Moment & Moment Generating Function

$$M_x(t) = \sum e^{tx} f(x) \rightarrow \text{Discrete}$$
$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \rightarrow \text{continuous}$$

$*$ $M_x(t) = 1 + t\mu_1' + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \cdots$

This is M.G.F. in series form.

$*$ Moment Formula $\mu_r' = \frac{d^r}{dt^r}\left[M_x(t)\right]_{\text{at } t=0}$

about the origin

put $r=1$

$$\text{mean} = \mu = \mu_1' = \frac{d}{dt}\left[M_x(t)\right]$$

put $r=2$

$$\text{Var} = \mu_2' = \frac{d^2}{dt^2}\left[M_x(t)\right] \quad \text{at } t=0$$

$$\text{S.D.} = \sigma = \sqrt{\mu_2} = \sqrt{\mu_2' - (\mu_1')^2}$$

Que $\rightarrow$ Find the moment generating function of the exponential distribution

$$f(x) = \frac{1}{c} e^{-x/c}, \quad 0 \le x \le \infty$$

Find mean and S.D.

Sol^n $\Rightarrow$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} e^{tx} \cdot \frac{1}{c} e^{-x/c}\, dx$$

$$= \frac{1}{c} \int_{0}^{\infty} e^{(t - 1/c)x}\, dx$$

$$= \frac{1}{c}\left[\int_{0}^{\infty} e^{-\left(\frac{1}{c} - t\right)x}\, dx\right]$$

$$\Rightarrow \frac{1}{c}\left[\frac{-e^{-\left(\frac{1}{c}-t\right)x}}{\frac{1}{c}-t}\right]_0^\infty$$

$$= \frac{1}{c}\left[\frac{1}{\frac{1}{c}-t}\right]$$

$$= \frac{\frac{1}{c}}{\frac{1-tc}{c}} = \frac{1}{1-tc}$$

$$= (1-tc)^{-1}$$ By Binomial thm;

$$\therefore M_x(t) = 1 + tc + (tc)^2 + (tc)^3 + \cdots\cdots\infty$$

Now, $\mu_r' = \frac{d^r}{dt^r} M_x(t)$ (at $t=0$)

$r=1 \rightarrow$ $\mu_1' = \frac{d}{dt}\left\{1 + tc + (tc)^2 + (tc)^3 + \cdots\cdots\infty\right\}$

Mean

$$= \left[c + 2tc^2 + 3t^2c + \cdots\cdots\infty\right] \quad \text{(1)}$$

Put $t=0$

$$= c$$

$r=2 \rightarrow$ $\mu_2' = \frac{d^2}{dt^2} M_x(t)$ at $t=0$

Variance

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]$$

By (1)

$$= \frac{d}{dt}\left[c + 2tc^2 + 3t^2c + \cdots\cdots\infty\right]$$

$$= \left(2c^2 + 6tc + \cdots\cdots\infty\right)$$

Put $t=0$

$$= 2c^2$$

Now, Standard deviation

S.D.

$$\sigma \text{ or } \mu_2 = \sqrt{\mu_2' - (\mu_1')^2}$$

$$= \sqrt{\text{Var.} - (\text{Mean})^2}$$

$$= \sqrt{2c^2 - c^2}$$

$$= \sqrt{c^2}$$

S.D $\boxed{\sigma = c}$

Binomial Distribution

** mean = np
variance = npq
S.D. = $\sqrt{npq}$ **

n → minimum Between (1 to 50)

" In some trials the probability will not change from 1 trial to the next. Such trials are known as "Bernoulli's Trial" or "Binomial distribution".

p = Prob. of successful event
q = Prob. of faluire event
n = Total trials.

Formula for Binomial Distribution →

$$p(X=x) = {}^nC_x \, p^x q^{n-x}$$

Que① → An ordinary six faced die is thrown 4 times what are the prob. of obtaining 4,3,2,1,0 faces.

Solⁿ → p = prob. of obtaining any single face

$$= \frac{1}{6}$$

$$q = ~~pro~~ \; (1-p) = \frac{5}{6}$$

n = total trial = 4

$$* \quad \boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}} \quad *$$

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$$P(x=4) = {}^4C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^0$$

$$= 1 \times \left(\frac{1}{6}\right)^4 =$$

$$P(x=3) = 4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{4-3}$$

$$= 4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1$$

$$=$$

$$P(x=2) = 4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{4-2}$$

$$= 4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 =$$

$$P(x=1) = {}^4C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 =$$

$$P(x=0) = {}^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 =$$

x ——— x ——— x

Que → Imp

If the chance that 1 of the 10. telephone lines is busy at an instant is 0.2

(a) What is the chance that 5 of the lines are busy.

(b) What is the ~~chance that 5 of the lines~~ most probable number of busy lines, and what is probability of this number.

(c) What is the probability that all the lines are busy.

Solⁿ →

p = prob. of 1 telephone lines is busy

$= 0.2$ (given)

$q = (1-p) = 0.8$

$n = 10$

(a) $p(x=5) = {}^{10}C_5 \, (0.2)^5 \, (0.8)^{10-5}$

$= {}^{10}C_5 \, (0.2)^5 \, (0.8)^5$

$= 0.026$

(b) Most probable no. of busy lines = Expectation (mean) $= np$

$= 10 \times 0.2 = 2$

probability of most probable no. is

$P(X=2) = {}^{10}C_2 \, (0.2)^2 \, (0.8)^8$

$=$

(c) $P(X=10) = {}^{10}C_{10} \, (0.2)^{10} \, (0.8)^0$

$=$

Ques. If the prob. that a new born child is a male is 0.6, find the prob. that in a family of 5 children there are exactly 3 boys.

Soln. ⟹ prob. that a new born child is a male $= p = 0.6$

$q = 1 - p = 0.4$

$n = 5$

Then Prob. of exactly 3 boys

$P(X=3) = {}^5C_3 \, (0.6)^3 \, (0.4)^2$

$= 0.3456$

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Que ⟹ If on an average 1 vessel in every 10 is wrecked, find the prob that out of 5 vessels expected to arrive at least 4 will arrive safely?

Solⁿ ⟹ Prob. of 1 vessel is wrecked out of 10, i.e nine vessels out of 10 are safe

$$= p = \frac{9}{10}$$

and $q = 1 - p = \frac{1}{10}$

Required prob. is → At least 4 ~~will~~ are safe ~~out~~ of 5

$= P(x=4) + P(x=5)$

or

$= 1 - \{P_0 + P_1 + P_2 + P_3\}$

$\Rightarrow P(x=4) + P(x=5)$

$\Rightarrow {}^5C_4 \left(\frac{9}{10}\right)^4 \left(\frac{1}{10}\right)^1 + {}^5C_5 \left(\frac{9}{10}\right)^5 \left(\frac{1}{10}\right)^0$

$=$

Que ⟹ The bomb that a bomb dropped from a plane will strike the target is 1/5 if six bombs are dropped find the prob. that

(a) exactly 2 will strike the target.

(b) At least 2 will strike the target.

Solⁿ ⟹ Prob. that a bomb dropped from a plane will strike the target $p = \frac{1}{5}$

$q = 1 - \frac{1}{5} = \frac{4}{5}$

$n = 6$

(a) exactly 2 :- $P(x=2) = {}^6C_2 \cdot \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$

$= 0.245$

(b) At least 2 :- $1 - [P_0 + P_1]$

$\Rightarrow 1 - \left[{}^6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^6 + {}^6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5\right]$

$= 0.3447$

Que ⟹ A Sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane flails to return is 5%. Find the prob that (a) One plane does not return. (b) At the most 5 planes do not return. (c) What is the most probable no. of returns

Sol^n ⟹ prob. of an aeroplane fails to return (do not return = means went successfully)

$p = \frac{5}{100} = \frac{1}{20}$

return $= q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

$n = 20$

(a) $p(x=1) = {}^{20}C_1 \left(\frac{1}{20}\right)^1 \left(\frac{19}{20}\right)^{19} =$

(b) At Most 5 $= 1 - [P_6 + P_7 + P_8 + \cdots + P_{20}]$

or $P_0 + P_1 + P_2 + P_3 + P_4 + P_5$

$=$

(c) most probable number to return = Expectation :

mean $= nq$

$= 20 \times \frac{19}{20}$

$= 19$

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Que ⟹ The prob. that an entering Student will graduate is 0.4, determine the prob. that out of 5 Student (a) None (b) One (c) At least 1 will graduate.

Sol^n ⟹ prob that entering student will graduate, $p = 0.4$

$q = 0.6$

(a) None → $P(x=0) = {}^5C_0 (0.4)^0 (0.6)^5$

$=$

(b) one → $P(x=1) = {}^5C_1 (0.4)^1 (0.6)^4$

$=$

(c) at least one $= 1 - P_0$

$= 1 - \{{}^5C_0 (0.4)^0 (0.6)^5\}$

$= 0.92$

Imp

Que ⟹ out of 800 families with 5 children each how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys. Assume equal prob. for boys & girls.

Sol^n ⟹ prob of girls and boys $= \frac{1}{2}$

$n = 5$

Total family $= 800$

(a) 3 boys → $P(x=3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$

$= \frac{\lfloor 5}{\lfloor 3 \cdot \lfloor 2} \times \frac{1}{8} \times \frac{1}{4}$

$= \frac{10}{32}$

Since, Total 800 families $= \frac{10}{32} \times 800 = 250$

(b) 5 girls $\Rightarrow$ $p(x=5) = \left[{}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \right] \times 800$

$= 25$

(c) either 2 or 3 boys $\Rightarrow$

$P(x = 2 \text{ or } 3) =$

$= 800 [P(x=2) + P(x=3)]$

$= 800 \left[{}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 \right]$

$= 500$

Que $\Rightarrow$ If 10 % of the rivets produced by a machine are defective, find the prob. that out of 5 rivets. choose at random

(a) None will be defective.

(b) 1 will be defective.

(c) at least 2 will be defective.

Solⁿ $\Rightarrow$ Prob of the rivets produced are defective $p = \frac{10}{100} = 0.1$

$q = 1 - 0.1 = 0.9$, $n = 5$

(a) none $\rightarrow$ $P(x=0) = {}^5C_0 (0.1)^0 (0.9)^5 = 0.59$

(b) one $\rightarrow$ $P(x=1) = {}^5C_1 (0.1)^1 (0.9)^4 = 0.328$

(c) At least 2 $= 1 - [P_0 + P_1]$

$= 1 \left[{}^5C_0 (0.1)^2 (0.9)^5 + {}^5C_1 (0.1)^1 (0.9)^4 \right]$

$= 0.08146$

Que → fit a Binomial distribution to these data

x:	0	1	2	3	4	5	6	7	8	9	10
f:	6	20	28	12	8	6	0	0	0	0	0

Sol^n → Formula for Best fit of Binomial distribution, $\boxed{N(P+q)^n}$

We have $x \to$ 0 to 10 i.e total $11 = n$ term, but we will take $n = 10$ Because in Binomial thm we know that

Ex: $(a+b)^2 = \underbrace{a^2+b^2+2ab}_{3\ terms}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \infty$

$N = \Sigma f_i = 80$

mean $= \mu = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{20+56+36+32+30}{80}$

$\mu = 2.175$

$np = \mu = 2.175 \Rightarrow 10p = 2.175$

so, $p = 0.2175$

& $q = 1 - p = 0.782$

Best fit $\to \underset{N}{80}\left[\underset{a}{0.2175} + \underset{b}{0.7825}\right]^{10 \to n}$

By Binomial thm; $(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots\cdots$

$$\Rightarrow 80\left[{}^{10}C_0(0.2175)^{10}(0.7825)^{0} + {}^{10}C_1(0.2175)^{9}(0.7825)^{1} + {}^{10}C_2(0.2175)^{8}(0.7825)^{2} + {}^{10}C_3(0.2175)^{7}(0.7825)^{3} + \cdots\right)$$

After solving, we get,

$$\Rightarrow \underset{\substack{\downarrow\\ 10\text{th}\\ \text{value}}}{0.0002} + \underset{\substack{\downarrow\\ 9\text{th}}}{0.0007} + \underset{\substack{\downarrow\\ 8\text{th}}}{\cdots} + 19.13 + 5.42$$

$\cdots\cdots$ 1^{st} value.