# Dynamic Programming

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## What is Dynamic programming?

- It is **algorithm design technique** where the subproblems of the **same form** occurs many time.
- We **store the result** and use it to **avoid** re-computation of sub-problems **repeatedly** encountered in getting solution to **original problem**.
- It is mostly used for **optimization problems**, problems where multiple solutions are there and we have to opt for **best solution ( define optimization problem, linear and non linear programming)**
- Storing the results of subproblems and using the tabulated/stored results of them is key feature of **dynamic programming.**

## Dynamic programming-general strategy

- Dynamic programming, like the **divide-and-conquer** method, solves problems by **combining** the **solutions to subproblems**.
- "**Programming**" in this context refers to a **tabular** method.
- **divide-and-conquer** algorithms **partition** the problem into disjoint subproblems, solve the subproblems recursively, and then **combine** their solutions to solve the **original problem**.
- In contrast, dynamic programming applies when the subproblems **overlap**—that is, **when subproblems share subsubproblems.** In this context, a divide-and-conquer algorithm does more work than necessary, repeatedly solving the common subsubproblems.
- A dynamic-programming algorithm solves each subsubproblem **just once** and then saves its answer in a **table**, thereby **avoiding** the work of recomputing the answer every time it solves each **subsubproblem**.

## Dynamic programming-general strategy

- We typically apply dynamic programming to ***optimization problems***. Such problems can have many possible solutions. Each solution has a value, and we wish to find a solution with the optimal (minimum or maximum) value.
- When developing a **dynamic-programming** algorithm, we follow a sequence of four steps:

1. Characterize the **structure of an optimal solution**.
2. **Recursively** define the value of an **optimal solution**.
3. **Compute** the value of an optimal solution, typically in a **bottom-up fashion.**
4. **Construct** an optimal solution from computed information

## Fibonacci sequence algorithm-Recursive version

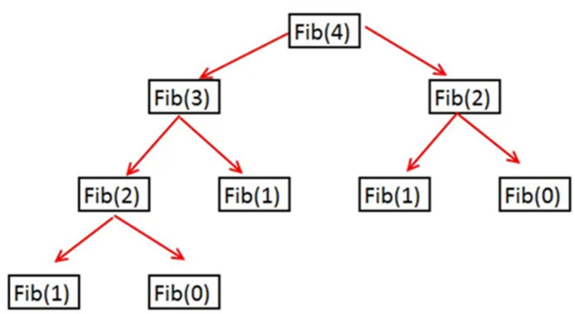
```
Int FibRec (int n)
{
    if (n==0)
       return 0;
    if(n==1)
       return 1;
    else
        f=FibRec(n-1)+FibRec(n-2);
          return f;
}
```

F={0,1,1,2,3,5,8,13,…….}

Given F(0)=0 and F(1)=1

F(n)=F(n-1)+F(n-2)

## Fibonacci sequence algorithm-Recursive version

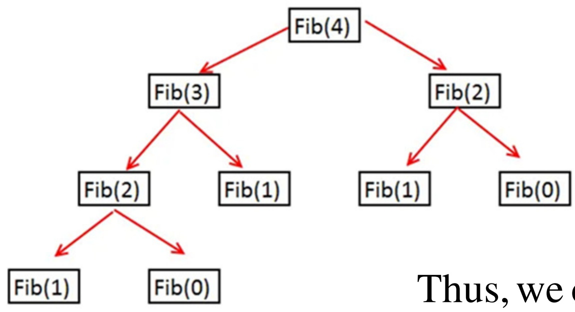



Fib(2) is called 2 times, thus recursive version is doing unnecessary work

**T(n) = T(n-1) + T(n-2) + 1**

## Fibonacci sequence algorithm-Recursive version performance

For computing Fib(n) we need $(2^{n-1}+1)$ calls




**T(n) = T(n-1) + T(n-2) + 1**

Thus, we can write

$$T(n) = O(2^{n-1}) + O(2^{n-2}) + k$$

$$\Rightarrow O(2^n)$$

## Fibonacci sequence algorithm- Dynamic prog.

```
Int FibDP(int n)
{
  int Fib[n+1]; //array of n+1
  Fib[0]=0; Fib[1]=1; //base
  for i=2 to n
     Fib[i]=Fib[i-1]+Fib[i-2];
  return Fib[n];
}
```

- **Fib[ ] is an array of size n+1**
- **At each iteration i ,previous 2 values (i-1) and (i-2) are added**
- **O(n) time and space complexity**

| Recursive | Dynamic Programming |
|---|---|
| Time exponential O(2^n) | Time linear O(n) |
| Space O(2^n) | Space O(n) |

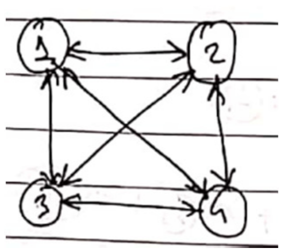
https://algorithms.tutorialhorizon.com/introduction-to-dynamic-programming-fibonacci-series/

## Travelling salesman Problem- Dynamic prog.

- A **salesman** has to visit **set of n** cities. Each city has to be visited **only once** and has to return to the same starting city, such that the distance travelled should be **minimum**.

## Travelling salesman Problem (TSP)- Dynamic prog.

- The dynamic programming approach is to solve the smaller subproblems first and use their stored solutions to get solutions for bigger problems, there by avoiding the repeated computations.




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 10 | 15 | 20 |
| 2 | 5 | 0 | 9 | 10 |
| 3 | 6 | 13 | 0 | 12 |
| 4 | 8 | 8 | 9 | 0 |

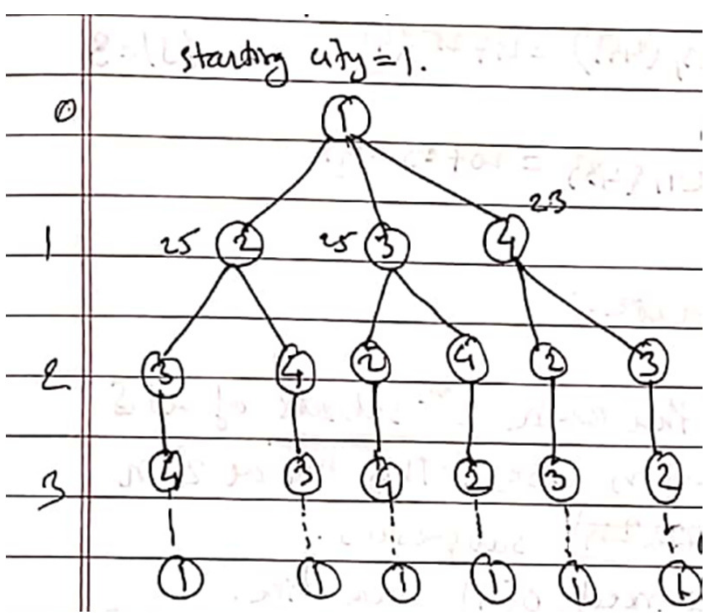
Cost adjacency matrix.

Matrix C

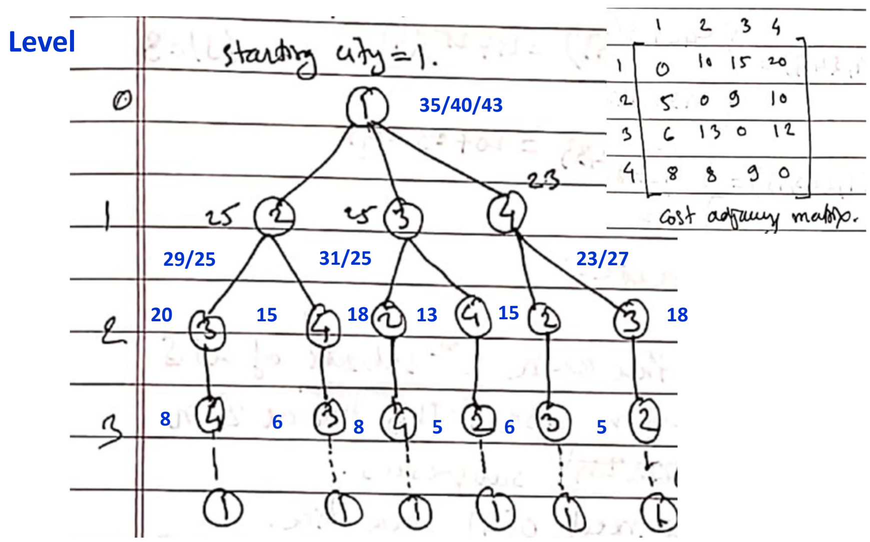
## Travelling salesman Problem (TSP)- Dynamic prog.

- Way of working/understanding
  - Will solve by using brute-force
  - Derive the formula
  - Apply the formula

## TSP- Using Dynamic prog.

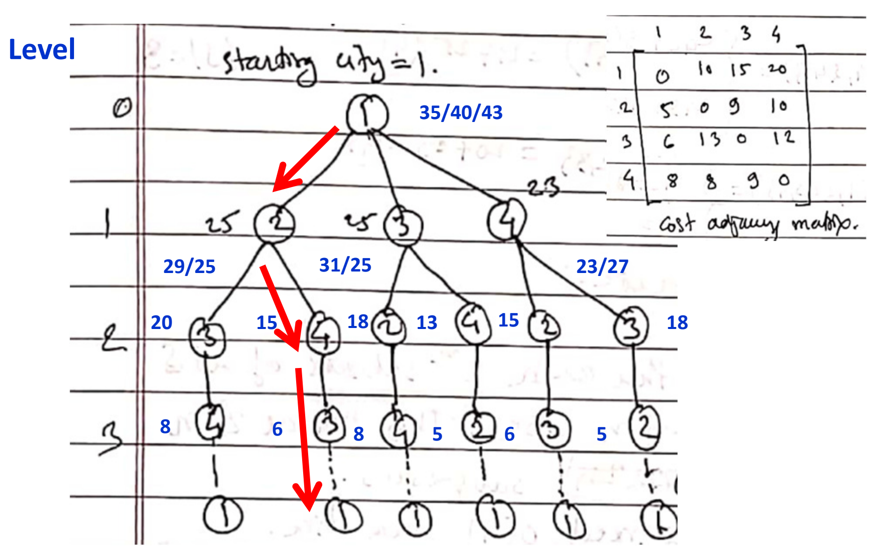


## Understanding TSP- Using Dynamic prog.




Need to be solved bottom up manner-brute force

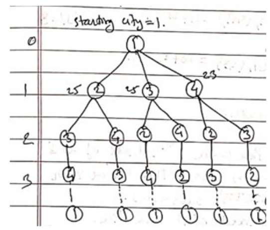
## Understanding TSP- Using Dynamic prog.




Need to be solved bottom up manner-brute force

## Understanding TSP- Using Dynamic prog.

- Lets generate formula for the above problem and then we will generalize it to any starting node.




$$g(1,\{2,3,4\}) = \min\begin{cases} C_{1,2} + g(2,\{3,4\}) \\ C_{1,3} + g(3,\{2,4\}) \\ C_{1,4} + g(4,\{2,3\}) \end{cases}$$

**g(1,{2,3,4})-** cost of going from node 1 and visiting each of the node from set s={2,3,4} once and returning back to node 1.

$$g(2,\{3,4\}) = \min\begin{cases} C_{2,3} + g(3,\{4\}) \\ C_{2,4} + g(4,\{3\}) \end{cases}$$

$$g(3,\{2,4\}) = \min\begin{cases} C_{3,2} + g(2,\{4\}) \\ C_{3,4} + g(4,\{2\}) \end{cases}$$

$$g(4,\{2,3\}) = \min\begin{cases} C_{4,2} + g(2,\{3\}) \\ C_{4,3} + g(3,\{2\}) \end{cases}$$

$$g(3,\{4\}) = C_{3,4} + g(4,\phi)$$

$g(4,\phi)$ – cost of going from 4 through each node in empty set and returning to node 1, which is same as $C_{4,1}$

$$g(4,\{3\}) = C_{4,3} + g(3,\phi)$$

## Understanding TSP- Using Dynamic prog.

$$g(i,S) = \min_{j \in S} \{c_{ij} + g(j, S - \{j\})\}$$

where

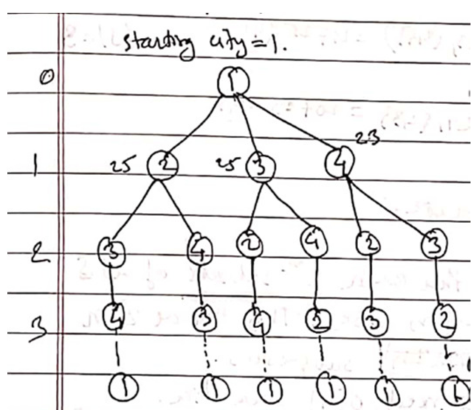
$g(i,S)$ - is the shortest path starting from $i$ and going through all the vertices in set $S$ and terminate at node $i$.

https://www.youtube.com/watch?v=XaXsJJh-Q5Y

## Understanding TSP- Using Dynamic prog.

**Starting from level 3 compute the costs, there are links between each of the node 2,3,4 to the city 1.**

$$g(2,\phi) = c_{21} = 5$$
$$g(3,\phi) = c_{31} = 6$$
$$g(4,\phi) = c_{41} = 8$$



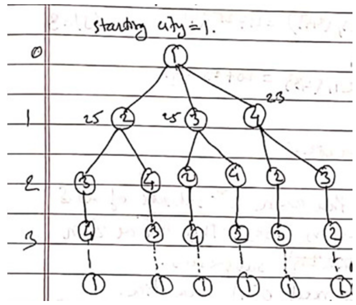

$$g(i,S) = \min_{j \in S} \{c_{ij} + g(j, S - \{j\})\}$$

$g(2,\phi)$ - means there are no intermediate nodes bet 2 and 1

# Understanding TSP- Using Dynamic prog.

**Computing costs at level 2**

$$g(i,S) = \min_{j \in S} \{ c_{ij} + g(j, S - \{j\}) \}$$




$$g(2,\{3\}) = c_{23} + g(3,\phi) = 9 + 6 = 15$$
$$g(2,\{4\}) = c_{24} + g(4,\phi) = 10 + 8 = 18$$
$$g(3,\{2\}) = c_{32} + g(2,\phi) = 13 + 5 = 18$$
$$g(3,\{4\}) = c_{34} + g(4,\phi) = 12 + 8 = 20$$
$$g(4,\{3\}) = c_{43} + g(3,\phi) = 9 + 6 = 15$$
$$g(4,\{2\}) = c_{42} + g(2,\phi) = 8 + 5 = 13$$

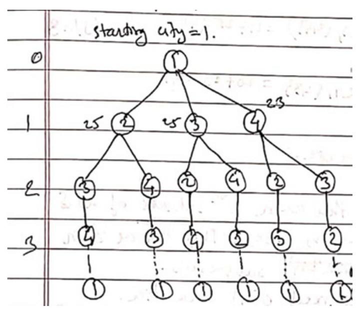
| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 10 | 15 | 20 |
| 2 | 5 | 0 | 9 | 10 |
| 3 | 6 | 13 | 0 | 12 |
| 4 | 8 | 8 | 9 | 0 |

Cost adjacency matrix.

# Understanding TSP- Using Dynamic prog.

**Computing costs at level 1**

$$g(i,S) = \min_{j \in S} \{ c_{ij} + g(j, S - \{j\}) \}$$




$$g(2,\{3,4\}) = \min \begin{Bmatrix} j=3 & c_{23} + g(3,\{4\}) = 9 + 20 = 29 \\ j=4 & c_{24} + g(4,\{3\}) = 10 + 15 = 25 \end{Bmatrix}$$

$$g(2,\{3,4\}) = 25$$

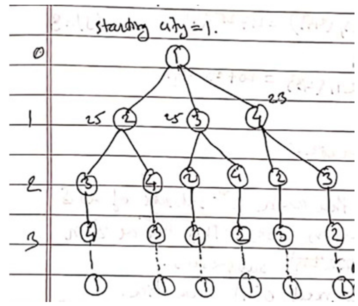
$$g(3,\{2,4\}) = \min \begin{Bmatrix} j=2 & c_{32} + g(2,\{4\}) = 13 + 18 = 31 \\ j=4 & c_{34} + g(4,\{2\}) = 12 + 13 = 25 \end{Bmatrix}$$

$$g(3,\{2,4\}) = 25$$

## Understanding TSP- Using Dynamic prog.

**Computing costs at level 1**

$$g(i,S)=\min_{j\in S}\{c_{ij}+g(j,S-\{j\})\}$$



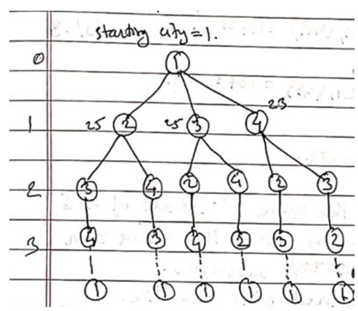

$$g(4,\{2,3\})=\min\begin{cases} j=2 & c_{42}+g(2,\{3\})=8+15=23 \\ j=3 & c_{43}+g(3,\{2\})=9+18=27 \end{cases}$$

$$g(4,\{2,3\})=23$$

## Understanding TSP- Using Dynamic prog.

**Computing costs at level 0**

$$g(i,S)=\min_{j\in S}\{c_{ij}+g(j,S-\{j\})\}$$

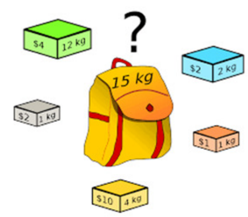



$$g(1,\{2,3,4\})=\min\begin{cases} j=2 & c_{12}+g(2,\{3,4\})=10+25=35 \\ j=3 & c_{13}+g(3,\{2,4\})=15+25=40 \\ j=4 & c_{14}+g(4,\{2,3\})=20+23=43 \end{cases}$$

$$g(1,\{2,3,4\})=35$$

**Final answer**

## 0/1 knapsack problem




- We have a knapsack with a capacity of weight W and a set of objects with their corresponding profits. We have to choose the set of objects with weights <=W and has to **maximize the profit.**
- 0/1 means we can either pick an object (1) or not pick(0) it. That is **partial object** is not allowed.
- An object can be **picked only once**.

## 0/1 knapsack problem

| Object | ob1 | ob2 | ob3 |
|---|---|---|---|
| weight | 2 | 4 | 8 |
| profit | 20 | 25 | 60 |

Given knapsack capacity M=12, find the objects to be put in the sack to get **maximum profit** with weight<=12

For each object two possibilities are there 1 or 0. Thus, for n objects we have 2^n possibilities.

# 0/1 knapsack problem-brute force approach

| Object | ob1 | ob2 | ob3 |
|---|---|---|---|
| weight | 2 | 4 | 8 |
| profit | 20 | 25 | 60 |

In this case 2^3=8 possibilities are there.

| objects | Weight | profit |
|---|---|---|
| 000 | 0 | 0 |
| 001 | 8 | 60 |
| 010 | 4 | 25 |
| 011 | 12 | 85 |
| 100 | 2 | 20 |
| 101 | 10 | 80 |
| 110 | 6 | 45 |
| 111 | 14 >12 (not possible) | |

# 0/1 knapsack problem- Dynamic Prog.

Weight W ->

M

Items considered-i

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

Increasing order

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | | | | | | | | |
| 2 | 0 | | | | | | | | |
| 3 | 0 | | | | | | | | |
| 4 | 0 | | | | | | | | |

M[i, w]=profit

**For i=0, no item is considered and thus, first row of M will be zero, since no profit from 0 objects**

**Weights={3,4,6,5}, Profit={2,3,1,4}, capacity W=8 and N=4**

## 0/1 knapsack problem- Dynamic Prog.

W ->

i

M

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | | | | | | | | |

Increasing order

Considering first object and ignore remaining objects

M[i, w]=profit

Weights={3,4,6,5}, Profit={2,3,1,4}, capacity W=8 and N=4

## 0/1 knapsack problem- Dynamic Prog.

W ->

i

M

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 0 | 0 | 0 | 2 | 3 | 3 | 3 | 5 | 5 |
| 3 | 0 | | | | | | | | |
| 4 | 0 | | | | | | | | |

Increasing order

Considering 1 and 2 nd object

M[i, w]=profit

Weights={3,4,6,5}, Profit={2,3,1,4}, capacity W=8 and N=4

## 0/1 knapsack problem- Dynamic Prog.

W -> i

**M**

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | **2** | **2** | **2** | **2** | **2** | **2** |
| 2 | 0 | 0 | 0 | 2 | **3** | **3** | **3** | **5** | **5** |
| 3 | 0 | 0 | 0 | 2 | 3 | **4** | **4** | **5** | **6** |
| 4 | 0 | | | | | | | | |

Increasing order

Considering 1,2 and 3rd object

**M[i, w]=profit**

**Weights={3,4,6,5}, Profit={2,3,1,4}, capacity W=8 and N=4**

## 0/1 knapsack problem- Dynamic Prog.

W -> i

**M**

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | **2** | **2** | **2** | **2** | **2** | **2** |
| 2 | 0 | 0 | 0 | 2 | **3** | **3** | **3** | **5** | **5** |
| 3 | 0 | 0 | 0 | 2 | 3 | **4** | **4** | **5** | **6** |
| 4 | 0 | 0 | 0 | 2 | 3 | 4 | **4** | **5** | **6** |

Final answer

Increasing order

Considering 1,2, 3 and $4^{th}$ object

**M[i, w]=profit**

**Weights={3,4,6,5}, Profit={2,3,1,4}, capacity W=8 and N=4**

## 0/1 knapsack problem- Dynamic Prog.-Philosophy

- The optimal solution to the overall problem depends upon the optimal solution to its subproblems. This simple optimization reduces time complexities from exponential to polynomial.

## 0/1 knapsack problem- Dynamic Prog.

W ->

i ↓

M

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 0 | 0 | 0 | 2 | 3 | 3 | 3 | 5 | 5 |
| 3 | 0 | 0 | 0 | 2 | 3 | 4 | 4 | 5 | 6 |
| 4 | 0 | 0 | 0 | 2 | 3 | 4 | 4 | 5 | 6 |

$$M[i,w] = \max\left(M[i-1,w], M[i-1,w-w[i]] + p[i]\right)$$

Where **p[i]-** is **profit** from **i th object**

$$M[i,w] = \max\left(M[i-1,w], M[i-1,w-w[i]] + p[i]\right)$$
$$M[2,4] = \max\left(M[1,4], M[1,4-4] + p[2]\right)$$
$$M[2,4] = \max\left(M[1,4], M[1,0] + 3\right)$$
$$M[2,4] = \max\left(2, 0+3\right)$$
$$M[2,4] = 3$$

$$M[i,w] = \max\left(M[i-1,w], M[i-1,w-w[i]] + p[i]\right)$$
$$M[2,7] = \max\left(M[1,7], M[1,7-4] + p[2]\right)$$
$$M[2,7] = \max\left(M[1,7], M[1,3] + 3\right)$$
$$M[2,7] = \max\left(2, 2+3\right)$$
$$M[2,7] = 5$$

# 0/1 knapsack problem- Dynamic Prog.

W ->

i

| pi | wi |
|---|---|
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 1 | 6 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| 2 | 0 | 0 | 0 | 2 | 3 | 3 | 3 | 5 | 5 |
| 3 | 0 | 0 | 0 | 2 | 3 | 4 | 4 | 5 | 6 |
| 4 | 0 | 0 | 0 | 2 | 3 | 4 | 4 | 5 | 6 |

Deciding the objects in the sack

**X={0,0,0,0}** **X={0,0,1,0}**

**Profit from 3rd object is 4.**
**6-4=2 profit, check 2 in 2nd row**

**X={1,0,1,0}**

https://www.youtube.com/watch?v=nLmhmB6NzcM

Time complexity of Dynamic programming is
O(n*w) (polynomial time), where
n- number of objects and W is the knapsack capacity. Using
Brute-force approach is O(2^n) (exponential time)

**Solve 0/1 knapsack problem using Dynamic Prog.**

**Weights={2,3,4,5}, Profit={1,2,5,6}, capacity W=8 and N=4**

## TSP-Time complexity

For a pnroblem of size n cities, there can be $2^n$ subsets of set S.
Each subset contains max n cities. Thus, there are $2^n.n$ subproblems.
To solve each problem we need O(n) linear time, thus

$$\mathrm{T(n)} = O(2^n.n.n) = O(2^n.n^2)$$