SHRI SHANKARACHARYA TECHNICAL CAMPUS
SHRI SHANKARACHARYA GROUP OF INSTITUTIONS
BHILAI
(An Autonomous Institute affiliated to CSVTU,Bhilai)
Scheme of Examination and Syllabus 2021

Second Year B. Tech. -CSE

3rd semester Subject Code:AM102301

Subject Code	APPLIED MATHEMATICS-III	L = 3	T = 1	P = 0	Credits = 4
Evaluation Scheme	ESE	CT	TA	Total	ESE Duration
	100	20	30	150	3 Hours

Course Objectives	Course Outcomes
The objective of this course is to familiarize the prospective engineers with techniques in calculus of multivariable and infinite series expansion of continuous function as well as some statistical treatment of discrete functions. More precisely, the objectives are: • To investigate a thorough knowledge of partial differential equations which arise in mathematical descriptions of situations in engineering. • To develop the tool of Fourier series for learning advanced Engineering Mathematics. • To provide knowledge of Laplace transform of elementary functions including its properties and applications to solve ordinary differential equations. • To originate a thorough study about random quantities and their description in terms of their probability. • To provide a thorough understanding of interpolation.	**On successful completion of the course, the student will be able to:** CO 1. To have a thorough knowledge of PDE which arise in mathematical descriptions of situations in Engineering. CO 2. To make the students understand that Fourier series analysis is powerful methods where the formulas are integrals and to have knowledge of expanding periodic functions that explore variety of applications of Fourier series. CO3. To provide knowledge of Laplace transform of elementary functions including its properties and applications to solve ordinary differentials equations. CO4. To study about a quantity that may take any of a given range of values that can't be predicted as it is but can be described in terms of their probability CO5. To study the technique of estimating the values of a function for any intermediate value of the independent variable.

UNIT – I Partial differential equation: Formation, Solution by direct integration method, Linear equation of first order, Homogeneous linear equation with constant coefficients, Non-homogeneous linear equations, Method of separation of variables; Equation of vibrating string (wave equation). **[10 Hrs]**

UNIT – II Fourier Series- Euler's formula; Functions having point of discontinuity; Change of interval; Even and Odd function; Half range series; Harmonic Analysis. **[10Hrs]**

UNIT – III Laplace transform: Definition; Transform of elementary functions; Properties of Laplace transform; Inverse Laplace Transform (Method of partial fraction, using properties and Convolution theorem); Transform of Unit step function and Periodic functions; Application to the solution of ordinary differential equations. **[10Hrs]**

UNIT – IV Probability distributions: Random variable; Discrete and continuous probability distributions; Mathematical expectation; Mean, Variance and Moments; Moment generating functions; Probability distribution (Binomial, Poisson, and Normal distributions). **[10Hrs]**

		October 2020	1.00	Applicable for AY 2020-21 Onwards
Chairman (AC)	Chairman (BoS)	Date of Release	Version	

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UNIT – V Interpolation with equal and unequal intervals: Finite difference, Newton's Forward and Backward Difference Formulae, Central Difference Formula, Stirling's Formula, Bessel's Formula, Langrange's Formula and Newton's Divided Difference Formula.

[10Hrs]

Text Books:

S. No.	Title	Authors	Publisher
1)	Higher Engineering Mathematics	Dr. B.S. Grewal	Khanna Publishers
2)	Numerical Methods in Engineering and Science	Dr. B.S. Grewal	Khanna Publishers
3)	Advanced Engineering Mathematics	Erwin Kreyszig	John Wiley & Sons
4)	Applied Engineering Mathematics	Madan Mohan Singh	BS Publications

Reference Books:

S. No.	Title	Authors	Publisher
1)	Calculus and Analytic geometry	G. B. Thomas and R. L. Finney	Pearson, Reprint
2)	Engineering Mathematics for first year	T. Veerarajan	Tata McGraw-Hill, New Delhi
3)	Higher Engineering Mathematics	B. V. Ramana	Tata McGraw Hill New Delhi
4)	A text book of Engineering Mathematics	N.P. Bali and Manish Goyal	Laxmi Publications

Dr. M M Singh, Chairman(BOS)

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Laplace Transforms :

Transformation :- Transformation is a mathematical process by which one function converts into another function.

For example :- $D(\sin x) = \cos x$

$\sin x \downarrow$ odd fuⁿ. $\cos x \downarrow$ even fuⁿ.

Here, D is Transformation operator.

Laplace Transforms :- Let $f(t)$ be a fuⁿ of t, $t \geq 0$, then Laplace Tr. is defined by $L[f(t)]$ which is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt \quad , t \geq 0$$

Here, s is a parameter which may be real or complex.

$$L[f(t)] = \bar{f}(s)$$

Laplace Tr. : developed by 'oliver Heaviside'.

Application of Laplace Transforms :-

① Laplace Trasform directly gives the Solution of Differential eqⁿ.

② Electric signals → To Solve problems of Electric signals.

Properties of L.T.

① **Linearity property :-** If c_1 and c_2 are constants and f and g are functions of t then

$$L[c_1 f(t) + c_2 f_2(t)] = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

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* Existence of Laplace Transforms :-

The L.T. of $f(t)$, i.e. $\int_0^\infty e^{-st} f(t)\,dt$ exists for $s > a$ if

(i) $f(t)$ is continuous
(ii) $\lim_{t \to \infty} e^{-at} f(t)$ is finite.

* Change of Scale Property :—

If $L[f(t)] = \bar{f}(s)$ then
$L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

* I^{st} Shifting Property :—

If $L[f(t)] = \bar{f}(s)$ then
$L[e^{at} f(t)] = \bar{f}(s-a)$

Standard Formula for L.T.

① $\boxed{L(1) = \frac{1}{s}}$, $s>0$

$$L(1) = \int_0^\infty e^{-st} \cdot 1\, dt = \int_0^\infty e^{-st} dt$$

$$= \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s} \qquad \text{(Here, } s>0)$$

② $L(t^n) = \int_0^\infty e^{-st} t^n dt$

$$= \int_0^\infty e^{-p}\left(\frac{p}{s}\right)^n \frac{dp}{s}$$

let $st = p$
or $t = \frac{p}{s}$
$dt = \frac{dp}{s}$
limits are 0 to ∞

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p} \cdot p^n dp$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-p}\, p^{(n+1)-1} dp$$

By ~~Be~~ Gamma fu^n formula :—

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \frac{1}{s^{n+1}} \Gamma n+1 \quad \text{or} \quad \frac{n!}{s^{n+1}}, \; s>0$$

$$\boxed{L(t^n) = \frac{\Gamma n+1}{s^{n+1}} \quad \text{or} \quad \frac{n!}{s^{n+1}}}$$

(3) $L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} \, dt$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a} \right]_0^\infty = \frac{1}{s-a}$$

Here. $s>0$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

(4) $L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$

By Using Formula :- $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

$$= \left[\frac{e^{-st}}{s^2+a^2} \left[-s \sin at - a \cos at \right] \right]_0^\infty$$

$$= \frac{-1}{s^2+a^2} \{-a\} = \frac{a}{s^2+a^2}$$

$$\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$$

(5) $L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$

$$= \left\{ \frac{e^{-st}}{s^2+a^2} \left[-s\cos at + a\sin at \right] \right\}_0^\infty$$

$$= \frac{+s}{s^2+a^2}$$

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2}}$$

(6) $L(\sinh at) = \int_0^\infty e^{-st} \sinh at \, dt$

Since, $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$= \int_0^\infty e^{-st} \left[\frac{e^{at} - e^{-at}}{2} \right] dt$$

$$= \frac{1}{2} \int_0^\infty \left\{ e^{-(s-a)t} - e^{-(s+a)t} \right\} dt$$

$$= \frac{1}{2} \left[-\frac{e^{-(s-a)t}}{s-a} + \frac{e^{-(s+a)t}}{s+a} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{\cancel{s}+a - \cancel{s}+a}{s^2-a^2} \right]$$

$$= \frac{\cancel{2} a}{\cancel{2}(s^2-a^2)} = \frac{a}{s^2-a^2}$$

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}}$$

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(7) $L(\cosh at) = \int_0^\infty e^{-st} \cosh at \, dt$

Since, $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$= \int_0^\infty e^{-st} \left\{ \frac{e^{at} + e^{-at}}{2} \right\} dt$$

$$= \frac{1}{2} \int_0^\infty \left\{ e^{-(s-a)t} + e^{-(s+a)t} \right\} dt$$

$$= \frac{1}{2} \left[\frac{-e^{-(s-a)t}}{s-a} - \frac{e^{-(s+a)t}}{s+a} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+\cancel{a} + s-\cancel{a}}{s^2-a^2} \right]$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}s}{s^2-a^2} = \frac{s}{s^2-a^2}$$

$$\boxed{L(\cosh at) = \frac{s}{s^2-a^2}}$$

Example of Linearity property

(1) $L\left[2e^{3t} + \frac{5}{2}\sin 5t\right] =$

$2L\{e^{3t}\} + \frac{5}{2} L\{\sin 5t\}$ by Linearity Prop.

$$= 2\left[\frac{1}{s-3}\right] + \frac{5}{2}\left\{\frac{5}{s^2+25}\right\} \quad \text{Ans.}$$

* [By Using formula, $L(e^{at}) = \frac{1}{s-a}$ & $L(\sin at) = \frac{a}{s^2+a^2}$]

Some Questions

① $\cos(at+b)$ find L.T. of given f^n.

Sol^n ⟹ let, $f(t) = \cos(at+b)$ Taking L.T. operator on both sides

$$L[f(t)] = L[\cos(at+b)]$$

$$L\{f(t)\} = L[\cos at \cdot \cos b - \sin at \cdot \sin b]$$

* (by Using formula, $\cos(A+B) = \cos A \cdot \cos B - \sin A \sin B$)

$$= \cos b\{L(\cos at)\} - \sin b\{L(\sin at)\}$$

{By linearity prop}

$$= \cos b\left[\frac{s}{s^2+a^2}\right] - \sin b\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{s^2+a^2}\{s\cos b - a\sin b\} \text{ Ans.}$$

② $(\sin t - \cos t)^2$ Find L.T.

Sol^n ⟹ let, $f(t) = (\sin t - \cos t)^2$

Taking L.T. operator on both sides,

$$L\{f(t)\} = L[(\sin t - \cos t)^2]$$

$$= L[\sin^2 t + \cos^2 t - 2\sin t \cos t]$$

$$= L(\sin^2 t) + L(\cos^2 t) - L[\sin 2t]$$

* (By Using formula $2\sin A \cdot \cos B = \sin 2A$)

or, $= L[\sin^2 t + \cos^2 t] - L[\sin 2t]$

$= L(1) - L(\sin 2t)$

$= \frac{1}{s} - \frac{2}{s^2+4}$

or, $\frac{s^2+4-2s}{s(s^2+4)}$ Ans..

(3) find L.T. of $\cos^3 2t$

Soln, let $f(t) = \cos^3 2t$

Taking L.T. operator on both sides.

$L\{f(t)\} = L[\cos^3 2t]$

By Using Formula, $\cos^3 x = \frac{1}{4}[3\cos x + \cos 3x]$

$L[f(t)] = L[\cos^3 2t]$

$= L\frac{1}{4}[3\cos 2t + \cos 6t]$

$= \frac{3}{4}L(\cos 2t) + \frac{1}{4}L(\cos 6t)$

(By Linearity Prop.)

$= \frac{3}{4}\left[\frac{s}{s^2+4}\right] + \frac{1}{4}\left[\frac{s}{s^2+6}\right]$

$= \frac{s}{4}\left[\frac{3}{s^2+4} + \frac{3}{s^2+6}\right]$

Ans..

H.W: Find L.T. of $\sin^3 2t$.

use this formula → $\sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$

Q. Find L.T. of $L\{\sin\sqrt{t}\}$.

Sol^n → We have to solve $L\{\sin\sqrt{t}\}$ by using expansion of $\sin\sqrt{t}$.

expansion is →
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots \infty$$

So, $$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots \infty$$

Taking L.T. operator on both sides;

$$L[\sin\sqrt{t}] = L(t^{1/2}) - \frac{1}{6}L(t^{3/2}) + \frac{1}{120}L(t^{5/2}) - \cdots \infty$$

$$= \frac{\Gamma(1/2+1)}{s^{3/2}} - \frac{1}{6}\frac{\Gamma(\frac{3}{2}+1)}{s^{5/2}} + \frac{1}{120}\frac{\Gamma(5/2+1)}{s^{7/2}} - \cdots \infty$$

since, $\Gamma(n+1) = n\Gamma(n)$ so,

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\Gamma(3/2)}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\Gamma(\frac{5}{2})}{s^{7/2}} - \cdots \infty$$

since, $\Gamma(1/2) = \sqrt{\pi}$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{4}\frac{\Gamma(1+\frac{1}{2})}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\Gamma(1+\frac{3}{2})}{s^{7/2}} - \cdots \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{1}{4}\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}} + \frac{1}{120}\times\frac{5}{2}\times\frac{3}{2}\frac{\Gamma(3/2)}{s^{7/2}} - \cdots \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\Gamma 3/2}{32s^{7/2}} - - - - \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\Gamma 1+1/2}{32s^{7/2}} - - - - - \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\frac{1}{2}\Gamma\frac{1}{2}}{32s^{7/2}} - - - - \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{64s^{7/2}} - - - - \infty$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 - - - - - - - \infty\right]$$

(We know that the expansion of e^{-x} is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - - - - - - - - \infty$$)

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[e^{-1/4s}\right]$$ Ans..

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I^{st} shifting Property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L[e^{at} f(t)] = \bar{f}(s-a)$$

Useful Results :—

① $L[e^{at} t^n] = \dfrac{n!}{(s-a)^{n+1}}$

② $L[e^{at} \sin at] = \dfrac{a}{(s-a)^2+b^2}$

③ $L[e^{at} \cos at] = \dfrac{s-a}{(s-a)^2+b^2}$

④ $L[e^{at} \sinh at] = \dfrac{a}{(s-a)^2-b^2}$

⑤ $L[e^{at} \cosh at] = \dfrac{s-a}{(s-a)^2-b^2}$

Que ① Find L.T. $L[\cosh 3t \cdot \cos 2t]$

Sol^n let $f(t) = \cosh 3t \cdot \cos 2t$

Taking L.T. operator on both sides, $L[f(t)] = L[\cosh 3t \cdot \cos 2t]$

$$L[f(t)] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right)\cos 2t\right]$$

$$= \frac{1}{2}\left[L(e^{3t}\cos 2t) + L(e^{-3t}\cos 2t) \right]$$

By Linearity prop.

$$= \frac{1}{2}\left[\frac{s-3}{(s-3)^2+4} + \frac{s+3}{(s+3)^2+4} \right]$$

Ans..

Que Find L.T. of $e^{-t}\sin^2 t$.

Soln ⟹ let, $f(t) = e^{-t}\sin^2 t$

Taking L.T. on both sides;

$$L[f(t)] = L[e^{-t}\sin^2 t]$$

$$L[f(t)] = L\left[e^{-t}\left\{\frac{1-\cos 2t}{2}\right\}\right]$$

$$= \frac{1}{2} L\left[e^{-t} - e^{-t}\cos 2t\right]$$ By Linearity Prop

$$= \frac{1}{2}\left\{ L\{e^{-t}\} - L\{e^{-t}\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]$$

$$= \frac{2}{(s+1)(s^2+2s+5)}$$ Ans..

Laplace Transforms for functions which are divided into subintervals

Q(1) $f(t) = \begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Find Laplace Transforms.

Sol^n

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \left[\int_0^1 e^{-st}.e^t\,dt + \int_1^\infty e^{-st}.0\,dt\right]$$

$$= \int_0^1 e^{-(s-1)t}\,dt$$

$$= \left[\frac{-e^{(s-1)t}}{s-1}\right]_0^1$$

$$= \frac{-e^{(s-1)}}{s-1} + \frac{1}{s-1}$$

$$= \frac{1}{s-1}\left\{1-e^{(s-1)}\right\} \text{ Ans..}$$

Q(2) $f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

Find Laplace Transforms.

Sol^n

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st}.\sin t\,dt + \int_\pi^\infty e^{-st}.0\,dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt$$

By Using formula, $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left\{a\sin bx - b\cos bx\right\}$

$$= \left[\frac{e^{-st}}{s^2+1} \left\{ -s \sin t - \cos t \right\} \right]_0^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+1} \left\{ +1 \right\} - \frac{1}{s^2+1} \left\{ -1 \right\} \quad \begin{array}{l} (\sin \pi = 0) \\ (\cos \pi = -1) \end{array}$$

$$= \frac{1}{s^2+1} \left\{ 1 + e^{-s\pi} \right\} \quad \text{Ans..}$$

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* H.W. Q $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$

*

Ans :- $\bar{f}(s) = \frac{2}{s^3} - \frac{e^{-2s}}{s^3} \left\{ 2 + 3s + 2s^2 \right\} -$

$$\frac{e^{-3s}}{s^2} \left\{ -4s + 1 \right\}$$

Que → Imp Find Laplace transform of

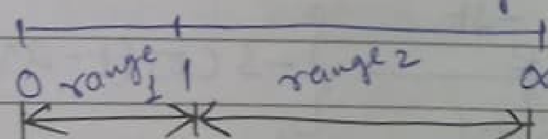
$$f(t) = |t-1| + |t+1| \quad , t \geq 0$$

Sol^n ⇒ Since mod of any f^n is defined by

$$|x| = \begin{cases} -x, & x < 0 \\ +x, & x > 0 \end{cases}$$

Since, Given range is $t \to 0$ to ∞

So we distribute the range in 2 parts Such that,



$$f(t) = -(\cancel{t}-1) + \cancel{t} + 1 \quad , \quad 0 < t < 1$$
$$= 2$$

and,

$$f(t) = t - 1 + t + 1 \quad , \quad t > 1$$
$$= 2t$$

So,

$$L\{f(t)\} = \int_0^1 e^{-st}\, 2\,dt + \int_1^\infty e^{-st} \cdot 2t\,dt$$

$$= 2\left\{-\frac{e^{-st}}{s}\right\}_0^1 + 2\left[t\left\{\frac{-e^{-st}}{s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_1^\infty$$

$$= \left(-\frac{2e^{-s}}{s} + \frac{2}{s}\right) - 2\left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right)$$

$$= \cancel{-\frac{2e^{-s}}{s}} + \frac{2}{s} + \cancel{\frac{2e^{-s}}{s}} + \frac{2e^{-s}}{s^2}$$

$$= \frac{2}{s}\left\{1 + \frac{e^{-s}}{s}\right\} \quad \text{Ans..}$$

Imp

Que ⟹ Find L.T. of

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

Sol^n ⟹

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{2\pi/3} e^{-st} \cdot 0\,dt + \int_{2\pi/3}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

* By using $\int e^{ax}\cos bx\,dx$ * →

$$= 0\left[\frac{e^{-st}}{s^2+1}\left\{-s\cos(t - 2\pi/3) + \sin\left(t - \frac{2\pi}{3}\right)\right\}\right]_{2\pi/3}^\infty$$

$$= -\left[\frac{e^{-\frac{2\pi s}{3}}}{s^2+1}\{-s\}\right]$$

$$= \frac{s e^{-\frac{2\pi s}{3}}}{s^2+1} \quad \text{Ans..}$$

* Some Different Questions *

① Find $L f(t) = 2^t$

Sol^n ⟹ let, $f(t) = 2^t$

$$f(t) = e^{\log 2^t}$$

$$f(t) = e^{t\log 2}$$

$$f(t) = e^{(\log 2)t} \qquad (a = \log 2)$$

$$L\{f(t)\} = L\{e^{(\log 2)t}\}$$

$$= \frac{1}{s - \log 2} \quad \underline{\text{Ans}}$$

② Find $L(f(t)) = \sin^5 t$

Sol^n let

$$f(t) = \sin^5 t$$

$$= \sin^4 t \cdot \sin t$$

$$= (\sin^2 t)^2 \sin t$$

$$= \left(\frac{1 - \cos 2t}{2}\right)^2 \sin t$$

$$= \frac{1}{4}\left[(1 + \cos^2 2t - 2\cos 2t)\sin t\right]$$

$$= \frac{1}{4}\left[\left\{1 + \left(\frac{1 + \cos 2t}{2}\right) - 2\cos 2t\right\}\sin t\right]$$

$$= \frac{1}{4}\left[\sin t + \frac{\sin t}{2} + \frac{\sin t\cos 2t}{2} - 2\sin t\cos 2t\right]$$

$$= \frac{1}{4}\left[\frac{3\sin t}{2} + \frac{2\sin t\cos 2t}{4} - 2\sin t\cos 2t\right]$$

$$= \frac{1}{4}\left[\frac{3\sin t}{2} + \frac{\sin 3t - \sin t}{4} - (\sin 3t - \sin t)\right]$$

$$= \frac{1}{4}\left[\frac{3\sin t}{2} + \frac{\sin 3t}{4} - \frac{\sin t}{4} - \sin 3t + \sin t\right]$$

$$= \frac{9}{16}\sin t - \frac{3}{16}\sin 3t$$

Taking L.T. on both sides;

$$L\{f(t)\} = \frac{9}{16} L\{\sin t\} - \frac{3}{16} L\{\sin 3t\}$$

$$= \frac{9}{16}\left\{\frac{1}{s^2+1}\right\} - \frac{3}{16}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{9}{16}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$= \frac{9}{16}\left[\frac{s^2+9-(s^2+1)}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{9}{\cancel{16}\,2}\left[\frac{\cancel{8}\,1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{9}{2(s^2+1)(s^2+9)} \text{ Ans.}$$

— x —

Change of Scale Property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

Que ⟹ Solve $L\{\cos 7x\}$ by change of scale property.

Sol ⟹ We know that $L(\cos x) = \frac{s}{s^2+1} = \bar{f}(s)$

then

$$L(\cos 7x) = \frac{1}{7}\bar{f}\left(\frac{s}{7}\right)$$ By change of Scale Prop.

$$= \frac{1}{7}\left\{\frac{s/7}{(s/7)^2+1}\right\}$$

$$= \frac{1}{7}\left\{\frac{s/7}{s^2/49+1}\right\} = \frac{1}{7}\left\{\frac{s/7}{\frac{s^2+49}{49}}\right\}$$

$$= \frac{s}{s^2+49} = \frac{s}{s^2+(7)^2} \text{ Ans.}$$

Multiplication by t

If $L\{f(t)\} = \bar{f}(s)$ Then

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)}$$

① Find $L\{f(t)\} = L\{t \sin^2 t\}$

Sol^n ⇒ 1^{st} we have to find $L\{\sin^2 t\}$ which is $\bar{f}(s)$.

let, $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2}$$

$$L\{f(t)\} = L\left[\frac{1-\cos 2t}{2}\right]$$ Taking L.T. on both sides

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$ By Linearity Prop.

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Now by multiplication by t :—

$$L\{t^1 \sin^2 t\} = (-1)^1 \frac{d^1}{ds^1}\left[\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}\right]$$

(Here $n=1$)

$$= -\frac{1}{2}\left[\frac{-1}{s^2} - \left\{\frac{(s^2+4).1 - s(2s)}{(s^2+4)^2}\right\}\right]$$

$$= -\frac{1}{2}\left[\frac{-1}{s^2} - \left\{\frac{s^2+4-2s^2}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \left\{\frac{-s^2+4}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+4)^2 + s^2(-s^2+4)}{s^2(s^2+4)^2}\right]$$ By ~~Taking~~ Calculation

$$= \cancel{\tfrac{1}{2}}\left[\frac{6s^2+8}{s^2(s^2+4)^2}\right]$$

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Find,

(2) $L[t\sin 3t \cdot \cos 2t]$

Sol^n ⟹

let, $f(t) = \sin 3t \cdot \cos 2t$

$f(t) = \frac{1}{2} \times 2 \sin 3t \cdot \cos 2t$

$f(t) = \frac{1}{2}[\sin 5t + \sin t]$

Taking L.T. operator on both sides;

$L\{f(t)\} = \frac{1}{2} L(\sin 5t + \sin t)$ By Linearity Prop.

$= \frac{1}{2}[L(\sin 5t) + L(\sin t)]$

$\bar{f}(s) = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

Now, by multiplication by t :—

$L[t^1 \sin 3t \cdot \cos 2t] = (-1)^1 \frac{d^1}{ds^1} \bar{f}(s)$

$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25}\right) + \frac{1}{2}\frac{1}{s^2+1}\right]$

$$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25}\right)+\frac{1}{2}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{-5\times 2s}{(s^2+25)^2}\right]+\frac{-1}{2}\left(\frac{-2s}{(s^2+1)^2}\right)\Big]$$

$$= \frac{5s}{(s^2+25)^2}+\frac{s}{(s^2+1)^2} \quad \text{Ans..}$$

Que. $L[t^2 \cos at]$

Sol^n $L\{\cos at\} = \frac{s}{s^2+a^2} = \bar{f}(s)$

then, $L[t^2 \cos at] = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2\cdot(-2s)-(a^2-s^2)\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

This is easiest way to solve

$$= \frac{(s^2+a^2)\{(s^2+a^2)(-2s)-(a^2-s^2)\cdot 4s\}}{(s^2+a^2)^4}$$

$$= \frac{\{-2s^3-2a^2s-4a^2s+4s^3\}}{(s^2+a^2)^3}$$

$$= \frac{2s^3-6sa^2}{(s^2+a^2)^3} \quad \text{Ans..}$$

Que $\Rightarrow$ $L\{t e^{-2t}\sin 2t\}$

Sol^n $\Rightarrow$ let $f(t) = \sin 2t$

$$L\{f(t)\} = \bar{f}(s) = \frac{2}{s^2+4}$$

Now by 1st shifting prop.

$$L[e^{-2t}\sin 2t] = \frac{2}{(s+2)^2+4} \qquad (a=-2)$$

again,

$$L[t^1 e^{-2t}\sin 2t] = (-1)^1 \frac{d^1}{ds^1}\left[\frac{2}{(s+2)^2+4}\right]$$

$$= -2\left[\frac{-2\{2(s+2)\}}{\{(s+2)^2+4\}^2}\right]$$

$$= \frac{8(s+2)}{\{(s+2)^2+4\}^2} \quad \text{Ans}$$

* *

Division By t

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$$

Que ⟹ Find L.T. of the fu^n

$$\frac{e^{-at} - e^{-bt}}{t}$$

Sol^n ⟹ $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

By Division by t

$$= \int_s^\infty \bar{f}(s)\,ds \quad \text{---- (1)}$$

let, $f(t) = e^{-at} - e^{-bt}$

$$\bar{f}(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

By (1) :—

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \left[\log \frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

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Que Find L.T. $\frac{\sin at}{t}$

Sol^n ⟹ let, $f(t) = \sin at$

$\bar{f}(s) = L\{f(t)\} = L\{\sin at\}$

$= \frac{a}{s^2+a^2}$

Now, By Division by t :-

$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2} ds$

We know that, $\int \frac{1}{s^2+a^2} ds = \frac{1}{a} \tan^{-1} s/a$

$= \left[\tan^{-1} s/a\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} s/a$

$= \cot^{-1} s/a$ Ans

Que $\frac{\cos 2t - \cos 3t}{t}$ Find Laplace Transform

Sol^n let, $f(t) = \cos 2t - \cos 3t$

$L\{f(t)\} = \bar{f}(s) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$

$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$

let, $s^2+4 = u$ & $s^2+9 = v$

$2s\,ds = du$ | $2s\,ds = dv$

limits are :- s to ∞ (no change)

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$$= \frac{1}{2}\int_s^\infty \frac{du}{u} - \frac{1}{2}\int_s^\infty \frac{dv}{v}$$

$$= \left[\frac{1}{2}\log u - \frac{1}{2}\log v\right]_s^\infty$$

$$= \left[\frac{1}{2}\log \frac{s^2+4}{s^2+9}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left\{\frac{\cancel{s^2}(1+4/s^2)}{\cancel{s^2}(1+9/s^2)}\right\}\right]_s^\infty$$

$$= 0 - \frac{1}{2}\log\frac{s^2+4}{s^2+9}$$

$$= \frac{1}{2}\log\frac{s^2+9}{s^2+4}$$ Ans.

Que. Find Laplace Transforms of $\dfrac{1-\cos t}{t^2}$

Sol^n ⟹ let, $f(t) = 1-\cos t$

$$L\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+1} = \bar{f}(s)$$

Now, $$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_0^\infty$$

$$= \left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+1)\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2(1+1/s^2)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\log \frac{1}{1+1/s^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2}{1+s^2}\right]$$

$$= \frac{1}{2}\log \frac{1+s^2}{s^2}$$

again, $L\left[\frac{1-\cos t}{t^2}\right] = \frac{1}{2}\int_s^\infty \log\left(\frac{1+s^2}{s^2}\right) ds$

$$= \frac{1}{2}\int_s^\infty \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log} \frac{1+s^2}{s^2}\, ds$$

Integration by Part

$$= \frac{1}{2}\left[\log\left(\frac{1+s^2}{s^2}\right) \cdot s\right]_s^\infty - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left\{\frac{1+s^2}{s^2}\right\} \cdot s\, ds$$

$$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}(1/s^2+1)}{\cancel{s^2}}\right) \cdot s\right]_s^\infty - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left\{\frac{1+s^2}{s^2}\right\} \cdot s\, ds \; *$$

$$= \frac{1}{2}\left[0 - s \cdot \log\left(\frac{1+s^2}{s^2}\right)\right] - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left(\frac{1+s^2}{s^2}\right) \cdot s\, ds$$

$$= -\frac{1}{2}\, s\log\left[\frac{1+s^2}{s^2}\right] - \frac{1}{2}\int_s^\infty \frac{1}{\frac{1+s^2}{s^2}} \left\{\frac{s^2(2s)-(1+s^2)\cdot 2s}{s^4}\right\}\cdot s\,ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) - \frac{1}{2}\int_s^\infty \frac{s^2\times s}{1+s^2}\left\{\frac{\cancel{2s^3}-2s-\cancel{2s^3}}{s^4}\right\}ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) - \frac{1}{2}\int_s^\infty \frac{1}{1+s^2}\left\{\frac{-2\cancel{s}}{\cancel{s}}\right\}ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \frac{1}{\cancel{2}}\int_s^\infty \frac{\cancel{2}}{1+s^2}\,ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \left(\tan^{-1}s\right)_s^\infty$$

$$= -\frac{s}{2}\log\frac{1+s^2}{s^2} + \frac{\pi}{2} - \tan^{-1}s$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \cot^{-1}s$$

$$= \frac{s}{2}\log\left(\frac{s^2}{1+s^2}\right) + \cot^{-1}s$$

Variety

Que → Evaluate, $\int_0^\infty t\,e^{-2t}\cos t\,dt$

Sol^n → ~~Here, s=~~

Since, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ ①

By comparing given integral with ① :-

$s=2$ and $f(t) = t\cos t$

Now, we have to ~~to~~ find

$$\bar{f}(s) = L\{f(t)\} = L[t\cos t]$$

$$= (-1)^1 \frac{d^1}{ds^1}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)\cdot 1 - s(2s)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= -\left[\frac{-s^2+1}{(s^2+1)^2}\right]$$

put $\xrightarrow{s=2}$ $= -\left[\frac{-(2)^2+1}{(2^2+1)^2}\right]$

$$= -\left[\frac{-4+1}{(5)^2}\right]$$

$$= 3/25 \text{ Ans.}$$

Que Evaluate, $\int_0^\infty e^{-3t}\, t \sin t \, dt$

Solⁿ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

By comparing, $s = 3$, $f(t) = t \sin t$

we have to find $\bar{f}(s)$

$$\bar{f}(s) = L[t^1 \sin t]$$

$$= (-1)^1 \frac{d^1}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[-\frac{2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$$

put $s=3 \rightarrow \frac{6}{(9+1)^2} = \frac{6}{100} = \frac{3}{50}$

$$= 3/50 \quad \text{Ans}$$

Transforms of Integration

If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\bar{f}(s)}$$

Que ① Find L.T. of $\int_0^t \frac{\cos at - \cos bt}{t} dt$

Solⁿ $L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \cos at - \cos bt$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Now, $L\left[\dfrac{\cos at - \cos bt}{t}\right] =$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

let, $s^2+a^2 = u$ | $s^2+b^2 = v$

$2s\,ds = du$ | $2s\,ds = dv$

limits are s to ∞

$$= \int_{s \to}^{\infty} \frac{du}{2u} - \frac{dv}{2v}$$

$$= \frac{1}{2}\left[\log u - \log v\right]$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right.$$

$$= \left\{\frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]\right\}_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log \frac{s^2+b^2}{s^2+a^2}$$

Now, $L\left[\int_0^t \dfrac{\cos at - \cos bt}{t}\, dt\right]$

By L.T. of Integral,

$$= \frac{1}{s}\bar{f}(s)$$

$$= \frac{1}{2s}\log \frac{s^2+b^2}{s^2+a^2}$$

(2) Find $L\left[\int_0^t \frac{\sin t}{t}\,dt\right]$

Sol^n $\Rightarrow$ $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \frac{\sin t}{t}$

$$L\{f(t)\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= (\tan^{-1}s)_s^\infty = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

So, $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{\cot^{-1}s}{s}$ Ans.

(By Transforms of Integral)

Another variety — (Type of evaluation) Prove that

1) Prove that, $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt = \log 3$

Sol^n $\Rightarrow$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ — ①

By comparing given Integral by ①

Such that,

$$\int_0^\infty e^{-t}\cdot\frac{1}{t}\,dt - \int_0^\infty e^{-3t}\cdot\frac{1}{t}\,dt$$

Here, $s = 1$, $f_1(t) = \frac{1}{t}$ & $s = 3$, $f_2(t) = \frac{1}{t}$

Ist we will find $\bar{f}(s)$

So $L(1) = \frac{1}{s}$

$\Rightarrow L\left(\frac{1}{t}\right) = \int_s^\infty \frac{1}{s}\,ds = (\log s)_s^\infty$

$= \log\infty - \log s$ — ②

Here, $s \to 1$ $= \log\infty - \log 1$

again, for 2^{nd} Integral,

$$= \log\infty - \log s$$

$$s \to 3 \qquad = \log\infty - \log 3 \quad \text{---} \; (3)$$

By (2) & (3) :—

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt = \cancel{\log\infty} - \log 1 - \cancel{\log\infty} + \log 3$$

$(\log 1 = 0)$

$$= \log 3$$

$$= \underline{\underline{RHS.}}$$

Que Prove that, $\int_0^\infty \frac{e^{-t}\sin^2 t}{t} dt = \frac{1}{4}\log 5$

Sol^n $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

By comparing, $\int_0^\infty e^{-t} \frac{\sin^2 t}{t}$

$s = 1$ & $f(t) = \frac{\sin^2 t}{t}$ or $\frac{1-\cos 2t}{2t}$

$\bar{f}(s) = ?$

$$\bar{f}(s) = L\{f(t)\} = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+4/s^2}\right)\right]_s^{\infty}$$

$$= 0 - \frac{1}{4}\log\frac{s^2}{s^2+4}$$

$$= +\frac{1}{4}\log\frac{s^2+4}{s^2}$$

Now, put $s=1 \Rightarrow \frac{1}{4}\log\frac{1^2+4}{1^2}$

$$= \frac{1}{4}\log 5 \quad \text{Ans}$$

Periodic Function

If $f(t)$ be a function then it is called "periodic function" if it repeats its values in any particular interval.

* Laplace Transform of a periodic function :-

If $f(t)$ be a periodic function then,

$$\boxed{L\{f(t)\} = \int_0^T \frac{e^{-st} f(t)\,dt}{1-e^{-sT}}}$$

where, T = Total Period.

Q① Find L.T. of given Periodic f^n.

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Sol^n $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

Periodic f^n

Here, T = Total period $= 2\pi/\omega$.

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}\{-s\sin\omega t - \omega\cos\omega t\}\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\{-s\sin\omega\times\pi/\omega - \omega\cos\omega\times\pi/\omega\} - \frac{e^{-s\times 0}}{s^2+\omega^2}\{-s\sin 0\omega - \omega\cos 0\omega\}\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\{\omega\} - \frac{1}{s^2+\omega^2}\{-\omega\}\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{\omega}{s^2+\omega^2}\{e^{-s\pi/\omega}+1\}\right]$$

$$= \frac{\omega\,(1+e^{-s\pi/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}$$

Que $\Rightarrow$ Find L.T. of given Periodic fn.

$$f(t) = \begin{cases} 1, & 0 < t < a/2 \\ -1, & a/2 < t < a \end{cases}$$

Soln $\Rightarrow$ $T =$ Total Period $= a$

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{\int_0^{a/2} e^{-st}(1)\,dt + \int_{a/2}^{a} e^{-st}(-1)\,dt}{1-e^{-sa}}$$

$$= \frac{1}{1-e^{-sa}}\left[\left\{-\frac{e^{-st}}{s}\right\}_0^{a/2} + \left\{+\frac{e^{-st}}{s}\right\}_{a/2}^{a}\right]$$

$$= \frac{1}{1-e^{-sa}}\left[\left(-\frac{e^{-as/2}}{s} + \frac{1}{s}\right) + \left(\frac{e^{-sa}}{s} - \frac{e^{-sa/2}}{s}\right)\right]$$

$$= \frac{1}{1-e^{-sa}}\left[-\left(\frac{e^{-as/2} + e^{-sa/2}}{s}\right) + \left(\frac{e^{-sa}+1}{s}\right)\right]$$

$$= \frac{1}{1-e^{-sa}}\left[-\frac{2e^{-as/2}}{s} + \frac{e^{-sa}+1}{s}\right]$$

$$= \frac{1}{1-e^{-sa}}\left[\frac{1+e^{-sa} - 2e^{-as/2}}{s}\right]$$

Transforms of Integration

If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s)}$$

Que ① Find L.T. of $\int_0^t \frac{\cos at - \cos bt}{t}\,dt$

Sol^n $L\left[\int_0^t f(u)\,du\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \cos at - \cos bt$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Now, $L\left[\frac{\cos at - \cos bt}{t}\right] =$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

let, $s^2+a^2 = u$ | $s^2+b^2 = v$

$2s\,ds = du$ | $2s\,ds = dv$

limits are s to ∞

$$= \int_{s\to}^{\infty} \frac{du}{2u} - \frac{dv}{2v}$$

$$= \frac{1}{2}\left[\log u - \log v\right]$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right)$$

$$= \left\{\frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]\right\}_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log \frac{s^2+b^2}{s^2+a^2}$$

Now, $L\left[\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right]$

By L.T. of Integral.

$$= \frac{1}{s}\bar{f}(s)$$

$$= \frac{1}{2s}\log \frac{s^2+b^2}{s^2+a^2}$$

(2) Find $L\left[\int_0^t \frac{\sin t}{t}\, dt\right]$

Sol^n $\Rightarrow$ $L\left[\int_0^t \frac{\sin t}{t}\, dt\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \frac{\sin t}{t}$

$$L\{f(t)\} = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left(\tan^{-1}s\right)_s^\infty = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

So, $L\left[\int_0^t \frac{\sin t}{t}\, dt\right] = \frac{\cot^{-1}s}{s}$ A

(By Transforms of Integral)

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$$= \frac{-2t\cosh t + 2\sinh t}{t^2}$$ Ans

* Convolution Theorem *

Convolution means "Extremly complex" or "Twisted" in the complex way.

Imp

Statement $\Rightarrow$ If $L^{-1}\bar{f}(s) = f(t)$ and $L^{-1}\bar{g}(s) = g(t)$

Then

$$\boxed{L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\,du}$$

Que ① $\Rightarrow$ Find, $L^{-1}\left(\frac{1}{s+a}\cdot\frac{1}{s+b}\right)$

Sol^n $\Rightarrow$ let, $\bar{f}(s) = \frac{1}{s+a}$ and $\bar{g}(s) = \frac{1}{s+b}$

so, $L^{-1}\bar{f}(s) = L^{-1}\frac{1}{s+a} = e^{-at} = f(t)$

$L^{-1}\bar{g}(s) = L^{-1}\frac{1}{s+b} = e^{-bt} = g(t)$

$$L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = \int_0^t e^{-au}\cdot e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{-u(a-b)}\,du$$

$$= e^{-bt}\left[\frac{-e^{-(a-b)u}}{a-b}\right]_0^t$$

$$= e^{-bt}\left[\frac{-e^{-(a-b)t}+1}{a-b}\right]$$

$$= \frac{1}{a-b}\left[-e^{-at}+e^{-bt}\right]$$ Ans

(2) $L^{-1} \dfrac{1}{s(s^2+4)}$

Sol^n $\Rightarrow$ $L^{-1}\left(\dfrac{1}{s}\right) = 1$

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \int_0^t 1 \cdot \frac{1}{2}\sin 2u \, du$$

$$= \frac{1}{2}\left\{-\frac{\cos 2u}{2}\right\}_0^t$$

$$\Rightarrow -\frac{\cos 2t + 1}{4}$$

$$= \frac{1 - \cos 2t}{4}$$

Ans.

(3) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Sol^n $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\,du$

$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

$= \frac{1}{2a^2}\int_0^t 2\sin au \cdot \sin a(t-u)\,du$

$= \frac{1}{2a^2}\int_0^t \cos\{au - a(t-u)\} - \cos\{au + a(t-u)\}\,du$

$= \frac{1}{2a^2}\int_0^t \cos\{au - at + au\} - \cos\{au + at - au\}\,du$

$= \frac{1}{2a^2}\int_0^t \{\cos(2au - at) - \cos at\}\,du$

$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - u\cos at\right]_0^t$

$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$ Ans

Que $\Rightarrow$ $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Solⁿ $\Rightarrow$ $L^{-1}\left(\frac{1}{s^2}\right) = t$

$L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t (t-u)\times\frac{1}{a}\sin au\, du$

$= \frac{1}{a}\int_0^t (t\sin au - \underset{u}{u}\,\underset{v}{\sin au})\, du$

$= \frac{1}{a}\left[t\left(\frac{-\cos au}{a}\right)_0^t - \left\{u\left(\frac{-\cos au}{a}\right) - \left(\frac{-\sin au}{a^2}\right)\right\}_0^t\right]$

$= \frac{1}{a}\left[t\left(-\frac{\cos at}{a} + \frac{1}{a}\right) - \left\{t\left(-\frac{\cos at}{a}\right) + \frac{\sin at}{a^2}\right\}\right]$

$= \cancel{\frac{-t\cos at}{a^2}} + \frac{t}{a^2} + \cancel{\frac{t\cos at}{a^2}} - \frac{\sin at}{a^3}$

$= \left(\frac{t}{a^2} - \frac{\sin at}{a^3}\right)$ Ans

Que $\Rightarrow$ $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$

Solⁿ $\Rightarrow$ $L^{-1}\frac{1}{s^2+1} = \sin t$

$L^{-1}\frac{s}{s^2+4} = \cos 2t$

$L^{-1}\frac{s}{(s^2+1)(s^2+4)} = \int_0^t \sin(t-u)\cdot\cos 2u\, du$

$= \frac{1}{2}\int_0^t 2\sin(t-u)\cdot\cos 2u\, du$

$= \frac{1}{2}\int_0^t \sin(t-u+2u) + \sin(t-u-2u)\, du$

$$\Rightarrow \frac{1}{2}\int_0^t \left[\sin(t+u) + \sin(t-3u)\right] du$$

$$\Rightarrow \frac{1}{2}\left[-\cos(t+u) + \left(-\frac{\cos(t-3u)}{-3}\right)\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left[-\cos 2t + \cos t + \frac{\cos 2t}{3} - \frac{\cos t}{3}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{-2\cos 2t}{3} + \frac{2\cos t}{3}\right]$$

$$\Rightarrow \frac{1}{3}\left[\cos t - \cos 2t\right]$$

Show that

(1) $L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} \ldots \infty$

Sol^n $\Rightarrow$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots \infty$

$$\sin\frac{1}{s} = \frac{1}{s} - \frac{(1/s)^3}{3!} + \frac{(1/s)^5}{5!} \ldots \infty$$

Now,

$$\frac{1}{s}\sin\frac{1}{s} = \frac{1}{s^2} - \frac{1}{(3!)^2 s^4} + \frac{1}{5!\, s^5} \ldots \infty$$

$$L^{-1}\left[\frac{1}{s}\sin\frac{1}{s}\right] = \frac{t}{1!} - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} \ldots \infty$$

Application of Laplace Transform

Transform of Derivative

$L(x) = \bar{x}$
$L(y) = \bar{y}$
$L(f(t)) = \bar{f}$

$$L\{f(t)\} = \bar{f}(s)$$

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

$$\boxed{L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - - - - \infty}$$

Q (1)

$$\frac{d^2y}{dx^2} + w^2 y = 0 \quad , \text{where } y(0) = A \, , \left(\frac{dy}{dx}\right)_0 = B$$

Sol^n Taking L.T. on both sides.

$$L[y''] + w^2 L(y) = 0$$

$$s^2\bar{y} - s\,y(0) - y'(0) + w^2\bar{y} = 0$$

(Given) put $y(0) = A$ and $y'(0) = B$

$$s^2\bar{y} - As - B + w^2\bar{y} = 0$$

$$\bar{y}(s^2 + w^2) = As + B$$

$$\bar{y} = \frac{As}{s^2+w^2} + \frac{B}{s^2+w^2}$$

Taking Inverse LT :-

$$y = A\cos wx + \frac{B}{w}\sin wx$$

Ans

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Que. Solve, $(D^2+9)x = \cos 2t$
if $x(0)=1$, $x\left(\frac{\pi}{2}\right) = -1$

Sol^n $\Longrightarrow$ $(D^2+9)x = \cos 2t$

Taking L.T. on both sides;

$$L[D^2(x)] + 9L(x) = L[\cos 2t]$$

$$s^2\bar{x} - sx(0) - x'(0) + 9\bar{x} = \frac{s}{s^2+4}$$

by Putting Initial value $\Longrightarrow$

$$s^2\bar{x} - s - A + 9\bar{x} = \frac{s}{s^2+4}$$

* Given, $x(0)=1$ but $x'(0)$ is no given in the Que. So assume that $x'(0)=A$

$$(s^2+9)\bar{x} - (s+A) = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{x} = (s+A) + \frac{s}{s^2+4}$$

$$\bar{x} = \frac{A}{s^2+9} + \frac{s}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

Taking Inverse L.T:-

$$x = L^{-1}\frac{A}{s^2+9} + L^{-1}\frac{s}{s^2+9} + L^{-1}\frac{s}{(s^2+4)(s^2+9)}$$

By solving $\frac{s}{(s^2+4)(s^2+9)}$ —— (1)

solving by Convolution thm;

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$$

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{1}{2}\int_0^t \sin 2u \cdot \cos 3(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos 3(t-u)\,du$$

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$$= \frac{1}{4}\int_0^t \sin\{2u+3t-3u\} + \sin\{2u-(3t-3u)\}\,du$$

$$= \frac{1}{4}\int_0^t \sin\{3t-u\} + \sin\{5u-3t\}\,du$$

$$= \frac{1}{4}\left[\frac{+\cos(3t-u)}{+1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\left(\cos 2t - \cos 3t\right) - \left(\frac{\cos 2t - \cos 3t}{5}\right)\right]$$

$$= \frac{1}{4}\left[\frac{4}{5}\cos 2t - \frac{4}{5}\cos 3t\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

So, By ①

$$x = L^{-1}\frac{A}{s^2+9} + L^{-1}\frac{s}{s^2+9} + L^{-1}\frac{s}{(s^2+4)(s^2+9)}$$

$$x = \frac{A}{3}\sin 3t + \cos 3t + \frac{1}{5}\{\cos 2t - \cos 3t\}$$

$$x = \frac{A}{3}\sin 3t + \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t \quad \text{------ ②}$$

Given, $x\left(\frac{\pi}{2}\right) = -1$ Putting this value

$x = -1$ when $t = \pi/2$ in eqn ② :-

$$-1 = \frac{A}{3}\sin\frac{3\pi}{2} + \frac{4}{5}\cos\frac{3\pi}{2} + \frac{1}{5}\cos 2\frac{\pi}{2}$$

$\sin\frac{3\pi}{2} = -1$

$\cos\frac{3\pi}{2} = 0$

$\cos\pi = -1$

$$-1 = -\frac{A}{3} + 0 - \frac{1}{5}$$

$$-1 = \frac{-A}{3} + 0 - \frac{1}{5}$$

$\sin(180+90) =$

$\sin(180+\theta) = -\sin\theta$

So, $\sin(180+90) = -\sin 90 = -1$

$$+1 = +\left(\frac{A}{3} + \frac{1}{5}\right)$$

$$\frac{A}{3} = 1 - \frac{1}{5}$$

$$\frac{A}{3} = \frac{4}{5} \Rightarrow \boxed{A = \frac{12}{5}}$$

$\cos(180+\theta) = -\cos\theta$

So, $\cos(180+90) = -\cos 90 = \boxed{0}$

Find Ans →

$$x = \frac{12}{15}\sin 3t + \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t$$

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Que :- Solve, $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$

Given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

Solve $\Rightarrow$

$$(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$$

Taking ~~Inverse~~ L.T. on both sides ;

$$L[D^3 y] - 3L[D^2 y] + 3L[Dy] - L(y) = L[t^2 e^t]$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$\text{So, } \underset{\text{I}^{\text{st}}\text{ shifting}}{L(t^n e^{at})} = \frac{n!}{(s-a)^{n+1}}$$

$$\Rightarrow [s^3 \bar{y} - s^2 y(0) - s y'(0) - y''(0)] - 3[s^2 \bar{y} - s y(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Putting Initial values $\Rightarrow$

$$[s^3 \bar{y} - s^2(1) - 2] - 3[s^2 \bar{y} - s(1)] + 3[s\bar{y} - 1] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow [s^3 \bar{y} - s^2 - 2] - 3[s^2 \bar{y} - s] + 3[s\bar{y} - 1] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y}(s^3 - 3s^2 + 3s - 1) - (s^2 + 2 + 3s - 3) = \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y}(s^3 - 3s^2 + 3s - 1) - (s^2 - 3s + 1) = \frac{2}{(s-1)^3}$$

$$\Rightarrow \cancel{\boxed{\bar{y} = \frac{2}{(s-1)^3}}} \qquad {}^{**}(a-b)^3 = a^3 - b^3 - 3a^2 b + 3ab^2$$

$$\Rightarrow \bar{y}(s-1)^3 = \frac{2}{(s-1)^3} + s^2 - 3s + 1$$

$$\bar{y} = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3}$$

solving by Partial fraction

$$\bar{y} = \frac{2}{(s-1)^6} + \frac{s^2-3s+1}{(s-1)^3} \quad \text{--- (1)}$$

By Partial Fraction;

$$\frac{s^2-3s+1}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \quad \text{--- (2)}$$

$$\frac{s^2-3s+1}{(s-1)^3} = \frac{A(s-1)^2 + B(s-1) + C}{(s-1)^3}$$

$$s^2-3s+1 = A(s-1)^2 + B(s-1) + C \quad \text{--- (3)}$$

Put $\boxed{s=1}$ $\quad 1-3+1 = C \Rightarrow \boxed{C=-1}$

$$s^2-3s+1 = A[s^2+1-2s] + Bs - B - 1$$

$$s^2-3s+1 = As^2 + s(-2A+B) + A - B - 1$$

By comparing :—

Coeff. of $s^2 \rightarrow$ $\boxed{A=1}$

By comparing :—

Coeff. of $s \rightarrow$ $-3 = -2A + B$

$$-3 = -2 + B \Rightarrow \boxed{B=-1}$$

By (2) $\Rightarrow$ $\frac{s^2-3s+1}{(s-1)^3} = \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$

By (1) $\Rightarrow$ $\bar{y} = \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3}$

Taking Inverse LT.

$$\bar{y} = \frac{2t^5e^t}{5!} + e^t - e^t t - e^t \frac{t^2}{2}$$

$$\bar{y} = \frac{e^t t^5}{60} + e^t - e^t t - \frac{e^t t^2}{2}$$

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Que Solve, $y''' + 2y'' - y' - 2y = 0$
given, $y(0) = y'(0) = 0$, $y''(0) = 6$

Sol^n $y''' + 2y'' - y' - 2y = 0$
Taking L.T.

$$\left[s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)\right] + 2\left[s^2\bar{y} - s y(0) - y'(0)\right] - \left[s\bar{y} - y(0)\right] - 2\bar{y} = 0$$

$$\Rightarrow s^3\bar{y} - 6 + 2s^2\bar{y} - s\bar{y} - 2\bar{y} = 0$$

$$\bar{y}\left[s^3 + 2s^2 - s - 2\right] = 6$$

$$\bar{y} = \frac{6}{s^3 + 2s^2 - s - 2}$$

$$\bar{y} = \frac{6}{(s-1)(s+1)(s+2)}$$

$\left\{ \text{Factors of } s^3 + 2s^2 - s - 2 = (s-1)(s+1)(s+2) \right.$

By Partial fraction:—

$$\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{6}{\cancel{(s-1)(s+1)(s+2)}} = \frac{A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)}{\cancel{(s-1)(s+1)(s+2)}}$$

$$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

Put $s = +1$ $\rightarrow$ $6 = A(s+1)(s+2)$
$6 = A(2)(3) \Rightarrow \boxed{A = 1}$

Put $s = -1$ $\rightarrow$ $6 = B(s-1)(s+2)$
$6 = B(-1-1)(-1+2)$
$6 = -2B \Rightarrow \boxed{B = -3}$

Put $s = -2$ $\rightarrow$ $6 = C(s-1)(s+1)$
$6 = C(-3)(-1) \Rightarrow \boxed{C = 2}$

$$\bar{y} = \frac{6}{(s+1)(s-1)(s+2)}$$

$$\bar{y} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$$

Taking Invese L.T.

$$y = e^{t} - 3e^{-t} + 2e^{-2t}$$ Ans

Inverse Laplace Transforms

If $L\{f(t)\} = \bar{f}(s)$ then Inverse Laplace Transform is defined by

$$\boxed{f(t) = L^{-1}\,\bar{f}(s)}.$$

Here, L^{-1} is inverse L.T. operator.

Standard Formula

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

* (2) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3 \dots$

(3) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(4) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

(5) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(6) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

(7) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$

(8) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(9) $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{(n-1)!}$

(10) $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

(11) $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\{\sin at - at\cos at\}$

Some Rules of Partial Fraction

(1) $\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$

(2) $\frac{1}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2}$

(3) $\frac{1}{s^2+as+b^2} = \frac{As+B}{s^2+as+b^2}$

(4) $\frac{1}{(s^2+as+b^2)^2} = \frac{As+B}{s^2+as}$.

Q(1) Find Inverse L.T.

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$$\Rightarrow \frac{3e^{-t}}{2}\left[\cancel{e^{3t}} + e^{-3t} - \cancel{e^{3t}} + e^{-3t}\right]$$

$$\Rightarrow \frac{3e^{-t}}{\cancel{2}}\left[\cancel{2}e^{-3t}\right]$$

$$\Rightarrow 3e^{-4t} \quad Ans$$

Que $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Sol^n $\Rightarrow$ By Using <u>Partial fraction</u> :

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$s^2+s-2 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3)$$

Put $s=0$, $s=-3$, $s=2$

$\xrightarrow{s=0}$ $-2 = -6A \Rightarrow \boxed{A = 1/3}$

$\xrightarrow{s=-3}$ $9-3-2 = -3B\times -5$

$4 = 15B \Rightarrow \boxed{B = \frac{4}{15}}$

$\xrightarrow{s=2}$ $4+\cancel{2}-\cancel{2} = 2C\times 5$

$4 = 10C \Rightarrow C = \frac{\cancel{4}^{2}}{\cancel{10}_{5}} = \frac{2}{5}$

$\boxed{C = \frac{2}{5}}$

$$= \frac{1/3}{s} + \frac{4/15}{s+3} + \frac{2/5}{s-2}$$

$$L^{-1}\left[\frac{1}{3s} + \frac{4}{15(s+3)} + \frac{2}{5(s-2)}\right] =$$

$$\frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

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(2) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$

Sol^n By Partial fraction:—

$= \dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \cancel{\dfrac{A}{s-7} + \dfrac{Bs+C}{s^2-5s+6}}$

Split the term (s^2-5s+6) $\cancel{= \dfrac{A}{s-7} + }$

$\Rightarrow \dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)} = \dfrac{A}{s-7} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$\Rightarrow s^2-10s+13 = A(s-2)(s-3) + B(s-7)(s-3) + C(s-7)(s-2)$

Put, $s=7$, $s=2$ and $s=3$

$s=7 \longrightarrow 49-70+13 = A(7-2)(7-3)$
$-8 = A\times5\times4$
$-8 = 20A \Rightarrow A = \dfrac{-8}{20} = \dfrac{-2}{5}$
$\boxed{A=-2/5}$

$s=2 \longrightarrow 4-20+13 = B(2-7)(2-3)$
$-3 = B\times-5\times-1$
$-3 = 5B \Rightarrow \boxed{B = \dfrac{-3}{5}}$

$s=3 \longrightarrow 9-30+13 = C(3-7)(3-2)$
$-8 = C\times-4\times1$
$-8 = -4C \Rightarrow \boxed{C=2}$

So $L^{-1}\left[\dfrac{A}{s-7} + \dfrac{B}{s-2} + \dfrac{C}{s-3}\right] = L^{-1}\left[\dfrac{-2/5}{s-7} + \dfrac{-3/5}{s-2} + \dfrac{2}{s-3}\right]$

$\Rightarrow \dfrac{-2e^{7t}}{5} - \dfrac{3}{5}e^{2t} + 2e^{3t}$

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(3) $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

(1)

Sol^n $\Rightarrow$ $\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$

$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2}{(s+2)^2(s-1)^2}$

$\Rightarrow 1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2$ — (ii)

Put $\boxed{s=-2}$

$\Rightarrow 1+2(-2) = B(-2-1)^2$

$-3 = B\times 9 \Rightarrow \boxed{B = -1/3}$

again, put $\boxed{s=1}$

$1+2(1) = D(s+2)^2$

$3 = D(1+2)^2 \Rightarrow 3 = 9D \Rightarrow \boxed{D = 1/3}$

putting the values of B and D in eq^n (ii)

$\Rightarrow 1+2s = A(s+2)(s-1)^2 + \left(-\frac{1}{3}\right)(s-1)^2 + C(s-1)(s+2)^2 + \frac{1}{3}(s+2)^2$

$\Rightarrow 1+2s = A[(s+2)(s^2+1-2s)] - \frac{1}{3}[s^2+1-2s] + C[(s-1)\times(s^2+4+4s)] + \frac{1}{3}(s^2+4+4s)$

$= A[s^3+s-\cancel{2s^2}+\cancel{2s^2}+2-4s] - \frac{1}{3}s^2 - \frac{1}{3} + \frac{2}{3}s + C[s^3+4s+4s^2-s^2+4+4s] + \frac{1}{3}s^2 + \frac{4}{3} + \frac{4}{3}s$

$= A[s^3-3s+2] - \frac{s^2}{3} - \frac{1}{3} + \frac{2}{3}s + C(s^3+8s+3s^2+4) + \frac{s^2}{3} + \frac{4}{3} + \frac{4}{3}s$

$1+2s = s^3(A+C) + s^2\left(-\cancel{\frac{1}{3}} + 3C + \cancel{\frac{1}{3}}\right) + s\left(-3A + \frac{2}{3} + 8C + \frac{4}{3}\right) + 2A - \frac{1}{3} + 4C + \frac{4}{3}$

By comparing s^2's coeff. with LHS ; —

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i.e. $1+2s+0s^2 = s^2(3C)$

$3C=0 \Rightarrow \boxed{C=0}$

Now by comparing coeff of s^3 with LHS $\rightarrow$

i.e. $1+2s+0s^3 = s^3(A+C)$

$0 = A+C$, since $C=0$

$\boxed{A=0}$

By eqn (1) :-

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

put $A=0,\ C=0,\ B=-1/3\ \&\ D=1/3$

$$= \frac{-1/3}{(s+2)^2} + \frac{1/3}{(s-1)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = L^{-1}\left[\frac{-1}{3(s+2)^2} + \frac{1}{3(s-1)^2}\right]$$

Formula $\rightarrow$ $\left[L^{-1}\frac{1}{(s-a)^n} = \frac{e^{at}t^{n-1}}{(n-1)!}\right]$

$$= \frac{-1}{3}\frac{e^{-2t}t}{1!} + \frac{1}{3}\frac{e^{t}t}{1!}$$

$$= \frac{t}{3}\left\{e^{t} - e^{-2t}\right\}$$

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Que $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

Soln $\Rightarrow$ $L^{-1}\left[\frac{s}{\{(s+1)(s-1)\}^2}\right]$

$$L^{-1}\frac{s}{(s+1)^2(s-1)^2}$$

— (I)

$$\frac{s}{(s+1)^2(s-1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

$$\frac{s}{(s+1)^2(s-1)^2}=\frac{A(s+1)(s-1)^2+B(s-1)^2+C(s-1)(s+1)^2+D(s+1)^2}{(s+1)^2(s-1)^2}$$

$$s=A(s+1)(s-1)^2+B(s-1)^2+C(s-1)(s+1)^2+D(s+1)^2 \quad \text{— (II)}$$

put $\boxed{s=1}$

$$1=D(1+1)^2 \Rightarrow 1=4D \Rightarrow \boxed{D=1/4}$$

again, put $\boxed{s=-1}$

$$-1=B(-1-1)^2 \Rightarrow -1=4B \Rightarrow \boxed{B=-1/4}$$

By putting the values of B and D in eqn (II).

$$s=A(s+1)(s-1)^2-\frac{1}{4}(s-1)^2+C(s-1)(s+1)^2+\frac{1}{4}(s+1)^2$$

$$\Rightarrow s=A(s+1)(s^2-2s+1)-\frac{1}{4}(s^2+1-2s)+C(s-1)(s^2+1+2s)+$$
$$\frac{1}{4}(s^2+1+2s)$$

$$\Rightarrow s=A\left[s^3-2s^2+s+s^2-2s+1\right]-\frac{1}{4}s^2-\frac{1}{4}+\frac{1}{2}s+C\left[s^3+s+2s^2-s^2-1-2s\right]+$$
$$\frac{1}{4}s^2+\frac{1}{4}+\frac{s}{2}$$

$$\Rightarrow s=A\left[s^3-s^2-s+1\right]-\frac{1}{4}s^2-\frac{1}{4}+\frac{1}{2}s+C\left[s^3-s+s^2-1\right]$$
$$+\frac{1}{4}s^2+\frac{1}{4}+\frac{s}{2}$$

$$s=s^3(A+C)-s^2(A+\cancel{\tfrac{1}{4}}-C-\cancel{\tfrac{1}{4}})-s(A-\tfrac{1}{2}+C-\tfrac{1}{2})$$
$$A-\cancel{\tfrac{1}{4}}-C+\cancel{\tfrac{1}{4}}$$

By comparing coeff. of s^3 & s^2 with LHS: —

$A+C=0$ — (3) & $-A+C=0$ — (4)

By eqⁿ (3) and (4) :-

$$A + C = 0$$
$$-A + C = 0$$
$$2C = 0 \Rightarrow \boxed{C = 0}$$

Put $C = 0$ in eqⁿ (3) :-

$$A + C = 0$$
$$A + 0 = 0 \Rightarrow \boxed{A = 0}$$

Now put All the values i.e $A = 0$, $C = 0$, $B = -1/4$, $D = 1/4$ in eqⁿ (1)

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = L^{-1}\left[\frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{-1/4}{(s+1)^2} + \frac{1/4}{(s-1)^2}\right]$$

* Formula, $\left\{ L^{-1}\frac{1}{(s-a)^n} = \frac{e^{at}\,t^{n-1}}{(n-1)!} \right\}$

$$= \frac{-1}{4}e^{-t}t + \frac{1}{4}e^{t}t$$

$$\Rightarrow \frac{1}{4}t\left\{e^{t} - e^{-t}\right\}$$

or $\rightarrow \frac{t}{2}\left\{\frac{e^{t} - e^{-t}}{2}\right\}$

$$\Rightarrow \frac{t}{2}\sinh t$$

Que $\Rightarrow$ $L^{-1}\left(\frac{1}{s^3-a^3}\right)$

Solⁿ $\Rightarrow$ $\frac{1}{s^3-a^3} = \frac{1}{(s-a)(s^2+as+a^2)}$

By Partial fraction ;

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A}{s-a} + \frac{Bs+c}{s^2+as+a^2} \quad \text{-----} (1)$$

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A(s^2+as+a^2)+(s-a)(Bs+c)}{(s-a)(s^2+as+a^2)}$$

$$1 = A(s^2+as+a^2)+(s-a)(Bs+c) \quad \text{-----} (2)$$

Put $\xrightarrow{s=a}$

$$1 = A(a^2+a^2+a^2) \Rightarrow 1 = 3a^2A \Rightarrow \boxed{A = \frac{1}{3a^2}}$$

Put $A = \frac{1}{3a^2}$ in eqⁿ (2) :—

$$1 = \frac{1}{3a^2}\left[s^2+as+a^2\right] + (s-a)(Bs+c)$$

$$1 = \frac{1}{3a^2}(s^2+as+a^2) + Bs^2 + cs - Bas - ca$$

$$0s+0s^2+1 = s^2\left(\frac{1}{3a^2}+B\right) + s\left(\frac{1}{3a}+c-Ba\right) + \frac{1}{3}$$

comparing coeffi. of s^2 with LHS :—

$$\frac{1}{3a^2}+B = 0 \Rightarrow \boxed{B = \frac{-1}{3a^2}}$$

again, comparing coeffi. of s with LHS.

$$\frac{1}{3a}+c-Ba = 0 \Rightarrow \frac{1}{3a}+c-\left(\frac{-1}{3a^2}\right)a = 0$$

$$\Rightarrow \boxed{c = \frac{-2}{3a}}$$

By Putting values of A, B, C in (1)

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{1}{3a^2(s-a)} + \frac{-1/3a^2 \div 2/3a}{s^2+as+a^2}$$

$$= \frac{1}{3a^2(s-a)} - \frac{1}{3a^2(s^2+as+a^2)} - \frac{2}{3a(s^2+as+a^2)}$$

These 2 terms are not in standard form of Inverse L.T..

(3)

$$(a+b)^2 = a^2 + 2ab + b^2$$

** $$s^2+as+a^2 = s^2 + 2s\boxed{?} + \boxed{?} + a^2$$

$$= s^2 + 2s\times\frac{a}{2} + \underbrace{\frac{a^2}{4} + \frac{3a^2}{4}}$$

$$= \left[s^2+2s\times\frac{a}{2} + \left(\frac{a}{2}\right)^2\right] + \left(\frac{\sqrt{3}a}{2}\right)^2$$

$$= \left(s+\frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2$$

$$\frac{1}{(s-a)(s^2+as+a^2)} = \cancel{\frac{1}{3a^2(s^2+as}}$$

$$= \frac{1}{3a^2(s-a)} - \frac{1}{3a^2\left[\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$- \frac{2}{3a\left[\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

Taking Inverse L.T. operator on both sides

$$L^{-1}\frac{1}{(s-a)(s^2+as+a^2)} = L^{-1}\left\{\frac{1}{3a^2(s-a)}\right\} - L^{-1}\frac{1}{3a^2\left[(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$- L^{-1}\frac{2}{3a\left[(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-\frac{a}{2}t}\left(\frac{1}{\frac{\sqrt{3}a}{2}}\right)\sin\frac{\sqrt{3}at}{2}$$

$$- \frac{2}{3a}e^{-\frac{a}{2}t}\frac{1}{\left(\frac{\sqrt{3}a}{2}\right)}\sin\frac{\sqrt{3}at}{2}$$

$$= \frac{e^{at}}{3a^2} - \frac{2}{3\sqrt{3}a^3}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2} - \frac{4}{3\sqrt{3}a^2}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2}$$

$$= \frac{e^{at}}{3a^2} - \frac{2}{3\sqrt{3}a^3}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2} - \frac{4}{3\sqrt{3}a^2}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2}$$

: Some Questions :

(1) $$\frac{2s-3}{s^2+4s+13} = \frac{2s-3}{s^2+4s+9+\underbrace{13-9}}$$
$$= \frac{2s-3}{(s^2+4s+4)+9}$$
$$= \frac{2s-3}{(s+2)^2+3^2}$$
$$= \frac{2s}{(s+2)^2+3^2} - \frac{3}{(s+2)^2+3^2}$$

Now solve Inverse L.T. by your own, using standard formula.

(2) $$\frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{(s^2+4s+4+1)^2}$$
$$= \frac{s+2}{[(s+2)^2+1^2]^2}$$

This is in standard form.
Apply formula and find Inverse L.T.

(3) $$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1)^2-s^2}$$
$$= \frac{s}{(s^2+1+s)(s^2+1-s)}$$
$\left\{ \text{By, } a^2-b^2 = (a-b)(a-b) \right.$

$*$ $s^2+1+s = (s+1)^2-\left(\frac{\sqrt{3}}{2}\right)^2$

and $**$ $s^2-s+1 = (s-1)^2-\left(\frac{\sqrt{3}}{2}\right)^2$

(4) $$\frac{a(s^2-2a^2)}{s^4+4a^4}$$

$$s^4+4a^4 = (s^2)^2+(2a^2)^2+4a^2s^2-4a^2s^2$$
$$= (s^2+2a^2+4a^2s^2)-(2as)^2$$
$$= (s^2+2a^2)^2-(2as)^2$$
$$= (s^2+2a^2-2as)(s^2+2a^2+2as)$$

Try to factorise by your own.

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Que $\Rightarrow$ $L^{-1}\left(\frac{1}{s^3-a^3}\right)$

Sol^n $\frac{1}{s^3-a^3} = \frac{1}{(s-a)(s^2+as+a^2)}$

By Partial fraction ;

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A}{s-a} + \frac{Bs+c}{s^2+as+a^2} \quad \text{------ (1)}$$

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A(s^2+as+a^2)+(s-a)(Bs+c)}{(s-a)(s^2+as+a^2)}$$

$$1 = A(s^2+as+a^2)+(s-a)(Bs+c) \quad \text{------ (2)}$$

Put $s=a \rightarrow$

$$1 = A(a^2+a^2+a^2) \Rightarrow 1 = 3a^2A \Rightarrow \boxed{A = \frac{1}{3a^2}}$$

put $A = \frac{1}{3a^2}$ in eq^n (2) ; —

$$1 = \frac{1}{3a^2}\left[s^2+as+a^2\right] + (s-a)(Bs+C)$$

$$1 = \frac{1}{3a^2}(s^2+as+a^2) + Bs^2 + cs - Bas - ca$$

$$0.s+0s^2+1 = s^2\left(\frac{1}{3a^2}+B\right) + s\left(\frac{1}{3a}+c-Ba\right) + \frac{1}{3}$$

comparing coeffi. of s^2 with LHS : —

$$\frac{1}{3a^2}+B = 0 \Rightarrow \boxed{B = \frac{-1}{3a^2}}$$

again, comparing coeff. of s with LHS.

$$\frac{1}{3a}+c-Ba = 0 \Rightarrow \frac{1}{3a}+c-\left(\frac{-1}{3a^2}\right)a = 0$$

$$\Rightarrow \boxed{C = \frac{-2}{3a}}$$

By Putting values of A, B, c in (1)

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{1}{3a^2(s-a)} + \frac{-1/3a^2 s - 2/3a}{s^2+as+a^2}$$

$$= \frac{1}{3a^2(s-a)} - \frac{s}{3a^2(s^2+as+a^2)} - \frac{2}{3a(s^2+as+a^2)}$$

(3)

These 2 terms are not in standard form of Inverse L.T.

$$s^2+as+a^2 = s^2+as+\left(\tfrac{1}{4}+\tfrac{3}{4}\right)a^2$$

(split)

$$= \underbrace{s^2+\left(\tfrac{1}{2}a\right)^2+2\times s\times\tfrac{a}{2}}+\left(\tfrac{\sqrt{3}}{2}a\right)^2$$

$$= \left(s+\tfrac{a}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}a\right)^2$$

$$\Rightarrow \frac{1}{(s-a)(s^2+as+a^2)} = \frac{1}{3a^2(s-a)} - \frac{s}{3a^2\left\{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}} - \frac{2}{3a}\frac{1}{(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}$$

$$= \frac{1}{3a^2(s-a)} - \frac{s+\frac{a}{2}-\frac{a}{2}}{3a^2\left\{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}} - \frac{2}{3a}\frac{1}{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2}$$

$$= \frac{1}{3a^2(s-a)} - \frac{s+a/2}{3a^2\left\{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}} + \frac{a/2}{3a^2\left\{\left(s+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}} - \frac{2}{3a}\frac{1}{\left\{(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}}$$

(2) $$= \frac{1}{3a^2(s-a)} - \frac{s+a/2}{3a^2\left\{(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}} + \frac{1}{6a}\frac{1}{(s+a/2)+\left(\frac{\sqrt{3}a}{2}\right)^2} - \frac{2}{3a}\frac{1}{\left\{(s+a/2)^2+\left(\frac{\sqrt{3}a}{2}\right)^2\right\}}$$

Inverse L.T.

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}at}{2} + \frac{1}{3\cancel{6}a}\times\frac{e^{-\frac{at}{2}}}{\frac{\sqrt{3}a}{2}}\sin\frac{\sqrt{3}at}{2} - \frac{2}{3a}e^{-at/2}\times\frac{1}{\frac{\sqrt{3}a}{2}}\sin\frac{\sqrt{3}at}{2}$$

(3) $$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}}{2}at + \frac{1}{3\sqrt{3}a^2}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}}{2}at - \frac{4}{3\sqrt{3}a^2}e^{-at/2}\sin\frac{\sqrt{3}at}{2}$$

$$= \frac{e^{at}}{3a^2} - \frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}}{2}at - \frac{3}{3\sqrt{3}a^2}e^{-at/2}\sin\frac{\sqrt{3}}{2}at$$

$$= \frac{e^{at}}{3a^2} - \frac{1}{3a^2}e^{-at/2}\cos\frac{\sqrt{3}}{2}at - \frac{1}{\sqrt{3}a^2}e^{-at/2}\sin\frac{\sqrt{3}}{2}at$$

(4)

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Other Method for Inverse L.T.

① **Multiplication by S :—**

If $L^{-1}\bar{f}(s) = f(t)$ then

$$\boxed{L^{-1} s\bar{f}(s) = \frac{d}{dt} f(t)}$$

② **Division by S :—**

If $L^{-1}\bar{f}(s) = f(t)$ then

$$\boxed{L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(t)\,dt}$$

Que. Find Inverse Laplace Tr. of given f^n :—

(a) $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Sol^n → By using division by S :—

$$\bar{f}(s) = \frac{1}{s^2+a^2}$$

then, $L^{-1}\bar{f}(s) = \frac{1}{a}\sin at$

Now, $L^{-1}\frac{1}{s^2(s^2+a^2)} = \int_0^t\int_0^t \frac{1}{a}\sin at\,dt\,dt$

$$= \int_0^t \frac{1}{a}\left\{-\frac{\cos at}{a}\right\}_0^t dt$$

$$= \frac{-1}{a^2}\int_0^t \{\cos at - 1\}\,dt$$

$$= \frac{-1}{a^2}\left[\frac{\sin at}{a} - t\right]_0^t$$

$$= \frac{-1}{a^2}\left[\frac{\sin at}{a} - t\right]$$

$$= \frac{t}{a^2} - \frac{\sin at}{a^3} \quad \text{Ans}$$

(b) $L^{-1} \frac{s}{(s+a)^2}$

Sol^n By Multiplication by s? —

$\bar{f}(s) = \frac{1}{(s+a)^2}$

$L^{-1}[\bar{f}(s)] = e^{-at}\, t$

Now, $L^{-1}\left(\frac{s}{(s+a)^2}\right) = \frac{d}{dt}\left[\underset{I}{e^{-at}}\ \underset{II}{t}\right]$

$= e^{-at}\cdot 1 + t\,a(-e^{-at})$

$= e^{-at} - t e^{-at} a$

$= e^{-at} a (1-t)$ Ans.

** Such type of Questions are solved by multiplication by t method **

(c) $L^{-1} \log\left(\frac{1+s}{s}\right)$

$\Rightarrow$ since, $L^{-1} \log\left(\frac{1+s}{s}\right) = f(t)$

By multiplication by t method

we know that

$L[t f(t)] = -\frac{d}{ds}\bar{f}(s)$

$\xrightarrow{or}$ $t f(t) = L^{-1}\left[-\frac{d}{ds}\bar{f}(s)\right]$

$t f(t) = -L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{s}\right)\right]$

$= -L^{-1}\left[\frac{d}{ds}\log(1+s) - \frac{d}{ds}\log s\right]$

$= -L^{-1}\left[\frac{1}{1+s} - \frac{1}{s}\right]$

$= -\left[e^{-t} - 1\right] = 1 - e^{-t}$

So, $f(t) = \frac{1-e^{-t}}{t}$ Ans.

Que ⟹ $L^{-1} \log\left\{\frac{s+a}{s+b}\right\}$

Sol^n ⟹ Using multiplication by t ;

$$L[t f(t)] = -\frac{d}{ds}\bar{f}(s)$$

$$t f(t) = L^{-1}\left[-\frac{d}{ds}\bar{f}(s)\right]$$

$$t f(t) = -L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)\right]$$

$$= -L^{-1}\left\{\frac{d}{ds}\log(s+a) - \frac{d}{ds}\log(s+b)\right\}$$

$$= -L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= -\left(e^{-at} - e^{-bt}\right)$$

So, $f(t) = \frac{e^{-bt} - e^{-at}}{t}$

Que ⟹ $L^{-1} \log\left[\frac{(s+1)}{(s+2)(s+3)}\right]$

Sol^n ⟹ $t f(t) = L^{-1} -\frac{d}{ds}\bar{f}(s)$

$$= L^{-1}\frac{d}{ds}\left[\log(s+1) - \{\log(s+2) + \log(s+3)\}\right]$$

$$= L^{-1}\left[\frac{1}{s+1} - \left\{\frac{1}{s+2} + \frac{1}{s+3}\right\}\right]$$

$$= -\left\{e^{-t} - e^{-2t} - e^{-3t}\right\}$$

So $f(t) = \frac{-e^{-t} + e^{-2t} + e^{-3t}}{t}$

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Que $L^{-1} \log\left(1-\frac{a^2}{s^2}\right)$

Sol^n $L^{-1} \log\left\{\frac{s^2-a^2}{s^2}\right\}$

$$tf(t) = L^{-1}\left(-\frac{d}{ds}\bar{f}(s)\right)$$

$$tf(t) = L^{-1}\left[-\frac{d}{ds}\log\left(\frac{s^2-a^2}{s^2}\right)\right]$$

$$= -L^{-1}\left[\frac{d}{ds}\log(s^2-a^2) - \frac{d}{ds}\log s^2\right]$$

$$= -L^{-1}\left[\frac{2s}{s^2-a^2} - \frac{2s}{s^2}\right]$$

$$tf(t) = -2\cosh at + 2$$

$$f(t) = \frac{2}{t}(1-\cosh at) \text{ Ans}$$

Que $L^{-1} \tan^{-1}\frac{2}{s}$

Sol^n $tf(t) = L^{-1} -\frac{d}{ds}\bar{f}(s)$

$$= -L^{-1}\frac{d}{ds}\tan^{-1}2/s$$

$$= -L^{-1}\left\{\frac{1}{1+(2/s)^2} \times \frac{-2}{s^2}\right\}$$

$$= +L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$tf(t) = \sin 2t$$

So, $f(t) = \frac{\sin 2t}{t}$ Ans

Imp

Que $L^{-1}\ s\log\frac{s-1}{s+1}$

Sol^n $\Rightarrow$ I^{st} we have to solve $L^{-1}\left(\log\frac{s-1}{s+1}\right)$ then apply multiplication by s :-

let, $\bar{f}(s) = \log\frac{s-1}{s+1}$

$$t f(t) = L^{-1}\ \frac{-d}{ds}\bar{f}(s)$$

$$= L^{-1}\ \frac{-d}{ds}\left(\log\frac{s-1}{s+1}\right)$$

$$= -L^{-1}\left[\frac{d}{ds}\log(s-1) - \frac{d}{ds}\log(s+1)\right]$$

$$= -L^{-1}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$$

$$= -\{e^{t} - e^{-t}\}$$

$$= e^{-t} - e^{t}$$

So, $f(t) = \dfrac{e^{-t} - e^{t}}{t}$

Now, multiplication by s

$$L^{-1}\left(s\log\frac{s-1}{s+1}\right) = \frac{d}{dt}\left[\frac{e^{-t}-e^{t}}{t}\right]$$

$$= \frac{t\{-e^{-t} - e^{t}\} - (e^{-t} - e^{t}).1}{t^2}$$

$$= \frac{-t\{e^{t} + e^{-t}\} + \{e^{t} - e^{-t}\}}{t^2}$$

$$= \frac{-2t\left\{\frac{e^{t}+e^{-t}}{2}\right\} + 2\left\{\frac{e^{t}-e^{-t}}{2}\right\}}{t^2}$$

Application of L.T. - 2nd form

Que ① $ty'' + 2y' + ty = \sin t$

when $y(0) = 1$.

Sol^n $\Rightarrow$ Taking L.T. on both sides;

$$\Rightarrow -\frac{d}{ds}[L(y'')] + 2L(y') + t\,L(y) = L(\sin t)$$

$$\Rightarrow -\frac{d}{ds}\left[s^2\bar{y} - sy(0) - y'(0)\right] + 2\left[s\bar{y} - y(0)\right]$$

$$-\frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

$\left\{\begin{array}{l}\text{Given, } y(0)=1 \\ \& \ y'(0) = A \text{ (let)}\end{array}\right.$

$$\Rightarrow -\frac{d}{ds}\left[\underset{\mathrm{I}}{s^2}\,\underset{\mathrm{II}}{\bar{y}} - s - A\right] + 2\left[s\bar{y} - y(0)\right] - \frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

$$\Rightarrow -\left[s^2\frac{d\bar{y}}{ds} + \bar{y}(2s) - 1\right] + 2s\bar{y} - 2 - \frac{d\bar{y}}{ds} = \frac{1}{s^2+1}$$

$$\Rightarrow \frac{d\bar{y}}{ds}\left[-s^2 - 1\right] + \bar{y}\left[\cancel{-2s} + \cancel{2s}\right] + 1 - 2 = \frac{1}{s^2+1}$$

$$\Rightarrow \frac{d\bar{y}}{ds}(-s^2-1) - 1 = \frac{1}{s^2+1}$$

$$\Rightarrow -\frac{d\bar{y}}{ds} = \frac{1}{(s^2+1)^2} + \frac{1}{(s^2+1)} \quad \text{—— ①}$$

** $-\frac{d\bar{y}}{ds} = ?$

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we know that multiplication by t :-

$$L[t f(t)] = -\frac{d}{ds}\bar{f}(s)$$

$\xrightarrow{\text{or}}$ $t f(t) = L^{-1}\left(\frac{-d\bar{f}(s)}{ds}\right)$

So we will put $-\frac{d\bar{y}}{ds} = t y(t)$ — ②

$\xrightarrow{\text{By ①}}$ Taking Inverse L.T :-

$$L^{-1}\left[\frac{-d\bar{y}}{ds}\right] = L^{-1}\frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$$

$$\xrightarrow{\text{by ②}} t y(t) = L^{-1}\frac{1}{(s^2+1)^2} + \frac{1}{s^2+1}$$

Formula $\longrightarrow$ $\boxed{L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\,\sin at - at\cos at}$

So,

$$t y(t) = \left\{\frac{1}{2}(\sin t - t\cos t)\right\} + \sin t$$

$$t y(t) = \frac{3}{2}\sin t - \frac{t\cos t}{2}$$

therefore, $y(t) = \frac{1}{2t}\left[3\sin t - t\cos t\right]$

Imp

Que, $ty'' + (1-2t)y' - 2y = 0$

when $y'(0) = 1$, $y(0) = 2$.

Solⁿ $\Longrightarrow$ $ty'' + (1-2t)y' - 2y = 0$

Taking L.T. on both sides;

$\Rightarrow -\frac{d}{ds}L(y'') + L(y') + 2\frac{d}{ds}L(y') - 2L(y) = 0$

$\Rightarrow -\frac{d}{ds}[s^2\bar{y} - s(y(0)) - y'(0)] + [s\bar{y} - y(0)] + \frac{2d}{ds}[s\bar{y} - y(0)] - 2\bar{y} = 0$

$\Rightarrow -\frac{d}{ds}[\underset{I}{s^2}\underset{II}{\bar{y}} - 2s - 1] + (s\bar{y} - 2) + 2\frac{d}{ds}[\underset{I}{s}\underset{II}{\bar{y}} - 2] - 2\bar{y} = 0$

$\Rightarrow -[s^2\frac{d\bar{y}}{ds} + 2s\bar{y} - 2] + s\bar{y} - 2 + 2[s\frac{d\bar{y}}{ds} + \bar{y}] - 2\bar{y} = 0$

$\Rightarrow \frac{d\bar{y}}{ds}[-s^2 + 2s] + \bar{y}[-2s + s + \cancel{2} - \cancel{2}] + \cancel{2} - \cancel{2} = 0$

$\Rightarrow (2s - s^2)\frac{d\bar{y}}{ds} - s\bar{y} = 0$

$\Rightarrow \frac{d\bar{y}}{ds} = \frac{s\bar{y}}{2s - s^2}$

$\Rightarrow \frac{d\bar{y}}{ds} = \frac{\bar{y}}{2-s}$ This is form of Linear diffⁿ eqⁿ of Ist order

$*$ $\frac{dy}{dx} + Py = Q$

I.F. $\rightarrow e^{\int P dx}$

Solⁿ $\rightarrow y(\text{I.F}) = \int Q(\text{I.F}) + c$

$\Rightarrow \frac{d\bar{y}}{ds} - \frac{1}{2-s}\bar{y} = 0$

or $\Rightarrow \frac{d\bar{y}}{ds} + \frac{1}{s-2}\bar{y} = 0$

$\Rightarrow$ I.F. $\rightarrow e^{\int \frac{1}{s-2}ds} \Rightarrow e^{\log(s-2)} = (s-2)$

Sol^n $\bar{y} \cdot (s-2) = \int 0(s-2) + c$

or, $\bar{y}(s-2) = c$

$\bar{y} = \frac{c}{s-2}$ Taking Inverse L.T.

$\boxed{y = ce^{2t}}$ —— (1)

Now we have to find $c = ?$

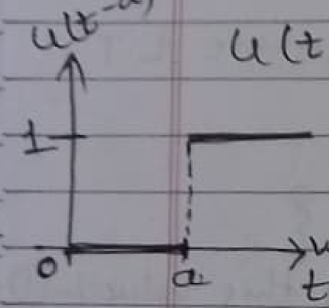
Given, $y(0) = 2$ putting this value in (1)

$2 = ce^0 \Rightarrow \boxed{c=2}$ Put $c=2$ in (1)

$\boxed{y = 2e^{2t}}$ — Ans..

~~Unit Step function~~

Definition → The Unit function $u(t-a)$ is defined by as follows:



$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

where a is always +ve

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Properties :- (2nd shifting Property)

If $L\{f(t)\} = \bar{f}(s)$

then, $L\{f(t-a) \cdot u(t-a)\} = e^{-as}\,\bar{f}(s)$

~~Que ① Find L.T. of $(t-1)^2\, u(t-1)$~~

1st Form

Find L.T. of

Que → $\sin t \cdot u(t-\pi)$

Sol^n → Changing in standard form of

* $L[(t-a)\, u(t-a)]$

$\sin t \cdot u(t-\pi)$

$\Rightarrow \sin(t-\pi+\pi) \cdot u(t-\pi)$

$\Rightarrow \sin[\pi + (t-\pi)] \cdot u(t-\pi)$ $\left(\sin(180+\theta) = -\sin\theta\right)$

$\Rightarrow -\sin(t-\pi) \cdot u(t-\pi)$

$\Rightarrow -L[\sin(t-\pi) \cdot u(t-\pi)]$

$\Rightarrow e^{-as}\bar{f}(s)$ (Formula)

$\Rightarrow -e^{-\pi s} \cdot \dfrac{1}{s^2+1}$

$\Rightarrow \dfrac{-e^{-\pi s}}{s^2+1}$ Ans

Here $a = \pi$
and $f(t) = \sin t$
so $L\{f(t)\} = \dfrac{1}{s^2+1} = \bar{f}(s)$

Question in the form of Intervals

(2nd form)

$$F(t) = \begin{cases} f_1(t), & 0 < t < a_1 \\ f_2(t), & a_1 < t < a_2 \\ f_3(t), & a_2 < t < a_3 \\ f_4(t), & a_3 < t \end{cases}$$

$$f_1(t) \cdot u(t-0) + [f_2(t) - f_1(t)] \cdot u(t-a_1) + [f_3(t) - f_2(t)] \cdot u(t-a_2) + [f_4(t) - f_3(t)] \cdot u(t-a_3)$$

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Que ① $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

Find L.T. of unit step function;

Sol^n $\Rightarrow$

$$f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$$

$$f(t) = \begin{cases} 0; & 0<t<1 \\ t-1, & 1<t<2 \\ 3-t, & 2<t<3 \\ 0, & t>3 \end{cases}$$

$L[f(t-a).u(t-a)] = e^{-as}\bar{f}(s)$

$$= 0.u(t-0) + [t-1-0].u(t-1) + [3-t-(t-1)].u(t-2) + [0-(3-t)].u(t-3)$$

$$= (t-1)\,u(t-1) + (4-2t)\,u(t-2) + (t-3)\,u(t-3)$$

$$= \underbrace{(t-1)}_{a=1}u(t-1) - 2\underbrace{(t-2)}_{a=2}u(t-2) + (t-3)\,u(\underbrace{t-3}_{a=3})$$

$(L(t) = \frac{1}{s^2})$

$$= \frac{e^{-s}}{s^2} - 2e^{-2s} \times \frac{1}{s^2} + e^{-3s} \times \frac{1}{s^2}$$

$$= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2}$$

Ans