Solutions:

Unit - 2

Partial differential Equations.

Ans:- 1 Two marks Questions:-

(i) Given $z = e^{my}\,\phi(x-y)$ —— ①

Diff w.r.t. 'x'

$$\frac{\partial z}{\partial x} = \phi'(x-y)\,e^{my} \quad \text{—— ②}$$

Diff ① w.r.t. 'y'

$$\frac{\partial z}{\partial y} = me^{my}\,\phi(x-y) - e^{my}\,\phi'(x-y) \quad \text{—— ③}$$

from ①, ② & ③

$$\frac{\partial z}{\partial y} = mz - \frac{\partial z}{\partial x}$$

$$\Rightarrow \boxed{p + q = mz}$$

(ii) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— ①

Diff w.r.t. x

$$\frac{\partial z}{\partial x} = 0 + 2f'(1/x + \log y)\cdot(-1/x^2)$$

$$= -\frac{2}{x^2}\,f'(1/x + \log y) \quad \text{—— ②}$$

Diff ① w.r.t. 'y'

$$\frac{\partial z}{\partial y} = 2y + 2f'(1/x + \log y)\cdot 1/y$$

$$y\frac{\partial z}{\partial y} = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right) \quad \text{—— ③}$$

from ② & ③

$$y\frac{\partial z}{\partial y} = 2y^2 - x^2\frac{\partial z}{\partial x}$$

$$\Rightarrow \boxed{x^2 p + y\cdot q = 2y^2} \longrightarrow \text{Ans}$$

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(iii) Lagrange's Linear Equation :- The Partial Equation of the type $Pp+Qq=R$ is the standard form of the linear partial differential equation of order one and is called Lagrange's linear eq. where P, Q, R are functions of x, y, z.

(iv) Given eq^n: $xp+yq=3z$.

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}.$$

The first two fractions gives

$$\int \frac{dx}{x} = \int \frac{dy}{y} + \log c_1$$

$$\Rightarrow \log x - \log y = \log c_1$$

$$\boxed{c_1 = x/y}$$

Again the second and third fractions gives

$$\frac{dy}{y} = \frac{dz}{3z} \Rightarrow 3\int \frac{dy}{y} = \int \frac{dz}{z} + \log c_2$$

$$\Rightarrow 3\log y - \log z = \log c_2$$

$$\boxed{c_2 = y^3/z}$$

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Ans 1

(v) Given equation in symbolic form is

$$D^3 - 4D^2 D' + 4D'^2 D = 0$$

Auxiliary Equation:

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m = 0, 2, 2.$$

$\therefore$ Here the complete solⁿ is

$$z = f_1(y) + f_2(y+2x) + x f_3(y+2x)$$

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Q 2 (i) We have given.

$$\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0 \quad \text{--- (1)}$$

We can be solved eqn (1) by direct integration method.

So integrating eqn (1) w.r.t 'x' keeping 'y' fixed.

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} + \frac{18}{2}x^2y^2 - \frac{1}{2}\cos(2x-y) = c_1(y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = c_1(y) \quad \text{--- (2)}$$

Again integrating eqn (2) w.r.t. 'x' keeping 'y' fixed.

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = \int c_1(y)\,dx + c_2(y)$$

$$= c_1(y)x + c_2(y) \quad \text{--- (3)}$$

Finally integrating eqn (3) w.r.t 'y' keeping 'x' fixed we have,

$$\Rightarrow z + x^3y^3 - \frac{1}{4}\cos(2x-y) = \cancel{\int c_1(y)x\,dx + \int c_2(y)\,dx}$$

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$$= \int c_1(y)\, x\, dy + \int c_2(y)\, dy + c_3(x).$$

$$= x\int c_1(y)\, dy + \int c_2(y)\, dy + c_3(x)$$

$$= x\,\phi_1(y) + \phi_2(y) + c_3(x).$$

where $\int c_1(y)dy = \phi_1(y)$ and $\int c_2(y)dy = \phi_2(y)$

i.e. $z = \frac{1}{4}\cos(2x-y) - x^3y^3$
$\quad + x\,\phi_1(y) + \phi_2(y) + c_3(x)$

which is required soln P.d.

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Q 2

(ii) we have given $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$ —— (1)

∴ Now the above question we can be solved by direct integration method.

So integrating eqn (1) w.r.t. 'x' keeping y fixed we have,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\sin(2x+3y) + c_1(y) \quad \text{—— (2)}$$

Again integrating eqn (2) w.r.t. 'x' keeping 'y' fixed we have,

$$\frac{\partial z}{\partial y} = \frac{1}{4}(-\cos(2x+3y)) + \int c_1(y)\,dx + c_2(y)$$

$$= -\frac{1}{4}\cos(2x+3y) + c_1(y)x + c_2(y) \quad \text{—— (3)}$$

finally integrating eqn (3) w.r.t. 'y', we have, keeping (x fixed), we have

$$z = -\frac{1}{12}\sin(2x+3y) + \int x\,c_1(y)\,dy + \int c_2(y)\,dy + c_3(x)$$

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2(ii)

$\Rightarrow z = -\frac{1}{12} \sin(2x+3y) + x\, \phi_1(y) + \phi_2(y) + c_3(x)$ — Ans

where $\int c_1(y)\,dy = \phi_1(y)$

and $\int c_2(y)\,dy = \phi_2(y)$.

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Q (2) (iii) We have given $z = f(x^2+y^2, z-xy)$.

Let $x^2+y^2 = u$, and $z-xy = v$,

so that $f(u,v) = 0$.

Differentiating partially w.r.t 'x' and 'y'

we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0.$$

or $\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y+p) = 0$ —— (i)

and $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0.$

or $\frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x+q) = 0.$ —— (ii)

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from eqn (i) and (ii)

we have $\begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix} = 0$

or $xq - yp = x^2 - y^2$

which is the required p.d.e.

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Q 2 (iv) We have given: $p - q = \log(x+y)$

We have by Lagrange's linear p.d.e.

i.e. $Pp + Qq = R$.

$\therefore$ A.E. $= \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad \text{--- (1)}$$

from 1st two fraction of eq

$dx = -dy$

On integrating we have,

$x = -y + c_1$

$\Rightarrow x + y = c_1$.

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and taking from 1st 3rd fraction of eqn ① we have

$$dx = \frac{dz}{\log(x+y)}$$

$\Rightarrow dx \log(x+y) = dz.$

On integrating we have,

$\Rightarrow \log c_1 \, dx = dz \quad (\because x+y = c_1)$

integrating we have

$x \log c_1 = z + c_2.$

$\Rightarrow c_2 = x \log c_1 - z.$

$\therefore$ The complete soln $f(c_1, c_2) = 0$

$\Rightarrow f(x+y, \; x\log c_1 - z) = 0$

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Q2

(v) We have given $p\tan x + q\tan y = \tan z$.

Here is the Lagrange's linear p.d.e.

Comparing $Pp + Qq = R$.

$\therefore$ subsidiary / A.E is: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$. ——— (1)

from the 1st two fraction of eqn (1), we have

$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \cot x\, dx = \cot y\, dy$

integrating $\Rightarrow \log\sin x = \log\sin y + \log c_1$

$\Rightarrow \frac{\sin x}{\sin y} = c_1$

Again from last two fraction of eqn (1)

we have $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

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$\Rightarrow \cot y\, dy = \cot z\, dz$

On integrating we have

$\log \sin y = \log \sin z + \log c_2$

$\Rightarrow \dfrac{\sin y}{\sin z} = c_2$

$\therefore$ The complete sol^n: $f(c_1, c_2) = 0$

$\Rightarrow f\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

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Qu. 3. 8 marks

(i) Solve :- $x^2(y-z)p+y^2(z-x)q = z^2(x-y)$.

Sol^n :- Here the Lagrange's subsidiary equ^n are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \text{--- (1)}$$

Taking multiplier $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ each fraction of equ^n (1)

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{(y-z)+(z-x)+(x-y)}$$

$$= \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

$$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -c_1$$

$$\Rightarrow \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1} \quad \text{--- (2)}$$

Again, taking multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ each fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

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$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating

$\log(xyz) = \log c_2$

$\therefore \boxed{xyz = c_2}$ ——— (3)

Hence, from equ^n (2) and (3) the required solution of the given equation is

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$$

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(ii) Solve:- $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

Sol^n:- The given diff. equ^n

$Px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

It's subsidiary equ^n

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \text{--- (1)}$$

Taking 2^{nd} and 3^{rd} fraction of (1)

$$\frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating

$$\log y = \log z + \log c_1$$

$$\boxed{\frac{y}{z} = c_1} \quad \text{--- (2)}$$

Taking 1^{st} and 3^{rd} fraction of (1)

$$\frac{dx}{x(z-zc_1^2z^2)} = \frac{dz}{z(z-c_1^2z^2-2x^3)}$$

$$\frac{dx}{x(1-2c_1^2z)} = \frac{dz}{z-c_1^2z^2-2x^3}$$

or $(1-2c_1^2z) \frac{dz}{dx} = \frac{z-c_1^2z^2}{x} - 2x^2$

or $\frac{du}{dx} = \frac{u}{x} - 2x^2$ on putting $\boxed{u = z-c_1^2z^2}$

$$\frac{du}{dx} - \frac{u}{x} = -2x^2 \quad \text{—— (3)}$$

which is linear eqn in u

$\therefore$ It's I.F. $= e^{\int(-\frac{1}{x})dx} = e^{-\log x} = \frac{1}{x}$

Hence solution of (3) is

$$u \cdot \frac{1}{x} = \int \frac{1}{x} \cdot (-2x^2)\,dx + C_2$$

$$\frac{u}{x} = -x^2 + C_2$$

or $\dfrac{z - C_1^2 z^2}{x} + x^2 = C_2$

or $\dfrac{z - y^2}{x} + x^2 = C_2$

or $\boxed{\dfrac{z}{x} - \dfrac{y^2}{x} + x^2 = C_2}$. —— (4)

Hence the Required soln

$$\phi\left(\frac{y}{z}, \frac{z}{x} - \frac{y^2}{x} + x^2\right) = 0.$$

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Q4.3 (iii) **Solve:-** $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Sol^n:- Here the subsidiary equ^n

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

Using the multiplier systems

1, -1, 0 ; 0, 1, -1 and -1, 0, 1, we have

$$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}$$

taking 1st, two fraction

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

$$\Rightarrow \log(x-y)=\log(y-z)+\log c_1$$

$$\therefore \boxed{\frac{x-y}{y-z}=c_1}$$

similarly, taking last. two fraction

$$\boxed{\frac{z-x}{y-z}=c_2}$$

$\therefore$ The Required sol^n

~~$\phi\left(\frac{y-z}{\ }\right.$~~

$$\phi\left(\frac{x-y}{y-z},\frac{z-x}{y-z}\right)=0$$

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Solve (IV) $(x^2-y^2-z^2)p+2xyq=2xz$

Here the subsidiary equations are

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$$

from last two fraction

$$\Rightarrow \quad \frac{dy}{y}=\frac{dz}{z}$$

$$\Rightarrow \quad \log y-\log z=\log c_1$$

$$\therefore \quad \boxed{c_1=y/z}$$

Using multipliers. x, y, z, we have

$$\frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}=\frac{dz}{2xz}$$

$$\Rightarrow \frac{2x\,dx+2y\,dy+2z\,dz}{(x^2+y^2+z^2)}=\frac{dz}{z}$$

$$\Rightarrow \quad \log(x^2+y^2+z^2)-\log z=\log c_2$$

$$\Rightarrow \quad c_2=\frac{x^2+y^2+z^2}{z}$$

The required solution. is

$$x^2+y^2+z^2=z\,f(y/z)$$

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Qu.5. Solve:-

(v) $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$.

Solⁿ:- Its subsidiary eqⁿ

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad -(1)$$

Taking multipliers, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ in (1)

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2-z^2)+(z^2-x^2)+(x^2-y^2)}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\log(xyz) = \log c_1$$

$$\boxed{xyz = c_1}$$

Taking multi x, y, z in (1)

$$\frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore x\,dx + y\,dy + z\,dz = 0$$

$$\boxed{x^2 + y^2 + z^2 = c_2}$$

$\therefore$ Required solⁿ

$$\phi(xyz,\ x^2+y^2+z^2) = 0.$$

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Q4.6 Solve:- $(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}$
$= ly-mx$

Solⁿ:- Subsidiary eqⁿ are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}.$$

Using x, y, z as multipliers,

each fraction $= \frac{x\,dx+y\,dy+z\,dz}{0}$

$\therefore x\,dx+y\,dy+z\,dz=0.$

Integrating

$$\boxed{x^2+y^2+z^2=c_1}$$

Again using l, m, n as multipliers.

each fraction

$$= \frac{l\,dx+m\,dy+n\,dz}{0}$$

$\therefore l\,dx+m\,dy+n\,dz=0$

Integrating

$$\boxed{lx+my+nz=c_2}$$

Hence Required solution is

$$\boxed{\phi(lx+my+nz,\ x^2+y^2+z^2)=0}$$

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Question:- Solve $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

Solution:- Writing given equation in symbolic form

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

$\therefore$ A.E is

$$4D^2 - 4DD' + D'^2 = 0$$

$$\Rightarrow 4m^2 - 4m + 1 = 0$$

$$\Rightarrow (2m-1)^2 = 0$$

$$m = 1/2, 1/2$$

$$\therefore C.F = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right)$$

Now. $P.I = \frac{1}{4D^2 - 4DD' + D'^2} 16\log(x+2y)$

$$= 16\frac{1}{(2D-D')^2}\log(x+2y)$$

$$= 16\frac{1}{(2D-D')(2D-D')}\log(x+2y)$$

$$= 16\frac{1}{2\times 2\left(D-\frac{1}{2}D'\right)\left(D-\frac{1}{2}D'\right)}\log(x+2y)$$

$$= 4\times\frac{1}{\left(D-\frac{1}{2}D'\right)}\left[\frac{1}{\left(D-\frac{1}{2}D'\right)}\log(x+2y)\right]$$

$$= \frac{4}{\left(D-\frac{1}{2}D'\right)}\int \log\left[x+2\left(c-\frac{1}{2}x\right)\right]dx$$

$$= \frac{4}{\left(D-\frac{1}{2}D'\right)}\int \log[\cancel{x}+2c-\cancel{x}]\,dx \qquad \left[\because m = \frac{1}{2}\right]$$

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$$P.I = \frac{4}{(D - \frac{1}{2}D')} \int \log 2c \, dx$$

$$= 4 \times \frac{1}{(D - \frac{1}{2}D')} x \log 2c \quad , \text{Put } c = y + \frac{1}{2}x$$

$$= 4 \frac{1}{(D - \frac{1}{2}D')} x \log 2 (y + \frac{1}{2}x)$$

$$P.I = 4 \frac{1}{(D - \frac{1}{2}D')} x \log (x + 2y)$$

$$= 4 \int x \log [x + 2(c - \frac{1}{2}x)] \, dx$$

$$= 4 \int x \log [x + 2c - x] \, dx$$

$$= 4 \int x \log 2c \, dx$$

$$= \frac{4x^2}{2} \log 2c \quad , \text{put } c = y + \frac{1}{2}x$$

$$= \frac{4x^2}{2} \log 2 (y + \frac{1}{2}x)$$

$$P.I = 2x^2 \log (x + 2y)$$

$$\therefore Z = C.F + P.I$$

$$z = f_1 (y + \frac{x}{2}) + x f_2 (y + \frac{x}{2}) \quad \underline{\text{Ans}}$$

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Question:- Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution:- given

$$(D^2 - DD' - 2D'^2)z = (y-1)e^x$$

$\therefore$ A.E is

$$D^2 - DD' - 2D'^2 = 0$$

$$\Rightarrow m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$\Rightarrow m = 2, -1$$

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$\therefore$ $C.F = f_1(y+2x) + f_2(y-x)$

Now $P.I = \dfrac{1}{D^2 - DD' - 2D'^2}(y-1)e^x$

$P.I = \dfrac{1}{(D-2D')(D+D')}(y-1)e^x$

$= \dfrac{1}{(D-2D')}\left[\dfrac{1}{D+D'}(y-1)e^x\right]$

$= \dfrac{1}{(D-2D')}\left[\int (c+x-1)e^x dx\right]$ $[\because m=-1]$

$= \dfrac{1}{(D-2D')}\left[(c+x-1)e^x - 1.e^x\right]$

Put $c = y-x$

$P.I = \dfrac{1}{(D-2D')}\left[(y-1)e^x - e^x\right]$

Again

$P.I = \int \left[(c-2x-1)e^x - e^x\right] dx$ $[\because m=2]$

$= \int (c-2x-1)e^x dx - \int e^x dx$

$= \left[(c-2x-1)e^x - (-2)e^x\right] - e^x$

Put $c = y+2x$

$= (y-1)e^x + 2e^x - e^x$

$P.I = ye^x - \cancel{e^x} + \cancel{e^x} = ye^x$

$\therefore$ $z = C.F + P.I$

$\Rightarrow z = f_1(y+2x) + f_2(y-x) + ye^x$ Ans

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Question:- Solve $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$

Solution:- given equation

$(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$ is a Non-Homogeneous PDE.

Now $f(D,D') = D^2+2DD'+D'^2-2D-2D'$

$= (D+D')^2 - 2(D+D')$

$f(D,D') = (D+D')(D+D'-2)$

$\therefore$ $CF = e^{c_1 x}\phi_1(y+m_1x) + e^{c_2 x}\phi_2(y+m_2x)$

[Comparing factors with $D-mD'-c$]

$CF = e^{0x}\phi_1(y-x) + e^{2x}\phi_2(y-x)$

$\Rightarrow CF = \phi_1(y-x) + e^{2x}\phi_2(y-x)$.

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Now $P.I = \frac{1}{D^2+2DD'+D'^2-2D-2D'} \sin(x+2y)$

Put $D^2=-1^2=-1,\ DD'=-1\times2=-2,\ D'^2=-2^2=-4$

$$P.I = \frac{1}{-1-4-4-2D-2D'} \sin(x+2y)$$

$$= -\frac{1}{9+(2D+2D')} \sin(x+2y)$$

$$= -\frac{[9-(2D+2D')]}{81-(2D+2D')^2} \sin(x+2y)$$

$$= -\frac{[9-(2D+2D')]}{81-4D^2-4D'^2-8DD'} \sin(x+2y)$$

Put $D^2=-1^2=-1,\ D'^2=-2^2=-4,\ DD'=-1\times2=-2$

$$= -\frac{[9-2D-2D']}{81+4+16+16} \sin(x+2y)$$

$$= \frac{-1}{117}\left[9\sin(x+2y)+2D\sin(x+2y)+2D'\sin(x+2y)\right]$$

$$= \frac{-1}{117}\left[9\sin(x+2y)+2\cos(x+2y)+4\cos(x+2y)\right]$$

$$= \frac{-1}{117}\left[-9\sin(x+2y)+6\cos(x+2y)\right]$$

$$= \frac{-3}{-117}\left[3\sin(x+2y)-2\cos(x+2y)\right]$$

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$$P.I = \frac{1}{39}\left[3\sin(x+2y) - 2\cos(x+2y)\right]$$

Hence $z = CF + PI$

$$\Rightarrow z = \phi_1(y-x) + e^{2x}\phi_2(y-x) + \frac{1}{39}\left[3\sin(x+2y) - 2\cos(x+2y)\right]$$

Ans

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Question:- Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

Solution:- writing given equation in symbolic form

$$(D^2+DD'-6D'^2)z = \cos(2x+y)$$

$\therefore$ A.E is

$$(D^2+DD'-6D'^2)=0$$

$$\Rightarrow m^2+m-6=0$$

$$\Rightarrow m=2,-3$$

$$\therefore C.F = f_1(y+2x)+f_2(y-3x)$$

Now P.I $= \frac{1}{D^2+DD'-6D'^2}\cos(2x+y)$

Put $D^2=-2^2=-4$, $DD'=-2\times1=-2$, $D'^2=-1^2=-1$

$$P.I = \frac{1}{-4-2+6}\cos(2x+y)$$

$$P.I = \frac{1}{0}\cos(2x+y)$$

which is failure of second case. we will use general case (i.e case-IV).

$$P.I = \frac{1}{D^2+DD'-6D'^2}\cos(2x+y)$$

$$P.I = \frac{1}{(D-2D')(D+3D')}\cos(2x+y)$$

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$$P.I = \frac{1}{(D-2D')}\left[\frac{1}{(D+3D')}\cos(2x+y)\right]$$

$$\left[\because \frac{1}{D-mD'} f(x,y) = \int f(x, c-mx)\,dx\right]$$

$$= \frac{1}{(D-2D')}\left[\int \cos(2x+c+3x)\,dx\right]$$

$[\because m=-3]$

$$= \frac{1}{(D-2D')}\int \cos(5x+c)\,dx$$

$$= \frac{1}{(D-2D')}\left[\frac{\sin(5x+c)}{5}\right], \text{ Put } c = y+mx$$

$\Rightarrow c = y-3x$

$$= \frac{1}{D-2D'}\frac{\sin(2x+y)}{5}$$

$$= \frac{1}{5}\frac{1}{(D-2D')}\sin(2x+y)$$

$$= \frac{1}{5}\int \sin(2x+c-2x)\,dx$$

$$= \frac{1}{5}\int \sin c\,dx$$

$$= \frac{1}{5}x\sin c, \text{ Put } c = y+2x$$

$$P.I = \frac{1}{5}x\sin(2x+y)$$

$$\therefore z = C.F + P.I$$

$$\Rightarrow z = f_1(y+2x)+f_2(y-3x)+\frac{x\sin(2x+y)}{5} \text{ Ans}$$

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Question:- Solve $(D^2+3DD'+2D'^2)z = 24xy$

Solution:- given

$$(D^2+3DD'+2D'^2)z = 24xy$$

$\therefore$ A.E is

$$D^2+3DD'+2D'^2 = 0$$

$$\Rightarrow m^2+3m+2 = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m = -1, -2$$

$$\therefore CF = f_1(y-x) + f_2(y-2x)$$

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Now

$$P.I = \frac{1}{D^2+3DD'+2D'^2} 24xy$$

$$= \frac{1}{D^2\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]} 24xy$$

$$= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1} 24xy$$

Now by Binomial series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{\lfloor 2}x^2+ \cdots$$

$$P.I = \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^2+\cdots\right](24xy)$$

[Since $f(x,y) = 24xy$. Hence it can be diff w.r.to x and y only once]

$$\Rightarrow P.I = \frac{1}{D^2}\left[1-\frac{3D'}{D}\right] 24xy$$

$$= \frac{1}{D^2}\left[24xy - \frac{3}{D}(D' 24xy)\right]$$

$$= \frac{1}{D^2}\left[24xy - \frac{3}{D}\frac{\partial}{\partial y}(24xy)\right] \quad \left[\because D' = \frac{\partial}{\partial y}\right]$$

$$= \frac{1}{D^2}\left[24xy - \frac{3}{D}\times 24x\right]$$

$$= \frac{1}{D^2}\left[24xy - 72\int x\,dx\right] \quad \left[\because \frac{1}{D} = \int dx\right]$$

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$$= \frac{1}{D^2}\left[24xy - 72\times \cancel{2xy}\cdot\frac{x^2}{2}\right]$$

$$= \frac{1}{D^2}\left[24xy - 36x^2\right]$$ 16

$$= \frac{1}{D}\int (24xy - 36x^2)\,dx$$

$$= \frac{1}{D}\left[\frac{24x^2y}{2} - \frac{36x^3}{3}\right]$$

$$= \frac{1}{D}\left[12x^2y - 12x^3\right]$$

$$= \int (12x^2y - 12x^3)\,dx$$

$$= \frac{12x^3y}{3} - \frac{12x^4}{4}$$

$$P.I = 4x^3y - 3x^4$$

$$z = C.F + P.I$$

$$z = f_1(y-x) + f_2(y-2x) + 4x^3y - 3x^4$$

(11)

Question — Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, for which

$\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$

Also $z = 0$, when y is an odd multiple of $\pi/2$.

Solution — given

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \quad \text{--- (1)}$$

Integrating eqn (1) w.r.to x, keeping y constant

$$\frac{\partial z}{\partial y} = -\cos x \cdot \sin y + f(y) \quad \text{--- (2)}$$

Now given $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$

$\therefore$ Put $x = 0$ in eqn (2)

$$-2\sin y = -\cos(0) \cdot \sin y + f(y)$$

$$\Rightarrow f(y) = -\sin y$$

Put this value in eqn (2)

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$$

Now Integrating w.r.to y, keeping x constant.

$$z = +\cos x \cos y + \cos y + g(x) \quad \text{--- (3)}$$

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Again given $z = 0$, when y is an odd multiple of $\pi/2$.

$\therefore$ Put $y = \pi/2$ in eqn (3)

$$0 = \cos \pi/2 \cos x + \cos \pi/2 + g(x)$$

$$0 = 0 + 0 + g(x) \Rightarrow \boxed{g(x) = 0}$$

Put this value in eqn (3)

$$z = \cos x \cos y + \cos y$$

$$\Rightarrow z = (1 + \cos x) \cos y.$$ Ans

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$\underline{\text{Sol}^n}$ (xiii) Given eq^n $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ — ①

and $u(0,y) = 8e^{-3y}$ — ②

Assume the sol^n

$u(x,y) = XY$ — ③

Substituting in the Given Eq^n, we have

$$X'Y = 4Y'X$$

$\Rightarrow$ $\frac{X'}{X} = 4\frac{Y'}{Y} = k$ (Let)

$\Rightarrow$ $\frac{X'}{X} = k$ and $4\frac{Y'}{Y} = k$

$\Rightarrow$ $\log X = kx + \log c_1$

$\Rightarrow$ $\log\frac{X}{c_1} = kx \Rightarrow X = c_1 e^{kx}$ — ④

Now $4\frac{Y'}{Y} = k$

$\Rightarrow$ $4\log Y = ky + \log c_2$

$\Rightarrow$ $\log\frac{Y}{c_2} = \frac{ky}{4}$

$\Rightarrow$ $Y = c_2 e^{ky/4}$ — ⑤

$\therefore$ $sol^n.$ $u(x,y) = c_1 c_2 e^{kx + ky/4}$

$= c\, e^{kx+ky/4}$ — ⑥

Using $cond^n.$ $u(0,y) = 8e^{-3y}$ in ⑥

$\Rightarrow$ $8e^{-3y} = c\, e^{ky/4} \Rightarrow c = 8, k = -12$

$\therefore$ solution of eq^n ①

$$\underline{\underline{u(x,y) = 8\, e^{-12x-3y}.}}$$

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Question:- Solve the equation by the method of separation of variables:

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u, \text{ given that } u(0,y) = 3e^{-y} - e^{-5y}$$

Solution:- given $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ — (1)

Let trial solution of eqn (1) is

$$u(x,y) = X(x).Y(y) \quad \text{— (2)}$$

Hence eqn (1) can be written as

$$4X'Y + XY' = 3XY$$

$\left[\because \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XY) = X'Y,\ \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = XY',\ u = XY\right]$

$$4X'Y = 3XY - XY'$$

$$\Rightarrow 4X'Y = X(3Y - Y')$$

$$\Rightarrow \frac{4X'}{X} = \frac{3Y - Y'}{Y} = k \text{ (say)}$$

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Let $\frac{4X'}{X} = k$

$\Rightarrow 4X' - kX = 0$

$\therefore$ A.E is

$4m - k = 0$

$\Rightarrow m = k/4$

$\therefore CF = c_1 e^{\frac{kx}{4}}$

$\Rightarrow X = c_1 e^{kx/4}$ — (3)

Again

$\frac{3Y - Y'}{Y} = k$

$\Rightarrow 3Y - Y' - kY = 0$

$\Rightarrow Y' + (k-3)Y = 0$

$\therefore$ A.E is

$m + (k-3) = 0$

$m = -(k-3)$

$m = 3-k$

$\therefore C.F = c_2 e^{(3-k)y}$

$\Rightarrow Y = c_2 e^{(3-k)y}$ [$\because$ Y is function of y]

— (4)

Put these values in equation (2)

$u(x,y) = c_1 e^{kx/4} \cdot c_2 e^{(3-k)y}$

$\Rightarrow u(x,y) = c_1 c_2 e^{\frac{kx}{4} + (3-k)y}$ — (5)

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Now given

$$u(0,y) = 3e^{-y} - e^{-5y}$$

$\therefore$ Put $x=0$ in eqn (5)

$$u(0,y) = c_1 c_2 e^{(3-k)y}$$

$$\Rightarrow \quad 3e^{-y} - e^{-5y} = c_1 c_2 e^{(3-k)y}$$

Now comparing with first term

$$c_1 c_2 = 3$$

$$3-k = -1 \Rightarrow k = 4$$

Again compairing with second term

$$c_1 c_2 = -1$$

$$3-k = -5 \Rightarrow k = 8$$

Hence from (5)

$$u = 3e^{\frac{4x}{4} + (3-4)y} - 1 \cdot e^{\frac{8x}{4} + (3-8)y}$$

$$u = 3e^{x-y} - e^{2x-5y}$$

Ans

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Question:- Using the method of separation of variables, Solve:

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

given that $u(x,0) = 6e^{-3x}$

Solution:- given that

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{---} (1)$$

Let the trial solution of (1) is

$$u(x,t) = X(x).T(t) \quad \text{---} (2)$$

then eqn (1) can be written as

$$X'T = 2XT' + XT$$

$$\left[u = XT \Rightarrow \frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = X\frac{\partial}{\partial t}T = XT'\right]$$

$$\Rightarrow X'T = X(2T' + T)$$

$$\Rightarrow \frac{X'}{X} = \frac{2T' + T}{T} = k \text{ (say)}$$

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Let $\frac{X'}{X} = k$

$\Rightarrow X' = kX$

$\Rightarrow X' - kX = 0$

$\therefore$ A.E is

$m - k = 0$

$\Rightarrow m = k$

$\therefore C.F = c_1 e^{kx}$ — (3) [$\because$ X is function of x]

$\Rightarrow X = c_1 e^{kx}$ [$\because$ there is no function is R.H.S]

Again Let $\frac{2T' + T}{T} = k$

$\Rightarrow 2T' + T - kT = 0$

$\Rightarrow 2T' + (1-k)T = 0$

$\therefore$ A.E is

$2m + (1-k) = 0$

$m = -\frac{(1-k)}{2}$

$m = \frac{k-1}{2}$

$\therefore C.F = c_2 e^{\left(\frac{k-1}{2}\right)t}$ [$\because$ T is function of t]

$\Rightarrow T = c_2 e^{\left(\frac{k-1}{2}\right)t}$ — (4)

Put values in eqn (2), from (3) and (4)

$u = c_1 e^{kx} \cdot c_2 e^{\left(\frac{k-1}{2}\right)t}$

$\Rightarrow u = c_1 \cdot c_2 e^{kx + \left(\frac{k-1}{2}\right)t}$ — (5)

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given $u(x,0) = 6e^{-3x}$

$\therefore$ Put $t=0$ in eqn (5)

$$6e^{-3x} = c_1c_2\, e^{kx}$$

Comparing both sides

$$c_1c_2 = 6, \quad k = -3$$

Put these values in eqn (5)

$$u = 6\, e^{-3x + \left(\frac{-3-1}{2}\right)t}$$

$$u = 6\, e^{-3x-2t}$$

$$\Rightarrow u = 6\, e^{-(3x+2t)}$$

Ans

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