = y^2,\ R = z^2$

$$\underset{\text{I}}{\frac{dx}{x^2}} = \underset{\text{II}}{\frac{dy}{y^2}} = \underset{\text{III}}{\frac{dz}{z^2}}$$

By Ist 2 members;

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} - c_1$$

then; $\boxed{c_1 = \frac{1}{x} - \frac{1}{y}}$

By last 2 members;

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

By Integration, $\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$

$$-\frac{1}{y} = \frac{-1}{z} - c_2$$

So, $\boxed{c_2 = \frac{1}{y} - \frac{1}{z}}$

then, $\phi\left\{\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right\} = 0$ Ans

Que 3 Solve, $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$

Sol^n → $P = y^2 + z^2 - x^2$, $Q = -2xy$, $R = 2zx$

* different

$$\underset{\text{I}}{\frac{dx}{y^2 + z^2 - x^2}} = \underset{\text{II}}{\frac{dy}{-2xy}} = \underset{\text{III}}{\frac{dz}{2zx}} \quad \text{—— (1)}$$

By (II) & (III): —

$$\frac{dy}{-\cancel{2x}y} = \frac{dz}{\cancel{2}z\cancel{x}}$$

$$\int -\frac{dy}{y} = \int \frac{dz}{z}$$

$$-\log y = \log z - \log c_1$$

$$\log c_1 = \log z + \log y$$

So, $\boxed{c_1 = yz}$

~~Now By both members (I) & (III):~~

Now It is difficult to find another Answer, So we will Solve such that

Again, Using x, y, z as multipliers, each fraction of (1) and we make a new member;

$$\cancel{\underset{\text{I}}{\frac{dx}{y^2+z^2-x^2}} = \underset{\text{III}}{\frac{dz}{2zx}}}$$

~~It is difficult to calculate & find Sol^n, So we multiply.~~

$$\frac{x\,dx + y\,dy + z\,dz}{-xy^2 - xz^2 - x^3} \quad \text{IV} \text{ (This is new member)}$$

Now By equating IV th member with II or III, such that,

$$\underset{\text{IV}}{\frac{x\,dx + y\,dy + z\,dz}{-x(y^2+z^2+x^2)}} = \underset{\text{III}}{\frac{dz}{2zx}}$$

Now, $$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2} = \frac{-dz}{z}$$

here we derivative of denominator is its numerator so we can write

By Integ^n, $$\log(x^2+y^2+z^2) = -\log z + \log c_2$$

$$z(x^2+y^2+z^2) = c_2$$

So, $$\phi\{yz, z(x^2+y^2+z^2)\} = 0$$

Que (6) Solve, $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol^n Given eq^n is in the form of Langranges LDE :-

So, $$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \text{--- (1)}$$

By Taking multiplier System;

$1/x^2$, $1/y^2$, $1/z^2$ multiply in each fraction of (1) and form a new member such that

dx $\quad$ dy $\quad$ dz

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} =$$

I II III

$$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y}$$

or $$\frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

IV

By Taking new member;

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

By Integration, $-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -C_1$

or $\rightarrow$ $\boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = C_1}$

again, Taking another multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)}$$

I II III or, $\rightarrow$ $$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

IV

By Taking new member;

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

By Integration, $\log x + \log y + \log z = \log C_2$

$\boxed{xyz = C_2}$

So, $\phi\left\{\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right\} = 0$

Ques (7) $(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=ly-mx$

$(mz-ny)p+(nx-lz)q=ly-mx$

Sol → This is Langrange's LDE;

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

By Taking multiplier, x,y,z

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}=\frac{xdx+ydy+zdz}{mxz-nxy+nxy-lyz+lzy-mxz}$$

or → $\frac{xdx+ydy+zdz}{0}$ IV

By Taking new member;

$xdx+ydy+zdz=0$

By Integration, $\boxed{\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c_1}$

again, By Taking multiplier, l,m,n.

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}=\frac{lx\,dx+my\,dy+nz\,dz}{mlz-nly+nmx-lmz+lny-mnx}$$

or → $\frac{lx+my+nz}{0}$ IV

By taking new member;

~~lx+my+nz=0~~

$lxdx+mydy+nzdz=0$

By Integration; $\boxed{\frac{lx^2}{2}+m\frac{y^2}{2}+n\frac{z^2}{2}=c_2}$

So, $\phi\left\{\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ \frac{lx^2}{2}+\frac{my^2}{2}+\frac{nz^2}{2}\right\}=0$

Ans..

Que (8) $p\tan x + p\tan y = \tan z$

Sol^n This is Langrange's LPDE;

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

I, II, III

Taking 1^{st} 2 members;

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

By Integration; $\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log \sin x - \log \sin y = \log c_1$$

$$\boxed{\frac{\sin x}{\sin y} = c_1}$$

again By Taking last 2 members;

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

By Integration; $\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$

$$\log \sin y = \log \sin z + \log c_2$$

$$\boxed{\frac{\sin y}{\sin z} = c_2}$$

Now, $\phi\left\{\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right\} = 0$ Ans.

Que (9) $x(y-z)p + y(z-x)q = z(x-y)$

Sol^n This is in the form of L.LPDE:-

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Using multipliers 1, 1, 1 :-

$$= \frac{dx+dy+dz}{xy - xz + yz - xy + zx - yz} \quad \text{IV} \quad (\text{This new member})$$

Or, $\frac{dx+dy+dz}{0}$

So, we get, $\int dx + \int dy + \int dz = 0$

$\boxed{x+y+z = c_1}$ —— (i)

Again, Using multipliers $1/x, 1/y, 1/z$;

then new member is

$= \dfrac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$

So, we get, $\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz$ (new member) $= 0$

$\log x + \log y + \log z = \log c_2$

or $\Rightarrow$ $\boxed{xyz = c_2}$ —— (ii)

then,

$\phi\{x+y+z,\ xyz\} = 0$ Ans.

Different

Que (10) Solve, $(x^2-yz)p+(y^2-zx)q = z^2-xy$

Sol^n $\Rightarrow$ This is Laugranges L.PDE^n

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

Using multiplier Systems;

$(1,-1,0)$, $(0,1,-1)$ & $(-1,0,1)$

$$\frac{dx-dy}{(x-y)(x+y+z)}_{IV} = \frac{dy-dz}{(y-z)(x+y+z)}_{V} = \frac{dz-dx}{(z-x)(x+y+z)}_{VI}$$

These are new members, which formed by multiplying $(1,-1,0)$, $(0,1,-1)$ & $(-1,0,1)$ in each fraction.

Taking 1st 2 members;

$$\frac{dx-dy}{(x-y)\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}}$$

$$\int\frac{dx-dy}{x-y} = \int\frac{dy-dz}{y-z}$$

By Integration, $\log(x-y) = \log(y-z)+\log c_1$

$$\frac{x-y}{y-z} = c_1 \quad \text{--- (1)}$$

again, By last 2 members;

$$\frac{dy-dz}{(y-z)\cancel{(x+y+z)}} = \frac{dz-dx}{(z-x)\cancel{(x+y+z)}}$$

By Integration, $\int\frac{dy-dz}{y-z} = \int\frac{dz-dx}{z-x}$

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$$\log(y-z) = \log(z-x) + \log c_2$$

$$\therefore \frac{y-z}{z-x} = c_2 \quad \text{—— (2)}$$

So, $\phi\left\{\frac{x-y}{y-z}, \frac{y-z}{z-x}\right\} = 0$ Ans.

Que (11) Solve, $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

Sol^n ⇒ $\cancel{\frac{dx}{x(z-2y^2)} = \frac{dy}{}}$

$$px(z-2y^2) + y(z-y^2-2x^3)q = z(z-y^2-2x^3)$$

Now, This is in the standard form of Langrange's LPDE :-

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

I II III

Taking last 2 members ;

$$\frac{dy}{y(\cancel{z-y^2-2x^3})} = \frac{dz}{z(\cancel{z-y^2-2x^3})}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

By Integration, $\int \frac{dy}{y} = \int \frac{dz}{z}$

$$\log y = \log z + \log c_1$$

$$\frac{y}{z} = c_1 \quad \text{—— (1)}$$

Now Taking I st and III rd (or) You can take II nd and III rd also.

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}$$

I III

$$\frac{dx}{x(z-\cancel{y^2}-2x^2)} = \frac{dz}{z(z-y^2-2x^3)}$$

Here, we put, $\boxed{\frac{y}{z} = c_1}$

$\xrightarrow{or}$ $y = c_1 z$

$$\frac{dx}{x(z-2(c_1z)^2)} = \frac{dz}{z(z-(c_1z)^2-2x^3)}$$

$$\frac{dx}{x(z-2c_1^2z^2)} = \frac{dz}{z(z-c_1^2z^2-2x^3)}$$

$$\frac{dx}{x\cancel{z}(1-2c_1^2z)} = \frac{dz}{\cancel{z}(z-c_1^2z^2-2x^3)}$$

$$\frac{dz}{dx} = \frac{z-c_1^2z^2-2x^3}{x(1-2c_1^2z)}$$

* Here we want to form linear diff^n eq^n of Ist order

$\frac{dy}{dx}+Py=Q$

sol^n $\rightarrow$ $y(I.F) = \int Q(I.F)+C$

$\xrightarrow{Or}$ $(1-2c_1^2z)\frac{dz}{dx} = \frac{z-c_1^2z^2}{x} - \frac{2x^3}{x}$

$$\left\{\begin{aligned} \text{let, } u &= z-c_1^2z^2 \\ du &= dz - c_1^2 \cdot 2z \\ du &= (1-2c_1^2z) \end{aligned}\right.$$

$$\frac{du}{dx} = \frac{u}{x} - 2x^2$$

This is Linear diff^n eq^n of 1st order i $\rightarrow$

$$\frac{du}{dx} - \frac{1}{x}u = -2x^2$$

$\xrightarrow{I.F}$ $e^{\int -\frac{1}{x}dx} = e^{-\log x} = x^{-1}$ or $\frac{1}{x}$

So $\xrightarrow{sol^n}$ $u(I.F) = \int(-2x^2)(I.F)\,dx +$

$$\frac{u}{x} = \int -2x^2 \times \frac{1}{x}\,dx + c_2$$

$$\frac{u}{x} = \int -2x\,dx + c_2$$

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$$\frac{u}{x} = -2 \times \frac{x^2}{x} + c_2$$

$$\frac{u}{x} = -x^2 + c_2$$

Put, $u = z - c_1^2 z^2$

$$\frac{z - c_1^2 z^2}{x} + x^2 = c_2$$

again, put, $c_1 = y/z$

$$\frac{z - \frac{y^2}{z^2} \times z^2}{x} + x^2 = c_2$$

$$\frac{z - y^2}{x} + x^2 = c_2$$

Or, $\boxed{\frac{z}{x} - \frac{y}{x} + x^2 = c_2}$

Finally we get, $\phi\left\{\frac{y}{z}, \frac{z}{x} - \frac{y}{x} + x^2\right\} = 0$

Ans.

Method of Separation of Variable

In this method, we assume that the dependent variable is the product of 2 functions, each of which involves only one of the independent variables. As a consequence, 2 ordinary differential eqⁿ's are formed.

Q(1) Solve the diffⁿ eqⁿ, by using method of separation of variable.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

given that, $u(x,0) = 6e^{-3x}$

Solⁿ ⟹ Given, $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ ------(i)

Assume that the trial solⁿ of (i) is

Here, $X(x)$ is fⁿ of x only and $T(t)$ is fⁿ of t only ⟶ $u(x,t) = X(x) \cdot T(t)$ ------(ii)

diffⁿ w.r.t x:- $\frac{\partial u}{\partial x} = X'T$ ------(iii)

diffⁿ w.r.t t:- $\frac{\partial u}{\partial t} = XT'$ ------(iv)

By putting values of (ii), (iii) and (iv) in eqⁿ (i)

$$X'T = 2XT' + XT$$

$$X'T - XT = 2XT'$$

$$T(X' - X) = 2XT'$$

$$\underset{\text{I}}{\frac{X'-X}{X}} = \underset{\text{II}}{\frac{2T'}{T}} = \underset{\text{III}}{K \text{ (let)}}$$

By equating (I) and (III) :-

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$$\frac{X'-X}{X} = K$$

$$\frac{X'}{X} - 1 = K$$

integrating w.r.t x:—

$$\int \frac{X'}{X} - \int dx = \int K\,dx$$

$$\log X - x = Kx + \log c_1$$

$$\log X = Kx + x + \log c_1$$

$$\log X = x(K+1) + \log c_1$$

$$\text{So, } \frac{X}{c_1} = e^{(K+1)x} \Rightarrow X = c_1 e^{(K+1)x} \quad \text{—— (v)}$$

again, By (II) and (III) :—

$$\frac{2T'}{T} = K$$

integrating w.r.t. t.

$$\cancel{2\log T}\; 2\int \frac{T'}{T} = \int K\,dt$$

$$2\log T = Kt + \log c_2$$

$$\log T = \frac{Kt}{2} + \frac{1}{2}\log c_2 \Rightarrow \log T = \frac{Kt}{2} + \log c_3$$

$$\text{So, } \frac{T}{c_3} = e^{Kt/2} \Rightarrow T = c_3 e^{Kt/2} \quad \text{---- (vi)}$$

By putting the values of (v) & (vi) in eqn (ii) :—

$$u = X \cdot T$$

$$\boxed{u = c_1 c_3 e^{(K+1)x} \cdot e^{Kt/2}} \quad \text{—— (vii)}$$

By Putting initial conditions;

$$u(x,0) = 6e^{-3x}$$

By putting $u = 6e^{-3x}$ when $t=0$ in (vii)

$$6e^{-3x} = e^{(K+1)x} \cdot e^{0}$$

$$6e^{-3x} = e^{(K+1)x}$$

By Comparing; $c_1c_3 = 6$ and

$-3 = k+1$

and $c_1c_3=6$ $\quad k=-4$

Put $k=-4$ in eq^n (vii)

$$u = 6e^{(-4+1)x} \cdot e^{-4t/2}$$

$$u = 6e^{-3x} \cdot e^{-2t}$$ Ans

* ________ *

(2) Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given that $u(0,y) = 3e^{-y} - e^{-5y}$ by method of Separation of variable.

Sol^n ⟹ Given, $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ ------ (1)

let, trial sol^n of, ~~$X'Y + XY' =$~~

$u(x,y) = X(x) \cdot Y(y)$ ------ (2)

diff^n w.r.t x ⟶ $\frac{\partial u}{\partial x} = X'Y$ ------ (3)

diff^n w.r.t y ⟶ $\frac{\partial u}{\partial y} = XY'$ ------ (4)

Putting the eq^n of (2), (3), (4) in eq^n (1) :-

$$4X'Y + XY' = 3XY$$

$$4X'Y = 3XY - XY'$$

$$4X'Y = X(3Y - Y')$$

$$\underbrace{\frac{4X'}{X}}_{\text{I}} = \underbrace{\frac{3Y-Y'}{Y}}_{\text{II}} = \underbrace{k \text{ (let)}}_{\text{III}}$$

By (I) & (III) :- ⟶ $\frac{X'}{X} = \frac{k}{4}$

By Integration, $\int \frac{X'}{X} = \int \frac{k}{4} dx$

$$\log X = \frac{k}{4}x + \log c_1$$

$$\log \frac{x}{c_1} = \frac{kx}{4}$$

So, $$\frac{x}{c_1} = e^{\frac{kx}{4}}$$

$$\Rightarrow x = c_1 e^{\frac{kx}{4}} \quad \text{------ (6)}$$

again by II & III :—

$$\frac{3y - y'}{y} = k$$

$$\frac{3y}{y} - \frac{y'}{y} = k$$

By Integration, $$3 - \frac{y'}{y} = k$$

$$\cancel{\Rightarrow \int 3\,dy - \int \frac{y'}{y} = \int k\,dy}$$

$$\cancel{3y - \log y = ky + \log c_2}$$

$$\frac{y'}{y} = 3 - k$$

$$\int \frac{y'}{y} = \int (3-k)\,dy$$

$$\log y = (3-k)y + \log c_2$$

$$\frac{y}{c_2} = e^{(3-k)y}$$

$$y = c_2 e^{(3-k)y} \quad \text{------ (7)}$$

Putting the values of (6) & (7) in (2)

$$u = x \cdot y$$

$$u = c_1 e^{\frac{kx}{4}} \cdot c_2 e^{(3-k)y}$$

$$\boxed{u = c_1 c_2 e^{\frac{kx}{4}} \cdot e^{(3-k)y}} \quad \text{— (8)}$$

Given that, $u(0,y) = 3e^{-y} - e^{-5y}$

By putting $u = 3e^{-y} - e^{-5y}$ at $x=0$ in eqⁿ (8)

$$3e^{-y} - e^{-5y} = c_1 c_2 e^{(3-k)y}$$

Now equating, 1st term of left with Right side;

$$3e^{-y} = c_1 c_2 e^{(3-k)y}$$

$$\boxed{c_1 c_2 = 3} \quad \& \quad -1 = 3-k$$
$$\boxed{k=4}$$

again, equating, 2nd term of left with right side;

$$-e^{-5y} = c_1 c_2 e^{(3-k)y}$$

$$\boxed{c_1 c_2 = -1}, \quad -5 = 3-k$$
$$\boxed{k=8}$$

So required Solⁿ,

$$u = 3e^{\cancel{x}} \cdot e^{(3-4)y} + (-1) e^{\frac{8}{4}x} \cdot e^{(3-8)y}$$

$$u = 3e^{x} \cdot e^{-y} - e^{2x} \cdot e^{-5y}$$

$$u = 3e^{x-y} - e^{2x-5y} \quad \text{Ans.}$$

Wave Equation (one dimensional)

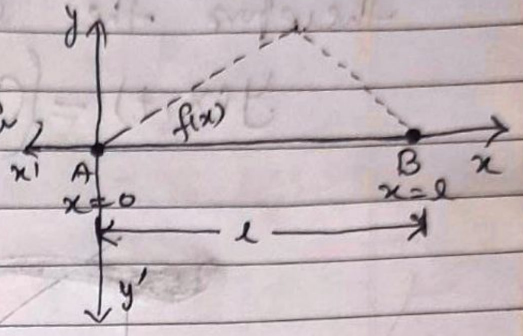
consider a tightly stretched elastic string of length l and fixed ends A and B. T = Tension which will be considered to be large as compared to the weight of the string so that effects of gravity are negligible.

then,

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ is

"wave eq^n," where, T = Tension of a string, m = mass per unit

$c^2 = T/m$ = constant.



subject to the condition;—

Boundary Condition $u(0,t) = u(l,t) = 0$
and Initial condition $u(x,0) = f(x)$ and
$\frac{\partial u}{\partial t} = 0$ at $t=0$, where l is length of string or wire.

Que ① A string is streched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by $y(x,t) = a\sin\left(\frac{\pi x}{l}\right).\cos\left(\frac{\pi c t}{l}\right)$

Sol^n The vibration of the string is given by

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \quad \text{—— ①}$$

we have to apply Boundary and initial Conditions one by one such that;

As the end of the string are fixed for all time then

$$y(0,t) = 0 \quad \text{—— ②}$$
$$\& \quad y(l,t) = 0 \quad \text{—— ③}$$

Since the transverse velocity of any point of the string is zero, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — (4)

Also, $y(x,0) = y = a\sin\frac{\pi x}{l}$ — (5)

Now we have to solve, (1) subject to the boundary conditions (2) and (3) and initial condition (4) and (5)

Since vibration of the string is periodic, therefore the sol^n of (1) is of the form

$$y(x,t) = \{C_1\cos px + C_2\sin px\}\cdot\{C_3\cos cpt + C_4\sin cpt\} \quad (6)$$

By (2) $y(0,t) = C_1[C_3\cos cpt + C_4\sin cpt]$ (7)

here, $C_3\cos cpt + C_4\sin cpt \neq 0$

so, $\boxed{C_1 = 0}$

put $C_1 = 0$ in eq^n (6) :—

$$y(x,t) = C_2\sin px\,\{C_3\cos cpt + C_4\sin cpt\} \quad (8)$$

Now by eq^n (4) :— $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

differentiating (8) w.r.t t :—

$$\frac{\partial y}{\partial t} = C_2\sin px\,\{-C_3\sin cpt\,(cp) + cp(C_4)\cos cpt\}$$

put $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2\sin px\,\{C_4 cp\} = 0$$

whence $C_2C_4cp = 0$

If $C_2 = 0$ in (8) will lead to the trivial sol^n $y(x,t) = 0$.

Only possiblity is that $C_4 = 0$

Thus (8) becomes $y(x,t) = C_2 C_3 \sin px \cos cpt$ —— (9)

Now, by eq^n (3), ~~y(l,t)=~~

$$y(l,t) = C_2 C_3 \sin pl \cdot \cos cpt = 0, \quad \forall t$$

Since, $C_2 \neq 0$ & $C_3 \neq 0$

we have $\sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

where $n =$ an integer

Hence, (1) reduces to $y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}$ —— (10)

Finally imposing the last condition (5) :—

$$y(x,0) = C_2 C_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c(0)}{l}$$

$$y(x,0) = C_2 C_3 \sin \frac{n\pi x}{l}$$

let, $C_2 C_3 = a$, $n = 1$

Hence the required sol^n is {By 10}

$$\boxed{y(x,t) = a \sin \frac{\pi x}{l} \cdot \cos \frac{\pi ct}{l}}$$ —— (11)

—x—