# P.D.E.

Scanned by CamScanner

# * Partial Diff. Equation *



A partial D.E. is an equation involving two (or more) independent variable $x, y$ and a dependent variable $z$ and it's partial derivatives such as, $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 x}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}$ ---etc.

i.e.
$$\boxed{f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \cdots\right) = 0}$$

For eg: $\Rightarrow$

(1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin(\cancel{z}, y)$

(2) $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial z}{\partial y} + 2\frac{\partial^2 z}{\partial y^2}\cdot\frac{\partial z}{\partial x} = 0$ etc.

Some standard notations :

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$
etc

---

## Formation of Partial D.E :$\Rightarrow$

**Method (1) : By Elimination of arbitrary constants.**

let $f(x, y, z, a, b) = 0$ ——— (1)

be an eqn involving two arbitrary const $\underline{a}$. and $\underline{b}$ where $x, y$ are independent variable & $z$ is dependent.

Diff. eqn (1) w.r.to $x$ and $y$ partially we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \quad \text{——— (2)}$$

& $$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0 \quad \text{——— (3)}$$

Solving eqn ② & ③, To ~~elliminate~~ elliminate a, & b

we get eqn of the type

$$f(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$$

or $f(x, y, z, p, q) = 0$

which is the partial diff. eqn of first order.

Note: ① If the number of orbitrary constants are equals to no. of independent variable appearing in an eqn. then P.D.E obtained by ellimination is of first orde

(ii) If the number of arbitrary const are more than no. independent variable then P.D.E. obtained by ~~elliminat~~ ellimination is of order two, or higher order.

---

Ex: Form a partial D.E. for

$$x^2+y^2+(z-c)^2 = a^2$$

soln: Given eqn is

$$x^2+y^2+(z-c)^2 = a^2 \quad \text{—— (1)}$$

where a, and c are arbitrary consts.

Diff eqn (1) w.r.to x. partially we get

$$2x + 0 + 2(z-c)\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad x + (z-c)\cdot p = 0 \quad \text{—— (2)} \quad \text{where } p = \frac{\partial z}{\partial x}$$

Similarly Diff. eqn (1) partially w.r.to. y. we get

$$0 + 2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad y + (z-c)\, q = 0 \quad \text{—— (3)}$$

To eliminate c,

from eqn (2) $(z-c) = -x/p$

$\therefore$ (3) $\Rightarrow$ $y + \left(-\frac{x}{p}\right)\cdot q = 0$

$\Rightarrow \quad yp - xq = 0$

$$\Rightarrow \quad \boxed{y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0}$$

is the reqd. P.D.E.

From (3)

$(z-c) = \frac{-y}{q}$

$\therefore$ (2) $\Rightarrow$ $x + \frac{-y}{q}p = 0$

$\Rightarrow \quad qx - yp = 0$

$\therefore \quad x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$

$\Rightarrow \quad y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

## Solution of Partial Diff. eqn :

### I) By Method of Direct Integration :

**Ex:** Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

**Soln:** Given P.D.E is

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right) = \cos(2x+3y)$$

integrating w.r.to $x$ (partially i.e. keeping $y$ const). we get

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\sin(2x+3y)}{2} + \underbrace{f(y)}_{\text{const.}}$$

again integrating w.r.to $x$ partially,

$$\frac{\partial z}{\partial y} = \frac{-\cos(2x+3y)}{4} + \int f(y)\,dx + g(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{\cos(2x+3y)}{4} + f(y)\cdot x + g(y)$$

Now integrating w.r.to. $y$ partially we get

$$z = \frac{-\sin(2x+3y)}{12} + x\int f(y)\,dy + \int g(y)\,dy + h(x)$$

$$\therefore \boxed{z = \frac{-\sin(2x+3y)}{12} + x\,\phi_1(y) + \phi_2(y) + \psi(x)}$$

where $\phi_1(y)$, $\phi_2(y)$ and $\psi(x)$ are arbitrary function.

**Ex:** Solve

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = x^2 y$$

subject to the condⁿ $z(x,0) = x^2$, $z(1,y) = \cos y$

Soln: Given D.E is

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y \quad \text{—— (1)}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$$

integrating w.r.to. x we get

$$\frac{\partial z}{\partial y} = \frac{y x^3}{3} + f(y)$$

again integrating w.r.to y. we get

$$z = \frac{y^2 x^3}{6} + \int f(y)\,dy + g(x)$$

Here. f(y). & g(x) are arbitrary function

let $\int f(y)\,dy = \phi(y)$

∴ Soln is $\boxed{z = \frac{y^2 x^3}{6} + \phi(y) + g(x)}$ —— (2)

using initial condⁿ $z(x,0) = x^2$

$\Rightarrow z = x^2$ when $x = x$, & $y = 0$

$$\therefore (2) \Rightarrow x^2 = 0 + \phi(0) + g(x)$$

i.e. $g(x) = \phi(0) - x^2$

Hence eqⁿ (2) ⇒

$$z = \frac{y^2 x^3}{6} + \phi(y) + \phi(0) - x^2 \quad \text{—— (3)}$$

also given that $z(1,y) = \cos y$

ie $z = \cos y$ when $x=1$, $y=y$

$\therefore$ (3) $\Rightarrow$

$$\cos y = \frac{y^2}{6} + \phi(y) + \phi(0) - 1$$

$$\Rightarrow \phi(y) = \cos y - \frac{y^2}{6} - \phi(0) + 1$$

Hence eqn (2) $\Rightarrow$

$$z = \frac{y^2x^3}{6} + \cos y - \frac{y^2}{6} + 1 - \cancel{\phi(0)} + \cancel{\phi(0)} - x^2$$

Thus reqd soln is

$$\boxed{z = \frac{y^2x^3}{6} + \cos y - \frac{y^2}{6} - x^2 + 1}$$

## Linear partial Diff. Equation of first order

The general form of Linear partial D.E. of first order is

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

OR $\boxed{Pp + Qq = R}$ ——— (1)

This is called as Lagrange's linear P.D.E.

and the soln of P.D.E (1) is given by

$$f(u,v) = 0$$

where $u = u(x,y,z)$, & $v = v(x,y,z)$ are specific function of $x, y$ & $z$.

### ☆ Method of obtaining General soln :⇒

step (1) Rewrite the given P.D.E in standard form

$$Pp + Qq = R$$

Step (II): ~~Form a~~ formed Lagrange's Auxiliary eqn (A.E)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{——— (2)}$$

Step (III): Nature of soln to the simultaneous equation of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —

$u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are said to be the complete soln (provided $u$ & $v$ are linearly independent ie $u/v \neq$ const.)

Case (I): one of the variables i.e either absent or cancells out from the set of auxiliary equation.

i.e. grouping any two ratio find the ~~first~~ first sol^n's $u(x,y,z) = c_1$ or $v(x,y,z) = c_2$

Case (II): If $u(x,y,z) = c_1$ is known, but $v(x,y,z) = c_2$ is not possible by case (I), then use $\overset{\text{First sol}^n}{u = c_1}$ to find $v = c_2$.

Case (III). Introduced lagranges multiplyers $l, m, n$ which are functions of $x, y, \& z$ or constents. so that each fraction in A.E (2) is equal to

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \quad \text{--- (3)}$$

choose $l, m, n$. in such a way that $lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$ which can be integrated

Case (IV). multipliers $l, m, n$ may be chosen (more than once) such that numerator $l\,dx + m\,dy + n\,dz$ is an exact differential of the denomenator $lP + mQ + nR$. Now combine (3) with a fraction of (2) to get an integral.

Step (IV): General sol^n of eq^n (1) is

$$f(u, v) = 0$$

or $u = \phi(v)$

etc.

⑥

**Ex:** Solve $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$ or $x^2y\frac{\partial z}{\partial x} + y^2x\frac{\partial z}{\partial y} = 3xyz$

**Soln:** Given D.E. is

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z \quad \text{—— (1)}$$

This is of the form

$$Pp + Qq = R$$

where $P = x,\ Q = y,\ R = 3z$ & $p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$

Therefore A.E $\Rightarrow$ $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z} \quad \text{—— (1)}$$

solving First and 2nd Fraction.

$$\frac{dx}{x} = \frac{dy}{y}$$

on integrating we get

$$\log x = \log y + c_1$$

$$\Rightarrow \log(x/y) = c_1$$

$$\Rightarrow x/y = e^{c_1} = A$$

$\therefore$ First soln is $\boxed{u = x/y = A}$ —— (2)

Also solving 2nd and 3rd ratio.

$$\frac{dy}{y} = \frac{dz}{3z}$$

on integrating $\log y = \frac{1}{3}\log(z) + c_2$

$$\Rightarrow \log\left(\frac{y}{z^{1/3}}\right) = c_2$$

$$\text{ie} \left(\frac{y}{z^{1/3}}\right) = e^{c_2} = B$$

$$\Rightarrow \frac{y}{z^{1/3}} = B$$

Hence second solⁿ is $V = \frac{y}{z^{1/3}} = B$

Hence complete solⁿ is

$$f(u, v) = 0$$

$$\Rightarrow \boxed{f\left(x/y,\ y/z^{1/3}\right) = 0}$$

or $u = \phi(v)$

ie $\boxed{x/y = \phi\left(y/z^{1/3}\right)}$

---

Ex: solve $z(z^2 + xy)(px - qy) = x^4$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Solⁿ: given D.E. is

$$z(z^2 + xy)(px - qy) = x^4$$

$$\Rightarrow \cancel{(x\ z^3 + xyz)p - z^3}$$

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4 \quad \text{--- (1)}$$

This is of the form

$$Pp + Qq = R$$

where $P = xz(z^2 + xy)$, $Q = -yz(z^2 + xy)$, $R = x^4$

∴ The auxiliary eqⁿ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4} \quad \text{--- (2)}$$

Solving 1st and 2nd fraction.

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{-y}$$

or $$\frac{dx}{x} + \frac{dy}{y} = 0$$

on integrating we get

$$\log x + \log y = \log c_1$$

$$\Rightarrow \log(x.y) = \log c_1$$

$$\Rightarrow \boxed{xy = c_1}$$

Hence first soln is $u(x,y,z) = xy = c_1$ — ③

Now from ① 1st and 3rd fraction

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{x^4}{x} dx = z(z^2+xy)\,dz$$

$$\Rightarrow x^3\,dx = (z^3+xyz)\,dz$$

puting $xy = c_1$ we get

$$x^3\,dx = (z^3 + c_1 z)\,dz$$

on integrating we get

$$\frac{x^4}{4} = \frac{z^4}{4} + c_1\frac{z^2}{2} + c_2$$

$$\Rightarrow x^4 = z^4 + 2z^2 c_1 + c_2$$

$$\Rightarrow x^4 - z^4 + 2z^2 c_1 = c_2$$

$\Rightarrow \quad x^4 - z^4 - 2zq = c_2$

$\Rightarrow \quad x^4 - z^4 - 2z\,xy = c_2$

Hence $\quad v(x,y,z) = x^4 - z^4 - 2zxy = c_2$

Hence complete solⁿ is

$$f(u,v) = 0$$

i.e. $\boxed{f(xy,\ x^4 - z^4 - 2xyz^2) = 0}$

---

Ex: Solve $\quad xq = yp + x\,e^{(x^2+y^2)}$

Solⁿ: Given P.D.E is

$$yp - xq = -x\,e^{(x^2+y^2)} \quad \text{--- (1)}$$

where $\quad p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$

This is of the form $\boxed{Pp + Qq = R}$

where $\quad P = y,\ Q = -x,\ R = -x\,e^{(x^2+y^2)}$

∴ The auxiliary eqⁿ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{-x\,e^{(x^2+y^2)}} \quad \text{--- (2)}$$

Solving 1st and 2nd fraction of eqⁿ (2).

i.e. $\quad \frac{dx}{y} = \frac{dy}{-x}$

$$\Rightarrow \quad -x\,dx = y\,dy$$

$$\Rightarrow \quad x\,dx + y\,dy = 0$$

on integrating we get

$$x^2 + y^2 = c_1$$

$\therefore$ Ist solⁿ is

$$u(x,y,z) = x^2+y^2 = c_1 \quad \text{--- (3)}$$

Now from 2nd and 3rd ratio

$$\frac{dy}{-x} = \frac{dz}{-x \cdot e^{(x^2+y^2)}}$$

$$\Rightarrow \quad \frac{dy}{1} = \frac{dz}{e^{(x^2+y^2)}}$$

$$\Rightarrow \quad e^{(x^2+y^2)}\,dy = dz$$

$$\Rightarrow \quad e^{c_1}\,dy = dz \qquad (\because \text{from } (3))$$

on integrating we get

$$e^{c_1} \cdot y = z + c_2$$

$$\Rightarrow \quad y\,e^{c_1} - z = c_2$$

$$\Rightarrow \quad y\,e^{(x^2+y^2)} - z = c_2$$

$\therefore$ second solⁿ is $v(x,y,z) = y\,e^{(x^2+y^2)} - z = c_2$

Hence the complete solⁿ is

$$f(u,v) = 0 \quad \text{or} \quad u = \phi(v)$$

$$\Rightarrow \boxed{f\left(x^2+y^2,\; y\,e^{(x^2+y^2)} - z\right) = 0} \quad \text{or} \quad \boxed{x^2+y^2 = \phi\left(y\,e^{(x^2+y^2)} - z\right)}$$

Ex :⇒ Solve

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z - a(\sqrt{x^2+y^2+z^2})$$

Soln: Given. P.D.E is

$$xp + yq = z - a(\sqrt{x^2+y^2+z^2}) \quad \text{--- (1)}$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

This is of the form

$$\boxed{Pp + Qq = R}$$

Where

$$P = x, \quad Q = y, \quad R = z - a(\sqrt{x^2+y^2+z^2})$$

∴ The auxiliary eqn is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a(\sqrt{x^2+y^2+z^2})} \quad \text{--- (2)}$$

Solving Ist and 2nd ratio.

i.e. $\frac{dx}{x} = \frac{dy}{y}$

on integrating we get

$$\log x = \log y + \log c_1$$

$$\Rightarrow \log(x/y) = \log c_1$$

i.e. $x/y = c_1$ $\Rightarrow \boxed{x = y c_1}$

∴ Ist soln is $u(x,y,z) = x/y = c_1$ --- (3)

Now, from 2nd and 3rd ratio

$$\frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2+y^2+z^2}}$$

Put $x = yc_1$ we get (from ③)

$$\frac{dy}{y} = \frac{dz}{z - a\sqrt{y^2c_1^2+y^2+z^2}}$$

$$\Rightarrow \frac{dz}{dy} = \frac{z - a\sqrt{y^2(1+c_1^2)+z^2}}{y}$$

To being Homogeneous Diff. eqn. we put $z = v.y$

so that.

$$v + y\frac{dv}{dy} = \frac{dz}{dy} =$$

← put $z = vy$

$$\Rightarrow v + y\frac{dv}{dy} = \frac{vy - a\sqrt{y^2(1+c_1^2)+v^2y^2}}{y}$$

$$\Rightarrow \cancel{v} + y\frac{dv}{dy} = \cancel{v} - a\sqrt{(1+c_1^2)+v^2}$$

$$\Rightarrow y\frac{dv}{dy} = -a\sqrt{(1+c_1^2)+v^2}$$

by separation of variable., we get

$$\frac{dv}{\sqrt{(\sqrt{1+c_1^2})^2+v^2}} = -a\frac{dy}{y}$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log(x+\sqrt{a^2+x^2})$$

on integrating we get

$$\log\left(v + \sqrt{(1+c_1^2)+v^2}\right) = -a\log y + \log c_2$$

Scanned by CamScanner

$$\Rightarrow \log\left(v+\sqrt{(1+c_1^2)+v^2}\right)+\log(y^a) = c_2 \log c_2$$

$$\Rightarrow \log\left(y^a\left(\frac{v+\sqrt{1+c_1^2+v^2}}{\cancel{\ }}\right)\right) = \log c_2$$

$$\Rightarrow \cancel{\ }\ \phi\ \ \cancel{z/y + \sqrt{\ }}$$

$$y^a\left(\frac{v+\sqrt{(1+c_1^2)+v^2}}{\cancel{y^a}}\right) = c_2$$

$$\Rightarrow y^a\left(z/y+\sqrt{1+\frac{x^2}{y^2}+\frac{z^2}{y^2}}\right)=c_2$$

$$\Rightarrow y^a\left(\frac{z}{y}+\frac{1}{y}\sqrt{x^2+y^2+z^2}\right)=c_2$$

$$\Rightarrow y^{a-1}\left(\cancel{\ }z+\sqrt{x^2+y^2+z^2}\right)=c_2$$

$$\therefore \cancel{\phi}\quad v(x,y,z) = y^{a-1}\left(z\cancel{\ }+\sqrt{x^2+y^2+z^2}\right)=c_2$$

$$\text{or } v(x,y,z) = \frac{y^{1-a}}{z+\sqrt{x^2+y^2+z^2}} = c_2 \quad \text{(Takin Reci}$$

$\therefore$ The complete soln is

$$f(u,v)=0$$

$$\Rightarrow \boxed{f\left(x/y,\ \frac{y^{1-a}}{z+\sqrt{x^2+y^2+z^2}}\right)=0}$$

Ex: Solve.

$$x(z-2y^2)p = (z-qy)(z-y^2-2x^3)$$

Soln: Given D.E is

$$x(z-2y^2)p = (z-yq)(z-y^2-2x^3)$$

$$\Rightarrow x(z-2y^2)p = -y(z-y^2-2x^3)q + z\cdot(z-y^2-2x^3)$$

$$\Rightarrow [x(z-2y^2)]p + [y(z-y^2-2x^3)]q = z(z-y^2-2x^3) \quad \text{--- (1)}$$

This is of the form.

$$\boxed{Pp + Qq = R}$$

where $P = x(z-2y^2)$, $Q = y(z-y^2-2x^3)$

$R = z(z-y^2-2x^3)$

$\therefore$ The auxiliary eqn's are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \text{--- (2)}$$

Solving 2nd and 3rd ratio. we get

$$\frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

on integrating we get

$$\log y = \log z + \log c_1$$

$$\Rightarrow \quad y/z = c_1$$

$$\therefore \quad u(x, y, z) = y/z = c_1 \quad \text{—— (3)}$$

$$\Rightarrow \boxed{y = c_1 z}$$

Now, from Ist and 3rd ration

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)}$$

put $y = c_1 z$. , we get

$$\therefore \frac{dx}{x(z-2c_1^2z^2)} = \frac{dz}{z(z-c_1^2z^2-2x^3)}$$

$$\Rightarrow \frac{dx}{xz(1-2c_1^2z)} = \frac{dz}{z(z-c_1^2z^2-2x^3)}$$

$$\Rightarrow \frac{dx}{x(1-2c_1^2z)} = \frac{dz}{z-c_1^2z^2-2x^3}$$

$$\Rightarrow (z-c_1^2z^2-2x^3)dx = x(1-2c_1^2z)\,dz$$

$$\Rightarrow z\,dx - c_1^2z^2dx - 2x^3dx = x\,dz - 2c_1^2 zx\,dz$$

$$\Rightarrow (z\,dx - x\,dz) + (-c_1^2z^2dx + 2c_1^2zx\,dz) = +2x^3dx$$

Dividing both side by $x^2$ we get

$$\left(\frac{z\,dx - x\,dz}{x^2}\right) - c_1^2\left(\frac{z^2dx - 2zx\,dz}{x^2}\right) = 2x\,dx$$

$$\Rightarrow -\left[\frac{x\,dz - z\,dx}{x^2}\right] + c_1^2\left[\frac{2z\cdot x\,dz - z^2 dx}{x^2}\right] = 2x\,dx$$

$$\Rightarrow -d\left(\frac{z}{x}\right) + c_1^2\, d\left(\frac{z^2}{x}\right) = 2x\,dx$$

on integrating we get

$$-\left(z/x\right) + c_1^2\left(z^2/x\right) = x^2 + c_2$$

ie. $$-x^2 - \frac{z}{x} + \frac{z^2c_1^2}{x} = c_2$$

$$\Rightarrow -x^2 - \frac{z}{x} + \frac{y^2}{x} = c_2$$

ie. $$v(x,y,z) = x^2 + \frac{z}{x} - \frac{y^2}{x} = -c_2$$

nce The complete soln is

$$f(u,v) = 0$$

$$\Rightarrow \boxed{f\left(y/z,\ x^2 + z/x - y^2/x\right) = 0}$$

Ex: Solve.

$$p + 3q = 5z + \tan(y-3x) \quad \text{——— (1)}$$

Soln: Given D.E is of form

$$Pp + Qq = R$$

where $P = 1, \quad Q = 3, \quad R = 5z + \tan(y-3x)$

∴ The auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)} \quad \text{——— (2)}$$

Solving 1st and 2nd ratio:

$$\frac{dx}{1} = \frac{dy}{3}$$

on integrating $\Rightarrow 3x = y + c_1$

$$\Rightarrow 3x - y = c_1$$

ie $u(x,y,z) = 3x - y = c_1$ ——— (3)

or $\Rightarrow (y - 3x = c_1)$

Now from 1st and 3rd ratio

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)}$$

$$\Rightarrow dx = \frac{dz}{5z + \tan(c_1)}$$

on integrating $\Rightarrow$

$$x = \frac{\log(5z + \tan(c_1))}{5} + \log c_2$$

$\Rightarrow$ ~~or~~

$$\Rightarrow \quad x = \log\left[\left(5z + \tan c_1\right)^{1/5} \cdot c_2\right]$$

$$\Rightarrow \quad e^{x} = \left(5z + \tan c_1\right)^{1/5} \cdot c_2$$

$$\Rightarrow \quad \frac{e^{x}}{\left(5z + \tan c_1\right)^{1/5}} = c_2$$

$$\Rightarrow \quad \frac{e^{x}}{\left[5z + \tan(y-3x)\right]^{1/5}} = c_2$$

i.e. $\quad v(x,y,z) = e^{x} \cdot \left(5z + \tan(y-3x)\right)^{-1/5} = c_2$

Hence complete soln is

$$f(u, v) = 0$$

i.e. $\quad f\left(3x-y, \; e^{x}\left(5z + \tan(y-3x)\right)^{-1/5}\right) = 0.$

---

Ex: Solve $x(z^2-y^2)\frac{\partial z}{\partial x} + y(x^2-z^2)\frac{\partial z}{\partial y} = z(y^2-x^2)$

Soln: Given P.D.E. is

$$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2) \quad \text{--- (1)}$$

where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

This is of the form $Pp + Qq = R$

where $P = x(z^2-y^2)$, $Q = y(x^2-z^2)$, $R = z(y^2-x^2)$

$\therefore$ The auxiliary eqn are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} \quad \text{--- (2)}$$

$$\Rightarrow \frac{\frac{1}{x}dx}{(z^2-y^2)} = \frac{\frac{1}{y}dy}{(x^2-z^2)} = \frac{1/z\, dz}{(y^2-x^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(z^2-y^2)+(x^2-z^2)+(y^2-x^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore$ ie. $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

on integrating we get

$$\log x + \log y + \log z = \log c_1$$

$$\Rightarrow \log(x.y.z) = \log c_1$$

Scanned by CamScanner

$$\Rightarrow \boxed{xyz = c_1}$$

ie $\quad u(x,y,z) = x \cdot y \cdot z = c_1 \quad \text{———} (3)$

Now: Choose lagrange's multipliers $l = x,\ m = y,\ n = z$. we get

$$(2) \Rightarrow \frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} = \frac{l\,dx + m\,dy + N\,dz}{lP + mQ + nR}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(z^2-y^2) + y^2(x^2-z^2) + z^2(y^2-x^2)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

i.e. $\quad x\,dx + y\,dy + z\,dz = 0$

on integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

i.e. $\quad x^2 + y^2 + z^2 = 2c_2 = c$

$\therefore \quad v(x,y,z) = x^2 + y^2 + z^2 = c$

Thus the complete soln is

$$f(u, v) = 0$$

$$\Rightarrow f(xyz,\ x^2+y^2+z^2) = 0$$

Ex: Solve

$$(x^2-y^2-yz)\frac{\partial z}{\partial x}+(x^2-y^2-zx)\frac{\partial z}{\partial y}=z(x-y)$$

Soln: Given P.D.E. is

$$(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y) \quad \text{—(1)}$$

Which is of the type $Pp+Qq=R$

where

$$P=x^2-y^2-yz$$

$$Q=x^2-y^2-zx$$

$$R=z(x-y)=zx-zy$$

∴ The auxiliary Eqn's are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{zx-zy} \quad \text{—(2)}$$

Adding all the ratio (using multipliers 1, -1, -1) we get

$$\text{Each ratio}=\frac{dx-dy-dz}{(x^2-y^2-yz)-(x^2-y^2-zx)-(zx-zy)}$$

$$=\frac{dx-dy-dz}{0}$$

i.e. $dx-dy-dz=0$

on integrating we get

$$x-y-z=c_1$$

Thus $u(x,y,z)=x-y-z=c_1$

Now from 1st and 2nd ratio. ( $l = x, m = -y$ , )

( ~~use multiplier~~ $\cancel{l = x, m = y, n = 0}$ )

each ratio $= \dfrac{x\,dx - y\,dy}{x(x^2-y^2-yz) - y(x^2-y^2-zx)}$

$$= \frac{x\,dx - y\,dy}{x^3 - xy^2 - \cancel{xyz} - yx^2 + y^3 + \cancel{xyz}}$$

$$= \frac{x\,dx - y\,dy}{x(x^2-y^2) - y(x^2-y^2)}$$

from (2).

$$= \frac{x\,dx - y\,dy}{(x^2-y^2)\cdot(x-y)} = \frac{dz}{z(x-y)}$$

$$\Rightarrow \frac{x\,dx - y\,dy}{(x^2-y^2)} = \frac{dz}{z}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{(2x\,dx - 2y\,dy)}{(x^2-y^2)} = \frac{dz}{z}$$

$d \log(x^2-y^2)$

$= \left(\frac{1}{x^2+y^2}\right)\cdot d(x^2-y^2)$

$= \frac{2x\,dx - 2y\,dy}{x^2+y^2}$

on integrating we get

$$\frac{1}{2}\log(x^2-y^2) = \log z + \log c_2$$

$$\Rightarrow \log\left(\sqrt{x^2-y^2}\,/\,z\right) = \log c_2$$

$$\Rightarrow \frac{1}{z}\sqrt{x^2-y^2} = c_2$$

$$\therefore\ v(x,y,z) = \frac{1}{z}\sqrt{x^2-y^2} = c_2$$

Hence complete soln is

$$f(u,v) = 0$$

$$\Rightarrow \boxed{f\left(x-y-z,\ \frac{1}{z}\sqrt{x^2+y^2}\right) = 0}$$

## ~~Non~~ Homogeneous linear P.D.E. with const coefficient

The P.D.E. of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y} + \cdots$$

$$\cdots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{—(1)}$$

where $a_0, a_1, \ldots a_n$ are consts is called as nth order ~~non~~ Homogeneous P.D.E. with const. coefficient.

( Eqn (1) is called as Homogeneous as all the differential coefficients ~~in~~ are of same same order ).

~~To~~ we write $\frac{\partial}{\partial x} \equiv D$ , $\frac{\partial}{\partial y} = D'$

$\therefore$ (1) $\Rightarrow$ $[a_0 D^n + a_1 D^{n-1} \cdot D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n] z = f(x,y)$

or $F(D, D') z = f(x,y)$

Where $F(D,D')$ is the polynomial in D and D'

Then general soln of P.D.E (1) is given by

$$\boxed{z = C.F + P.I}$$

where C.F $\Rightarrow$ complementary factor

P.I $\Rightarrow$ perticular Integral.

## Rules to find C.F

let $f(D, D')z = \phi(x,y)$ be the P.D.E.

Then Auxiliary eqn is obtained by replacing $\underline{D \text{ by } m}$ and $\underline{D' \text{ by } 1}$ and equating ~~this~~ L.H.S. to zero

i.e. $f(D\,D') = f(m, 1) = 0$

Solving this eqn we get root of auxiliary eqn. a and C.F is depend on nature of the roots.

**Case (I):** If Root are real and distinct.

for eg: let $m_1 \neq m_2 \neq m_3$ are real roots then

$$CF = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \phi_3(y+m_3x).$$

**Case (II).** If some roots are real and equal

for eg: $m_1 = m_2 = m_3$, and $m_4$, $m_5$ are distinct

$$C.F = \left[\phi_1(y+m_1x) + x\,\phi_2(y+m_1x) + x^2\phi_3(y+m_1x)\right] + \phi_4(y+m_4x) + \phi_5(y+m_5x)$$

**Case (IV)** If two roots are imaginary of the type $\alpha \pm i\beta$

ie $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

then $C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x)$

Scanned by CamScanner

Rules To find P.I.

Consider a P.D.E.

$$F(D,D')z = f(x,y)$$

then

$$\boxed{P.I = \frac{1}{F(D,D')} \cdot f(x,y)}$$

Thus P.I is depend on nature of function $f(x,y)$.

**Type ①**: If $f(x,y) = e^{ax+by}$

Then $P.I = \frac{1}{F(D,D')} e^{ax+by}$

Replace D by a and D' by b

$\therefore$ $P.I = \frac{1}{F(a,b)} \cdot e^{ax+by}$, provided $F(a,b) \neq 0$

If $F(a,b) = 0$ the above case is fail.

In such situation,

$$P.I = x \cdot \left[\frac{1}{\frac{\partial}{\partial D} F(D,D')} \cdot e^{ax+by}\right]$$

again Replace D by a and D' by b

$\therefore$ $P.I = x \left[\frac{1}{F'(a,b)} e^{(ax+by)}\right]$, provided $F'(a,b) \neq 0$

and so on.

Ex :⇒ Solve

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

Soln : Given P.D.E is

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}.$$

$$\Rightarrow (D^2 - 4DD' + 4D'^2)z = e^{2x+y} \quad \text{——— (1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

To find C.F : Replace D by m and D' by 1

Auxiliary eqn is

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)(m-2) = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore C.F = \phi_1(y+2x) + x\,\phi_2(y+2x) \quad \text{——— (2)}$$

To find P.I :

$$\text{Let } P.I = \frac{1}{F(D, D')} f(x, y)$$

$$= \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+y} \quad \text{(Type (1)}$$

Replace D by 2 and D' by 1

$$\therefore PI = \frac{1}{4 - 4(2)(1) + 4(1)} e^{2x+1}$$

$$= \frac{1}{0} e^{2x+1}$$

Case is fails.

$\therefore P.I = x \cdot \left[ \frac{1}{\frac{\partial}{\partial D}(D^2 - 4DD' + 4D'^2)} \cdot e^{2x+y} \right]$

$= x \left[ \frac{1}{2D - 4D' + 0} e^{2x+y} \right]$

Replace D by 2 and D' by 1.

$\therefore P.I = x \left[ \frac{1}{2(2) - 4(1)} e^{2x+y} \right]$

$= x \left[ \frac{1}{0} e^{2x+y} \right]$ , case is fail

$\therefore P.I = x^2 \left[ \frac{1}{\frac{\partial}{\partial D}(2D - 4D')} e^{2x+y} \right]$

$\boxed{P.I = x^2 \left[ \frac{1}{2} e^{(2x+y)} \right]}$

$\therefore$ Hence general soln is

$z = C.F + P.I$

$\boxed{z = \phi_1(y+2x) + x\phi_2(y+2x) + \frac{x^2}{2} e^{(2x+y)}}$

**Ex:** Solve

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

**Soln:** Given P.D.E is

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

$$\Rightarrow (D^3 - 3D^2D' + 4D'^3) z = e^{x+2y} \quad \text{--- (1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

**To find C.F:** Replace $D$ by $m$ and $D'$ by $1$

$\therefore$ A.E $\Rightarrow m^3 - 3m^2 + 4 = 0$

$\Rightarrow m = -1, 2, 2$

$$\therefore \text{C.F} = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) \quad \text{--- (2)}$$

**To find P.I.**

Let $\text{P.I} = \frac{1}{f(D,D')} f(x,y)$

$$= \frac{1}{D^3 - 3D^2D' + 4D'^3} \cdot e^{x+2y}$$

Replace $D$ by $1$ and $D'$ by $2$

$$\therefore \text{P.I} = \frac{1}{1^3 - 3(1^2)(2) + 4(2^3)} e^{x+2y} = \frac{1}{27} e^{x+2y}$$

$\therefore$ The general soln is

$z = \text{C.F} + \text{P.I}$

$$\boxed{z = \phi(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{1}{27} e^{x+2y}}$$

Type (II): If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

Then

$$P.I = \frac{1}{F(D,D')} \cdot \sin(ax+by) \text{ or } \cos(ax+by)$$

~~Replace~~ $= \dfrac{1}{F(D^2, DD', D'^2 \ldots)} \sin(ax+by)$

replace $D$ by $-(a^2)$

$DD'$ by $-(a \cdot b)$ & $D'^2$ by $-(b^2)$

$$\therefore P.I = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$$

provided $F(-a^2, -ab, -b^2) \neq 0$

If $F(-a^2, -ab, -b^2) = 0$ then case is fail

In this condn.

$$P.I = x \cdot \left[ \frac{1}{\frac{\partial}{\partial D} F(D^2, DD', D'^2)} \sin(ax+by) \right]$$

and repeat the same procedure as explain ~~as~~ earlier.

Ex: Solve

$$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$$

Soln: Given D.E is

$$(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x+2y) \quad \text{--- (1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

To find C.F.:

Replace D by m and D' by 1

$\therefore$ Auxiliary eqns. is

$$m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$\Rightarrow m = 0, 2, 2$$

$$\therefore C.F \Rightarrow = \phi_1(y - 0x) + \phi_2(y+2x) + x\,\phi_3(y+2x)$$

$$\boxed{C.F = \phi_1(y) + \phi_2(y+2x) + x\,\phi_3(y+2x)}$$

To find P.I:

$$P.I = \frac{1}{D^3 - 4D^2D' + 4DD'^2}\, 2\sin(3x+2y)$$

$$= 2\left[\frac{1}{D[D^2 - 4DD' + 4D'^2]}\sin(3x+2y)\right]$$

Replace $D^2$ by $-(3^2) = -9$

$DD'$ by $-(3)(2) = -6$

$D'^2$ by $-(2^2) = -4$

Scanned by CamScanner

$$\therefore P.I = 2 \cdot \left[ \frac{1}{D\,[(-9)-4(-6)+4(-4)]} \sin(3x+2y) \right]$$

$$= 2 \frac{1}{D\,(-1)} \sin(3x+2y)$$

$$= -2 \int \sin(3x+2y)\, dx$$

$$= (-2) \frac{-\cos(3x+2y)}{3} = \frac{2}{3}\cos(3x+2y)$$

$\therefore$ General soln is

$$z = C.F + P.I$$

$$\Rightarrow \boxed{z = \phi_1(y) + \phi_2(y+2x) + x\,\phi_3(y+2x) + \frac{2}{3}\cos(3x+2y)}$$

---

Ex: Solve $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

soln.

Given D.E is

$$(D^2 + DD' - 6D'^2)z = \cos(2x+y) \quad \text{--- (1)}$$

To find C.F:

Replace D by m and D' by 1

$\therefore$ Auxiliary eqn is

$$m^2 + m - 6 = 0$$

$$\Rightarrow (m+3)(m-2) = 0$$

$$\Rightarrow m = -3, 2$$

$$\therefore \boxed{C.F = \phi(y-3x) + \phi_2(y+2x)}$$

To find P.I.

let $P.I = \dfrac{1}{D^2+DD'-6D'^2} \cdot \cos(2x+y)$

Replace $D^2$ by $-(2^2) = -4$

$DD'$ by $-(2)(1) = -2$

$D'^2$ by $-(1^2) = -1$

$$\therefore P.I. = \frac{1}{-4-2-6(-1)} \cos(2x+y) = \frac{1}{0} \cos(2x+y)$$

case is fail

$$\therefore P.I = x \cdot \left[\frac{1}{\frac{\partial}{\partial D}(D^2+DD'-6D'^2)} \cos(2x+y)\right]$$

$$= x \left[\frac{1}{2D+D'} \cos(2x+y)\right]$$

$$= x \left[\frac{D}{D(2D+D')} \cos(2x+y)\right]$$

$$= x \left[\frac{D}{2D^2+D'D} \cos(2x+y)\right]$$

Replace $D^2$ by $-(2^2) = -4$

$D'D$ by $-(2)(1) = -2$

$$\therefore P.I = x \left[\frac{D}{2(-4)-2} \cos(2x+y)\right]$$

$$\therefore P.I = x \cdot \left[ \frac{D}{-10} \cos(2x+y) \right]$$

$$= -\frac{x}{10} \left[ \frac{\partial}{\partial x} \cos(2x+y) \right]$$

$$= \frac{-x}{10} \left[ 2\sin(2x+y) \right] = \frac{-x}{5} \sin(2x+y)$$

$$\therefore \boxed{P.I = -\frac{x}{5} \sin(2x+y).}$$

Hence general soln is :

$$z = C.F + P.I$$

$$\boxed{z = \phi_1(y-3x) + \phi_2(y+2x) - \frac{x}{5}\sin(2x+y)}$$

HW

H.W Solve the following P.D.E.

① $(D^2 - 3DD' + 2D'^2) z = e^{2x+3y} + \sin(x-2y)$

② $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y$

Hint: $\sin x \cdot \cos 2y = \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$

③ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = \cos x \cdot \cos 2y$

④ $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} + 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y)$

Scanned by CamScanner

## Type (III):

If $f(x,y) = x^m y^n$

where m and n are positive integers

Then

$$P.I. = \frac{1}{F(DD')} x^m y^n$$

$$= [F(DD')]^{-1} (x^m y^n)$$

Expand $[F(D,D')]^{-1}$ by using binomial expansion and then operate on algebraic function / polynomial $x^m y^n$. and solve

---

**Note.** (For $\phi(x,y) = x^m y^n$)

(i) if $m<n$ then expand $[F(D,D')]^{-1}$ in ascending power of $D/D'$

(ii) If $m>n$ then expand $[F(D,D')]^{-1}$ in ascending power of $\frac{D'}{D}$.

---

Scanned by CamScanner

Ex: Solve

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$$

Soln: Given D.E. is

$$(D^3 - 2D^2 D') z = 2e^{2x} + 3x^2 y \quad \text{--- } (1)$$

where

$$D \equiv \frac{\partial}{\partial x}, \quad D \equiv \frac{\partial}{\partial y}$$

To find C.F: Replace D by m and D' by 1

$\therefore$ A.E $\Rightarrow$

$$m^3 - 2m^2 = 0$$

$$\Rightarrow m^2(m^2 - 2) = 0$$

$$\Rightarrow m = 0, 0, 2$$

$$\therefore \boxed{C.F = \phi_1(y) + x\,\phi_2(y) + \phi_3(y+2x)}$$

To find P.I:

$$P.I = \frac{1}{D^3 - 2D^2 D'}\left(2e^{2x} + 3x^2 y\right)$$

$$= 2\left[\frac{1}{D^3 - 2D^2 D'} e^{2x+0y}\right] + 3\left[\frac{1}{D^3 - 2D^2 D'} x^2 y\right]$$

$$P.I = 2\,PI_1 + 3\,PI_2$$

Here

$$PI_1 = \frac{1}{D^3 - 2D^2 D'} e^{2x+0}$$

Replace D by 2, D' by 0

$$\therefore P.I_1 = \frac{1}{2^3 - 0} e^{2x} = \frac{1}{8} e^{2x}$$

Now.

$$PI_2 = \frac{1}{D^3 - 2D^2D'} x^2y$$

Here $m > n$

$$= \frac{1}{D^3\left[1 - 2\frac{D'}{D}\right]} x^2y$$

$$= \frac{1}{D^3}\left[1 - 2\frac{D'}{D}\right]^{-1} (x^2y)$$

$$= \frac{1}{D^3}\left[1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 + \cdots\right](x^2y)$$

$$= \frac{1}{D^3}\left[x^2y + \frac{2}{D}D'(x^2y) + \frac{4D'^2}{D^2}(x^2y) + \cdots\right]$$

$$= \frac{1}{D^3}\left[x^2y + \frac{2}{D}(x^2) + 0\right]$$

$$= \frac{1}{D^3}\left[x^2y + \frac{2}{3}x^3\right]$$

$$= \iiint \left(x^2y + \frac{2}{3}x^3\right) dx\, dx\, dx$$

$$= \iint \left[\frac{x^3y}{3} + \frac{2}{3}\frac{x^4}{4}\right] dx\, dx$$

$$= \int \left(\frac{x^4}{12}y + \frac{2}{60}x^5\right) dx$$

$$PI_2 = \frac{x^5}{60}y + \frac{2}{360}x^6 = \frac{1}{60}x^5\left[y + \frac{1}{3}x\right]$$

$\therefore P.I. = 2 P.I_1 + 3 \, P.I_2$

$$= 2\left(\frac{1}{8}e^{2x}\right) + 3\left[\frac{1}{60}x^5\left(y+\frac{1}{3}x\right)\right]$$

$$= \frac{1}{4}e^{2x} + \frac{1}{20}x^5\left(y+\frac{1}{3}x\right)$$

$$P.I. = \frac{1}{4}e^{2x} + \frac{1}{20}x^5 y + \frac{1}{60}x^6$$

$\therefore$ Complete soln is

$$z = C.F. + P.I.$$

$$\Rightarrow \boxed{z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) + \frac{1}{4}e^{2x} + \frac{1}{20}x^5 y + \frac{1}{60}x^6}$$

Ex: Solve
$$(D^2+3DD'+2D'^2)z = 24xy$$

Soln: Given D.E is
$$(D^2+3DD'+2D'^2)z = 24xy \quad \text{--- (1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

To find C.F:

Replace D by m and D' by 1

$\therefore$ Auxiliary eqn is $\Rightarrow$ $m^2+3m+2=0$

$\Rightarrow$ $m = -2, -1$

$$\therefore \boxed{C.F = \phi_1(y-2x) + \phi_2(y-x)}$$

To find P.I:

let $P.I = \frac{1}{D^2+3DD'+2D'^2} \cdot 24xy$

$$= 24\left[\frac{1}{D^2+3DD'+2D'^2} xy\right]$$

Type III
$m<n$

$$= 24\left[\frac{1}{2D'^2}\left(\frac{1}{1+\left(\frac{D^2+3DD'}{2D'^2}\right)}\right)\right](xy)$$

$$= \cancel{24}\ 12 \frac{1}{D'^2}\left[1+\left(\frac{3}{2}\frac{D}{D'}+\frac{1}{2}\left(\frac{D}{D'}\right)^2\right)\right]^{-1} (x.y)$$

$$= 12 \cdot \frac{1}{D'^2}\left[1 - \frac{3}{2}\frac{D}{D'} - \frac{1}{2}\frac{D^2}{D'^2} + \cdots\right](x.y)$$

Scanned by CamScanner

$$= 12 \frac{1}{D'^2}\left[(x \cdot y) - \frac{3}{2}\frac{D}{D'}(xy) + 0\right]$$

$$= 12 \frac{1}{D'^2}\left[xy - \frac{3}{2}\frac{1}{D'}(y)\right] \qquad \because D \equiv \frac{\partial}{\partial x}$$

$$= 12 \frac{1}{D'^2}\left[xy - \frac{3}{2}\left(\frac{y^2}{2}\right)\right] \qquad \because \frac{1}{D'}y = \int y \, dy$$

$$= 12 \iint \left[(x \cdot y) - \frac{3y^2}{4}\right] dy \, dy$$

$$= 12 \int \left[\frac{xy^2}{2} - \frac{3y^3}{4 \times 3}\right] dy$$

$$P.I. = 12\left[\frac{xy^3}{6} - \frac{y^4}{16}\right]$$

$$PI = 2xy^3 - \frac{3}{4}y^4$$

$\therefore$ Complete soln is

$$z = C.F. + P.I$$

$$\therefore \boxed{z = \phi_1(y-2x) + \phi_2(y-x) + 2xy^3 - \frac{3}{4}y^4}$$

## Type IV :

If $f(x,y) = \phi(v) = \phi(ax+by)$

where $\phi(v) =$ any function of $v = ax+by$

for eg: $(ax+by)^m$, $\log(ax+by)$, $\tan(ax+by)$ etc.

Then.

$$P.I = \frac{1}{F(D,D')} \phi(ax+by)$$

Put $v = ax+by$. and Replace D by a & D' by b and then integrate $\phi(v)$ w.r.to. $v$, m times (If m is order of P.D.E.)

ie.
$$\boxed{P.I = \frac{1}{F(a,b)} \underbrace{\int\int \cdots \int}_{m \cdot time} \phi(v)\,(dv)^m}$$

provided $F(a,b) \neq 0$

If $F(a,b) = 0$ the case is fail.

In such situation.

$$\boxed{P.I = \frac{x^m}{b^m \cdot m!} \cdot \phi(v)}$$

---

Note let $F(D,D') = (bD-aD')^m \cdot G(DD')$ with order p

Then $PI = \frac{1}{F(DD')} \phi(ax+by) = \frac{1}{(bD-aD')^m \cdot G(DD')} \cdot \phi(ax+by)$

If we replace D by a and D' by b ~~then~~ and put $v = ax+by$

Then $PI = \frac{1}{(ba-ab)^m \cdot G(a,b)} \underbrace{\int\int \cdots \int}_{p \cdot time} \phi(v)\, dv^p$

$= \frac{1}{0}$ case is fail.

Then

$PI = \frac{1}{(bD-aD')^m} \cdot \phi(ax+by) \underset{i.e}{\Rightarrow} PI = \frac{x^m}{b^m \cdot m!} \phi(ax+by)$

Ex Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = \sqrt{x+3y}$

Soln: Given P.D.E is

$$(D^2 - 4DD' + 3D'^2)z = \sqrt{x+3y} \quad \text{--- (1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

To find C.F Replace D by m and D' by 1

$\therefore$ A.E $\Rightarrow$ $m^2 - 4m + 3 = 0$

$\Rightarrow m = 1, 3$

$$\Rightarrow \boxed{C.F = \phi_1(y+x) + \phi_2(y+3x)}$$

To find P.I

$$P.I = \frac{1}{D^2 - 4DD' + 3D'^2}\cdot\left(\sqrt{x+3y}\right)$$ Imp

let us put $v = x+3y$ and Hence replace D by 1 and D' by 3 and then integrate $\phi(v) = \sqrt{v}$ w.r.to. v twice times as order of P.D.E is 2

$$\therefore P.I = \frac{1}{1^2 - 4(1)(3) + 3(3)^2}\iint(\sqrt{v})\cdot dv\, dv$$

$$= \frac{1}{16} \iint v^{1/2} \, dv \, dv$$

$$= \frac{1}{16} \int \left[ \frac{v^{3/2}}{3/2} \right] dv$$

$$= \frac{1}{16}\left(\frac{2}{3}\right) \cdot \left[ \frac{v^{5/2}}{5/2} \right]$$

$$= \frac{4}{16 \times 3 \times 5} (v)^{5/2}$$

$$\boxed{P.I = \frac{1}{60} (x+3y)^{5/2}} \qquad \because v = x+3y$$

$\therefore$ general soln is

$$z = C.F. + P.I$$

$$\Rightarrow z = \phi_1 (y+x) + \phi_2 (y+3x) + \frac{1}{60} (x+3y)^{5/2}$$

Ex :⇒ Solve, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 8 \log(x+5y)$

Soln: let given P.D.E is

$$(D^2 + DD' - 2D'^2) z = 8 \log(x+5y) \quad \text{—(1)}$$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

To find CF:

Replace D by m and D' by 1

∴ A.E ⇒ $m^2 + m - 2 = 0$

⇒ $m = -2, 1$

∴ $\boxed{C.F = \phi_1(y-2x) + \phi_2(y+x)}$

To find PI:

$$P.I. = \frac{1}{D^2 + DD' - 2D'^2} \cdot 8 \log(x+5y)$$

let $v = x+5y$, Replace D by 1 & D' by 5 and integrating log(v) two time as order of P.D.E is 2 (i.e. Type-(IV))

$$\therefore P.I = 8 \left[\frac{1}{1^2 + 1(5) - 2(5)^2} \cdot \iint \log(v)\, dv.dv\right]$$

$$= 8 \left[\frac{1}{-44} \iint \log v \, dv \, dv\right]$$

$$= \frac{2}{-11} \cdot \int (v \log v - v)\, dv$$

(∵ Integrating by parts.)

$$= \frac{-2}{11} \left[\frac{v^2}{2} \log v - \frac{v^2}{4} - \frac{v^2}{2}\right] dv$$

$\therefore$ P.I $= \frac{-2}{11} \frac{v^2}{4}\left[2\log v - 3\cancel{\ }\right]$

$= \frac{-v^2}{22} \cdot \left[2\log v - 3\cancel{\ }\right]$

$= -\frac{(x+5y)^2}{22}\left[2\log(x+5y) - 3\right]\cancel{(\ )}^2]$

$\therefore$ The general soln is

$z = C.F + P.I$

$\Rightarrow \boxed{z = \phi_1(y-2x) + \phi_2(y+x) - \frac{(x+5y)^2}{22}\left[2\log(x+5y) - 3\right]}$

---

**Ex:** Solve

$$\left(D^2 - 2DD' + D'^2\right)z = \tan(x+y)$$

where. $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

**Soln** :⇒ given P.D.E is

$$\left(D^2 - 2DD' + D'^2\right)z = \tan(x+y)$$

$\therefore$ To find CF :

Replace D by m and D' by 1

$\therefore$ A.E ⇒ $m^2 - 2m + 1 = 0$

⇒ $(m-1)(m-1) = 0$

$\therefore m = 1, 1$

$\therefore \boxed{C.F = \phi_1(y+x) + x\,\phi_2(y+x)}$

Scanned by CamScanner

To find P.I : $\Rightarrow$ $P.I = \frac{1}{(D-D')^2} \tan(x+y)$

$$P.I = \frac{1}{D^2 - 2DD' + D'^2} \cdot \tan(x+y)$$

[ Put $(x+y) = v$ and replace D by 1 & D' by 1 and then integrate $\phi(v) = \tan v$ w.r.to v two times as order of P.D.E. is 2 ]

$$\therefore P.I = \frac{1}{1^2 - 2(1)(1) + 1^2} \iint \tan v \, dv$$

$$= \frac{1}{0} \iint \tan v \, dv$$

Case is fail

$$\therefore P.I = \frac{x^2}{(1)^2 \cdot 2!} \tan(x+y)$$

$$\boxed{P.I = \frac{x^2}{2!} \tan(x+y)}$$

OR

$$P.I = x \cdot \frac{1}{2D - 2D'} \tan(x+y)$$

put $D = 1$ & $D' = 1$

case is fail again

$$\therefore P.I = x^2 \cdot \frac{1}{2} \tan(x+y)$$

Hence complete soln is

$$z = C.F + P.I$$

$$\boxed{z = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^2}{2!} \tan(x+y)}$$

**Type (V)**. If $f(x,y) = e^{ax+by} \cdot \phi(x,y)$

where. $\phi(x,y)$ is any function of $x$ & $y$.

Then $P.I = \frac{1}{F(D D')} \left( e^{ax+by} \cdot \phi(x,y) \right)$

Replace $D$ by $(D+a)$ and $D'$ by $(D'+b)$

$\Rightarrow P.I = e^{ax+by} \left[ \frac{1}{F(D+a, D'+b)} \phi(x,y) \right]$

$P.I = e^{ax+by} \left[ \frac{1}{G(D, D')} \phi(x,y) \right]$

Now. Solve integral $\left[ \frac{1}{G(D D')} \phi(x,y) \right]$ as per cases explain earlier.

---

Ex: Solve $(D^2 - 3DD' + 2D'^2) z = e^{2x} \cdot \sin(x+3y)$

Soln: Given P.D.E

$(D^2 - 3DD' + 2D'^2) z = e^{2x} \cdot \sin(x+3y)$ ← (1)

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

∴ To find C.F.. Replace D by m & D' by 1

∴ A.E ⇒ $m^2 - 3m + 2 = 0$

⇒ $m = 1, 2$

$$\boxed{C.F = \phi_1(y+x) + \phi_2(y+2x)}$$

To find P.I :

let $P.I = \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{2x} \sin(x+3y)$

Type (IV)

Replace $D$ by $D+2$ & $D'$ by $D'+0$

$$\therefore P.I = e^{2x}\left[\frac{1}{(D+2)^2 - 3(D+2)D' + 2D'^2} \sin(x+3y)\right]$$

Replace $D^2$ by $= -(1^2) = -1$   $DD'$ by $= -(1)(3) = -3$

& $D'^2$ by $-(3^2) = -9$

$D^2 + 4D + 4 - 3DD' - 6D' + 2D'^2$

$$= e^{2x}\left[\frac{1}{4D - 6D' - 6} \cdot \sin(x+3y)\right]$$

$$= e^{2x}\left[\frac{D}{D(4D - 6D' - 6)} \sin(x+3y)\right]$$

$$= e^{2x}\left[\frac{D}{4D^2 - 6DD' - 6D} \sin(x+3y)\right]$$

Replace $D^2$ by $-(1^2) = -1$

$DD'$ by $-(1)(3) = -3$

Type (III)

$$\therefore P.I = e^{2x}\left[\frac{D}{4(-1) - 6(-3) - 6D} \sin(x+3y)\right]$$

$$= e^{2x}\left[\frac{1}{14 - 6D} \cos(x+3y)\right]$$

$$= \frac{e^{2x}}{2}\left[\frac{1}{7 - 3D} \cos(x+3y)\right]$$

$$= \frac{e^{2x}}{2}\left[\frac{1}{7-3D} \times \frac{7+3D}{7+3D} \cos(x+3y)\right]$$

$$= \frac{e^{2x}}{2}\left[\frac{7+3D}{49-9D^2} \cos(x+3y)\right]$$

replace $D^2$ by $-(1^2) = -1$

$$\therefore P.I = \frac{e^{2x}}{2}\left[\frac{7+3D}{49-9(-1)} \cos(x+3y)\right]$$

$$= \frac{e^{2x}}{2}\left[\frac{1}{58} (7+3D) \cos(x+3y)\right]$$

$$= \frac{e^{2x}}{116}\left[7 \cos(x+3y) + 3 \sin(x+3y)\right]$$

$\therefore$ General soln is

$$z = C.F + P.I$$

$$z = \phi_1(y+x) + \phi_2(y+2x) + \frac{e^{2x}}{116}\left[7 \cos(x+3y) - 3 \sin(x+3y)\right]$$

Ex: Solve $$\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} = e^{x+2y}\cdot(x+4y)$$

Soln, Given P.D.E is

$$(D^3 - D^2D' - 2DD'^2)z = e^{x+2y}\cdot(x+4y) \quad \text{—— (1)}$$

To find C.F :

Replace D by m and D' by 1

$\therefore$ A.E is $m^3 - m^2 - 2m = 0$

$\Rightarrow m = 0, \ (m^2 - m - 2) = 0$

$\Rightarrow m = 0, \ 2, -1$

$-6D'$ $10'$

$$\therefore \boxed{C.F = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x)}$$

To find. P.I :

let $P.I = \dfrac{1}{D^3 - D^2D' - 2DD'^2}\, e^{x+2y}\cdot(x+4y)$

Replace D by D+1 & D' by D'+2 Type (IV)

$$\therefore P.I = e^{x+2y}\left[\frac{1}{(D+1)^3 - (D+1)^2(D'+2) + 2(D+1)(D'+2)^2}\cdot(x+4y)\right]$$

let us consider $v = x+4y$ Type (V)

Replace D by 1 and D' by 4

$$\therefore P.I = e^{x+2y}\left[\frac{1}{2^3 + (2)^2(4+2) - 2(2)+(4+2)^2}\underbrace{\iiint}_{3\,time}(v)'\,dv\,dv\,dv\right]$$

$$= e^{x+2y}\left[\frac{1}{16}\iiint (v)\,dv\,dv\,dv\right]$$

$$= e^{x+2y} \cdot \left[ \frac{1}{160} \cdot \iint \frac{v^2}{2}\, dv\, dv \right]$$

$$= e^{x+2y} \left[ \frac{1}{160} \int \frac{v^3}{6}\, dv \right]$$

$$= e^{x+2y} \left[ \frac{1}{160} \left( \frac{v^4}{24} \right) \right]$$

$$P.I = \frac{e^{x+2y}}{3840} (x+4y)^4$$

$\therefore$ General soln is

$$z = C.F + P.I.$$

$$z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x) + \frac{1}{3840} e^{x+2y} \cdot (x+4y)^4.$$

---

HW

① solve $\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{2x+y} \cdot \cosh x$

② Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x^2y^2 + \sinh(x+y)$

Hint $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

③ solve

$$(D^3 + 6D^2D_1 + 12DD_1^2 + 8D_1^3)\, z = \cos x \cdot \sin 2y$$

Hint $\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

## General Method.

let $f(x,y)$ be any function of $x$ & $y$ and let $F(D,D')$ can be factorised into linear factor of the type $(D-mD')$ etc. then Resolve. $\frac{1}{F(D,D')}$ in to partial fraction.

and then.

$$\frac{1}{D-mD'} f(x,y) = \int f(x, c-mx)\, dx$$

i.e. Replace $y$ by $c-mx$ and integrate w.r.to $x$ where $c$ should be replaced by $y+mx$ after integration

For eg

$$P.I = \frac{1}{F(D,D')} f(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')} f(x,y)$$

$$= \left[\frac{A}{(D-m_1D')} + \frac{B}{(D-m_2D')}\right] f(x,y)$$

$$= A\left[\frac{1}{D-m_1D'} f(x,y)\right] + B\left[\frac{1}{D-m_2D'} f(x,y)\right]$$

$$= A\int f(x, c-m_1x)\, dx + B\int f(x, c-m_2x)\, dx$$

and then integrate w.r.to $x$. to get a desired soln

After integration Replace $C$ by $y+m_1x$ & $y+m_2x$ respt.

**Ex:** Solve

$$4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log(x+2y)$$

**Soln.** Given P.D.E is

$$(4D^2 - 4DD' + D'^2) z = 16 \log(x+2y)$$

$\therefore$ To find C.F:

A.E is obtained by replacing D by m and D' by 1 and equating to zero.

$$\therefore A.E \Rightarrow (4m^2 - 4m + 1) = 0$$

$$(2m-1)^2 = 0$$

$$\Rightarrow m = 1/2,\ 1/2$$

$$\therefore C.F = \phi_1(y + x/2) + x\,\phi_2(y + x/2)$$

To find P.I

$$P.I = \frac{1}{(4D^2 - 4DD' + D'^2)} \cdot 16 \log(x+2y)$$

$$= 16 \left[ \frac{1}{(2D - D')^2} \log(x+2y) \right]$$

$$= \cancel{16} \left[ \frac{1}{4\left(D - \frac{D'}{2}\right)^2} \log(x+2y) \right]$$

$$= 4 \left[ \frac{1}{\left(D - \frac{D'}{2}\right)\left(D - \frac{D'}{2}\right)} \log(x+2y) \right]$$

$$= 4\,\frac{1}{\left(D-\frac{D'}{2}\right)}\cdot\left[\frac{1}{\left(D-\frac{D'}{2}\right)}\log(x+2y)\right]$$

(by general method)

$$= 4\,\frac{1}{(D-D'/2)}\cdot\int\log\left(x+2\left(c-x/2\right)\right)dx$$

$y = c - mx$

$= c - x/2$

$$= 4\,\frac{1}{(D-D'/2)}\int\log(x+2c-x)\,dx$$

$$= 4\,\frac{1}{(D-D'/2)}\int\log 2c\,.\,dx$$

$$= 4\,\frac{1}{(D-D'/2)}\cdot\log 2c\cdot[x]$$

$$= 4\,\frac{1}{(D-D'/2)}\cdot x\log 2(y+x/2)$$

$$= 4\,\frac{1}{(D-D'/2)}\cdot x\log(x+2y)$$

~~Genes~~

$$= 4\int x\log\left(x+2\left(c-\tfrac{1}{2}x\right)\right)\cdot dx$$

$$= 4\int x\log(2c)\cdot dx$$

$$= 4\log(2c)\cdot\left[\frac{x^2}{2}\right]$$

$$= 2x^2\log 2c$$

$$= 2x^2\log 2\left(y+\tfrac{1}{2}x\right)$$

$$PI = 2x^2\log(x+2y)$$

OR

$$PI = \frac{1}{4D^2-4DD'+D'^2}\cdot 16\log(x+2y)$$

let $v = x+2y$. $\Rightarrow \phi(v) = \log v$

Replace $D$ by 1 and $D'$ by 2

$$\therefore PI = 16\cdot\frac{1}{4(1)^2-4(1)(2)+(2)^2}\iint\log v\,dv\,dv$$

$$= 16\,\frac{1}{0}\iint\log v\,dv\,dv\cdot \text{ case fails}$$

$$\therefore PI = 16\cdot\frac{x^2}{(2)^2(2!)}\cdot\log(v)$$

$$\boxed{PI = 2x^2\log(x+2y).}$$

$\therefore$ complete soln is

$$z = CF + PI = \phi_1(y+x/2) + x\phi_2(y+x/2) + 2x^2\log(x+2y)$$

Ex: Solve

$$(D^2 - DD' - 2D'^2) z = (y-1) e^x$$

where $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

Soln: Given P.D.E is

$$(D^2 - DD' - 2D'^2) z = (y-1) e^x \longrightarrow (1)$$

$\therefore$ To find C.F Replace D by m and D' by 1

A.E $\Rightarrow$ $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m = 2, -1$

$\therefore$ C.F $= \phi_1(y+2x) + \phi_2(y-x) \longrightarrow (2)$

Now To find P.I

$$P.I = \frac{1}{(D^2 - DD' - 2D'^2)} (y-1) e^x$$

$$= \frac{1}{(D-2D')(D+D')} (y-1) e^x$$

$$= \frac{1}{(D-2D')} \left[ \frac{1}{(D+D')} (y-1) e^x \right]$$

$\frac{1}{D-m_1D'} f(x,y) = \int f(x, c-mx)\, dx$

$$= \frac{1}{(D-2D')} \int e^x (c+x-1)\, dx$$

$$= \frac{1}{D-2D'} \; c\int e^x dx + \int x e^x dx - \int e^x dx$$

$$= \frac{1}{D-2D'} \left( c e^x + (x e^x - e^x) - e^x \right)$$

$$= \frac{1}{D-2D'} \left(ce^{x} + xe^{x} - 2e^{x}\right)$$

$$= \frac{1}{D-2D'} \cdot (c + x - 2)\, e^{x}$$

$$= \frac{1}{D-2D'} (y - x + x - 2)\, e^{x}$$

$$= \frac{1}{(D-2D')} \cdot (y-2)\, e^{x}$$

$$= \int (c - 2x - 2)\, e^{x}\, dx \qquad \because \text{by general meth}$$

$$= (c-2x-2) \cdot e^{x} - (0-2-0)\, e^{x}$$

$$= (c-2x-2)\, e^{x} + 2e^{x}$$

$$= ce^{x} - 2xe^{x} - 2e^{x} + 2e^{x}$$

$$= (c-2x)\, e^{x}$$

$$= \left[(y+2x) - 2x\right] e^{x}$$

$$P.I. = ye^{x}$$

Hence complete soln is

$$z = C.F + P.I$$

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + ye^{x}}$$

OR

$$PI = \frac{1}{D^2 - DD' - 2D'^2} (y-1) e^x$$

$$= \frac{1}{D^2 - DD' - 2D'^2} [e^x \cdot y - e^x]$$

$$= \left[\frac{1}{D^2 - DD' - 2D'^2} e^{x+0y} \cdot (x^0 \cdot y^1)\right] - \left[\frac{1}{D^2 - DD' - 2D'^2} e^{x+0y}\right]$$

$$PI = PI_1 + PI_2 \quad \text{——} (3)$$

Here, $PI_1 = \frac{1}{D^2 - DD' - 2D'^2} e^{x+0y} \cdot (x^0 y^1)$ (Type (V))

Replace $D$ by $D+1$, and $D'$ by $D'+0$

$$\therefore PI_1 = e^{x+0y} \cdot \left[\frac{1}{(D+1)^2 - (D+1)D' - 2D'} (x^0 y^1)\right]$$

$$= e^x \left[\frac{1}{D^2 + 2D + 1 - DD' - D' - 2D'^2} (x^0 y^1)\right]$$

$$= e^x \left[\frac{1}{1 + (D^2 + 2D - DD' - D' - 2D'^2)} (x^0 y^1)\right]$$

$$= e^x [1 + (D^2 + 2D - DD' - D' - 2D'^2)]^{-1} (x^0 y^1)$$

$$= e^x [1 - (D^2 + 2D - DD' - D' - 2D'^2) + \cdots] (x^0 y^1)$$

$$= e^x [x^0 y^1 - (0 + 0 - 0 - 1 - 0) + 0 \cdots]$$

$$PI_1 = e^x y + e^x$$

Now: $PI_2 = \frac{1}{D^2 - DD' - 2D'^2} e^{x+0y}$ (Type-1)

Replace $D$ by $1$ and $D'$ by $0$

$\therefore PI_2 = \frac{1}{(1)^2 - 0 - 0} e^{x+0y} = e^x$

Hence

$PI = PI_1 - PI_2$

$= (ye^x + e^x) - (e^x)$

$\boxed{PI = ye^x}$

Hence complete soln is

$z = CF + PI$

$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + ye^x}$

# Method of Separation of variables :

In this method . A dependent variable is a soln and it is considered as product of two function, each of which involves only one of the ~~integ~~ independent variables . so two ordernary diff. eqns are formed

**for ex :**

step-① Consider a ~~total soln~~ $u = u(x, y)$ be the dependent variable and $x, y$ are independent variables.

Then consider total soln is

$$\boxed{u(x,y) = X(x) \cdot Y(y)}$$

Step(II) Then find all differential coefficient appearing in given P.D.E.

i.e.
$$\frac{\partial u}{\partial x} = \frac{\partial (XY)}{\partial x} = Y \frac{dX}{dx} = YX'$$

$$\frac{\partial u}{\partial y} = \frac{\partial (XY)}{\partial y} = X \frac{dY}{dy} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 (XY)}{\partial x^2} = Y \frac{d^2X}{dx^2} = YX''$$

etc.

step(III) Substitute all the values in given P.D.E. we will get a D.E containing $X', X'', Y, Y'', X, Y$ etc.

Step(IV) Now separate the variable and equate to any const. k

In this way we get two distinct ordinary diff. eqns and on solving them we get ~~the~~ soln $X(x)$ & $Y(y)$

Now substitute these values in total soln.

so final
$$\boxed{u(x,y) = X(x) \cdot Y(y)}$$
is reqd soln.

Ex: Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

where $u(x,0) = 6e^{-3x}$, by method separation of variable.

Soln: Given P.D.E.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{--- (1)}$$

Consider a Trial soln is

$$u(x,t) = X(x) \cdot T(t). \quad \text{--- (2)}$$

Substituting these value in eqn (1). we get

$$\frac{\partial (XT)}{\partial x} = 2\frac{\partial (XT)}{\partial t} + XT$$

$$\Rightarrow T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

$$\Rightarrow TX' = X(2T' + T)$$

Separating variable we get

$$\frac{X'}{X} = \frac{2T'}{T} + 1 = k \quad \text{(any const.)}$$

$$\text{--- (3)}$$

From eqn (3). we have

$$\frac{X'}{X} = K$$

on integrating we get

$$\log X = Kx + \log c_1$$

$$\Rightarrow \log\left(\frac{X}{c_1}\right) = Kx$$

$$\Rightarrow \boxed{X = c_1 e^{Kx}}$$

also. from (3).

$$2\frac{T'}{T}+1=k$$

$$\Rightarrow \frac{T'}{T}+\frac{1}{2}=k/2 \quad \Rightarrow \frac{T'}{T}=\frac{(k-1)}{2}$$

on integrating we get

$$\log T=\left(\frac{k-1}{2}\right)\cdot t+\log c_2$$

$$\Rightarrow \log\left(T/c_2\right)=\left(\frac{k-1}{2}\right)t$$

$$\boxed{T=c_2\,e^{\left(\frac{k-1}{2}\right)\cdot t}}$$

Hence reqd soln. is

$$u(x,t)=X(x)\cdot T(t)$$

$$=c_1\,e^{kx}\cdot c_2\,e^{\frac{1}{2}(k-1)t}$$

$$\boxed{u(x,t)=A\cdot e^{kx+\frac{1}{2}(k-1)t}} \quad \text{--- (4)}$$

It is given that $u(x,0)=6e^{-3x}$

$\therefore$ put $t=0$ in eqn (4) we get

$$u(x,0)=A\,e^{kx}$$

$$\Rightarrow 6e^{-3x}=Ae^{kx}$$

on equating we get $\boxed{A=6}$ & $\boxed{k=-3}$

$\therefore$ Hence soln is

$$u(x,t)=6\,e^{-3x+\frac{1}{2}(-3-1)t}$$

$$\boxed{u(x,t)=6e^{-(3x+2t)}}$$

Scanned by CamScanner

Ex :⇒ Solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ given $u(0,y) = 8e^{-3y}$

by method of separation of variables.

Soln : Given PDE is

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \text{—— (1)}$$

Consider A trial soln is

$$u(x,y) = X(x) \cdot Y(y) \quad \text{—— (2)}$$

Substituting these value in eqn (1) we get

$$(1) \Rightarrow \frac{\partial (XY)}{\partial x} = 4\frac{\partial (XY)}{\partial y}$$

$$\Rightarrow Y\frac{dX}{dx} = 4X\frac{dY}{dy}$$

$$\Rightarrow YX' = 4XY'$$

by separating ~~a~~ variables.

$$\frac{X'}{X} = 4\frac{Y'}{Y} = k \;(\text{const}) \quad \text{—— (3)}$$

from eqn (3) : we have

$$\frac{X'}{X} = k$$

on integrating we get

$$\log X = kx + \log c_1$$

$$\Rightarrow \log (X/c_1) = kx$$

$$\Rightarrow \boxed{X = c_1 e^{kx}}$$

Scanned by CamScanner

From eqn (3): we have

$$\frac{Y'}{Y} = \frac{k}{4}$$

on integrating ~~we get~~ w.r.to. y. we get

$$\log Y = \frac{k}{4}y + \log c_2$$

$$\Rightarrow \log\left(Y/c_2\right) = \frac{k}{4}y$$

$$\Rightarrow \boxed{Y = c_2 e^{k/4\, y}}$$

∴ The reqd soln is

$$u(x,y) = X(x) \cdot Y(y)$$

$$= c_1 e^{kx} \cdot c_2 e^{k/4\, y}$$

$$\boxed{u(x,y) = A \cdot e^{k(x + y/4)}}$$ where $A = c_1 \cdot c_2$ — (4)

~~Now using given condn that~~ It is

It is given that $u(0,y) = 8e^{-3y}$

put $x=0$ in eqn (4) we get

$$u(0,y) = A \cdot e^{k/4\, y}.$$

$$\Rightarrow 8e^{-3y} = Ae^{k/4\, y}$$

$$\Rightarrow \boxed{A=8} \quad \& \quad k/4 = -3 \quad \Rightarrow \boxed{k=-12}$$

∴ The reqd soln is

$$u(x,y) = 8e^{-12(x + y/4)}$$

$$\boxed{u(x,y) = 8e^{-(12x+3y)}}$$

Scanned by CamScanner

Ex: Using method of separation of variable

Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 3u$

given $u = 3e^{-t} - e^{-5t}$ when $x=0$

Soln: Given P.D.E is

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 3u \quad \text{--- (1)}$$

Consider a trial soln

$$u(x,t) = X(x)\, T(t) \quad \text{--- (2)}$$

$\therefore$ Substituting in eqn (1) we get

$$4\frac{\partial}{\partial x}(XT) + \frac{\partial}{\partial t}(XT) = 3(XT)$$

$$\Rightarrow 4T\frac{dX}{dx} + X\frac{dT}{dt} = 3XT$$

$$\Rightarrow 4TX' + XT' = 3XT$$

$$\Rightarrow 4TX' = X(3T - T')$$

separating variables we get

$$4\frac{X'}{X} = \frac{3T - T'}{T}$$

$$\Rightarrow 4\frac{X'}{X} = -\frac{T'}{T} + 3 = k \text{ (const)} \quad \text{--- (3)}$$

from eqn (3) we get

$$4\frac{X'}{X} = k$$

$$\Rightarrow \frac{X'}{X} = k/4$$

on integrating w.r.t. x we get

Scanned by CamScanner

$$\log X = \tfrac{k}{4}x + \log c_1$$

$$\Rightarrow \log (X/c_1) = \frac{k}{4}x$$

$$\Rightarrow \boxed{X = c_1 e^{k/4 x}}$$

Also from (3),

$$-\frac{T'}{T} + 3 = k$$

$$\Rightarrow \frac{T'}{T} - \cancel{3} = (3-k)$$

on integrating w.r.to $t$, we get

$$\log T = (3-k)t + \log c_2$$

$$\Rightarrow \log\left(\frac{T}{c_2}\right) = (3-k)t$$

$$\Rightarrow \boxed{T = c_2 e^{(3-k)t}}$$

Hence the reqⁿ solⁿ is

$$(2) \Rightarrow u(x,t) = c_1 e^{k/4 x} \cdot c_2 e^{(3-k)t}$$

$$\Rightarrow \boxed{u(x,t) = A e^{\frac{k}{4}x + (3-k)t}}$$

It is given that $u(0,t) = 3e^{-t} - e^{-5t}$

having two terms, Hence by using Superposition thm

we consider solⁿ is

$$u(x,t) = \sum_{n=1}^{2} A_n e^{\frac{k_n}{4}x + (3-k_n)t}$$

$$= A_1 e^{\frac{k_1}{4}x + (3-k_1)t} + A_2 e^{\frac{k_2}{4}x + (3-k_2)t} \quad \text{——(4)}$$

Put $x=0$ we get

$$\therefore u(0,t) = A_1 e^{(3-k_1)t} + A_2 e^{(3-k_2)t}$$

$$\Rightarrow 3e^{-t} - e^{-5t} = A_1 e^{(3-k_1)t} + A_2 e^{(3-k_2)t}$$

$\Rightarrow A_1 = 3, \quad A_2 = -1$

$3 - k_1 = -1 \quad \& \quad 3 - k_2 = -5$

$\Rightarrow \boxed{k_1 = 4} \quad \& \quad \boxed{k_2 = 8}$

$\therefore$ The reqd soln is

$$u(x,t) = 3\, e^{4/4 x + (3-4)t} + (-1)\, e^{8/4 x + (3-8)t}$$

$$\boxed{u(x,t) = 3\, e^{x-t} - e^{2x-5t}}$$

---

xoox

---

**Ex** :⇒ Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$

by method of seperation of variable.

if $u = 0$, $\dfrac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x = 0$

**Soln** :⇒ Given P.D.E is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{——} (1)$$

Given cond

$$\left.\begin{aligned} u(0,y) &= 0 \\ u'(0,y) = \frac{\partial u}{\partial x} &= 0 \text{ when } x = 0 \end{aligned}\right] \text{——} (2)$$

let us put Consider a trial soln is

$$u(x,y) = X(x) \cdot Y(y) \quad \text{——} (3)$$

$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = XY'$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XY) = X'Y$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(X'Y) = X''Y$

putting these values in eqn (1) we get

$$X''Y = XY' + 2XY$$

$$\Rightarrow X''Y = X(Y' + 2Y)$$

$$\Rightarrow \frac{X''}{X} = \frac{Y' + 2Y}{Y} = K$$

$$\text{ie} \Rightarrow \frac{X''}{X} = \left(2 + \frac{Y'}{Y}\right) = K \quad \text{——— (4)}$$

we get two ordinary D.Eqn's. as follows

$$\frac{X''}{X} = K$$

$$\Rightarrow X'' - KX = 0 \quad \Rightarrow (D^2 - K)X = 0$$

$\therefore$ A.E: $m^2 - K = 0$

$$\Rightarrow m = \pm\sqrt{K}$$

$\therefore$ soln is $X = C.F = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$

$$\boxed{X(x) = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}} \quad \text{——— (5)}$$

another D.E is

$$2 + \frac{Y'}{Y} = K$$

$$\Rightarrow \frac{Y'}{Y} = (K-2)$$

on integrating we get

$$\log Y = (K-2)y + \log c_3$$

$$\Rightarrow \frac{Y}{c_3} = e^{(K-2)y}$$

$$\Rightarrow \boxed{Y = c_3 e^{(K-2)y}} \quad \text{——— (6)}$$

Scanned by CamScanner

Hence soln of given diff. eqn become (from ②)

$$u(x,y) = \left(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}\right) \cdot c_3 e^{(k-2)y}$$

$$= c_1 c_3 \, e^{\sqrt{k}x + (k-2)y} + c_2 c_3 \, e^{-\sqrt{k}x + (k-2)y}$$

$$u(x,y) = A \, e^{\sqrt{k}x + (k-2)y} + B \, e^{-\sqrt{k}x + (k-2)y}$$

$$= \left(A \, e^{\sqrt{k}x} + B \, e^{-\sqrt{k}x}\right) \cdot e^{(k-2)y} \longrightarrow ⑦$$

given condition is

~~u(0,y) = 0~~

$u = 0$ when $x = 0$

i.e. $u(0,y) = 0$

$\therefore$ put $x=0$ in eqn ⑦ we get

$$u(0,y) = (A+B) \, e^{(k-2)y}$$

$$\Rightarrow \quad 0 = (A+B) \, e^{(k-2)y}$$

$$\Rightarrow \quad A+B = 0 \qquad \left(\because e^{(k-2)y} \neq 0\right)$$

$$\Rightarrow \quad \boxed{A = -B}$$

⑦ $\Rightarrow$

$$\therefore u(x,y) = \left(A e^{\sqrt{k}x} - A e^{-\sqrt{k}x}\right) \cdot e^{(k-2)y}$$

$$u(x,y) = A \cdot \left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right) \cdot e^{(k-2)y} \longrightarrow ④$$

Diff. partially w.r.to. $x$ we get

$$\frac{\partial u}{\partial x} = A\left[\sqrt{k}\, e^{\sqrt{k}x} + \sqrt{k}\, e^{-\sqrt{k}x}\right] \cdot e^{(k-2)y}$$

$$u'(x,y) = A\sqrt{k}\left(e^{\sqrt{k}x} + e^{-\sqrt{k}x}\right) \cdot e^{(k-2)y} \longrightarrow ⑧$$

using ~~cond~~ 2nd condition

ie $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ when $x=0$

ie $u'(0,y) = 1 + e^{-3y} = e^{0y} + e^{-3y}$

~~Since it contain two distinct~~

put $x=0$ in eqn (3)

$$u'(0,y) = 2A\sqrt{k} \cdot e^{(k-2)y}.$$

but given condn contain two distinct term. thus by using superposition thm. we consider

$$u'(0,y) = 2\sum_{n=1}^{2} A_n \sqrt{k_n}\, e^{(k_n-2)y}$$

$$\Rightarrow e^{0y} + e^{-3y} = 2A_1\sqrt{k_1}\, e^{(k_1-2)y} + 2A_2\sqrt{k_2}\, e^{(k_2-2)y}$$

Comparing coefficient of we get

$2A_1\sqrt{k_1} = 1$, $k_1 - 2 = 0$

$\Rightarrow k_1 = 2$

$\therefore A_1 = \frac{1}{2\sqrt{k_1}} = \frac{1}{2\sqrt{2}}$

also $2A_2\sqrt{k_2} = 1$ & $k_2 - 2 = -3$

$\Rightarrow k_2 = -1$

$\therefore A_2 = \frac{1}{2\sqrt{k_2}} = \frac{1}{2\sqrt{-1}} = \frac{1}{2i}$

Now the reqd soln by ~~metho~~ superposition meth

$$u(x,y) = A_1\left(e^{\sqrt{k_1}x} - e^{-\sqrt{k_1}x}\right)e^{(k_1-2)y} + A_2\left(e^{\sqrt{k_2}x} - e^{-\sqrt{k_2}x}\right)e^{(k_2-2)y}$$

$$= \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right)e^{(\cancel{2}-2)y} + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\cdot e^{-3y}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{e^{\sqrt{2}x} - e^{-\sqrt{2}x}}{2}\right) + \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\cdot e^{-3y}$$

$$\boxed{u(x,y) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x + \sin x \cdot e^{-3y}}$$