* no. of features = 2
* no. of examples = 4

① **Implementation of mean of variables :-**

$$\bar{x} = (4+8+13+7)/4 = 8$$

$$\bar{y} = (11+4+5+14)/4 = 8.50$$

② **Computation of Covariance Matrix :-**

no. of features = 2 ⟹ $(x, y)$

∴ ordered pairing : $(x,x)$ $(x,y)$ $(y,x)$ $(y,y)$

$$\boxed{Cov(a,b) = \underbrace{\frac{1}{N-1}}_{\text{no. of examples}} \sum_{k=1}^{N} (\underbrace{a_k}_{\text{example } k \text{ of } a} - \overbrace{\bar{a}}^{\text{mean of } a})(\underbrace{b_k}_{\text{example } k \text{ of } b} - \underbrace{\bar{b}}_{\text{mean of } b})}$$

* $$Cov(x,x) = \frac{1}{4-1}\left[(4-8)^2 + (8-8)^2 + (13-8)^2 + (7-8)^2\right]$$

$$= \frac{1}{3}\left[16+0+25+1\right] = \frac{42}{3}$$

$$Cov(x,x) = 14$$

* $$Cov(x,y) = \frac{1}{4-1}\left[(4-8)(11-8.5) + 0 + (13-8)(5-8.5) + (1)(14-8.5)\right]$$

$$= \frac{1}{3}\left[-10 - 17.5 + 5.5\right] = \frac{-22}{3}$$

$$Cov(x,y) = -11$$

$\ast$ $Cov(y,x) = Cov(x,y)$

$\Rightarrow \boxed{Cov(y,x) = -11}$

$\ast$ $Cov(y,y) = \frac{1}{3}\left[(2.5)^2 + (-3.5)^2 + (5.5)^2 + (-0.5)^2\right]$

$= \frac{1}{3}(6.25 + 12.25 + 30.25 + 0.25)$

$= \frac{69}{3} \Rightarrow \boxed{Cov(y,y) = 23}$

$\therefore$ Covariance matrix is :-

$$\begin{bmatrix} Cov(x,x) & Cov(x,y) \\ Cov(y,x) & Cov(y,y) \end{bmatrix} \Rightarrow \begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix}$$

③ Calculate Eigen values :-

$$\det(\underbrace{S}_{\text{Covariance matrix}} - \underbrace{\lambda}_{\text{Eigen values}} I) = 0$$

$\ast$ $$\det\left(\begin{bmatrix} 14 & -11 \\ -11 & 23 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} 14-\lambda & -11 \\ -11 & 23-\lambda \end{vmatrix} = 0$$

$\Rightarrow (14-\lambda)(23-\lambda) - (121) = 0$

$\Rightarrow 322 - 14\lambda - 23\lambda - \lambda^2 - 121 = 0$

$\Rightarrow \lambda^2 - 37\lambda + 201 = 0$ → solve this eqn for $\lambda_1$ & $\lambda_2$

roots $= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\boxed{\lambda_1 = 30.3849}$ $\boxed{\lambda_2 = 6.6181}$

* first principle component ↓

$$\lambda_1 > \lambda_2$$

∴ first principle component (largest eigen value)

## (4) Calculate Eigen vector for FPC :-

formula :- $(S-\lambda_1 I)(U_1) = 0$ → Eigen vector for $\lambda_1$

$$\begin{bmatrix} 14-\lambda_1 & -11 \\ -11 & 23-\lambda_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} (14-\lambda_1) u_1 - 11 u_2 \\ -11 u_1 + (23-\lambda_1) u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (14-\lambda_1) u_1 - 11 u_2 = 0$$

$$\Rightarrow -11 u_1 + (23-\lambda_1) u_2 = 0$$

$$\Rightarrow (14-\lambda_1) u_1 = 11 u_2$$

$$\Rightarrow \boxed{\frac{u_1}{11} = \frac{u_2}{14-\lambda_1} = t}$$

for t = 1 :

$$u_1 = 11 \quad , \quad u_2 = 14-\lambda_1$$

∴ Eigen vector $(U_1)$ : $\begin{bmatrix} 11 \\ 14-\lambda_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 11 \\ -16.3849 \end{bmatrix}$

## (5) Normalize the eigen value $(U_1)$ :-

$$e_1 = \begin{bmatrix} \dfrac{11}{\sqrt{11^2 + (-16.3849)^2}} \\ \dfrac{-16.3849}{\sqrt{11^2 + (-16.3849)^2}} \end{bmatrix} = \begin{bmatrix} -0.5574 \\ -0.8303 \end{bmatrix}$$

## ⑥ Derive New dataset :-

| | Ex1 | Ex2 | Ex3 | Ex4 |
|---|---|---|---|---|
| first principle Component | $P_{11}$ | $P_{12}$ | $P_{13}$ | $P_{14}$ |

$$P_{ij} = e_i^T \begin{bmatrix} (\text{Example}_i)_a - \text{mean}(\text{Example})_a \\ (\text{Example}_i)_b - \text{mean}(\text{Example})_b \end{bmatrix}$$

* $P_{11} = e_1^T \begin{bmatrix} 4 - \text{mean}(x) \\ 11 - \text{mean}(y) \end{bmatrix} = e_1^T \begin{bmatrix} 4-8 \\ 11-8.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.5574 & -0.8303 \end{bmatrix} \begin{bmatrix} -4 \\ 2.5 \end{bmatrix}$

* $\boxed{P_{11} = -4.3052}$

IIIly * $\boxed{P_{12} = 3.7361}$ * $\boxed{P_{13} = 5.6928}$ * $\boxed{P_{14} = -5.1238}$

∴ new dataset

| | Ex1 | Ex2 | Ex3 | Ex4 |
|---|---|---|---|---|
| $PC_1$ | −4.3052 | 3.7361 | 5.6928 | −5.1238 |