$$= \frac{-2t\cosh t + 2\sinh t}{t^2}$$ Ans

* Convolution Theorem *

Convolution Means "Extremly complex" or "Twisted" in the complex way.

Imp

Statement → If $L^{-1}\bar{f}(s) = f(t)$ and $L^{-1}\bar{g}(s) = g(t)$

Then

$$\boxed{L^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\cdot g(t-u)\,du}$$

Que ① ⟹ Find, $L^{-1}\left(\frac{1}{s+a}\cdot\frac{1}{s+b}\right)$

Sol^n ⟹ let, $\bar{f}(s) = \frac{1}{s+a}$ and $\bar{g}(s) = \frac{1}{s+b}$

so, $L^{-1}\bar{f}(s) = L^{-1}\frac{1}{s+a} = e^{-at} = f(t)$

$L^{-1}\bar{g}(s) = L^{-1}\frac{1}{s+b} = e^{-bt} = g(t)$

$$L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s+b}\right\} = \int_0^t e^{-au}\cdot e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{-u(a-b)}\,du$$

$$= e^{-bt}\left[\frac{-e^{-(a-b)u}}{a-b}\right]_0^t$$

$$= e^{-bt}\left[\frac{-e^{-(a-b)t}+1}{a-b}\right]$$

$$= \frac{1}{a-b}\left[-e^{-at}+e^{-bt}\right]$$ Ans

(2) $L^{-1}\dfrac{1}{s(s^2+4)}$

Sol^n $\Rightarrow$ $L^{-1}\left(\dfrac{1}{s}\right) = 1$

$L^{-1}\left(\dfrac{1}{s^2+4}\right) = \dfrac{1}{2}\sin 2t$

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \int_0^t 1\cdot\frac{1}{2}\sin 2u \, du$$

$$= \frac{1}{2}\left\{-\frac{\cos 2u}{2}\right\}_0^t$$

$$\Rightarrow -\frac{\cos 2t + 1}{4}$$

$$= \frac{1-\cos 2t}{4}$$

Ans

(3) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Sol^n $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$.

$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{a^2}\int_0^t \sin au \cdot \sin a(t-u)\,du$

$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

$= \frac{1}{2a^2}\int_0^t 2\sin au \cdot \sin a(t-u)\,du$

$= \frac{1}{2a^2}\int_0^t \cos\{au - a(t-u)\} - \cos\{au + a(t-u)\}\,du$

$= \frac{1}{2a^2}\int_0^t \cos\{au - at + au\} - \cos\{au + at - au\}\,du$

$= \frac{1}{2a^2}\int_0^t \{\cos(2au - at) - \cos at\}\,du$

$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - u\cos at\right]_0^t$

$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right]$ Ans..

Que $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Soln $L^{-1}\left(\frac{1}{s^2}\right) = t$

$L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t (t-u)\times\frac{1}{a}\sin au\,du$$

$$= \frac{1}{a}\int_0^t (t\sin au - u\sin au)\,du$$

$$= \frac{1}{a}\left[t\left(\frac{-\cos au}{a}\right)_0^t - \left\{u\left(\frac{-\cos au}{a}\right) - \left(\frac{-\sin au}{a^2}\right)\right\}_0^t\right]$$

$$= \frac{1}{a}\left[t\left(-\frac{\cos at}{a}+\frac{1}{a}\right) - \left\{t\left(-\frac{\cos at}{a}\right)+\frac{\sin at}{a^2}\right\}\right]$$

$$= \cancel{\frac{-t\cos at}{a^2}} + \frac{t}{a^2} + \cancel{\frac{t\cos at}{a^2}} - \frac{\sin at}{a^3}$$

$$= \left(\frac{t}{a^2} - \frac{\sin at}{a^3}\right) \quad \text{Ans}$$

Que $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$

Soln $L^{-1}\frac{1}{s^2+1} = \sin t$

$L^{-1}\frac{s}{s^2+4} = \cos 2t$

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)} = \int_0^t \sin(t-u)\cdot\cos 2u\,du$$

$$= \frac{1}{2}\int_0^t 2\sin(t-u)\cdot\cos 2u\,du$$

$$= \frac{1}{2}\int_0^t \sin(t-u+2u) + \sin(t-u-2u)\,du$$

$$\Rightarrow \frac{1}{2}\int_0^t \left[\sin(t+u) + \sin(t-3u)\right] du$$

$$\Rightarrow \frac{1}{2}\left[-\cos(t+u) + \left(-\frac{\cos(t-3u)}{-3}\right)\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left[-\cos 2t + \cos t + \frac{\cos 2t}{3} - \frac{\cos t}{3}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{-2\cos 2t}{3} + \frac{2\cos t}{3}\right]$$

$$\Rightarrow \frac{1}{3}\left[\cos t - \cos 2t\right]$$

Show that

(1) $L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} = t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} \cdots \infty$

Solⁿ $\Longrightarrow$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots \infty$

$$\sin\frac{1}{s} = \frac{1}{s} - \frac{(1/s)^3}{3!} + \frac{(1/s)^5}{5!} \cdots\cdots \infty$$

Now,

$$\frac{1}{s}\sin\frac{1}{s} = \frac{1}{s^2} - \frac{1}{(3!)^2 s^4} + \frac{1}{5!\, s^5} \cdots\cdots \infty$$

$$L^{-1}\left[\frac{1}{s}\sin\frac{1}{s}\right] = \frac{t}{1!} - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} \cdots\cdots \infty$$

Application of Laplace Transforms

Transform of Derivative

$L(x) = \bar{x}$
$L(y) = \bar{y}$
$L(f(t)) = \bar{f}$

$$L\{f(t)\} = \bar{f}(s)$$

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

$$\boxed{L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots \infty}$$

Q (1)

$$\frac{d^2y}{dx^2} + w^2 y = 0 \quad , \text{ where } y(0) = A \ , \left(\frac{dy}{dx}\right)_0 = B$$

Sol^n Taking L.T. on both sides.

$$L[y''] + w^2 L(y) = 0$$

$$s^2\bar{y} - s\,y(0) - y'(0) + w^2\bar{y} = 0 \qquad \text{(Given)}$$

$$s^2\bar{y} - AS - B + w^2\bar{y} = 0 \qquad \text{put } y(0) = A$$

and $y'(0) = B$

$$\bar{y}(s^2 + w^2) = AS + B$$

$$\bar{y} = \frac{AS}{s^2+w^2} + \frac{B}{s^2+w^2}$$

Taking Inverse LT :-

$$y = A\cos wx + \frac{B}{w}\sin wx$$

Ans

Que. Solve, $(D^2+9)x = \cos 2t$
if $x(0) = 1$, $x(\frac{\pi}{2}) = -1$

Sol^n $\Longrightarrow$ $(D^2+9)x = \cos 2t$

Taking L.T. on both sides;

$$L[D^2(x)] + 9L(x) = L[\cos 2t]$$

$$s^2\bar{x} - sx(0) - x'(0) + 9\bar{x} = \frac{s}{s^2+4}$$

by Putting Initial value $\Longrightarrow$

$$s^2\bar{x} - s - A + 9\bar{x} = \frac{s}{s^2+4}$$

* Given, $x(0)=1$ but $x'(0)$ is no given in the Que. So assume that $x'(0) = A$

$$(s^2+9)\bar{x} - (s+A) = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{x} = (s+A) + \frac{s}{s^2+4}$$

$$\bar{x} = \frac{A}{s^2+9} + \frac{s}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

Taking Inverse L.T :-

$$x = L^{-1}\frac{A}{s^2+9} + L^{-1}\frac{s}{s^2+9} + L^{-1}\frac{s}{(s^2+4)(s^2+9)}$$

By solving $\frac{s}{(s^2+4)(s^2+9)}$ —— (1)

solving by Convolution thm;

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t$$

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{1}{2}\int_0^t \sin 2u \cdot \cos 3(t-u)\,du$$

$$= \frac{1}{4}\int_0^t 2\sin 2u \cdot \cos 3(t-u)\,du$$

$$= \frac{1}{4}\int_0^t \sin\{2u + 3t - 3u\} + \sin\{2u - (3t - 3u)\} \, du$$

$$= \frac{1}{4}\int_0^t \sin\{3t - u\} + \sin\{5u - 3t\} \, du$$

$$= \frac{1}{4}\left[\frac{+\cos(3t-u)}{+1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[(\cos 2t - \cos 3t) - \left(\frac{\cos 2t - \cos 3t}{5}\right)\right]$$

$$= \frac{1}{4}\left[\frac{4}{5}\cos 2t - \frac{4}{5}\cos 3t\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

So, By ①

$$x = L^{-1}\frac{A}{s^2+9} + L^{-1}\frac{s}{s^2+9} + L^{-1}\frac{s}{(s^2+4)(s^2+9)}$$

$$x = \frac{A}{3}\sin 3t + \cos 3t + \frac{1}{5}\{\cos 2t - \cos 3t\}$$

$$x = \frac{A}{3}\sin 3t + \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t \quad \text{----- ②}$$

Given, $x\left(\frac{\pi}{2}\right) = -1$ Putting this value

$x = -1$ when $t = \pi/2$ in eq^n ② :-

$$-1 = \frac{A}{3}\sin\frac{3\pi}{2} + \frac{4}{5}\cos\frac{3\pi}{2} + \frac{1}{5}\cos 2\frac{\pi}{2}$$

$\sin\frac{3\pi}{2} = -1$

$\cos\frac{3\pi}{2} = 0$

$\cos\pi = -1$

$$-1 = -\frac{A}{3} + 0 - \frac{1}{5}$$

$$-1 = \frac{-A}{3} + 0 - \frac{1}{5}$$

$\sin(180+90) =$

$\sin(180+\theta) = -\sin\theta$

so, $\sin(180+90) = -\sin 90 = -1$

$$+1 = +\left(\frac{A}{3} + \frac{1}{5}\right)$$

$$\frac{A}{3} = 1 - \frac{1}{5}$$

$$\frac{A}{3} = \frac{4}{5} \Rightarrow \boxed{A = \frac{12}{5}}$$

$\cos(180+\theta) = -\cos\theta$

so, $\cos(180+90) = -\cos 90 = \boxed{0}$

Final Ans →

$$x = \frac{12}{15}\sin 3t + \frac{4}{5}\cos 3t + \frac{1}{5}\cos 2t$$

Que :- Solve, $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$
Given that $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$

Solve $\Rightarrow$

$$(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$$

Taking ~~Inverse~~ L.T. on both sides ;

$$L[D^3 y] - 3L[D^2 y] + 3L[Dy] - L(y) = L[t^2 e^t]$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$\text{So, } \underset{\text{I}^{st} \text{ shifting}}{L(t^n e^{at})} = \frac{n!}{(s-a)^{n+1}}$$

$$\Rightarrow [s^3 \bar{y} - s^2 y(0) - s y'(0) - y''(0)] - 3[s^2 \bar{y} - s y(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Putting Initial values $\Rightarrow$

$$[s^3 \bar{y} - s^2(1) - 2] - 3[s^2 \bar{y} - s(1)] + 3[s\bar{y} - 1] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow [s^3 \bar{y} - s^2 - 2] - 3[s^2 \bar{y} - s] + 3[s\bar{y} - 1] - \bar{y} = \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y}(s^3 - 3s^2 + 3s - 1) - (s^2 + 2 - 3s \; 3) = \frac{2}{(s-1)^3}$$

$$\Rightarrow \bar{y}(s^3 - 3s^2 + 3s - 1) - (s^2 - 3s + 1) = \frac{2}{(s-1)^3}$$

$$\Rightarrow \cancel{\boxed{\bar{y} = \frac{2}{(s-1)^3}}}$$

$$^{**}(a-b)^3 = a^3 - b^3 - 3a^2 b + 3ab^2$$

$$\Rightarrow \bar{y}(s-1)^3 = \frac{2}{(s-1)^3} + s^2 - 3s + 1$$

$$\bar{y} = \frac{2}{(s-1)^6} + \frac{s^2 - 3s + 1}{(s-1)^3}$$

solving by Partial fraction

$$\bar{y} = \frac{2}{(s-1)^6} + \frac{s^2-3s+1}{(s-1)^3} \quad \text{---} (1)$$

By Partial Fraction;

$$\frac{s^2-3s+1}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} \quad \text{---} (2)$$

$$\frac{s^2-3s+1}{(s-1)^3} = \frac{A(s-1)^2 + B(s-1) + C}{(s-1)^3}$$

$$s^2-3s+1 = A(s-1)^2 + B(s-1) + C \quad \text{---} (3)$$

Put $\boxed{s=1}$ $\quad 1-3+1 = C \Rightarrow \boxed{C=-1}$

$$s^2-3s+1 = A[s^2+1-2s] + Bs - B - 1$$

$$s^2-3s+1 = As^2 + s(-2A+B) + A - B - 1$$

By comparing :—

Coeff. of s^2 $\longrightarrow$ $\boxed{A=1}$

By comparing :—

Coeff. of s $\longrightarrow$ $-3 = -2A+B$

$$-3 = -2+B \Rightarrow \boxed{B=-1}$$

By (2) $\Longrightarrow$ $\dfrac{s^2-3s+1}{(s-1)^3} = \dfrac{1}{(s-1)} - \dfrac{1}{(s-1)^2} - \dfrac{1}{(s-1)^3}$

By (1) $\Longrightarrow$ $\bar{y} = \dfrac{2}{(s-1)^6} + \dfrac{1}{s-1} - \dfrac{1}{(s-1)^2} - \dfrac{1}{(s-1)^3}$

Taking Inverse LT.

$$y = \frac{2t^5e^t}{5!} + e^t - e^t t - e^t \frac{t^2}{2}$$

$$y = \frac{e^t t^5}{60} + e^t - e^t t - \frac{e^t t^2}{2}$$

Que Solve, $y''' + 2y'' - y' - 2y = 0$
given, $y(0) = y'(0) = 0$, $y''(0) = 6$

Sol^n $y''' + 2y'' - y' - 2y = 0$
Taking L.T.

$$\left[s^3\bar{y} - s^2 y(0) - s y'(0) - y''(0)\right] + 2\left[s^2\bar{y} - s y(0) - y'(0)\right] - \left[s\bar{y} - y(0)\right] - 2\bar{y} = 0$$

$$\Rightarrow s^3\bar{y} - 6 + 2s^2\bar{y} - s\bar{y} - 2\bar{y} = 0$$

$$\bar{y}\left[s^3 + 2s^2 - s - 2\right] = 6$$

$$\bar{y} = \frac{6}{s^3 + 2s^2 - s - 2}$$

$$\bar{y} = \frac{6}{(s-1)(s+1)(s+2)}$$

$\left\{\begin{array}{l}\text{Factors of} \\ s^3 + 2s^2 - s - 2 = \\ (s-1)(s+1)(s+2)\end{array}\right.$

By Partial fraction:—

$$\frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{6}{\cancel{(s-1)(s+1)(s+2)}} = \frac{A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)}{\cancel{(s-1)(s+1)(s+2)}}$$

$$6 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$$

Put $s = +1$ $\rightarrow$ $6 = A(s+1)(s+2)$
$6 = A(2)(3) \Rightarrow \boxed{A = 1}$

Put $s = -1$ $\rightarrow$ $6 = B(s-1)(s+2)$
$6 = B(-1-1)(-1+2)$
$6 = -2B \Rightarrow \boxed{B = -3}$

Put $s = -2$ $\rightarrow$ $6 = C(s-1)(s+1)$
$6 = C(-3)(-1) \Rightarrow \boxed{C = 2}$

$$\bar{y} = \frac{6}{(s+1)(s-1)(s+2)}$$

$$\bar{y} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$$

Taking Inverse L.T.

$$y = e^{t} - 3e^{-t} + 2e^{-2t}$$ Ans