$\rightarrow$ $a_1 = a > 0$ $\qquad a_{n+1} = \dfrac{3+2a_n}{2+a_n} \quad \forall n \quad$ —(1)

Step I If $\langle a_n \rangle$ converges to say l, then

$a_{n+1} \to l$ & by AOL

then $3+2a_n \to 3+2l$

$2+a_n \to 2+l \qquad (\because l+2 \neq 0)$

Reason $a_n > 0 \quad \forall n$

Proof $a_1 = a > 0$ Let $a_k > 0$ (by Induction hypo)

TS $a_{k+1} > 0$

ie $\dfrac{3+2a_k}{2+a_k} > 0$

ie $3+2a_k > 0$ (mpm $\because a_k > 0$, $2+a_k > 0$ $(\because 2 > 0)$)

ie $a_k > -\dfrac{3}{2}$

which is true $\because a_k > 0 > -\dfrac{3}{2}$

$\therefore$ by PMI $a_n > 0 \; \forall n$

$\therefore l \geq 0$ (limits preserve order)

$\therefore l+2 > 0 \quad (\because 2 > 0)$

taking limits in (1)

$$l = \dfrac{3+2l}{2+l}$$

$$l^2 + 2l = 3 + 2l$$

$$l = \pm\sqrt{3}$$

But $l \geq 0 \quad \therefore l = \sqrt{3}$

Thus a_n converges $\Rightarrow$ its limit is $\sqrt{3}$

* We now show $\langle a_n \rangle$ indeed converges

Step II (a) ~~Since~~ Case (i) $a_1 > \sqrt{3}$

Since $a_1 > \sqrt{3}$

We show $a_n > \sqrt{3}$ $\forall n$ by Inductn

Assuming $a_k > \sqrt{3}$

T.S $a_{k+1} > \sqrt{3}$

$$\frac{3+2a_k}{2+a_k} > \sqrt{3}$$

$$3+2a_k > 2\sqrt{3} + a_k\sqrt{3}$$

$$a_k(2-\sqrt{3}) > 2\sqrt{3}-3$$

$$a_k(2-\sqrt{3}) > \sqrt{3}(2-\sqrt{3})$$

$$a_k > \sqrt{3} \quad (\because 2-\sqrt{3} > 0)$$

which is true (by Inductn hypo)

$\therefore$ by PMI $a_n > \sqrt{3}$ $\forall n$

$\therefore$ a_n is b b — (2)

(b) T.S a_n is $\downarrow$

ie $a_n > a_{n+1}$ $\forall n$

ie $a_n > \dfrac{3+2a_n}{2+a_n}$ $\forall n$

ie $2a_n + a_n^2 > 3 + 2a_n$ $\forall n$

ie $(a_n+\sqrt{3})(a_n-\sqrt{3}) > 0$ $\forall n$

which is true [$\because a_n > \sqrt{3}$ $\forall n$ proved above]

$\therefore$ $\langle a_n \rangle$ is $\downarrow$ — (3)

by (2), (3) & cor to MCT

$\langle a_n \rangle$ converges

$\therefore$ by Step I its limit is $\sqrt{3}$

Case (ii) $a_1 < \sqrt{3}$

Since $a_1 < \sqrt{3}$

We show $a_n < \sqrt{3}$ $\forall n$ by Induction

Assuming $a_k < \sqrt{3}$ (by inductn hypo)

T.S $a_{k+1} < \sqrt{3}$

$= IB$

ie $\dfrac{3+2a_k}{2+a_k} < \sqrt{3}$

ie $3+2a_k < 2\sqrt{3} + a_k\sqrt{3}$

ie $a_k(2-\sqrt{3}) < \sqrt{3}(2-\sqrt{3})$

ie $a_k < \sqrt{3}$ (MPM $\because 2-\sqrt{3} > 0$)

which is true

$\therefore \langle a_n \rangle$ is ba — (4)

~~Step~~ T.S $\langle a_n \rangle$ is $\uparrow$

i.e $a_n < a_{n+1}$ $\forall n$

ie $a_n < \dfrac{3+2a_n}{2+a_n}$ $\forall n$

ie $2a_n + a_n^2 < 3 + 2a_n$ $\forall n$

ie $a_n^2 < 3$ $\forall n$

ie $(a_n+\sqrt{3})(a_n-\sqrt{3}) < 0$ $\forall n$

which is true [Reason $a_n > 0 > -\sqrt{3}$ $\forall n$ (proved earlier)

$a_n < \sqrt{3}$ $\forall n$ (proved above)

$\therefore \langle a_n \rangle$ is $\uparrow$ — (5)

by (4) & (5) & MCT

$\langle a_n \rangle$ converges

by Step I its limit is $\sqrt{3}$