Application of L.T. - 2nd form

Que ① $ty'' + 2y' + ty = \sin t$

when $y(0) = 1$.

Sol^n Taking L.T. on both sides;

$$\Rightarrow -\frac{d}{ds}\left[L(y'')\right] + 2L(y') + t\,L(y) = L(\sin t)$$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y} - sy(0) - y'(0)\right] + 2\left[s\bar{y} - y(0)\right]$$

$$-\frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

{ Given, $y(0)=1$ & $y'(0) = A$ (let)

$$\Rightarrow -\frac{d}{ds}\left[\underset{\mathrm{I}}{s^2}\underset{\mathrm{II}}{\bar{y}} - s - A\right] + 2\left[s\bar{y} - y(0)\right] - \frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

$$\Rightarrow -\left[s^2\frac{d\bar{y}}{ds} + \bar{y}(2s) - 1\right] + 2s\bar{y} - 2 - \frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

$$\Rightarrow \frac{d\bar{y}}{ds}\left[-s^2 - 1\right] + \bar{y}\left[\cancel{-2s} + \cancel{2s}\right] + 1 - 2 = \frac{1}{s^2+1}$$

$$\Rightarrow \frac{d\bar{y}}{ds}(-s^2 - 1) - 1 = \frac{1}{s^2+1}$$

$$\Rightarrow -\frac{d\bar{y}}{ds} = \frac{1}{(s^2+1)^2} + \frac{1}{(s^2+1)} \quad \text{①}$$

$$** \quad -\frac{d\bar{y}}{ds} = ?$$

We know that multiplication by t :-

$$L[t f(t)] = -\frac{d}{ds}\bar{f}(s)$$

or $\rightarrow$ $t f(t) = L^{-1}\left(\frac{-d\bar{f}(s)}{ds}\right)$

So we will put $-\frac{d\bar{y}}{ds} = t y(t)$ — (2)

By (1) $\rightarrow$ Taking Inverse L.T :-

$$L^{-1}\left[\frac{-d\bar{y}}{ds}\right] = L^{-1}\frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$$

by (2) $\rightarrow$ $t y(t) = L^{-1}\frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$

Formula $\Longrightarrow$ $\boxed{L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\sin at - at\cos at}$

So,

$$t y(t) = \left\{\frac{1}{2}(\sin t - t\cos t)\right\} + \sin t$$

$$t y(t) = \frac{3}{2}\sin t - \frac{t\cos t}{2}$$

therefore, $y(t) = \frac{1}{2t}\left[3\sin t - t\cos t\right]$

Imp

Que, $ty'' + (1-2t)y' - 2y = 0$

when $y'(0) = 1$, $y(0) = 2$.

Sol^n $\Rightarrow$ $ty'' + (1-2t)y' - 2y = 0$

Taking L.T. on both sides;

$\Rightarrow -\frac{d}{ds}L(y'') + L(y') + 2\frac{d}{ds}L(y') - 2L(y) = 0$

$\Rightarrow \frac{-d}{ds}[s^2\bar{y} - s(y(0)) - y'(0)] + [s\bar{y} - y(0)] +$

$\frac{2d}{ds}[s\bar{y} - y(0)] - 2\bar{y} = 0$

$\Rightarrow \frac{-d}{ds}[\underset{I}{s^2}\underset{II}{\bar{y}} - 2s - 1] + (s\bar{y} - 2) + 2\frac{d}{ds}[\underset{I}{s}\underset{II}{\bar{y}} - 2]$

$- 2\bar{y} = 0$

$\Rightarrow -[s^2\frac{d\bar{y}}{ds} + 2s\bar{y} - 2] + s\bar{y} - 2 + 2[s\frac{d\bar{y}}{ds} + \bar{y}]$

$- 2\bar{y} = 0$

$\Rightarrow \frac{d\bar{y}}{ds}[-s^2 + 2s] + \bar{y}[-2s + s + \cancel{2} - \cancel{2}] + \cancel{2} - \cancel{2} = 0$

$\Rightarrow (2s - s^2)\frac{d\bar{y}}{ds} - s\bar{y} = 0$

$\Rightarrow \frac{d\bar{y}}{ds} = \frac{s\bar{y}}{2s - s^2}$

$\Rightarrow \frac{d\bar{y}}{ds} = \frac{\bar{y}}{2-s}$ This is form of Linear diff^n eq^n of Ist order

* $\frac{dy}{dx} + Py = Q$

I.F. $\rightarrow$ $e^{\int P dx}$

Sol^n $\rightarrow$ $y(\text{I.F}) = \int Q(\text{I.F}) + c$

$\Rightarrow \frac{d\bar{y}}{ds} - \frac{1}{2-s}\bar{y} = 0$

or $\Rightarrow$ $\frac{d\bar{y}}{ds} + \frac{1}{s-2}\bar{y} = 0$

$\Rightarrow$ I.F. $\rightarrow$ $e^{\int \frac{1}{s-2}ds} \Rightarrow e^{\log(s-2)} = (s-2)$

Sol^n $\bar{y} \cdot (s-2) = \int 0(s-2) + c$

or, $\bar{y}(s-2) = c$

$\bar{y} = \frac{c}{s-2}$ Taking Inverse L.T.

$\boxed{y = ce^{2t}}$ —— ①

Now we have to find $c = ?$

Given, $y(0) = 2$ putting this value in ①

$2 = ce^{0} \Rightarrow \boxed{c=2}$ Put $c=2$ in ①

$\boxed{y = 2e^{2t}}$ — Ans..