FOURIER TRANSFORM

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Introduction :-

The fourier series methods provides a technique for the representation of periodic functions. Since, of course, many practical problems do not involve periodic function. It is desirable to generalise the method of fourier series to include non-periodic function. Hence we introduce consept of Fourier Transform.

Fourier Integral :-

In many practical problems. function like the impressed force (or (voltage in non-periodic rather than periodic i.e. a signal unrepeated pulse) occure in some finite time interval.

Function of this sort can not be handled directly through the use of fourier series. since such series necessarily define only periodic functions. However, the limiting form of the Fourier series as the period is made to approch infinity. a suitable representation for non-periodic function can perhaps be obtained and it called Fourier integral of function.

* Fourier integral thm

Statement : If the function $f(x)$ is such that

(i) it self and it's derivative $f'(x)$ are piecewise contin in $(-l, l)$

(ii) it is absolutely integrable, so that

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \quad \text{(i.e. integral is finite)}$$

Then

$$f(x) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cdot e^{+i\lambda(t-x)}\, dt\, d\lambda$$

Proof :⇒ We start with the complex form of the Fourier series expansion of the periodic function $f_l(x)$:

$$f_l(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{+in\pi x}{l}} \quad \text{——— (1)}$$

where,

$$C_n = \frac{1}{2l} \int_{-l}^{l} f_l(t) \cdot e^{\frac{-in\pi t}{l}}\, dt \quad \text{——— (2)}$$

using (2) in eqn (1), we get

$$f_l(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2l} \int_{-l}^{l} f_l(t)\, e^{\frac{-in\pi t}{l}}\, dt \right] e^{\frac{+in\pi x}{l}} \quad \text{——— (3)}$$

let $\lambda = \frac{n\pi}{l}$ and we note that $\frac{n\pi}{l}$'s are called the frequencies and the set of all the frequencies $0, \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \ldots\ldots$ is called the freq. spectrum.

$$\therefore \Delta\lambda = \left[\frac{(n+1)\pi}{l} - \frac{n\pi}{l}\right] = \frac{\pi}{l}$$

$$\therefore \frac{1}{l} = \frac{\Delta\lambda}{\pi}$$

Here we observe that as l increases the discrete spectrum becomes more and more dense, and approaches a continuous spectrum as $l \to \infty$. Therefore as $l \to \infty$ the summation in eq^n (3), on the discrete variable n, will give way to an ~~a cont~~ Integration on a continuous variable λ.

Now, as $l \to \infty$, $f_l(x) = f(x)$ everywhere & $\Delta\lambda \to 0$

$\therefore$ from eq^n (3), we have:

$$\lim_{l\to\infty} f_l(x) = \lim_{l\to\infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{2l} \int_{-l}^{l} f_l(t) \cdot e^{\mp\frac{in\pi t}{l}} dt\right] e^{\pm\frac{in\pi x}{l}}$$

$$f(x) = \lim_{\Delta\lambda\to 0} \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\lambda}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot e^{\mp i\lambda t} dt\right] e^{\pm i\lambda x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot \left[\lim_{\Delta\lambda\to 0} \sum_{n=-\infty}^{\infty} e^{\mp i\lambda(t-x)} \Delta\lambda\right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[\int_{-\infty}^{\infty} e^{\mp i\lambda(t-x)} d\lambda\right] dt$$

$$\boxed{f(x) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cdot e^{\mp i\lambda(t-x)} \, dt \, d\lambda} \quad \text{--- (4)}$$

is known as the Fourier integral representation of $f(x)$.

This is also known as the complex or exponential form of the fourier integral.

If the function $f(x)$ which satisfies the Dirichlet cond^n (i) & (ii) then the fourier integral representation of $f(x)$, given in eq^n (4) converges to $f(x)$ at all points where $f(x)$ is continuous. and to the average of left hand limit and right hand limit of $f(x)$ where $f(x)$ is discontinuous. i.e at pt of discontinuity say x_0, the Fourier integral
$$= \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$$

Trignometric form of fourier integral

From the complex form of the Fourier integral we have

$$f(x) = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\infty} \int_{t=-\infty}^{\infty} f(t)\, e^{-i\lambda(t-x)}\, dt \cdot d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot [\cos \lambda(t-x) - i \sin \lambda(t-x)]\, dt\, d\lambda$$

$$\left[\because \cos\theta - i\sin\theta = e^{-i\theta}\right]$$
$\because$ Eulers Thm

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt\, d\lambda$$

$$- \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x)\, dt\, d\lambda . \quad (5)$$

Since $\sin \lambda(t-x)$ is an odd function of λ and $\cos\lambda(t-x)$ is an even function of λ.

$$\therefore \int_{-\infty}^{\infty} \sin\lambda(t-x)\, d\lambda = 0$$

$$\int_{-\infty}^{\infty} \cos \lambda(t-x)\, d\lambda = 2\int_{0}^{\infty} \cos\lambda(t-x)\, d\lambda$$

$\therefore$ eqn (5) becomes

$$f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt \right] d\lambda \quad \text{—— (6)}$$

This is the required trignometric form of Fourier integral of f(x)

* Fourier sine and cosine integral

From the trignometric form of the Fourier integral of f(x) we have

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \left[\cos \lambda t \cos \lambda x + \sin \lambda t \cdot \sin \lambda x \right] dt\, d\lambda$$

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a}$$
$$= 0$$

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos \lambda x \left[\int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt \right] d\lambda$$

$$+ \frac{1}{\pi} \int_{0}^{\infty} \sin \lambda x \left[\int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt \right] d\lambda \quad \text{—— (7)}$$

Case (1): If f(t) is an odd function of t then.

$f(t) \cos \lambda t$ is also an odd function. but $f(t) \sin \lambda t$ is an even function of t

$\therefore$ By the properties of definite integral, we have

$$\int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt = 0$$

& $$\int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt = 2 \int_{0}^{\infty} f(t) \sin \lambda t\, dt.$$

$\therefore$ ⑦ $\Rightarrow$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left[\int_0^\infty f(t)\sin\lambda t \, dt\right] d\lambda$$

which is called as Fourier sine integral of $f(x)$

Case ②: If $f(t)$ is an even function of t, then

$f(t)\cos\lambda t$ is an even function and $f(t)\sin\lambda t$ is a odd function of t.

$\therefore$ By property of definite integral.

$$\int_{-\infty}^{\infty} f(t)\cos\lambda t \, dt = 2\int_0^\infty f(t)\cos\lambda t \, dt$$

$$\& \int_{-\infty}^{\infty} f(t)\sin\lambda t \, dt = 0$$

$\therefore$ ⑦ $\Rightarrow$ $$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \left[\int_0^\infty f(t)\cos\lambda t \, dt\right] d\lambda$$

which is called as the fourier cosine integral of $f(x)$

standerd formula:

① Fourier integral: (Trigonometric form)

$$f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cdot \cos \lambda (t-x) \, dt \, d\lambda .$$

(ii) Fourier cosine integral (when f(t) is even)

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \cos \lambda x \left[\int_{t=0}^{\infty} f(t) \cos \lambda t \, dt \right] d\lambda$$

(iii) Fourier sine integral (when f(t) is odd)

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \sin \lambda x \left[\int_{t=0}^{\infty} f(t) \sin \lambda t \, dt \right] d\lambda$$

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Ex :⇒ Express the function

$$f(x) = \begin{cases} 1, & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

as a Fourier integral, Hence evaluate

(i) $\displaystyle\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda$ (ii) $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda$.

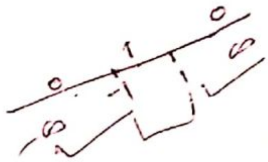
$f(x) = \begin{cases} 1 & |x| \le 1 \\ & -1 \le x \le 1 \end{cases}$

$f(x) = \{ 1 \quad -1 < x < 1$

$-1 < x < 1$

Sol^n: By def^n, Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_{\lambda=0}^{\infty} \int_{t=-\infty}^{\infty} f(t) \cos\lambda(t-x)\, dt\, d\lambda .$$



Here. $f(t) = 1$, $|t| \le 1$ i.e $(-1 \le t \le 1$

and $f(t) = 0$, $|t| > 1$ $(-\infty < t < -1$ & $1 < t < \infty)$

Hence, $f(x) = \dfrac{1}{\pi} \displaystyle\int_{\lambda=0}^{\infty} \left[\int_{-1}^{1} 1 \cdot \cos\lambda(t-x)\, dt \right] d\lambda$

$$= \cancel{\frac{1}{\pi} \int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda} \right]_{-1}^{1}} d\lambda$$

$\int \cos\lambda t\, dt = \dfrac{\sin\lambda t}{\lambda}$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda} \right]_{t=-1}^{1} d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\lambda} \left[\sin\lambda(1-x) - \sin\lambda(-1+x) \right] d\lambda$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{\lambda} \left[\sin(\lambda - \lambda x) + \sin(\lambda + \lambda x) \right] d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda$$

$\because 2\sin A \cos B = \sin(A-B) + \sin(A+B)$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda}\, d\lambda \quad \text{—— (1)}$$

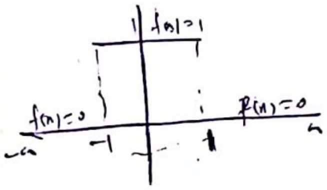
which is the required Fourier integral of $f(x)$

Now (i), from eq^n (1) we have

$$\boxed{\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x)} \quad \text{—— (2)}$$

Clearly $|x| = 1$ is a point of discontinuity in this case

$$\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda} = \frac{\pi}{2}\left[\frac{f(x^+) + f(x^-)}{2}\right]$$



$$= \frac{\pi}{2}\left[\frac{f(|x|^+) + f(|x|^-)}{2}\right]$$

$$= \pi/2 \left[\frac{1+0}{2}\right]$$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda} = \pi/4$$

(ii) Now putting $x=0$ in eq^n (2) and note that $x=0$ lies in $[-1,1]$ where $f(x)=1$ (i.e. $f(0)=1$)

$$\therefore \int_0^\infty \frac{\sin\lambda \cdot \cos(0)}{\lambda} d\lambda = \frac{\pi}{2} f(0)$$

Hence (2) $\Rightarrow$ $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \pi/2$

$$\int_0^\infty e^{-at}\left(\frac{\sin t}{t}\right) dt$$

$$\int_0^\infty \frac{1}{1+t^2}\, dt = \tan^{-1}\infty - \tan^{-1}(0)$$
$$= \pi/2 - \tan^{-1}(0) = \pi/2$$

Ex :- Find Fourier sine integral for

(i) $f(x) = \begin{cases} 1 & , 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$

Hence evaluate $\displaystyle\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \sin\lambda x \, d\lambda$.

Sol^n :- By def^n of Fourier sine integral, we have

$$f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \sin\lambda x \left[\int_{t=0}^{\infty} f(t) \cdot \sin\lambda t \, dt \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\int_0^\pi 1 \cdot \sin\lambda t \, dt \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[-\frac{\cos\lambda t}{\lambda} \right]_0^\pi d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda x}{\lambda} \cdot \left[-\cos\lambda\pi + \cos 0 \right] d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda x}{\lambda} \left[1 - \cos\lambda\pi \right] d\lambda \qquad \because \cos 0 = 1$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \cdot \sin\lambda x \, d\lambda \quad \text{——— (1)}$$

Now, from eq^n (1), we have

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \cdot \sin\lambda x \, d\lambda = \frac{\pi}{2} f(x)$$

$$\Rightarrow \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \cdot \sin\lambda x = \begin{cases} \pi/2\, f(x), & 0 \le x \le \pi \\ 0 & x > \pi \end{cases}$$

clearly $x = \pi$ is a point of discontinuity of $f(x)$

$\therefore$ value of integral is

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda} \sin\lambda x = \frac{\pi/2 + 0}{2}$$

$$= \pi/4.$$

$$f(\pi) = \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{\cancel{\pi/4}\ 0 \ 1}{2} = 1/2$$

Ex: Find Fourier cosine integral for

(i) $f(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

(ii) $f(x) = e^{-x}\cos x$

Soln (i): $\Rightarrow$ By defn. of Fourier cosine Integral, we have

$$f(x) = \frac{2}{\pi}\int_{\lambda=0}^{\infty} \cos\lambda x \left[\int_{t=0}^{\infty} f(t)\cos\lambda t \, dt\right] d\lambda.$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \cos\lambda x \left[\int_0^1 t\cos\lambda t\,dt + \int_1^2 (2-t)\cos\lambda t\,dt + \int_2^{\infty} 0\cos\lambda t\,dt\right] d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \cos\lambda x \left\{\left[t\,\frac{\sin\lambda t}{\lambda} - (1)\,\frac{-\cos\lambda t}{\lambda^2}\right]_0^1 + \left[(2-t)\,\frac{\sin\lambda t}{\lambda} - (-1)\left(\frac{-\cos\lambda t}{\lambda^2}\right)\right]_1^2 + 0\right\} d\lambda$$

$$= \frac{2}{\pi}\int_{\lambda=0}^{\infty} \cos\lambda x \left\{\left[\cancel{\frac{\sin\lambda}{\lambda}} + \frac{\cos\lambda}{\lambda^2} - \frac{1}{\lambda^2}\right] + \left[-\frac{\cos 2\lambda}{\lambda^2} - \cancel{\frac{\sin\lambda}{\lambda}} + \frac{\cos\lambda}{\lambda^2}\right]\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \cdot \frac{1}{\lambda^2}\left(2\cos\lambda - 1 - \cos 2\lambda\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \cdot \frac{1}{\lambda^2}\left(2\cos\lambda - 2\cos^2\lambda\right) d\lambda \qquad \left(\because \cos 2\lambda = 2\cos^2\lambda - 1\right)$$

$$= \frac{2}{\pi}\int_0^{\infty} \cos\lambda x \; \frac{2\cos\lambda}{\lambda^2}\left(1 - \cos\lambda\right) d\lambda$$

$$\therefore f(x) = \frac{4}{\pi}\int_0^\infty \frac{1}{\lambda^2}\cos\lambda\,(1-\cos\lambda)\cos\lambda x\,d\lambda$$

which is the required Fourier cosine integral for f(x)

Sol^n (2) let $f(x) = e^{-x}\cos x$

By def^n of Fourier's cosine integral, we have

$$f(x) = \frac{2}{\pi}\int_{\lambda=0}^\infty \cos\lambda x\cdot\left[\int_{t=0}^\infty f(t)\cos\lambda t\,dt\right]d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left[\int_0^\infty e^{-t}\cos t\cdot\cos\lambda t\,dt\right]d\lambda$$

$$= \frac{\cancel{2}}{\pi}\int_0^\infty \cos\lambda x\left\{\int_0^\infty e^{-t}\cdot\frac{1}{\cancel{2}}\left[\cos(t+\lambda t)+\cos(t-\lambda t)\right]dt\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\cdot\left\{\underbrace{\int_0^\infty e^{-t}\cos(1+\lambda)t\,dt}_{= L\{\cos(1+\lambda)t\}_{s=1}} + \int_0^\infty e^{-t}\cos(1-\lambda)t\,dt\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\left\{\frac{1}{1+(1+\lambda)^2}+\frac{1}{1+(1-\lambda)^2}\right\}d\lambda$$

$$\left(\because \underbrace{\int_0^\infty e^{-ax}\cos bx\,dx}_{L\{\}} = \frac{a}{a^2+b^2}\right)$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\left\{\frac{1}{2+2\lambda+\lambda^2}+\frac{1}{2-2\lambda+\lambda^2}\right\}d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\cdot\left\{\frac{4+2\lambda^2}{4+\lambda^4}\right\}d\lambda = \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)\cos\lambda x}{\lambda^4+4}\,d\lambda$$

This is required Fourier cosine integral for f(x).

Ex :⇒ using Fourier integral, show that

(i) $$\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda = e^{-ax} - e^{-bx}, \quad a,b>0$$

(ii) $$\int_0^\infty \frac{1+\cos\lambda\pi}{1-\lambda^2}\cos\lambda x \cdot d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0\le x\le \pi \\ 0, & x>\pi. \end{cases}$$

Soln (i) since the integrand contains the term of $\sin\lambda x$, we use Fourier sine integral.

Let $f(x) = e^{-ax} - e^{-bx}$

By defn of fourier sine integral, we have,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\cdot\sin\lambda t\,dt\right]d\lambda$$

$$\Rightarrow e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\left[\int_0^\infty (e^{-at}-e^{-bt})\sin\lambda t\,dt\right]d\lambda.$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty e^{-at}\sin\lambda t\,dt - \int_0^\infty e^{-bt}\cdot\sin\lambda t\,dt\right]dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{a^2+\lambda^2} - \frac{\lambda}{b^2+\lambda^2}\right]d\lambda$$

$\because \int_0^\infty e^{-sx}\sin bx\,dx$

$L\{\sin bx\} = \frac{b}{s^2+b^2}$

when $s=a$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\cdot\lambda\left[\frac{b^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right]d\lambda$$

$$\therefore e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi} \int_0^\infty \frac{\lambda \sin\lambda x}{(\lambda^2+a^2)\cdot(\lambda^2+b^2)}\, d\lambda$$

(ii) Since the integrand contains the term $\underline{\cos\lambda x}$, we use fourier cosine integral.

let $f(x) = \begin{cases} \pi/2 \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$

By defn of fourier cosine integral, we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos\lambda x \cdot \left[\int_0^\infty f(t) \cdot \cos\lambda t \, dt \right] d\lambda$$

$$= \cancel{\frac{2}{\pi}} \int_0^\infty \cos\lambda x \cdot \left[\int_0^\pi \cancel{\pi/2} \sin t \cdot \cos\lambda t \, dt \right] d\lambda$$

$$= \int_0^\infty \cos\lambda x \left[\int_0^\pi \sin t \cdot \cos\lambda t \, dt \right] d\lambda$$

$$= \int_0^\infty \cos\lambda x \cdot \left\{ \int_0^\pi \frac{1}{2} \left[\sin(1+\lambda)t + \sin(1-\lambda)t \right] dt \right\} d\lambda$$

$$= \frac{1}{2} \int_0^\infty \cos\lambda x \cdot \left[-\frac{\cos(1+\lambda)t}{(1+\lambda)} - \frac{\cos(1-\lambda)t}{(1-\lambda)} \right]_0^\pi d\lambda$$

$$= \frac{1}{2} \int_0^\infty \cos\lambda x \cdot \left[-\frac{\cos(1+\lambda)\pi}{(1+\lambda)} - \frac{\cos(1-\lambda)\pi}{(1-\lambda)} + \frac{1}{(1+\lambda)} + \frac{1}{(1-\lambda)} \right] d\lambda$$

$$= \frac{1}{2} \int_0^\infty \cos\lambda x \left[\frac{\cos\lambda\pi}{1+\lambda} + \frac{\cos\lambda\pi}{1-\lambda} + \frac{1}{1+\lambda} + \frac{1}{1-\lambda} \right] d\lambda \quad \because \cos(\pi \pm \theta) = -\cos\theta$$

$$= \frac{1}{2} \int_0^\infty \cos\lambda x \left[\frac{1}{1+\lambda}(1+\cos\pi\lambda) + \frac{1}{(1-\lambda)}(1+\cos\pi\lambda) \right] d\lambda$$

$$= \int_0^\infty \frac{1+\cos\lambda\pi}{(1-\lambda^2)} \cos\lambda x \, d\lambda$$

$$\therefore \int_0^\infty \frac{1+\cos\lambda\pi}{(1-\lambda^2)} \cos\lambda x \, d\lambda = f(x) = \begin{cases} \pi/2 \sin x, & 0 \le x \le \pi \\ 0, & x > \pi. \end{cases}$$

HW

(1) Solve the integral eqn.

$$\int_0^\infty f(x) \cdot \sin x t \, dx = \begin{cases} 1, & 0 \le t \le 1 \\ 2 & 1 \le t \le 2 \\ 0 & t > 2 \end{cases}$$

Note. Here $\lambda = x$ and variable is t.
(use sine integral)

(2) using fourier integral, show that

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda} \cdot \sin\lambda x \, d\lambda = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

(use sine integral)

let $f(x) = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{cases} \Rightarrow f(t) = \begin{cases} \pi/2, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

$\therefore$ By Defn of sine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(\int_0^\infty f(t) \cdot \sin\lambda t \, dt \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left(\int_0^\pi \pi/2 \sin\lambda t \, dt + \int_\pi^\infty 0 \right) d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \left[\frac{-\cos\lambda t}{\lambda} \right]_0^\pi d\lambda = \int_0^\infty \sin\lambda x \left(\frac{1-\cos\lambda\pi}{\lambda} \right) d\lambda -$$

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Integral transform :→

The integral transform of function $f(x)$ is denoted by $I[f(x)]$ and defined as.

$$I\{f(x)\} = \bar{f}(s) = \int_{x_1}^{x_2} f(x) \cdot k(s,x)\, dx$$

where, $k(s,x)$ is called the kernel of the transform and is a function of s and x. The function $f(x)$ is called the inverse transform of $\bar{f}(s)$.

① when $k(s,x) = e^{-sx}$ the above integral transform lead to the Laplace Transform.

i.e $$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx$$

② when $k(s,x) = e^{-isx}$, we have the Fourier transform of $f(x)$ i.e.

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{-isx}\, dx$$

③ when $k(s,x) = x^{s-1}$, it leads to the Mellin transform of $f(x)$ i.e. $$M\{f(x)\} = \int_0^\infty f(x) \cdot x^{s-1}\, dx$$

④ when. $k(s,x) = J_n(sx)$. we get the Hankel Transform of $f(x)$. i.e.

$$H\{f(x)\} = \int_0^\infty x f(x) \cdot J_n(sx)\, dx$$

⑤ when $k(s,x) = \sin sx$, OR $\cos sx$. then special transform arises and it lead to Fourier sine or Fourier cosine transform.

* ~~Inverse~~ Fourier Transform :⇒

The complex form of fourier integral for a function $f(x)$ is given by (Complex form / Exponential form)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot e^{-i\lambda(t-x)} \, dt \, . d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{+i\lambda x} \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-i\lambda t} \, dt \right] d\lambda \quad \text{①}$$

$\frac{1}{\sqrt{2\pi}}$ $\frac{1}{\sqrt{2\pi}}$

We introduce the symbol F ~~to~~ denoting ~~the~~ fourier transform operator. Then from eqn ①, the Fourier transform of $f(x)$ is denoted by $F\{f(x)\}$ and it's defined as

$$\boxed{F\{f(x)\} = \bar{F}(\lambda) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda x} \, . dx}$$

whenever integral exist

Now eqn ① reduces to

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(\lambda) \, . e^{i\lambda x} \, d\lambda}$$

The function $f(x)$ is called the inverse Fourier transform of $\bar{F}(\lambda) = F\{f(x)\}$ ie.

$$\boxed{f(x) = F^{-1}\{\bar{F}(\lambda)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(\lambda) \cdot e^{+i\lambda x} \, . d\lambda}$$

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Fourier sine Transform of $f(x)$

It is denoted and defined as

$$\boxed{F_S\{f(x)\} = \bar{f}_S(\lambda) = \int_{x=0}^{\infty} f(x)\ \sin \lambda x.\ dx}$$

and inverse fourier sine transform of $f(x)$ is given by

$$F^{-1}[f_S] \quad \boxed{f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \bar{f}_S(\lambda).\ \sin \lambda x\ d\lambda}$$

Fourier cosine Transform of $f(x)$

It is denoted and defined as

$$\boxed{F_C\{f(x)\} = \bar{f}_C(\lambda) = \int_{x=0}^{\infty} f(x) \cdot \cos \lambda x\ dx}$$

and ~~its~~ inverse fourier cosine transform of $f(x)$ is given by

$$F^{-1}(f_C) = \boxed{f(x) = \frac{2}{\pi} \int_{\lambda=0}^{\infty} \bar{f}_C(\lambda) \cdot \cos \lambda x\ d\lambda}$$

Ex : Find Fourier Transform of

$$f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a. \end{cases} \quad \text{where } a > 0$$

Hence show that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2$$

Soln : Given function is

$$f(x) = \begin{cases} a - |x|, & -a < x < a \\ 0, & \text{otherwise} \end{cases} \quad \text{--- (1)}$$

By defn of Fourier Transform,

$$F\{f(x)\} = \bar{f}(\lambda) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda x} \, dx$$

$$= \int_{-a}^{a} (a - |x|) \cdot e^{-i\lambda x} \, dx$$

$$\left(\int_{-\infty}^{\infty} = \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\infty} = 0 + \int_{-a}^{a} + 0 \right)$$

$$= \int_{-a}^{a} (a - |x|) (\cos \lambda x - i \sin \lambda x) \, dx$$

$$= \underbrace{\int_{-a}^{a} (a - |x|) \cdot \cos \lambda x \, dx}_{\text{(even function)}} - i \underbrace{\int_{-a}^{a} (a - |x|) \sin \lambda x \, dx}_{\text{(odd function)}}$$

$$= 2 \int_0^a (a - |x|) \cos \lambda x \, dx - 0$$

$$= 2 \int_0^a (a - x) \cos \lambda x \, dx \qquad \because |x| = x \text{ for +ve range}$$

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$$\overline{f(\lambda)} = 2\left[(a-x)\left(\frac{\sin\lambda x}{\lambda}\right) - (-1)\left(\frac{-\cos\lambda x}{\lambda^2}\right)\right]_0^a$$

$$= 2\left[(a-x)\left(\frac{\sin\lambda x}{\lambda}\right) - \frac{\cos\lambda x}{\lambda^2}\right]_0^a$$

$$= 2\left[\left(0 - \frac{\cos\lambda a}{\lambda^2}\right) - \left(0 - \frac{1}{\lambda^2}\right)\right]$$

$$\boxed{\overline{f(\lambda)} = \frac{2}{\lambda^2}(1-\cos\lambda a)}$$ This is Fourier Transform of $f(x)$

<u>Now</u>: By defn of <u>inverse fourier transform</u>

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f(\lambda)}\cdot e^{i\lambda x}\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{\lambda^2}(1-\cos\lambda a)\left(\cos\lambda x + i\sin\lambda x\right)d\lambda$$

$$= \frac{1}{\pi}\left[\underbrace{\int_{-\infty}^{\infty}\frac{(1-\cos\lambda a)}{\lambda^2}\cdot\cos\lambda x\,d\lambda}_{\text{even function}} + i\underbrace{\int_{-\infty}^{\infty}\frac{(1-\cos\lambda a)}{\lambda^2}\sin\lambda x\,d\lambda}_{\text{odd function}}\right]$$

$$= \frac{1}{\pi}\,2\int_{0}^{\infty}\frac{1-\cos\lambda a}{\lambda^2}\cdot\cos\lambda x\,d\lambda + i(0)$$

$$= \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{2\sin^2\left(\frac{\lambda a}{2}\right)}{\lambda^2}\right)\cdot\cos\lambda x\,d\lambda$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\frac{\sin^2\left(\frac{\lambda a}{2}\right)}{\lambda^2}\cos\lambda x\,d\lambda$$

$$\therefore \int_0^\infty \frac{\sin^2\left(\frac{\lambda a}{2}\right)}{\lambda^2} \cos \lambda x \, d\lambda = \pi/4 \, f(x)$$

Now put $x = 0$, we get

$$\int_0^\infty \frac{\sin^2(\lambda a/2)}{\lambda^2} \, d\lambda = \pi/4 \, f(0)$$

$f(x) = a - |x|$ for $-a < x < a$

$\therefore \boxed{f(0) = a}$

$$= \pi/4 \, (a)$$

$$\Rightarrow \boxed{\int_0^\infty \frac{\sin^2(\lambda a/2)}{\lambda^2} \, d\lambda = \frac{\pi a}{4}} \longrightarrow (2)$$

put $\lambda a/2 = x \Rightarrow d\lambda = \frac{2}{a} dx$

as $\lambda \to 0 \Rightarrow x \to 0$

$\lambda \to \infty \Rightarrow x \to \infty$

$$\therefore (2) \Rightarrow \int_0^\infty \frac{\sin^2 x}{(4x^2/a^2)} \frac{2}{a} dx = \pi/4 \, a$$

$$\Rightarrow \frac{a}{2} \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/4 \, a$$

$$\Rightarrow \boxed{\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \pi/2}$$

H.P.

Ex. Find the fourier Transform of

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

~~Also~~, Hence evaluate

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) \cos(x/2)\, dx.$$

Soln: Given function is $f(x) = \begin{cases} 1-x^2 & -1 \le x \le 1 \\ 0 & \text{, otherwise} \end{cases}$ — (1)

By defn of Fourier Transform.

$$f(\lambda) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\lambda x}\, dx$$

$$= \int_{-1}^{1} (1-x^2) \cdot (\cos\lambda x - i\sin\lambda x)\, dx$$

$$= \int_{-1}^{1} (1-x^2)\cos\lambda x\, dx - i\int_{-1}^{1}(1-x^2)\cdot\sin\lambda x\, dx$$

even function   odd function

$$= 2\int_0^1 (1-x^2)\cos\lambda x\, dx - i(0)$$

$$f(\lambda) = 2\left[(1-x^2)\frac{\sin\lambda x}{\lambda} - (-2x)\left(\frac{-\cos\lambda x}{\lambda^2}\right) + (-2)\left(\frac{-\sin\lambda x}{\lambda^3}\right)\right]_0^1$$

$$= 2\left[(1-x^2)\frac{\sin\lambda x}{\lambda} - 2x\cdot\frac{\cos\lambda x}{\lambda^2} + 2\frac{\sin\lambda x}{\lambda^3}\right]_0^1$$

$$\cancel{= 2\left[\left(0 - \frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right) - \right.}$$

$$= 2\left[\left(0 - 2\frac{\cos\lambda}{\lambda^2} + 2\frac{\sin\lambda}{\lambda^3}\right) - (0)\right] = \frac{4}{\lambda^3}\left[\sin\lambda - \lambda\cos\lambda\right]$$

$$\bar{f}(\lambda) = \frac{4}{\lambda^3}\left(\sin\lambda - \lambda\cos\lambda\right)$$

Now. By defn. of inverse Fourier Transform. we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\lambda)\cdot e^{i\lambda x}\cdot d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\lambda^3}\left(\sin\lambda - \lambda\cos\lambda\right)\left(\cos\lambda x + i\sin\lambda x\right) d\lambda$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \underline{\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x}\, d\lambda + \frac{2i}{\pi}\int_{-\infty}^{\infty} \underline{\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\sin\lambda x}\, d\lambda$$

even function of λ odd function of λ

$$= \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x + 0$$

$$(x) = \frac{4}{\pi}\int_{0}^{\infty} \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x\, d\lambda$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x\cdot d\lambda = \frac{\pi}{4} f(x).$$

How putting $x = 1/2$ we get.

$$\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\frac{\lambda}{2}\cdot d\lambda = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\, dx = \frac{\pi}{4}\cdot\frac{3}{4}$$

$\because$ $1/2$ lies in $-1<x<$ ...

when $f(x) = 1 - x^2$

$= 1 - \left(\frac{1}{2}\right)^2$

$= 1 - \frac{1}{4} =$ ...

$$\therefore \int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos\frac{x}{2}\, dx = \frac{3\pi}{16}$$

HW

(1) Find the fourier transform of

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$$

(2) Find the fourier transform of

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

hence evaluate $\int_0^\infty \frac{\sin ax}{x}\,dx$ & $\int_0^\infty \frac{\sin x}{x}\,dx$

Ex: Find the fourier sine transform of

i) $f(x) = \begin{cases} x , & 0 \le x \le 1 \\ 1 ; & 1 < x \le 2 \\ 0 , & x > 2 \end{cases}$

Soln :→ By defn of fourier sine transform, we have

$$F_s\{f(x)\} = \int_0^\infty f(x) \sin \lambda x \, dx$$

$$= \int_0^1 x \sin\lambda x \, dx + \int_1^2 1 \cdot \sin\lambda x \, dx + \int_2^\infty 0 \cdot \sin\lambda x \, dx$$

$$= \left[x\left(-\frac{\cos\lambda x}{\lambda}\right) - (1)\, \frac{-\sin\lambda x}{\lambda^2}\right]_0^1 + \left(-\frac{\cos\lambda x}{\lambda}\right)_1^2 + 0$$

$$= -\frac{\cos\lambda}{\lambda} + \frac{\sin\lambda}{\lambda^2} - \frac{\cos 2\lambda}{\lambda} + \frac{\cos\lambda}{\lambda}$$

$$= \frac{\sin\lambda - \lambda\cos 2\lambda}{\lambda^2}$$

Ex :→ Find the fourier sine transform of

$$f(x) = e^{-|x|} , \quad -\infty < x < \infty$$

$\frac{2}{\pi} = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}}$ Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$

Soln: By defn of Fourier sine transform, we have $\sqrt{2/\pi}$

$$F_s[f(x)] = \bar{f_s}(\lambda) = \int_0^\infty f(x) \sin\lambda x \, dx$$

$$= \int_0^\infty e^{-|x|} \sin\lambda x \, dx$$

$$= \int_0^\infty e^{-x} \sin \lambda x \, dx$$ $\left[\because \text{ when } x > 0, \; |x| = x \right.$

$$= L\{\sin \lambda x\}_{s=1}$$

$$F_s\{f(x)\} = \frac{\lambda}{1+\lambda^2}$$

$\because \int_0^\infty e^{-ax} \sin bx \, dx = L\{\sin bx\}_{s=a} = \frac{b}{a^2+b^2}$

Now, By definition of inverse Fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\lambda) \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} \, d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\lambda \sin \lambda x}{1+\lambda^2} \, d\lambda = \frac{\pi}{2} f(x)$$

$$= \pi/2 \; e^{-|x|}$$

putting $x = m$, we get

$$\int_0^\infty \frac{\lambda \sin m\lambda}{1+\lambda^2} \, d\lambda = \frac{\pi}{2} e^{-|m|}$$

Now Replace λ by x

$$\Rightarrow \int_0^\infty \frac{x \sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-|m|}$$

Ex: Find the Fourier cosine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

Soln: By defn. of Fourier cosine transform, we have

$$F_c[f(x)] = \int_0^\infty f(x) \cdot \cos\lambda x \, dx$$

$$= \int_0^1 x \cos\lambda x \, dx + \int_1^2 (2-x)\cos\lambda x \, dx + \int_2^\infty 0 \cos\lambda x \, dx$$

$$= \left[x \frac{\sin\lambda x}{\lambda} - (1) \cdot \frac{-\cos\lambda x}{\lambda^2}\right]_0^1 + \left[(2-x)\frac{\sin\lambda x}{\lambda} - (-1)\frac{-\cos\lambda x}{\lambda^2}\right]_1^2 + 0$$

$$= \cancel{\frac{1}{\lambda}\sin\lambda} + \frac{1}{\lambda^2}\cos\lambda - \frac{1}{\lambda^2} - \frac{1}{\lambda^2}\cos 2\lambda \cancel{- \frac{1}{\lambda}\sin\lambda} + \frac{1}{\lambda^2}\cos\lambda$$

$$F_c[f(x)] = \frac{2\cos\lambda - \cos 2\lambda - 1}{\lambda^2}$$

Ex. Find the Fourier cosine transform of

$$f(x) = e^{-x^2}, \quad \text{given that } \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Soln : By defn of Fourier cosine transform we have

$$F_c\{f(x)\} = \bar{f}_c(\lambda) = \int_0^\infty f(x) \cdot \cos \lambda x \, dx$$

$$\Rightarrow \bar{f}_c(\lambda) = \int_0^\infty e^{-x^2} \cos \lambda x \cdot dx \quad \text{——— (1)}$$

Differentiating both sides of eqn (1) w.r.to λ, we get

$$\frac{d}{d\lambda} \bar{f}_c(\lambda) = \frac{d}{d\lambda} \int_0^\infty e^{-x^2} \cos \lambda x \, dx$$

$$= \int_0^\infty e^{-x^2} \left(\frac{\partial}{\partial \lambda} \cos \lambda x \right) dx$$ [∵ by Leibnitz's rule for differentiation under the integral sign.]

$$= \int_0^\infty e^{-x^2} \cdot (-x \sin \lambda x) \, dx$$

$$= \frac{1}{2} \int_0^\infty \sin \lambda x \cdot \left[-2x e^{-x^2} \right] dx \quad \text{(\# note)}$$

$$= \frac{1}{2} \int_0^\infty \underset{\text{I}}{\sin \lambda x} \cdot \underset{\text{II}}{\left(\frac{d}{dx} e^{-x^2} \right)} dx$$

Now integrating by parts, we get

$$= \frac{1}{2}\left\{\left[\sin\lambda x \cdot e^{-x^2}\right]_0^\infty - \int_0^\infty \lambda\cos\lambda x \cdot e^{-x^2}\,dx\right\}$$

$$= \frac{1}{2}\left\{0 - \lambda\int_0^\infty e^{-x^2}\cos\lambda x\,dx\right\}$$

$$\frac{d}{d\lambda}\bar{f}_c(\lambda) = \frac{1}{2}\left\{-\lambda\bar{f}_c(\lambda)\right\} \quad \because \text{ by eq}^n \text{ (1)}$$

Now separating the variables and integrating

$$\int \frac{d\bar{f}_c(\lambda)}{\bar{f}_c(\lambda)} = \frac{-1}{2}\int \lambda\,d\lambda$$

$$\Rightarrow \log\bar{f}_c(\lambda) = -\frac{\lambda^2}{4} + \log c_1$$

$$\Rightarrow \log\frac{\bar{f}_c(\lambda)}{c_1} = \frac{-\lambda^2}{4}$$

$$\Rightarrow \bar{f}_c(\lambda) = c_1 e^{-\lambda^2/4} \quad \text{—— (2)}$$

To find c_1, we use eq^n (1) and (2)

we choose. $\lambda = 0$, (1) $\Rightarrow$ $\bar{f}_c(0) = \int_0^\infty e^{-x^2}\cos 0\,dx$

$$= \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \quad \text{(given)}$$

and (2) $\Rightarrow$ $\bar{f}_c(0) = c_1$

$$\therefore c_1 = \sqrt{\pi}/2$$

Hence eq^n (2) become, $\bar{f}_c(\lambda) = \frac{\sqrt{\pi}}{2}\cdot e^{-\lambda^2/4}$

i.e. $F_c\{e^{-x^2}\} = \frac{\sqrt{\pi}}{2}e^{-\lambda^2/4}$

OR Find value of f(x) for the following integral eqns

Ex: Solve for $f(x)$, the Integral equation

(i) $$\int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} -\lambda & , 0<\lambda<1 \\ 1 & 1<\lambda<2 \\ 0 & \text{otherwise} \end{cases}$$

Ex

(ii) $$\int_0^\infty f(x)\cos\alpha x\,dx = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$$

Hence evaluate. $$\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$$

Soln. (i) Given that

$$\bar{f}_s(\lambda) = \int_0^\infty f(x)\sin\lambda x\,dx = \begin{cases} -\lambda, & 0<\lambda<1 \\ 1, & 1<\lambda<2 \\ 0, & \text{otherwise} \end{cases}$$

By defn of inverse fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(\lambda)\cdot\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\int_0^1 -\lambda\sin\lambda x\,d\lambda + \int_1^2 1\cdot\sin\lambda x\,d\lambda\right]$$

$$= \frac{2}{\pi}\left\{\left[-\lambda\left(\frac{-\cos\lambda x}{x}\right) - (-1)\frac{-\sin\lambda x}{x^2}\right]_0^1 + \left[\frac{-\cos\lambda x}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\cos x}{x} - \frac{\sin x}{x^2} - \frac{\cos 2x}{x} + \frac{\cos x}{x}\right\}$$

$$f(x) = \frac{2}{\pi x^2}\left[x(2\cos x - \cos 2x) - \sin x\right]$$

(ii) Given

$$\bar{f}_c(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases} \quad \text{(1)}$$

By defn. of inverse fourier cosine transform, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_c(\alpha)\cdot\cos\alpha x\,d\alpha$$

$$= \frac{2}{\pi}\int_0^1 (1-\alpha)\cos\alpha x\,d\alpha$$

$$= \frac{2}{\pi}\left[(1-\alpha)\frac{\sin\alpha x}{x} - (-1)\frac{-\cos\alpha x}{x^2}\right]_0^1$$

$$= \frac{2}{\pi}\left(-\frac{\cos x}{x^2} + \frac{1}{x^2}\right)$$

$$f(x) = \frac{2(1-\cos x)}{\pi x^2} \quad \text{(2)}$$

Hence part: using (2) in eqn (1), we obtain

$$\bar{f}_c(\alpha) = \int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\cdot\cos\alpha x\,dx = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$$

$$\Rightarrow \frac{2}{\pi}\int_0^\infty \frac{2\sin^2 x/2}{x^2}\cdot\cos\alpha x\,dx = \begin{cases} 1-\alpha, & 0\le\alpha\le 1 \\ 0, & \alpha>1 \end{cases}$$

putting $\alpha=0$, we get

$$\frac{4}{\pi}\int_0^\infty \frac{\sin^2 x/2}{x^2}\,dx = 1$$

put $x/2 = t$
$\Rightarrow dx = 2dt$
as $x \to 0,\ t \to 0$
$x \to \infty,\ t \to \infty$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{4t^2}\cdot 2\,dt = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \pi/2$$

HW

① Find Fourier sine transform of $\frac{e^{-ax}}{x}$

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★ Parsevals Identity

(1) For Fourier Transform

let $F\{f(x)\} = f(\lambda)$ and $F\{g(x)\} = g(\lambda)$

Then

(i) $$\int_{-\infty}^{\infty} f(x)\cdot \overline{g(x)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)\cdot \overline{g(\lambda)}\, d\lambda \quad \left(\text{If } f(x) \neq g(x)\right)$$

(ii) $$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda \quad \left(\text{if } f(x) = g(x)\right)$$

Note that • If $g(x)$ is Real function then $g(x) = \overline{g(x)}$, (Here $\overline{g(x)}$ is conjugate of $g(x)$)

For ex. $g(x) = e^{-x} \Rightarrow \overline{g(x)} = e^{-x}$

• If $g(x)$ is imaginary then $\overline{g(x)}$ is conjugate of $g(x)$ ie $g(x) \neq \overline{g(x)}$

For eg: $g(x) = e^{-ix} \Rightarrow \overline{g(x)} = e^{ix}$

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(2) For Fourier sine Transform

let $F_s\{f(x)\} = f_s(\lambda)$ and $F_s\{g(x)\} = g_s(\lambda)$

Then

(i) $$\int_{0}^{\infty} f(x)\cdot \overline{g(x)}\, dx = \frac{2}{\pi}\int_{0}^{\infty} f_s(\lambda)\cdot \overline{g_s(\lambda)}\, d\lambda \quad \left(\text{If } f(x) \neq g(x)\right)$$

(ii) $$\int_{0}^{\infty} |f(x)|^2\, dx = \frac{2}{\pi}\int_{0}^{\infty} |f_s(\lambda)|^2\, d\lambda \quad \left(\text{if } f(x) \neq g(x)\right)$$

(3) For Fourier cosine Transform

let $F_c\{f(x)\} = f_c(\lambda)$ and $F_c\{g(x)\} = g_c(\lambda)$

Then

(i) $$\int_{0}^{\infty} f(x)\cdot \overline{g(x)}\, dx = \frac{2}{\pi}\int_{0}^{\infty} f_c(\lambda)\cdot \overline{g_c(\lambda)}\, d\lambda \quad \left(\text{if } f(x) \neq g(x)\right)$$

(ii) $$\int_{0}^{\infty} |f(x)|^2\, dx = \frac{2}{\pi}\int_{0}^{\infty} |f_c(\lambda)|^2\, d\lambda \quad \left(\text{if } f(x) = g(x)\right)$$

Note: ① If $f_s(\lambda) = \frac{\lambda}{s^2+\lambda^2}$ (where s is any const)

$$\Rightarrow \frac{\lambda}{s^2+\lambda^2} = f_s(\lambda) = \int_0^\infty f(t)\cdot \sin\lambda t \, dt \quad \text{(By def}^n \text{ F.S.T)} \quad \text{— ①}$$

also

$$\frac{\lambda}{s^2+\lambda^2} = L\{\sin\lambda t\} = \int_0^\infty e^{-st}\cdot \sin\lambda t \, dt \quad \text{(By def}^n \text{ of L.T)} \quad \text{— ②}$$

Comparing both eqn, we conclude that

$$\boxed{f(t) = e^{-st}}$$

∴ Whenever

$$f_s(\lambda) = \frac{\lambda}{s^2+\lambda^2} \Rightarrow f(t) = e^{-st}$$

For eg: If

$$f_s(\lambda) = \frac{\lambda}{4+\lambda^2} \Rightarrow f(t) = e^{-2t}$$

Note (ii): If $f_c(\lambda) = \frac{1}{s^2+\lambda^2}$ (where s is const.)

$$\Rightarrow \frac{1}{s^2+\lambda^2} = f_c(\lambda) = \int_0^\infty f(t)\cdot \cos\lambda t \, dt \quad \text{(}\because \text{By def}^n \text{ of F.C.T)} \quad \text{— ①}$$

also

$$\frac{1}{s^2+\lambda^2} = \frac{1}{s}\cdot\left[\frac{s}{s^2+\lambda^2}\right] = \frac{1}{s} L\{\cos\lambda t\}$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\cos\lambda t \, dt$$

$$= \int_{t=0}^\infty \left(\frac{e^{-st}}{s}\right)\cos\lambda t \, dt \quad \text{— ②}$$

Comparing eqn ① & ② we conclude that

$$f(t) = \frac{e^{-st}}{s}$$

Thus whenever

$$f_c(\lambda) = \frac{1}{s^2+\lambda^2} \Rightarrow f(t) = \frac{e^{-st}}{s}$$

For eg. let $f_c(\lambda) = \frac{1}{4+\lambda^2} \Rightarrow f(t) = \frac{e^{-2t}}{2}$

Ex: Using parsewal Identity, show that

$$\int_0^\infty \frac{t^2}{(4+t^2)\cdot(9+t^2)}\,dt = \pi/10$$

Soln: let $I = \int_{t=0}^{\infty} \frac{t^2}{(4+t^2)\cdot(9+t^2)}\,dt$

put $t=\lambda \Rightarrow dt = d\lambda$

and as $t=0 \Rightarrow \lambda=0$

as $t=\infty \Rightarrow \lambda=\infty$

$$\therefore I = \int_{\lambda=0}^{\infty} \frac{\lambda^2}{(4+\lambda^2)(9+\lambda^2)}\,d\lambda$$

$$= \int_0^\infty \left(\frac{\lambda}{4+\lambda^2}\right)\cdot\left(\frac{\lambda}{9+\lambda^2}\right) d\lambda$$

$$I = \int_0^\infty f_s(\lambda)\cdot g_s(\lambda)\,d\lambda \quad \text{——— (1)}$$

$\therefore$ By using parsewal identity for fourier sine Transform

$\left(\text{ie} \quad \int_0^\infty f(x)\cdot g(x)\,dx = \frac{2}{\pi}\int_0^\infty f_s(\lambda)\cdot g_s(\lambda)\,d\lambda\right)$

eqn (1) $\Rightarrow$

$$I = \frac{\pi}{2}\int_0^\infty f(x)\cdot g(x)\,dx \quad \text{——— (2)}$$

Now Here $f_s(\lambda) = \frac{\lambda}{4+\lambda^2} = \int_0^\infty f(t)\cdot\sin\lambda t\,dt$ ——— (3) ($\because$ By defn of F.S.T.)

also $\frac{\lambda}{4+\lambda^2} = \frac{\lambda}{2^2+\lambda^2} = L\{\sin\lambda t\}_{s=2} = \int_0^\infty e^{-2t}\cdot\sin\lambda t\,dt$ ——— (4)

Comparing both eqn's (3) & (4), we get $f(t) = e^{-2t}$

Similarly, Here $g_s(\lambda) = \frac{\lambda}{9+\lambda^2} \Rightarrow g(t) = e^{-3t}$.

Hence eqn (2) $\Rightarrow$

$$I = \pi/2 \int_0^\infty e^{-2x}\cdot e^{-3x}\,dx$$

$$I = \pi/2 \int_0^\infty e^{-5x}\, dx$$

$$= \pi/2 \left[\frac{e^{-5x}}{-5}\right]_0^\infty = \pi/2 \left[0 + \frac{1}{5}\right]$$

$$= \pi/10$$

Hence $I = \pi/10$

ie $$\int_0^\infty \frac{t^2}{(4+t^2)\cdot(9+t^2)}\, dt = \pi/10$$

H.P

Ex: Using parsewals identity Evaluate $$\int_0^\infty \frac{x^2}{(1+x^2)^2}\, dx$$

Soln: let $$I = \int_{x=0}^\infty \frac{x^2}{(1+x^2)^2}\, dx$$

put $x = \lambda \Rightarrow dx = d\lambda$

and as $x=0 \Rightarrow \lambda = 0$
$x = \infty \Rightarrow \lambda = \infty$

$$\therefore I = \int_0^\infty \frac{\lambda^2}{(1+\lambda^2)^2}\, d\lambda = \int_0^\infty \underbrace{\left(\frac{\lambda}{1+\lambda^2}\right)}_{f_s(\lambda)} \cdot \underbrace{\left(\frac{\lambda}{1+\lambda^2}\right)}_{g_s(\lambda)} d\lambda$$

$$= \int_0^\infty \cancel{f_s(\lambda)\cdot f_s(\lambda)}$$

since $f_s(\lambda) = g_s(\lambda)$
$\Rightarrow f(x) = g(x)$

$$= \int_0^\infty \left[\frac{\lambda}{1+\lambda^2}\right]^2 d\lambda$$

$$= \int_0^\infty |f_s(\lambda)|^2\, d\lambda$$

$$I = \frac{\pi}{2}\int_0^\infty |f(x)|^2\, dx \qquad (1)$$

(∵ By Parseval Identity of Fourier sine Transf

Here $f_s(\lambda) = \frac{\lambda}{1+\lambda^2} = \int_0^\infty f(t) \cdot \sin \lambda t \, dt$ (∵ By defn of F.S.T) —②

also $\frac{\lambda}{1+\lambda^2} = L\{\sin \lambda t\}_{S=1} = \int_0^\infty e^{-t} \cdot \sin\lambda t \, dt$ —③

comparing eqn ② & ③ we get

$$f(t) = e^{-t} \Rightarrow f(x) = e^{-x}$$

∴ ① ⇒ $I = \frac{\pi}{2}\int_0^\infty |f(x)|^2 dx = \pi/2 \int_0^\infty |e^{-x}|^2 dx$

$= \pi/2 \int_0^\infty e^{-2x} dx$

$= \pi/2 \left[\frac{e^{-2x}}{-2}\right]_0^\infty = \pi/2 \left[0 + \frac{1}{2}\right] = \pi/4$

$$\therefore \boxed{I = \int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \pi/4}$$

Ex: Using parseval identity

evaluate $\int_0^\infty \frac{1}{(a^2+x^2)\cdot(b^2+x^2)} dx$ where a & b are const.

Soln: let $I = \int_{x=0}^\infty \frac{1}{(a^2+x^2)\cdot(b^2+x^2)} dx$

put $x = \lambda \Rightarrow dx = d\lambda$

as $x = 0 \Rightarrow \lambda = 0$

$x = \infty \Rightarrow \lambda = \infty$

$\therefore I = \int_0^\infty \frac{1}{(a^2+\lambda^2)\cdot(b^2+\lambda^2)} d\lambda$

$= \frac{1}{ab} \cdot \int_0^\infty \left(\frac{a}{a^2+\lambda^2}\right)\cdot\left(\frac{b}{a^2+\lambda^2}\right) d\lambda$ (dividing and multiplying by ab.)

$= \frac{1}{ab}\left[\int_0^\infty f_c(\lambda)\cdot g_c(\lambda)\, d\lambda\right]$

$$I = \frac{1}{ab} \cdot \left[\frac{\pi}{2} \int_0^\infty f(x) \cdot g(x)\, dx \right]$$ (∵ By parsewal identity For fourier cosine Term) —— ①

Here $f_c(\lambda) = \frac{a}{a^2+\lambda^2} = \int_0^\infty f(t) \cdot \cos \lambda t \; dt$ —— ②

also $\frac{a}{a^2+\lambda^2} = L\{\cos \lambda t\}_{s=a} = \int_0^\infty e^{-at} \cdot \cos \lambda t \, dt$ —— ③

comparing eqn ② & ③, we get

$$f(t) = e^{-at} \Rightarrow f(x) = e^{-ax}$$

Similarly $g_c(\lambda) = \frac{b}{b^2+\lambda^2} \Rightarrow g(t) = e^{-bt}$

i.e. $g(x) = e^{-bx}$

From eqn ①

$$I = \frac{1}{ab} \cdot \frac{\pi}{2} \int_0^\infty f(x) \cdot g(x)\, dx$$

$$= \frac{\pi}{2ab} \int_0^\infty e^{-ax} \cdot e^{-bx}\, dx$$

$$= \frac{\pi}{2ab} \left[\int_0^\infty e^{-(a+b) \cdot x}\, dx \right]$$

$$= \frac{\pi}{2ab} \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^\infty = \frac{\pi}{2ab} \left[0 + \frac{1}{(a+b)} \right]$$

$$= \frac{\pi}{2ab \cdot (a+b)} = \frac{\pi}{2a^2 b + 2ab^2}$$

i.e. $$\boxed{\int_0^\infty \frac{1}{(a^2+x^2)(b^2+x^2)}\, dx = \frac{\pi}{2ab\,(a+b)}}$$

Ex: Using parsewal identity

show that $\int_0^\infty \frac{1}{(1+t^2)^2}\, dt = \pi/4$

Soln: let $I = \int_{t=0}^{\infty} \frac{1}{(1+t^2)^2}\, dt$

put $t = \lambda \Rightarrow dt = d\lambda$

and as $t=0 \Rightarrow \lambda=0$

$t=\infty \Rightarrow \lambda=\infty$

$$\therefore I = \int_{\lambda=0}^{\infty} \frac{1}{(1+\lambda^2)^2}\, d\lambda$$

$$= \int_{\lambda=0}^{\infty} \left[\frac{1}{(1+\lambda^2)}\right]^2 d\lambda .$$

$$= \int_0^\infty |f_c(\lambda)|^2\, d\lambda \quad , \quad \left(\text{where } f_c(\lambda) = \frac{1}{1+\lambda^2}\right)$$

$$I = \frac{\pi}{2}\left[\int_0^\infty |f(x)|^2\, dx\right] \quad \text{——— (1)}$$

$\left(\because\right.$ By Parsewal identity of fourier cosine Transform$\left.\right)$

Here $f_c(\lambda) = \frac{1}{1+\lambda^2} = \int_0^\infty f(t)\cdot \cos\lambda t\, dt$ ——— (2) ;

also $\frac{1}{1+\lambda^2} = L\{\cos\lambda t\}_{s=1} = \int_0^\infty e^{-t}\cdot\cos\lambda t\, dt$ ——— (3)

Comparing eqn (2) & (3), we get

$$f(t) = e^{-t} \Rightarrow f(x) = e^{-x}$$

$\therefore$ eqn (1) $\Rightarrow$ $I = \pi/2 \int_0^\infty (e^{-x})^2\, dx$

$$= \pi/2 \int_0^\infty e^{-2x}\, dx = \pi/2 \left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$= \pi/2\left[0 + \frac{1}{2}\right] = \pi/4$$

$$\boxed{I = \int_0^\infty \frac{1}{(1+t^2)^2}\, dt = \pi/4}$$

H.P.

Ex: Using Parseval identity on function

$$f(x) = \begin{cases} 1-|x| , & |x| \le 1 \\ 0 , & |x| > 1 \end{cases}$$

Find the value of $\int_0^\infty \frac{\sin^4 x}{x^4} dx$

Soln: (Note. To use Parseval identity we should find Fourier Transform of given function first).

Given function is

$$f(x) = \begin{cases} 1-|x| , & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{--- (1)}$$

∴ By defn of Fourier Transform.

$$f(\lambda) = F\{f(x)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\lambda t} dt$$

$$\Rightarrow f(\lambda) = \int_{-1}^{1} (1-|t|) e^{-i\lambda t} dt$$

$$= \int_{-1}^{1} (1-|t|) (\cos\lambda t - i \sin\lambda t) dt$$

$$= \left[\int_{-1}^{1} (1-|t|) \cos\lambda t \, dt \right] - i \left[\int_{-1}^{1} (1-|t|) \sin\lambda t \right] dt$$

(Even function) (odd function)

$$= 2 \int_0^1 (1-t) \cos\lambda t \, dt + 0$$

$$\therefore f(\lambda) = 2 \left[(1-t) \frac{\sin\lambda t}{\lambda} - (-1) \left(\frac{-\cos\lambda t}{\lambda^2} \right) \right]_0^1$$

$$= 2 \left[\frac{(1-t)\sin\lambda t}{\lambda} - \frac{\cos\lambda t}{\lambda^2} \right]_0^1$$

$$f(\lambda) = 2 \left[\left(0 - \frac{\cos\lambda}{\lambda^2} \right) - \left(0 - \frac{1}{\lambda^2} \right) \right]$$

$$f(\lambda) = 2\left[\frac{1-\cos\lambda}{\lambda^2}\right] = 2\left[\frac{2\sin^2\lambda/2}{\lambda}\right]$$

$$\Rightarrow \boxed{f(\lambda) = 4\,\frac{\sin^2(\lambda/2)}{\lambda^2}} \quad \text{—— (2)}$$

Now. By using parsewal identity for Fourier Transform. we have

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\lambda)|^2\,d\lambda$$

$$\Rightarrow \int_{-1}^{1} \underbrace{(1-|x|)^2}_{\text{even function.}}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{4\left(\frac{\sin^2(\lambda/2)}{\lambda^2}\right)^2}_{\text{even function}}\cdot d\lambda$$

$$\Rightarrow 2\int_0^1 (1-x)^2\,dx = \frac{1}{2\pi}\,2\int_0^{\infty}\left(4\,\frac{\sin^2(\lambda/2)}{\lambda^2}\right)^2 d\lambda$$

$$\Rightarrow 2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{16}{\pi}\int_0^{\infty}\frac{\sin^4(\lambda/2)}{\lambda^4}\,d\lambda$$

$$\Rightarrow \frac{2}{-3}\left[(1-x)^3\right]_0^1 = \frac{16}{\pi}\int_0^{\infty}\frac{\sin^2(\lambda/2)}{\lambda^2}\,d\lambda$$

$$\Rightarrow \int_0^{\infty}\frac{\sin^4(\lambda/2)}{\lambda^4}\,d\lambda = \frac{\pi}{16}\cdot\left(\frac{-2}{3}\right)[0-1] = \frac{\pi}{24}$$

$$\Rightarrow \int_{\lambda=0}^{\infty}\frac{\sin^4(\lambda/2)}{\lambda^4}\,d\lambda = \pi/24 \quad \text{—— (3)}$$

Now put $\lambda/2 = x \Rightarrow d\lambda = 2dx$

as $\lambda=0 \Rightarrow x=0$

$\lambda=\infty \Rightarrow x=\infty$

$$\therefore (3) \Rightarrow \int_{x=0}^{\infty}\frac{\sin^4 x}{(2x)^4}\cdot 2dx = \pi/24$$

$$\Rightarrow \int_{x=0}^{\infty}\frac{\sin^4 x}{x^4}\left(\frac{2}{2^4}\right)dx = \pi/24$$

$$\Rightarrow \frac{1}{8}\int_{x=0}^{\infty}\frac{\sin^4 x}{x^4}\,dx = \pi/24 \Rightarrow \boxed{\int_0^{\infty}\frac{\sin^4 x}{x^4}\,dx = \pi/3}$$

Ex: Verify Parsewals identity for the function

$$f(x) = \begin{cases} 0 & x<0 \\ e^{-x}, & x>0 \end{cases}$$

Soln: ~~I~~ Since $f(x)$ is real function (and also $g(x)$ is not given) (i.e. $f(x)=g(x)$)

~~Hence To very Verify parselwa~~

Hence to very parsewals identity we need to prove that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\lambda)|^2 d\lambda$$

let $f(x) = \begin{cases} 0, & x<0 \text{ ie } (-\infty<x<0) \\ e^{-x}, & x>0 \text{ ie } (0<x<\infty) \end{cases}$

$\therefore$ ~~By~~

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) (f(x))^2 dx$$

$$= \int_{0}^{\infty} (e^{-x})^2 dx = \int_{0}^{\infty} e^{-2x} dx$$

$$= \left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = \left[0+\frac{1}{2}\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = 1/2 \quad \text{--- (1)}$$

Now. By defn of Fourier transform we have

$$f(\lambda) = F\{f(x)\} = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\lambda t} dt$$

$$\Rightarrow f(\lambda) = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) f(t)\cdot e^{-i\lambda t}\, dt$$

$$= \int_0^\infty (e^{-t})\, e^{-i\lambda t}\, dt$$

$$= \int_0^\infty e^{-(1+i\lambda)t}\, dt$$

$$= \left[\frac{e^{-(1+i\lambda)t}}{-(1+i\lambda)}\right]_0^\infty = \left(0 + \frac{1}{1+i\lambda}\right)$$

$$\therefore f(\lambda) = \frac{1}{1+i\lambda}$$

Since $|f(\lambda)|^2 = f(\lambda)\cdot \overline{f(\lambda)} = \left(\frac{1}{1+i\lambda}\right)\cdot\left(\frac{1}{1-i\lambda}\right)$

$$\Rightarrow \boxed{|f(\lambda)|^2 = \frac{1}{1+\lambda^2}}$$

Now $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{1+\lambda^2}\right) d\lambda$ (even function)

$$= \frac{1}{2\pi}\cdot 2\int_0^\infty \left(\frac{1}{1+\lambda^2}\right) d\lambda$$

$$= \frac{1}{\pi}\left[\tan^{-1}\lambda\right]_0^\infty = \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= 1/\pi\,[\pi/2 - 0]$$

$$\Rightarrow \boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda = 1/2} \quad \text{——— (2)}$$

from eqn (1) & (2)

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\lambda)|^2\, d\lambda .$$

This is parsevals identity   Hence verified.

Example for practice

① using parseval identity evaluate following integral

① $$\int_0^\infty \frac{x^2}{(25+x^2)^2}\,dx$$

② $$\int_0^\infty \frac{1}{(3+x^2)^2}\,dx$$

③ $$\int_0^\infty \frac{1}{(4+t^2)(9+t^2)}\,dt$$

④ $$\int_0^\infty \frac{t^2}{(36+t^2)(16+t^2)}\,dt$$