Variety

Que→ Evaluate, $\int_0^\infty t\, e^{-2t} \cos t \, dt$

Sol^n→ ~~Here, s=~~

Since, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ — ①

By comparing given integral with ① :-

$s = 2$ and $f(t) = t \cos t$

Now, we have to ~~to~~ find

$$\bar{f}(s) = L\{f(t)\} = L[t \cos t]$$

$$= (-1)^1 \frac{d^1}{ds^1}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)\cdot 1 - s(2s)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= -\left[\frac{-s^2+1}{(s^2+1)^2}\right]$$

put $\xrightarrow{s=2}$ $= -\left[\frac{-(2)^2+1}{(2^2+1)^2}\right]$

$$= -\left[\frac{-4+1}{(5)^2}\right]$$

$$= 3/25 \text{ Ans.}$$

Que Evaluate, $\int_0^\infty e^{-3t}\, t \sin t \, dt$

Sol^n $\Rightarrow$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

By comparing, $s = 3$, $f(t) = t \sin t$

we have to find $\bar{f}(s)$

$\bar{f}(s) = L[t^1 \sin t]$

$= (-1)^1 \frac{d^1}{ds}\left[\frac{1}{s^2+1}\right]$

$= -\left[-\frac{2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$

put $\xrightarrow{s=3}$ $\frac{6}{(9+1)^2} = \frac{6}{100} = \frac{3}{50}$

$= 3/50$ Ans.

Transforms of Integration

If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\bar{f}(s)}$$

Que ① Find L.T. of $\int_0^t \frac{\cos at - \cos bt}{t}\, dt$

Sol^n $\Rightarrow$ $L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \cos at - \cos bt$

$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

Now, $L\left[\frac{\cos at - \cos bt}{t}\right] =$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

let, $s^2+a^2 = u$ | $s^2+b^2 = v$
$2s\,ds = du$ | $2s\,ds = dv$
limits are s to ∞

$$= \int_{s\to}^{\infty} \frac{du}{2u} - \frac{dv}{2v}$$

$$= \frac{1}{2}\left[\log u - \log v\right]$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right.$$

$$= \left\{\frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]\right\}_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log \frac{s^2+b^2}{s^2+a^2}$$

Now, $L\left[\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right]$

By L.T. of Integral.

$$= \frac{1}{s}\bar{f}(s)$$

$$= \frac{1}{2s}\log \frac{s^2+b^2}{s^2+a^2}$$

(2) Find $L\left[\int_0^t \frac{\sin t}{t}\,dt\right]$

Sol^n $\Rightarrow$ $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s}\bar{f}(s)$

let, $f(t) = \frac{\sin t}{t}$

$$L\{f(t)\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= (\tan^{-1}s)_s^\infty = \frac{\pi}{2} - \tan^{-1}s$$

$$= \cot^{-1}s$$

So, $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{\cot^{-1}s}{s}$ Ans.

(By Transforms of Integral)

Another variety (Type of evaluation) Prove that

1) Prove that, $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt = \log 3$

Sol^n $\Rightarrow$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ — (1)

By comparing given Integral by (1)

Such that,

$$\int_0^\infty e^{-t}\cdot\frac{1}{t}\,dt - \int_0^\infty e^{-3t}\cdot\frac{1}{t}\,dt$$

Here, $s=1$, $f_1(t) = \frac{1}{t}$ & $s=3$, $f_2(t) = \frac{1}{t}$

Ist we will find $\bar{f}(s)$

So $L(1) = \frac{1}{s}$

$\Rightarrow L\left(\frac{1}{t}\right) = \int_s^\infty \frac{1}{s}\,ds = (\log s)_s^\infty$

$= \log\infty - \log s$ — (2)

Here, $s \to 1$ $= \log\infty - \log 1$

again, for 2^{nd} Integral,

$$= \log\infty - \log s$$

$$s \to 3 \qquad = \log\infty - \log 3 \quad \text{——} (3)$$

By (2) & (3) :—

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = \cancel{\log\infty} - \log 1 - \cancel{\log\infty} + \log 3$$

$(\log 1 = 0)$ $$= \log 3$$

$$= \underline{\underline{RHS.}}$$

Que $\Rightarrow$ Prove that, $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\, dt = \frac{1}{4}\log 5$

Sol^n $\Rightarrow$ $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

By comparing, $\displaystyle\int_0^\infty e^{-t}\, \frac{\sin^2 t}{t}$

$s = 1$ & $f(t) = \dfrac{\sin^2 t}{t}$ or $\dfrac{1-\cos 2t}{2t}$

$\bar{f}(s) = ?$

$$\bar{f}(s) = L\{f(t)\} = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+4/s^2}\right)\right]_s^\infty$$

$$= 0 - \frac{1}{4}\log\frac{s^2}{s^2+4}$$

$$= +\frac{1}{4}\log\frac{s^2+4}{s^2}$$

Now, put $s=1$ $\Rightarrow \frac{1}{4}\log\frac{1^2+4}{1^2}$

$$= \frac{1}{4}\log 5 \quad \text{Ans}$$

Periodic Function

If $f(t)$ be a function then it is called "periodic function" if it repeats its values in any particular interval.

* Laplace Transform of a periodic function :-

If $f(t)$ be a periodic function then,

$$\boxed{L\{f(t)\} = \int_0^T \frac{e^{-st} f(t)\,dt}{1-e^{-sT}}}$$

where, T = Total Period.

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Q① Find L.T. of given Periodic fnⁿ.

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Solⁿ ⟹ $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

Periodic fnⁿ

Here, T = Total Period $= 2\pi/\omega$.

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}\cdot 0\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}\{-s\sin\omega t - \omega\cos\omega t\}\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\{-s\sin\omega\times\pi/\omega - \omega\cos\omega\times\pi/\omega\} - \frac{e^{-s\times 0}}{s^2+\omega^2}\{-s\sin 0\omega - \omega\cos 0\omega\}\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}\{\omega\} - \frac{1}{s^2+\omega^2}\{-\omega\}\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{\omega}{s^2+\omega^2}\{e^{-s\pi/\omega}+1\}\right]$$

$$= \frac{\omega\,(1+e^{-s\pi/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}$$

Que ⟹ Find L.T. of given Periodic fn.

$$f(t) = \begin{cases} 1, & 0<t<a/2 \\ -1, & a/2<t<a \end{cases}$$

Soln ⟹ $T =$ Total Period $= a$

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{\int_0^{a/2} e^{-st}(1)\,dt + \int_{a/2}^{a} e^{-st}(-1)\,dt}{1-e^{-sa}}$$

$$= \frac{1}{1-e^{-sa}}\left[\left\{-\frac{e^{-st}}{s}\right\}_0^{a/2} + \left\{+\frac{e^{-st}}{s}\right\}_{a/2}^{a}\right]$$

$$= \frac{1}{1-e^{-sa}}\left[\left(-\frac{e^{-as/2}}{s} + \frac{1}{s}\right) + \left(\frac{e^{-sa}}{s} - \frac{e^{-sa/2}}{s}\right)\right]$$

$$= \frac{1}{1-e^{-sa}}\left[-\left(\frac{e^{-as/2} + e^{-sa/2}}{s}\right) + \left(\frac{e^{-sa}+1}{s}\right)\right]$$

$$= \frac{1}{1-e^{-sa}}\left[-\frac{2e^{-as/2}}{s} + \frac{e^{-sa}+1}{s}\right]$$

$$= \frac{1}{1-e^{-sa}}\left[\frac{1+e^{-sa}-2e^{-as/2}}{s}\right]$$

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$$= \int_{s\to}^{\infty} \frac{du}{2u} - \frac{dv}{2v}$$

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So, $L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{\cot^{-1}s}{s}$ A

(By Transforms of Integral)