Unit-4

Random Variable

* **Sample Space →** A set S that consists of all possible outcomes of a random experiments are ~~is~~ called "Sample Space".

Example → If we have a coin then Sample Space $S = \{H, T\}$.

* **Random Variable →** A real valued function defined on a sample space is called "Random variable". It is called "Simply Variant". It is denoted by x or y.

Example:→ Suppose that a coin is tossed twice So that Sample space,

Sample Space-$S = \{HH, TT, HT, TH\}$ | If Favorable outcome is H

~~Assign a real Valued~~ no. $= \{2, 0, 1, 1\}$

So, Random Variable $X = \{0, 1, 2\}$

Random Variable

Discreate R.V.

I Probability Distribution

x	x_1	x_2	x_3	x_4
$P(x)$	p_1	p_2	p_3	p_4

II Distribution Function

$$F(x) = \begin{cases} 0 & , -\infty \leq x \leq 0 \\ 0 + f_1(x) & , 0 \leq x \leq 1 \\ 0 + f_1(x) + f_2(x) & , 1 \leq x \leq 2 \\ 0 + f_1(x) + f_2(x) + f_3(x) & , 2 \leq x \leq 3 \\ 0 + f_1(x) + f_2(x) + f_3(x) + f_4(x) = 1 & , 3 \leq x \leq \infty \end{cases}$$

III Density Function

(a) $f(x) \geq 0$

(b) $\sum_{\forall i} f(x_i) = 1$

Continuous R.V

I

Not possible to show in tabular form.

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II Distribution function

"Cummulative (or) Dist. fun^n"

$$\boxed{P(x) = \int_{-\infty}^{x} f(u)\,du}$$

III Density Function

(a) $f(x) \geq 0$

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

IV

If we want to find probability b/w 2 values in continuous R.V ($-\infty$ to ∞) then

$$\boxed{P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx}$$

Discrete R.V

(2)

Q. A Coin tossed 3 times. If X is R.V. giving the no. of Heads which arise, then Construct a table Showing the Prob. distr. fun, also find Density function

Soln

Sample space $\rightarrow$ $2^n = 2^3 = 8$

$S = \{HHH, HTT, THT, TTH, HHT, HTH, THH, TTT\}$

Total 8 Combinations.

$X = R.V. = \{0, 1, 2, 3\}$

then

$P(X=0) = 1/8$

$P(X=1) = 3/8$

$P(X=2) = 3/8$

$P(X=3) = 1/8$

Probability Distribution $\rightarrow$

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

Distribution fun $\rightarrow$

$$F(x) = \begin{cases} 0 & , -\infty \leq x \leq 0 \\ 0 + 1/8 & , 0 \leq x \leq 1 \\ 0 + 1/8 + 3/8 & , 1 \leq x \leq 2 \\ 0 + 1/8 + 3/8 + 3/8 & , 2 \leq x \leq 3 \\ 0 + 1/8 + 3/8 + 3/8 + 1/8 = 8/8 = 1 & , 3 \leq x < \infty \end{cases}$$

Density function $\longrightarrow$

(i) $f(x) \geq 0$

It is obvious that fn is $f(x) \geq 0$.

(ii) $\sum_{\forall i} f(x_i) = 1$

$\Rightarrow 1/8 + 3/8 + 3/8 + 1/8 = 1$

Then given fn is a Density fn.

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Q. The probability Density fn of a variate (R.V.) X is

X:	0	1	2	3	4	5	6
$P(X)$:	K	3K	5K	7K	9K	11K	13K

(a) Find $P(X<4)$, $P(X \geq 5)$, $P(3<x\leq 6)$

(b) What will be the minimum value of K so that $P(X\leq 2) > 0.3$

Soln

I^{st} we will find $K = ?$

for this,

$$\sum_{i=1}^{6} P(x_i) = K + 3K + 5K + 7K + 9K + 11K + 13K$$

Sum of all probabilities $= 1$

then, $1 = 49K$

then $\boxed{K = \frac{1}{49}}$

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(a) $P(x<4) = K+3K+5K+7K$

$= 16K$

since, $K = \frac{1}{49}$

then $= 16 \times \frac{1}{49} = \frac{16}{49}$

(2) $P(x \geq 5) = 11\cancel{K}+13K$

$= 24K$

$= 24 \times \frac{1}{49}$

$= \frac{24}{49}$

$P(3 < x \leq 6) = 9K + 11K + 13K$

$= 33K$

$= 33 \times \frac{1}{49}$

$= \frac{33}{49}$

(b) $P(x \leq 2) > 0.3$

$K + 3K + 5K > 0.3$

$9K > 0.3$

then $K > \frac{0.3}{9}$

$K > \frac{1}{30}$

Then the minimum value of K is $\Rightarrow$ $\boxed{K > 1/30}$.

Continuous R.V

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~~fwp~~.

Q.① Find the constant c such that the fu^n $f(x) = cx^2, \; 0 < x < 3$
$0, \;$ otherwise.

is ⓐ Density fu^n.
ⓑ compute $(1 < x < 2)$
ⓒ cummulative Dist. fu^n.

Soln → ("In continuous fu^n Question is always in the form of interval.")

ⓐ for Density fu^n. Ist we will find $c = ?$
for this,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$\boxed{\cancel{\int_{-\infty}^{\infty} f(x)\, dx}}$ Break ~~in~~ the fu^n in subintervals

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx = 1$$

$$0 + \int_0^3 cx^2\, dx + 0 = 1$$

$$c\left(\frac{x^3}{3}\right)_0^3 = 1$$

$$\frac{c}{3} \times 27 = 1$$

$$\boxed{c = 1/9}$$

since, $c = 1/9$ then it is clear that $f(x) > 0$
So given fu^n is Density fu^n.

(b) $P(1 < x < 2) = \int_1^2 cx^2\,dx$

$$= c\left(\frac{x^3}{3}\right)_1^2$$

$$= \frac{c}{3} \times (8-1)$$

$$= \frac{c}{3} \times 7$$

Put, $\boxed{c = 1/9}$ $= \frac{1}{9 \times 3} \times 7$

$$= \frac{7}{27}$$

(c) Cummulative Distribution →

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

Since, A/c to the Question

$$f(x) = cx^2, \quad 0 < x < 3$$
$$0, \quad \text{otherwise}$$

then $\boxed{x = 3}$ = upper limit

then, $F(x) = \int_{-\infty}^{x=3} f(u)\,du$

$$= \int_{-\infty}^{3} f(u)\,du$$

we break the fun in sub intervals → $= \int_{-\infty}^{0} + \int_{0}^{3}$

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$$= \int_{-\infty}^{0} 0\, dy + \int_{0}^{3} cx^2\, dx$$

$$= c\left(\frac{x^3}{3}\right)_0^3 = \frac{c}{3} \times 27$$

$$= \frac{1}{9 \times 3} \times 27 = 1$$

then $F(x) = 1$

Que Is the function defined as follows a density fu^n.

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(a) If so determine the Probability that the variate has this density will fall in the interval (1, 2)

(b) Also find the cummulative prob. ~~&~~ fu^n. $F(2)$.

Sol^n Ist we will prove that Density fu^n,

(i) It is clear that $f(x) \geq 0$

and (ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty}$$

$$ + \int_{0}^{\infty} e^{-x}\, dx = 1$$

$$= \left[-e^{-x}\right]_0^{\infty} = 1$$

(a) $P(1 \leq x \leq 2) = \int_1^2 e^{-x}\,dx$

$$= \left(-e^{-x}\right)_1^2$$

$$= e^{-2} - e^{-1}$$

$$= 0.233 \text{ //.}$$

(c) Cummulative Distribution

$$fu^n = \int_{-\infty}^{x} f(x)\,dx$$

$$F(2) = \int_{-\infty}^{2} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^2 e^{-x}\,dx$$

$$= \left(-e^{-x}\right)_0^2$$

$$= -e^{-2} + 1$$

$$= 0.865 \text{ //.}$$

*

E(X) Mathematical Expectation

Discrete R.V.	Continuous R.V

x	x_1	x_2	x_3	x_4
$P(x)$	P_1	P_2	P_3	P_4

$E(X) = x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4$

expectation

uncountable terms

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

It is also known as "Expectation".

Q(1) Discrete R.V.

Suppose that a game is to be played with a single die assumed fair. In this game a player wins \$ 20, if 2 turns up, \$ 40 if 4 turns up, loses \$ 30 if 6 turns up, while the player neither wins nor losses if any other face turns up. Find the expected sum of money to be won.

Solⁿ → Let X be the R.V. giving the amount of money won on any toss. The possible amounts won when the die turns up.

outcomes:	1	2	3	4	5	6
R.V.:	x_1	x_2	x_3	x_4	x_5	x_6
prob:	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$

S	1	2	3	4	5	6
x_i	0	+20	0	+40	0	-30
$f(x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(x) = 0 \times 1/6 + 20 \times 1/6 + 0 \times 1/6 + 40 \times 1/6 + 0 \times 1/6 + (-30) \times 1/6$$

$$\boxed{E(x) = 5}$$

Continuous R.V.

Q. The Density fn of a R.V. X is given by

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $E(X)$.

Soln

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} + \int_{0}^{2} + \int_{2}^{\infty}$$

$$= \int_0^2 x \times \frac{x^2}{2}\, dx = \left(\frac{x^3}{6}\right)_0^2$$

$$= \frac{8}{6} = \frac{4}{3}$$

V.V Imp

Q The probability Density function

$$p(x) = y_0 e^{-|x|}, \quad -\infty \text{ to } \infty$$

Then Prove that $\boxed{y_0 = 1/2}$

also find mean and variance.

Sol^n Mod function is Defined by

$$|x| = \begin{cases} +x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\boxed{** \; e^{-|x|} = \begin{cases} e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases}}$$

Given that $p(x)$ is Density fu^n;

$$\Rightarrow \int_{-\infty}^{\infty} p(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} y_0 e^{-|x|}\, dx = 1$$

$$\Rightarrow \cancel{\int_{-\infty}^{0} y_0 e^{+(x)} dx + \int_0^{\infty} y_0 e^{-(x)} dx = 1}$$

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Since, $|x|$ is an even fu^n and range is $-\infty$ to ∞ then By Definite Integral

$$p(x) = 2\int_0^{\infty} y_0 e^{-x}\, dx = 1$$

$$\Rightarrow 2y_0 \left[-e^{-x}\right]_0^{\infty} = 1$$

$$\Rightarrow 2y_0 [0+1] = 1$$

$$\Rightarrow 2y_0 = 1$$

then $\boxed{y_0 = 1/2}$

Now, we have to find mean (μ) and Variance

By Formula

$$\boxed{\text{mean} = E(x) = \int_{-\infty}^{\infty} x\, p(x)\, dx}$$

$$E(x) = \int_{-\infty}^{0} y_0\, x e^{x}\, dx + \int_0^{\infty} y_0\, x e^{-x}\, dx$$

$$= y_0 \left[x e^{x} - e^{x}\right]_{-\infty}^{0} + y_0 \left[x(-e^{-x}) - (e^{-x})\right]_0^{\infty}$$

$$= y_0 [-1+1] = 0$$

$$\boxed{E(x) = 0}$$

Note:- $\int_{-\infty}^{\infty} \underbrace{x}_{\text{odd } fu^n}\, \underbrace{p(x)}_{\text{even } fu^n}\, dx = 0 \times e = 0$

$= 0$

Variance $= E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, p(x)\, dx$

$= \int_{-\infty}^{0} (x-\mu)^2 y_0 e^{x} dx + \int_{0}^{\infty} (x-\mu)^2 y_0 e^{-x} dx$

$\boxed{\mu = 0}$

$= \int_{-\infty}^{0} x^2 y_0 e^{+x} dx + \int_{0}^{\infty} x^2 y_0 e^{-x} dx$

$= y_0 \Big[\left(x^2 e^{x} - 2x e^{x} + 2e^{x} \right)_{-\infty}^{0} + \left\{ x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x}) \right\}_{0}^{\infty} \Big]$

Var. $= 2$

Put $y_0 = 1/2$

then, Std Deviation

$\sigma = \sqrt{2}$

—x—

Q. From an urn containing 3 red, 2 white balls, a man is to draw 2 balls at random without replacement being promised Rs. 20, for each red ball, and Rs. 10 for each white one. Find his expectation.

Sol^n

3 red 2 white = 5 Balls

S =	RR	WW	RW
X →	40/-	20/-	30/-
$P(x)$ =	?	?	?

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$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

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$$P(X=40) = \frac{^3C_2}{^5C_2} = \frac{\frac{3!}{1!\,2!}}{\frac{5!}{2!\,3!}}$$

$$= \frac{3}{10}$$

★★ This is Permutation / Combination, we use this formula when we have different color balls. But If we have Identical ball then we use prob. $\frac{3 \to \text{red}}{5 \to \text{Total}}$.

$$P(X=20) = \frac{^2C_2}{^5C_2} = \frac{\frac{2!}{0!\,2!}}{\frac{5!}{3!\,2!}}$$

$$= \frac{1}{10}$$

$$P(X=30) = \frac{^3C_1 \cdot {^2C_1}}{^5C_2}$$

$$= \frac{\frac{3!}{2!\,1!} \times \frac{2!}{1!}}{\frac{5!}{3!\,2!}}$$

$$= \frac{3}{5}$$

S	RR	WW	RW
X	40	20	30
P(X)	3/10	1/10	3/5

then expectation

$$E(X) = 40 \times \frac{3}{10} + 20 \times \frac{1}{10} + 30 \times \frac{3}{5}$$

$$= 32$$

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Q A Continuous Distribution of a R.V. x in the range $(-3,3)$ is defined as

$$f(x) = \frac{1}{16}(3+x)^2, \quad -3 \le x \le -1$$
$$= \frac{1}{16}(6-2x^2), \quad -1 \le x \le 1$$
$$= \frac{1}{16}(3-x^2), \quad 1 \le x \le 3$$

Verify that the area under the curve is unity. Show that the mean is 0.

Sol^n Area under the curve:—

$$= \frac{1}{16}\int_{-3}^{-1}(x+3)^2\,dx + \frac{1}{16}\int_{-1}^{1} \cancel{(2-6x)^2}\; 6-2x^2\,dx$$
$$+ \frac{1}{16}\int_{1}^{3}(3-x^2)^2\,dx$$

$$= \frac{1}{16}\times 16 = 1$$

Mean $\mu = \frac{1}{16}\int_{-3}^{-1} x(x+3)^2\,dx$
$$+ \frac{1}{16}\int_{-1}^{1} x(2-6x)^2\,dx$$
$$+ \frac{1}{16}\int_{1}^{3} x(3-1)^2\,dx$$

$= 0$

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Q. X is a random variable giving time (in min) during which a certain electrical equipment is used at max. load in a specified time period if the probability distribution function is given by,

$$f(x) = \frac{x}{(1500)^2}, \quad 0 \le x \le 1500$$

$$= \frac{-(x-3000)}{(1500)^2}, \quad 1500 \le x \le 3000$$

$$= 0, \quad \text{otherwise}$$

Find the expected value of x.

Soln: $E(x) = \int_0^{1500} x \times \frac{x}{(1500)^2} dx - \int_{1500}^{3000} x \times \frac{x-3000}{(1500)^2} dx$

$$= \frac{1}{(1500)^2} \left[\frac{x^3}{3}\right]_0^{1500} - \frac{1}{(1500)^2} \left\{\frac{x^3}{3} - 1500x^2\right]_{1500}^{3000}$$

$$= 500 - (-1,000)$$

$$= 1500 \; //$$

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The End