I^{st} Shifting Property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\left[e^{at} f(t)\right] = \bar{f}(s-a)$$

Useful Results :—

① $L\left[e^{at} t^n\right] = \dfrac{n!}{(s-a)^{n+1}}$

② $L\left[e^{at} \sin at\right] = \dfrac{a}{(s-a)^2 + b^2}$

③ $L\left[e^{at} \cos at\right] = \dfrac{s-a}{(s-a)^2 + b^2}$

④ $L\left[e^{at} \sinh at\right] = \dfrac{a}{(s-a)^2 - b^2}$

⑤ $L\left[e^{at} \cosh at\right] = \dfrac{s-a}{(s-a)^2 - b^2}$

Que ① // Find L.T. $L\left[\cosh 3t \cdot \cos 2t\right]$

Sol^n Let $f(t) = \cosh 3t \cdot \cos 2t$

Taking L.T. operator on both sides,

$$L[f(t)] = L\left[\cosh 3t \cdot \cos 2t\right]$$

$$L[f(t)] = L\left[\left(\frac{e^{3t} + e^{-3t}}{2}\right) \cos 2t\right]$$

$$= \frac{1}{2}\left[L(e^{3t}\cos 2t) + L(e^{-3t}\cos 2t) \right]$$

By Linearity prop.

$$= \frac{1}{2}\left[\frac{s-3}{(s-3)^2+4} + \frac{s+3}{(s+3)^2+4} \right]$$

Ans..

Que Find L.T. of $e^{-t}\sin^2 t$.

Soln ⟹ let, $f(t) = e^{-t}\sin^2 t$

Taking L.T. on both sides;

$$L[f(t)] = L[e^{-t}\sin^2 t]$$

$$L[f(t)] = L\left[e^{-t}\left\{\frac{1-\cos 2t}{2}\right\}\right]$$

$$= \frac{1}{2} L\left[e^{-t} - e^{-t}\cos 2t\right]$$ By Linearity Prop

$$= \frac{1}{2}\left\{ L\{e^{-t}\} - L\{e^{-t}\cos 2t\} \right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4} \right]$$

$$= \frac{2}{(s+1)(s^2+2s+5)}$$ Ans..

Laplace Transforms for functions which are divided into subintervals

Q(1) $f(t) = \begin{cases} e^t, & 0<t<1 \\ 0, & t>1 \end{cases}$

Find Laplace Transforms.

Sol^n $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L\{f(t)\} = \left[\int_0^1 e^{-st}.e^t\,dt + \int_1^\infty e^{-st}.0\,dt\right]$$

$$= \int_0^1 e^{-(s-1)t}\,dt$$

$$= \left[\frac{-e^{(s-1)t}}{s-1}\right]_0^1$$

$$= \frac{-e^{(s-1)}}{s-1} + \frac{1}{s-1}$$

$$= \frac{1}{s-1}\left\{1-e^{(s-1)}\right\}$$ Ans..

Q(2) $f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$

Find Laplace Transforms.

Sol^n $L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\pi e^{-st}.\sin t\,dt + \int_\pi^\infty e^{-st}.0\,dt$$

$$= \int_0^\pi e^{-st} \sin t\,dt$$

By using formula, $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left\{a\sin bx - b\cos bx\right\}$

$$= \left[\frac{e^{-st}}{s^2+1}\{-s\sin t - \cos t\}\right]_0^{\pi}$$

$$= \frac{e^{-s\pi}}{s^2+1}\{+1\} - \frac{1}{s^2+1}\{-1\} \qquad \begin{pmatrix}\sin\pi = 0\\ \cos\pi = -1\end{pmatrix}$$

$$= \frac{1}{s^2+1}\{1+e^{-s\pi}\} \quad \text{Ans..}$$

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* H.W. Q *

$$f(t) = \begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & t>3 \end{cases}$$

Ans:- $$\bar{f}(s) = \frac{2}{s^3} - \frac{e^{-2s}}{s^3}\{2+3s+2s^2\} - \frac{e^{-3s}}{s^2}\{-4s+1\}$$

Que Imp → Find Laplace transform of

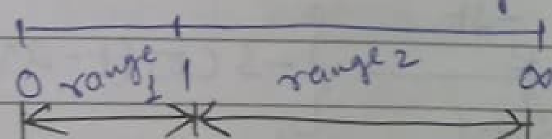
$$f(t) = |t-1| + |t+1| \quad , t \geq 0$$

∴ Sol^n ⟹ Since mod of any f^n is defined by

$$|x| = \begin{cases} -x, & x<0 \\ +x, & x>0 \end{cases}$$

Since, Given range is $t \to 0$ to ∞

So we distribute the range in 2 parts such that,



$$f(t) = -(\cancel{t}-1) + \cancel{t}+1 \quad , \quad 0<t<1$$
$$= 2$$

and,

$$f(t) = t-1+t+1 \quad , \quad t>1$$
$$= 2t$$

So,

$$L\{f(t)\} = \int_0^1 e^{-st}\, 2\,dt + \int_1^\infty e^{-st} \cdot 2t\,dt$$

$$= 2\left\{\frac{-e^{-st}}{s}\right\}_0^1 + 2\left[t\left\{\frac{-e^{-st}}{s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}\right]_1^\infty$$

$$= \left(\frac{-2e^{-s}}{s} + \frac{2}{s}\right) - 2\left(\frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2}\right)$$

$$= \cancel{\frac{-2e^{-s}}{s}} + \frac{2}{s} + \cancel{\frac{2e^{-s}}{s}} + \frac{2e^{-s}}{s^2}$$

$$= \frac{2}{s}\left\{1 + \frac{e^{-s}}{s}\right\}$$ Ans..

Imp

Que ⟹ Find L.T. of

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

Soln ⟹

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{2\pi/3} e^{-st} \cdot 0\,dt + \int_{2\pi/3}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

* By using $\int e^{ax}\cos bx\,dx$ * →

$$= 0 \left[\frac{e^{-st}}{s^2+1}\left\{-s\cos(t - 2\pi/3) + \sin\left(t - \frac{2\pi}{3}\right)\right\}\right]_{2\pi/3}^\infty$$

$$= -\left[\frac{e^{-\frac{2\pi s}{3}}}{s^2+1}\{-s\}\right]$$

$$= \frac{s e^{-\frac{2\pi s}{3}}}{s^2+1}$$ Ans..

* Some Different Questions *

① Find $L\, f(t) = 2^t$

Sol^n ⟹ let, $f(t) = 2^t$

$f(t) = e^{\log 2^t}$

$f(t) = e^{t\log 2}$

$f(t) = e^{(\log 2)t}$ $(a = \log 2)$

$L\{f(t)\} = L\{e^{(\log 2)t}\}$

$= \dfrac{1}{s - \log 2}$ Ans

② Find $L(f(t)) = \sin^5 t$

Sol^n let

$f(t) = \sin^5 t$

$= \sin^4 t \cdot \sin t$

$= (\sin^2 t)^2 \sin t$

$= \left(\dfrac{1 - \cos 2t}{2}\right)^2 \sin t$

$= \dfrac{1}{4}\left[(1 + \cos^2 2t - 2\cos 2t)\sin t\right]$

$= \dfrac{1}{4}\left[\left\{1 + \left(\dfrac{1 + \cos 2t}{2}\right) - 2\cos 2t\right\}\sin t\right]$

$= \dfrac{1}{4}\left[\sin t + \dfrac{\sin t}{2} + \dfrac{\sin t \cos 2t}{2} - 2\sin t \cos 2t\right]$

$= \dfrac{1}{4}\left[\dfrac{3\sin t}{2} + \dfrac{2\sin t \cos 2t}{4} - 2\sin t \cos 2t\right]$

$= \dfrac{1}{4}\left[\dfrac{3\sin t}{2} + \dfrac{\sin 3t - \sin t}{4} - (\sin 3t - \sin t)\right]$

$= \dfrac{1}{4}\left[\dfrac{3\sin t}{2} + \dfrac{\sin 3t}{4} - \dfrac{\sin t}{4} - \sin 3t + \sin t\right]$

$= \dfrac{9}{16}\sin t - \dfrac{3}{16}\sin 3t$

Taking L.T. on both sides;

$$L\{f(t)\} = \frac{9}{16} L\{\sin t\} - \frac{3}{16} L\{\sin 3t\}$$

$$= \frac{9}{16}\left\{\frac{1}{s^2+1}\right\} - \frac{3}{16}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{9}{16}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$= \frac{9}{16}\left[\frac{s^2+9-(s^2+1)}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{9}{\cancel{16}\,2}\left[\frac{\cancel{8}\,1}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{9}{2(s^2+1)(s^2+9)} \quad \text{Ans}$$

— x —

Change of Scale Property

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Que ⟹ Solve $L\{\cos 7x\}$ by change of scale property.

Sol^n ⟹ We know that $L(\cos x) = \frac{s}{s^2+1} = \bar{f}(s)$

then

$$L(\cos 7x) = \frac{1}{7}\bar{f}\left(\frac{s}{7}\right) \quad \text{By change of scale Prop.}$$

$$= \frac{1}{7}\left\{\frac{s/7}{(s/7)^2+1}\right\}$$

$$= \frac{1}{7}\left\{\frac{s/7}{s^2/49+1}\right\} = \frac{1}{\cancel{7}}\left\{\frac{s/\cancel{7}}{\frac{s^2+49}{49}}\right\}$$

$$= \frac{s}{s^2+49} = \frac{s}{s^2+(7)^2} \quad \text{Ans}$$

Multiplication by t

If $L\{f(t)\} = \bar{f}(s)$ Then

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)}$$

① Find $L\{f(t)\} = L\{t \sin^2 t\}$

Sol^n ⟹ 1st we have to find $L\{\sin^2 t\}$ which is $\bar{f}(s)$.

let, $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2}$$

$$L\{f(t)\} = L\left[\frac{1-\cos 2t}{2}\right] \quad \text{Taking L.T. on both sides}$$

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right] \quad \text{By Linearity Prop.}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Now by multiplication by t :—

$$L\{t^1 \sin^2 t\} = (-1)^1 \frac{d^1}{ds^1}\left[\frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}\right]$$

(Here $n=1$)

$$= -\frac{1}{2}\left[\frac{-1}{s^2} - \left\{\frac{(s^2+4).1 - s(2s)}{(s^2+4)^2}\right\}\right]$$

$$= -\frac{1}{2}\left[\frac{-1}{s^2} - \left\{\frac{s^2+4-2s^2}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} + \left\{\frac{-s^2+4}{(s^2+4)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s^2+4)^2 + s^2(-s^2+4)}{s^2(s^2+4)^2}\right]$$

$$= \cancel{\tfrac{1}{2}}\left[\frac{6s^2+8}{s^2(s^2+4)^2}\right] \quad \text{By calculation}$$

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Find,

(2) $L[t \sin 3t \cdot \cos 2t]$

Sol^n $\Rightarrow$

let, $f(t) = \sin 3t \cdot \cos 2t$

$f(t) = \frac{1}{2} \times 2 \sin 3t \cdot \cos 2t$

$f(t) = \frac{1}{2}[\sin 5t + \sin t]$

Taking L.T. operator on both sides;

$L\{f(t)\} = \frac{1}{2} L(\sin 5t + \sin t)$ By linearity prop.

$= \frac{1}{2}[L(\sin 5t) + L(\sin t)]$

$\bar{f}(s) = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$

Now, by multiplication by t :—

$L[t^1 \sin 3t \cdot \cos 2t] = (-1)^1 \frac{d^1}{ds^1} \bar{f}(s)$

$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25}\right) + \frac{1}{2}\frac{1}{s^2+1}\right]$

$$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{5}{s^2+25}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{-5 \times 2s}{(s^2+25)^2}\right] + \frac{-1}{2}\left(\frac{-2s}{(s^2+1)^2}\right)]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2} \quad \text{Ans..}$$

Que. $L\left[t^2 \cos at\right]$

Sol^n $L\{\cos at\} = \frac{s}{s^2+a^2} = \bar{f}(s)$

then, $L\left[t^2 \cos at\right] = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$

$$= \frac{d}{ds}\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2\cdot(-2s) - (a^2-s^2)\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

This is easiest way to solve

$$= \frac{(s^2+a^2)\left\{(s^2+a^2)(-2s) - (a^2-s^2)\cdot 4s\right\}}{(s^2+a^2)^4}$$

$$= \frac{\left\{-2s^3 - 2a^2s - 4a^2s + 4s^3\right\}}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6sa^2}{(s^2+a^2)^3} \quad \text{Ans..}$$

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Que $\Rightarrow$ $L\{t e^{-2t} \sin 2t\}$

Sol^n $\Rightarrow$ let $f(t) = \sin 2t$

$$L\{f(t)\} = \bar{f}(s) = \frac{2}{s^2+4}$$

Now by 1^{st} shifting prop.

$$L[e^{-2t} \sin 2t] = \frac{2}{(s+2)^2+4} \qquad (a=-2)$$

again,

$$L[t^1 e^{-2t} \sin 2t] = (-1)^1 \frac{d^1}{ds^1}\left[\frac{2}{(s+2)^2+4}\right]$$

$$= -2\left[\frac{-2\{2(s+2)\}}{\{(s+2)^2+4\}^2}\right]$$

$$= \frac{8(s+2)}{\{(s+2)^2+4\}^2} \quad \text{Ans}$$

* *

Division By t

If $L\{f(t)\} = \bar{f}(s)$ then
$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$$

Que ⟹ Find L.T. of the f^n
$$\frac{e^{-at} - e^{-bt}}{t}$$

Sol^n ⟹ $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

By Division by t

$$= \int_s^\infty \bar{f}(s)\,ds \quad \text{----} (1)$$

let, $f(t) = e^{-at} - e^{-bt}$

$$\bar{f}(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

By (1) :—

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \left[\log \frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

Que Find L.T. $\frac{\sin at}{t}$

Sol^n ⟹ let, $f(t) = \sin at$

$\bar{f}(s) = L\{f(t)\} = L\{\sin at\}$

$= \frac{a}{s^2+a^2}$

Now, By Division by t :-

$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2} ds$

We know that, $\int \frac{1}{s^2+a^2} ds = \frac{1}{a} \tan^{-1} s/a$

$= \left[\tan^{-1} s/a\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} s/a$

$= \cot^{-1} s/a$ Ans

Que $\frac{\cos 2t - \cos 3t}{t}$ Find Laplace Transform

Sol^n let, $f(t) = \cos 2t - \cos 3t$

$L\{f(t)\} = \bar{f}(s) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$

$L\left[\frac{\cos 2t - \cos 3t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$

let, $s^2+4 = u$ & $s^2+9 = v$

$2s\,ds = du$ | $2s\,ds = dv$

limits are :- s to ∞ (no change)

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$$= \frac{1}{2}\int_s^\infty \frac{du}{u} - \frac{1}{2}\int_s^\infty \frac{dv}{v}$$

$$= \left[\frac{1}{2}\log u - \frac{1}{2}\log v\right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{s^2+4}{s^2+9}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left\{\frac{\cancel{s^2}(1+4/s^2)}{\cancel{s^2}(1+9/s^2)}\right\}\right]_s^\infty$$

$$= 0 - \frac{1}{2}\log\frac{s^2+4}{s^2+9}$$

$$= \frac{1}{2}\log\frac{s^2+9}{s^2+4} \quad \text{Ans.}$$

Que. Find Laplace Transforms of $\dfrac{1-\cos t}{t^2}$

Solⁿ ⟹ let, $f(t) = 1-\cos t$

$$L\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+1} = \bar{f}(s)$$

Now, $$L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_0^\infty$$

$$= \left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+1)\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2(1+1/s^2)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\log \frac{1}{1+1/s^2}\right]_0^\infty$$

$$= \frac{1}{2}\left[0 - \log \frac{s^2}{1+s^2}\right]$$

$$= \frac{1}{2}\log \frac{1+s^2}{s^2}$$

again, $L\left[\frac{1-\cos t}{t^2}\right] = \frac{1}{2}\int_s^\infty \log\left(\frac{1+s^2}{s^2}\right) ds$

$$= \frac{1}{2}\int_s^\infty \underset{\text{II}}{1} \cdot \underset{\text{I}}{\log} \frac{1+s^2}{s^2}\, ds$$

Integration by part

$$= \frac{1}{2}\left[\log\left(\frac{1+s^2}{s^2}\right) \cdot s\right]_s^\infty - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left\{\frac{1+s^2}{s^2}\right\} \cdot s\, ds$$

$$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}(1/s^2+1)}{\cancel{s^2}}\right) \cdot s\right]_s^\infty - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left\{\frac{1+s^2}{s^2}\right\} \cdot s\, ds \quad *$$

$$= \frac{1}{2}\left[0 - s \cdot \log\left(\frac{1+s^2}{s^2}\right)\right] - \frac{1}{2}\int_s^\infty \frac{d}{ds}\log\left(\frac{1+s^2}{s^2}\right) \cdot s\, ds$$

$$= \frac{-1}{2} s\log\left[\frac{1+s^2}{s^2}\right] - \frac{1}{2}\int_s^{\infty} \frac{1}{\frac{1+s^2}{s^2}} \left\{\frac{s^2(2s)-(1+s^2)\cdot 2s}{s^4}\right\} \cdot s\,ds$$

$$= \frac{-s}{2}\log\left(\frac{1+s^2}{s^2}\right) - \frac{1}{2}\int_s^{\infty} \frac{s^2 \times s}{1+s^2} \left\{\frac{\cancel{2s^3}-2s-\cancel{2s^3}}{s^4}\right\} ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) - \frac{1}{2}\int_s^{\infty} \frac{1}{1+s^2} \left\{\frac{-2\cancel{s}}{\cancel{s}}\right\} ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \frac{1}{\cancel{2}}\int_s^{\infty} \frac{\cancel{2}}{1+s^2} ds$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \left(\tan^{-1}s\right)_s^{\infty}$$

$$= \frac{-s}{2}\log\frac{1+s^2}{s^2} + \frac{\pi}{2} - \tan^{-1}s$$

$$= -\frac{s}{2}\log\left(\frac{1+s^2}{s^2}\right) + \cot^{-1}s$$

$$= \frac{s}{2}\log\left(\frac{s^2}{1+s^2}\right) + \cot^{-1}s$$