2- Marks

Q.① Write the condition for existence of Laplace transform.

Solⁿ:- $L\{f(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$ —— ①

exist if the Integral ① converges for some values of s.

Q.② Define unit impulse function.

Solⁿ:- The unit impulse function $\delta(t-a)$ is defined as follows:-

$\delta(t-a) = \infty$ for $t = a$; $= 0$ for $t \neq a$

Q.③ Find $L^{-1}\left\{\frac{1}{(s+3)^5}\right\}$

Solⁿ:- $\because L^{-1}\left\{\frac{1}{(s+a)^n}\right\} = e^{-at}\frac{t^{n-1}}{\lceil n}$

$\therefore L^{-1}\left\{\frac{1}{(s+3)^5}\right\} = e^{-3t}\frac{t^4}{\lceil 5} = \frac{e^{-3t}t^4}{24}$

Q.④ Find $L(4^t)$

Solⁿ:- $\because L(a^t) = \frac{1}{s-\log a}$, $s > \log a$, $a > 0$

$\therefore L(4^t) = \frac{1}{s-\log 4}$, $s > \log 4$, $4 > 0$

Q.⑤ If $f(t)$ is a periodic function with period T, then $L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$

4- Marks

Q.(1) Evaluate $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$

Solⁿ:- We know that $L(\sin t) = \frac{1}{s^2+1}$

$$\Rightarrow \quad L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}\,ds$$

$$\Rightarrow L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$$

using shifting property;

$$L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right\} = \frac{1}{s+1}\cot^{-1}(s+1).$$

Q.2 Find $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

Solⁿ: Since $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$

Hence $L^{-1}\left(\frac{1}{s^2+1}\right) = 1.\sin t$

$\because$ If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

Hence $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin t\,dt$

$$= (-\cos t)_0^t$$

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

Q.③ Express the following function in terms of unit step function and find its Laplace transform:

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$$

Solⁿ:— Given

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$$

then $L\{f(t)\} = \int_0^1 0 \cdot e^{-st}\,dt + \int_1^2 (t-1)\,e^{-st}\,dt$

$+ \int_2^\infty 1 \cdot e^{-st}\,dt$

$\Rightarrow L\{f(t)\} = 0 + \left| -\frac{(t-1)e^{-st}}{s} - \frac{e^{-st}}{s^2} \right|_1^2 + \left| -\frac{e^{-st}}{s} \right|_2^\infty$

$\Rightarrow L\{f(t)\} = \left| \frac{-(2-1)e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} \right| + \left| \frac{-e^{-\infty} + e^{-2s}}{s} \right|$

$\Rightarrow L\{f(t)\} = \cancel{-\frac{e^{-2s}}{s}} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} + \cancel{\frac{e^{-2s}}{s}}$

$\Rightarrow L\{f(t)\} = \frac{e^{-s} - e^{-2s}}{s^2}$

Q④ ~~Let~~ If $L\{f(t)\} = \bar{f}(s)$. then Prove that
$L\{e^{at} f(t)\} = \bar{f}(s-a)$.

Solⁿ:- Given $L\{f(t)\} = \bar{f}(s)$
Now By definition

$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$L\{e^{at} f(t)\} = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

(2)

1. . . .

(4)

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-rt} f(t)\,dt \text{ where } r = s-a$$

$$\Rightarrow L\{e^{at}f(t)\} = \bar{f}(r)$$

$$\Rightarrow L\{e^{at}f(t)\} = \bar{f}(s-a).$$

Q.(5) Find the inverse transform of

$$\frac{4s+5}{(s-1)^2(s+2)}$$

Sol^n:- Given,

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{\cancel{(-2-1)^2}(s+2)} \quad \text{--- (1)}$$

$$\Rightarrow \frac{4s+5}{(s-1)^2(s+2)} = \frac{A(s-1)(s+2) + B(s+2) + C(s-1)^2}{(s-1)^2(s+2)}$$

$$\Rightarrow 4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

for $s=1$, $B = 3$

for $s=-2$, $C = -\frac{1}{3}$

for $s=0$, $5 = -2A + 2B + C$

$$\Rightarrow 5 = -2A + 6 - \tfrac{1}{3}$$

$$\Rightarrow A = \tfrac{1}{3}$$

from eq^n (1)

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$$

$$\Rightarrow L^{-1}\left(\frac{4s+5}{(s-1)^2(s+2)}\right) = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow = \frac{1}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$$

Q. (1)(i) Find the Laplace transform of $(e^{-t}\sin t)\,t$.

Sol^n:- Let $f(t) = e^{-t}\sin t$

then $L\{f(t)\} = \dfrac{1}{(s-1)^2+1} = \bar{f}(s)$

Now $L(t.\, e^{-t}\sin t) = (-1)\dfrac{d}{ds}\left(\dfrac{1}{(s-1)^2+1}\right)$

$\Rightarrow L(t.\, e^{-t}\sin t) = -\dfrac{d}{ds}\left(\dfrac{1}{s^2-2s+2}\right)$

$= \dfrac{-[-(2s-2)]}{(s^2-2s+2)^2}$

$\Rightarrow L(t.\, e^{-t}\sin t) = \dfrac{2s-2}{(s^2-2s+2)^2}$

Q.(1)(ii) Find the Laplace transform of

$\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$

Sol^n:- Given $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$\Rightarrow L\left(\sqrt{t} - 1/\sqrt{t}\right)^3 = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$

$= \dfrac{\Gamma\frac{3}{2}+1}{s^{3/2+1}} - 3\,\dfrac{\Gamma\frac{1}{2}+1}{s^{1/2+1}} + 3\,\dfrac{\Gamma-\frac{1}{2}+1}{s^{-1/2+1}} - \dfrac{\Gamma-\frac{3}{2}+1}{s^{-3/2+1}}$

$= \dfrac{\frac{3}{2}\Gamma\frac{3}{2}}{s^{5/2}} - 3.\dfrac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} + 3.\dfrac{\Gamma\frac{1}{2}}{s^{1/2}} - \dfrac{\Gamma-\frac{1}{2}}{s^{-1/2}}$

$= \dfrac{3}{4}\dfrac{\sqrt{\pi}}{s^{5/2}} - \dfrac{3}{2}\dfrac{\sqrt{\pi}}{s^{3/2}} + \dfrac{3\sqrt{\pi}}{s^{1/2}} + \dfrac{2\sqrt{\pi}}{s^{-1/2}}$

$= \dfrac{\sqrt{\pi}}{4}\left(\dfrac{3}{s^{5/2}} - \dfrac{6}{s^{3/2}} + \dfrac{12}{s^{1/2}} + \dfrac{8}{s^{-1/2}}\right)$

Q.2(i) Find Laplace transform of $\frac{\cos at - \cos bt}{t}$

Sol^n:- $L\left(\frac{\cos at - \cos bt}{t}\right)$

Let $f(t) = \cos at - \cos bt$

$L(f(t)) = L(\cos at) - L\{\cos bt\}$

$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

Now $L\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$\Rightarrow$ $= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\lim_{s\to\infty}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$

$= -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$

$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$

Q.2.(ii) Find laplace transform of $\cos^3 2t$

Sol^n:- Given $\cos^3 2t = \frac{\cos 6t + 3\cos 2t}{4}$

$[\cos 3A = 4\cos^3 A - 3\cos A]$

$L(\cos^3 2t) = \frac{1}{4}[L(\cos 6t) + 3L(\cos 2t)]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$

$= \frac{1}{4}\left(\frac{4s+112s}{(s^2+4)(s^2+36)}\right]$

$L(\cos^3 2t) = \frac{s^2+28s}{(s^2+4)(s^2+36)}$

Q.3 - Find the laplace transform of
$\frac{1-\cos t}{t^2}$

Sol^n:— Let $f(t) = 1-\cos t$

$L\{f(t)\} = \frac{1}{s} - \frac{s}{s^2+1}$

Now $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$

$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$

$= -\log s + \frac{1}{2}\log(s^2+1)$

$= \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

Again Now $L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\int_s^\infty 1 \cdot \log\left(\frac{s^2+1}{s^2}\right) ds$

$L\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}\left[s \cdot \log\left(\frac{s^2+1}{s^2}\right) - \int \frac{s^2}{s^2+1} \cdot \frac{s^2(2s)-(s^2+1) \cdot 2s}{s^4} \, s\,ds\right]_s^\infty$

$= \frac{1}{2}\left(s\log\left(\frac{s^2+1}{s^2}\right) + 2\int \frac{ds}{s^2+1}\right]_s^\infty$

$= \frac{1}{2}\left[-s\log\left(\frac{s^2+1}{s^2}\right) + 2\left(\frac{\pi}{2} - \tan^{-1}s\right)\right]$

$= \cot^{-1}s - \frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$

Q.4 (i) Find the inverse laplace transform of
$\frac{s^2+6}{(s^2+1)(s^2+4)}$

Sol^n:— Given $\frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{s^2+4+2}{(s^2+1)(s^2+4)}$

$\Rightarrow \quad = \frac{1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$

$= \frac{1}{s^2+1} + \frac{2}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$

$$\Rightarrow \frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{5}{3}\left(\frac{1}{s^2+1}\right) - \frac{2}{3}\left(\frac{1}{s^2+4}\right)$$

$$L^{-1}\left\{\frac{s^2+6}{(s^2+1)(s^2+4)}\right\} = \frac{5}{3} L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{2}{3} L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Q.(4)(ii) Find the inverse laplace transform of $\frac{s}{(s^2+1)(s^2+4)}$ using Convolution theorem.

Sol:-

$$L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right) = L^{-1}\left\{\frac{s}{s^2+1}\right\} * L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$\Rightarrow \quad = L^{-1}\left\{\frac{s}{s^2+1}\right\} * L^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2+4}\right\}$$

$$= \cos t * \frac{\sin 2t}{2}$$

$$\left[f(t) * g(t) = \int_0^t f(x)\cdot g(t-x)\,dx\right]$$

$$= \int_0^t \cos x \cdot \frac{\sin(2(t-x))}{2}\,dx$$

$$= \frac{1}{2}\int_0^t \cos x\,\sin(2t-2x)\,dx$$

$$= \frac{1}{4}\int_0^t \sin(2t-x) - \sin(3x-2t)\,dx$$

$$= \frac{1}{4}\left[\cos(2t-x) + \frac{\cos(3x-2t)}{3}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t + \frac{\cos t}{3} - \left(\cos 2t + \frac{\cos 2t}{3}\right)\right]$$

$$= \frac{1}{4}\left[\frac{4\cos t}{3} - \frac{4\cos 2t}{3}\right]$$

$$= \frac{1}{3}(\cos t - \cos 2t).$$

Q. 5 (i) Find the inverse Laplace transform of
$$\frac{s^2+s-2}{s(s+3)(s-2)}$$

Solⁿ:- Given $\frac{s^2+s-2}{s(s+3)(s-2)} = \frac{s}{(s+3)(s-2)} + \frac{1}{s(s+3)}$

Now $\frac{s}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$

$$s = A(s-3) + B(s+3)$$

for $s=2$, $B = 2/5$

for $s=-3$, $A = 3/5$

then,
$$\frac{s}{(s+3)(s-2)} = \frac{3}{5}\left(\frac{1}{(s+3)}\right) + \frac{2}{5}\left(\frac{1}{s-2}\right)$$

$$L^{-1}\left\{\frac{s}{(s+3)(s-2)}\right\} = \frac{3}{5}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{2}{5}L^{-1}\left\{\frac{1}{s-2}\right\}$$
$$= \frac{3}{5}e^{-3t} + \frac{2}{5}e^{2t}$$

Again Now
$$\frac{1}{s.(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{1}{3s} - \frac{1}{3(s+3)}$$

$$L^{-1}\left\{\frac{1}{s(s+3)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+3}\right\}$$
$$= \frac{1}{3} - \frac{1}{3}e^{-3t}$$

Hence.
$$L^{-1}\left\{\frac{s^2+s-2}{s(s+3)(s-2)}\right\} = \frac{3}{5}e^{-3t} + \frac{2}{5}e^{2t} + \frac{1}{3} - \frac{1}{3}e^{-3t}$$
$$= \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t} + 1/3$$

Q. 5 (ii) Find the inverse Laplace transform of $\tan^{-1}\frac{2}{s^2}$.

Solⁿ:- Let $L^{-1}\left\{\tan^{-1}\frac{2}{s^2}\right\} = F(t)$

$$tF(t) = L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$$

$$= L^{-1}\left\{\frac{4s}{s^4+4}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= e^{t}\sin t - e^{-t}\sin t$$

$$= (e^{t} - e^{-t})\sin t$$

$$= 2\left(\frac{e^{t}-e^{-t}}{2}\right)\sin t$$

$$= 2\sinh t \cdot \sin t \; //$$

Q.6 Apply convolution theorem to prove that

$$L^{-1}\left\{\frac{8}{(s^2+1)^3}\right\} = (3-t^2)\sin t - 3t\cos t.$$

Solution:- we have

$$L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t$$

$\therefore$ By the convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+1)}\right\} = \int_0^t \sin x \cdot \sin(t-x)\,dx$$

$$= \int_0^t \sin x(\sin t\cos x - \cos t\sin x)\,dx$$

$$= \sin t\int_0^t \sin x\cos x\,dx - \cos t\int_0^t \sin^2 x\,dx$$

$$= \frac{\sin t}{2}\int_0^t \sin 2x\,dx - \frac{\cos t}{2}\int_0^t (1-\cos 2x)\,dx$$

$$= \frac{\sin t}{2}\left(\frac{1-\cos 2t}{2}\right) - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

$$= \frac{1}{2}\sin t\cdot\sin^2 t - \frac{t}{2}\cos t + \frac{\cos t}{2}\sin t\cos t$$

$$= \frac{1}{2}\sin t(1-\cos^2 t) - \frac{t}{2}\cos t + \frac{\sin t}{2}\cos^2 t$$

$$L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}\sin t - \frac{t}{2}\cos t.$$

$$\therefore L^{-1}\left\{\frac{8}{(s^2+1)^3}\right\} = 8L^{-1}\left\{\frac{1}{(s^2+1)^2}\cdot\frac{1}{(s^2+1)}\right\}$$

By the convolution theorem,

$$= 8\int_0^t \left(\frac{1}{2}\sin x - \frac{x}{2}\cos x\right)\sin(t-x)\,dx$$

$$= 4\int_0^t (\sin x - x\cos x)(\sin t\cos x - \cos t\sin x)\,dx$$

$$= 4\sin t\int_0^t (\sin x\cos x - x\cos^2 x)\,dx - 4\cos t\int_0^t (\sin^2 x - x\sin x\cos x)\,dx$$

$$= 2\sin t\int_0^t \{\sin 2x - x(1+\cos 2x)\}\,dx - 2\cos t\int_0^t \{(1-\cos 2x) - x\sin 2x\}\,dx$$

$$= 2\sin t\left[-\frac{t^2}{2} + \frac{1-\cos 2t}{2} - \frac{t}{2}\sin 2t + \frac{1-\cos 2t}{4}\right]$$
$$- 2\cos t\left[t - \frac{1}{2}\sin 2t + \frac{t\cos 2t}{2} - \frac{\sin 2t}{4}\right]$$

$$= -t^2 \sin t + \frac{3}{2}\sin t - \frac{3}{2}\sin t \cos t - t \sin t \sin 2t - 2t \cos t + \frac{3}{2}\sin 2t \cos t - t \cos t \cos 2t$$

$$= -t^2 \sin t + \frac{3}{2}\sin t + \frac{3}{2}(-\sin t \cos 2t + \sin 2t \cos t) - t(\cos 2t \cos t + \sin t \sin 2t) - 2t \cos t$$

$$= (3 - t^2)\sin t - 3t \cos t. \quad // \text{Proved}$$

Q.7 Use the method of partial fraction to find the inverse transform of $\frac{s}{s^4+s^2+1}$

Solution:-

We have,

$$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+s+1)(s^2-s+1)} = F(s)$$

$$= \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{\sqrt{3}}{\sqrt{3}.(2)}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\sqrt{3}/2}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

So,

$$L^{-1}[F(s)] = \frac{1}{\sqrt{3}} e^{1/2 t} \sin \frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \cdot \sin \frac{\sqrt{3}}{2}t$$

$$= \frac{1}{\sqrt{3}}\left(e^{1/2 t} - e^{-1/2 t}\right) \sin \frac{\sqrt{3}}{2}t$$

$$= \frac{2}{\sqrt{3}} \sinh \frac{t}{2} \cdot \sin \frac{\sqrt{3}}{2}t \quad // \text{Ans.}$$

Q.8 (a). Find the Laplace transform of $\frac{1-\cos 2t}{t}$

Solution:-

Let $F(t) = 1-\cos 2t$. Then

$$\underset{t\to 0}{Lt}\ \frac{F(t)}{t} = \underset{t\to 0}{Lt}\ \frac{1-\cos 2t}{t} = 0$$

and

$$f(s) = L\{F(t)\} = L\{1-\cos 2t\}$$
$$= L\{1\} - L\{\cos 2t\}$$
$$= \frac{1}{s} - \frac{s}{s^2+2^2} = \frac{4}{s(s^2+4)}, \quad s>0.$$

We know that if $L\{F(t)\} = f(s)$, then,

$$L\left\{\frac{1}{t}F(t)\right\} = \int_s^\infty f(x)\,dx.$$

Provided $\underset{t\to 0}{Lt}\ \frac{F(t)}{t}$ exists.

Therefore,

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty f(x)\,dx$$
$$= \int_s^\infty \frac{4}{x(x^2+4)}\,dx$$
$$= \int_s^\infty \left[\frac{1}{x} - \frac{x}{x^2+2^2}\right] dx$$
$$= \left[\log x - \frac{1}{2}\log(x^2+2^2)\right]_s^\infty$$
$$= \left[\frac{1}{2}\log\frac{x^2}{x^2+2^2}\right]_s^\infty$$
$$= \frac{1}{2}\underset{x\to\infty}{Lt}\ \log\frac{x^2}{x^2+2^2} - \frac{1}{2}\log\frac{s^2}{s^2+a^2}$$

$$= \frac{1}{2}\log(1) + \frac{1}{2}\log\frac{s^2+2^2}{s^2}$$

$[\because \log(1) = 0]$

$$= \frac{1}{2}\log\left(1+\frac{4}{s^2}\right)$$

// Ans.

(b). Evaluate the following: $\int_0^\infty te^{-3t}\sin t\,dt$.

Solution:- Given that

$$\int_0^\infty te^{-3t}\sin t\,dt$$

(where $s=3$)

$$= \int_0^\infty e^{-st}(t\sin t)\,dt$$

$= L(t\sin t)$; by definition.

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$= \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50} \quad //$$

Ans.

Q.9 (a). Find the Laplace transform of $\sin 2t \cdot \sin 3t + \cos^2 t$.

Solution:-

Since

$$\sin 2t \sin 3t + \cos^2 t = \frac{1}{2}[\cos t - \cos 5t] + \frac{1}{2}(1+\cos 2t)$$

$\left[\begin{array}{l} \because \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\ \& \quad 2\cos^2 t = 1 + \cos 2t \end{array}\right]$

$$\therefore L(\sin 2t \sin 3t + \cos^2 t) = \frac{1}{2}\left[L(\cos t) - L(\cos 5t)\right] + \frac{1}{2}\left[L(1) + L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+2^2}\right]$$

$$= \frac{12s}{(s^2+1)(s^2+25)} + \frac{2s^2+4}{2s(s^2+4)}$$

$$= \frac{12s}{(s^2+1)(s^2+25)} + \frac{\cancel{2}(s^2+2)}{\cancel{2}s(s^2+4)}$$

$$= \frac{12s}{(s^2+1)(s^2+25)} + \frac{(s^2+2)}{s(s^2+4)}$$ // Ans.

(b) Show that $\int_0^\infty t e^{-2t} \cos t \, dt = \frac{3}{25}$

Solution:-

we have

$$L\{t \cos t\} = -\frac{d}{ds} L\{\cos t\}$$

or $$\int_0^\infty e^{-st} t \cos t \, dt = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{s^2-1}{(s^2+1)^2} \quad -(1)$$

taking $s=2$ in (1), we get

$$\int_0^\infty t e^{-2t} \cos t \, dt = \frac{3}{25}$$ // Ans.

Q.10 Solve the differential equation by transform method $\frac{d^2x}{dt^2} + 9x = \cos 2t$, when $x(0) = 1$, $x\left(\frac{\pi}{2}\right) = -1$.

Solution:- Since $x'(0)$ is not given, we assume $\boxed{x'(0) = a}$

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L\{\cos 2t\}$$

i.e., $$\left[s^2\bar{x} - s x(0) - x'(0)\right] + 9\bar{x} = \frac{s}{s^2+4}$$

$$\Rightarrow (s^2+9)\bar{x} = s + a + \frac{s}{s^2+4}$$

$$\Rightarrow \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or $$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\,\frac{s}{s^2+9}.$$

On inversion (Laplace), we get

$$x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t \quad \text{— (1)}$$

When $t = \pi/2$, $\boxed{\because x\left(\frac{\pi}{2}\right) = -1}$

$$-1 = -\frac{a}{3} - \frac{1}{5}$$

or $\boxed{\frac{a}{3} = \frac{4}{5}}$

Hence, the solution is, from (1)

$$x = \frac{1}{5}\left[\cos 2t + 4\sin 3t + 4\cos 3t\right]. \;//$$

Ans.

Q.11 Solve the differential equation by transform method

$ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.

Solution:- Taking Laplace transform of both sides of the equation and noting that

$$L\{t(f(t))\} = -\frac{d}{ds}\left[L\{f(t)\}\right], \text{ we get}$$

$$-\frac{d}{ds}\left[s^2\bar{y} - sy(0) - y'(0)\right] + 2\left[s\bar{y} - y(0)\right] - \frac{d}{ds}(\bar{y}) = \frac{1}{s^2+1}$$

or $$-\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{1}{s^2+1}$$

or $$-(s^2+1)\frac{d\bar{y}}{ds} - 1 = \frac{1}{s^2+1} \qquad \boxed{\because y(0) = 1}$$

$$-(s^2+1)\frac{d\bar{y}}{ds} = \frac{1}{s^2+1} + 1$$

$$\frac{d\bar{y}}{ds} = \frac{-1}{(s^2+1)^2} - \frac{1}{(s^2+1)}$$

on inversion and noting that

$$L^{-1}[\bar{f}'(s)] = -tf(t), \text{ we get}$$

$$-ty = -\sin t - \left(\frac{1}{2}\sin t - \frac{t\cos t}{2}\right)$$

$$= \frac{1}{2}(-3\sin t + t\cos t)$$

$$\Rightarrow y = \frac{1}{2}\left(\frac{3\sin t}{t} - \cos t\right)$$

which is the desired solution. // Ans.

Q.12 Solve the differential equation by transform method $ty'' + (1-2t)y' - 2y = 0$, when $y(0) = 1$ and $y'(0) = 2$.

Solution:- The given equation can be written as

$$ty'' + y' - 2ty' - 2y = 0$$

Taking Laplace transforms of both sides, we have

$$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$$

or $$-\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L\{y'\} - 2L\{y\} = 0$$

or $$-\frac{d}{ds}[s^2L\{y\} - sy(0) - y'(0)] + [sL\{y\} - y(0)] + 2\frac{d}{ds}[sL\{y\} - y(0)] - 2L\{y\} = 0$$

or $$-\frac{d}{ds}(s^2z - s - 2) + (sz - 1) + 2\frac{d}{ds}(sz - 1) - 2z = 0 \quad \text{where } z = L\{y\}$$

or $$-(s^2 - 2s)\frac{dz}{ds} - sz = 0$$

or $$\frac{dz}{z} + \frac{1}{s-2}\,ds = 0$$

Integrating, we get

$$\log z + \log(s-2) = \log c_1$$

or $$z = \frac{c_1}{s-2} \quad \text{or} \quad L\{y\} = \frac{c_1}{s-2}$$

Now taking inverse Laplace transform, we get

$$y = c_1 L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y = c_1 e^{2t}$$

But $y(0) = 1$ $\therefore 1 = c_1$

hence $\boxed{y = e^{2t}}$ //

which is the required solution.

Q.13 Solve the differential equation by transform method $y'' - 3y' + 2y = 4e^{2t}$, when $y(0) = -3$ and $y'(0) = 5$.

Solution:- Given eqn

$$y'' - 3y' + 2y = 4e^{2t}$$

The Laplace transform of eqn

$$(s^2\bar{y} - sy(0) - y'(0)) - 3(s\bar{y} - y(0)) - 2\bar{y} = 4\left(\frac{1}{s-2}\right)$$

$$(s^2 - 3s + 2)\bar{y} + 3s - 5 - 9 = \frac{4}{s-2}$$

$$[(s-1)(s-2)]\bar{y} = \frac{4}{s-2} - 3s + 14$$

$$\bar{y} = \frac{4}{(s-1)(s-2)^2} - \frac{3s-14}{(s-1)(s-2)} = \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

$$\cancel{\bar{y} = 4L^{-1}\left(\frac{1}{(s-1)(s-2)^2}\right) - L^{-1}\left(\frac{11}{s-1} - \frac{8}{s-2}\right)}$$

$$y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\bar{y} = L^{-1}(y) = L^{-1}\left[\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right]$$

$$= -7e^{t} + 4e^{2t} + 4te^{2t}$$ Ans.

Q. 14 Solve $(D^2+m^2)x = a\cos nt$, $t>0$, when $x = x_0$ and $Dx = x_1$, when $t=0$, $m \neq n$.

Solution:- The given differential equation can be written as

$$x'' + m^2 x = a\cos nt \quad , t>0 \qquad \text{—①}$$

with the initial conditions

$$x(0) = x_0 \text{ and } x'(0) = x_1.$$

Taking the Laplace transform of both sides of ① and by linearity, we have

$$L\{x''\} + m^2 L\{x\} = aL\{\cos nt\}$$

or $$s^2 L\{x\} - s.x(0) - x'(0) + m^2 L\{x\} = a.\frac{s}{s^2+n^2}$$

or $$s^2 L\{x\} - s.x_0 - x_1 + m^2 L\{x\} = \frac{as}{s^2+n^2}$$

or $$(s^2+m^2)L\{x\} = sx_0 + x_1 + \frac{as}{s^2+n^2}$$

or $$L\{x\} = x_0 \frac{s}{s^2+m^2} + x_1 \frac{1}{s^2+m^2} + \frac{as}{(s^2+m^2)(s^2+n^2)}$$

$$= x_0 \frac{s}{s^2+m^2} + x_1 . \frac{1}{s^2+m^2} + \frac{a}{m^2-n^2}\left(\frac{s}{s^2+m^2} + \frac{s}{s^2+n^2}\right)$$

Taking the inverse Laplace transform, we have

$$x = x_0 L^{-1}\left\{\frac{s}{s^2+m^2}\right\} + x_1 L^{-1}\left\{\frac{1}{s^2+m^2}\right\}$$

$$+ \frac{a}{m^2-n^2}\left[-L^{-1}\left\{\frac{s}{s^2+m^2}\right\} + L^{-1}\left\{\frac{s}{s^2+n^2}\right\}\right]$$

or $$x = x_0 \cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}(-\cos mt + \cos nt) \;//$$

which is the required solution. Ans.

Q. 15 Solve $(D^3 - 3D^2 + 3D - 1)y = t^2 e^t$, when $y(0) = 1$, $Dy(0) = 0$ and $D^2 y(0) = -2$.

Solution:- Taking the Laplace transforms of both sides we get

$$[s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y} - sy(0) - y'(0)] +$$
$$3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

using the given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain,

$$y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-1)^3}\right\} + 2L^{-1}\left(\frac{1}{(s-1)^6}\right)$$

$$y = e^t\left\{1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right\} \;//$$

Ans.