Fourier Series

Periodic function → A function $f(x)$ is said to be periodic function if it repeats its value after a certain period.

Ex:- $\sin x$ and $\cos x$ are Periodic for $2n\pi$ period
$\tan x$ is Periodic for $n\pi$ Period.

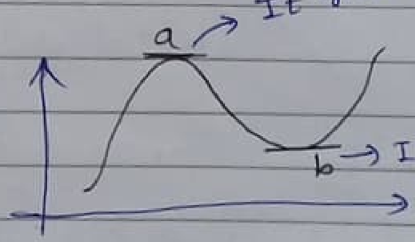
Dirichlet Condition → A function is defined in the interval ($-\pi$ to π) can be expressed in the fourier series. If the following Conditions are satisfy in the interval ($-\pi$ to π)

1. $f(x)$ is peridic, ~~sig~~ single valued and finite.
2. $f(x)$ ~~is~~ has finite no. of discontinuity.
3. $f(x)$ has at most finite no. of maxima and minima.

Continuous function → If $f(x)$ be a function between two points a and b then $\exists$ a Point c, $a<c<b$ Such that

$$\lim_{x \to c} f(x) = f(c)$$

Maxima/Minima →



a → It gives max value of function

b → It gives min value of function

Multivalued and single valued Function →

a fix value, $x=2$, $x=-1$, $x=0$ ..... are single valued fu^n.
but more than one value of any fu^n like $x^2=4 \Rightarrow x=\pm 2$ are multi-valued function.

Introduction of fourier Series

Fourier series are infinite series of cosines and sines.

In many engineering problems like ~~engine~~ electro magnatic field, electro dynamics and Heat conduction, we need such type of series to express the function.

The series Introduce by french mathematician "Jacques fourier".

The Fourier Series can be expressed in the form $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Euler's formula

$$\left\{\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right.$$

This are also known as fourier constants

classmate

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Que① $f(x) = e^x$, $(0 \text{ to } 2\pi)$ Expand the fn in the form of fourier Series.

Soln $f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx + \sum_{n=1}^{\infty} b_n \sin nx$ —①

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^x dx \Rightarrow \frac{1}{\pi}\left[e^x\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[e^{2\pi} - 1\right] \text{ —— ②}$$

By using formula
$\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left\{\cos nx + n\sin nx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}\{1\} - \frac{1}{1+n^2}\{1\}\right]$$

$$= \frac{1}{\pi(1+n^2)}\left\{e^{2\pi} - 1\right\} \text{ —— ③}$$

Using formula,
$\int e^{ax}\sin nx \, dx = \frac{e^{ax}}{a^2+b^2}\{a\sin nx - n\cos nx\}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left\{\sin nx - n\cos nx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}\{-n\} - \frac{1}{1+n^2}\{-n\}\right]$$

$$= \frac{n}{\pi(1+n^2)}\left\{1 - e^{2\pi}\right\} \text{ —— ④}$$

By Putting values of eqn ②, ③ & ④ in ①:—

$$f(x) = e^x = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty} \frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{2\pi})\sin nx}{\pi(1+n^2)}$$

Put $n = 1, 2, 3 \ldots\ldots$

$$e^x = \frac{e^{2\pi}-1}{2\pi} + \frac{e^{2\pi}-1}{2\pi}\cos x + \frac{e^{2\pi}-1}{5\pi}\cos 2x + \frac{e^{2\pi}-1}{10\pi}\cos 3x + \ldots\ldots\infty + \frac{(1-e^{2\pi})}{2\pi}\sin x + \frac{2(1-e^{2\pi})}{5\pi}\sin 2x + \ldots\ldots\infty$$

Ans.

* $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$
$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

Que (2) $f(x) = x\sin x$, $(0 \text{ to } 2\pi)$
Expand the fu^n by fourier Series.

Sol^n $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ——— (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx \Rightarrow \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x}\,\underset{v}{\sin x}\,dx$$

Formula $\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 - \cdots \infty$

$$\Rightarrow \frac{1}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\right] = -2 \quad \text{——— (2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cos nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \underset{u}{x}\,\underset{v}{\sin(1+n)x}\,dx + \int_0^{2\pi} \underset{u}{x}\,\underset{v}{\sin(1-n)x}\,dx\right]$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(1+n)x}{1+n}\right) - \left(\frac{-\sin(1+n)x}{(1+n)^2}\right) + x\left(\frac{-\cos(1-n)x}{1-n}\right) - \left(\frac{-\sin(1-n)x}{(1+n)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-1}{1+n}\right) + 2\pi\left(\frac{-1}{1-n}\right)\right]$$

Formula
$\cos(2\pi+\theta) = \cos\theta$
$\cos(2\pi-\theta) = \cos\theta$
$\sin(2\pi+\theta) = \sin\theta$
$\sin(2\pi-\theta) = -\sin\theta$

$$= -\left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}$$

$$= -\left\{\frac{1-n+1+n}{1-n^2}\right\} = -\left\{\frac{2}{1-n^2}\right\}$$

$$= \frac{-2}{1-n^2}, \quad \text{Here } n \neq 1$$

when $n=1 \longrightarrow$ $a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x(\cos x)\,dx$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cos x)\,dx$$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} \underset{u}{x}\,\underset{v}{\sin 2x}\,dx$$

$$a_1 = \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\cancel{\frac{-\sin 2x}{4}}\right)\right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right] = -\frac{1}{2} \quad \text{———} \quad (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cdot\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{\cos(1-n)x - \cos(1+n)x\right\}dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \underset{u}{x}\,\underset{v}{\cos(1-n)x}\,dx - \int_0^{2\pi} \underset{u}{x}\,\underset{v}{\cos(1+n)x}\,dx\right]$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(1-n)x}{1-n}\right\} - \left\{\frac{-\cos(1-n)x}{(1-n)^2}\right\} - x\left\{\frac{\sin(1+n)x}{1+n}\right\} - \left\{\frac{-\cos(1+n)x}{(1+n)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{\cancel{\frac{1}{(1-n)^2}} + \cancel{\frac{1}{(1+n)^2}}\right\} - \left\{\cancel{\frac{1}{(1-n)^2}} + \cancel{\frac{1}{(1+n)^2}}\right\}\right]$$

$$= 0 \quad \text{———} \quad (4)$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cdot\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{2\pi} x\left\{\frac{1-\cos 2x}{2}\right\}dx\right.$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\,dx - \int_0^{2\pi} \underset{u}{x}\,\underset{v}{\cos 2x}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \left\{x\left(\frac{\sin 2x}{2}\right) - \left(\frac{-\cos 2x}{4}\right)\right\}\right]_0^{2\pi}$$

$$\cancel{= \frac{1}{2\pi}\left[\left(\frac{4\pi^2}{2} - \frac{\sin 2\pi}{8}\right) - \left(\frac{1}{4}\right)\right]}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}}\right]$$

$$= \pi \quad \text{—— (5)}$$

Put the values of (2) (3), (4) and (5) in (1):—

$$f(x) = x\sin x = -1 + \overset{-\frac{1}{2}\cos x}{\sum_{n=2}^{\infty}}\left\{\frac{-2}{1-n^2}\cos nx\right\} + \pi\sin x$$

$$= -1 - \frac{1}{2}\cos x + \frac{2}{3}\cos 2x + \frac{2}{10}\sin 3x + \cdots \infty + \pi\sin x$$

Ans..

Que $\Rightarrow$ $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

Sol^n $\Rightarrow$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ —— (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 + x^2 - 2x\pi)\, dx$$

$$= \frac{1}{4\pi}\left[\pi^2(x) + \left(\frac{x^3}{3}\right) - \cancel{2}\pi\left(\frac{x^2}{\cancel{2}}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2\pi^3 + \frac{8}{3}\pi^3 - 4\pi^3\right]$$

$$= \frac{1}{4\pi}\left[+\frac{2}{3}\pi^3\right] \Rightarrow +\frac{\pi^2}{6} \quad \text{—— (2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 + x^2 - 2x\pi)\cos nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 \cos nx + \underset{u}{x^2}\underset{v}{\cos nx} - 2\pi \underset{u}{x}\underset{v}{\cos nx})\, dx$$

$$= \frac{1}{4\pi}\left[\pi^2\frac{\sin nx}{n} + \left\{x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right\} - 2\pi\left\{x\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right\}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[2x\frac{\cos nx}{n^2} - 2\pi\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{4\pi}{n^2} - \cancel{\frac{2\pi}{n^2}}\right) - \left(0 - \cancel{\frac{2\pi}{n^2}}\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2} \quad \text{—— (3)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 + x^2 - 2\pi x)\sin nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi^2 \sin nx + \underset{u}{x^2}\underset{v}{\sin nx} - 2\pi \underset{u}{x}\underset{v}{\sin nx})\, dx$$

$$= \frac{1}{4\pi}\left[\pi^2\left(\frac{-\cos nx}{n}\right) + \left\{x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right\} - 2\pi\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right\}\right]_0^{2\pi}$$

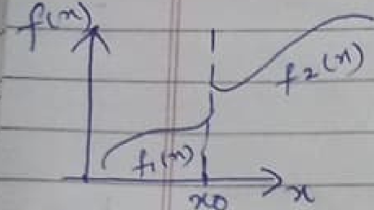
$$= \frac{1}{4\pi}\left[\cancel{\frac{-\pi^2}{n}} - \cancel{\frac{4\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{4\pi^2}{n}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right]$$

$$= 0 \quad \text{—— (4)}$$

So, $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$ expand the series by

putting $n = 1, 2, 3, 4 \ldots \ldots \infty$

Function having Point of Discontinuity



Point of Discontinuity

At the point of discontinuity the function is

$$f(x) = \frac{1}{2}\left[\underbrace{f(x-0)}_{\text{fn in lower range}} + \underbrace{f(x+0)}_{\text{fn in upper range}}\right]$$

it is also called

Arithmetic mean $\rightarrow f(x) = \dfrac{f(x-0)+f(x+0)}{2}$

(1) $f(x) = \begin{cases} x, & 0 \leq x < \pi \\ 2\pi - x, & \pi < x \leq 2\pi \end{cases}$

also Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Soln $\Rightarrow$ $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left(4\pi^2 - \frac{4\pi^2}{2}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{4\pi^2}{2} - \frac{3\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right] = \pi \quad \text{— (2)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\,\underset{v}{\cos nx}\,dx + \int_{\pi}^{2\pi} \underset{u}{(2\pi - x)}\,\underset{v}{\cos nx}\,dx\right.$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

** $\boxed{\cos n\pi = (-1)^n}$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)_0^{\pi} - \left(\frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right\} - \left\{\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right\}\right]$$

**

$$= \frac{1}{\pi}\left[\left\{\frac{(-1)^n - 1}{n^2}\right\} - \left\{\frac{1 - (-1)^n}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\{(-1)^n - 1\}}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \quad \text{—③}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \underset{u}{x}\,\underset{v}{\sin nx}\,dx + \int_{\pi}^{2\pi} \underset{u}{(2\pi - x)}\,\underset{v}{\sin nx}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - \cancel{\left(-\frac{\sin nx}{n^2}\right)}\right\}_0^{\pi} + \left\{(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\cancel{\left(-\frac{\sin nx}{n^2}\right)}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\left(\frac{-\pi(-1)^n}{n}\right)} - \cancel{\left(\frac{-\pi(-1)^n}{n}\right)}\right] = 0$$

So, $f(x) = \dfrac{\pi}{2} + \displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n^2}\{(-1)^n - 1\}\cos nx$

Put $n = 1, 2, 3 \ldots$

$$f(x) = \frac{\pi}{2} + \left\{\frac{-4}{1^2\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x \ldots \infty\right\} \quad \text{—④}$$

we have put a value of x in eqⁿ ④ by which we will get required series.

The value is $x = \pi$ but this is point of discontinuity

So we will find mean :—

$$f(x) = \frac{f(x-0) + f(x+0)}{2} = \frac{x + (2\pi - x)}{2}$$

Put $x = \pi$

$$= \frac{\cancel{\pi} + 2\pi - \cancel{\pi}}{2} = \pi = \text{Arithmetic Mean.}$$

Put $x = \pi$ and $f(x) = \pi$ in eq^n (4)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \infty\right]$$

$$\pi - \frac{\pi}{2} = \frac{-4}{\pi}\left[-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \cdots \infty\right]$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$$

Que i $f(x) = \begin{cases} I_0 \sin x, & 0 \le x < \pi \\ 0, & \pi < x \le 2\pi \end{cases}$

Sol^n i $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ --- (1)

$$a_0 = \frac{1}{\pi}\left(\int_0^{\pi} I_0 \sin x \, dx + \int_{\pi}^{2\pi} 0 \, dx\right)$$

$$= \frac{I_0}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{I_0}{\pi}\{1+1\} = \frac{2I_0}{\pi} \quad \text{--- (2)}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x \cos nx \, dx + \int_{\pi}^{2\pi} 0 \, dx\right]$$

$$= \frac{I_0}{2\pi}\left[\int_0^{\pi} 2 \sin x \cos nx \, dx\right]$$

$$= \frac{I_0}{2\pi}\left[\sin(1+n)x + \sin(1-n)x\right]_0^{\pi}$$

$\sin(\pi+\theta) = -\sin\theta$

$\sin(\pi-\theta) = \sin\theta$

$$= \frac{I_0}{2\pi}\left[(\sin n\pi + \sin n\pi) - (\ldots\right.$$

Interchange the Place

$$= \frac{I_0}{2\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^\pi$$

$$= \frac{I_0}{2\pi}\left[\left(\frac{\cos n\pi}{1+n} - \frac{\cos n\pi}{1-n}\right) - \left(\frac{-1}{1+n} - \frac{1}{1-n}\right)\right]$$

$$= \frac{I_0}{2\pi}\left[\left(\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n}\right) + \left(\frac{1}{1+n} + \frac{1}{1-n}\right)\right]$$

$$= \frac{I_0}{2\pi}\left[(-1)^n\left\{\frac{(1-n)+(1+n)}{1-n^2}\right\} + \left\{\frac{1-n+1+n}{1-n^2}\right\}\right]$$

$$= \frac{I_0}{\cancel{2}\pi}\left[(-1)^n\left\{\frac{+\cancel{2}}{1-n^2}\right\} + \frac{\cancel{2}}{1-n^2}\right]$$

$$= \frac{I_0\{(-1)^n+1\}}{1-n^2} \quad \text{------} \; (3)$$

but $n \neq 1$

$$b_n = \frac{1}{\pi}\left[\int_0^\pi I_0 \sin x \cdot \sin nx\,dx + \int_\pi^{2\pi} 0\,dx\right]$$

$$= \frac{I_0}{2\pi}\int_0^\pi 2\sin x \cdot \sin nx\,dx$$

$$= \frac{I_0}{2\pi}\int_0^\pi \left[\cos(1-n)x - \cos(1+n)x\right]dx$$

$$= \frac{I_0}{2\pi}\left[\int_0^\pi \cos(1-n)x\,dx - \int_0^\pi \cos(1+n)x\,dx\right]$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^\pi$$

$$= \frac{I_0}{2\pi}\left[\left(\frac{\sin n\pi}{1-n} + \frac{\sin n\pi}{1+n}\right)\right] = 0 \quad \text{---} \; (5)$$

Interchange the place

Now, $$a_1 = \frac{1}{\pi}\left[\int_0^\pi I_0 \sin x \cdot \cos x\,dx\right]$$

$$= \frac{I_0}{2\pi}\int_0^\pi 2\sin x \cos x\,dx$$

$$= \frac{I_0}{2\pi}\int_0^\pi\left[\sin 2x\right]dx = \frac{I_0}{2\pi}\left(\frac{-\cos 2x}{2}\right)_0^\pi$$

$$= \frac{-I_0}{4\pi}\left[-1+1\right] = 0 \quad \text{---} \; (4)$$

Now,

$$b_1 = \frac{1}{\pi}\int_0^\pi I_0 \sin^2 x\,dx = \frac{I_0}{\pi}\int_0^\pi \left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{I_0}{2\pi}\left[\pi\right] = \frac{I_0}{2} \quad \text{---} \; (6)$$

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Putting all the values of ②,③,④,⑤,⑥ in ①:—

$$i(x) = \frac{I_0}{\pi} + \sum_{n=2}^{\infty} \frac{I_0\{(-1)^n+1\}}{1-n^2}\cos nx + \frac{I_0}{2}\sin x.$$

Put, $n = 2, 3, 4 \dots$ for final expansion. Ans

Que ⟹ If $f(x) = 0$; $-\pi$ to 0
$\sin x$, 0 to π.

Hence, show that,

(a) $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots \infty = \frac{\pi-2}{4}$

(b) $\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \dots \infty = \frac{1}{2}$

Solⁿ ⟹ $f(x) = 0$, $-\pi < x < 0$
$\sin x$, $0 < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—①}$$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}(-\cos x)_0^{\pi} = \frac{2}{\pi} \quad \text{—②}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} \sin x \cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n}\right) - \left(\frac{-1}{1+n} - \frac{1}{1-n}\right)\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left\{\frac{1-n+1+n}{1-n^2}\right\} + \left\{\frac{1-n+1+n}{1-n^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n+1}{1-n^2}\right], \quad n \neq 1 \quad \text{---} (3)$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{4\pi}\left\{-1+1\right\} = 0 \quad \text{---} (4)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cdot \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right] dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n}\right) - \left(\frac{-1}{1+n} - \frac{1}{1-n}\right)\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(-1)^n\left\{\frac{1-n+1+n}{1-n^2}\right\} + \left\{\frac{1-n+1+n}{1-n^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left\{\frac{2}{1-n^2}\right\} + \left\{\frac{2}{1-n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n+1}{1-n^2}\right\}, \quad n \neq 1 \quad \text{----} (4)$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{1}{\pi}\int_0^{\pi}\left[\frac{1-\cos 2x}{2}\right] dx$$

$$= \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{2\pi}\left[\pi\right] = 1/2$$

$$f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{(-1)^n+1}{\pi(1-n^2)} \cos nx + \frac{1}{2}\sin x$$

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi(-3)}\cos 2x + \frac{2}{\pi(-15)}\cos 4x + \frac{2}{\pi(-35)}\cos 6x + \text{-----}\infty + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \text{---}\infty\right] + \frac{1}{2}\sin x$$

$$\text{---} (5)$$

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for (a)

put $\boxed{x=0}$ in eqⁿ (5) :-

$x=0$ = Point of Discontinuity

So we have to find $f(x)$ by (Arithmetic mean)

$$A.M = \frac{f(x-0) + f(x+0)}{2}$$

$$f(x) = \frac{0 + \sin x}{2} \quad \xrightarrow{\text{Put } x=0}$$

$$= 0$$

So we will put $x=0$ & $f(x)=0$ in eqⁿ (5)

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots \infty\right] + \frac{1}{2}\sin 0$$

$$\frac{-1}{\pi} = \frac{-2}{\pi}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \infty\right]$$

$$\frac{1}{2} = \left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \infty\right]$$ Hence proved

for (b) put $\boxed{x = \pi/2}$

at $x = \pi/2$

$f(x) = \sin x = \sin \pi/2$

$= 1$

By eqⁿ (5)

$$1 = \frac{1}{\pi} - \frac{2}{\pi}\left[-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} \cdots \infty\right] + \frac{1}{2}$$

$$\frac{1}{1} - \frac{1}{2} - \frac{1}{\pi} = +\frac{2}{\pi}\left[\frac{1}{3} - \frac{1}{15} + \frac{1}{35} + \cdots \infty\right]$$

$$\frac{2\pi - \pi - 2}{2\pi} = \frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \cdots \infty\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \cdots \infty$$ Hence proved

Even & Odd Function

* Even function → A function $f(x)$ is said to be an even function if $f(-x) = f(x)$, $\forall x$ (for all)
* Example:- x^2, $\cos x$, x^4 ... etc.
* Definite Integral → $\int_{-a}^{a} \underset{(\text{even})}{f(x)}\,dx = 2\int_0^a f(x)\,dx$

* Odd Function → A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$, $\forall x$
* Example → x^3, x^5, $\sin x$ etc.
* Definite Integral → $\int_{-a}^{a} \underset{(\text{odd})}{f(x)}\,dx = 0$

In Even fu^n → calculate a_0, a_n $\boxed{b_n = 0}$

In Odd fu^n → calculate b_n $\boxed{a_0 = 0, a_n = 0}$

Que(1) ⇒ $f(x) = x\sin x$, $-\pi$ to π

also deduce that $\frac{\pi-2}{4} = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots \infty$

Sol^n → $f(x) = x\sin x$

$f(-x) = x\sin x = f(x)$ this is an even fu^n.

we have to find a_0 and a_n

$b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underset{v}{\sin x}\,dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) - (-\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\big[\pi\big] = 2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\cos nx\,dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x\,(2\sin x\cos nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,(2\sin x\cos nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left\{\sin(1+n)x + \sin(1-n)x\right\}dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(1+n)x}{1+n}\right) - \left(\frac{-\sin(1+n)x}{(1+n)^2}\right) + x\left(\frac{-\cos(1-n)x}{1-n}\right) - \left(\frac{-\sin(1-n)x}{(1-n)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{+\pi}{1+n}\cos n\pi + \frac{\pi}{1-n}\cos n\pi\right]$$

$$= (-1)^n\left\{\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right\} = \frac{2(-1)^n}{1-n^2}$$

$n \neq 1$

$$a_1 = \frac{2}{2\pi}\int_0^{\pi} x\,(2\sin x\cdot\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right] = -1/2$$

$$f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^n}{1-n^2}\cos nx$$

$$f(x) = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x - \frac{2}{15}\cos 4x - \frac{2}{35}\cos 6x \cdots\cdots \infty \quad \text{——— (1)}$$

Now we have to prove that

$$\frac{\pi-2}{4} = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} \text{———} \infty$$

Put, $x = \pi/2$ in eqn (1) :—

$$\underset{x=\pi/2}{x\sin x} = 1 + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} \cdots\cdots \infty$$

$$\frac{\pi}{2} = 1 + 2\left\{\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - - - \infty\right\}$$

$$\frac{\pi}{2} - 1 = 2\left\{\frac{1}{3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - - - - \infty\right\}$$

$$\frac{\pi - 2}{4} = \frac{1}{3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - - - - \infty$$

Hence proved

* Introduction of Fourier Series *

Fourier series are Infinite series of sines and cosines.

In many engineering problems like electro magnatic field electro dynamics and Heat conduction etc. we need such type of series to express the function.

The series Introduce by french mathematician "Jacques fourier"

The fourier series can be expressed in the form in the interval 0 to 2π

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

If $\boxed{\alpha = 0}$

Here,

$$* \quad a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

If $\boxed{\alpha = -\pi}$

$\int_{-\pi}^{\pi} f(x)dx$

$$* \quad a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot \cos nx\,dx$$

$$* \quad b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cdot \sin nx\,dx$$

These are known as fourier Constants and also known as Euler's formula when the interval is

$$\alpha \le x \le \alpha + 2\pi$$

$3.5 \quad 5.7 \quad ---\infty$

Hence Proved

Q $f(x) = |x|$, $-\pi$ to π also P.T. $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + --\infty$

Sol^n $f(x) = \begin{cases} -x, & -\pi \text{ to } 0 \\ x, & 0 \text{ to } \pi \end{cases}$

$|x|$ is an even fu^n. So we have to find a_0 and a_n , $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$ (By Definite Integral)

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$ by definite Integral

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\{(-1)^n - 1\}$$ $\forall n = 1, 2, 3 --$

So, $$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\{(-1)^n - 1\}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x ---\infty$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + --\infty\right] \quad (1)$$

Put $x=0$, Point of Discontinuity :—

$$\text{A.M. } f(x) = \frac{f(x-0) + f(x+0)}{2}$$

$$f(x) = 0$$

By (1) $$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{9} + \frac{1}{25} + -----\infty\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots \infty$$

Hence Proved

Que. $f(x) = |\sin x|$, $-\pi$ to π

Solⁿ $f(x) = |\sin x|$

is an even fuⁿ.

we have a_0, a_n, $b_n = 0$

we know that $|\,|$ is defined by,

$$f(x) = \begin{cases} -\sin x, & -\pi \text{ to } 0 \\ \sin x, & 0 \text{ to } \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\, dx$$

By Definite Integral.

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi}\left(-\cos x\right)_0^{\pi} = \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} 2\sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(1+n)x + \sin(1-n)x\right] dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} - \left(\frac{-1}{1+n} - \frac{1}{1-n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} + \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1-n+1+n}{1-n^2}\right\} + \left\{\frac{1-n+1+n}{1-n^2}\right\}\right]$$

$$= \frac{2\{(-1)^n+1\}}{\pi(1-n^2)} \qquad n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos x \, dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} 2\sin x \cdot \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{\pi}\left\{-\frac{\cos 2x}{2}\right\}_0^{\pi}$$

$$= \frac{1}{2\pi}\left\{-1+1\right\} = 0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2\left[(-1)^n+1\right]}{\pi(1-n^2)}$$

put $n = 1, 2, 3 \ldots$ For final series.

Que **

$f(x) = |\cos x|$, $-\pi$ to π

$f(x) = |\cos x| \longrightarrow$ even $f^n \longrightarrow a_0, a_n$ $(b_n = 0)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx\right]$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi}$$

* 0 — $\pi/2$ — π

$\underbrace{\quad}_{+\cos x}$ $\underbrace{\quad}_{-\cos x}$

There are 2 different values of $\cos x$ between 0 to π

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx\right]$$

Formula
$* \; 2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$
$* \; \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$, $\cos\left(\frac{\pi}{2}-\theta\right) = \cos\theta$
$* \; \sin(\pi/2+\theta) = \cos\theta$, $\sin(\pi/2-\theta) = \cos\theta$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cdot \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cdot \cos nx \, dx\right]$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi/2} \cos(1+n)x + \cos(1-n)x \, dx - \int_{\pi/2}^{\pi} \cos(1+n)x + \cos(1-n)x \, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1+n)x}{1-n}\right)_0^{\pi/2} - \left(\frac{\sin(1+n)x}{1+n} + \frac{\sin(1-n)x}{1-n}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos\frac{n\pi}{2}}{1-n}\right) + \left(\frac{\cos n\pi/2}{1+n} + \frac{\cos n\pi/2}{1-n}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos\frac{n\pi}{2}}{1-n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right]\cos n\pi/2$$

$$= \frac{2\cos n\pi/2}{\pi}\left[\frac{2}{1-n^2}\right] = \frac{4}{\pi(1-n^2)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx + \int_{\pi/2}^{\pi} -\cos^2 x \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{1+\cos 2x}{2} dx - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2} dx\right]$$

$$= \frac{2}{2\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+0\right) - \left(\pi - \frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{4}{\pi(1-n^2)} \cos\frac{n\pi}{2} \cdot \cos nx$$

put $n = 1, 2, 3 \ldots$

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Q $f(x) = -x+1 \quad , -\pi \text{ to } 0$
$\qquad\quad\; x+1 \quad , \; 0 \text{ to } \pi$

Sol^n $f(x) = -x+1$
$\qquad\quad\; x+1$ when we put $x=-x$
again we get $f(x)$. So $f(x)$ is an even fu^n. a_0, a_n, $\boxed{b_n=0}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi}(x+1)\,dx$$
$$= \frac{2}{\pi}\left(\frac{x^2}{2}+x\right)_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2}+\pi\right]$$
$$= \frac{2\cancel{\pi}}{\cancel{\pi}}\left[\frac{\pi}{2}+1\right] = \cancel{2}\left[\frac{\pi+2}{\cancel{2}}\right]$$
$$= \pi+2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi}\underset{u}{(x+1)}\,\underset{v}{\cos nx}\,dx$$
$$= \frac{2}{\pi}\left[(x+1)\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\{(-1)^n-1\}$$

$$f(x) = \frac{\pi+2}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\{(-1)^n-1\}\cos nx$$

Put, $n=1,2,3 \text{ ---}$

$$f(x) = \frac{\pi+2}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x \text{ ------ } \infty \quad (1)$$

Put $x=0$ $\longrightarrow$ A.M. $\rightarrow f(x) = \dfrac{f(c-0)+f(c+0)}{2}$

$$= \frac{(-x+1)+(x+1)}{2} \qquad \text{Put } x=0$$

$$f(x) = \frac{2}{2} = 1$$

By eq^n (1)

$$1 = \frac{\pi+2}{2} - \frac{4}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\text{ ----- }\infty\right\}$$
$$\cancel{1} - \frac{\pi}{2} - \cancel{1} = -\frac{4}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\text{ ---- }\infty\right\}$$
$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\text{ ---- }\infty$$

Hence proved

Half Range Series

(1) $f(x) = x \quad (0, \pi)$

Find Half range Cosine Series.

* Half Range cosine Series: a_0 and a_n

i.e. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Formula

$a_0 = 2 \times \frac{1}{\pi} \int_0^{\pi} f(x)\,dx$

$a_n = 2 \times \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$

* Half Range Sine series: only b_n

i.e. $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Formula

$b_n = 2 \times \frac{1}{\pi} \int_0^{\pi} f(x)\,dx$

Sol^n → We have to find HRC so we calculate only a_0 and a_n.

$$a_0 = 2 \times \frac{1}{\pi} \int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = 2 \times \frac{1}{\pi} \int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\{(-1)^n - 1\} \cos nx$$

Put $n = 1, 2, \ldots$

x

(2) $f(x) = x \quad , \quad 0 < x < \pi/2$

$\pi - x \quad , \quad \pi/2 < x < \pi$

find half range sine series.

Solⁿ $\quad b_n = 2 \times \frac{1}{\pi}\left[\int_0^{\pi/2} x \sin nx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$

$b_n = \frac{2}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$

$= \frac{2}{\pi}\left[\left(\cancel{-\frac{\pi}{2n}\cos n\pi/2} + \frac{\sin\frac{n\pi}{2}}{n^2}\right) + \cancel{\frac{\pi}{2}\cos\frac{n\pi}{2}} + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$

$= \frac{2}{\pi}\left[\frac{2\sin\frac{n\pi}{2}}{n^2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$

So, $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$= \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin\frac{n\pi}{2} \sin nx$

Put $n = 1, 2, 3 \dots$

Change of Interval

In many engineering problems It is not necessary to expansion of any function in 2π, but some other interval like $2c$.

$\boxed{\alpha \text{ to } \alpha+2c}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

$$a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

* When the range is $(0 \text{ to } 2c)$ then c is calculated by using formula :—

$$2c = \text{Total length}$$

* When the ~~$2c = 2l$~~ range is $(-c \text{ to } c)$ then no need to calculate c i.e $\underline{c = c}$.

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Q. $f(x) = x^2$, $(0 \text{ to } 2l)$

Sol^n $f(x) = x^2$

$2c =$ Total length

$2c = 2l \Rightarrow \boxed{c = l}$

$$a_0 = \frac{1}{l}\int_0^{2l} x^2 dx$$

$$= \frac{1}{l}\left(\frac{x^3}{3}\right)_0^{2l} = \frac{1}{3l} \times 8l^3 = \frac{8l^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} x^2 \cos\frac{n\pi x}{l} dx$$

$$= \frac{1}{l}\left[x^2\left\{\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right\} - 2x\left\{\frac{-\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right\} + 2\left\{\frac{-\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right\}\right]_0^{2l}$$

$\sin 2n\pi = 0$

$\cos 2n\pi = 1$

$$= \frac{1}{l}\left[\frac{4l \times l^2}{n^2\pi^2}\right] = \frac{4l^2}{n^2\pi^2}, \quad \forall\, n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{l}\int_0^{2l} \underset{u}{x^2} \underset{v}{\sin\frac{n\pi x}{l}} dx$$

$$= \frac{1}{l}\left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{-4l^2 \times l}{n\pi} \cancel{- \frac{2l^3}{n^3\pi^3}} \cancel{+ \frac{2l^3}{n^3\pi^3}}\right]$$

$$= \frac{-4l^2}{n\pi}$$

So, $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{4l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left\{\frac{-4l^2}{n\pi}\right\}\sin\frac{n\pi x}{l}$$

Put $n = 1, 2, \ldots$ for Final Series.

Q $\Rightarrow$ $$f(x) = \begin{cases} \pi x & , 0<x<1 \\ \pi(2-x) & , 1<x<2 \end{cases}$$

also show that,

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$$

Sol^n $\Rightarrow$ $$f(x) = \begin{cases} \pi x & , 0<x<1 \\ \pi(2-x) & , 1<x<2 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$2l$ = Total length

$2l = 2 \Rightarrow \boxed{l=1}$

$$a_0 = \frac{1}{1}\left[\int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx\right]$$

$$= \pi\left[\left\{\frac{x^2}{2}\right\}_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2\right]$$

$$= \pi\left[\frac{1}{2} + \left\{(4-2) - \left(2-\frac{1}{2}\right)\right\}\right]$$

$$= \pi$$

$$a_n = \frac{1}{1}\left[\int_0^1 \pi x \cos\frac{n\pi x}{1}\, dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{1}\, dx\right]$$

$$= \pi\left[\left\{x\left(\frac{\sin n\pi x}{n\pi}\right) - \left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$= \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\} \quad \forall\ n = 1, 2, 3 \dots$$

$$b_n = \frac{1}{1}\left[\int_0^1 \pi x \sin\frac{n\pi x}{1}\,dx + \int_1^2 \pi(2-x)\sin\frac{n\pi x}{1}\,dx\right]$$

$$b_n = \pi\left[\left\{x\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\cancel{\frac{-(-1)^n}{n\pi}} + \cancel{\frac{(-1)^n}{n\pi}}\right]$$

$$= 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\{(-1)^n - 1\}}{n^2\pi}\cos n\pi x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos\pi x - \frac{4}{\pi 3^2}\cos 3\pi x - \frac{4}{\pi 5^2}\cos 5\pi x - - - \infty$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos\pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + - - - \infty\right] \quad \boxed{(1)}$$

put $x=0$ $\Longrightarrow$ then $f(x) = \pi x$
$= \pi \times 0 = 0$

So put $x=0$ & $f(x)=0$ in eqn (1) :-

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - \infty\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - - \infty$$

*

Que $v(t) = 0 \quad , -T/2 < t < 0$
$= E\sin\omega t \quad , 0 < t < T/2$
where $T = 2\pi/\omega$

Solⁿ $v(t) = 0 \quad , -T/2 < t < 0$
$E\sin\omega t \quad , 0 < t < T/2$

Here $\boxed{c = T/2}$ and $T = \frac{2\pi}{\omega}$ (Given)

$$a_0 = \frac{1}{T/2}\left[\int_{-T/2}^{0} 0\, dx + \int_0^{T/2} E\sin\omega t\, dt\right]$$

$$= \frac{2E}{T}\left[\frac{-\cos\omega t}{\omega}\right]_0^{T/2}$$

$$= \frac{2E}{T\omega}\left[-\cos\omega T/2 + 1\right] \qquad \text{put } T = 2\pi/\omega$$

$$= \frac{2E}{\frac{2\pi}{\omega}\times\omega}\left[-\cos\frac{\omega\times 2\pi}{2\,\omega} + 1\right]$$

$$= \frac{E}{\pi}[2] = \frac{2E}{\pi} \quad --- (1)$$

$$a_n = \frac{1}{T/2}\left[\int_{-T/2}^{0} 0\, dx \cos\frac{n\pi x}{T/2}\, dx + \int_0^{T/2} E\sin\omega t\cdot\cos\frac{n\pi t}{T/2}\right]$$

$$= \frac{2}{T}\left[\int_0^{T/2} E\sin\omega t\cdot\cos\frac{n\pi t\times 2}{\frac{2\pi}{\omega}}\, dt\right]$$

$$= \frac{2E}{2T}\left[\int_0^{T/2} 2\sin\omega t\, \cos n\omega t\, dt\right.$$

$$= \frac{E}{T}\left[\int_0^{T/2} \sin(1+n)\omega t + \sin(1-n)\omega t\, dt\right]$$

$$= \frac{E}{T}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{T/2}$$

$$= \frac{E}{\omega T}\left[\left\{\frac{-\cos(1+n)\omega\times T/2}{(1+n)} - \frac{\cos(1-n)\omega\times T/2}{1-n}\right\} + \left\{\frac{1}{(1+n)} + \frac{1}{(1-n)}\right\}\right]$$

$$\left\{\begin{array}{l}\cos(\pi+\theta) = -\cos\theta \\ \cos(\pi-\theta) = -\cos\theta\end{array}\right\}$$

$$\Rightarrow \frac{E}{\omega \times \frac{2\pi}{\omega}}\left[\frac{-\cos(1+n)\omega\times\frac{2\pi}{2\omega}}{1+n} - \frac{\cos(1-n)\omega\times\frac{2\pi}{2\omega}}{1-n} + \frac{1}{(1+n)} + \frac{1}{(1-n)}\right]$$

$$= \frac{E}{2\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{(1+n)} + \frac{1}{(1-n)}\right]$$

$$= \frac{E}{2\pi}\left[(-1)^n\left\{\frac{1-n+1+n}{1-n^2}\right\} + 1\left\{\frac{1-n+1+n}{1-n^2}\right\}\right]$$

$$= \frac{E}{\cancel{2}\pi}\left[\frac{(-1)^n}{1-n^2}\times\cancel{2} + 1\times\frac{2}{1-n^2}\right]$$

$$= \frac{E}{\pi(1-n^2)}\left\{(-1)^n + 1\right\} \quad n \neq 1$$

$$a_1 = \frac{1}{T/2}\left[\int_{-T/2}^{0} 0 \, dt + \int_0^{T/2} E\sin\omega t \cdot \sin\frac{n\pi t}{T/2}\, dt\right]$$

$$= \frac{2E}{T}\left[\int_0^{T/2} \sin\omega t \cdot \sin\frac{n\pi t}{\frac{2\pi}{\omega 2}}\, dt\right]$$

$$= \frac{2E}{T}\left[\int_0^{T/2} 2\sin\omega t \cdot \sin n\omega t \, dt\right]$$

$$= \frac{2E}{2T}\left[\int_0^{T/2} \cos(1-n)\omega t - \cos(1+n)\omega t \, dt\right]$$

$$= \frac{E}{T}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{T/2} \quad (T = 2\pi/\omega)$$

$$= \frac{E}{T}\left[\frac{\sin(1-n)\pi}{(1-n)\omega} - \frac{\sin(1+n)\pi}{(1+n)\omega}\right]_0^{T/2}$$

$$= 0$$

$$b_n = \frac{1}{T/2}\left[\int_0^{T/2} E\sin\omega t \cdot \sin\frac{n\pi t}{T/2}\, dt\right]$$

$$= \frac{2E}{2T}\left[\int_0^{T/2} 2\sin\omega t \cdot \sin \omega n t \, dt\right]$$

$$= \frac{E}{T}\left[\int_0^{T/2} \cos(1-n)\omega t - \cos(1+n)\omega t \, dt\right]$$

$$= \frac{E}{T}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{T/2}$$

by Putting limit and also put $T = \frac{2\pi}{\omega}$ we get,

$b_n = 0$

Now, $b_1 = \frac{1}{T/2}\left[\int_0^{T/2} E \sin \omega t \cdot \sin \frac{\pi t}{T/2}\, dt\right]$

$$b_1 = \frac{2E}{T}\left[\int_0^{T/2} \sin \omega t \cdot \sin \frac{\pi t}{\frac{2\pi}{2\omega}}\, dt\right]$$

$$= \frac{2E}{T}\left[\int_0^{\pi/2} \sin^2 \omega t\, dt\right]$$

$$= \frac{2E}{T}\left[\int_0^{T/2} \frac{1-\cos 2t}{2}\, dt\right].$$

$$= \frac{2E}{T}\left[t - \frac{\sin 2t}{2}\right]_0^{T/2}$$

$$= \frac{E}{T}\left[\frac{T}{2} - \frac{\sin T}{2}\right]$$

$$= \frac{E}{T}\left[\frac{2\pi}{\omega 2} - \frac{\sin \frac{2\pi}{2\omega}}{}\right]$$

$$= \frac{2E\pi}{T\omega^2} = \frac{\cancel{2}E\cancel{\pi}}{\frac{\cancel{2\pi}}{\omega}\cdot \omega^2} = \frac{E}{\omega}$$

$$f(x) = \frac{E}{\pi} + \sum_{n=2}^{\infty} \frac{E}{\pi(1-n^2)}\left\{(-1)^n+1\right\} \cos \frac{n\pi t}{T/2}$$

$$+ \frac{E}{\omega} \sin \frac{n\pi t}{T/2}$$

Put $n = 1, 2, \ldots$

Half Range Series

imp

Que $f(x) = \begin{cases} kx, & 0 < x < l/2 \\ k(l-x), & l/2 < x < l \end{cases}$

Find Half Range Cosine Series.

Sol^n $\Rightarrow$

H Range

$$a_0 = 2 \times \frac{1}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$\boxed{\begin{array}{l} 2c = \text{Total length} \\ 2c = 2l \rightarrow \text{full range} \\ \Rightarrow c = l \\ \text{since, given range is} \\ \text{Half range} \end{array}}$

$$\Rightarrow 2 \times \frac{1}{l}\left[k\left(\frac{x^2}{2}\right)_0^{l/2} + k\left(lx - \frac{x^2}{2}\right)_{l/2}^{l}\right]$$

$$\Rightarrow 2 \times \frac{1}{l}\left[\frac{kl^2}{8} + k\left\{\left(l^2 - \frac{l^2}{2}\right) - \left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right\}\right]$$

$$\Rightarrow 2 \times \frac{1}{l}\left[\frac{kl^2}{8} + \frac{kl^2}{2} - \frac{3l^2k}{8}\right]$$

$$\Rightarrow 2 \times \frac{1}{l} \times \frac{1}{4}kl^2 = 2 \times \frac{kl}{4} = \frac{kl}{2}$$

$$a_n = 2 \times \frac{1}{l}\left[\int_0^{l/2} \underset{u}{kx}\, \underset{v}{\cos\frac{n\pi x}{l}}\,dx + \int_{l/2}^{l} \underset{u}{k(l-x)}\, \underset{v}{\cos\frac{n\pi x}{l}}\,dx\right]$$

$$= \frac{2k}{l}\left[\left\{x\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - \left(\frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right\}_0^{l/2} + \right.$$

$$\left.\left\{(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2k}{l}\left[\left\{\frac{l}{2} \times \frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\right\} + \right.$$

$$\left.\left\{-\frac{l}{2} \times \frac{l}{n\pi}\sin\frac{n\pi}{2} - \frac{l^2(-1)^n}{n^2\pi^2} + \cos\frac{n\pi}{2} \times \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{2k}{l}\left[\cancel{\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2} - \cancel{\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}} \right.$$

$$\left. - \frac{l^2(-1)^n}{n^2\pi^2} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2K}{l}\left[\frac{l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2} - \frac{l^2(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2Kl}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{2Kl}{n^2\pi^2} - \frac{2Kl(-1)^n}{n^2\pi^2}$$

Or $\rightarrow$ $$a_n = \frac{2Kl}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1 - (-1)^n\right].$$

$$n = 1, 2, 3 \ldots$$

$$a_1 = \frac{2Kl}{\pi^2}\{0 - 1 + 1\} = 0$$

$$a_2 = \frac{2Kl}{2^2\pi^2}\{-1 - 1 - 1\} = \frac{6Kl}{4\pi^2}$$

$$= \frac{3Kl}{2\pi^2}$$

$$a_3 = 0$$

$$a_4 = \frac{2Kl}{4^2\pi^2}\{\cancel{1} - \cancel{1} - 1\}$$

$$= \frac{-2Kl}{16\pi^2} = \frac{-Kl}{8\pi^2}$$

$$f(x) = \frac{Kl}{4} - \frac{3Kl}{2\pi^2}\cos\frac{2\pi x}{l} - \frac{Kl}{8\pi^2}\cos\frac{4\pi x}{l} - \ldots \infty \quad (1)$$

Also P.T. $\rightarrow$ $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty$$

Put $l/2$ which is point of Discontinuity

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and $f(x) = \dfrac{f(x-0)+f(x+0)}{2}$

$= \dfrac{kx + k(l-x)}{2}$ $\left(\begin{array}{c}\text{put} \\ x = l/2\end{array}\right)$

$= \dfrac{\frac{kl}{2} + k\left(\frac{l}{2}\right)}{2} = \dfrac{kl}{2}$

Put $x = l/2$ & $f(x) = kl/2$

Try to ~~p~~ required proof.

*

(2) Prove that in the interval $-\pi < x < \pi$

$$x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=1}^{\infty}\frac{n(-1)^n}{n^2-1}\sin nx.$$

Solⁿ :- We have $f(x) = x\cos x$

$$f(-x) = -x\cdot\cos(-x)$$

$$= -x\cos x$$

Which is odd fuⁿ,

then we can ~~use~~ find only ~~...~~ b_n, $a_n = 0$ and $a_0 = 0$

Fourier Series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$a_0 = 0$

$a_n = 0$

then

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x}{2}\, 2\cos x \sin nx \, dx$$

$$= \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} x\left[\sin(1+n)x - \sin(1-n)x\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(1+n)x \, dx - \int_0^{\pi} x\sin(1-n)x \, dx$$

$$= \frac{1}{\pi}\left\{\left[\frac{x(-\cos(1+n)x)}{1+n} - \frac{1\cdot(-\sin(1+n)x)}{(1+n)^2}\right]_0^{\pi} - \left[\frac{x(-\cos(1-n)x)}{1-n} - \frac{(-\sin(1-n)x)}{1-n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{+\pi\cos n\pi}{1+n} + 0 + 0 - \frac{\pi\cos n\pi}{1-n} + 0\right\}$$

$$= \frac{\cancel{\pi}\cos n\pi}{\cancel{\pi}}\left\{\frac{1}{1+n} - \frac{1}{1-n}\right\}$$

$$= (-1)^n\left\{\frac{\cancel{1} - n - \cancel{1} - n}{1-n^2}\right\}$$

$$= \frac{2(-1)^n \cdot n}{1-n^2}, \text{ when } n \neq 1.$$

when $n = 1$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos x \sin x \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{x}{2} 2\cos x \sin x\right] dx$$

$$= \frac{2}{2\pi}\left\{\int_{0}^{\pi} x \cdot \sin 2x - \sin 0 \cdot x\right\} dx$$

$$= \frac{1}{\pi}\left\{x \frac{(-\cos 2x)}{2} - \frac{(-\sin 2x)}{4}\right\}_0^{\pi}$$

$$= \frac{1}{\cancel{\pi}}\left\{-\frac{\cancel{\pi}}{2}(1) + 0\right\}$$

$$= -1/2$$

By substituting b_1 and b_n in Fourier Series

$$f(x) = \frac{-1}{2}\sin x + 2 \cdot \sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{1-n^2} \sin nx$$

Practical Harmonic Analysis

Harmonic Analysis :- The process of finding the fourier series for a function given by a table of corresponding values of the function and independent variable is known as "Harmonic Analysis".

$$a_0 = 2 \times \frac{\sum f(x)}{N}$$

$$a_n = 2 \times \frac{\sum f(x) \cos nx}{N}$$

$$b_n \quad 2 \times \frac{\sum f(x) \sin nx}{N}$$

Ist Harmonic $\rightarrow \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$

IInd Harmonic $\rightarrow \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$

IIIrd Harmonic $\rightarrow \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x$

Que ①

x	y
0	1.80
30	1.10
60	0.30
90	0.16
120	1.50
150	1.30
180	2.16
210	1.25
240	1.30
270	1.52
300	1.76
330	2.00

Find 3rd Harmonic

we will form following table:

$\cos x$	$\cos 2x$	$\cos 3x$	$y\cos x$	$y\cos 2x$
			Σ	Σ

$y\cos 3x$	$\sin x$	$\sin 2x$	$\sin 3x$	$y\sin x$
Σ				Σ

$y\sin 2x$	$y\sin 3x$
Σ	Σ

$\boxed{N=12}$

$$a_0 = 2 \times \frac{\Sigma y}{N}$$

$$b_1 = 2 \times \frac{\Sigma y \sin x}{N}$$

$$a_1 = 2 \times \frac{\Sigma y \cos x}{N}$$

$$a_2 = 2 \times \frac{\Sigma y \cos 2x}{N}$$

$$b_2 = 2 \times \frac{\Sigma y \sin 2x}{N}$$

$$a_3 = 2 \times \frac{\Sigma y \cos 3x}{N}$$

$$b_3 = 2 \times \frac{\Sigma y \sin 3x}{N}$$

Que 2

x:	0	1	2	3	4	5
y:	4	8	15	7	6	12

Taking $\theta = 60^\circ$ Find 1^{st} 3 terms of cosine series.

$\theta =$	0	60°	120°	180°	240°	300°
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

$\boxed{N = 6}$ we will form following tables

$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
			Σ	Σ	Σ

$$a_0 = 2 \times \frac{\Sigma y}{N}, \quad a_1 = 2 \times \frac{\Sigma y\cos\theta}{N},$$

$$a_2 = 2 \times \frac{\Sigma y \cos 2\theta}{N}, \quad a_3 = 2 \times \frac{\Sigma y \cos 3\theta}{N}$$

Que 3

θ:	0	30	60	90	120	150	180
T:	0	5224	8094	7850	5499	2626	0

find 1^{st} 4 terms of sine series.

Sol^n In the Given data $\theta = 180$ gives repeated value so leave the term. also Find T, when $\theta = 75^\circ$

$$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta$$

Tables are

θ	T	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	$b_1 \sin\theta$	$b_2 \sin 2\theta$	$b_3 \sin 3\theta$	$b_4 \sin 4\theta$

$$b_1 = 2 \times \frac{\Sigma T \sin\theta}{N}$$

$$b_2 = 2 \times \frac{\Sigma T \sin 2\theta}{N}$$

$$b_3 = 2 \times \frac{\Sigma T \sin 3\theta}{N}$$

$$b_4 = 2 \times \frac{\Sigma T \sin 4\theta}{N}$$

$T = ?$

Que ⇒ t (sec) : 0, $T/6$, $T/3$, $T/2$, $2T/3$, $5T/6$, T

A (Amp) : 1.98, 1.30, 1.05, 1.30, -0.88, -0.25, 1.98

Find I^{st} Harmonic. Also find Amplitude of I^{st} harmonic

Sol^n ⇒ $2c = T \Rightarrow c = T/2$

$$A = \frac{a_0}{2} + a_1 \cos\frac{\pi t}{T/2} + b_1 \sin\frac{\pi t}{T/2}$$

* last term i.e. $t = T$ gives repeated value so we skip this.

$$A = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T}$$

$$\boxed{N = 6}$$

$$a_0 = 2 \times \frac{\Sigma A}{N}, \quad a_1 = 2 \times \Sigma A \cos\frac{2\pi t}{T}$$

$$b_1 = 2 \times \Sigma A \sin\frac{2\pi t}{T}$$

Tables ⇒

t	A	$\frac{2\pi t}{T}$	$\cos\frac{2\pi t}{T}$	$A\cos\frac{2\pi t}{T}$	$\sin\frac{2\pi t}{T}$	$A\sin\frac{2\pi t}{T}$

Amplitude of I^{st} harmonic $= \sqrt{a_1^2 + b_1^2}$

$$= 1.072$$