2. Laplace Transform

Laplace Transform $f(t)$ is defined by $L\{f(t)\} =$

$$= \int_0^\infty e^{-st} f(t)\, dt$$

$$= F(s)$$

$$L(t) = 1/s$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\cos^2 t) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$$

$$L(\sin^2 2t) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$L(\sin at) = \frac{a}{s^2+a^2} \qquad L^{-1}\{e^{-2t}\} = \frac{1}{s+2}$$

$$L(\sin 3t) = \frac{3}{s^2+9}$$

$$L(\sinh at) = \frac{e^{at} - e^{-at}}{2}$$

1) Find Laplace transform of $L\{\sin 3t . \cos 2t\}$

$$\sin at . \cos bt = \frac{1}{2}\left[\sin(a+b)t + \sin(a-b)t\right]$$

$$= \frac{1}{2}(\sin 5t + \sin t)$$

$$\therefore L\{\sin 3t . \cos 2t\} = \frac{1}{2} L\{\sin 5t + \sin t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+25} + \frac{1}{s^2+1}\right]$$

Properties of Laplace Transform

1) Change of Scale Property

if $l\{f(t)\} = F(s)$,

then $l\{f(at)\} = \frac{1}{a} F(s/a)$

Ex: If $l\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, find $l\{J_0(3t)\}$

We have, $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$ and $a=3$

and $J(s) = \frac{1}{\sqrt{s^2+1}}$

We know that,

$$l\{f(at)\} = \frac{1}{a} F(s/a)$$

$$\therefore l\{J_0(3t)\} = \frac{1}{3} \cdot \frac{1}{\sqrt{(s/3)^2+1}}$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{\frac{s^2}{9}+1}} = \frac{\frac{1}{3} \cdot \frac{1}{\sqrt{s^2+9}}}{3} = \frac{1}{\sqrt{s^2+9}}$$

$$= \frac{1}{\sqrt{s^2+9}}$$

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Ex. If $l\{\cos t\} = \frac{s}{s^2+1}$, find $l\{\cos 3t\}$

We have $a = 3$, $F(s) = \frac{s}{s^2+1}$

$$\frac{1}{a} F(s/a) = \frac{1}{a} \frac{s/a}{(s/a)^2 + 1}$$

$$= \frac{1}{3} \frac{s/3}{(s/3)^2+1}$$

$$= \frac{1}{\cancel{9}} \frac{s}{\frac{s^2+9}{\cancel{9}}}$$

$$= \frac{s}{s^2+9}$$

2) First shifting property:

if $l\{f(t)\} = F(s)$,

then $l\{e^{\pm at} f(t)\} = F(s \mp a)$

Find $l\{e^{at}\}$, $l\{e^{4t} \sin 3t\}$, $l\{\sinh at . \cos at\}$

$\Rightarrow$ $l\{e^{at}\} = ?$

We have, $l\{e^{at} f(t)\} = l\{e^{at} . 1\}$

$a = a$, $f(t) = 1$

$$l\{f(t)\} = l\{1\} = 1/s$$

$$F(s) = \frac{1}{s}$$

$$F(s-a) = \frac{1}{s-a}$$

We know that,

$$l\{e^{at} f(t)\} = F(s-a)$$

$$l\{e^{at}\} = \frac{1}{s-a}$$

$\Rightarrow$ $l\{e^{4t} \sin 3t\} = ?$

We have $l\{e^{at} f(t)\} = l\{e^{4t} \sin 3t\}$

$$a = 4 \quad , \quad f(t) = \sin 3t$$

$$l\{f(t)\} = l\{\sin 3t\}$$

$$= \frac{3}{s^2+9}$$

$$F(s-a) = \frac{3}{(s-4)^2+9}$$

$$F(s-4) = \frac{3}{(s-4)^2+9}$$

$$\therefore l(e^{4t} \sin 3t) = \frac{3}{(s-4)^2+9}$$

$\Rightarrow$ $l\{\sinh at . \cos at\} = ?$

$$\sinh at . \cos at$$

$$= \frac{e^{at} - e^{-at}}{2} . \cos at$$

$$= \frac{1}{2}\left[e^{at} \cos at - e^{-at} \cos at\right] \quad \ldots (1)$$

$$l\{e^{at} \cos at\} = l\{e^{at} f(t)\}$$

$$f(t) = \cos at \quad , \quad a = a$$

$L\{f(t)\} = L(\cos at)$

$$F(s) = \frac{s}{s^2+a^2}$$

$$F(s-a) = \frac{s-a}{(s-a)^2+a^2}$$

$$\therefore L\{e^{at}\cos at\} = \frac{s-a}{(s-a)^2+a^2}$$

Similarly,

$$L\{e^{-at}\cos at\} = \frac{s+a}{(s+a)^2+a^2}$$

from ①,

$$L\{\sinh at \cdot \cos at\} = \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{s+a}{(s+a)^2+a^2}\right]$$

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Homework

1) Evaluate $\int_0^\infty e^{st^2}\,dt$ and hence show that

$\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$

put $t^2 = u$ and $dt = \frac{du}{2t} = \frac{du}{2\sqrt{u}}$

when $t=0,\ u=0$ and
$t=\infty,\ u=\infty$

$$\therefore \int_0^\infty e^{-st^2}\,dt = \int_0^\infty e^{-su}\frac{1}{2\sqrt{u}}\,du$$

$$= L\left\{\frac{1}{2\sqrt{u}}\right\}$$

$$= \frac{1}{2}L\left\{u^{-1/2}\right\}$$

$$= \frac{1}{2}\frac{\Gamma(-1/2)+1}{s^{(-1/2)+1}}$$

$$= \frac{1}{2}\frac{\Gamma 1/2}{\sqrt{s}}$$

$$= \frac{1}{2}\sqrt{\pi/s}$$

putting $s=1$, we get

$$\int_0^\infty e^{-st^2}\,dt = \frac{\sqrt{\pi}}{2}$$

2) Given $L\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$, show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

Let $f(t) = 2\sqrt{\frac{t}{\pi}}$

$$\therefore \bar{f}(s) = L\{f(t)\} = \frac{1}{s^{3/2}}$$

Now $f'(t) = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{\sqrt{\pi t}}$

We have,

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = s \cdot \frac{1}{s^{3/2}} - 0$$

$$\therefore L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$$

3) Find L.T. of $t^2 \cos 3t \cdot e^{-t}$

We have $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} \bar{f}(s)$

Let $f(t) = \cos 3t$

$$\therefore \bar{f}(s) = \frac{s}{s^2+9}$$

$$\therefore L\{t^2 \cos 3t\} = \frac{d^2}{ds^2}\left\{\frac{s}{s^2+9}\right\} = \frac{d}{ds}\left\{\frac{(s^2+9)(1) - s(2s)}{(s^2+9)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{9-s^2}{(s^2+9)^2}\right\}$$

$$= \frac{2s(s^2-27)}{(s^2+9)^2}$$

we get,

$$L\{e^{-t}\cdot t^2 \cos 3t\} = \frac{2(s+1)\left((s+1)^2 - 27\right)}{\left[(s+1)^2 + 9\right]^3}$$

4) Find L.T. of $\int_0^t e^t \frac{\sin t}{t}\, dt$

$$L\left\{\int_0^t f(u)\, du\right\} = \frac{\bar{f}(s)}{s}$$

Here,

$f(t) = e^t \frac{\sin t}{t}$ and $\bar{f}(s) = L\left\{e^t \frac{\sin t}{t}\right\}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s$$

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5) Find L.T. of $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

Let $g(t) = \frac{\sin t}{t}$

$\bar{g}(s) = L\{g(t)\} = L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s$

We have, $L\left\{\frac{d}{dt}g(t)\right\} = s.\bar{g}(s) - g(0)$

where $g(0) = \lim_{t\to 0} g(t) = \lim_{t\to 0} \frac{\sin t}{t} = 1$

$\therefore\ L\left\{\frac{d}{dt}\frac{\sin t}{t}\right\} = s\cot^{-1}s^{-1}$

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Inverse Laplace Transform

1) convolution theorem

if $L^{-1}\{F(s)\} = f(t)$ and

$L^{-1}\{G(s)\} = g(t)$

then $L^{-1}\{F(s)\,G(s)\} = \int_0^t f(x)\,g(t-x)\,dx$

$= f * g$

$= \int_0^t g(x)\,f(t-x)\,dx = g * t$

Q.1 Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ by convolution theorem method.

To find:

$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

$= \frac{s}{(s^2+a^2)^2} = \frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}$

Let, $F(s) = \frac{s}{(s^2+a^2)}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$f(t) = \cos at$

$f(x) = \cos ax$

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Let $G(s) = \frac{1}{(s^2+a^2)} = \frac{1}{a} \times \frac{a}{(s^2+a^2)}$

$$L^{-1}\{G(s)\} = \frac{1}{a} L^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$

$$g(t) = \frac{1}{a} \sin at$$

$$g(t-x) = \frac{1}{a} \sin(at - ax)$$

By Convolution theorem,

we know that,

$$L^{-1}\{F(s)\, G(s)\} = \int_0^t f(x)\, g(t-x)\, dx$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos ax \cdot \frac{1}{a} \sin(at - ax)\, dx$$

$$= \frac{1}{a} \int_0^t \sin(at - ax) \cos ax\, dx$$

~~$\sin(at-ax)$~~

$$\sin at . \cos bt = \frac{1}{2}[\sin(a+b)t + \sin(a-b)t]$$

$$\sin(at-ax) \cdot \cos(ax) = \frac{1}{2}[\sin at + \sin(at - 2ax)]$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} \int_0^t [\sin at + \sin(at - 2ax)]\, dx$$

$$= \frac{1}{2a}\left[\int_0^t \sin at \cancel{dt}\, dx + \int_0^t \sin(at - 2ax)\, dx\right]$$

Formula:- $\int e^{ax}\sin bx\,dx$
$= \frac{e^{ax}}{a^2+b^2}(a\sin bx \cancel{dx} - b\cos bx)$

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$= \frac{1}{2a}\left[\sin at\,(x)_0^t - \frac{\cos(at-2ax)_0^t}{-2a}\right]$ $at-2at-at=$

$\cos[(at-2at)-(at-0)]=$

$= \frac{1}{2a}\left\{\sin at\,[t-0] + \frac{1}{2a}[\cos(at) - \cos(at)]\right\}$

$= \frac{1}{2a}\left\{\sin at\,[t-0] + \frac{1}{2a}[\cos(at) - \cos(at)]\right\}$

$= \frac{1}{2a}\, t\sin at$

$= \frac{t\sin at}{2a}$

Q.2 Find $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

$\frac{1}{(s+1)(s^2+1)} = \frac{1}{(s^2+1)} \cdot \frac{1}{(s+1)} = F(s)\cdot G(s)$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$f(t) = \sin t$
$f(x) = \sin x$

Let, $G(s) = \frac{1}{(s+1)}$

~~Let~~ $L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}$

$g(t) = e^{-t}$

$$g(t-x) = e^{-(t-x)}$$

$$= e^{-t} \cdot e^{x}$$

By Convolution theorem,

$$L^{-1}\{F(s)\,G(s)\} = \int_0^t f(x)\, g(t-x)\, dx$$

$$= \int_0^t \sin x \cdot e^{-t} \cdot e^{x}\, dx$$

$$= e^{-t} \int_0^t e^{x} \sin x\, dx$$

$$= e^{-t}\left[\frac{e^{x}}{1^2+1^2}(\sin x - \cos x)\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[e^{t}(\sin t - \cos t) - (0-1)\right]$$

$$= \frac{e^{-t}}{2}\left[e^{t}(\sin t - \cos t) + 1\right]$$

$$= \frac{1}{2}(\sin t - \cos t + e^{-t})$$

(1) ~~Find~~ $L\{f(t)u(t-a)\} = e^{-as} L\{f(t+a)\}$

(2) $L\{f(t-a)u(t-a)\} = e^{-as} L\{f(t)\}$

$= e^{-as} F(s)$

Find (1) $\sin t \, u(t-2)$
(2) $t^3 u(t-1)$
(3) $e^{-t} \cos t \, u(t-2\pi)$

Ans→ (1) $\sin t \, u(t-2)$

To find → $\sin t \, u(t-2)$

$L\{\sin t \, u(t-2)\}$

$= L\{f(t)u(t-a)\}$ — L is small always

$a = 2$

$f(t) = \sin t$

$\therefore f(t+a) = \sin(t+a)$

$f(t+2) = \sin(t+2)$ .... $\sin A \cos B + \cos A \sin B$

$f(t+2) = \sin t \cos 2 + \cos t \sin 2$

$L\{f(t+2)\} = L\{\sin t(\cos 2) + \cos t(\sin 2)\}$

$= \cos 2 \, L\{\sin t\} + \sin 2 \, L\{\cos t\}$

$= \cos 2 \left[\frac{1}{s^2+1}\right] + \sin 2 \left[\frac{s}{s^2+1}\right]$

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express $f(t-a)\,u(t-a) = e^{as}\,\mathcal{L}\{f(t)\}$

$$f(t) = \begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$$

in terms of unity terms function and hence find its Laplace transform.

$$f(t) = (t-1)\,[u(t-1) - u(t-2)] + (3-t)\,[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) - (3-t)\,u(t-3)$$

$$= (t-1)\,u(t-1) - [(t-2)+1]\,u(t-2) - (t-2)\,u(t-2) - u(t-2) + (t-3)\,u(t-3)$$

$$= e^{-s}\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - u(t-2) - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2}$$

$$= \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2}$$

~~$(t+3)\,u(t-2)$~~

$$= -(t-3)\,u(t-2)$$

$$= -(t-3-1)$$