Inverse Laplace Transforms

If $L\{f(t)\} = \bar{f}(s)$ then Inverse Laplace Transform is defined by

$$\boxed{f(t) = L^{-1}\,\bar{f}(s)}$$

Here, L^{-1} is inverse L.T. operator.

Standard Formula

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

* (2) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$, $n = 1, 2, 3 ----$

(3) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(4) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

(5) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(6) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

(7) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$

(8) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(9) $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{(n-1)!}$

(10) $L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt$

(11) $L^{-1}\frac{1}{(s^2+a^2)^2} = \frac{1}{2a^3}\{\sin at - at\cos at\}$

Some Rules of Partial Fraction

(1) $$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

(2) $$\frac{1}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2}$$

(3) $$\frac{1}{s^2+as+b^2} = \frac{AS+B}{s^2+as+b^2}$$

(4) $$\frac{1}{(s^2+as+b^2)^2} = \frac{AS+B}{s^2+as}$$

Q(1) Find Inverse L.T.

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$$\Rightarrow \frac{3e^{-t}}{2}\left[\cancel{e^{3t}}+e^{-3t}-\cancel{e^{3t}}+e^{-3t}\right]$$

$$\Rightarrow \frac{3e^{-t}}{\cancel{2}}\left[\cancel{2}e^{-3t}\right]$$

$$\Rightarrow 3e^{-4t}$$ Ans

Que $$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$$

Sol^n $\Rightarrow$ By Using Partial fraction :

$$\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$$

$$s^2+s-2 = A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$$

Put $s=0$, $s=-3$, $s=2$

$s=0 \Rightarrow$ $-2=-6A$ $\Rightarrow \boxed{A=1/3}$

$s=-3 \Rightarrow$ $9-3-2=-3B\times-5$

$4=15B$ $\Rightarrow \boxed{B=\frac{4}{15}}$

$s=2 \Rightarrow$ $4+\cancel{2}-\cancel{2}=2C\times5$

$4=10C$ $\Rightarrow C=\frac{\cancel{4}^{2}}{\cancel{10}_{5}}=\frac{2}{5}$

$\boxed{C=\frac{2}{5}}$

$$=\frac{1/3}{s}+\frac{4/15}{s+3}+\frac{2/5}{s-2}$$

$$L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]=$$

$$\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$ Ans

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(2) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$

Sol^n By Partial fraction:—

$$= \frac{s^2-10s+13}{(s-7)(s^2-5s+6)} \cancel{= \frac{A}{s-7} + \frac{Bs+C}{s^2-5s+6}}$$

Split the term (s^2-5s+6) $\cancel{= \frac{A}{s-7} + }$

$$\Rightarrow \frac{s^2-10s+13}{(s-7)(s-2)(s-3)} = \frac{A}{s-7} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow s^2-10s+13 = A(s-2)(s-3) + B(s-7)(s-3) + C(s-7)(s-2)$$

Put, $s=7$, $s=2$ and $s=3$

$s=7 \rightarrow$ $49-70+13 = A(7-2)(7-3)$

$-8 = A\times 5\times 4$

$-8 = 20A \Rightarrow A = \frac{-8}{20} = \frac{-2}{5}$

$\boxed{A = -2/5}$

$s=2 \rightarrow$ $4-20+13 = B(2-7)(2-3)$

$-3 = B\times -5\times -1$

$-3 = 5B \Rightarrow \boxed{B = \frac{-3}{5}}$

$s=3 \rightarrow$ $9-30+13 = C(3-7)(3-2)$

$-8 = C\times -4\times 1$

$-8 = -4C \Rightarrow \boxed{C=2}$

So $L^{-1}\left[\frac{A}{s-7} + \frac{B}{s-2} + \frac{C}{s-3}\right] = L^{-1}\left[\frac{-2/5}{s-7} + \frac{-3/5}{s-2} + \frac{2}{s-3}\right]$

$$\Rightarrow \frac{-2e^{7t}}{5} - \frac{3}{5}e^{2t} + 2e^{3t}$$

Ans

(3) $L^{-1}\left[\dfrac{1+2s}{(s+2)^2(s-1)^2}\right]$ (1)

Sol^n $\Rightarrow$ $\dfrac{1+2s}{(s+2)^2(s-1)^2} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2}$

$\dfrac{1+2s}{(s+2)^2(s-1)^2} = \dfrac{A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2}{(s+2)^2(s-1)^2}$

$\Rightarrow 1+2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s-1)(s+2)^2 + D(s+2)^2$ — (ii)

put $\boxed{s=-2}$

$\Rightarrow 1+2(-2) = \cancel{B(s+2)^2}\ B(-2-1)^2$

$-3 = B \times 9 \Rightarrow \boxed{B = -1/3}$

again, put $\boxed{s=1}$

$1+2(1) = D(s+2)^2$

$3 = D(1+2)^2 \Rightarrow 3 = 9D \Rightarrow \boxed{D = 1/3}$

putting the values of B and D in eq^n (ii)

$\Rightarrow 1+2s = A(s+2)(s-1)^2 + \left(-\frac{1}{3}\right)(s-1)^2 + C(s-1)(s+2)^2 + \frac{1}{3}(s+2)^2$

$\Rightarrow 1+2s = A[(s+2)(s^2+1-2s)] - \frac{1}{3}[s^2+1-2s] + C[(s-1)\times(s^2+4+4s)] + \frac{1}{3}(s^2+4+4s)$

$= A[s^3+s-\cancel{2s^2}+\cancel{2s^2}+2-4s] - \frac{1}{3}s^2 - \frac{1}{3} + \frac{2}{3}s + C[s^3+4s+4s^2-s^2+4+4s] + \frac{1}{3}s^2 + \frac{4}{3} + \frac{4}{3}s$

$= A[s^3-3s+2] - \frac{s^2}{3} - \frac{1}{3} + \frac{2}{3}s + C(s^3+8s+3s^2+4) + \frac{s^2}{3} + \frac{4}{3} + \frac{4}{3}s$

$1+2s = s^3(A+C) + s^2\left(-\cancel{\tfrac{1}{3}} + 3C + \cancel{\tfrac{1}{3}}\right) + s\left(-3A + \frac{2}{3} + 8C + \frac{4}{3}\right) + 2A - \frac{1}{3} + 4C + \frac{4}{3}$

By comparing s^2's coeff. with LHS ; —

i.e. $1+2s+0s^2 = s^2(3C)$

$3C=0 \Rightarrow \boxed{C=0}$

Now by comparing coeff of s^3 with LHS

i.e. $1+2s+0s^3 = s^3(A+C)$

$0 = A+C$, since $C=0$

$\boxed{A=0}$

By eqn (1) :-

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

Put $A=0$, $C=0$, $B=-1/3$ & $D=1/3$

$$= \frac{-1/3}{(s+2)^2} + \frac{1/3}{(s-1)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = L^{-1}\left[\frac{-1}{3(s+2)^2} + \frac{1}{3(s-1)^2}\right]$$

Formula $\rightarrow$ $\left[L^{-1}\frac{1}{(s-a)^n} = \frac{e^{at}t^{n-1}}{(n-1)!}\right]$

$$= \frac{-1}{3}\frac{e^{-2t}t}{1!} + \frac{1}{3}e^{t}\frac{t}{1!}$$

$$= \frac{t}{3}\left\{e^{t} - e^{-2t}\right\}$$

Ans

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Que $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

Sol^n $\Rightarrow$ $L^{-1}\left[\frac{s}{\{(s+1)(s-1)\}^2}\right]$

$$L^{-1}\frac{s}{(s+1)^2(s-1)^2}$$

$$\frac{s}{(s+1)^2(s-1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2} \quad \text{— (I)}$$

$$\frac{s}{(s+1)^2(s-1)^2}=\frac{A(s+1)(s-1)^2+B(s-1)^2+C(s-1)(s+1)^2+D(s+1)^2}{(s+1)^2(s-1)^2}$$

$$s=A(s+1)(s-1)^2+B(s-1)^2+C(s-1)(s+1)^2+D(s+1)^2 \quad \text{— (II)}$$

put $\boxed{s=1}$

$$1=D(1+1)^2 \Rightarrow 1=4D \Rightarrow \boxed{D=1/4}$$

again, put $\boxed{s=-1}$

$$-1=B(-1-1)^2 \Rightarrow -1=4B \Rightarrow \boxed{B=-1/4}$$

By putting the values of B and D in eq^n (II).

$$s=A(s+1)(s-1)^2-\frac{1}{4}(s-1)^2+C(s-1)(s+1)^2+\frac{1}{4}(s+1)^2$$

$$\Rightarrow s=A(s+1)(s^2-2s+1)-\frac{1}{4}(s^2+1-2s)+C(s-1)(s^2+1+2s)+\frac{1}{4}(s^2+1+2s)$$

$$\Rightarrow s=A[s^3-2s^2+s+s^2-2s+1]-\frac{1}{4}s^2-\frac{1}{4}+\frac{1}{2}s+C[s^3+s+2s^2-s^2-1-2s]+\frac{1}{4}s^2+\frac{1}{4}+\frac{s}{2}$$

$$\Rightarrow s=A[s^3-s^2-s+1]-\frac{1}{4}s^2-\frac{1}{4}+\frac{1}{2}s+C[s^3-s+s^2-1]+\frac{1}{4}s^2+\frac{1}{4}+\frac{s}{2}$$

$$s=s^3(A+C)-s^2(A+\cancel{\tfrac{1}{4}}-C-\cancel{\tfrac{1}{4}})-s(A-\tfrac{1}{2}+C-\tfrac{1}{2})$$

$$A-\cancel{\tfrac{1}{4}}-C+\cancel{\tfrac{1}{4}}$$

By comparing coeff. of s^3 & s^2 with LHS :—

$A+C=0$ — (3) & $-A+C=0$ — (4)

By eqⁿ (3) and (4) :-

$$A + C = 0$$
$$-A + C = 0$$
$$2C = 0 \Rightarrow \boxed{C = 0}$$

Put $C = 0$ in eqⁿ (3) :-

$$A + C = 0$$
$$A + 0 = 0 \Rightarrow \boxed{A = 0}$$

Now put All the values i.e $A = 0$, $C = 0$, $B = -1/4$, $D = 1/4$ in eqⁿ (1)

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = L^{-1}\left[\frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{-1/4}{(s+1)^2} + \frac{1/4}{(s-1)^2}\right]$$

* Formula, $\left\{ L^{-1} \frac{1}{(s-a)^n} = \frac{e^{at}\, t^{n-1}}{(n-1)!} \right\}$

$$= \frac{-1}{4} e^{-t} t + \frac{1}{4} e^{t} t$$

$$\Rightarrow \frac{1}{4} t \left\{ e^{t} - e^{-t} \right\}$$

or $\rightarrow$ $\frac{t}{2} \left\{ \frac{e^{t} - e^{-t}}{2} \right\}$

$$\Rightarrow \frac{t}{2} \sinh t$$

Que $\Rightarrow$ $L^{-1}\left(\frac{1}{s^3-a^3}\right)$

Solⁿ $\frac{1}{s^3-a^3} = \frac{1}{(s-a)(s^2+as+a^2)}$

By Partial fraction;

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2} \quad \text{-----} (1)$$

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{A(s^2+as+a^2)+(s-a)(Bs+C)}{(s-a)(s^2+as+a^2)}$$

$$1 = A(s^2+as+a^2)+(s-a)(Bs+C) \quad \text{-----} (2)$$

Put $s = a \rightarrow$

$$1 = A(a^2+a^2+a^2) \Rightarrow 1 = 3a^2A \Rightarrow \boxed{A = \frac{1}{3a^2}}$$

Put $A = \frac{1}{3a^2}$ in eqⁿ (2) :—

$$1 = \frac{1}{3a^2}\left[s^2+as+a^2\right] + (s-a)(Bs+C)$$

$$1 = \frac{1}{3a^2}(s^2+as+a^2) + Bs^2+Cs-Bas-Ca$$

$$0s+0s^2+1 = s^2\left(\frac{1}{3a^2}+B\right) + s\left(\frac{1}{3a}+C-Ba\right) + \frac{1}{3}$$

Comparing coeffi. of s^2 with LHS :—

$$\frac{1}{3a^2}+B = 0 \Rightarrow \boxed{B = \frac{-1}{3a^2}}$$

again, comparing coeffi. of s with LHS.

$$\frac{1}{3a}+C-Ba = 0 \Rightarrow \frac{1}{3a}+C-\left(\frac{-1}{3a^2}\right)a = 0$$

$$\Rightarrow \boxed{C = \frac{-2}{3a}}$$

By Putting values of A, B, C in (1)

$$\frac{1}{(s-a)(s^2+as+a^2)} = \frac{1}{3a^2(s-a)} + \frac{-1/3a^2 \cdot s - 2/3a}{s^2+as+a^2}$$

$$= \frac{1}{3a^2(s-a)} - \frac{s}{3a^2(s^2+as+a^2)} - \frac{2}{3a(s^2+as+a^2)}$$

These 2 terms are not in standard form of Inverse L.T..

(3)

$$(a+b)^2 = a^2 + 2ab + b^2$$

** $$s^2 + as + a^2 = s^2 + 2s\boxed{?} + \boxed{?} + a^2$$

$$= s^2 + 2s \times \frac{a}{2} + \frac{a^2}{4} + \frac{3a^2}{4}$$

$$= \left[s^2 + 2s \times \frac{a}{2} + \left(\frac{a}{2}\right)^2\right] + \left(\frac{\sqrt{3}a}{2}\right)^2$$

$$= \left(s + \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2$$

$$\frac{1}{(s-a)(s^2+as+a^2)} = \cancel{\frac{1}{3a^2(s^2+as}}$$

$$= \frac{1}{3a^2(s-a)} - \frac{1}{3a^2\left[\left(s+\frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$- \frac{2}{3a\left[\left(s+\frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

Taking Inverse L.T. operator on both sides

$$L^{-1}\frac{1}{(s-a)(s^2+as+a^2)} = L^{-1}\left\{\frac{1}{3a^2(s-a)}\right\} - L^{-1}\frac{1}{3a^2\left[\left(s+a/2\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$- L^{-1}\frac{2}{3a\left[\left(s+a/2\right)^2 + \left(\frac{\sqrt{3}a}{2}\right)^2\right]}$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}e^{-a/2\,t}\left(\frac{1}{\frac{\sqrt{3}a}{2}}\right)\sin\frac{\sqrt{3}at}{2}$$

$$- \frac{2}{3a}e^{-\frac{a}{2}t}\frac{1}{\left(\frac{\sqrt{3}a}{2}\right)}\sin\frac{\sqrt{3}at}{2}$$

$$= \frac{e^{at}}{3a^2} - \frac{2}{3\sqrt{3}a^3}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2} - \frac{4}{3\sqrt{3}a^2}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2}$$

$$= \frac{e^{at}}{3a^2} - \frac{2}{3\sqrt{3}a^3}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2} - \frac{4}{3\sqrt{3}a^2}e^{-\frac{at}{2}}\sin\frac{\sqrt{3}at}{2}$$

: Some Questions :

(1) $$\frac{2s-3}{s^2+4s+13} = \frac{2s-3}{s^2+4s+9+\underbrace{13-9}}$$

$$= \frac{2s-3}{(s^2+4s+4)+9}$$

$$= \frac{2s-3}{(s+2)^2+3^2}$$

$$= \frac{2s}{(s+2)^2+3^2} - \frac{3}{(s+2)^2+3^2}$$

Now Solve Inverse L.T. by your own, using standard formula.

(2) $$\frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{(s^2+4s+4+1)^2}$$

$$= \frac{s+2}{[(s+2)^2+1^2]^2}$$

This is in standard form. Apply formula and find Inverse L.T.

(3) $$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+1)^2-s^2}$$

$$= \frac{s}{(s^2+1+s)(s^2+1-s)}$$

By, $a^2-b^2 = (a-b)(a-b)$

* $s^2+1+s = (s+1)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

and ** $s^2-s+1 = (s-1)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

(4) $$\frac{a(s^2-2a^2)}{s^4+4a^4}$$

$$s^4+4a^4 = (s^2)^2+(2a^2)^2+4a^2s^2-4a^2s^2$$

$$= (s^2+2a^2+4a^2s^2) - (2as)^2$$

$$= (s^2+2a^2)^2 - (2as)^2$$

$$= (s^2+2a^2-2as)(s^2+2a^2+2as)$$

Try to factorise by your own.