Wave Equation (one dimensional)

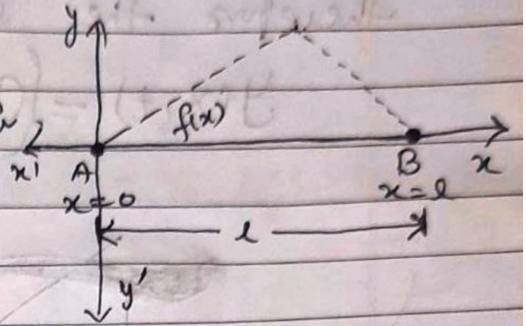
Consider a tightly stretched elastic string of length l and fixed ends A and B. T = Tension which will be considered to be large as compared to the weight of the string so that effects of gravity are negligible.

then,

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ is

"wave eq^n," where, $c^2 = T/m$ = constant.

T = Tension of a string
m = mass per unit



Subject to the condition;—

Boundary Condition $u(0,t) = u(l,t) = 0$
and Initial Condition $u(x,0) = f(x)$ and
$\frac{\partial u}{\partial t} = 0$ at $t=0$, where l is length of string or wire.

Que ① A string is streched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a\sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by $y(x,t) = a\sin\left(\frac{\pi x}{l}\right)\cdot\cos\left(\frac{\pi c t}{l}\right)$

Sol^n ⟹ The vibration of the string is given by

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \quad \text{—— ①}$$

we have to apply Boundary and initial Conditions one by one such that;

As the end of the string are fixed for all time then

$y(0,t) = 0$ —— ②
& $y(l,t) = 0$ —— ③

Since the transverse velocity of any point of the string is zero, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — (4)

Also, $y(x,0) = y = a \sin \frac{\pi x}{l}$ — (5)

Now we have to solve,

(1) subject to the boundary conditions (2) and (3) and initial condition (4) and (5)

Since vibration of the string is periodic, therefore the sol^n of (1) is of the form

$$y(x,t) = \{C_1 \cos px + C_2 \sin px\} \cdot \{C_3 \cos cpt + C_4 \sin cpt\}$$ — (6)

By (2) $y(0,t) = C_1 [C_3 \cos cpt + C_4 \sin cpt]$ — (7)

here, $C_3 \cos cpt + C_4 \sin cpt \neq 0$

so, $\boxed{C_1 = 0}$

put $C_1 = 0$ in eq^n (6) :—

$$y(x,t) = C_2 \sin px \{C_3 \cos cpt + C_4 \sin cpt\}$$ — (8)

Now by eq^n (4) :— $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

differentiating (8) w.r.t t :—

$$\frac{\partial y}{\partial t} = C_2 \sin px \{-C_3 \sin cpt \,(cp) + cp (C_4) \cos cpt\}$$

put $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \{C_4 cp\} = 0$$

whence $C_2 C_4 cp = 0$

If $C_2 = 0$ in (8) will lead to the trivial sol^n $y(x,t) = 0$.

Only possiblity is that $C_4 = 0$

Thus (8) becomes $y(x,t) = C_2C_3 \sin px \cos cpt$ — (9)

Now, by eqn (3), ~~y(l,t)=~~

$y(l,t) = C_2C_3 \sin pl \cdot \cos cpt = 0$, $\forall t$

Since, $C_2 \neq 0$ & $C_3 \neq 0$

we have $\sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

where n = an integer

Hence, (1) reduces to $y(x,t) = C_2C_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l}$ — (10)

Finally imposing the last condition (5) :—

$y(x,0) = C_2C_3 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c(0)}{l}$

$y(x,0) = C_2C_3 \sin \frac{n\pi x}{l}$ ✓

let, $C_2C_3 = a$, $n = 1$

Hence the required soln is {By 10}

$$\boxed{y(x,t) = a \sin \frac{\pi x}{l} \cdot \cos \frac{\pi ct}{l}}$$ — (11)

—x—