Partial Differential Equation ("PDE")

Def^n ⟹ A differential eq^n which involves partial derivatives is called "PDE".

Ex:-
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ are partial derivatives.

Notation ⟹ let, $z = f(x, y)$

$$\left\{ \begin{aligned} \frac{\partial z}{\partial x} &= p \\ \frac{\partial z}{\partial y} &= q \\ \frac{\partial^2 z}{\partial x^2} &= r \\ \frac{\partial^2 z}{\partial y^2} &= t \\ \frac{\partial^2 z}{\partial x \partial y} &= s \end{aligned} \right.$$

Formation of PDE by Eliminating arbitary constant

① Form PDE, $z = ax + by + ab$ by eliminating arbitary constant.

Sol^n ⟹ $z = ax + by + ab$

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$ {differentiating w.r.t x}

$\frac{\partial z}{\partial y} = b \Rightarrow q = b$ {differentiating w.r.t y}

by putting $p = a$ and $q = b$ in given eq^n

$z = ax + by + ab$

$z = px + qy + pq$

(2) $z = (x^2+a)(y^2+b)$

diffⁿ w.r.t (x) :—

$$\frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\Rightarrow p = 2x(y^2+b) \Rightarrow (y^2+b) = \frac{p}{2x}$$

diffⁿ w.r.t (y) :—

$$\frac{\partial z}{\partial y} = 2y(x^2+a)$$

$$q = 2y(x^2+a) \Rightarrow (x^2+a) = \frac{q}{2y}$$

by putting the values of (x^2+a) & (y^2+b) in given eqⁿ; —

$$z = \frac{q}{2y} \times \frac{p}{2x}$$

$$4xyz = pq$$ Ans

(3) $z = (x-a)^2 + (y-b)^2$

diffⁿ w.r.t (x) :—

$$\frac{\partial z}{\partial x} = 2(x-a) \Rightarrow p = 2(x-a)$$

$$\Rightarrow (x-a) = \frac{p}{2}$$

diffⁿ w.r.t (y) :—

$$\frac{\partial z}{\partial y} = 2(y-b) \Rightarrow q = 2(y-b)$$

$$\Rightarrow (y-b) = \frac{q}{2}$$

by putting the values of $(x-a)$ & $(y-b)$ in given eqⁿ;

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$4z = p^2 + q^2$$ Ans

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Formation of PDE by eliminating arbitary function :-

(1) $z = f\left(\frac{y}{x}\right)$

diffⁿ w.r.t x :-

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right)$$

$$p = f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-x^2 p}{y} \quad \text{--- (1)}$$

diffⁿ w.r.t y :-

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right)$$

$$\Rightarrow q = f'\left(\frac{y}{x}\right)\cdot\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{y}{x}\right) = xq \quad \text{--- (2)}$$

By ~~putting~~ equating the values of eqⁿ (1) & (2) :-

$$\frac{-x^2 p}{y} = xq \Rightarrow -xp = yq$$

or, $xp + yq = 0$ Ans

(2) $z = f(x^2 - y^2)$ --- (1)

diffⁿ w.r.t x :-

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\cdot 2x$$

$$\frac{p}{2x} = f'(x^2-y^2) \quad \text{--- (2)}$$

diffⁿ w.r.t y :-

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)\cdot(-2y)$$

$$q = f'(x^2-y^2)\cdot(-2y)$$

$$\frac{-q}{2y} = f'(x^2-y^2) \quad \text{--- (3)}$$

By eqⁿ (2) & (3) :→ (equating)

$$\frac{p}{2x} = \frac{-q}{2y}$$

$py + qx = 0$ Ans

Q → $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

diffⁿ (x) $\frac{\partial z}{\partial x} = 2f'\left\{\frac{1}{x} + \log y\right\} \cdot \left(\frac{-1}{x^2}\right)$

$$\frac{-px^2}{2} = f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (1)}$$

diffⁿ (y) $\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$

$$\frac{y}{2}(q - 2y) = f'\left(\frac{1}{x} + \log y\right)$$

$$\frac{yq}{2} - y^2 = f'\left(\frac{1}{x} + \log y\right) \quad \text{--- (2)}$$

by equating eqⁿ (1) & (2) :—

$$\frac{-px^2}{2} = \frac{yq}{2} - y^2$$

$$-px^2 = yq - 2y^2$$

$$x^2p + yq = 2y^2$$ Ans

II^{nd} form

(1) $z = f(x+at) + x\,g(x+at)$ —— (1)

diff^n $\dfrac{\partial z}{\partial x} = f'(x+at) + x\,g'(x+at) + g(x+at)$

(x) $\rightarrow$ $p = f'(x+at) + x\,g'(x+at) + g(x+at)$ —— (2)

diff^y (t) $\rightarrow$ $q = \dfrac{\partial z}{\partial t} = a f'(x+at) + a x\,g'(x+at)$ —— (3)

diff^y again (x) $\rightarrow$ $r = \dfrac{\partial^2 z}{\partial x^2} = f''(x+at) + x\,g''(x+at) + g'(x+at) + g'(x+at)$ —— (4)

again diff (t) $\rightarrow$ $\dfrac{\partial^2 z}{\partial t^2} = t = a^2 f''(x+at) + a^2 x\,g''(x+at)$ —— (5)

diff^y eq^n (3) w.r.t (x) $\rightarrow$ $s = \dfrac{\partial^2 z}{\partial x \partial t} = a f''(x+at) + a\{x\,g''(x+at) + g'(x+at)\}$

$s = a f''(x+at) + a x\,g''(x+at) + a g'(x+at)$ —— (6)

Here, eq^n (2) (4) $\rightarrow$

$$\begin{cases} r = f''(x+at) + x\,g''(x+at) + 2g'(x+at) & \text{—— (7)} \\ t = a^2 f''(x+at) + a^2 x\,g''(x+at) & \text{—— (8)} \\ s = a f''(x+at) + a x\,g''(x+at) + a g'(x+at) & \text{—— (9)} \end{cases}$$

&

* [$a^2 \times$ eq^n (7) , $2a \times$ eq^n (9)] , we get

$$\boxed{a^2 r - 2as + t = 0}$$ Ans

Q $z = f(x+ay) + g(x-ay)$

then P.T. $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol^n

$\Rightarrow p = \frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay)$

$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$ —— (1)

$q = \frac{\partial z}{\partial y} = a f'(x+ay) - a g'(x-ay)$

$t = \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$

$s = \frac{\partial^2 z}{\partial x \partial y} = a f''(x+ay) - g''(x-ay)$ —— (2)

By eq^n (1) and (2) :-

It is clear that

$$\boxed{a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}}$$

Hence proved

Q $z = y f(x) + x g(y)$ —— 1)

$p = \frac{\partial z}{\partial x} = y f'(x) + g(y)$ —— 2)

$q = \frac{\partial z}{\partial y} = f(x) + x g'(y)$ —— 3)

$r = \frac{\partial^2 z}{\partial x^2} = y f''(x)$ —— 4)

$t = \frac{\partial^2 z}{\partial y^2} = x g''(y)$ —— 5)

$S = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$ —— 6)

By eqⁿ (6) $\rightarrow$ $s = f'(x) + g'(y)$

By eqⁿ (2), $f'(x) = \dfrac{p - g(y)}{y}$

& By eqⁿ (3) $\rightarrow$ $g'(y) = \dfrac{q - f(x)}{x}$

Putting the values of $f'(x)$ & $g'(x)$ in s.

$$s = \frac{p - g(y)}{y} + \frac{q - f(x)}{x}$$

$$s = \frac{xp - xg(y) + qy - yf(x)}{xy}$$

$$sxy = xp + qy - (xg(y) + yf(x))$$

$$\boxed{sxy = xp + qy - Z}$$ Ans (By (1))

Q $z = f(x) + e^y g(x)$ — 1)

$p = f'(x) + e^y g'(x)$ — 2)

$\frac{\partial z}{\partial y} = q = e^y g(x)$ — 3)

$r = f''(x) + e^y g''(x)$ — 4)

$\frac{\partial^2 z}{\partial y^2}$ $t = e^y g(x)$ — 5)

$s = e^y g'(x)$ — 6)

By eqⁿ (3) & eqⁿ (6) :—

$$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$$

or $q = t$ — Ans

Que If $u = f_1(x^2 + 2yz, y^2 + 2zx)$

P.T. $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$

Sol^n $\rightarrow$ $u = f(x^2 + 2yz, y^2 + 2zx)$

$u \to f \begin{matrix} \nearrow v \\ \searrow w \end{matrix} \begin{matrix} \nearrow x \\ \to y \\ \searrow z \end{matrix}$ Here, u = Dependent var.
x, y, z = Independent var.

let, $u = f(v, w)$

Here, $v = x^2 + 2yz$ & $w = y^2 + 2zx$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v}\cdot(2x) + \frac{\partial f}{\partial w}(2z) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial v}(2z) + \frac{\partial f}{\partial w}(2y) \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial v}(2y) + \frac{\partial f}{\partial w}(2x) \quad \text{--- (3)}$$

LHS $\rightarrow$ $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z}$ --- (4)

Putting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial z}$ in (4)

$= 0 = $ RHS.

2 Independent variables

Single Independent variable

Direct Integration Method

(1) Solve, $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

By direct Integration Method.

Sol^n ⇒ $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

* Integrating w.r.t x {Keeping y fixed}

$$\frac{\partial^2 z}{\partial x \partial y} + 18y^2\left(\frac{x^2}{2}\right) - \frac{\cos(2x-y)}{2} = f(y)$$

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{\cos(2x-y)}{2} = f(y)$$

* Integrating w.r.t x {Keeping y fixed}

$$\frac{\partial z}{\partial y} + 9y^2\left(\frac{x^3}{3}\right) - \frac{\sin(2x-y)}{4} = x f(y) + g(y)$$

* Integrating w.r.t y {Keeping x fixed}

$$z + 3x^3\left(\frac{y^3}{3}\right) + \frac{\cos(2x-y)}{-4} = x\int f(y)\,dy + \int g(y)\,dy$$

$$\Rightarrow z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\,\underset{(\text{let})}{u(y)} + \underset{(\text{let})}{v(y)}$$

(2) Solve, $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$ by direct Integration Method.

Sol^n ⇒ $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

* Integrating w.r.t x :— (Keeping y as fixed)

$$\frac{\partial z}{\partial y} = \frac{1}{y}\left(\frac{x^2}{2}\right) + ax + f(y)$$

* Integrating w.r.t y :— (Keeping x as fixed)

$$z = \frac{x^2}{2}\left(\frac{-1}{y^2}\right) + axy + \int f(y)\,dy + f(x)$$

$$z = -\frac{x^2}{2y^2} + axy + \underset{(\text{let})}{u(y)} + f(x)$$

Q $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \sin y$, for which $\frac{\partial z}{\partial y} = -2\sin y$, when $x = 0$ & $z = 0$ when $z=0$ when y is an odd multiple of $\pi/2$.

Sol^n $\Rightarrow$ $$\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \sin y$$

* Integrating w.r.t x :–

$$\frac{\partial z}{\partial y} = -\cos x \cdot \sin y + f(y) \quad \text{—— (1)}$$

* Integrating w.r.t. y :—

$$z = \cos x \cdot \cos y + \int f(y)\,dy + f(x)$$
$$z = \cos x \cdot \cos y + \underbrace{u(y)}_{(\text{let})} + f(x) \quad \text{—— (2)}$$

By eq^n (1) :–

Here, $\boxed{\frac{\partial z}{\partial y} = -2\sin y,\ x = 0 \text{ (Given)}}$

$$\frac{\partial z}{\partial y} = -\cos x \cdot \sin y + f(y)$$
$$-2\sin y = -\cos 0 \cdot \sin y + f(y)$$
$$-2\sin y + \sin y = f(y)$$

So, $\boxed{f(y) = -\sin y}$

Put $f(y) = -\sin y$ in eq^n (2) :—

$$z = \cos x \cdot \cos y + \int -\sin y\,dy + f(x)$$
$$z = \cos x \cdot \cos y + \cos y + f(x) \quad \text{—— (3)}$$

$\boxed{\text{put, } z = 0 \text{, at } y = \pi/2 \text{ (Given)}}$

$$0 = \cos x \cdot \cos\frac{\pi}{2} + \cos\frac{\pi}{2} + f(x)$$

$\boxed{f(x) = 0}$

So finally we get, By (3) $$z = \cos x \cdot \cos y + \cos y$$ Ans

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Q. Solve, $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given that $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Sol^n $\Rightarrow$

$$\frac{\partial^2 z}{\partial x^2} + z = 0$$

(Notation $\frac{\partial}{\partial z} = D$)

$$(D^2 + 1) z = 0$$

Auxiliary eq^n :— $D^2 + 1 = 0$

$D = \pm i$ or $0 \pm i$ $\alpha = 0, \beta = 1$

Sol^n is :— { * $c_1 \cos x + c_2 \sin x \rightarrow$ C.F. of the diff^n eq^n }
{ * Formula:- $e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$ }

So, sol^n is, $z = f_1(y) \cos x + f_2(y) \sin x$ —— (1)

Arbitary fu^n

Given, $x = 0$ & $z = e^y$

Put $x = 0$ & $z = e^y$ in eq^n (1) :—

$$e^y = f_1(y) \cos 0 + f_2(y) \sin 0$$

$$\Rightarrow \boxed{e^y = f_1(y)} \quad \text{—— (2)}$$

again, Given, $\frac{\partial z}{\partial x} = 1$, $x = 0$:—

diff^n eq^n (1) w.r.t (1) :—
we get,

$$\frac{\partial z}{\partial x} = -f_1(y) \sin x + f_2(y) \cos x$$

Put $\frac{\partial z}{\partial x} = 1$ & $x = 0$:—

$$\boxed{1 = f_2(y)} \quad \text{—— (3)}$$

Put $f_1(y) = e^y$ & $f_2(y) = 1$ in eq^n (1) :—

$$\boxed{z = e^y \cos x + \sin x}$$ Ans.

(6) Solve, $\frac{\partial^2 z}{\partial x^2} = a^2 z$, when $x=0$, $\frac{\partial z}{\partial x} = a\sin y$ and $\frac{\partial z}{\partial y} = 0$.

H.W.

Ans :- $z = \sin y + \sinh ax$

Langrange's Linear PDE.

The PDE of the form, $Pp + Qq = R$ is the standard form of Linear PDE of order one is called "Langrange's LDE".

Here, P, Q, R are any fn of x, y, z.

Working Rule :- To Solve $Pp + Qq = R$ by Langrage's LDE :-

Step (1) → $Pp + Qq = R$ (standard form)

Step (2) → $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step (3) → equate any 2 Independent terms each other and find the Sol^n.

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(1) Solve, $xzp + yzq = xy$

Sol^n ⟹ By Langrange's LDE :-

$Pp + Qq = R$

By comparing given eq^n we get,

$P = xz$, $Q = yz$ and $R = xy$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

I II III (members)

Taking Ist two members :-

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

By Integrating, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\log x = \log y + \log c_1$

$x = y c_1$ $\xrightarrow{or}$ $\boxed{c_1 = \frac{x}{y}}$ $\Rightarrow x = c_1 y$

again, by equating last 2 members, $\frac{dy}{\cancel{y}z} = \frac{dz}{xy}$

$$\frac{dy}{z} = \frac{dz}{x}$$

Here, variable x is different from derivatives

So put $x = c_1 y$

$$\frac{dy}{z} = \frac{dz}{c_1 y}$$

$\Rightarrow c_1 y\, dy = z\, dz \Rightarrow \int c_1 y\, dy = \int z\, dz$

$$c_1 \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

again put $c_1 = x/y$

$$\frac{x}{y} \times \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{\frac{xy}{2} + \frac{z^2}{2} = c_2}$$

So final Ans is

$$\phi\left\{\frac{xy}{2} + \frac{z^2}{2}, \frac{x}{y}\right\} = 0$$

Ans.

(2) Solve,

$$yzp + zxq = xy$$

Solⁿ $\Rightarrow$ $P = yz, \; Q = zx, \; R = xy$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

I II III

By Iˢᵗ 2 members :-

$$\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x} \Rightarrow \int x\,dx = \int y\,dy$$

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \xrightarrow{or} \boxed{c_1 = \frac{x^2}{2} - \frac{y^2}{2}}$$

By last 2 members :-

$$\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y}$$

$$\int y\,dy = \int z\,dz \Rightarrow \frac{y^2}{2} = \frac{z^2}{2} - c_2$$

$$\xrightarrow{or} \boxed{c_2 = \frac{y^2}{2} - \frac{z^2}{2}}$$

So, $\phi\left\{\frac{x^2}{2} - \frac{y^2}{2}, \frac{y^2}{2} - \frac{z^2}{2}\right\} = 0$ Ans

Que $\Rightarrow$ Solve, $y^2p - xyq = x(z-2y)$

Solⁿ $\Rightarrow$ $P = y^2, \; Q = -xy, \; R = x(z-2y)$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

I II III

By Iˢᵗ 2 members $\Rightarrow$ $\frac{dx}{y^2} = \frac{dy}{-xy}$

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$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy$$

By Integration; $-\int x\,dx = \int y\,dy$

$$-\frac{x^2}{2} = \frac{y^2}{2} - C_1 \rightarrow \text{(our assumption)}$$

$$\boxed{\frac{y^2}{2} + \frac{x^2}{2} = C_1}$$

By last 2 members;

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$(z-2y)\,dy = -y\,dz$$

$$\underbrace{z\,dy + y\,dz} - 2y\,dy = 0$$

This is an exact Diffⁿ eqⁿ.

$$d(yz) - 2y\,dy = 0$$

By Integration, $yz - 2\frac{y^2}{2} = C_2$

$$\boxed{yz - y^2 = C_2}$$

So, $\phi\left\{\frac{y^2}{2} + \frac{x^2}{2},\ yz - y^2\right\} = 0$ Ans.

Q ⇒ Solve, $x^2p + y^2q = z^2$

Solⁿ ⇒ $P = x^2,\ Q = y^2,\ R = z^2$

$$\underset{\text{I}}{\frac{dx}{x^2}} = \underset{\text{II}}{\frac{dy}{y^2}} = \underset{\text{III}}{\frac{dz}{z^2}}$$

By Iˢᵗ 2 members;

$$\int\frac{dx}{x^2} = \int\frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} - C_1$$

then, $\boxed{C_1 = \frac{1}{x} - \frac{1}{y}}$

By last 2 members;

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

By Integration, $\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$

$$-\frac{1}{y} = \frac{-1}{z} - C_2$$

So, $\boxed{C_2 = \frac{1}{y} - \frac{1}{z}}$

then, $\phi\left\{\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right\} = 0$